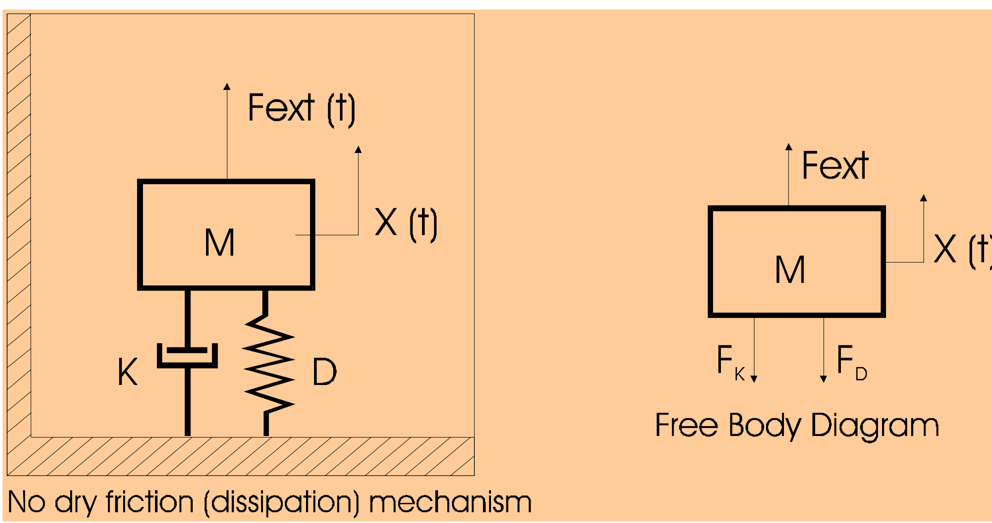
Appendix A: Conservation of Mechanical Energy = Conservation of Linear Momentum

Consider the dynamics of a 2nd order system composed of the fundamental mechanical elements, inertia or mass (*M*), stiffness (*K*), and viscous damping coefficient, (*D*). The **Principle of Conservation of Linear Momentum** (Newton's 2nd Law of Motion) leads to the following 2nd order differential equation:

$$M\ddot{X} + D\dot{X} + KX = F(t) \qquad (1)$$

where $X(t)$ represents the coordinate describing the system motion and $F(t)=F_{ext}$ is the external force applied to the system.



Now, integrate Eq. (1) between two displacements $X_1 = X(t_1)$ and $X_2 = X(t_2)$ occurring at times t_1 and t_2, respectively At these times the system velocities are also given by $\dot{X}_1 = \dot{X}(t_1),\ \dot{X}_2 = \dot{X}(t_2)$, respectively. From Eq. (1) obtain:

$$\int_{X_1}^{X_2} M\,\ddot{X}\,dX + \int_{X_1}^{X_2} D\,\dot{X}\,dX + \int_{X_1}^{X_2} K\,X\,dX = \int_{X_1}^{X_2} F(t)\,dX \qquad (2)$$

The acceleration and velocity are defined as $\ddot{X} = \frac{d\,\dot{X}}{d\,t}, \dot{X} = \frac{d\,X}{d\,t}$, respectively. Using these definitions, write Eq. (2) as:

$$\int_{t_1}^{t_2} M\,\frac{d\,\dot{X}}{dt}\frac{dX}{d\,t}dt + \int_{t_1}^{t_2} D\,\dot{X}\,\frac{dX}{d\,t}dt + \int_{X_1}^{X_2} K\,d\left(\frac{1}{2}X^2\right) = \int_{X_1}^{X_2} F(t)\,dX$$

or,

$$\int_{t_1}^{t_2} M\,\frac{d\,\dot{X}}{dt}\dot{X}\,dt + \int_{t_1}^{t_2} D\,\dot{X}\,\dot{X}\;dt + \int_{X_1}^{X_2} K\,d\left(\frac{1}{2}X^2\right) = \int_{X_1}^{X_2} F(t)\,dX$$

$$\int_{\dot{X}_1}^{\dot{X}_2} M\,d\left(\frac{1}{2}\dot{X}^2\right) + \int_{t_1}^{t_2} D\,\dot{X}\,\dot{X}\;dt + \int_{X_1}^{X_2} K\,d\left(\frac{1}{2}X^2\right) = \int_{X_1}^{X_2} F(t)\,dX \qquad (3)$$

and since *(M,K,D)* are constants, express Eq. (3) as:

$$\frac{1}{2}M\,(\,\dot{X}_2^2 - \dot{X}_1^2\,) + \int_{t_1}^{t_2} D\,\dot{X}^2\;dt + \frac{1}{2}K\,(\,X_2^2 - X_1^2\,) = \int_{X_1}^{X_2} F(t)\,dX \qquad (4)$$

Recognize several of the terms in equation above. These are known as

Change in kinetic energy,

$$T_2 - T_1 = \frac{1}{2} M \dot{X}_2^2 - \frac{1}{2} M \dot{X}_1^2 \qquad (5.a)$$

