ON SPECTRAL OPERATORS IN FINITE VON NEUMANN ALGEBRAS

A Dissertation

by

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ABSTRACT

An operator on a Hilbert space is said to be *spectral* if it has a suitably well-behaved 'idempotent-valued' spectral measure. Dunford introduced these operators and also provided the following characterization: An operator is spectral iff it is similar to the sum of a normal operator and a quasinilpotent operator that commute with each other. Operators in a von Neumann algebra with a normal, faithful, tracial state have an associated spectral measure (called the Brown measure) and invariant projections (the Haagerup-Schultz projections), which behave well with respect to the Brown measure. In this paper, we study the angles between the Haagerup-Schultz projections for such operators. We show that an operator in a finite von Neumann algebra is similar to the sum of a normal operator and a commuting s.o.t.-quasinilpotent operator iff the angles between its Haagerup-Schultz projections are uniformly bounded away from zero. This lets us provide a new characterization of spectral operators in finite von Neumann algebras.

We then estimate the angles between the Haagerup-Schultz projections for a class of operators from free probability called the DT-operators. These involve new estimates on the norms of algebra-valued circular operators. We then show, subject to some mild regularity conditions on the Brown measure of a DT-operator, that they fail to be spectral. This provides a large class of non-spectral but decomposable operators in a finite von Neumann algebra.

DEDICATION

To Prajakta, without whose constant encouragement and support this really would not have been written.

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1. INTRODUCTION AND BACKGROUND

1.1 Introduction

It is a well-known fact that every $n \times n$ matrix can be written as the sum of a diagonalizable matrix and a nilpotent matrix that commute with each other. On a Hilbert space, operators with an analogous property are known as spectral operators, and were first studied by Dunford ([4]). Spectral operators have an associated idempotent-valued spectral measure which behaves well with respect to the spectrum and restrictions. Spectral operators are part of a broader class of operators, called the decomposable operators.

In this work, we study spectral operators living in finite von Neumann algebras, and the angles between their associated invariant subspaces. Operators which live in a finite von Neumann algebra have an associated spectral measure, called the Brown measure. They also have a family of invariant projections, the Haagerup-Schultz projections, which behave well with respect to the Brown measure and restrictions. We prove that an operator in a finite von Neumann algebra is spectral if and only if it is decomposable and the angles between its Haagerup-Schultz projections are uniformly bounded away from zero. We then compute the angles between the Haagerup-Schultz projections for a class of operators with origins in free probability and random matrix theory, called the DT-operators. This provides a large class of non-spectral but decomposable operators in a finite von Neumann algebra.

This chapter contains a brief overview of the material we will require to prove the results in chapter 2.

1.2 Preliminaries and Notation

Throughout this paper, $\mathcal{H}$ will refer to a Hilbert space, and $B(\mathcal{H})$ will be the $*$-algebra of all bounded linear operators on it. A von Neumann algebra $\mathcal{M}$ of operators on $\mathcal{H}$ is a $*$-subalgebra of $B(\mathcal{H})$ which contains the identity operator and is closed in the strong operator topology. That is, if $T \in \mathcal{M}$, then $T^* \in \mathcal{M}$, and if S_n is a sequence of operators in $\mathcal{M}$ which converge pointwise on $\mathcal{H}$ to an operator $S \in B(\mathcal{H})$, then $S \in \mathcal{M}$. For a scalar λ, where the context is clear, $\lambda \in \mathcal{M}$ will refer to the scalar multiple of the identity operator 1.

Throughout, by a *idempotent*, we will mean a bounded operator $q \in \mathcal{M}$ such that $q^2 = q$. By a *projection*, we will mean a bounded, self-adjoint idempotent operator p ($p^2 = p, p^* = p$). We will also abbreviate the strong operator topology and the weak operator topology as s.o.t. an w.o.t. respectively.

We are interested in von Neumann algebras $\mathcal{M}$ which come equipped with a faithful, normal, tracial state τ. By this, we mean a function $\tau : \mathcal{M} \to \mathbb{C}$ which satisfies

(i) τ is linear.

(ii) $\tau(1) = 1$

(iii) τ has the trace property: $\tau(ST) = \tau(TS)$ for all $S, T \in \mathcal{M}$

(iv) τ is faithful: $\tau(T^*T) \geq 0$ for all $T \in \mathcal{M}$, and $\tau(T^*T) = 0$ only if $T = 0$.

(v) If $\{p_i\}_{i\in I} \subset \mathcal{M}$ is a family of pairwise orthogonal projections, then

$$\tau(\sum_{i\in I} p_i) = \sum_{i\in I} \tau(p_i),$$

where the sum on the left converges in the s.o.t. topology.

In the rest of this paper, $\mathcal{M}$ will denote a von Neumann algebra, τ will be a (faithful, normal) trace on it, and $\mathcal{H}$ will be the Hilbert space on which $\mathcal{M}$ acts. Oftentimes, this Hilbert space will be taken to be $L^2(\mathcal{M}, \tau)$, which is the completion of $\mathcal{M}$ with respect to the norm $\|x\|_2 := \tau(x^*x)^{1/2}$. We let $\hat{x} \in L^2(\mathcal{M}, \tau)$ denote the element corresponding to $x \in \mathcal{M}$.

The letters T, S, Z, X etc. will, unless specified, denote operators in $\mathcal{M}$, and $\sigma(T)$ will denote the spectrum of T. μ_T will denote the Brown measure of the operator T. $\mathbb{C}$ will denote the complex plane. For $0 \leq r < s \leq \infty$, $A(r, s)$ will denote the closed annulus with inner radius r and outer radius s, so $A(0, r)$ will denote the closed disc of radius r. $\mathfrak{A}$ will denote the Borel sigma algebra on $\mathbb{C}$.

1.3 Brown Measure and Haagerup-Schultz Projections

If $T \in \mathcal{M}$ is a normal operator, it has an associated projection-valued measure P_T, supported on $\sigma(T)$. By using the trace, we may convert this into a Borel probability measure on $\mathbb{C}$, given by

$$\mu_T(B) = \tau(P_T(B)).$$

L. Brown showed that any operator, not necessarily normal, in $\mathcal{M}$, can be associated with a spectral measure μ_T, which has the following characterization: [2]

Theorem 1.3.1. *Let $T \in \mathcal{M}$. Then there exists a unique probability measure μ_T such that for every $\lambda \in \mathbb{C}$,*

$$\int_{[0,\infty)} log(x) d\mu_{|T-\lambda|}(x) = \int_{\mathbb{C}} log|z-\lambda| d\mu_T(z),$$

where for a positive operator S, μ_S denotes the spectral distribution measure $\tau \circ P_S$, where P_S is the projection-valued spectral measure for S.

This measure μ_T is called the Brown measure of T. If T is normal, this is just the spectral distribution measure of T. Note that although the support of the Brown measure is contained in the spectrum of T, there are many operators for which $\sigma(T)$ is a much larger set than the support of μ_T.

Haagerup and Schultz later showed that the Brown measure does in fact arise from associated spectral projections [11].

Theorem 1.3.2 (Haagerup,Schultz)**.** *Let T be an operator in a finite von Neumann algebra $(\mathcal{M}, \tau)$ and $B \in \mathfrak{A}$. Then there exists a unique projection $P(T, B) \in \mathcal{M}$ such that the following conditions hold:*

(i) $\tau(P(T, B)) = \mu_T(B)$.

(ii) $P(T, B)\mathcal{H}$ *is an invariant subspace for* T.

(iii) The Brown measure of $TP(T,B)$ when computed in the corner $P(T,B)\mathcal{M}P(T,B)$ is supported inside the closure of B.

(iv) The Brown measure of $(1-P(T,B))T$ when computed in the corner $(1-P(T,B))\mathcal{M}(1-P(T,B))$ is supported inside the closure of B^c.

(v) If Q is another T-invariant projection with the property that the Brown measure of TQ (computed in $Q\mathcal{M}Q$) is concentrated in $\overline{B}$, then $Q \leq P(T,B)$.

For discs and complements of discs, the Haagerup-Schultz projections have the following explicit form:

Proposition 1.3.3. *Let $T \in (\mathcal{M}, \tau)$, and $r > 0$. Then*

$$P(T, A(0,r))\mathcal{H} = \left\{ \xi \in \mathcal{H} : \exists\, \xi_n \in \mathcal{H}, \lim_{n\to\infty} \xi_n = \xi, \limsup_n \|T^n \xi_n\|^{1/n} \leq r \right\}, \tag{1.1}$$

and

$$P(T, A(r,\infty))\mathcal{H} = \left\{ \eta \in \mathcal{H} : \exists\, \eta_n \in \mathcal{H}, \lim_{n\to\infty} T^n \eta_n = \eta, \limsup_n \|\eta_n\|^{1/n} \leq \frac{1}{r} \right\}. \tag{1.2}$$

In Theorem 8.1 of [11], Haagerup and Schultz also show the following convergence result:

Theorem 1.3.4. *Let $T \in \mathcal{M}$. Then the sequence $|T^n|^{1/n}$ has a strong operator limit A, and for every $r \geq 0$, the spectral projection of A associated with the interval r is $P(T, r\overline{\mathbb{D}})$, where $\mathbb{D}$ is the open disc of radius 1.*

It follows that $\mu_T = \delta_0$ is the point mass at 0 iff $|T^n|^{1/n}$ converges to 0 in the strong operator topology. Recall that for a quasinilpotent operator S ($\sigma(S) = \{0\}$), the spectral radius formula ensures that $|T^n|^{1/n}$ converges to 0 in norm. By analogy, operators with Brown measure supported only on $\{0\}$ are called *s.o.t.-quasinilpotent* operators.

The Haagerup-Schultz projections also satisfy nice lattice properties, as shown in Theorem 3.3 of [3].

Theorem 1.3.5. *Let $B_1, B_2 ...$ be Borel subsets of $\mathbb{C}$. Then*

$$\bigvee_{n=1}^{\infty} P(T, B_n) = P\left(T, \bigcup_{n=1}^{\infty} B_n\right),$$

and

$$\bigwedge_{n=1}^{\infty} P(T, B_n) = P\left(T, \bigcap_{n=1}^{\infty} B_n\right).$$

They also behave well with respect to compressions and similarities (Theorem 2.4.4, Theorem 12.3 in [3])

Theorem 1.3.6. *Let $Q \in \mathcal{M}$ be a non-zero T-invariant projection, and $S \in \mathcal{M}$ be invertible. Then, for all $B \in \mathfrak{A}$, we have*

(i) $P(T, B) \wedge Q = P^{(Q)}(TQ, B)$,

(ii) $\mu_{STS^{-1}} = \mu_T$,

(iii) $P(STS^{-1}, B)\mathcal{H} = \overline{SP(T, B)\mathcal{H}}$,

where $P^{(Q)}$ denotes the Haagerup-Schultz projection computed in the compression $Q\mathcal{M}Q$.

We will also need the following fact, which is well-known, and follows from the lattice properties, and property (v) of Theorem 1.3.2.

Proposition 1.3.7. *Let $p \in \mathcal{M}$ be a projection, and $T_{11} \in p\mathcal{M}p$, $T_{12} \in p\mathcal{M}(1-p)$, $T_{22} \in (1-p)\mathcal{M}(1-p)$, such that $\sigma_{p\mathcal{M}p}(T_{11}) \cap \sigma_{(1-p)\mathcal{M}(1-p)}(T_{22}) = \emptyset$. Let*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

be an operator in $\mathcal{M}$, expressed in the matrix notation with respect to p and $1-p$. Then, for $B \subset \sigma(T_{11})$, $P(T, B) = P(T_{11}, B)p$. In particular, $P(T, \sigma(T_{11})) = p$.

Brown measures and Haagerup-Schultz projections can also be defined for commuting tuples of operators. (See work by Charlesworth, Dykema, Sukochev and Zanin [3] and also by Schultz [16]).

