

COUPLING MISALIGNMENT FORCES

by

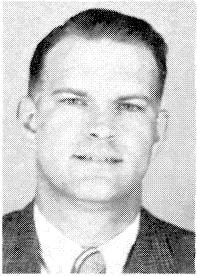
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C. B. Gibbons joined Bendix in 1957, having received his degree from the University of Rochester. The major portion of his career has involved the design of flexible diaphragm couplings for both aircraft and industrial service. He currently has the technical responsibility for the Bendix Contoured Flexible Diaphragm Couplings.

Mr. Gibbons' involvement with couplings has taken him into other disciplines, such as sales, manufacturing and field service. He is a member of the ASME Subcommittee on Couplings and has written and presented several papers on couplings to various organizations.

ABSTRACT

Misalignment of machinery shafts causes reaction forces to be generated in the coupling which affect the machines and are often a major cause of machinery vibration. The reaction forces generated by the couplings are described for each type of coupling in current use and especially for the several types of nonlubricated couplings which are seeing increased usage.

Comparative values of these forces are presented in graph and tabular form for ease of comprehension and for reference use by the reader to evaluate specific machinery/coupling applications.

The effect that these forces have upon machines is described in general and in certain specific field examples where even diaphragm coupling forces had to be reduced to permit satisfactory machinery operation.

Nomenclature

- Tq = Torque, Lb-In
- Cf = Coefficient of Friction
- PD = Gear Coupling Pitch Diameter, Inches
- W = Gear Coupling Tooth Width, Inches
- Kb = Flexure Coupling Bending Spring Rate per Diaphragm or per Disk Pack, Lb-In/Deg
- Ka = Flexure Coupling, Axial Spring Rate, Linear Portion, Lb/In
- KA = Flexure Coupling, Axial Spring Rate, Non-Linear Portion, Lb/In³
- M = Moment, Lb-In
- F = Force, Lb

INTRODUCTION

Flexible couplings are necessary to connect turbomachines to their drivers or loads. These couplings transmit the driving torque while accommodating the unavoidable misalignments between the two machines.

The effect that misalignment has on the machines is a function of the compliance of the flexible coupling. All torsionally loaded misaligned couplings have restoring moments which tend to bow the machine shafts. The amount of bowing, which can cause machine vibration, increases with higher speed shafts which are carrying higher torques for a given size machine.

The reaction forces from the couplings depend upon how they accommodate misalignment. The two basic groups of couplings are mechanical misalignment and flexing misalignment.

Mechanical Misalignment

This group comprises couplings which allow for misalignment by mechanical clearance between the driving elements. The only commonly used member of this group on turbomachinery is the lubricated gear coupling.

Gear couplings have no restoring moments and forces until they are torsionally loaded and their reaction forces are then a function of torque, misalignment angle and friction.

Flexing Misalignment

This group consists of couplings which allow for misalignment by flexure of an element within the coupling. This flexing element can be non-metallic or metallic, however, metallic flexures predominate on turbomachines.

Flexure couplings have an inherent restoring moment due to the spring rate of the flexure even at zero torque, however, their reaction forces do not increase markedly with torque and are not affected by friction.

TYPICAL PROBLEMS

Gear Coupling Forces

A steam turbine driven 13,000 HP/5700 RPM boiler feed pump was connected with a No. 5 continuous lube gear coupling with a 12 inch spacer. There are four of these pumps in a fossil fuel power plant with similar, but not identical piping arrangements. All four were running with excessive vibration, but one was unusually severe.

Misalignment was the obvious culprit, but attempts to position the pump to compensate for misalignment did not correct the vibration apparently because changing load conditions changed the pressure piping and exhaust duct piping forces, moving both the pump and the turbine.

The misalignment moment on the worst pump with the gear coupling was approximately 63,000 in-lb at the cold installed position and was still about 39,000 in-lb while hot. The resulting vibration was approximately 5.0 mils D.A. measured by a shaft proximity probe.

The pump manufacturer installed a diaphragm coupling on this pump, which reduced the misalignment moment to 1700 lb-in cold and 1100 lb-in hot for the same amount of misalignment.

This 35 fold decrease in misalignment force resulted in a reduction in pump vibration to less than 1.0 mil D.A. on the pump shaft probe.

Diaphragm couplings were installed on the remaining three pumps which corrected their high vibration levels as well.