Change in potential energy,

$$V_2 - V_1 = \frac{1}{2} K X_2^2 - \frac{1}{2} K X_1^2 \qquad (5.b)$$

Total work from external force input into the system,

$$W_{1-2} = \int_{X_1}^{X_2} F(t)\, dX \qquad (5.c)$$

With $P_v = D \dot{X}^2$ as the viscous power dissipation, Then

the dissipated energy (removed from system) is,

$$E_{v_{1-2}} = \int_{t_1}^{t_2} D \dot{X}^2 \; dt = \int_{t_1}^{t_2} P_v \; dt \qquad (5.d)$$

With these definitions, write Eq. (4) as

$$\left(T_2 - T_1\right) + \left(V_2 - V_1\right) + E_{v_{1-2}} = W_{1-2} \qquad (6)$$

That is, the change in (kinetic energy + potential energy) + the viscous dissipated energy = External work. This is also known as **the Principle of Conservation of Mechanical Energy.**

Note that Eq. (1) and Eq. (6) are **NOT** independent. They actually represent the same physical concept. Note also that Eq. (6) is not to be mistaken with the first-law of thermodynamics since it does not account for heat flow and/or changes in temperature.

One can particularize Eqn. (6) for the initial time t_0 with initial displacement and velocities given as $(X_0, \dot{X}_0)$, and at an arbitrary time (*t)* with displacements and velocities equal to $(X(t), \dot{X}(t))$, respectively, i.e.,

$$\left(T_t + V_t\right) + E_{v_t} = W_t + T_0 + V_0 \qquad (7)$$

or, using Eq. (4),

$$\frac{1}{2} M \dot{X}_{(t)}^2 + \frac{1}{2} K X_{(t)}^2 + \int_{t_0}^{t} D \dot{X}^2 \, dt = \int_{X_0}^{X(t)} F(t) dX + \frac{1}{2} M \dot{X}_0^2 + \frac{1}{2} K X_0^2 \qquad (8)$$

Note that the last two terms in the right hand side of equation (8) are constant and represent the initial state of (kinetic + potential) energy of the system.

Now, take the time derivative of Eq.. (8), i.e.

$$\frac{d}{dt}\left[\frac{1}{2} M \dot{X}_{(t)}^2 + \frac{1}{2} K X_{(t)}^2 + \int_{t_0}^{t} D \dot{X}^2 \, dt = \int_{X_0}^{X(t)} F(t) dX + \frac{1}{2} M \dot{X}_0^2 + \frac{1}{2} K X_0^2\right]$$

$$\frac{2}{2} M \dot{X}_{(t)} \frac{d\dot{X}_{(t)}}{dt} + \frac{2}{2} K X_{(t)} \frac{dX_{(t)}}{dt} + D \dot{X}^2 = F(t) \frac{dX_{(t)}}{dt} \qquad (9)$$

Recall that the derivative of an integral function is just the integrand.

Using well-known definitions $\ddot{X} = \frac{d\,\dot{X}}{d\,t}, \dot{X} = \frac{d\,X}{d\,t}$, then Eq. (9) is

$$M\,\dot{X}_{(t)}\,\ddot{X}_{(t)} + K\,X_{(t)}\,\dot{X}_{(t)} + D\,\dot{X}\,\dot{X}_{(t)} = F(t)\,\dot{X}_{(t)}$$

and factoring out the velocity, obtain

$$\left[M\ddot{X}_{(t)} + K X_{(t)} + D\dot{X}\right]\,\dot{X}_{(t)} = F(t)\dot{X}_{(t)}$$

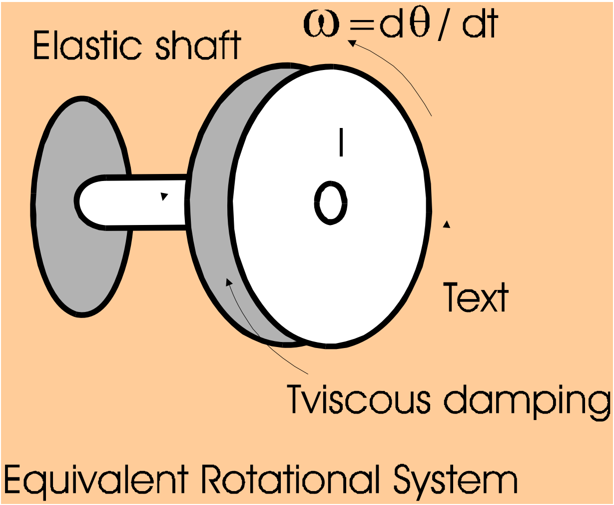
Since for most times the velocity is different from zero, i.e. system is moving; then

$$M\,\ddot{X} + D\,\dot{X} + K\,X = F(t) \qquad (1)$$

i.e., the equation for **conservation of linear momentum**.

Suggestion/recommended work:

Rework the problem for a rotational (torsional) mechanical system and show the equivalence of conservation of mechanical energy to the principle of angular momentum, i.e. start with the following Eqn.



$$I\,\ddot{\theta} + D_\theta\,\dot{\theta} + K_\theta\,\theta = T(t)$$

where (I, D_θ, K_θ) are the equivalent mass moment of inertia, rotational viscous damping and stiffness coefficients, $T(t)=T_{ext}$ is an applied external moment or torque, and $\theta(t)$ is the angular displacement of the rotational system.