Theorem 1.3.8. *Let $S, T \in \mathcal{M}$ be commuting operators. Then, there exists a unique compactly supported Borel probability measure $\mu_{S,T}$ on $\mathbb{C}^2$ such that*

$$\tau(\log|\alpha S + \beta T - 1|) = \int_{\mathbb{C}^2} \log|\alpha z + \beta w - 1| d\mu_{S,T}(z, w).$$

Theorem 1.3.9. *For commuting operators $S, T \in \mathcal{M}$, and a Borel set $B \subset \mathbb{C}^2$, there is a projection $P((S,T) : B) \in \mathcal{M}$ which is S, T-hyperinvariant, and which satisfies the following:*

(i) *For $B_1, B_2 \subseteq \mathbb{C}$, $P((S,T) : B_1 \times B_2) = P(S, B_1) \wedge P(T, B_2)$,*

(ii) *$P((S,T) : .)$ satisfies lattice properties analogous to Theorem 1.3.5,*

(iii) *For a Borel set B, with $p = P((S,T) : B)$, the Brown measure of (Sp, Tp) and $((1 - p)S, (1 - p)T)$, computed in the compressions $p\mathcal{M}p$ and $(1 - p)\mathcal{M}(1 - p)$ respectively, are concentrated in B, and B^c,*

(iv) *$\mu_{(S,T)}(B) = \tau(P((S,T) : B))$.*

The joint Brown measures and Haagerup-Schultz projections behave well under pushforwards. In particular, (Remark 6.5 in [16])

Proposition 1.3.10. *Let $S, T \in \mathcal{M}$ be commuting operators. Let $a : \mathbb{C}^2 \to \mathbb{C}$ denote the addition map. Then, for any Borel set $B \subset \mathbb{C}$, we have*

$$P(S + T, B) = P((S,T) : a^{-1}(B)).$$

1.4 Spectral and Decomposable Operators

Definition 1.4.1. A *bounded idempotent-valued spectral measure* in $\mathcal{M}$ is a mapping $B \mapsto E(B)$ that assigns to every $B \in \mathfrak{A}$ an idempotent operator $E(B) \in \mathcal{M}$ so that

(i) $E(\mathbb{C}) = 1$,

(ii) for all $B_1, B_2 \in \mathfrak{A}$, $E(B_1 \cap B_2) = E(B_1)E(B_2)$,

(iii) for all $B_1, B_2, \ldots \in \mathfrak{A}$ such that $B_i \cap B_j = \emptyset$ whenever $i \neq j$,

$$E(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} E(B_i),$$

where the sum converges with respect to $\|\cdot\|_2$,

(iv) $\sup_{B \in \mathfrak{A}} \|E(B)\| < \infty$.

Following Dunford [4], an operator $T \in B(\mathcal{H})$ is called a *spectral operator* if there exists a bounded idempotent-valued spectral measure E such that

(v) $E(B)T = TE(B)$, for every $B \in \mathfrak{A}$ (in particular, $E(B)\mathcal{H}$ is an invariant subspace for T),

(vi) the spectrum of T restricted to the range of $E(B)$ is contained in the closure of B:

$$\sigma(T, E(B)\mathcal{H}) \subseteq \overline{B}.$$

Unless otherwise mentioned, in the rest of this work, an *idempotent-valued spectral measure* will refer to a *bounded idempotent-valued spectral measure.* Of course, a bounded idempotent-valued spectral measure E where each $E(B)$ is self-adjoint (i.e. a projection) is just called a *spectral measure.* From Theorems 5 and 6 in [4], if T is a spectral operator, its idempotent-valued spectral measure is uniquely defined, and $E(B)$ belongs to the bicommutant of $\{T\}$, for every $B \in \mathfrak{A}$.

An operator $S \in B(\mathcal{H})$ is said to be of *scalar type* if S is spectral and also satisfies the equation

$$S = \int_{\sigma(S)} \lambda E(d\lambda), \tag{1.3}$$

where E is its associated idempotent-valued spectral measure, and the integral is in the uniform operator norm topology.

Recall that every normal operator S has an associated projection-valued spectral measure E, and also satisfies (1.3). So scalar type and spectral operators are natural generalizations of normal operators. In fact, as shown by Dunford in [4], scalar type operators can be characterised precisely as those operators which are similar to normal operators.

Theorem 1.4.2. *Let $\mathcal{M}$ be a von Neumann algebra. Then $S \in \mathcal{M}$ is a scalar type operator if and only if there exists an invertible element A in $\mathcal{M}$, such that $A^{-1}SA$ is a normal operator.*

Dunford further showed that spectral operators can be characterised by the following decomposition property:

Proposition 1.4.3. *If $S \in \mathcal{M}$ is a scalar type operator and $Q \in \mathcal{M}$ is quasinilpotent with $SQ = QS$, then $T = S + Q$ is a spectral operator. Conversely, if $T \in \mathcal{M}$ is a spectral operator, then T can be written as $T = S + Q$, where $S, Q \in \mathcal{M}$, Q is quasinilpotent, S is scalar type and $SQ = QS$. Moreover, we have*

$$S = \int_{\sigma(T)} \lambda E(d\lambda),$$

where E is the idempotent-valued spectral measure associated to T.

This is the analogue of the statement that every $n \times n$ matrix can be written as the sum of a commuting diagonalizable matrix and a nilpotent matrix.

Decomposable Operators

A broader class of operators are the decomposable operators, first introduced by Foias [10]. We will only require a few facts about these, which we state here. For a comprehensive overview of the subject, see the book by Laursen and Neumann [12]. An operator $T \in B(\mathcal{H})$ is said to be *decomposable* if it has a spectral capacity. A spectral capacity for T is a map

$$\{\text{closed sets in } \mathbb{C}\} \ni K \mapsto \mathcal{E}(K) \in \{\text{closed subspaces of } \mathcal{H}\}$$

so that

i) $\mathcal{E}(\emptyset) = \{0\}$, $\mathbb{C} = \mathcal{H}$,

ii) $\mathcal{E}(\overline{U_1}) + \mathcal{E}(\overline{U_2}) + ... + \mathcal{E}(\overline{U_n}) = \mathcal{H}$ (algebraic sum) for all open covers $\{U_1, ...U_n\}$ of $\mathbb{C}$,

iii) $\mathcal{E}(\cap_{k=1}^{\infty} K_n) = \cap_{k=1}^{\infty} \mathcal{E}(K_n)$,

iv) $\sigma(T\restriction_{\mathcal{E}(K)}) \subseteq K$.

If T is spectral with spectral measure E, then we can show that T is decomposable with spectral capacity $\mathcal{E}(K) = E(K)\mathcal{H}$. The local spectral subspaces of an operator play an important role in decomposability. We will not go into details here, (see [12] for more information), but we note the following result of Haagerup and Schultz (Proposition 9.2 of [11]), which we will use.

Proposition 1.4.4. *Suppose $T \in \mathcal{M}$ is decomposable. Then for every $B \in \mathfrak{A}$, the range of $P(T, B)$ is the closure of the local spectral subspace $\mathcal{H}_T(B)$. If B is a closed set, then we further have $P(T, B)\mathcal{H} = \mathcal{H}_T(B)$.*

As shown in [9], decomposable operators in a tracial von Neumann algebra also have 'full spectral support':

Proposition 1.4.5. *Let $T \in \mathcal{M}$ be a decomposable operator. Then* $\mathrm{supp}(\mu_T) = \sigma(T)$. *In particular, if T is decomposable and s.o.t.-quasinilpotent, T has to be quasinilpotent.*

The Haagerup-Schultz projections of spectral operators are determined by their idempotent-valued spectral measures.

Proposition 1.4.6. *Let $T \in \mathcal{M}$ be a spectral operator with idempotent-valued spectral measure E. Then, for every $B \in \mathfrak{A}$,*

$$P(T, B)\mathcal{H} = E(B)\mathcal{H}. \tag{1.4}$$

Proof. Since T is spectral, it is also decomposable. From Proposition 1.4.4, we have

$$P(T,K)\mathcal{H} = \mathcal{H}_T(K) = E(K)\mathcal{H}$$

for every closed set K. Thus, the desired equality (1.4) holds for closed sets B.

Now, given an arbitrary $B \in \mathfrak{A}$, using the inner regularity of μ_T, there is an increasing family $K_1 \subseteq K_2 \subseteq \cdots$ of closed subsets of B such that $\mu_T(B \setminus \bigcup_{k=1}^{\infty} K_n) = 0$. Together with the lattice property (Theorem 1.3.5), this implies

$$P(T,B) = P(T, \bigcup_{n=1}^{\infty} K_n) = \bigvee_{n=1}^{\infty} P(T, K_n).$$

Thus, we have

$$P(T,B)\mathcal{H} = \overline{\bigcup_{n=1}^{\infty} P(T,K_n)\mathcal{H}} = \overline{\bigcup_{n=1}^{\infty} E(K_n)\mathcal{H}} \subseteq E(B)\mathcal{H}. \tag{1.5}$$

Let p and, respectively, p' be the orthogonal projection from $\mathcal{H}$ onto $E(B)\mathcal{H}$ and, respectively, $E(B^c)\mathcal{H}$. Since $E(B)E(B^c) = 0$, we have $p \wedge p' = 0$. However, from (1.5) we have $P(T,B) \leq p$ and, likewise, $P(T,B^c) \leq p'$. We also have

$$\tau(P(T,B)) + \tau(P(T,B^c)) = \mu_T(B) + \mu_T(B^c) = 1.$$

Since $\tau(p \wedge p') \geq \tau(p) + \tau(p') - 1$, we cannot have $\tau(p) + \tau(p') > 1$. Thus, we must have $\tau(p) = \tau(P(T,B))$ and $\tau(p') = \tau(P(T,B^c))$. This implies $p = P(T,B)$, namely, that (1.4) holds. □

1.5 Free Probability

The pair $(\mathcal{M}, \tau)$ may be thought of as a *non-commutative probability space*, where τ plays the role of an 'expectation'. Operators in $\mathcal{M}$ are the 'free random variables'. This is the setting for free probability, first introduced by Voiculescu. We will (in Chapter 2) show that free probability provides us with a large set of non-spectral but decomposable operators in a finite von Neumann algebra. We compile a few basic definitions and some useful results here. For an overview of free probability theory, see [19].

Like in the classical case, we may define a non-commutative independence law. Let $(\mathcal{A}_s)_{s\in S}$ be a family of unital sub-algebras of $\mathcal{M}$. The algebras $(\mathcal{A}_s)_{s\in S}$ are said to be free if

$$\tau(x_1 x_2 .... x_n) = 0$$

holds whenever $s_1, ...s_n \in S$ such that $s_1 \neq s_2, s_2 \neq s_3, ...s_{n-1} \neq s_n$, $x_i \in \mathcal{A}_{s_i}$, and all the x_i are centered (i.e. $\tau(x_i) = 0$). We say that a family of sets of variables $A_i \subset \mathcal{M}, i \in S$ are free (respectively, $*$-free), if the algebras (respectively, $*$-algebras) generated by the A_i are free. A family of random variables $x_N, y_N \in (\mathcal{M}_N, \tau_N), N \in \mathbb{N}$ is said to be asymptotically free if the freeness criterion is satisfied in the limit $N \to \infty$.

Many constructions and results from classical probability have analogues in free probability. The $*$-distribution of an element $x \in \mathcal{M}$ will mean the set of all traces of non-commutative polynomials in words of x and x^*. That is, the $*$-distribution of x is

$$\left\{\tau\left(p(x^{\epsilon(1)}, x^{\epsilon(2)}, \cdots, x^{\epsilon(N)}\right) : N \in \mathbb{N}, \in \mathcal{P}_N, \epsilon(i) \in \{1, *\}\right\}$$

where $\mathcal{P}_N$ is the set of non-commutative polynomials in N random variables.

Let $(\mathcal{M}_n, \tau_n)$ be a sequence of tracial $*$-algebras. We say that a sequence of free random variables $(x_n) \in \mathcal{M}_n$ converges in $*$-distribution to $x \in \mathcal{M}$ if

$$\tau\left(x^{\epsilon(1)} x^{\epsilon(2)} ... x^{\epsilon(k)}\right) = \lim_n \tau_n\left(x_n^{\epsilon(1)} x_n^{\epsilon(2)} ... x_n^{\epsilon(k)}\right)$$

for every $k \in \mathbb{N}$ and $\epsilon(i) \in \{1, *\}$.

There is a free analogue of the real Gaussian random variable, which arises from a free central limit theorem - the semicircular operator. By a semicircular operator, we mean a self-adjoint element $x \in \mathcal{M}$ whose spectral measure is Wigner's semicircle distribution, which has the following density function:

$$s(t) = \frac{1}{2\pi}\sqrt{4-t^2}\,\chi_{\{-2\leq t\leq 2\}}.$$

The free analogue of the complex Gaussian is Voiculescu's circular operator Z, which is the sum of two free semicircular operators x_1 and x_2:

$$Z = \frac{1}{\sqrt{2}}\left(x_1 + ix_2\right).$$

Semicircular elements are the limiting $*$-distribution of many families of symmetric random matrices. For instance, for each $n \in \mathbb{N}$, let X_n be a self-adjoint $n \times n$ matrix of random variables. For $1 \leq i \leq n$, assume $X_n(i,i)$ is a real Gaussian $N(0,1)$ random variable, and for $i \neq j, 1 \leq i,j \leq n$, assume $X_n(i,j)$ is a complex Gaussian $N(0,1/2) + iN(0,1/2)$ random variable. Further, assume that $\{X_n(i,j) : 1 \leq i \leq j \leq n\}$ forms a set of (classically) independent random variables. $(X_n)_{n\in\mathbb{N}}$ is called a Gaussian Unitary Ensemble. Then, (after normalization), the family $(X_n)_{n\in\mathbb{N}}$ converges in $*$-distribution to a semicircular element $x \in \mathcal{M}$.

