STUDY OF SHEAR-DRIVEN UNSTEADY FLOWS OF A FLUID WITH A PRESSURE DEPENDENT VISCOSITY

A Thesis

by

SHRIRAM SRINIVASAN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

December 2008

Major Subject: Mechanical Engineering

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Approved by:

Chair of Committee,	K. R. Rajagopal
Committee Members,	Anastasia Muliana
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ABSTRACT

Study of Shear-Driven Unsteady Flows of a Fluid with a Pressure Dependent Viscosity. (December 2008)

Shriram Srinivasan, B.Tech, National Institute of Technology Tiruchirappalli

Chair of Advisory Committee: Dr. K. R. Rajagopal

In this thesis, the seminal work of Stokes concerning the flow of a Navier-Stokes fluid due to a suddenly accelerated or oscillating plate and the flow due to torsional oscillations of an infinitely long rod in a Navier-Stokes fluid is extended to a fluid with pressure dependent viscosity. The viscosity of many fluids varies significantly with pressure, a fact recognized by Stokes; and Barus, in fact, conducted experiments that showed that the variation of the viscosity with pressure was exponential. Given such a tremendous variation, we study how this change in viscosity affects the nature of the solution to these problems. We find that the velocity field, and hence the structure of the vorticity and the shear stress at the walls for fluids with pressure dependent viscosity, are markedly different from those for the classical Navier-Stokes fluid.

To all my teachers, especially Kaushik Sir and Dr. Salih

ACKNOWLEDGMENTS

First and foremost, I thank Dr. Rajagopal for being my advisor in the true sense of the term and I start my PhD sanguine that the next few years will enrich my life still further. I owe many thanks to the faculty members; their courses have been challenging and illuminating and have helped harden the bedrock of my understanding. Finally, I thank my colleagues Mr. Satish Karra, Mr. Saradhi Koneru and Dr Waqar Malik for being generous with their time and advice every time I faced a hurdle; their knowledge of things ranging from Mechanics and Mathematics to Linux has benefitted me immensely.

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CHAPTER I

INTRODUCTION

In deriving the momentum equations for an incompressible fluid, Stokes [1] had assumed the viscosity to be a constant and obtained the celebrated Navier-Stokes equations which bear his name. However, he was well aware of the fact that the viscosity could depend on the pressure (mean normal stress) and he delineated the conditions under which his assumption of constant viscosity was tenable. The functional form of the dependence of viscosity on the pressure was put forward by Barus [2] in 1893 as the Barus formula:

$$\mu = \mu_0 e^{\alpha p}.$$

Later, a more general expression was proposed by Andrade [3] which also accounted for the effects of temperature and density and was given by:

$$\mu(p, \rho, \theta) = A\rho^{\frac{1}{2}} \exp\left((p + \rho^2 r)\frac{s}{\theta}\right),$$

where θ denotes the temperature, A, r and s are constants and ρ denotes the density. A comprehensive literature review of the variation of viscosity with pressure, before 1931, can be found in the book by Bridgman [4]. Some recent experiments concerning fluids with pressure dependent viscosity are those carried out by Cutler et al. [5], Griest et al. [6], Johnson and Cameron [7] , Johnson and Greenwood [8], Johnson and Tevaarwerk [9], Bendler et al. [10], Paluch et al. [11], Bair and Kottke [12], Casalini and Bair [13], Harris and Bair [14] and Bair and Quareshi [15].

We shall merely mention in passing the rigorous studies concerning fluids with pressure dependent viscosity. Early studies by Renardy [16], Gazzola [17] and Gaz-

The journal model is *IEEE Transactions on Automatic Control.*

zola and Sechi [18] concerned existence of solutions to problems involving small data for short time. Existence of solutions for large data and long time have been established for various situations when the viscosity depends on both the pressure and the symmetric part of the velocity gradient, by Málek et al. [19], Hron et al. [20], Franta et al. [21] and Bulíček et al. [22]. However, all the above existence results were established under the condition $\frac{\mu(p)}{p} \to 0$ as $p \to \infty$, or $\frac{\mu(p)}{p} \to$ constant as $p \to \infty$. Unfortunately, experiments suggest $\frac{\mu(p)}{p} \to \infty$ as $p \to \infty$ (which is indeed the case for Barus' formula). Thus, rigorous results such as existence of solutions for fluid with pressure dependent viscosity are open with regard to the type of variation of the viscosity with pressure observed in experiment.

For many liquids, the density changes by only a few percent and hence the fluid can be considered incompressible. However, the flow of such fluids in certain cases may engender large variations of pressure in the flow domain which may lead to the viscosity varying by a factor as large as 10^8. Applications such as elastohydrodynamics immediately come to mind, where the high pressures in question make necessary the need to take the viscosity variations into account. Thus the constitutive equation for these fluids may be written as

$$\mathbf{T} = -p\mathbf{I} + 2\mu(p)\mathbf{D}, \tag{1.1}$$

with

$$\mathbf{D} = \frac{1}{2}\left[(\mathrm{grad}\mathbf{v}) + (\mathrm{grad}\mathbf{v})^T\right], \tag{1.2}$$

where $\mathbf{v}$ is the velocity and $-p\mathbf{I}$ is the response due to the constraint of incompressibility. Rewriting (1.1) by using the definition of pressure as the mean normal stress yields

$$\mathbf{T} = \left(-\frac{\mathrm{tr}\mathbf{T}}{3}\right)\mathbf{I} + 2\mu\left(\frac{\mathrm{tr}\mathbf{T}}{3}\right)\mathbf{D},$$

and allows us to note as an aside that this fluid model is markedly different from the classical Navier-Stokes model in the sense that it is a member of a more general set of models based on implicit constitutive theories which cannot be derived in the usual way by specifying that constraint forces do no work, since that procedure forbids material functions from depending on the constraint response (in this case the pressure). The reader may consult Rajagopal [23] for a detailed discussion of the relevant issues.

It is such an incompressible fluid with a pressure dependent viscosity that is considered in this study in two particular flow domains, namely, flows between two finitely spaced infinite parallel plates and flows in the annulus of two infinitely long coaxial cylinders.

Some special problems have been studied using the semi-inverse technique for the flows of fluid with pressure dependent viscosities. Hron et al. [24] obtained explicit exact solutions for Couette and Poiseuille flow between parallel plates. In marked contrast to the case of Navier-Stokes fluid, they were able to find multiple solutions for certain values of the parameters that appear in the problem. They also studied numerically the flow in the annular region between two cylinders that rotate about distinct axes, a problem that has technological relevance to the flow in a journal bearing. They also studied the flow of a fluid with pressure dependent viscosity across a slot. Vasudevaiah and Rajagopal [25] obtained explicit exact solutions for fully developed flow of fluids whose viscosity depends on both the pressure and the shear rate, Sharat and Rajagopal[26] considered the flow due to a boundary that is being stretched and more recently Rajagopal [27] studied the flow of fluids with pressure dependent viscosity down an inclined plane. The cases taken up in this study are also semi-inverse solutions, but unlike these earlier studies, we consider unsteady flows in all but one case.

CHAPTER II

FLOW BETWEEN INFINITE PARALLEL PLATES: A VARIANT OF STOKES' FIRST AND SECOND PROBLEMS

The flow between two finitely spaced infinite parallel plates is the first of the cases that we will study. The flow induced by a suddenly accelerating plate on the fluid above it, usually referred to as Stokes' first problem (see Stokes [28]) and the flow due to an oscillating flat plate, usually referred to as Stokes' second problem (see Stokes [28], Rayleigh [29]) are amongst a handful of unsteady flows of a Navier-Stokes fluid for which one can obtain an exact solution. Such exact solutions serve a dual purpose, that of providing an explicit solution to a problem that has physical relevance and as a means for testing the efficiency of complex numerical schemes for flows in complicated flow domains. These two problems have been extended to the case of a host of non-Newtonian fluids.

We shall consider the problem of a fluid with pressure dependent viscosity flowing between two parallel plates, one of which is fixed and the other either oscillating or suddenly accelerating (see Figure 1). We take into account the effect of gravity which we suppose acts perpendicular to the parallel plates. Unlike the problems considered by Stokes we cannot consider the problem due to suddenly accelerating plate or an oscillating plate wherein the fluid occupies the half-space above the plate. As we allow for the effects of gravity, the pressure and consequently the viscosity become unbounded when we allow the fluid to extend to infinity. We have to recognize that all these problems are approximations to real flow situations. Firstly, the assumption that the plates are of infinite extent allows us to introduce similarity transformations as we do not need to take into account edge effects. Secondly, Stokes ignored the effect of gravity which alllowed him to ignore the fact that the pressure at the plate