Diaphragm Coupling Forces

An electric motor-gear driven centrifugal air compressor train had a small high speed, 1190 HP/15,300 RPM, compressor at the end of the train. This compressor was driven by a small, high ratio, 1.8 to 1, step up gear box. The high speed pinion weighed 20 pounds and was supported in cylindrical journal bearings.

Vibration levels on the machine were too high with the original gear couplings, so the compressor manufacturer tried a diaphragm coupling and when the machine was started with no load, the vibration levels were even higher.

Subsequent investigation revealed that the misalignment force from the series 306 diaphragm coupling, approximately 23 pounds, was sufficient to unload the pinion journal bearing when there was little reaction force from the lightly loaded gear mesh.

The unloaded bearing caused whirling of the high speed pinion, resulting in excessive vibration. Since the machine had to be able to operate with minimal load, the condition was unacceptable.

A more flexible 206 series diaphragm coupling was installed, which reduced the misalignment force to about 6 pounds and a pressure dam was added to the bearing to augment the bearing load. These changes corrected the no load vibration conditions and enabled satisfactory operation of the machine.

GENERAL CASE OF REACTION FORCES

Figure 1 depicts two machine shaft centerlines, Z1 and Z2, which are misaligned both vertically, horizontally and are parallel to each other. The centerline of the coupling spacer is shown connecting the two shaft centerlines with the intersection points being the coupling centers of articulation, *not the shaft's ends*. For an existing machine, the values and directions of displacements ΔX1, ΔY1, ΔX2 and ΔY2 can be readily obtained from a graphical plot of reverse indicator readings. For a machine in the planning stage, realistic values should be chosen for comparative calculations.

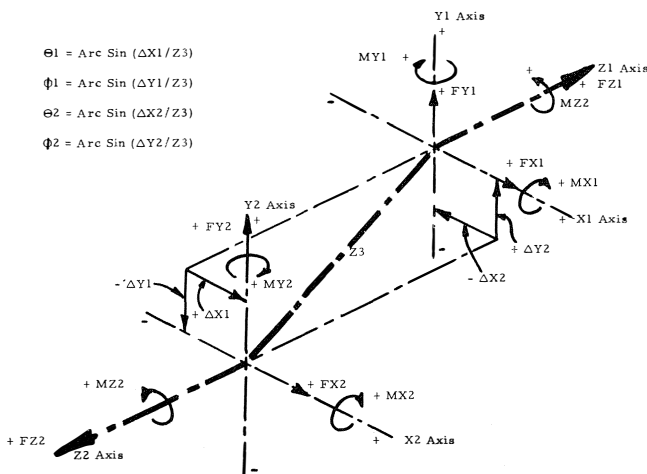


Figure 1. Coupling Coordinate System.

The misalignment angles Θ1, Φ1, Θ2 and Φ2 are computed using the equations of Figure 1. These misalignment angles are used to compute the three moments MX, MY and MZ and the three forces FX, FY and FZ which the coupling exerts on the machine's shafts. Assume that Z1 is the axis of the driving machine, that (+) torque is applied as shown and that rotation is in same direction as applied torque. Care must be used to follow the sign convention shown in Figure 1.

Gear Couplings

$$\begin{aligned}
 MX1 &= Tq \left\{ \sin \Theta 1 + Cf(\Theta 1 / \sqrt{\Theta 2^2 + \Phi 1^2}) + \right. \\
 &\quad \left. (W/PD)Q(\Phi 1 / \sqrt{\Theta 1^2 + \Phi 1^2}) \right\} \\
 MY1 &= Tq \left\{ \sin \Phi 1 + Cf(\Phi 1 / \sqrt{\Theta 1^2 + \Phi 1^2}) - \right. \\
 &\quad \left. (W/PD)Q(\Theta 1 / \sqrt{\Theta 1^2 + \Phi 1^2}) \right\} \\
 MZ1 &= Tq \\
 MX2 &= Tq \left\{ \sin \Theta 2 + Cf(\Theta 2 / \sqrt{\Theta 2^2 + \Phi 2^2}) - \right. \\
 &\quad \left. (W/PD)Q(\Phi 2 / \sqrt{\Theta 2^2 + \Phi 2^2}) \right\} \\
 MY2 &= Tq \left\{ \sin \Phi 2 + Cf(\Phi 2 / \sqrt{\Theta 2^2 + \Phi 2^2}) + \right. \\
 &\quad \left. (W/PD)Q(\Theta 2 / \sqrt{\Theta 2^2 + \Phi 2^2}) \right\} \\
 MZ2 &= -Tq \\
 FX1 &= (-MY1 - MY2)/Z3 \\
 FY1 &= (+MX1 + MX2)/Z3 \\
 FZ1 &= \pm Tq (Cf/PD), \text{ Dynamic only, sign depends upon} \\
 &\quad \text{direction of motion.} \\
 FX2 &= -FX1 \\
 FY2 &= -FY1 \\
 FZ2 &= \pm Tq (Cf/PD)
 \end{aligned}$$