More generally, we will require the following result, which follows almost directly from some random matrix results due to Voiculescu ([18]).

Lemma 1.5.1. *Let $p_1, ...p_N$ be projections in $\mathcal{M}$, with $\sum p_i = 1$, and $\tau(p_i) = t_i$. Let x, y be semicircular elements in $\mathcal{M}$, with $\tau(x^2) = \tau(y^2) = 1$, such that $(\{x\}, \{y\}, \{p_1, ...p_N\})$ is a free family. Then*

$$X = \sum_{i=1}^{N} p_i x p_i + \sum_{i<j} (p_i y p_j + p_j y p_i) \tag{1.6}$$

is a semicircular element in $\mathcal{M}$, with $\tau(X^2) = 1$

Proof. For $M \in \mathbb{N}$, let $k_M^{(0)}, ...k_M^{(N)}$ be integers such that

$$0 = k_M^{(0)} < k_M^{(1)} < k_M^{(2)} < ... < k_M^{(N)} = M$$

and for all $1 \leq i \leq N$

$$\lim_{M \to \infty} \frac{k_M^{(i)} - k_M^{(i-1)}}{M} = t_i.$$

Let $(e_M^{i,j})_{1 \leq i,j \leq N}$ denote the $M \times M$ matrix units, and let $d_M^{(i)}$ be the following diagonal matrices:

$$d_M^{(i)} = \sum_{i=k_M^{(i-1)}+1}^{k_M^{(i)}} e_M^{i,i} \quad ,1 \leq i \leq N.$$

Let $(X_M)_{M\in\mathbb{N}}, (Y_M)_{M\in\mathbb{N}}$ be independent Gaussian Unitary Ensembles. By a theorem of Voiculescu ([18]),

$$\{X_M, Y_M, \{d_M^{(1)}, ...d_M^{(N)}\}\}$$

form an asymptotically free family. Since the matrix

$$Z_M = \begin{pmatrix} d_M^{(1)}X_M d_M^{(1)} & d_M^{(1)}Y_M d_M^{(2)} & \cdots & \cdots & d_M^{(1)}Y_M d_M^{(N)} \\ d_M^{(2)}Y_M d_M^{(1)} & d_M^{(2)}X_M d_M^{(2)} & & \vdots & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \cdots & d_M^{(N-1)}X_M d_M^{(N-1)} & d_M^{(N-1)}Y_M d_M^{(N)} \\ d_M^{(N)}Y_M d_M^{(1)} & \cdots & \cdots & d_M^{(N)}Y_M d_M^{(N-1)} & d_M^{(N)}X_M d_M^{(N)} \end{pmatrix}$$

converges in $*$-moments to the expression in equation 1.6, it follows that Z is semicircular.

□

Algebra-valued Free Probability

Let B be a commutative, unital C^*-subalgebra of $\mathcal{M}$, with $\mathcal{E} : \mathcal{M} \to B$ the conditional expectation (i.e $\mathcal{E}$ is linear, satisfies $\mathcal{E}(1) = 1$ and $\mathcal{E}(d_1 x d_2) = d_1 \mathcal{E}(x) d2$ for every $d_1, d_2 \in B$ and $x \in \mathcal{M}$). The pair $(\mathcal{M}, \mathcal{E})$ is called a B-valued non-commutative probability space, and elements of $\mathcal{M}$ are called B-valued random variables. Let $(\mathcal{A}_s)_{s\in S}$ be a family of sub-algebras of $\mathcal{M}$ such that $B \subset \mathcal{A}_s$ for every $s \in S$. The algebras $(\mathcal{A}_s)_{s\in S}$ are said to be free over B if

$$\mathcal{E}(x_1 x_2 .... x_n) = 0 \tag{1.7}$$

holds whenever $s_1, ...s_n \in S$ such that $s_1 \neq s_2, s_2 \neq s_3, ...s_{n-1} \neq s_n$, and all the $x_i \in \mathcal{A}_{s_i}$ have $\mathcal{E}(x_i) = 0$. (When $B = \mathbb{C}$, note that this is just freeness, as defined earlier.) Let $(a_i)_{i\in I} \in \mathcal{M}$. Given $j = ((j(1), ...j(n)) \in \cup_{n\geq 1} I^n$, the corresponding cumulant map is a $\mathbb{C}$-multilinear map $\alpha_j : B^{n-1} \to \mathbb{C}$. These are defined,recursively, as follows:

$$\mathcal{E}(a_{j(1)} b_1 a_{j(2} ... b_{n-1} a_{j(n)}) = \sum_{\pi \in NC(n)} \hat{\alpha}_j(\pi)[b_1, ...b_{n-1}], \tag{1.8}$$

where $NC(n)$ denotes the set of non-crossing partitions of $\{1, ...n\}$, and for $\pi \in NC(n)$, $\hat{\alpha}_j(\pi)$ is a multilinear map defined in terms of the cumulant maps α'_j for the j' obtained by restricting j to the blocks of π. We omit the details here, as they are not necessary for what follows. See [1] or [17] for more details. An operator $T \in \mathcal{M}$ is said to be a B-valued circular operator, if all cumulants except for $\alpha_{1,2}$ and $\alpha_{2,1}$ vanish for the pair (T, T^*). To be precise, let $T \in \mathcal{M}$. Label $a_1 = T$,$a_2 = T^*$ and let $J = \cup_{n\geq 1} 1, 2^n$. Let $(\alpha_j)_{j\in J}$ be the family of cumulant maps for the pair (a_1, a_2). T is called a B-valued circular operator if $\alpha_j = 0$ whenever $j \in J$ and $j \notin \{(1, 2), (2, 1)\}$. When $B = \mathbb{C}$, and $\alpha_{1,2} = \alpha_{2,1}$, this is just Voiculescu's circular operator.

1.6 DT-Operators

In [6], Dykema and Haagerup introduced the class of DT-operators and proved that they are all strongly decomposable. For each compactly supported Borel probability measure μ on $\mathbb{C}$ and each $c > 0$, there is a DT(μ, c) operator Z, (or, more correctly, there is a DT(μ, c) $*$-distribution, and every element of a W^*-noncommutative probability space having this $*$-distribution is called a DT(μ, c) operator). The material in this section draws on [6] and

[7]. The $*$-distribution of a DT-operator arises as the limiting $*$-distribution of the sum of diagonal + upper triangular random matrices:

Theorem 1.6.1. *Let μ be a compactly supported Borel probability measure on $\mathbb{C}$. For $n \in \mathbb{N}$, let D_n be a $n \times n$ diagonal random matrix whose entries are i.i.d μ-distributed random variables and let T_n be strictly upper triangular $n \times n$ matrices whose non-zero entries are i.i.d-Gaussian random variables with mean 0 and variance $1/n$. Then the pair D_n, T_n converge jointly in $*$-moments as $n \to \infty$.*

Let $c > 0$, and μ be a compactly supported Borel probability measure on $\mathbb{C}$. An element Z of (M, τ) is said to be a *DT*(μ, c)-operator if its $*$-moment distribution is the limiting $*$-moment distribution of $D_n + cT_n$, with D_n and T_n as in Theorem 1.6.1 above.

DT operators can be realized as $Z = D + cT$, where D is a normal operator and T is the "upper triangular half" of a semicircular operator that is free from an abelian algebra containing D.

Theorem 1.6.2 (Theorem 4.4, [6])**.** *Let $X \in (\mathcal{M}, \tau)$ be a semicircular operator with $\tau(X^2) = 1$, $\tau(X) = 0$, and let*

$$\lambda : L^\infty([0,1]) \to \mathcal{M}$$

be a normal, unital, injective $*$-homomorphism such that X and the image of λ are free with respect to τ. Then, there exists an operator $T \in \mathcal{M}$, constructed in a prescribed manner, such that the following holds:

(i) For all $0 < t < 1$,

$$T\lambda(1_{[0,t]}) = \lambda(1_{[0,t]})T\lambda(1_{[0,t]}),$$

(ii) $X = T + T^*$,

(iii) For all $0 < t < 1$, $\lambda(1_{[0,t]})T\lambda(1_{[t,1]}) = \lambda(1_{[0,t]})X\lambda(1_{[t,1]})$,

(iv) If $f \in L^\infty([0,1])$, $c > 0$, and $D = \lambda(f)$, then $D + cT$ is a DT(μ, c) *operator, where μ is the push-forward of the Lebesgue measure by f,*

(v) T is a quasinilpotent $L^\infty([0,1])$-valued circular operator.

This operator T is denoted by $\mathcal{UT}(X,\lambda)$.

Dykema and Haagerup use this 'upper-triangular' result to express any DT-operator as an upper triangular matrix consisting of DT-operators (Theorem 4.12, [6]). This result can be generalized to the following version:

Theorem 1.6.3. *Let $c > 0$, and $\mu, \mu_1, ..\mu_N$ be compactly supported Borel probability measures on $\mathbb{C}$ such that μ is a convex combination of μ_i:*

$$\mu = \sum_{i=1}^{n} t_i \mu_i$$

for some $0 < t_i < 1$ such that $\sum_i t_i = 1$.

Then there is an example of a $\mathrm{DT}(\mu, c)$-operator Z in a von Neumann algebra $\mathcal{M}$ with the following properties:

(i) $\mathcal{M}$ has a $*$-subalgebra $\mathcal{N}$ and a semicircular element X normalized so that $\tau(X^2) = 1$, and $\mathcal{N}$ and $\{X\}$ are free.

(ii) There are projections $p_1, \ldots, p_n$ in $\mathcal{N}$ so that $\tau(p_j) = t_j$ and $\sum_i p_i = 1$.

(iii) For each j, there is a $\mathrm{DT}(\mu_j, c\sqrt{t_j})$-operator $a_j \in p_j \mathcal{N} p_j$, with respect to the renormalized trace $t_i^{-1}\tau$.

(iv) Letting $b_{ij} = cp_i X p_j$ for each $i < j$,

$$Z = \sum_{j=1}^{n} a_j + \sum_{1 \le i < j \le n} b_{ij} = \begin{pmatrix} a_1 & b_{12} & \cdots & \cdots & b_{1N} \\ 0 & a_2 & b_{23} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & a_{N-1} & b_{N-1,N} \\ 0 & \cdots & \cdots & 0 & a_N \end{pmatrix},$$

where we have used the natural matrix notation with respect to the projections $p_1, \ldots, p_n$.

Proof. This imitates the proof of Theorem 4.12 in [6], using Lemma 1.5.1. Take a von Neumann algebra $\mathcal{M}$ with a trace-preserving, normal, injective *-homomorphism, $\lambda : L^\infty([0,1]) \to \mathcal{M}$ and having $X, \tilde{X} \in \mathcal{M}$, centered semi-circular elements such that $\tau(X^2) = \tau(\tilde{X}^2) = 1$ and

$$\{X\},\ \{\tilde{X}\}, \lambda(L^\infty([0,1]))$$

is a free family. Choose $0 = s_0 < s_1 < \cdots s_n = 1$ such that $s_i - s_{i-1} = t_i$, for $1 \le i \le n$, and let $p_i = \lambda(1_{[s_{i-1}, s_i]})$. Then p_i are projections and $\tau(p_i) = t_i$. From Lemma 1.5.1,

$$Y = \sum_{i=1}^{n} p_i \tilde{X} p_i + \sum_{i \neq j} p_i X p_j$$

is a semicircular element with $\tau(Y^2) = 1$, and clearly Y is free from $\lambda(L^\infty([0,1])$. For $1 \le i \le n$, let $\lambda_i : L^\infty([0,1]) \to p_i \mathcal{M} p_i$ be the unital $*$-homomorphism given by $\lambda_i(f) = \lambda(g)$ where

$$g(x) = \begin{cases} f((x - s_{i-1})/t_i), & s_{i-1} \le x \le s_i \\ 0, & \text{otherwise.} \end{cases}$$

Let $f_i \in L^\infty([0,1])$ be such that the push-forward under f_i of Lebesgue measure is μ_i and let $f \in L^\infty([0,1])$ be defined by

$$\lambda(f) = \lambda_1(f_1) + \cdots + \lambda_n(f_n).$$

Then the push-forward of Lebesgue measure under f is μ. Let $T = \mathcal{UT}(Y, \lambda)$ be the operator from Theorem 1.6.2 corresponding to Y and λ. Let $Z = \lambda(f) + cT$. Then Z is a DT(μ, c)-operator. Let $\mathcal{N}$ be the von Neumann algebra generated by $\lambda(L^\infty([0,1]))$ and $\tilde{X}$, so that $\mathcal{N}$ and $\{X\}$ are free. Let $a_i = p_i Z p_i = \lambda_i(f_i) + p_i T p_i$ and, for $i < j$, let $b_{ij} = p_i T p_j$. Note that $p_i T p_i = p_i \mathcal{UT}(\tilde{X}, \lambda) p_i \in \mathcal{N}$ and $b_{ij} = p_i X p_j$. By Lemma 4.10 of [6], $p_i T p_i = \mathcal{UT}(p_i \tilde{X} p_i, \lambda_i),$. We have that $\lambda_i(L^\infty(0,1])$ and $p_i \tilde{X} p_i$ are free. Moreover, $p_i \tilde{X} p_i$ is with respect to the trace

$\tau(p_i)^{-1}\tau\restriction_{p_i\mathcal{N}p_i}$ a semicircular element with second moment $\tau(p_i) = t_i$. It follows that a_i is a DT($\mu_i, \sqrt{t_i}$)-element in $p_i\mathcal{N}p_i$ with respect to this trace. Thus, we have

$$\lambda(f) + cT = \sum_{i=1}^{n} a_i + \sum_{1<i<j<n} b_{ij}$$

as required.