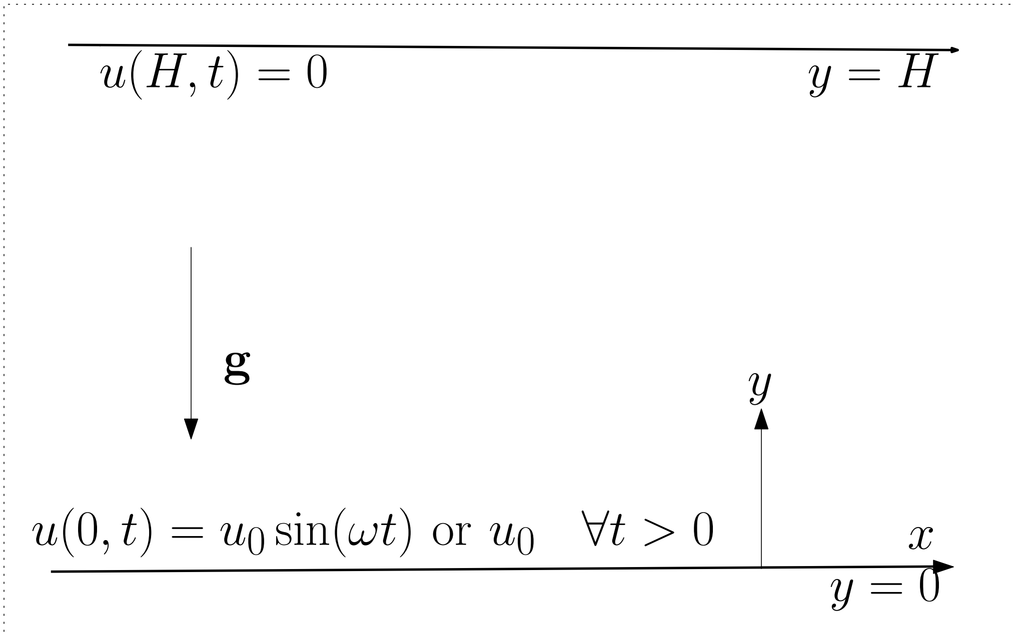

Fig. 1. Flow between a fixed plate and an oscillating/impulsively started plate

is infinite. To ignore the effect of gravity when the fluid above the plate is infinite could be viewed as an unacceptable approximation. However, the true test of such approximations is how well the results agree with the actual situation and in the case of the problems studied by Stokes they seem to provide reasonable approximations in a region of the flow domain. The test for our approximations will once again be some comparable application. One possibility that comes to mind is the flow in the annular region between two cylinders whose radii are sufficiently large so that the flow in a part of the annulus could be approximated as the flow between two parallel plates.

A. Governing Equations

We shall consider the flow of fluids whose Cauchy stress $\mathbf{T}$ is given by (1.1). Since the fluid is incompressible, it can only undergo isochoric motion and thus

$$\operatorname{div} \mathbf{v} = 0. \tag{2.1}$$

On substituting (1.1) into the balance of linear momentum

$$\rho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbf{T} + \rho \mathbf{b},$$

where $\mathbf{b}$ is the specific body force, we obtain

$$\rho \frac{d\mathbf{v}}{dt} = -\frac{\partial p}{\partial \mathbf{x}} + \rho \mathbf{b} + 2\mathbf{D} \frac{d\mu}{d\mathbf{x}} + \mu(p) \triangle \mathbf{v}. \tag{2.2}$$

As we shall be interested in unsteady unidirectional flows, we shall seek similarity solutions of the form

$$\mathbf{v} = u(y,t)\mathbf{i}, \quad p = p(y,t). \tag{2.3}$$

The above velocity field automatically meets (2.1). It immediately follows from (2.3) and the balance of linear momentum (2.2) that

$$\begin{aligned} &\rho \frac{du}{dt} = \mu'(p) \frac{\partial u}{\partial y} \frac{\partial p}{\partial y} + \mu(p) \frac{\partial^2 u}{\partial y^2}, \\ &\frac{\partial p}{\partial y} + \rho g = 0. \end{aligned} \tag{2.4}$$

Suppose the pressure at $y = H$ is p_H, then it immediately follows that

$$p - p_H = \rho g H \left(1 - \frac{y}{H}\right). \tag{2.5}$$

Also equation (2.4) simplifies to

$$\frac{\partial u}{\partial t} = \frac{1}{\rho} \frac{\partial}{\partial y} \left(\mu(p) \frac{\partial u}{\partial y} \right). \tag{2.6}$$

We shall consider the following two forms for the viscosity :

$$\mu(p) = \alpha e^{\beta p}, \quad \alpha > 0, \quad \beta > 0, \tag{2.7}$$

and

$$\mu(p) = \alpha(1 + \beta p), \quad \alpha > 0, \quad \beta > 0. \tag{2.8}$$

We notice that in both cases considered, the viscosity tends to infinity as p tends to infinity.In the case of (2.7), $\dfrac{\mu(p)}{p} \to \infty$ as $p \to \infty$ while in the case of (2.8),$\dfrac{\mu(p)}{p} \to$ constant as $p \to \infty$. The form (2.7) is used in problems in elastohydrodynamics (see Szeri [30]). Bair et al. [31], using Barus' formula for the viscosity (2.7), found that β^{-1} was approximately equal to 50 MPa for certain mineral oils.

1. Solution to the modified Stokes' first problem

Before we discuss the solution to Stokes' First problem, we shall appropriately non dimensionalize the governing equations. We introduce the appropriate non-dimensional variables indicated with an overbar through

$$\begin{aligned} &\bar{y} = \frac{y}{H}, \quad \bar{t} = \frac{tu_0}{H}, \quad \bar{u} = \frac{u}{u_0}, \\ &\bar{p} = \frac{p}{p_0}, \quad \bar{\mu} = \frac{\mu}{\mu_0}, \end{aligned} \tag{2.9}$$

where p_0 is a representative pressure and $\mu_0 = \mu(p_0)$. We shall choose $p_0 = p_H$. The above non-dimensionalization leads to the two non-dimensional parameters

$$Re = \frac{\rho u_0 H}{\mu_0}, \quad C = \frac{\rho g H}{\left(\dfrac{1}{\beta}\right)}.$$

The first of the above parameters is the Reynolds number and the second is the ratio of the pressure due to gravity to the pressure due to viscosity(as β is the pressure coefficient for the viscosity and $\left(\dfrac{1}{\beta}\right)$ has units of pressure). Notice that

$C = 0$ corresponds to the case of the classical Navier-Stokes fluid.

The appropriate boundary conditions are

$$u(0,t) = u_0 \qquad \forall \quad t > 0, \tag{2.10}$$

and

$$u(H,t) = 0 \qquad \forall \quad t > 0, \tag{2.11}$$

while the appropriate initial conditions are

$$u(y,0) = 0 \qquad \forall \quad 0 \leq y \leq H, \tag{2.12}$$

Thus,

$$\begin{aligned} \bar{u}\left(0,\bar{t}\right) = 1 \qquad & \forall \quad \bar{t} > 0, \\ \bar{u}\left(1,\bar{t}\right) = 0 \qquad & \forall \quad \bar{t} > 0, \\ \bar{u}\left(\bar{y},0\right) = 0 \qquad & \forall \quad 0 \leq \bar{y} \leq 1, \end{aligned} \tag{2.13}$$

The non-dimensional version of the governing equation is

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{1}{Re}\frac{\partial}{\partial \bar{y}}\left(\bar{\mu}(\bar{p})\frac{\partial \bar{u}}{\partial \bar{y}}\right), \tag{2.14}$$

where in virtue of (2.5),

$$\bar{p} = 1 + \frac{\rho g H}{p_H}\left(1 - \bar{y}\right).$$

If we use the viscosity given by (2.7), we find that

$$\bar{\mu}(\bar{p}) = e^{C(1-\bar{y})}, \tag{2.15}$$

while if the viscosity is given by (2.8),

$$\bar{\mu}(\bar{p}) = 1 + \frac{C\left(1-\bar{y}\right)}{1+\beta p_H}. \tag{2.16}$$

We set $\beta^{-1} = 50\,\mathrm{MPa}$ and $p_H = 100\mathrm{kPa}$, so that $\beta p_H \ll 1$. However, even for the extreme case of $\beta p_H = 1$, we found negligible change in the solution. We need to solve (2.14), subject to the boundary and initial conditions (2.13) and $\bar{\mu}$ given by (2.15) or (2.16).

We are not able to find an explicit exact solution and so we solve the system numerically, using a parabolic partial differential equation solver pdepe available in MATLAB.

Our aim is to show that the solution in the case of a fluid with pressure dependent viscosity is markedly different from that for the Navier-Stokes fluid. To illustrate this point we just provide the solution for the velocity, vorticity and the shear stress at the plate at $\bar{y} = 0$, at a representative time $\bar{t} = \dfrac{3\pi}{2}$ in Figures 2 to 4 respectively for the case of $\bar{\mu}(\bar{p})$ given by (2.15). We notice that the velocity for the Navier-Stokes fluid at a fixed value of $\bar{y}, 0 \leq \bar{y} \leq 1$, is lesser than that corresponding to the fluid with pressure dependent viscosity and more the value of C less than value of the velocity. This is perfectly in keeping with physical expectation as the viscosity in the case of the pressure dependent fluid, at a fixed $\bar{y}$, is greater than that for a fluid with constant viscosity equal to $\mu(p_H)$ and because of this the fluid with greater viscosity gets dragged further than that with less viscosity. This immediately leads to results for the vorticity and the shear stress at the wall as indicated. It is clear from Figures 3 and 4 that the variation of the vorticity and shear stress with $\bar{y}$ departs significantly from that for the Navier-Stokes case. We see that the magnitude of the vorticity at the bottom plate that is moving is over 300% larger in the classical Navier-Stokes case and even the qualitative structure of the vorticity is different; while in the case of the Navier-Stokes fluid the magnitude of the vorticity decreases monotonically from the bottom to the top plate, the variation is non-monotonic in the case of fluids with pressure dependent viscosity that have been considered. On the other hand, we find that the magnitude of the shear stress at the plate at $y = 0$ is nearly 400% greater than that for the Navier-Stokes fluid.