The value of Q is a function of the non-uniform tooth loading and can vary from 0 for a perfectly aligned coupling to 1.0 for a straight tooth coupling with significant misalignment.

The value of Q is affected by pitch diameter, misalignment angle and torque loading. Figure 2 is a family of curves of values of Q for differing pitch diameters. The curves are based upon a nominal torque loading of 600 pounds per linear inch of tooth. Figure 3 shows the increase in the value of Q for reduced torque levels.

Flexure Couplings

$$\begin{aligned}
 MX1 &= Tq \sin \Theta 1 + Kb \Phi 1 \\
 MY1 &= Tq \sin \Phi 1 - Kb \Theta 1 \\
 MZ1 &= Tq \\
 MX2 &= Tq \sin \Theta 2 - Kb \Phi 2
 \end{aligned}$$

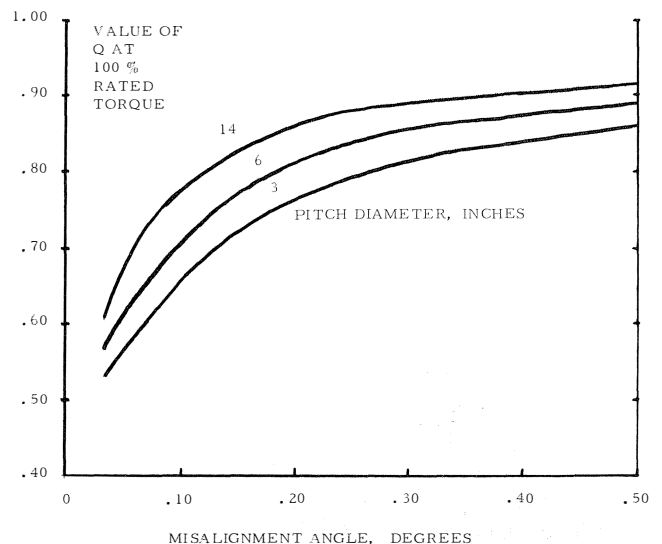


Figure 2. Alignment Load Factor (Q) as a Function of Misalignment Angle.

$$\begin{aligned}
 MY2 &= T_q \sin \Phi_2 + K_b \Theta_2 \\
 MZ2 &= -T_q \\
 FX1 &= (-MY1 - MY2)/Z3 \\
 FY1 &= (+MX1 + MX2)/Z3 \\
 FZ1 &= KA(\Delta Z)^3 + Ka\Delta Z, \Delta Z = \text{Stretch (+) or} \\
 &\quad \text{compression (-) of} \\
 &\quad \text{complete coupling from} \\
 &\quad \text{its free length.} \\
 FX2 &= -FX1 \\
 FY2 &= -FY1 \\
 FZ2 &= FZ1
 \end{aligned}$$

Many of the flexure couplings have non-linear spring rates in one or more deflection modes. The following description is general and specific information concerning the value and linearity of the spring rates of a specific coupling make, model and size must be obtained from the coupling manufacturer.

Diaphragm Couplings

Contoured thickness diaphragm couplings exhibit a bending spring rate that is linear. Such couplings with a flat midplane have a nonlinear axial spring rate while wavy midplaned couplings exhibit linear axial spring rates. Multi-disk, convoluted, constant thickness diaphragm couplings are nonlinear and linear in their bending and axial spring rates, respectively.

Annular Disk Couplings

These multi-disk, constant thickness couplings are of two types:

- a. Constant cross section, with nonlinear bending and axial spring rates.
- b. Tapered cross section, with nearly linear bending and axial spring rates.