□

All DT-operators are strongly decomposable, and their restriction to their Haagerup-Schultz subspaces are also DT-operators. These properties follow from Dykema and Haagerup's work in [6] and the Haagerup-Schultz projection properties listed in Theorem 1.3.2.

Theorem 1.6.4. *Let T be a $DT(\mu, c)$ operator. Then,*

1. *The Brown measure of T is μ*

2. *T is a strongly decomposable operator, and $\sigma(T) =$ support(μ)*

3. *If $B \subseteq \mathbb{C}$ is a Borel set with $\mu(B) \neq 0$, and $p = P(T, B)$ is the corresponding Haagerup-Schultz projection, then Tp is also a DT-operator. To be precise, Tp is a $DT(\mu(B)^{-1}\restriction_B, c\sqrt{\mu(B)})$ operator in $p\mathcal{M}p$.*

2. ANGLES BETWEEN HAAGERUP-SCHULTZ PROJECTIONS AND SPECTRALITY OF OPERATORS

This chapter is joint work with Ken Dykema. Section 2.1 and 2.2 of this chapter draws on part of our preprint [8], while the rest is work yet to be published.

2.1 Angles Between Subspaces

For two non-zero vectors $v, w \in \mathcal{H}$, $\alpha(v, w)$ will denote the (positive) angle between the vectors, defined as

$$\alpha(v, w) = \cos^{-1}\left(\frac{|\langle v, w\rangle|}{\|v\|\|w\|}\right).$$

If V and W are two subspaces of a Hilbert space, we define the angle $\alpha(V, W)$ between the subspaces as the infimum of the angles between vectors from the respective subspaces. That is,

$$\alpha(V, W) = \inf\{\alpha(v, w) : v \in V, w \in W, v \neq 0, w \neq 0\}.$$

Further abusing notation, if $p, q \in \mathcal{M}$ are projections, we denote by $\alpha(p, q)$ the angle between their corresponding images, $\alpha(p\mathcal{H}, q\mathcal{H})$. From consideration of the unital C*-algebra generated by two projections (see [15]), it is known that the cosine of the angle between $P\mathcal{H}$ and $Q\mathcal{H}$ is equal to the maximum element of the spectrum of PQP. Thus, $\alpha(P, Q)$ is well defined for projections P and Q in a unital C*-algebra, independently of the way the C*-algebra is non-degeneratly represented on a Hilbert space.

Non-zero angles between subspaces imply the existence of bounded idempotents: the following fact is well known, but we provide a proof for completeness.

Lemma 2.1.1. *Let V, W be closed subspaces of $\mathcal{H}$ with $V \cap W = \{0\}$ and $\overline{V + W} = \mathcal{H}$. Then the following are equivalent:*

(i) $\alpha(V, W) > 0$.

(ii) $V + W$ *is closed.*

(iii) There exists a bounded idempotent $e \in B(\mathcal{H})$ *such that*

$$e\mathcal{H} = V \quad \textit{and} \quad (1-e)\mathcal{H} = W.$$

Moreover, to refine the implication (i) $\Longrightarrow$ (iii)*, there is a continuous, strictly decreasing function* $f : (0,1] \to [1,\infty)$ *such that*

$$\|e\| \leq f\big(1 - \cos(\alpha(V,W))\big).$$

Proof. (i) implies (ii): Let $\epsilon = 1 - \cos(\alpha(V,W))$. Then $\epsilon > 0$. For $v \in V, w \in W$, we have $|\langle v, w\rangle| \leq (1-\epsilon)\|v\|\|w\|$ and thus,

$$\begin{aligned}\|v+w\|^2 &= \|v\|^2 + \|w\|^2 + 2\mathrm{Re}\,\langle v, w\rangle \\ &\geq \|v\|^2 + \|w\|^2 - 2(1-\epsilon)\|v\|\,\|w\| \geq 2\epsilon\|v\|\,\|w\|. \quad (2.1)\end{aligned}$$

So either $\|v\|$ or $\|w\|$ is $\leq \|v+w\|/\sqrt{2\epsilon}$. If $\|v\|$ is so bounded, then

$$\|w\| \leq \|v+w\| + \|v\| \leq \left(1 + \frac{1}{\sqrt{2\epsilon}}\right)\|v+w\|.$$

By symmetry, we also have

$$\|v\| \leq \left(1 + \frac{1}{\sqrt{2\epsilon}}\right)\|v+w\|. \quad (2.2)$$

Consider a sequence $(v_n + w_n)_{n=1}^{\infty}$ with $v_n \in V$ and $w_n \in W$ that converges in $\mathcal{H}$ to a vector z. We will show $z \in V + W$. Using (2.2), we have that the sequences $(v_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ are Cauchy, hence, converge to some elements $v \in V$ and $w \in W$, respectively. Hence, $z = v + w \in V + W$.

(ii) implies (iii): The map $e : V + W \to V$ which is the identity on V and has kernel equal to W is well defined. By the closed graph theorem, it is bounded.

(iii) implies (i): If the angle were zero, we would have unit vectors $v_n \in V$ and $w_n \in W$

such that $\langle v_n, w_n\rangle \to 1$. Then $\|v_n - w_n\| \to 0$, but $e(v_n - w_n) = v_n$. This contradicts that e is bounded.

In order to bound the norm of e, let $\epsilon = 1 - \cos(\alpha(V, W))$. Let $w \in W$ and $v \in V$ with $\|w\| = 1$ and $\|v\| = a$. Then, using the first inequality in (2.1), we get

$$\|v + w\|^2 \geq (1 - a)^2 + 2a\epsilon,$$

which yields

$$\frac{\|e(v + w)\|}{\|v + w\|} \leq \sqrt{\frac{a^2}{(1 - a)^2 + 2\epsilon a}}.$$

When $0 < \epsilon < 1$, the right hand side attains its maximum value of

$$f(\epsilon) := \frac{1}{\sqrt{\epsilon(2 - \epsilon)}}$$

when $a = 1/(1 - \epsilon)$. □

We now examine angles between Haagerup–Schultz projections of disjoint closed sets.

Theorem 2.1.2. *If $T \in \mathcal{M}$ is decomposable and if F_1 and F_2 are closed subsets of $\mathbb{C}$ with $F_1 \cap F_2 = \emptyset$, then*

$$\alpha(P(T, F_1), P(T, F_2)) > 0.$$

Proof. Let $G = F_1 \cup F_2$, $p = P(T, G)$, and consider the operator Tp. Since T is decomposable, its spectrum $\sigma_{p\mathcal{M}p}(Tp)$ (in the compression $p\mathcal{M}p$) is a subset of G. From Theorems 1.3.6 and 1.3.5,

$$P^{(p)}(Tp, F_i) = P(T, F_i)$$

Since $\sigma(Tp) \subset G$, we can apply the holomorphic functional calculus for the function 1_{F_1} to Tp and the resulting operator $e = 1_{F_1}(Tp)$ is a bounded idempotent. Since Tp restricted to the range of e has spectrum contained in F_1, we have $e \leq P(Tp, F_1)$ and we get

$$e(p\mathcal{H}) \subseteq P(Tp, F_1)(p\mathcal{H}) = P(T, F_1)(p\mathcal{H}) = P(T, F_1)\mathcal{H}. \tag{2.3}$$

Similarly, we have

$$(p - e)(p\mathcal{H}) = 1_{F_2}(Tp)(p(\mathcal{H})) \subseteq P(T, F_2)\mathcal{H}. \tag{2.4}$$

Since

$$e(p\mathcal{H}) + (p - e)(p\mathcal{H}) = p\mathcal{H} = \overline{P(T, F_1)\mathcal{H} + P(T, F_2)\mathcal{H}},$$

we must have equality for both inclusions in (2.3) and (2.4) and that the sum of subspaces $P(T, F_1)\mathcal{H} + P(T, F_2)\mathcal{H}$ is closed. By Lemma 2.1.1, this implies that the angle between the two projections is non-zero. □

The next example shows that the non-zero angle conclusion of the previous theorem may fail if T is not decomposable.

Example 2.1.3. Consider

$$T = \oplus_{k=1}^{\infty} T_k \in \bigoplus_{k=1}^{\infty} M_k(\mathbb{C}),$$

where the algebra on the right is the ℓ^∞ sum embedded into a finite von Neumann algebra $\mathcal{M}$ and where T_k is the $k \times k$ matrix

$$T_k = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ & & & & 0 \end{pmatrix}$$

consisting of $(-1, \ldots, -1, 0)$ on the main diagonal, all entries on the diagonal above it being 1, and all other entries of the matrix being 0. Note that the Brown measure of T is supported on $\{-1, 0\}$, but it is easy to see that the spectrum of T is the closed disk of radius 1 centered at -1. Indeed, if $d = -1 - z$ for $z \in \mathbb{C} \setminus \{-1, 0\}$, then considering the $k \times k$ diagonal matrix $D_k = \text{diag}(d, d, \ldots, d, -z)$ and the $k \times k$ Jordan block matrix J_k, so that $T_k - zI_k = D_k + J_k$, we have

$$(T_k - zI_k)^{-1} = \left(\sum_{j=0}^{k-1}(-D_k^{-1}J_k)^j\right) D_k^{-1} = \left(\sum_{j=0}^{k-1}(-d^{-1}J_k)^j\right) D_k^{-1}.$$

Note that this stays uniformly bounded in operator norm as $k \to \infty$ if and only if $|1 + z| > 1$. From this, the assertion about the spectrum of T follows. In particular, T is not decomposable, since the support of its Brown measure is much smaller than its spectrum. (Proposition 1.4.5)

On the other hand, the vector $v_k = (1, 1, \ldots, 1)^t$ lies in the kernel of T_k while the vector $w_k = (1, 1, \ldots, 1, 0)^t$ lies $\ker(T_k + I_k)^k)$ and the angle between v_k and w_k is $\arccos(\sqrt{1 - \frac{1}{k}})$. This implies that the angle between $P(T, \{-1\})$ and $P(T, \{0\})$ is zero.

Note that for normal operators (which are spectral, and hence decomposable), spectral subspaces corresponding to disjoint sets are orthogonal to each other.

2.2 The UNZA Property and Spectrality

In [5], Dunford showed an equivalence between the spectrality of an operator and four conditions, which he labeled (A) to (D). Of these, condition (B) actually translates to saying that the angles between the local spectral subspaces for disjoint sets is bounded away from zero. Conditions (A) and (C) are related to the decomposability of the operator - they are now known as the single-valued extension property and Dunford's property C respectively. However, condition (D) is rather contrived, and it is not clear if conditions (A), (C) and (D) together imply that the operator is decomposable.

With this and theorem 2.1.2 in mind, we define the *non-zero angle* (or NZA) property for an operator $T \in \mathcal{M}$. We will say that T has the NZA property if the angles between the Haagerup-Schultz projections corresponding to complementary sets is bounded away from 0. That is, T has the NZA property

$$\alpha\left(P(T, B), P(T, B^c)\right) > 0$$

for all Borel sets $B \subset \mathbb{C}$ such that $P(T, B) \neq 0$ and $P(T, B^c) \neq 0$.

We define, for an operator $T \in \mathcal{M}$,

$$\alpha(T) = \inf \{\alpha(P(T, B), P(T, B^c)) : B \in \mathfrak{A}, P(T, B) \neq 0, P(T, B^c) \neq 0\} \qquad (2.5)$$

We may also write $\alpha_{\mathcal{M}}(T)$ for $\alpha(T)$, in order to emphasize the von Neumann algebra we consider.

We say T has the *uniformly non-zero angle property* (or the UNZA property) if $\alpha(T) > 0$. It is currently unclear if, for $T \in \mathcal{M}$, having the NZA property implies having the UNZA property. We think this is not true, but have been unable to construct a counter-example.