A comparison of the solutions for the velocity, vorticity and shear stress for different forms for the viscosity, namely, the constant viscosity Navier-Stokes model, the model depending linearly on the pressure and finally the model depending exponentially on the pressure are provided in Figures 5 - 7. Once again, we find that the results are in keeping with physical expectation, with the velocity being lesser for fluids with lesser viscosity, at a fixed value of $\bar{y}$, leading to corresponding profiles for the vorticity and shear stress profiles, which are both qualitatively and quantitatively different than that for the Navier-Stokes fluid.

2. Solution to the modified Stokes' second problem

Here we shall consider the case wherein the plate at $y = 0$ is oscillating with speed $u_0 \sin \omega t$ while that at $y = H$ is held fixed. We shall find it convenient to non-dimensionalize the equation differently, namely by introducing

$$\begin{aligned} &\bar{y} = \frac{y}{H}, \quad \bar{t} = \omega t, \quad \bar{u} = \frac{u}{u_0}, \\ &\bar{p} = \frac{p}{p_0}, \quad \bar{\mu} = \frac{\mu}{\mu_0}, \end{aligned} \tag{2.17}$$

which lead to the non-dimensional parameters

$$Re = \frac{\rho \omega H^2}{\mu_0}, \quad C = \frac{\rho g H}{\left(\dfrac{1}{\beta}\right)}. \tag{2.18}$$

In this case we have to enforce the boundary conditions

$$u(0, t) = u_0 \sin \omega t \tag{2.19}$$

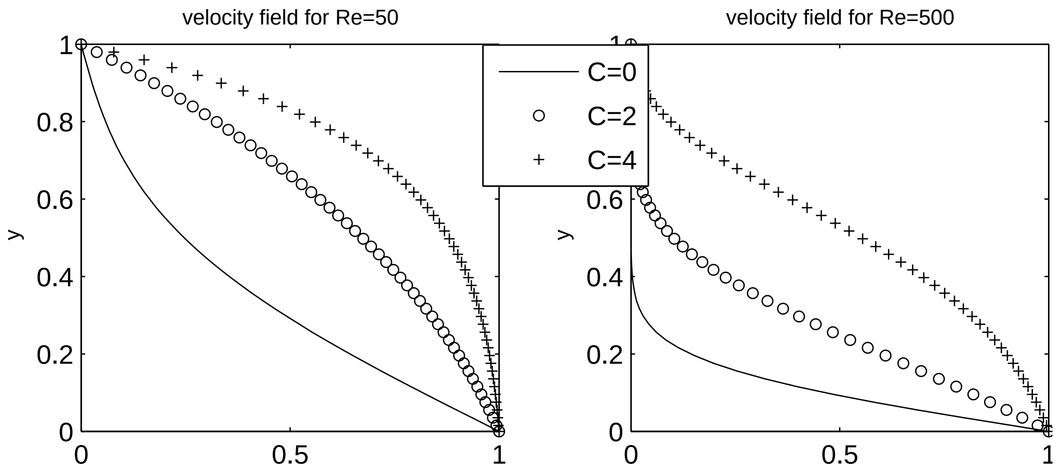

Fig. 2. Velocity $\bar{u}(\bar{y})$ for accelerated plate at time $\bar{t} = \dfrac{3\pi}{2}$

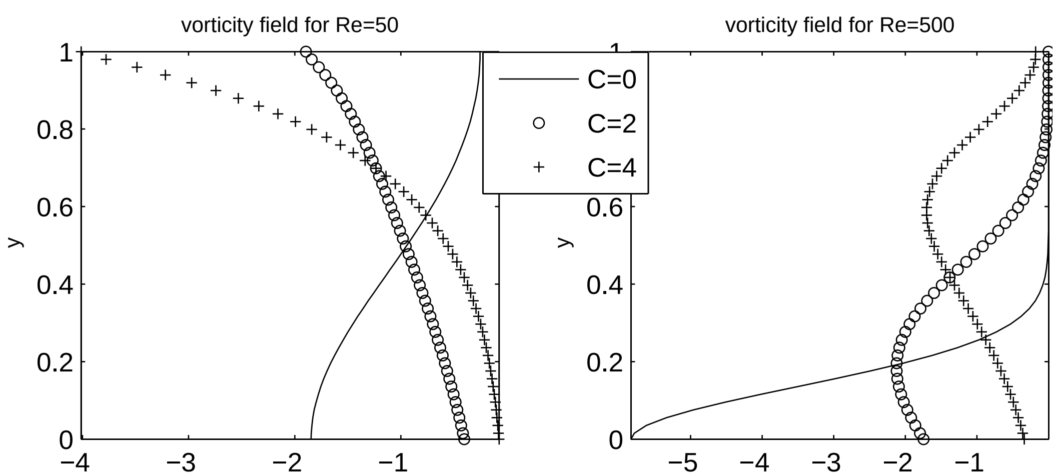

Fig. 3. Vorticity $\bar{\omega}(\bar{y})$ for accelerated plate at time $\bar{t} = \dfrac{3\pi}{2}$

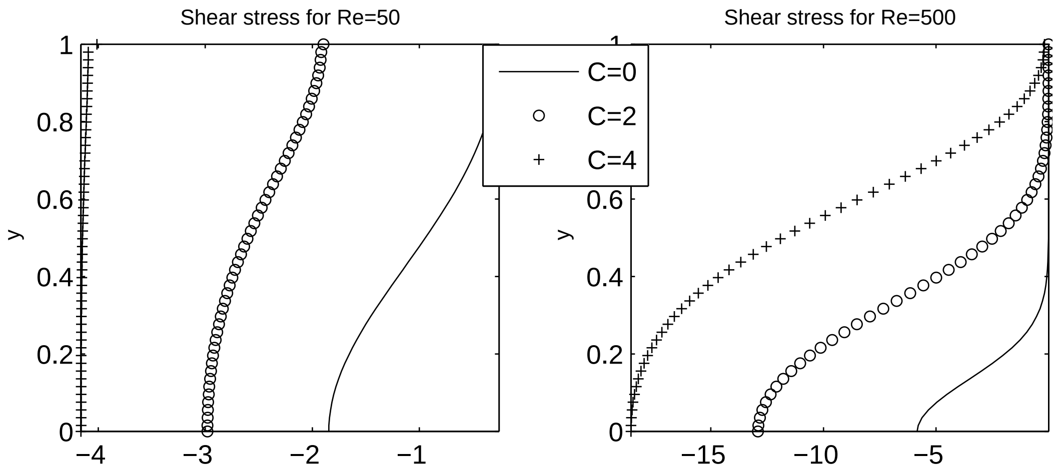

Fig. 4. Shear stress $\bar{\tau}(\bar{y})$ for accelerated plate at time $\bar{t} = \dfrac{3\pi}{2}$

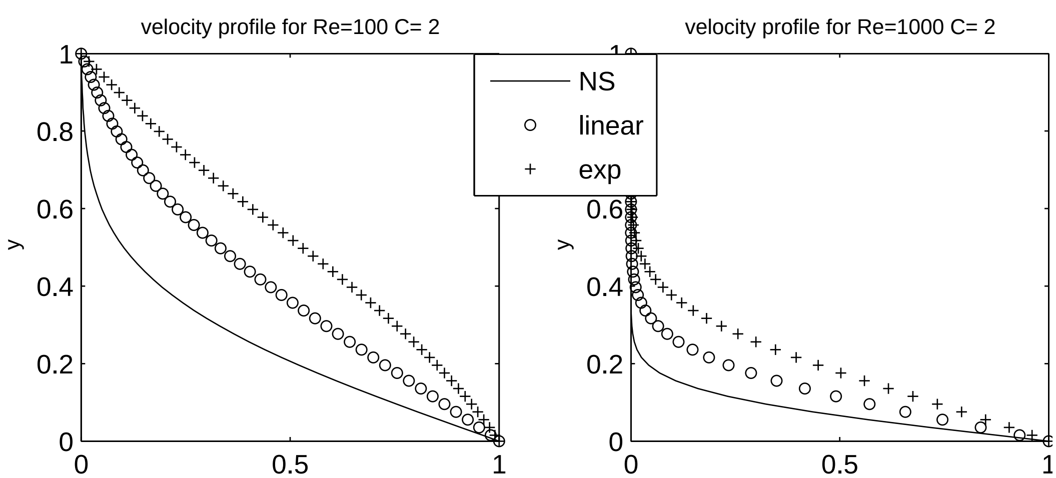

Fig. 5. Velocity $\bar{u}(\bar{y})$ for different viscosity models at $\bar{t} = \dfrac{3\pi}{2}$ for accelerated plate