CONCLUSIONS

An analysis of various types of couplings was made, as detailed in Appendix A. The couplings considered were:

- a. gear coupling,
- b. convoluted, multi-disk diaphragm coupling with constant thickness,
- c. annular disk coupling, and
- d. diaphragm coupling with contoured thickness and flat midplane.

Figure 4 shows the bending moment on the machine shaft at the hub/shaft junction. It is clear that the gear coupling (a) places the maximum bending moment on the machine shaft as

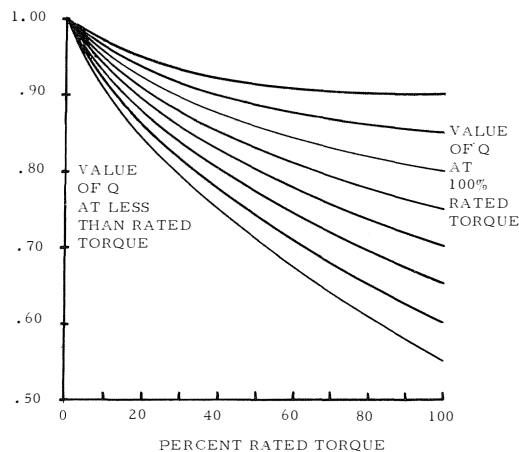


Figure 3. Alignment Load Factor Variations as a Function of Torque.

against the lowest moment due to the flexure coupling (d). Referring to Figure 5, the deflection (bowing) of the machine shaft from bearing center to the shaft end, when gear coupling is employed, is seen to be about ten times that when the coupling (c) is in use. Figure 6 is an account of the axial forces present, with the four types of couplings mentioned. Coupling (b) is seen to experience the least amount of axial forces. As shown in the figure, the axial force transmitted is not a function of deflection. The particular flexure couplings chosen for the comparison could be modified by the coupling manufacturers to further reduce the moments and forces for a specific application if required.

APPENDIX A

Effect of Forces and Moments

A comparative example of two machines connected by first, a gear coupling and secondly, a flexure coupling, provide a numerical example of the force and moment calculations and their effect.

A moderate sized, fictitious machine is chosen for the example.

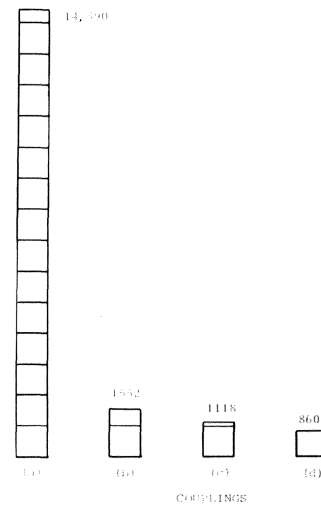


Figure 4. Bending Moment on Machine Shaft at Hub/Shaft Junction (lb-in).

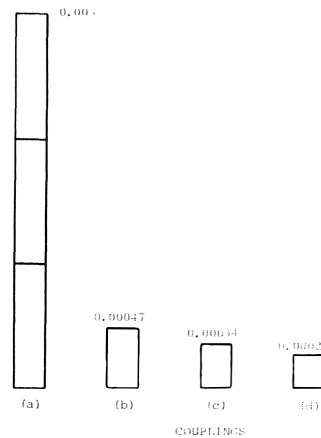


Figure 5. Deflection (bowing) of Machine Shaft From Bearing Center to Shaft End (inches).

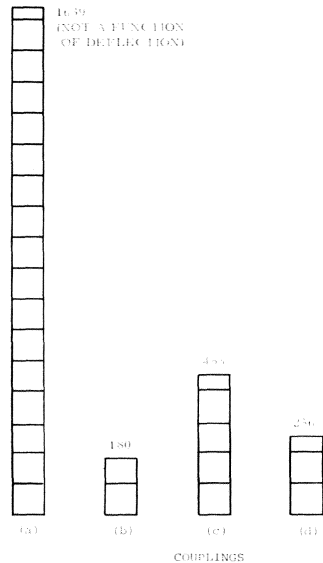


Figure 6. Axial Force at 0.050 Inch Axial Deflection (pounds).