Using Lemma 2.1.1, an operator T with the UNZA property has many associated idempotents. We can show that these idempotents in fact constitute an idempotent-valued spectral measure for T.

Lemma 2.2.1. *Assume $T \in \mathcal{M}$ has the uniformly non-zero angle property. Then there exists an idempotent-valued spectral measure E with the following properties, where $B \in \mathfrak{A}$.*

(a) $E(B)\mathcal{H} = P(T, B)\mathcal{H}$ *and* $\ker E(B) = P(T, B^c)\mathcal{H}$,

(b) $TE(B) = E(B)T$,

(c) The Brown measure of the restriction of T to the range of $E(B)$ is concentrated in B.

Proof. By Lemma 2.1.1, for any $B \in \mathfrak{A}$ there is a bounded idempotent $E(B)$ satisfying condition (a). We verify that E is indeed an idempotent-valued spectral measure by checking the conditions of Definition 1.4.1.

Clearly, $E(\mathbb{C}) = 1$, so 1.4.1(i) holds. By Lemma 2.1.1 and the UNZA hypothesis, we have uniform boundedness of the $E(B)$, so 1.4.1(iv) holds.

Note that if B_1 and B_2 are disjoint Borel subsets of $\mathbb{C}$, then it follows from the UNZA property that $\alpha(P(T,B_1),P(T,B_2))>0$ and, thus, by Theorem 1.3.5 and Lemma 2.1.1, that

$$\begin{aligned}P(T,B_1\cup B_2)\mathcal{H}&=(P(T,B_1)\vee P(T,B_2))\mathcal{H}\\&=\overline{P(T,B_1)\mathcal{H}+P(T,B_2)\mathcal{H}}=P(T,B_1)\mathcal{H}+P(T,B_2)\mathcal{H}.\end{aligned}$$

Iterating this, we see that if $B_1,\ldots,B_n$ are pairwise disjoint and $B=\bigcup_{j=1}^n B_j$, then

$$P(T,B)\mathcal{H}=P(T,B_1)\mathcal{H}+\cdots+P(T,B_n)\mathcal{H}. \tag{2.6}$$

We now show that property 1.4.1(ii) holds. Let $B_1,B_2\in\mathfrak{A}$. If $\xi\in\mathcal{H}$ then, by the above, we may write $\xi=\xi_{00}+\xi_{01}+\xi_{10}+\xi_{11}$, where

$$\begin{aligned}&\xi_{00}\in P(T,B_1^c\cap B_2^c)\quad \xi_{01}\in P(T,B_1^c\cap B_2)\\&\xi_{10}\in P(T,B_1\cap B_2^c)\quad \xi_{11}\in P(T,B_1\cap B_2).\end{aligned}$$

We have

$$E(B_1\cap B_2)\xi_{ij}=\begin{cases}\xi_{11} & \text{if } i=j=1,\\ 0 & \text{otherwise.}\end{cases}$$

We also have

$$\begin{aligned}E(B_1)E(B_2)\xi_{00}&=E(B_1)0=0\\E(B_1)E(B_2)\xi_{01}&=E(B_1)\xi_{01}=0\\E(B_1)E(B_2)\xi_{10}&=E(B_1)0=0\\E(B_1)E(B_2)\xi_{11}&=E(B_1)\xi_{11}=\xi_{11}.\end{aligned}$$

Thus, we have

$$E(B_1)E(B_2)\xi=\xi_{11}=E(B_1\cap B_2)\xi.$$

We now show that property 1.4.1(iii) holds. Let $B_1, B_2, \ldots \in \mathfrak{A}$ be pairwise disjoint. Let $A_n = \bigcup_{i=1}^n B_i$ and $A = \bigcup_{i=1}^\infty B_i$. Given $\xi \in \mathcal{H}$ and $n \in \mathbb{N}$, using the property proved at (2.6), we may write $\xi = \eta + \xi_1 + \cdots + \xi_n$, where $\eta \in P(T, A_n^c)\mathcal{H}$ and $\xi_j \in P(T, B_j)\mathcal{H}$. For each j, we have $E(B_j)\xi = \xi_j$ and

$$E(A_n)\xi = \xi_1 + \cdots + \xi_n = \sum_{j=1}^n E(B_j)\xi.$$

Thus $E(A_n) = \sum_{j=1}^n E(B_j)$ and in order to prove property 1.4.1(iii), it will suffice to show that $E(A_n)$ converges to $E(A)$ in strong operator topology as $n \to \infty$. Given $\xi \in \mathcal{H}$, we have $\xi = \xi_0 + \xi_1$ for $\xi_0 \in P(T, A^c)\mathcal{H}$ and $\xi_1 \in P(T, A)\mathcal{H}$. Then for all n, $E(A_n)\xi_0 = E(A)\xi_0 = 0$. Since $P(T, A_n)$ increases and converges in strong operator topology to $P(T, A)$ as $n \to \infty$, the vector $\xi_1^{(n)} := P(T, A_n)\xi_1$ converges to $P(T, A)\xi_1 = \xi_1$ as $n \to \infty$. Let $\epsilon > 0$. For all n sufficiently large, we have

$$\|\xi_1^{(n)} - \xi_1\| < \frac{\epsilon}{1 + \sup_B \|E(B)\|}$$

and for such n we have

$$\begin{aligned}
\|E(A_n)\xi - E(A)\xi\| &= \|E(A_n)\xi_1 - \xi_1\| \\
&\leq \|E(A_n)(\xi_1 - \xi_1^{(n)})\| + \|E(A_n)\xi_1^{(n)} - \xi_1\| \\
&= \|E(A_n)(\xi_1 - \xi_1^{(n)})\| + \|\xi_1^{(n)} - \xi_1\| \\
&\leq (\|E(A_n)\| + 1)\|\xi_1^{(n)} - \xi_1\| < \epsilon.
\end{aligned}$$

This completes the proof of property 1.4.1(iii).

We now prove (b). Given $\xi \in \mathcal{H}$, we write $\xi = \xi_0 + \xi_1$ where $\xi_0 \in P(T, B^c)\mathcal{H}$ and $\xi_1 \in P(T, B)\mathcal{H}$. Since $E(B)\mathcal{H} = P(T, B)\mathcal{H}$ and $P(T, B^c)\mathcal{H}$ are invariant subspaces for T, we have

$$E(B)T\xi = E(B)T(\xi_0 + \xi_1) = E(B)T\xi_1 = T\xi_1 = TE(B)(\xi_0 + \xi_1) = TE(B)\xi.$$

This proves that T and $E(B)$ commute.

The assertion (c) follows immediately from $E(B)\mathcal{H} = P(T,B)\mathcal{H}$ and the property of Haagerup–Schultz projections. $\square$

Dunford's proof of Theorem 1.4.2 relied on the fact that a bounded idempotent-valued measure in $B(\mathcal{H})$, is similar to a projection-valued measure in $B(\mathcal{H})$. This may be found in [13] (*cf* [20]), but we have not been able to obtain a copy of [13]. For completeness, we provide a proof of this result, when $B(\mathcal{H})$ is replaced with $\mathcal{M}$.

Proposition 2.2.2. *Suppose E is a bounded idempotent-valued spectral measure in $\mathcal{M}$. Then there is an invertible $A \in \mathcal{M}$ so that for every $\sigma \in \mathfrak{A}$, the idempotent $A^{-1}E(\sigma)A$ is self-adjoint.*

Proof. Fix a normal faithful representation $\mathcal{M} \hookrightarrow B(\mathcal{H})$. Given a finite Borel partition $\pi = \{\sigma_1, \ldots, \sigma_n\}$ of $\mathbb{C}$, we consider the sesquilinear form on $\mathcal{H}$ given by

$$\langle x, y\rangle_\pi = \sum_{i=1}^{n} \langle E(\sigma_i)x, E(\sigma_i)y\rangle$$

and denote the corresponding norm by

$$\|x\|_\pi = \left(\sum_{i=1}^{n} \|E(\sigma_i)x\|^2 \right)^{1/2}.$$

From Lemma 1 of [20], we have

$$\frac{1}{2M}\|x\| \le \|x\|_\pi \le 2M\|x\|, \tag{2.7}$$

for every $x \in \mathcal{H}$, where $M = \sup_{\sigma\in\mathfrak{A}} \|E(\sigma)\|$.

Let Ω be the directed set of all finite Borel partitions of $\mathbb{C}$, partially ordered by refinement. Consider the net

$$\Omega \ni \pi \mapsto \langle \cdot, \cdot \rangle_\pi. \tag{2.8}$$

We identify each sesquilinear form $\langle \cdot, \cdot \rangle_\pi$ with its restriction to the Cartesian product $\mathcal{S}_1 \times \mathcal{S}_1$ of the unit sphere of $\mathcal{H}$ with itself. Using the upper bound from (2.7), we have $|\langle x, y \rangle_\pi| \leq 2M$ for every $(x, y) \in \mathcal{S}_1 \times \mathcal{S}_1$. Thus, each sesquilinear form $\langle \cdot, \cdot \rangle_\pi$ is identified with an element of the product space $X = \prod_{\mathcal{S}_1 \times \mathcal{S}_1} 2M\mathbb{D}$ of copies of the closed disk of radius $2M$, which is compact, by Tychonoff's Theorem. Thus, the net (2.8) has an accumulation point in X, and this extends to a bounded sesquilinear form $\langle \cdot, \cdot \rangle_\alpha$ on $\mathcal{H}$.

Writing $\|x\|_\alpha = \langle x, x, \rangle_\alpha^{1/2}$, from (2.7) we have

$$\frac{1}{2M}\|x\| \leq \|x\|_\alpha \leq 2M\|x\|. \tag{2.9}$$

Let $x, y \in \mathcal{H}$. If $\pi = \{\sigma_1, \ldots, \sigma_n\} \in \Omega$ and $\sigma = \bigcup_{i \in I} \sigma_i$ for some $I \subseteq \{1, \ldots, n\}$, then $\langle E(\sigma)x, y \rangle_\pi = \langle x, E(\sigma)y \rangle_\pi$. This implies that, for every $\sigma \in \mathfrak{A}$,

$$\langle E(\sigma)x, y \rangle_\alpha = \langle x, E(\sigma)y \rangle_\alpha. \tag{2.10}$$

Since $\langle \cdot, \cdot \rangle_\alpha$ is a bounded sesquilinear form, there is $A \in B(\mathcal{H})$, $A \geq 0$, so that for all $x, y \in \mathcal{H}$, we have

$$\langle x, y \rangle_\alpha = \langle Ax, Ay \rangle.$$

Using (2.9), we see that A is invertible. From (2.10), we get $A^2 E(\sigma) = E(\sigma)^* A^2$, from which we get

$$(AE(\sigma)A^{-1})^* = AE(\sigma)A^{-1}.$$

It remains to show $A \in \mathcal{M}$. Suppose U is a unitary in the commutant of $\mathcal{M}$. We see immediately that for every $\pi \in \Omega$ and for all $x, y \in \mathcal{H}$ we have $\langle Ux, Uy \rangle_\pi = \langle x, y \rangle_\pi$, so we must have

$$\langle AUx, AUy \rangle = \langle Ux, Uy \rangle_\alpha = \langle x, y \rangle_\alpha = \langle Ax, Ay \rangle.$$

Thus, U commutes with A^2, so $A^2 \in \mathcal{M}$ and $A \in \mathcal{M}$. $\square$

We now use this result to provide a characterization for operators with the UNZA property.

Theorem 2.2.3. *Let $T \in \mathcal{M}$. Then the following are equivalent:*

(a) T has the UNZA property,

(b) there exist $S, Q \in \mathcal{M}$ with $[S, Q] = 0$, S a scalar type operator and Q s.o.t.-quasinilpotent, such that $T = S + Q$,

(c) there exist $A, N, Q' \in \mathcal{M}$, with $[N, Q'] = 0$, N normal, Q' s.o.t.-quasinilpotent, and A invertible, such that $ATA^{-1} = N + Q'$.

Proof. (a) $\Longrightarrow$ (b). Assume T has the UNZA property. Using the spectral measure E from Lemma 2.2.1, define

$$S = \int_{\mathbb{C}} \lambda E(d\lambda).$$

This integral exists, and S is a bounded operator, since by construction, $E(B) = 0$ for $B \subset \sigma(T)^c$. Moreover, by definition, S is an operator of scalar type.