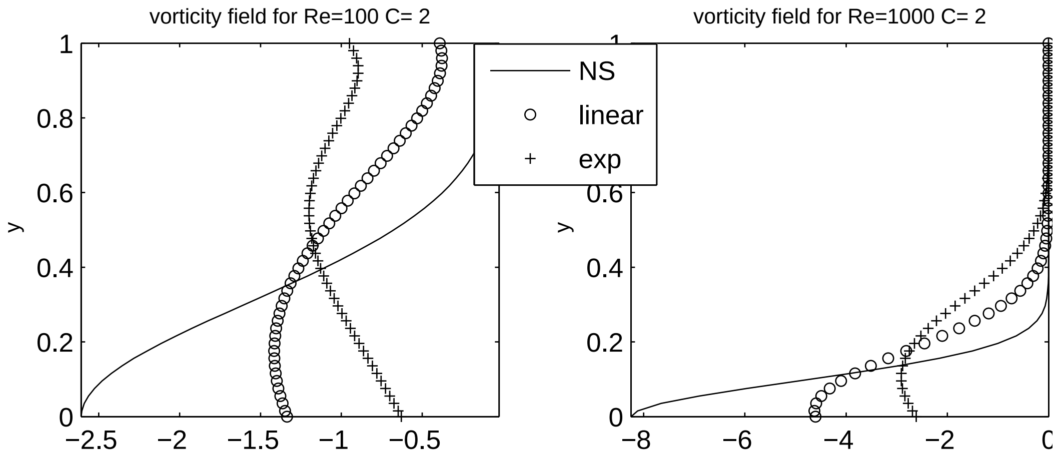

Fig. 6. Vorticity $\bar{\omega}(\bar{y})$ for different viscosity models at $\bar{t} = \dfrac{3\pi}{2}$ for accelerated plate

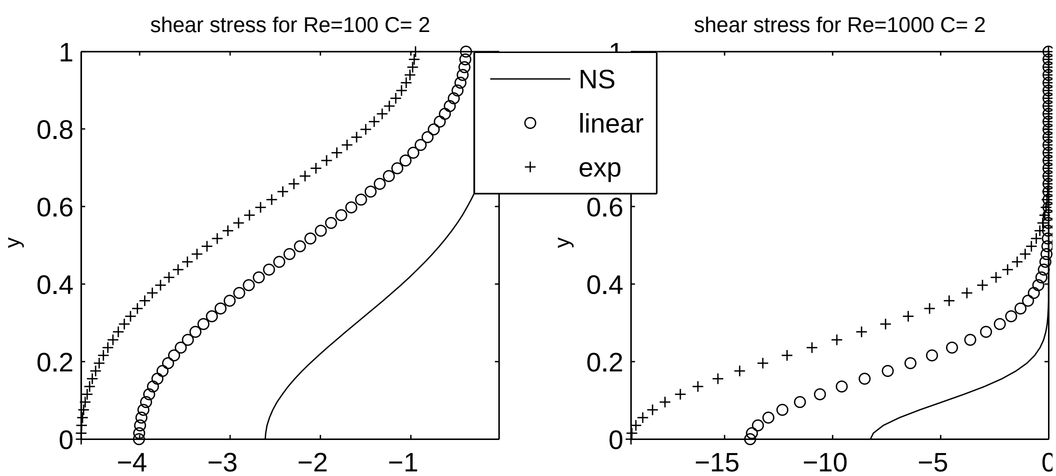

Fig. 7. Shear stress $\bar{\tau}(\bar{y})$ for different viscosity models at $\bar{t} = \dfrac{3\pi}{2}$ for accelerated plate

and

$$u(H,t) = 0, \tag{2.20}$$

which leads to

$$\begin{aligned} \bar{u}\,(0,\bar{t}) &= \sin\bar{t} \\ \bar{u}\,(1,\bar{t}) &= 0. \end{aligned} \tag{2.21}$$

The governing equation is once again given by (2.14)) with Re defined through (2.18). As before, we solve the system (2.14), (2.21), (2.15), (2.16), numerically using MATLAB. A representative solution for the velocity, vorticity and the shear stress at the bottom plate are given in Figures 8 - 10 respectively. Once again, the results for the velocity field are consistent with physical expectation in virtue of the forms for the viscosity, with the magnitude of the velocity being greater in the case of the fluid with pressure dependent viscosity being dragged along more due to its larger viscosity. The profiles for the vorticity and shear stress are a consequence of the solution for the velocity and as before the vorticity for the Navier-Stokes case, at the oscillating plate, at the instant of time being considered is much larger than the pressure dependent case and it is also qualitatively different as the Navier-Stokes case shows a monotonic variation while the pressure dependent case is non monotonic. Finally, the solution for different forms of viscosity are portrayed in Figures 11 - 13. The important point to observe is that the solutions corresponding to the viscosity depending on pressure are significantly different from that for the Navier-Stokes fluid.

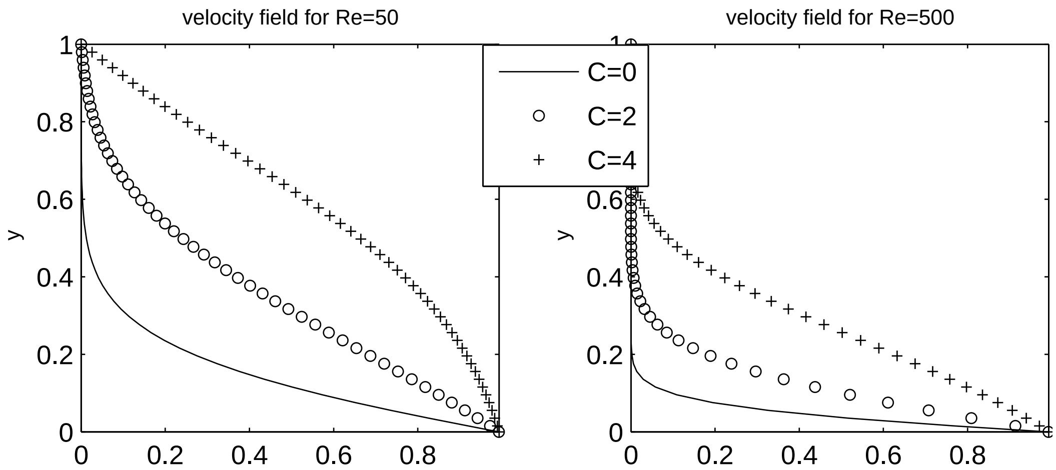

Fig. 8. Velocity $\bar{u}(\bar{y})$ for oscillating plate at time $\bar{t} = \dfrac{\pi}{2}$

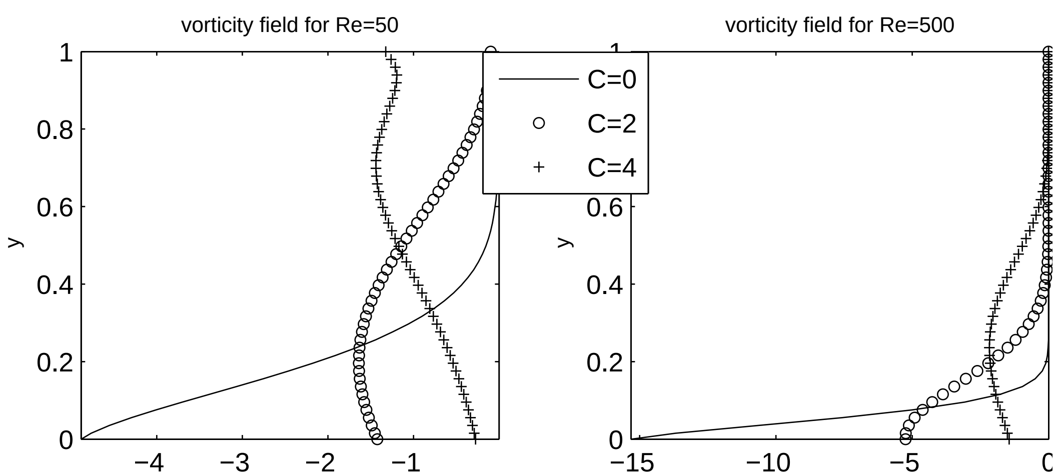

Fig. 9. Vorticity $\bar{\omega}(\bar{y})$ for oscillating plate at time $\bar{t} = \dfrac{\pi}{2}$

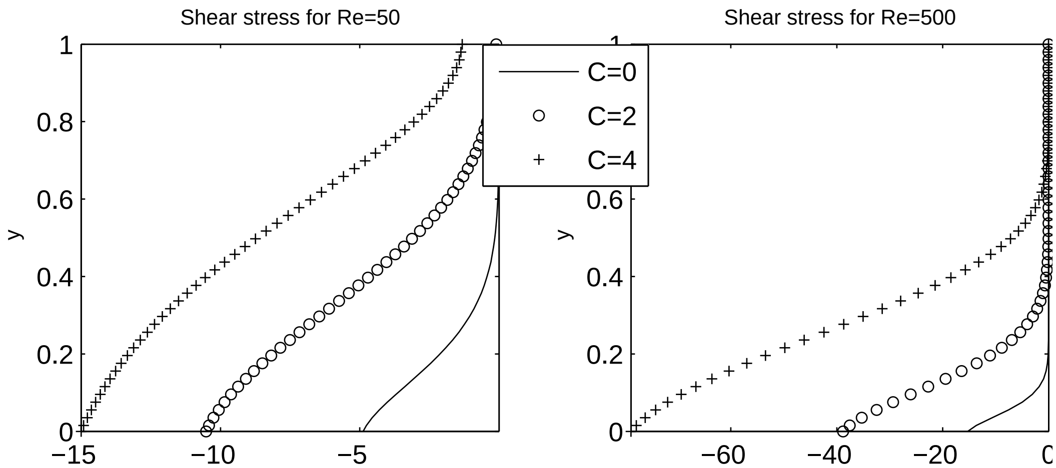

Fig. 10. Shear stress $\bar{\tau}(\bar{y})$ for oscillating plate at time $\bar{t} = \frac{\pi}{2}$