Power, 6500 HP/7500 RPM; 54,622 Lb-In Torque
 Shaft Diameters, 3.00 Inches Each Machine
 Shaft Separation, 10.00 Inches
 Shaft End to Bearing Center, 8.00 Inches
 Misalignment, Parallel Only,

$$\Delta X1 = .023 \text{ Inches} \quad \Delta X2 = -.023 \text{ Inches}$$

$$\Delta Y1 = -.031 \text{ Inches} \quad \Delta Y2 = +.031 \text{ Inches}$$

(a) Gear Coupling

5.00 Inch Pitch Diameter, 60 Teeth, 1.00 Inch Face Width, 3.60 Inch Hub Length, 1.65 Inch Tooth Centerline to Shaft End. The distance between tooth centerlines (centers of articulation), Z3 is 13.30 Inches.

$$\Theta 1 = \text{Arc Sin}(\Delta X1/Z3) = \text{Arc Sin}(.023/13.30) = +.099 \text{ Deg}$$

$$\Phi 1 = \text{Arc Sin}(\Delta Y1/Z3) = \text{Arc Sin}(-.031/13.30) = -.134 \text{ Deg}$$

$$\Theta 2 = \text{Arc Sin}(\Delta X2/Z3) = \text{Arc Sin}(-.023/13.30) = -.099 \text{ Deg}$$

$$\Phi 2 = \text{Arc Sin}(\Delta Y2/Z3) = \text{Arc Sin}(+.031/13.30) = +.134 \text{ Deg}$$

$$\Theta 1/\sqrt{\Theta 1^2 + \Phi 1^2} = .594$$

$$\Theta 2/\sqrt{\Theta 2^2 + \Phi 2^2} = -.594$$

$$\Phi 1/\sqrt{\Theta 1^2 + \Phi 1^2} = -.804$$

$$\Phi 2/\sqrt{\Theta 2^2 + \Phi 2^2} = .804$$

Coefficient of friction, Cf, is assumed at 0.15 for calculations. Values ranging from 0.05 to 0.35 have been suggested by others as being realistic.

Q from Figure 2 for a straight toothed coupling at 0.167 degrees and 364 pounds per linear inch of tooth is approximately 0.83 for a (W/PD) value of 0.20.

$$MX1 = 54622 \{ (.00173) + .15 (.594) + (.20) (.83) (-.804) \} = -2329 \text{ Lb-In}$$

$$MY1 = 54622 \{ (-.00234) + .15 (-.804) - (.20) (.83) (.594) \} = 12,101 \text{ Lb-In}$$

$$MZ1 = 54622 \text{ Lb-In}$$

$$MX2 = 54622 \{ (-.00173) + .15 (-.594) - (.20) (.83) (.804) \} = -12,251 \text{ Lb-In}$$

$$MY2 = 54622 \{ (.00234) + .15 (.804) + (.20) (.83) (-.594) \} = +1329 \text{ Lb-In}$$

$$MZ2 = 54622$$

$$FX1 = \{ -(-12101) - (+1329) \} / 13.30 = +810 \text{ Lbs}$$

$$FY1 = \{ -(-2329) - (-12251) \} / 13.30 = -1096 \text{ Lbs}$$

$$FX2 = -810 \text{ Lbs}$$

$$FY2 = +1096 \text{ Lbs}$$

$$FZ1 = \pm 54622 (.15/5) = \pm 1639 \text{ Lbs}$$

$$FZ2 = \pm 1639 \text{ Lbs}$$

Bending Moment at Hub End, M

$$M1 = \frac{\sqrt{\{FX1(3.60 - 1.65) - MY1\}^2 + \{FY1(3.60 - 1.65) + MX1\}^2}}$$

$$M1 = 14391 \text{ Lb-In at Z1 Axis}$$

$$M2 = \frac{\sqrt{\{FX2(3.60 - 1.65) + MY2\}^2 + \{FY2(3.60 - 1.65) - MX2\}^2}}$$