Let $Q = T - S$. Since $TE(B) = E(B)T$, it follows that Q and S commute. We claim that Q is s.o.t.-quasinilpotent. Using Proposition 1.4.6 and Lemma 2.2.1, for every $B \in \mathfrak{A}$ we have

$$P(S, B)\mathcal{H} = E(B)\mathcal{H} = P(T, B)\mathcal{H},$$

so the Haagerup–Schultz projections of S and T agree. Using the pushforward result from Proposition 1.3.10, we have

$$P(Q, B) = P((T, S) : \{(\lambda_1, \lambda_2) : \lambda_1 - \lambda_2 \in B\})$$

Since $P(T, \cdot) = P(S, \cdot)$, and $P((T, S) : B_1 \times B_2) = P(T, B_1) \wedge P(T, B_2)$, it follows that $\mu_{(T,S)}$ is concentrated on the set $\{(z, z) : z \in \mathbb{C}\}$. Hence, if $0 \notin B$, then $P(Q, B) = 0$. Thus implies that Q is s.o.t.-quasinilpotent.

(b) $\implies$ (c). Assuming $T = S + Q$ with S of scalar type and Q s.o.t.-quasinilpotent and commuting with S, let $A \in \mathcal{M}$ be the invertible operator from Theorem 1.4.2 so that $N := ASA^{-1}$ is normal. Let $Q' = AQA^{-1}$. Since similarity doesn't change the Brown measure, we have that Q' is s.o.t.-quasinilpotent. Moreover, N and Q' commute. We have $ATA^{-1} = N + Q'$, as required.

(c) $\implies$ (a). Assume $ATA^{-1} = N + Q'$ as described in (c). Let $B \in \mathfrak{A}$. Then, from Theorem 1.3.6 and Proposition 1.3.10, we get

$$P(N, B)\mathcal{H} = P(ATA^{-1}, B)\mathcal{H} = AP(T, B)\mathcal{H} \tag{2.11}$$

If T fails to have the UNZA property, there exist sets B_n so that

$$\alpha\left(P(T, B_n), P(T, B_n^c)\right) \to 0$$

Then, there exist $v_n \in P(T, B_n)\mathcal{H}$, $w_n \in P(T, B_n^c)\mathcal{H}$ such that $\langle v_n, w_n \rangle \to 1$, and $\|v_n\| = \|w_n\| = 1$. Since N is normal, its spectral subspaces are orthogonal. So, from (2.11), we have

$$\|A(v_n - w_n)\|^2 = \|Av_n\|^2 + \|Aw_n\|^2 \geq 2\|A^{-1}\|^{-2} > 0, \tag{2.12}$$

which contradicts the fact that $\|v_n - w_n\| \to 0$. □

Corollary 2.2.4. *Let $T \in \mathcal{M}$. Then $P(T, \cdot)$ defines a spectral measure if and only if $T = N + Q$ for some $N, Q \in \mathcal{M}$, where N is normal, Q is s.o.t.-quasinilpotent, and $NQ = QN$.*

Proof. If $T = N + Q$ as described, then, from Proposition 1.3.10, $P(T, \cdot) = P(N, \cdot)$ is a spectral measure. On the other hand, if $P(T, \cdot)$ is a spectral measure, $P(T, B)P(T, B^c) = 0$. Hence the corresponding subspaces are orthogonal, and so T has the UNZA property. The construction (a) $\implies$ (b) in the proof of Theorem 2.2.3 then yields $T = S + Q$ with S actually normal. □

It is well known and is also easily seen from the above that spectral operators are decomposable. With the help of Theorem 2.2.3, we get the following equivalence:

Corollary 2.2.5. *Let $T \in \mathcal{M}$. Then, T is spectral if and only if T is decomposable and satisfies the UNZA property.*

Proof. It is clear from definitions that spectrality implies decomposability and, from Theorem 2.2.3 and the characterization in Proposition 1.4.3, that spectrality implies the UNZA property.

To prove the converse, suppose that T is decomposable and has the UNZA property. Let E be the idempotent-valued spectral measure constructed in Lemma 2.2.1. By decomposability and Proposition 1.4.4, we see that for each $B \in \mathfrak{A}$, the spectrum of the restriction of T to $E(B)$ is contained in the closure of B. Hence T is in fact a spectral operator. □

2.3 Examples of Non-Spectral DT-operators

In this section, we estimate $\alpha(Z)$, for DT-operators Z with certain nice spectral measures, and use this to show that they fail to have the UNZA property. In conjunction with the characterization from Theorem 2.2.3, this will provide non-trivial examples of non-spectral operators in a finite von Neumann algebra.

In [8], we computed $\alpha(Z)$ for the circular free Poisson operators. The following is an extension of those results to other DT-operators. We will use two elements in our study of DT-operators - a slight generalization of the upper triangular decomposition shown in 1.6.3 for DT-operators, and a new estimate of the norms of powers of DT-operators.

Norm Estimates for DT-operators

Throughout this section B will be a commutative, unital C*-algebra and (A, E) a B-valued C*-noncommutative probability space. Recall that a B-valued circular element is $T \in A$ satisfying that $E(T) = 0$ and that the only nonvanishing noncrossing cumulants of the pair $(T_1, T_2) = (T, T^*)$ are the completely positive maps from B to B, given by $\alpha_{1,2}(b) = E(TbT^*)$ and $\alpha_{2,1}(b) = E(T^*bT)$. Note that these are positive maps from B to B

and must satisfy

$$|\alpha_{1,2}(b)| \leq \alpha_{1,2}(|b|), \qquad |\alpha_{2,1}(b)| \leq \alpha_{2,1}(|b|). \tag{2.13}$$

Indeed, if ϕ is a state of B, then $\phi \circ \alpha_{1.2}$ is a positive linear functional on B and is given by integration with respect to a measure on the spectrum of B, so satisfies

$$|\phi \circ \alpha_{1.2}(b)| \leq \phi \circ \alpha_{1.2}(|b|).$$

By way of notation, given a finite sequence $\epsilon(1), \epsilon(2), \ldots, \epsilon(n) \in \{1, *\}$, we will say the sequence is *balanced* if there are as many $*$'s as 1's, namely, if

$$\#\{j \mid \epsilon(j) = 1\} = \frac{n}{2}.$$

Lemma 2.3.1. *Let B be a commutative, unital C^*-algebra and let T be a B-valued circular element in some B-valued C^*-noncommutative probability space (A, E). Then for all $\epsilon(1), \ldots, \epsilon(n) \in \{1, *\}$, we have*

$$E(T^{\epsilon(1)} T^{\epsilon(2)} \cdots T^{\epsilon(n)}) \geq 0. \tag{2.14}$$

Proof. If the sequence $\epsilon(1), \ldots, \epsilon(n)$ is not balanced, then from the moment-cumulant formula, the expectation on the left hand side of (2.14) is zero. So we may assume the sequence is balanced. Let $\alpha_{1,2}$ and $\alpha_{2,1}$ be the cummulant maps for T, T^*. We proceed by induction on n. The case $n = 2$ is clear.

For the induction step, suppose $n \geq 4$. Let J be the set of all $j \in \{2, 3, \ldots, n\}$ such that $\epsilon(j) \neq \epsilon(1)$ and the sequence $\epsilon(1), \ldots, \epsilon(j)$ is balanced. Suppose for the moment $\epsilon(1) = 1$. Then by the moment-cumulant formula, we have

$$E(T^{\epsilon(1)} \cdots T^{\epsilon(n)}) = \sum_{j \in J} \alpha_{1,2}\big(E(T^{\epsilon(2)} \cdots T^{\epsilon(j-1)})\big) E(T^{\epsilon(j+1)} \cdots T^{\epsilon(n)}). \tag{2.15}$$

Using the induction hypothesis, the positivity of the map $\alpha_{1,2}$ and the commutativity of B, we see that for each $j \in J$ the corresponding term in the sum (2.15) is positive and, hence, we get $E(T^{\epsilon(1)}T^{\epsilon(2)}\cdots T^{\epsilon(n)}) \geq 0$.

The situation when $\epsilon(1) = 2$ is the same but with $\alpha_{2,1}$ replacing $\alpha_{1,2}$ in (2.15). □

Lemma 2.3.2. *Let B be a commutative, unital C^*-algebra and suppose T is a B-valued circular element in a C^*-noncommutative probability space (A, E). Then for every $n \in \mathbb{N}$, $\epsilon(1), \ldots, \epsilon(n) \in \{1, *\}$ and $b_1, \ldots, b_n \in B$, we have*

$$\left|E(T^{\epsilon(1)}b_1T^{\epsilon(2)}b_2\cdots T^{\epsilon(n)}b_n)\right| \leq \left(\prod_{j=1}^{n}\|b_j\|\right)E(T^{\epsilon(1)}T^{\epsilon(2)}\cdots T^{\epsilon(n)}). \tag{2.16}$$

Proof. Let $\alpha_{1,2}$ and $\alpha_{2,1}$ be the cumulant maps for the pair $(T_1, T_2) = (T, T^*)$, given by

$$\alpha_{1,2}(b) = E(TbT^*), \qquad \alpha_{2,1}(b) = E(T^*bT).$$

Note that, if the sequence $\epsilon(1), \ldots, \epsilon(n)$ is not balanced, then both sides of (2.16) are zero, so we assume the sequence is balanced and we proceed to prove (2.16) by induction on n.

For the case $n = 2$, using (2.13) we get

$$\begin{aligned}|E(T^{\epsilon(1)}b_1T^{\epsilon(2)}b_2)| &= |E(T^{\epsilon(1)}b_1T^{\epsilon(2)})|\,|b_2| \\ &\leq E(T^{\epsilon(1)}|b_1|T^{\epsilon(2)})\,|b_2| \leq \|b_1\|\,E(T^{\epsilon(1)}T^{\epsilon(2)})\,\|b_2\|. \end{aligned} \tag{2.17}$$

For the induction step, let $n \geq 4$ and suppose $\epsilon(1) = 1$. Letting J be as in the proof of Lemma 2.3.1, using the moment-cumulant formula for T we have

$$\begin{aligned}&\left|E(T^{\epsilon(1)}b_1T^{\epsilon(2)}b_2\cdots T^{\epsilon(n)}b_n)\right| \\ &= \left|\sum_{j\in J}\alpha_{1,2}\big(b_1E(T^{\epsilon(2)}b_2\cdots T^{\epsilon(j-1)}b_{j-1})\big)b_jE(T^{\epsilon(j+1)}b_{j+1}\cdots T^{\epsilon(n)}b_n)\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j\in J} \left|\alpha_{1,2}\big(b_1 E(T^{\epsilon(2)}b_2\cdots T^{\epsilon(j-1)}b_{j-1})\big)\right| \|b_j\| \left|E(T^{\epsilon(j+1)}b_{j+1}\cdots T^{\epsilon(n)}b_n)\right| \\
&\leq \sum_{j\in J} \alpha_{1,2}\big(\left|b_1 E(T^{\epsilon(2)}b_2\cdots T^{\epsilon(j-1)}b_{j-1})\right|\big) \left(\prod_{i=j}^{n}\|b_i\|\right) E(T^{\epsilon(j+1)}\cdots T^{\epsilon(n)}) \\
&\leq \sum_{j\in J} \alpha_{1,2}\big(\|b_1\|\left|E(T^{\epsilon(2)}b_2\cdots T^{\epsilon(j-1)}b_{j-1})\right|\big) \left(\prod_{i=j}^{n}\|b_i\|\right) E(T^{\epsilon(j+1)}\cdots T^{\epsilon(n)}) \\
&\leq \sum_{j\in J} \alpha_{1,2}\Big(\left(\prod_{i=1}^{j-1}\|b_i\|\right) E(T^{\epsilon(2)}\cdots T^{\epsilon(j-1)})\Big) \left(\prod_{i=j}^{n}\|b_i\|\right) E(T^{\epsilon(j+1)}\cdots T^{\epsilon(n)}) \\
&= \left(\prod_{i=1}^{n}\|b_i\|\right) \sum_{j\in J} \alpha_{1,2}\big(E(T^{\epsilon(2)}\cdots T^{\epsilon(j-1)})\big)\, E(T^{\epsilon(j+1)}\cdots T^{\epsilon(n)}) \\
&= \left(\prod_{i=1}^{n}\|b_i\|\right) E(T^{\epsilon(1)}\cdots T^{\epsilon(n)})
\end{aligned}
$$

When $\epsilon(1) = *$, the proof is the same but with $\alpha_{2,1}$ replacing $\alpha_{1,2}$. □

Lemma 2.3.3. *Let B be a commutative, unital C^*-algebra and suppose T is a B-valued circular element in a C^*-noncommutative probability space (A, E) with E faithful. Then for every $n \in \mathbb{N}$and $\epsilon(1), \dots, \epsilon(n) \in \{1, *\}$ and all $b_1, \dots, b_n \in B$, we have*

$$
\|T^{\epsilon(1)}b_1T^{\epsilon(2)}b_2\cdots T^{\epsilon(n)}b_n\| \leq \left(\prod_{j=1}^{n}\|b_j\|\right) \|T^{\epsilon(1)}T^{\epsilon(2)}\cdots T^{\epsilon(n)}\|.
$$

In particular, if T is quasinilpotent and $b_1, b_2 \in B$, then b_1Tb_2 is quasinilpotent.