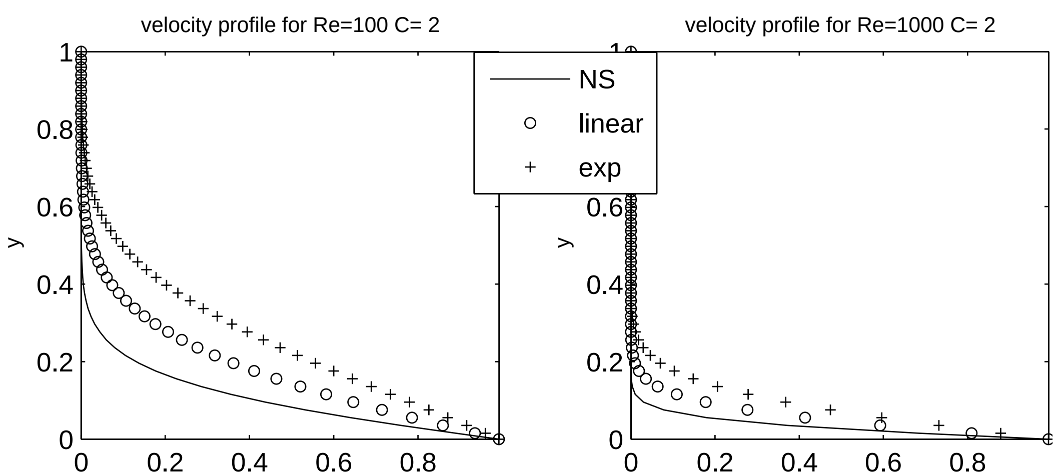

Fig. 11. Velocity $\bar{u}(\bar{y})$ for different viscosity models at $\bar{t} = \frac{\pi}{2}$ for oscillating plate

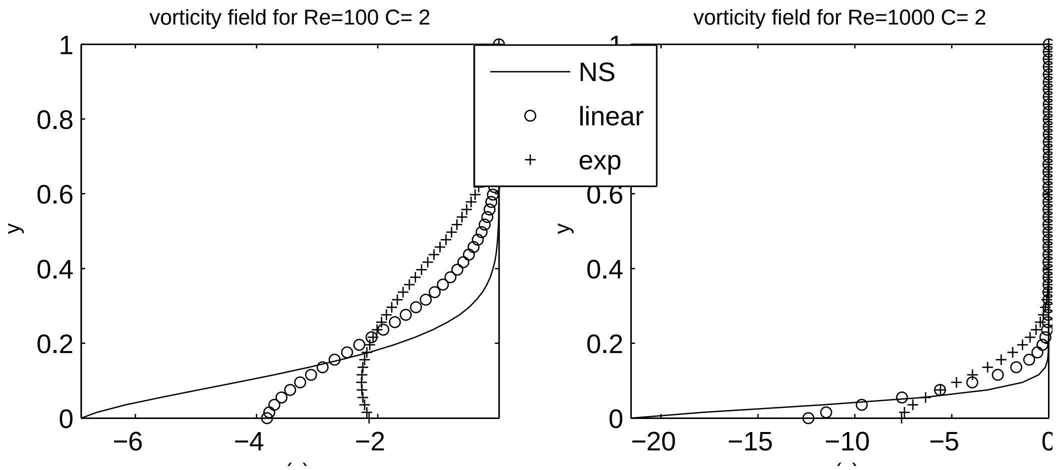

Fig. 12. Vorticity $\bar{\omega}(\bar{y})$ for different viscosity models at $\bar{t} = \dfrac{\pi}{2}$ for oscillating plate

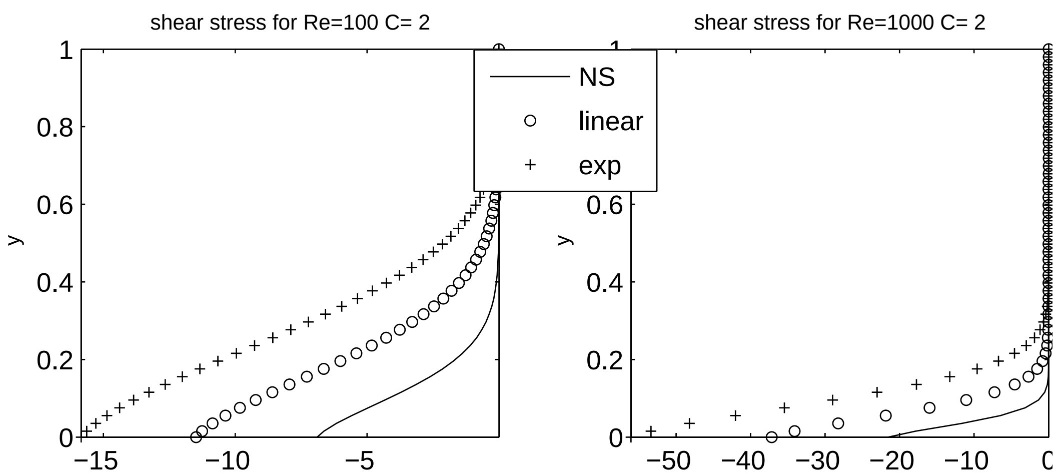

Fig. 13. Shear stress $\bar{\tau}(\bar{y})$ for different viscosity models at $\bar{t} = \dfrac{\pi}{2}$ for oscillating plate

CHAPTER III

FLOW IN THE ANNULUS OF INFINITELY LONG COAXIAL CYLINDERS

The next case that we study is flow in the annulus of two infinitely long coaxial cylinders. It should not surprise the reader to know that Stokes [32] was perhaps the first to interest himself in such a problem, though it is not eponymous unlike the ones solved earlier. Stokes however, confined his attention to the problem of rotational oscillations of an infinite rod immersed in a classical Navier-Stokes fluid. More recently, Casarella and Laura [33] dealt with the problem of a rod undergoing both torsional and longitudinal oscillations. The problem has been extended for a host of non-Newtonian fluids as evinced by the studies of Rajagopal [34], Rajagopal et al [35], Rajagopal and Bhatnagar [36], Maneschy and Massoudi [37] and Massoudi and Phuoc [38], to name a few. We shall address the problem in the context of an incompressible fluid fluid with a pressure dependent viscosity. The problem has special relevance in ocean engineering applications such as off shore drilling and towing operations where the viscous drag of the fluid is a quantity of interest.

The previous studies consider an infinite rod in an infinite extent of fluid, i.e, the outer cylinder is taken to be of infinite radius. But the pressure term appears explicitly in our equations and not just as a gradient and we cannot allow the outer radius to extend to infinity because then the pressure and consequently the viscosity become unbounded. We shall thus investigate flows of a fluid with pressure dependent viscosity in the annulus of two coaxial cylinders (see Figure 14). The radius of the inner and outer cylinders are denoted by R_{in} and R_{out} respectively. The cylinders are assumed to be infintely long so that end effects are neglected and the effect of gravity too is not considered here, so that the velocity fields may be assumed to be axisymmteric. The outer cylinder is assumed stationary while the inner cylinder is free to rotate or oscillate.

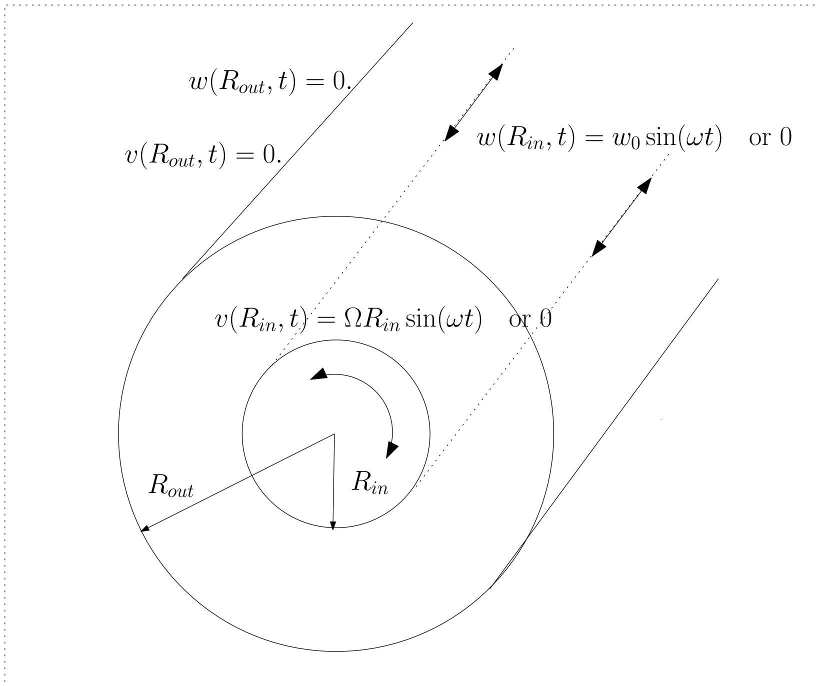

Fig. 14. Flow in the annulus of two infinitely long coaxial cylinders

A. Governing Equations

We shall consider the flow of fluids whose Cauchy stress $\mathbf{T}$ is given by (1.1) and assume the two forms for the viscosity given by (2.7) and (2.8).

Since the fluid is incompressible, it can only undergo isochoric motion and thus

$$\operatorname{div} \mathbf{v} = 0. \tag{3.1}$$

The balance of linear momentum is :

$$\rho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbf{T} + \rho \mathbf{b}, \tag{3.2}$$

where the specific body force is denoted by $\mathbf{b}$.

We shall exploit the cylindrical symmetry in the problem by taking recourse to cylindrical coordinates (r, θ, z), with z directed along the axis of the cylinder.