$$M2 = 14390 \text{ Lb-In at Z2 Axis}$$

Bending Stress at Hub End, Sb

$$Sb = Mr/I \quad r = \text{Shaft Radius } 1.50 \text{ In.}$$

$$I = \pi r^4/4, 3.976 \text{ In}^4$$

$$Sb = 5429 \text{ PSI at Z1 Axis}$$

$$Sb = 5429 \text{ PSI at Z2 Axis}$$

Deflection of Shaft From Bearing Center to Shaft End

$$\delta X1 = \{ FX1(Z1a)/3 - (MY1)/2 \} \{ Z1a^2 + Z1b \} / EI$$

Z1a = Center of Articulation to Bearing Center, 6.35 Inches

Z1b = Center of Articulation to Shaft End, 1.65 Inches

$$\delta X1 = +.0027$$

$$E = 30,000,000 \text{ PSI}$$

$$\delta Y1 = \{ FY1(Z1a)/3 + (MX1)/2 \} \{ Z1a^2 + Z1b \} / EI$$

$$\delta Y1 = -.0012$$

$$\text{Total Deflection } \delta 1 = \sqrt{X1^2 + Y1^2}$$

$$\delta 1 = .0030$$

$$\delta X2 = \{ FX2(Z2a)/3 + (MY2)/2 \} \{ Z2a^2 + Z2b \} / EI$$

$$\delta X2 = -.0004$$

$$\delta Y2 = \{ FY2(Z2a)/3 - (MX2)/2 \} \{ Z2a^2 + Z2b \} / EI$$

$$\delta Y2 = +.0030$$

$$\delta 2 = .0030$$

Diaphragm Coupling, Constant Thickness, Multi-Disk, Convolute

10.88 Inches O.D., 1/2 Degree Rated Misalignment Angle, 3.62 Inch Hub Length, 1.375 Inch Disk Pack Centerline to Shaft End. The distance between disk pack centerlines (Center of Articulation), Z3, is 7.25 inches.

$$\Theta 1 = \text{Arc Sin}(.023/7.25) = +.182$$

$$\Phi 1 = \text{Arc Sin}(-.031/7.25) = -.245$$

$$\Theta 2 = \text{Arc Sin}(-.023/7.25) = -.182$$

$$\Phi 2 = \text{Arc Sin}(+.031/7.25) = +.245$$

$$Kb = 2100 \text{ Lb-In/Deg Per Disk Pack}$$

$$Ka = 7200 \text{ Lb-In Per Disk Pack}$$

Assume 0.050 inch axial deflection, compressed, which is 0.025 inches per disk pack.

$$\begin{aligned} MX1 &= 54622 (+.00318 + 2100 (-.245)) = \\ &\quad -341 \text{ Lb-In} \\ MY1 &= 54622 (-.00428) - 2100 (+.182) = \\ &\quad -616 \text{ Lb-In} \\ MZ1 &= 54622 \text{ Lb-In} \\ MZ2 &= 54622 (-.00318) - 2100 (+.245) = \\ &\quad -688 \text{ Lb-In} \\ MY2 &= 54622 (+.00428) + 2100 (-.182) = \\ &\quad -149 \text{ Lb-In} \\ MZ2 &= -54622 \\ FX1 &= \{-(-616) - (-149)\} / 7.25 = +105.5 \text{ Lbs} \\ FY1 &= \{+(-341) + (-688)\} / 7.25 = -141.9 \text{ Lbs} \\ FX2 &= -105.5 \text{ Lbs} \\ FY2 &= +141.9 \text{ Lbs} \\ FZ1 &= 7200 (.025) = 180 \text{ Lbs} \\ FZ2 &= 180 \text{ Lbs} \end{aligned}$$

Bending Moment, M

$$\begin{aligned} M1 &= \sqrt{\{FX1 (3.62 + 1.375) - MY1\}^2 + \\ &\quad \{FY1 (3.62 + 1.375) + MX1\}^2} \\ M1 &= 1552 \text{ Lb-In at Z1 Axis} \\ M2 &= \sqrt{\{FX2 (3.62 + 1.375) + MY2\}^2 + \\ &\quad \{FY2 (3.62 + 1.375) - MX2\}^2} \\ M2 &= 1552 \text{ Lb-In at Z2 Axis} \end{aligned}$$

Bending Stress at Hub End, Sb

$$\begin{aligned} Sb &= Mr/I \quad r = \text{Shaft Radius,} \\ &\quad 1.50 \text{ Inches} \\ &\quad I = \pi r^4/4, 3.976 \text{ Inches} \\ Sb &= 586 \text{ PSI at Z1 Axis} \\ Sb &= 586 \text{ PSI at Z2 Axis} \end{aligned}$$