Proof. Let

$$
W = T^{\epsilon(1)}b_1T^{\epsilon(2)}b_2\cdots T^{\epsilon(n)}b_n, \qquad V = T^{\epsilon(1)}T^{\epsilon(2)}\cdots T^{\epsilon(n)}
$$

and let $m = \left(\prod_{j=1}^{n}\|b_j\|\right)$. For any element $x \in A$, we have

$$
\|x\| = \limsup_{n\to\infty} \|E((x^*x)^n)\|^{1/2n}.
$$

Indeed, if ϕ is a faithful state on B, then $\phi \circ E$ is a faithful state on A, and we have

$$\|x\| = \limsup_{n\to\infty} \left(\phi \circ E((x^*x)^n)\right)^{1/2n}.$$

But

$$\phi \circ E((x^*x)^n) \leq \|E((x^*x)^n)\| \leq \|x\|^{2n}.$$

Applying Lemma 2.3.2, for every $n \in \mathbb{N}$ we get

$$E((W^*W)^n) \leq m^{2n} E((V^*V)^n)$$

and we conclude $\|W\| \leq m\|V\|$. □

Lemma 2.3.4. *Let B be a commutative, unital C^*-algebra and suppose T is a quasinilpotent B-valued circular element. Take $Z = b + T$, where $b \in B$ is invertible. Then Z is invertible and for all $n \in \mathbb{N}$, we have*

$$E((Z^n)^*Z^n) \geq (b^n)^*b^n$$

and

$$E((Z^{-n})^*Z^{-n}) \geq (b^{-n})^*b^{-n}.$$

Proof. To prove the first inequality, we first expand Z^n as follows:

$$Z^n = b^n + \sum_{p=1}^{n} \sum_{\substack{q_0,\dots,q_p \geq 0 \\ q_0+\cdots+q_p = n-p}} b^{q_0}Tb^{q_1}\cdots Tb^{q_p}.$$

Since T is B-circular, $E(b_0Tb_1Tb_2\cdots Tb_p) = 0$ for all $p \in \mathbb{N}$ and $b_0, ...b_p \in B$.

If we let $Y_n = Z^n - b^n$, we have $E(b_0Y_n) = 0$, for every $b_0 \in B$. We then have

$$E((Z^n)^*Z^n) = E((b^n)^*b^n) + E((b^n)^*Y_n) + E(Y_n^*b^n) + E(Y_n^*Y_n) = (b^n)^*b^n + E(Y_n^*Y_n) \geq (b^n)^*b^n$$

The proof of the second inequality is largely along the same lines, using the power series expansion for Z^{-1}. We have $Z = b(1 + b^{-1}T)$. By Lemma 2.3.3, $b^{-1}T$ is quasinilpotent. Therefore, Z is invertible and its inverse has the power series expansion

$$Z^{-1} = (1 + b^{-1}T)^{-1}b^{-1} = b^{-1} + \sum_{k=1}^{\infty}(-1)^k(b^{-1}T)^k b^{-1},$$

which converges in norm. Let $Y_1 = Z^{-1} - b^{-1}$ be the summation found above. By using the power series expansion for Y_1 and the fact that T is B-circular, we see

$$E(b_0Y_1b_1Y_1b_2 \cdots Y_1b_p) = 0$$

for all $p \in \mathbb{N}$ and $b_0, b_1, \ldots, b_p \in B$. We have

$$Z^{-n} = b^{-n} + \sum_{p=1}^{n} \sum_{\substack{q_0,\ldots,q_p \geq 0 \\ q_0+\cdots+q_p=n-p}} b^{-q_0}Y_1b^{-q_1} \cdots Y_1b^{-q_p}.$$

Let $Y_n = Z^{-n} - b^{-n}$ be the summation found above. Then we have $E(bY_n) = 0$ for every $b \in B$. Thus, we have

$$\begin{aligned} E((Z^{-n})^*Z^{-n}) = (b^{-n})^*b^{-n} + E((b^{-n})^*Y_n) + E(Y_n^*b^{-n}) + E(Y_n^*Y_n) \\ = (b^{-n})^*b^{-n} + E(Y_n^*Y_n) \geq (b^{-n})^*b^{-n}. \end{aligned}$$

□

As a consequence of Lemma 2.3.4 and Theorem 1.6.4 and we get the following estimates for the traces of powers of a DT-operator Z.

Corollary 2.3.5. *Let $0 < r < s$, and let Z be a DT-operator whose spectrum is contained in the closed annulus $A(r, s)$. Then, we have the following estimates:*

$$\tau((Z^k)^*Z^k)) \geq r^{2k}, \ \forall k \geq 1, \tag{2.18}$$

$$\tau(Z^{*k}Z^k) \geq s^{2k},\ \forall k \leq -1 \tag{2.19}$$

Proof. We have $Z = b + cT$ where T is the quasinilpotent DT-operator and b is a normal operator whose spectrum lies in the closed annulus $A(r,s)$. Since the conditional expectation E is τ-preserving, from Lemma 2.3.4, we have

$$\tau((Z^k)^*Z^k) = \tau \circ E(((Z^k)^*Z^k) \geq \tau((b^k)^*b^k) \geq r^{2k},$$
$$\tau((Z^{-k})^*Z^{-k}) = \tau \circ E((Z^{-k})^*Z^{-k}) \geq \tau((b^{-k})^*b^{-k}) \geq s^{-2k}.$$

□

Angle Estimates for DT-operators

In order to estimate the infimum angle of an operator, it is useful to cut it down by a projection. Restricting an operator to a Haagerup-Schultz projection can only increase the minimum angle between its subspaces:

Lemma 2.3.6. *For an operator $T \in \mathcal{M}$, and a Borel set $B \subset \mathbb{C}$, let $p = P(T,B)$, and consider $Tp = pTp \in p\mathcal{M}p$. Then,*

$$\alpha_{p\mathcal{M}p}(Tp) \geq \alpha(T) \tag{2.20}$$

In particular, if Tp fails to have the UNZA property, then so does T.

Proof. This follows from Theorem 1.3.2 and Proposition 1.3.5. We have

$$P_{p\mathcal{M}p}(Tp, C) = P(T,C) \wedge P(T,B) = P(T, B \cap C) \leq P(T,C) \tag{2.21}$$

and

$$P_{p\mathcal{M}p}(Tp, C^c) = P(T,C^c) \wedge P(T,B) = P(T, B \cap C^c) \leq P(T,C^c) \tag{2.22}$$

so the angles obey the inequality above. □

Note that for any operator $T \in \mathcal{M}$, we have the following inequality:

$$\tau(T^{*n}T^n) \leq \|T^n\|^2 \tag{2.23}$$

Using the spectral radius formula on the right side of the above equation, we get

$$\limsup(\tau(T^{*n}T^n))^{1/2n} \leq r(T) \tag{2.24}$$

where $r(T)$ denotes the spectral radius of T.

We are now ready to estimate the angles between some Haagerup-Schultz subspaces of DT-operators with suitably nice Brown measures.

Lemma 2.3.7. *Let $0 \leq r < r' < s' < s$ and let $A(r, r')$ and $A(s', s)$ denote the closed annuli with inner and outer radii r, r' and s', s respectively. Let Z be a* $\mathrm{DT}(\mu, c)$*-operator whose measure μ is radially symmetric and concentrated in the union of these annuli.*

Let $t = \mu(A(r, r'))$, and assume $0 < t < 1$. Then

$$\cos(\alpha(Z)) \geq \max\left\{\left(1 + \frac{s^2 - r^2}{c^2 t}\right)^{-1/2}, \left(1 + \frac{s^2 - r^2}{c^2(1-t)}\right)^{-1/2}\right\} \geq \left(1 + \frac{2(s^2 - r^2)}{c^2}\right)^{-1/2}. \tag{2.25}$$

Proof. Let μ_1 and μ_2 denote the renormalized restrictions of μ to $A(r, r')$ and $A(s, s')$ respectively. From the upper-triangular model found in Theorem 1.6.3, there exists an example of a $\mathrm{DT}(\mu, 1)$ operator

$$Z = \begin{pmatrix} Z_1 & cpX(1-p) \\ 0 & Z_2 \end{pmatrix},$$

where $p \in \mathcal{M}$ is a projection with $\tau(p) = t$, $Z_1 \in p\mathcal{M}p$ is a $\mathrm{DT}(\mu_1, c\sqrt{t})$-operator, $Z_2 \in (1-p)\mathcal{M}(1-p)$ is a $\mathrm{DT}(\mu_2, c\sqrt{1-t})$-operator, X is a semicircular operator with $\tau(X^2) = 1$ and so that X and $\{Z_1, Z_2, p\}$ are $*$-free. Consider Z as an operator acting on the Hilbert space $L^2(\mathcal{M})$. For $x \in \mathcal{M}$, let $\hat{x}$ denote the corresponding vector in $L^2(\mathcal{M})$. From the description of Haagerup-Schultz projections noted in Proposition 1.3.3, we have

$$P(Z, A(s', s)) = \big\{x : \exists x_n, Z^n x_n \to x, \limsup \|x_n\|^{1/n} \le 1/s\big\}$$

Since DT operators have spectra equal to the supports of their Brown measures, Z_2 is invertible in $(1-p)\mathcal{M}(1-p)$. Let

$$\eta_n = Z_2^{-n}\widehat{(1-p)}$$
$$\xi_n = cpX(1-p)Z_2^{-n-1}\widehat{(1-p)}.$$

Then a direct computation shows

$$Z^n \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^n Z_1^k cpX(1-p)Z_2^{-k-1}\widehat{(1-p)} \\ \widehat{(1-p)} \end{pmatrix}. \tag{2.26}$$

Using $*$-freeness of X and $\{Z_2, p\}$, we get

$$\begin{aligned} \|\xi_n\|^2 &= c^2\tau\big((1-p)(Z_2^{-n-1})^*(1-p)XpX(1-p)Z_2^{-n-1}(1-p)\big) \\ &= c^2\tau(p)\tau(X^*X)\tau\big((1-p)(Z_2^{-n-1})^*(1-p)Z_2^{-n-1}(1-p)\big) \\ &= c^2\tau(p)\tau(1-p)\tau_{(1-p)}((Z_2^{-n-1})^*Z_2^{-n-1}). \end{aligned}$$

Since the trace of a positive operator is bounded above by its norm and since the spectral radius of Z_2^{-1} is $\le (s')^{-1}$, from the formula (2.24), for all $\epsilon > 0$ we have, for all n large enough,

$$\|\xi_n\|^2 \le c^2\frac{(1+\epsilon)^n}{(s')^{2n+2}}t(1-t) \tag{2.27}$$

Similarly, given $\epsilon > 0$, we have, for all suitably large n

$$\|\eta_n\|^2 = \tau((1-p)(Z_2^*)^{-n}Z_2^{-n}(1-p)) \le (1-t)\frac{(1+\epsilon)^n}{(s')^{2n}}. \tag{2.28}$$

Now combining (2.27) and (2.28), we have

$$\limsup_{n\to\infty} \left\| \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} \right\|_2^{1/n} \leq \frac{1}{s'} \tag{2.29}$$

From radial symmetry of μ_1, it follows that Z_1 and λZ_1 have the same $*$-moments for every complex number λ of modulus 1. Thus, $\tau((Z_1^j)^* Z_1^k) = 0$ whenever k and j are nonnegative integers with $j \neq k$. Now using $*$-freeness of X and $\{Z_1, Z_2, p\}$, we calculate, for $0 \leq m \leq n$,

$$\begin{aligned}
&\left\| \sum_{k=m}^{n} Z_1^k cpX(1-p)Z_2^{-k-1}\widehat{(1-p)} \right\|_2^2 \\
&= c^2 \sum_{m\leq k_1,k_2\leq n} \tau\big((1-p)(Z_2^{-k_1-1})^*(1-p)Xp(Z_1^{k_1})^* Z_1^{k_2} pX(1-p)Z_2^{-k_2-1}\big) \\
&= c^2 \sum_{m\leq k_1,k_2\leq n} \tau(1-p)\tau_{(1-p)}((Z_2^{-k_1-1})^* Z_2^{-k_2-1})\tau(X^2)\tau(p(Z_1^{k_1})^* Z_1^{k_2} p) \\
&= c^2\tau(p)\tau(1-p) \sum_{k=m}^{n} \tau_{(1-p)}((Z_2^{-k-1})^* Z_2^{-k-1})\tau_p((Z_1^k)^* Z_1^k)
\end{aligned} \tag{2.30}$$