We consider the most general velocity field

$$\mathbf{v} = v_r \mathbf{e_r} + v_\theta \mathbf{e}_\theta + v_z \mathbf{e_z}. \tag{3.3}$$

The equations (3.1) and (3.2), on neglecting body forces, are written out as

$$\begin{aligned}
&\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0, \\
&\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r}, \\
&\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2T_{r\theta}}{r}, \\
&\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = \frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{rz}}{r}.
\end{aligned} \tag{3.4}$$

The most general matrix representation for grad$\mathbf{v}$ is

$$\text{grad}\mathbf{v} = \begin{pmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - v_\theta\right) & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_z}{\partial \theta}\right) & \dfrac{\partial v_z}{\partial z} \end{pmatrix}, \tag{3.5}$$

using which, the matrix representation of $\mathbf{D}$ can be found from (1.2).

The assumption of axisymmetry implies that all quantities are independent of θ. Since the cylinders are infinitely long, we expect the solutions to be similar all along the length, i.e., to be independent of z as well. These observations considerably simplify the governing equations and allow us to solve them for some simple cases.

We will merely mention that the vorticity was computed as

$$\mathbf{W} = -\frac{\partial v_z}{\partial r}\mathbf{e}_\theta + \frac{1}{r}\frac{\partial\left(r v_\theta\right)}{\partial r}\mathbf{e}_\mathbf{z},$$

while the shear stress

$$T_{r\theta} = \mu(p)\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right] \quad \text{and} \quad T_{rz} = \mu(p)\frac{\partial v_z}{\partial r}.$$

The drag per unit length on the inner cylinder, which is often the quantity of interest is then simply

$$|F_D| = 2\pi R_{in}\sqrt{T_{r\theta}^2 + T_{rz}^2}\Big|_{r=R_{in}}$$

1. Steady rotation of inner cylinder

The first case we consider is one where the inner cylinder rotates with a constant angular velocity Ω while the outer cylinder is stationary.

As we shall be interested in steady flows, we shall seek similarity solutions of the form (see Figure 14)

$$\mathbf{v} = v(r)\mathbf{e}_\theta, \;\; p = p(r). \tag{3.6}$$

The solution sought automatically meets (3.1) and it immediately follows from (1.1), (1.2), (3.5) and the balance of linear momentum (3.4) that

$$\begin{aligned} &\frac{dp}{dr} - \frac{\rho v^2}{r} = 0, \\ &\frac{d}{dr}\left[\mu(p)\left(\frac{dv}{dr} - \frac{v}{r}\right)\right] + \frac{2\mu(p)}{r}\left(\frac{dv}{dr} - \frac{v}{r}\right) = 0. \end{aligned} \tag{3.7}$$

We will suppose the pressure at the inner cylinder surface $r = R_{in}$ is p_i, so that $p(R_{in}) = p_i$. The pressure at the outer cylinder surface $p(R_{out})$ is determined from this boundary condition.

We shall consider forms of the viscosity given earlier by (2.7) and (2.8).

Before we discuss the solution to the problem, we shall appropriately non dimensionalize the governing equations. We introduce the appropriate non-dimensional variables indicated with an overbar through

$$\begin{aligned} \bar{r} &= \frac{r}{r_0}, \quad \bar{v} = \frac{v}{v_0}, \\ \bar{p} &= \frac{p}{p_0}, \quad \bar{\mu} = \frac{\mu}{\mu_0}, \end{aligned} \tag{3.8}$$

where p_0, μ_0, r_0 and v_0 are representative quantities. We shall choose $r_0 = R_{in}$, $v_0 = \Omega R_{in}$, $p_0 = \rho v_0^2$ and $\mu_0 = \mu(p_0)$.

The above non-dimensionalization leads to the following three non-dimensional parameters which are pertinent to this problem:

$$\eta = \frac{R_{out}}{R_{in}}, \quad k = \frac{p_i}{p_0}, \quad C = \frac{\rho v_0^2}{\left(\frac{1}{\beta}\right)}.$$

The last of the above parameters, C, is the ratio of the dynamic pressure to the pressure due to viscosity(as β is the pressure coefficient for the viscosity and $\left(\frac{1}{\beta}\right)$ has units of pressure). Notice that $C = 0$ corresponds to the case of the classical Navier-Stokes fluid. The appropriate boundary conditions are

$$v(R_{in}) = v_0, \tag{3.9}$$

and

$$v(R_{out}) = 0. \tag{3.10}$$

Thus,

$$\begin{aligned} \bar{v}(1) &= 1, \\ \bar{v}(\eta) &= 0, \\ \bar{p}(1) &= k. \end{aligned} \tag{3.11}$$

The non-dimensional version of the governing equation is

$$\begin{aligned} &\frac{d\bar{p}}{d\bar{r}} - \frac{\bar{v}^2}{\bar{r}} = 0, \\ &\frac{d}{d\bar{r}}\left[\bar{\mu}(\bar{p})\left(\frac{d\bar{v}}{d\bar{r}} - \frac{\bar{v}}{\bar{r}}\right)\right] + \frac{2\bar{\mu}(\bar{p})}{\bar{r}}\left(\frac{d\bar{v}}{d\bar{r}} - \frac{\bar{v}}{\bar{r}}\right) = 0. \end{aligned} \tag{3.12}$$

If we use the viscosity given by (2.7), we find that

$$\bar{\mu}(\bar{p}) = e^{C(\bar{p}-1)}, \tag{3.13}$$

while if the viscosity is given by (2.8)

$$\bar{\mu}(\bar{p}) = \frac{1 + C\bar{p}}{1 + C} \tag{3.14}$$

We need to solve (3.12), subject to the boundary condition (3.11) and $\bar{\mu}$ given by (3.13) or (3.14).

For our problem, we select $\eta = \dfrac{R_{out}}{R_{in}} = 2$. We are not able to find an explicit exact solution and so we solve the system numerically, using a solver for boundary value problems in ordinary differential equations bvp4c available in MATLAB.

Our objective is to show that solutions to problems involving pressure dependent viscosity are markedly different from those for Navier-Stokes fluids. To illustrate, we provide the profiles for pressure, velocity, vorticity and shear stress in Figures 15 and 16 for the case when $\bar{\mu}(\bar{p})$ is given by (3.13). The non-dimensional drag force on the inner cylinder is also tabulated for reference in Tables I and II. The most striking feature we observe is that at a given value of $\bar{r}$, the velocity of the Navier-Stokes fluid is maximum and the velocity decreases as the value of C increases. This might appear counterintuitive at a first glance for we expect that as the value of C and consequently $\bar{\mu}(\bar{p})$ increases, the fluid in contact with the surface should get dragged along more than a Navier-Stokes fluid and hence have greater velocity. However, it does not happen because the pressure and hence the viscosity increases radially (see

(3.12)) Thus, if we consider a layer of fluid, it gets dragged in opposite directions at its top and bottom surface, but the viscosity at its top is higher. As a result, the more the value of C, the lesser the velocity at a particular $\bar{r}$. This immediately leads to a similar trend in pressures as seen in Figure 15. The maximum vorticity and shear stress increase with C and their variation with $\bar{r}$ departs significantly from that of a Navier-Stokes fluid. The drag force on the cylinder, tabulated in Tables I through III shows remarkable variation and justifies the need to take pressure dependence of viscosity into account.

A comparison of the solutions for different forms of the viscosity, namely, (3.13), (3.14) and the constant viscosity Navier-Stokes case $\bar{\mu}(\bar{p}) = 1$, is also given in Figure 17. The results are entirely in keeping with physical expectation.

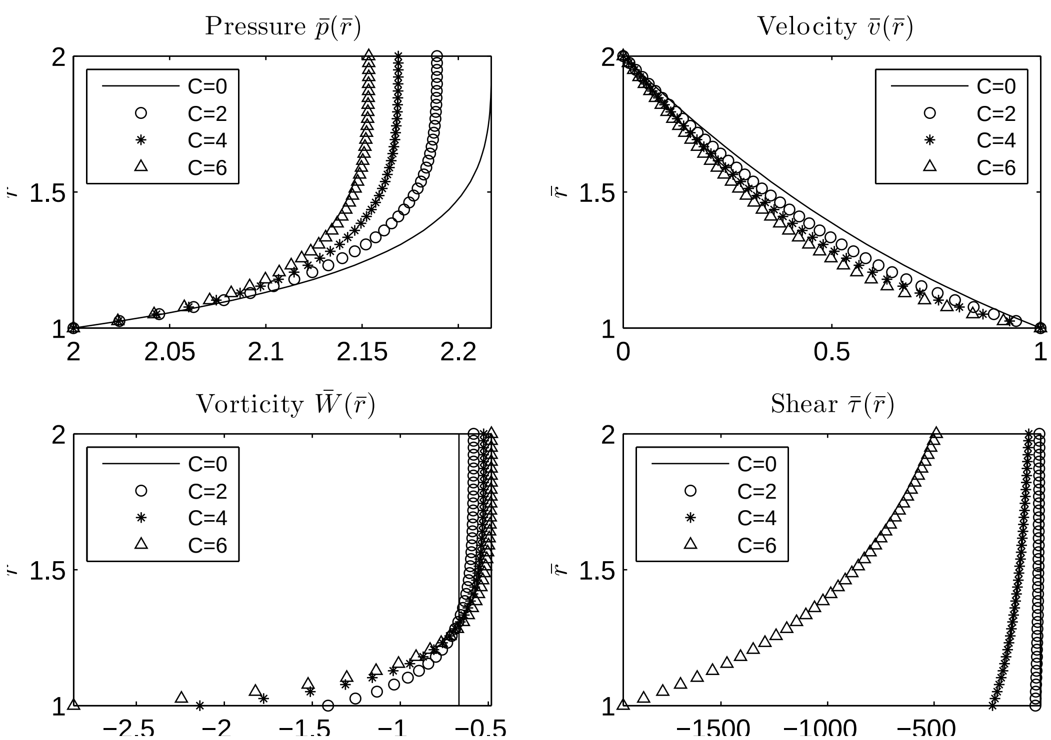

Fig. 15. Steady rotation ($k = 2$)