Deflection of Shaft From Bearing Center to Shaft End

$$\begin{aligned} \delta X1 &= \{FX1 (Z1a)/3 - (MY1)/2\} \\ &\quad \{Z1a - Z1b\}^2 / EI \\ Z1a &= \text{Center of Articulation to Bearing} \\ &\quad \text{Center, 9.375} \\ Z1b &= \text{Center of Articulation to Shaft End,} \\ &\quad 1.375 \\ \delta X1 &= +.00034 \\ \delta Y1 &= -.00033 \\ \delta 1 &= .00047 \\ \delta X2 &= -.00021 \\ \delta Y2 &= +.00042 \\ \delta 2 &= .00047 \end{aligned}$$

(c) Annular Disk Coupling

9.88 Inch O.D., 1/4 Degree Rated Misalignment Angle, 3.75 Inch Hub Length, Approximately 0.25 Inch Disk Pack Centerline to Shaft End. The distance between diaphragm centerlines (Centers of Articulation), Z3, is 10.50 inches.

$$\begin{aligned} \Theta 1 &= \text{Arc Sin } (+.023/10.5) = +.126 \text{ Deg} \\ \Phi 1 &= \text{Arc Sin } (-.031/10.5) = -.169 \text{ Deg} \\ \Theta 2 &= \text{Arc Sin } (-.023/10.5) = -.126 \text{ Deg} \\ \Phi 2 &= \text{Arc Sin } (+.031/10.5) = .169 \text{ Deg} \\ Kb &\text{ is Non-Linear:} \\ &\quad \text{Moment at .126 Deg} = 368 \text{ Lb-In} \\ &\quad \text{Moment at .169 Deg} = 556 \text{ Lb-In} \\ Ka &\text{ is Non-Linear: Axial Force at Assumed Deflection} \\ &\quad \text{of 0.050 Inches is 453 Lbs} \\ MX1 &= 54622 (+.00220) - 556 = -436 \text{ Lb-In} \end{aligned}$$

$$\begin{aligned} MY1 &= 54622 (-.00295) - 368 = -529 \text{ Lb-In} \\ MZ1 &= 54622 \\ MX2 &= 54622 (-.00220) - 556 = -676 \text{ Lb-In} \\ MY2 &= 54622 (+.00295) - 368 = -207 \text{ Lb-In} \\ MZ2 &= -54622 \\ FX1 &= \{-(-529) - (-207)\} / 10.5 = +70.1 \text{ Lbs} \\ FY1 &= \{+(-436) + (-676)\} / 10.5 = \\ &\quad -105.9 \text{ Lbs} \\ FX2 &= -70.1 \text{ Lbs} \\ FY2 &= +105.9 \text{ Lbs} \\ FZ1 &= \text{At 0.050 Inches} = 453 \text{ Lbs} \\ FZ2 &= \text{At 0.050 Inches} = 453 \text{ Lbs} \end{aligned}$$

Bending Moment at Hub End, M

$$\begin{aligned} M1 &= \sqrt{\frac{FX1 (3.75 - .25) - MY1}{FY1 (3.75 - .25) + MX1}^2 +} \\ M1 &= 1118 \text{ Lb-In at Z1 Axis} \\ M2 &= 1118 \text{ Lb-In at Z2 Axis} \end{aligned}$$

Bending Stress at Hub End, Sb

$$\begin{aligned} Sb &= Mr/I \\ Sb &= 422 \text{ PSI at Z1 Axis} \\ Sb &= 422 \text{ PSI at Z2 Axis} \end{aligned}$$

Deflection of Shaft From Bearing Center to Shaft End

$$\begin{aligned} \delta X1 &= \{FX1 (Z1a)/3 - (MY1)/2\} \\ &\quad \{Z1a^2 + Z1b\} / EI \\ Z1a &= \text{Center of Articulation of Bearing} \\ &\quad \text{Center, 7.75} \\ Z1b &= \text{Center of Articulation to Shaft End,} \\ &\quad .25 \\ \delta X1 &= +.00023 \\ \delta Y1 &= -.00025 \\ \delta 1 &= .00034 \\ \delta X2 &= -.00014 \\ \delta Y2 &= +.00031 \\ \delta 2 &= .00034 \end{aligned}$$

(d) Diaphragm Coupling, Contoured Thickness, Flat Midplane

10.625 In. O.D., 1/3 Degree Rated Misalignment Angle, 3.60 Inch Hub Length, 0.44 Inch Diaphragm Centerline to Shaft End. The distance between diaphragm centerlines (Centers of Articulation), Z3, is 9.12 inches.