Now from the bounds r' and $(s')^{-1}$, respectively, on the spectral radii of Z_1 and Z_2^{-1}, we get that for all $\epsilon > 0$, when m is sufficiently large the quantity (2.30) when m is sufficiently large is bounded above by

$$c^2 t(1-t) \sum_{k=m}^{n} (1+\epsilon)^{4k} \frac{(r')^{2k}}{(s')^{2k+2}}.$$

Define

$$\xi = \sum_{k=0}^{\infty} Z_1^k cpX(1-p)Z_2^{-k-1}\widehat{(1-p)}, \qquad \eta = \widehat{(1-p)}$$

The above norm estimate shows that this sum in fact converges in $L^2(\mathcal{M})$ and from (2.26) we have

$$\lim_{n\to\infty} Z^n \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Now using the inequality (2.29), we get

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \in P(Z, A(s', s))\mathcal{H}.$$

From Corollary 2.3.5, we have, for all $k \geq 0$,

$$\tau_{(p)}\left(Z_1^{*k} Z_1^k\right) \geq r^{2k},$$
$$\tau_{(1-p)}\left((Z_2^*)^{-k-1} Z_2^{-k-1}\right) \geq s^{-2k-2}.$$

So using (2.30) we get

$$\|\xi\|_2^2 \geq c^2 t(1-t) \sum_{k=0}^{\infty} \frac{1}{s^2}\left(\frac{r}{s}\right)^{2k} = c^2 \frac{t(1-t)}{s^2 - r^2}. \tag{2.31}$$

Moreover, we have

$$\begin{pmatrix} \xi \\ 0 \end{pmatrix} \in L^2(p\mathcal{M}) \oplus 0 = P(Z, B(r, r'))\mathcal{H}.$$

Computing the cosine of the angle between $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ and $\begin{pmatrix} \xi \\ 0 \end{pmatrix}$ gives

$$\cos\left(\alpha(P(Z, A(r, r')), P(Z, A(s', s))\right) \geq \frac{\|\xi\|}{((1-t) + \|\xi\|^2)^{1/2}} = \left(1 + \frac{(1-t)}{\|\xi\|^2}\right)^{-1/2}.$$

Now using the lower bound (2.31), we get

$$\cos(\alpha(Z)) \geq \left(1 + \frac{s^2 - r^2}{c^2 t}\right)^{-1/2}. \tag{2.32}$$

To get the other lower bound, we do a similar construction. There is an example of a

DT(μ,c) operator

$$Z=\begin{pmatrix} Z_2 & c(1-p)Xp \\ 0 & Z_1 \end{pmatrix},$$

where $p\in\mathcal{M}$ is a projection with $\tau(1-p)=1-t$, $Z_1\in p\mathcal{M}p$ is a DT$(\mu_1,c\sqrt{t})$-operator, $Z_2\in(1-p)\mathcal{M}(1-p)$ is a DT$(\mu_2,c\sqrt{1-t})$-operator and X is a semicircular operator with $\tau(X^2)=1$ and so that X and $\{Z_1,Z_2,p\}$ are $*$-free. As before, regard Z as an operator acting on the Hilbert space $L^2(\mathcal{M})$. From the description of Haagerup-Schultz projections noted in Proposition 1.3.3, we have

$$P(Z,A(r,r'))=\left\{x:\exists x_n, x_n\to x, \limsup\|Z^n x_n\|^{1/n}\le r'\right\}.$$

Let

$$\xi_n=-\sum_{k=0}^{n-1} Z_2^{-k-1}c(1-p)XpZ_1^k\widehat{p}$$

$$\eta_n=\widehat{p}.$$

Then a direct computation shows

$$Z^n\begin{pmatrix}\xi_n\\ \eta_n\end{pmatrix}=\begin{pmatrix}0\\ Z_1^n\widehat{p}\end{pmatrix}. \tag{2.33}$$

Since the spectral radius of Z_1 is $\le r'$,

$$\limsup_{n\to\infty}\left\|Z^n\begin{pmatrix}\xi_n\\ \eta_n\end{pmatrix}\right\|_2^{1/n}\le r'. \tag{2.34}$$

Define

$$\xi=-\sum_{k=0}^{\infty} Z_2^{-k-1}c(1-p)XpZ_1^k\widehat{p},\qquad \eta=\widehat{p}$$

A similar computation as before shows that ξ_n converges to ξ, and

$$\|\xi\|_2^2 \geq c^2 t(1-t) \sum_{k=0}^{\infty} \frac{1}{s^2} \left(\frac{r}{s}\right)^{2k} = c^2 \frac{t(1-t)}{s^2 - r^2}. \tag{2.35}$$

Therefore, it follows that

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \in P(Z, A(r, r'))\mathcal{H}.$$

Moreover, we have

$$\begin{pmatrix} \xi \\ 0 \end{pmatrix} \in L^2((1-p)\mathcal{M}) \oplus 0 = P(Z, A(s', s))\mathcal{H}.$$

Computing the cosine of the angle between $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ and $\begin{pmatrix} \xi \\ 0 \end{pmatrix}$ gives

$$\cos\left(\alpha(P(Z, A(r, r')), P(Z, A(s', s))\right) \geq \frac{\|\xi\|}{(t + \|\xi\|^2)^{1/2}} = \left(1 + \frac{t}{\|\xi\|^2}\right)^{-1/2}. \tag{2.36}$$

Now using the lower bound (2.35), we get

$$\cos(\alpha(Z)) \geq \left(1 + \frac{s^2 - r^2}{c^2(1-t)}\right)^{-1/2}. \tag{2.37}$$

Combining (2.32) and (2.37) gives us (2.25).

□

This lemma allows us to show that many DT-operators are non-spectral.

Theorem 2.3.8. *Let $c > 0$, and let μ be a radially symmetric, compactly supported Borel probability measure on $\mathbb{C}$, such that there exists $x_0 \geq 0$ with*

$$\lim_{\delta \to 0^+} \frac{\mu(A(x_0 - \delta, x_0 + \delta) \setminus A(x_0, x_0))}{\delta} = \infty.$$

Let Z be a $\mathrm{DT}(\mu, c)$ operator. Then Z fails to satisfy the UNZA property, and hence is not spectral.

Proof. Let $N \in \mathbb{N}$. Choose $\epsilon > 0$ such that for all $\delta \in (0, \epsilon]$,

$$\frac{\mu(A(x_0 - \delta, x_0 + \delta) \setminus A(x_0, x_0))}{\delta} > N. \tag{2.38}$$

Let $r = \max\{0, x_0 - \epsilon\}$ and $s = x_0 + \epsilon$. Since $\mu(A(x_0 - \delta, x_0 + \delta)) > 0$ for all $\delta \leq \epsilon$, we have

$$\sup\{x \in (r, s) : \mu(A(r, x)) = 0\} \leq x_0$$

and

$$\inf\{x \in (r, s) : \mu(A(x, s)) = 0\} \geq x_0.$$

Further, these two quantities cannot both be equal to x_0, as this would violate equation (2.38). Using this fact, and the fact that the set of circles centered at 0 with positive μ-measure is countable, we may choose $x_\epsilon \in (r, s)$ such that $\mu(A(r, x_\epsilon)) \neq 0$, $\mu(A(x_\epsilon, s)) \neq 0$, and $\mu(A(x_\epsilon, x_\epsilon)) = 0$. From (2.38), with $\delta = \epsilon$, we have $\mu(A(r, s) \setminus A(x_0, x_0) > \epsilon N$. Since $\mu(A(x_\epsilon, x_\epsilon)) = 0$, we may now choose r', s', with $r < r' < x_\epsilon < s' < s$, such that

$$\mu(A(r, r')) > 0, \mu(A(s', s)) > 0,$$

and

$$\mu(A(r, r') \cup A(s', s)) > \epsilon N. \tag{2.39}$$

Let B be the set $A(r, r') \cup A(s', s)$, $q = P(Z, B)$ and let $\tilde{Z}$ be the restriction Zq. Let $\tilde{\mu}$ be the renormalized restriction of μ to B. From Theorem 1.6.4, Zq is a $\mathrm{DT}(\tilde{\mu}, c\sqrt{\mu(B)})$ operator. Applying Lemma 2.3.7 and Lemma 2.3.6 now gives us the estimate

$$\cos(\alpha(Z)) \geq \cos(\alpha(\tilde{Z})) \geq \left(1 + \frac{2(s^2 - r^2)}{c^2 \mu(B)}\right)^{-1/2}.$$

Since x_0 is in the support of μ, $x_0 \leq \|Z\|$, so $s^2 - r^2 = 4x_0\epsilon \leq 4\epsilon\|Z\|$, if $x_0 - \epsilon > 0$. If $x_0 - \epsilon \leq 0$,

$$s^2 - r^2 = (x_0 + \epsilon)^2 - 0 \leq 4\epsilon^2 \leq 4\epsilon\|Z\|,$$

since we may safely assume that $\epsilon < \|Z\|$. Hence, using (2.39), we get

$$\frac{2(s^2 - r^2)}{c^2\mu(B)} \leq \frac{8\|Z\|}{c^2}\frac{1}{N}.$$

Letting N be arbitrarily large shows that $\cos(\alpha(Z))$ is arbitrarily close to 1, which implies $\alpha(Z) = 0$, and Z fails to have the UNZA property.

□

There are many examples of measures which satisfy the concentration hypothesis of Theorem 2.3.8. For instance, consider the following discrete measures:

Example 2.3.9. Let $a \geq 0$. Let μ be a radially symmetric Borel probability measure supported on the union of the circles C_n of radius $a + 1/n$ $(n \in \mathbb{N})$, with $\mu(C_n) = w_n$. If

$$\lim_{n\to\infty}\left(n\sum_{k=n}^{\infty} w_k\right) = \infty,$$

then μ satisfies the hypothesis of Theorem 2.3.8, so if Z is a $\mathrm{DT}(\mu, 1)$ operator, Z is non-spectral. This happens, for instance, when w_n is asymptotically equal to n^{-b}, for $1 < b < 2$.

Example 2.3.10. Let f be a non-zero compactly supported Lebesgue integrable function on $[0, \infty)$. Assume that $x_0 \in [0, \infty)$ is a point such that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{x_0-\epsilon}^{x_0+\epsilon} |f(x)|\, dx = \infty.$$

Let μ be a radially symmetric measure on $\mathbb{C}$ defined by $\mu(A(r, s)) = \frac{1}{\|f\|_1}\int_r^s |f(x)|\, dx$. Then μ satisfies the hypothesis of Theorem 2.3.8. For instance, this happens when $f(x) = x^{-a}$ in some neighborhood of 0, for $0 < a < 1$.

3. SUMMARY

We have proved that an operator in a finite von Neumann algebra can be expressed as the sum of a commuting spectral and s.o.t.-quasinilpotent operator if and only if the angles between its spectral subspaces are uniformly bounded away from zero (Theorem 2.2.3). This provides (Corollary 2.2.5) a new characterization for when a decomposable operator in a finite von Neumann algebra is also spectral, in terms of the angle between the operator's Haagerup-Schultz projections. We note here that there are more general characterizations for when a decomposable operator on a Banach space is spectral (see, for instance, [14]), involving the local spectra and local spectral subspaces of the operator. However, the advantage of our characterization is that we can estimate the angle between the Haagerup-Schultz subspaces for a large class of operators. This allows us to show that many strongly decomposable operators from free probability theory are non-spectral (Theorem 2.3.8). We believe this is interesting in and of itself, as typical examples of non-spectral operators in the literature involve weighted shift operators, which do not live in a finite von Neumann algebra.

We end by describing a few interesting questions which we were unable to resolve satisfactorily.

3.1 Unresolved Questions

As we noted in a previous section, we have been unable to construct an example of an operator which has the non-zero angle property but fails to have the uniformly non-zero angle property, but we believe that such an operator should exist.

Conjecture 3.1.1. *There is an operator $T \in \mathcal{M}$ which has the NZA property but not the UNZA property.*

In [8], we show that the circular free Poisson operators are non-spectral. The circular free Poisson operators are DT-operators which are also R-diagonal. R-diagonal operators have the same $*$-distribution as UH, where U is a Haar Unitary, $H \geq 0$, and U, H are $*$-free. As

Theorem 2.3.8 shows, we can get rid of the R-diagonal requirement, and extend this result to a broader class of DT-operators. This raises the question of whether there is a larger class of R-diagonal operators which are non-spectral. We formulate this as a conjecture:

Conjecture 3.1.2. *Let $U, H \in \mathcal{M}$ be $*$-free operators, with U unitary, $\tau(U^n) = 0$ for all $n \neq 0 \in \mathbb{Z}$, and $H \geq 0$. If U and H are not scalars, then UH is non-spectral.*

Finally, we note that our techniques say nothing about DT-operators whose Brown measures are supported on unions of finitely many circles. When such an operator is also R-diagonal, a result from [7] implies that the operator is in fact a scalar multiple of a unitary, and hence spectral. It is unclear if all such operators are in fact spectral.

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