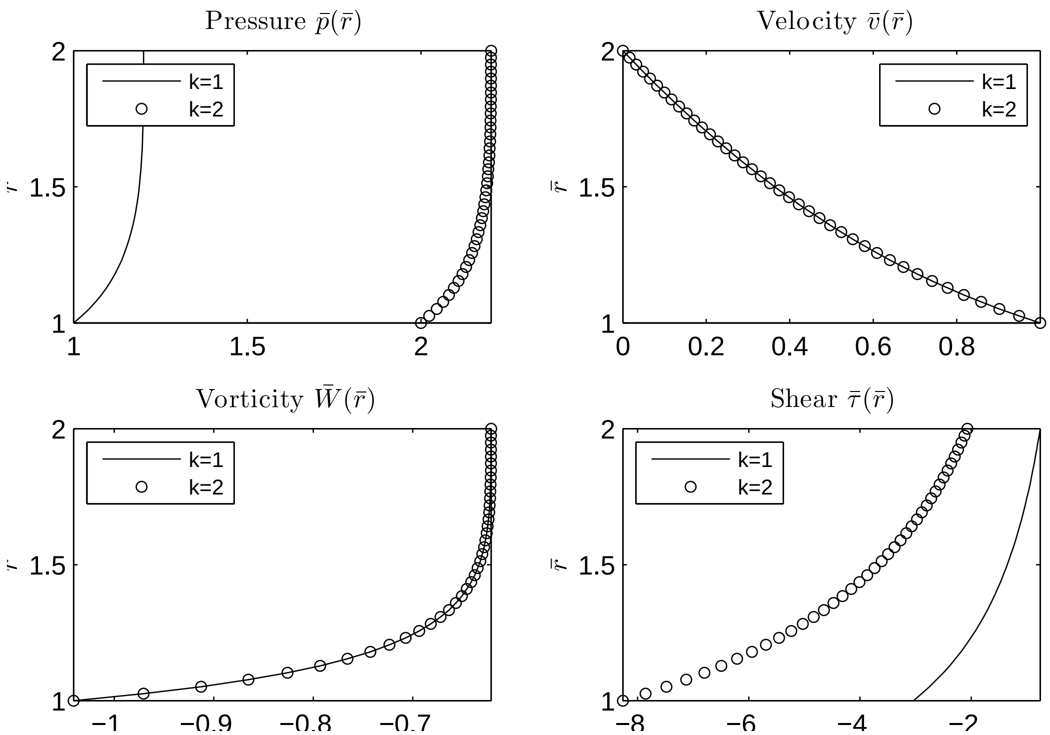

Fig. 16. Steady rotation ($C = 1$)

Table I. Drag force/length for steady state rotation ($Re = 100$, $k = 2$)

Drag on cylinder	$C = 0$	$C = 2$	$C = 4$	$C = 6$
$\bar{F}_D$	-2.66	-25.19	-2.25×10^2	-1.95×10^3

Table II. Drag force/length for steady state rotation ($Re = 100$, $C = 1$)

Drag on cylinder	$k = 1$	$k = 2$
$\bar{F}_D$	-3.04	-8.26

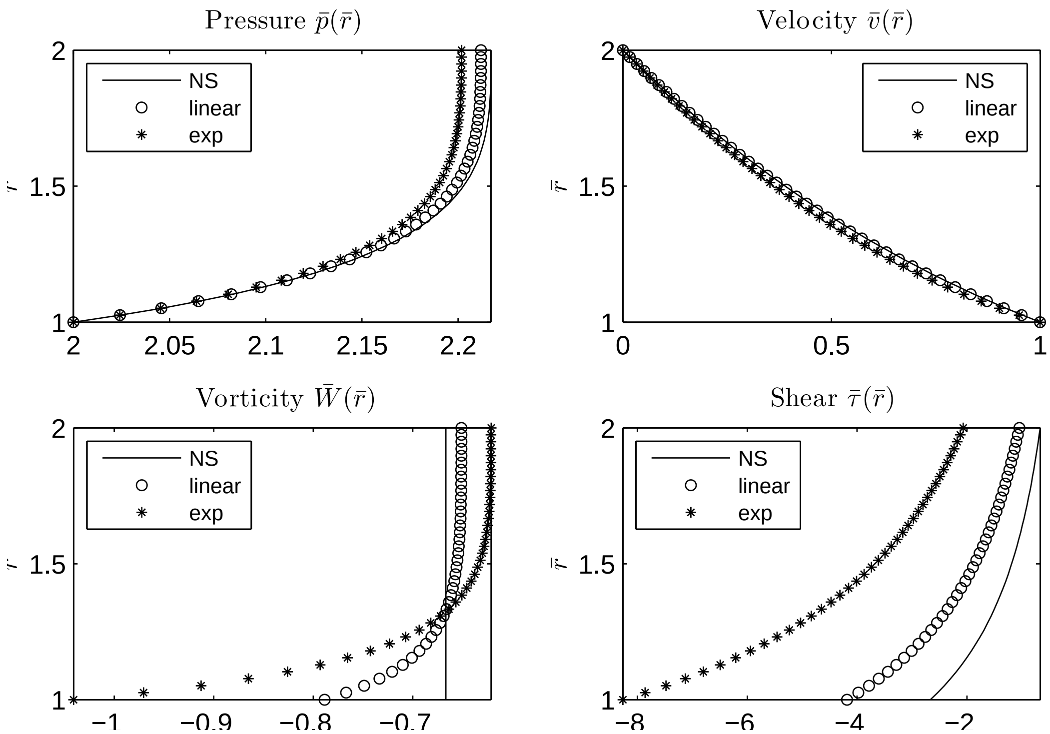

Fig. 17. Steady rotation ($C = 1$, $k = 2$)

Table III. Drag force/length for steady state rotation ($Re = 100$, $C = 1$, $k = 2$)

Drag on cylinder	$\mu(p) = \alpha$	$\mu(p) = \alpha(1 + \beta p)$	$\mu(p) = \alpha e^{\beta p}$
$\bar{F}_D$	-2.66	-4.18	-8.26

2. Torsional and longitudinal oscillations of inner cylinder

We now turn our attention to the problem of torsional and longitudinal oscillations of the inner cylinder while the outer cylinder remains stationary. Here we assume the torsional oscillations to be given by $\Omega \sin \omega_1 t$ while the longitudinal oscillations may be taken as $w_0 \sin \omega_2 t$.

We seek a velocity of the form

$$\mathbf{v} = v(r,t)\mathbf{e}_\theta + w(r,t)\mathbf{e}_\mathbf{z}, \;\; p = p(r,t). \tag{3.15}$$

By following the now familiar process of simplifying the momentum equation (3.4), it immediately follows that:

$$\begin{aligned} &\frac{\partial p}{\partial r} - \frac{\rho v^2}{r} = 0, \\ &\rho \frac{\partial v}{\partial t} = \frac{\partial}{\partial r}\left[\mu(p)\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\right] + \frac{2\mu(p)}{r}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \\ &\rho \frac{\partial w}{\partial t} = \frac{\partial}{\partial r}\left[\mu(p)\left(\frac{\partial w}{\partial r}\right)\right] + \frac{\mu(p)}{r}\left(\frac{\partial w}{\partial r}\right) \end{aligned} \tag{3.16}$$

The non-dimensionalisation here is done as follows:

$$\begin{aligned} &\bar{r} = \frac{r}{r_0}, \quad \bar{v} = \frac{v}{v_0}, \quad \bar{t} = \omega_1 t \\ &\bar{p} = \frac{p}{p_0}, \quad \bar{\mu} = \frac{\mu}{\mu_0}, \end{aligned} \tag{3.17}$$

where p_0, μ_0, r_0 and v_0 are representative quantities. We shall choose $r_0 = R_{in}$, $v_0 = \Omega R_{in}$, $p_0 = \rho v_0^2$ and $\mu_0 = \mu(p_0)$.