$$\begin{aligned} \Theta 1 &= \text{Arc Sin } (\Delta X1/Z3) = \text{Arc Sin } (+.023/9.12) = \\ &\quad +.144 \text{ Deg} \\ \Phi 1 &= \text{Arc Sin } (\Delta Y1/Z3) = \text{Arc Sin } (-.031/9.12) = \\ &\quad -.195 \text{ Deg} \\ \Theta 2 &= \text{Arc Sin } (\Delta X2/Z3) = \text{Arc Sin } (-.023/9.12) = \\ &\quad -.144 \text{ Deg} \\ \Phi 2 &= \text{Arc Sin } (\Delta X2/Z3) = \text{Arc Sin } (+.031/9.12) = \\ &\quad +.195 \text{ Deg} \\ Kb &= 1810 \text{ Lb-In/Deg Per Diaphragm} \\ Ka &= 8910 \text{ Lb-In Per Diaphragm (Linear Portion)} \\ KA &= 2.107 \times 10^6 \text{ Lb/In}^3 \text{ Per Diaphragm} \\ &\quad \text{(Non-Linear Portion)} \end{aligned}$$

Assume 0.050 Inch Axial Deflection, Compressed, Which is 0.025 Inches Per Diaphragm.

$$\begin{aligned} MX1 &= 54622 (+.00251) + 1810 (-.195) = \\ &\quad -216 \text{ Lb-In} \end{aligned}$$

$$MY1 = 54622 (-.00340) - 1810 (+.144) = -447 \text{ Lb-In}$$

$$MZ1 = 54622 \text{ Lb-In}$$

$$MX2 = 54622 (-.00251) - 1810 (+.195) = -490 \text{ Lb-In}$$

$$MY2 = 54622 (+.00340) + 1810 (-.144) = -75 \text{ Lb-In}$$

$$MZ2 = -54622$$

$$FY1 = -(-447) - (-75) / 9.12 = +57.1 \text{ Lbs}$$

$$FY1 = +(-216) + (-490) / 9.12 = -77.4 \text{ Lbs}$$

$$FX2 = -57.1 \text{ Lbs}$$

$$FY2 = +77.4 \text{ Lbs}$$

$$FZ1 = \frac{(2.107 \times 10^6) (.025)^3 + (8910) (.025)}{256} = \text{Lbs}$$

$$FZ2 = 256 \text{ Lbs}$$

Bending Moment at Hub End, M

$$M1 = \sqrt{\frac{FX1(3.60 + .44) - MY1^2 + FY1(3.60 + .44) + MX1^2}{}}$$

$$M1 = 860 \text{ Lb-In at Z1 Axis}$$

$$M2 = \sqrt{\frac{FX2(3.60 + .44) + MY2^2 + FY2(3.60 + .44) - MX2^2}{}}$$

$$M2 = 859 \text{ Lb-In at Z2 Axis}$$

Bending Stress at Hub End, Sb

$$Sb = Mr/I \quad r = \text{Shaft Radius, 1.50 Inches}$$

$$I = \pi r^4/4, 3.976 \text{ In}^4$$

$$Sb = 324 \text{ PSI at Z1 Axis}$$

$$Sb = 324 \text{ PSI at Z2 Axis}$$

Deflection of Shaft From Bearing Center to Shaft End

$$\delta X1 = \left\{ \frac{FX1(Z1a)/3 - (MY1)/2}{(Z1a - Z1b)^2} \right\} / EI$$

$$Z1a = \text{Center of Articulation to Bearing Center, 8.44 Inches}$$

$$Z1b = \text{Center of Articulation to Shaft End, .44 Inches}$$

$$\delta X1 = +.00020$$

$$\delta X1 = \left\{ \frac{FY1(Z1a)/3 + (MX1)/2}{(Z1a - Z1b)^2} \right\} / EI$$

$$\delta Y1 = -.00017$$

$$\text{Total Deflection } \delta 1 = X1^2 + Y1^2$$

$$\delta 1 = .00026$$

$$\delta X2 = \left\{ \frac{FX2(Z2a)/3 + (MY2)/2}{(Z2a - Z2b)^2} \right\} / EI$$

$$\delta X2 = -.00011$$

$$\delta Y2 = \left\{ \frac{FY2(Z2a)/3 - (MX2)/2}{(Z2a - Z2b)^2} \right\} / EI$$

$$\delta Y2 = +.00025$$

$$\delta 2 = .00027$$