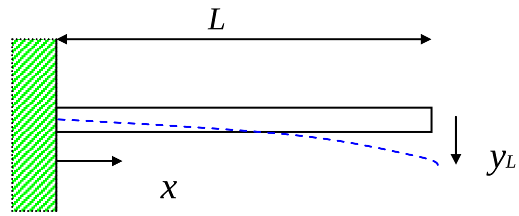
Appendix D. Technical Note on Assumed Modes

The **fundamental vibrating mode** of a cantilever beam and its associated natural frequency can be modeled as a single degree of freedom **lumped mass on a spring**.



The beam equivalent stiffness and mass can be determined by equating the beam strain energy *(V)* and kinetic energy *(T)* of the vibrating beam to the strain and kinetic energy of the lumped spring and mass, respectively. The equivalent displacement coordinate should be equal for both energies.

The beam has continuously distributed mass and elastic properties. Let

ρ : beam material density, *E:* beam elastic modulus,
A : beam cross section area, *L:* beam length,
I: area moment of inertia.

$y_{(x,t)}$ is the displacement of a beam material point, i.e. a function of its location *(x)* and time *(t)*. $y_{L(t)}$ denotes the beam dynamic displacement at $x=L$. The beam potential (strain) and kinetic energies, *V* and *T*, are defined as:

$$V = \frac{1}{2}\int_0^L E\,I\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx = \frac{1}{2}K_{eq}\,y_L^2;$$

(1)

$$T = \frac{1}{2}\int_0^L \rho\,A\left(\frac{\partial^2 y}{\partial t^2}\right)^2 dx = \frac{1}{2}M_{eq}\,\dot{y}_L^2$$

In practice, an assumed shape of vibration $\phi_{(x)}$ is used to estimate the equivalent stiffness (K_{eq}) and mass (M_{eq}).

Let $$y_{(x,t)} = \phi_{(x)}\, y_{L(t)} \qquad (2)$$

The **mode shape** $\phi_{(x)}$ must be twice differentiable and consistent with the essential boundary conditions of the **cantilever beam**, i.e. no displacement or slope at the fixed end. That is, from

$y_{(0,t)} = 0 \quad \rightarrow \quad \phi_{(x=0)} = 0$

$(\partial y/\partial x)_{x=0} = 0 \quad \rightarrow \quad d\phi/dx|_{x=0} = 0$ for all times $t>0$.

Substitution of $y_{(x,t)} = \phi_{(x)}\, y_{L(t)}$ into Eq. (1) gives:

$$K_{eq} = \int_0^L E\,I \left(\frac{d^2\phi}{d\,x^2}\right)^2 dx; \qquad M_{eq} = \int_0^L \rho\, A\,(\phi)^2\, dx \qquad (3)$$

The fundamental natural frequency of the vibrating beam is then given by

$$\omega_n = \sqrt{K_{eq}/M_{eq}} \qquad (4)$$

Using $\phi = (x/L)^2$, then $M_{eq} = \frac{1}{5}\rho\, A\, L$; $K_{eq} = 4\frac{E\,I}{L^3}$;

And $$\omega_n \approx \frac{1}{L^2}\left(20\frac{E\,I}{\rho\, A}\right)^{1/2} \qquad (5)$$

The equivalent mass of the beam, M_{eq}, is a fraction of the total mass (~1/5) since the material points composing the beam participate differently in the vibratory motion.

$K_{eq}=3EI/L^3$ (more exact value) follows if the **static deflection curve** for the beam with a point load at its free end is used as the assumed mode shape, i.e.

$$\phi(x)=\frac{1}{2}\left[3\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)^3\right]$$