The above non-dimensionalization leads to the following non-dimensional parameters which are pertinent to this problem:

$$\eta = \frac{R_{out}}{R_{in}}, \quad k = \frac{p_i}{p_0}, \quad \gamma = \frac{w_0}{v_0}$$
$$\alpha_0 = \frac{\omega_2}{\omega_1} \quad C = \frac{\rho v_0^2}{\left(\frac{1}{\beta}\right)}, \quad Re = \frac{\rho \omega_1 r_0^2}{\mu_0}$$

The boundary conditions are

$$\begin{aligned} v(R_{in}, t) &= v_0 \sin \omega_1 t, \quad v(R_{out}, t) = 0, \\ w(R_{in}, t) &= w_0 \sin \omega_2 t, \quad w(R_{out}, t) = 0. \end{aligned} \tag{3.18}$$

Thus,

$$\begin{aligned} &\bar{v}(1, \bar{t}) = \sin \bar{t}, \quad \bar{w}(1, \bar{t}) = \gamma \sin \alpha_0 \bar{t}, \\ &\bar{v}(\eta, \bar{t}) = 0, \quad \bar{w}(\eta, \bar{t}) = 0, \\ &\bar{p}(1, \bar{t}) = k. \end{aligned} \tag{3.19}$$

The non-dimensional version of the governing equation is

$$\begin{aligned} &\frac{\partial \bar{p}}{\partial \bar{r}} - \frac{\bar{v}^2}{\bar{r}} = 0, \\ &Re \frac{\partial \bar{v}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{r}} \left[\bar{\mu}(\bar{p}) \left(\frac{\partial \bar{v}}{\partial \bar{r}} - \frac{\bar{v}}{\bar{r}} \right) \right] + \frac{2\bar{\mu}(\bar{p})}{\bar{r}} \left(\frac{\partial \bar{v}}{\partial \bar{r}} - \frac{\bar{v}}{\bar{r}} \right) \\ &Re \frac{\partial \bar{w}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{r}} \left[\bar{\mu}(\bar{p}) \left(\frac{\partial \bar{w}}{\partial \bar{r}} \right) \right] + \frac{\bar{\mu}(\bar{p})}{\bar{r}} \left(\frac{\partial \bar{w}}{\partial \bar{r}} \right) \end{aligned} \tag{3.20}$$

For simplicity, we also assume that $w_0 = v_0$ and $\omega_2 = \omega_1$. This immediately gives $\gamma = \alpha_0 = 1$.

The equations (3.20) and (3.19) are solved numerically, using an implicit finite difference scheme for the velocity that employs forward differences for time derivatives and central differences for space derivatives while an explicit forward differencing is used to solve for the pressure field (see Appendix A for details).

As before the solutions are illustrated by displaying profiles for shear stress, pressure and the components of vorticity and velocity in Figures 18 - 20. It appears from Figure 18 that at a given $\bar{r}$, the velocity first increases with C and then decreases. This is perfectly in keeping with our explanation given earlier for the steady state case. However the trend is not monotonic because the boundary condition at $\bar{r} = 1$ in this problem is sinusoidal, which makes the boundary layer thinner. Hence the velocity profile we see is a result of this effect and at higher C, the velocity of a fluid layer becomes lesser due to the fluid being retarded by the layers on top with greater viscosity.

Different forms of the viscosity are used as before and results displayed in Figure 20. It clearly shows that there is sufficient departure from the Navier-Stokes case and a huge variation in the drag force (often by a factor of 10) that we witness in Tables IV through VI and it makes germane the study of flows in the context of pressure dependent viscosity.

Table IV. Drag force per unit length for oscillating cylinder ($Re = 100$, $k = 2$)

Drag on cylinder	$C = 0$	$C = 2$	$C = 4$	$C = 6$
$\bar{F}_D$	63.72	2.41×10^2	1.59×10^3	1.37×10^4

Table V. Drag force per unit length for oscillating cylinder ($Re = 100$, $C = 1$)

Drag on cylinder	$k = 1$	$k = 2$
$\bar{F}_D$	64.68	1.18×10^2

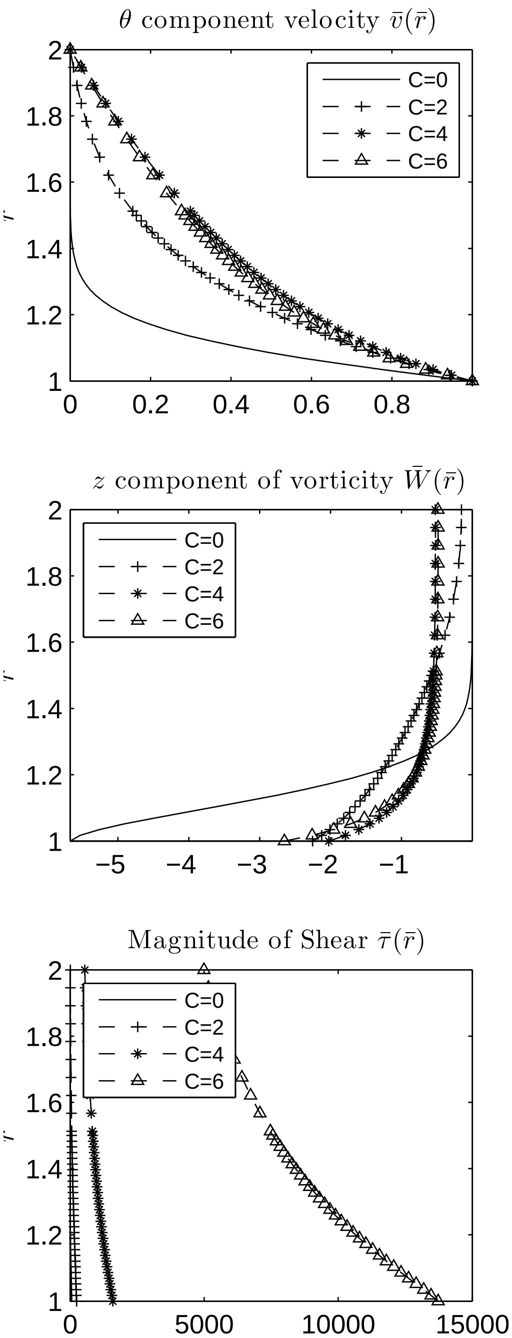

Fig. 18. Torsional and longitudinal oscillations ($Re = 100$, $k = 2$) at $\bar{t} = \dfrac{\pi}{2}$

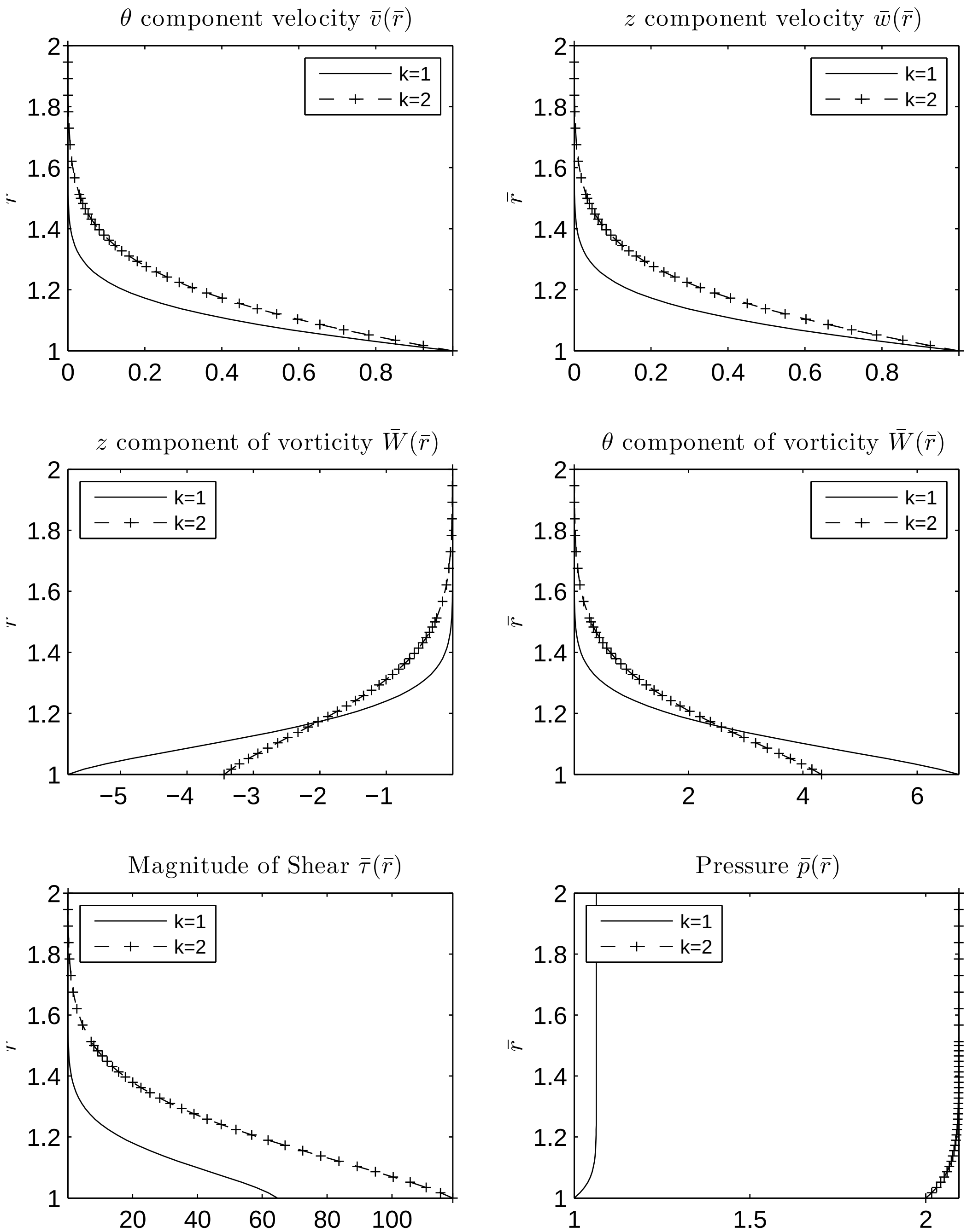

Fig. 19. Torsional and longitudinal oscillations ($Re = 100$, $C = 1$) at $\bar{t} = \dfrac{\pi}{2}$

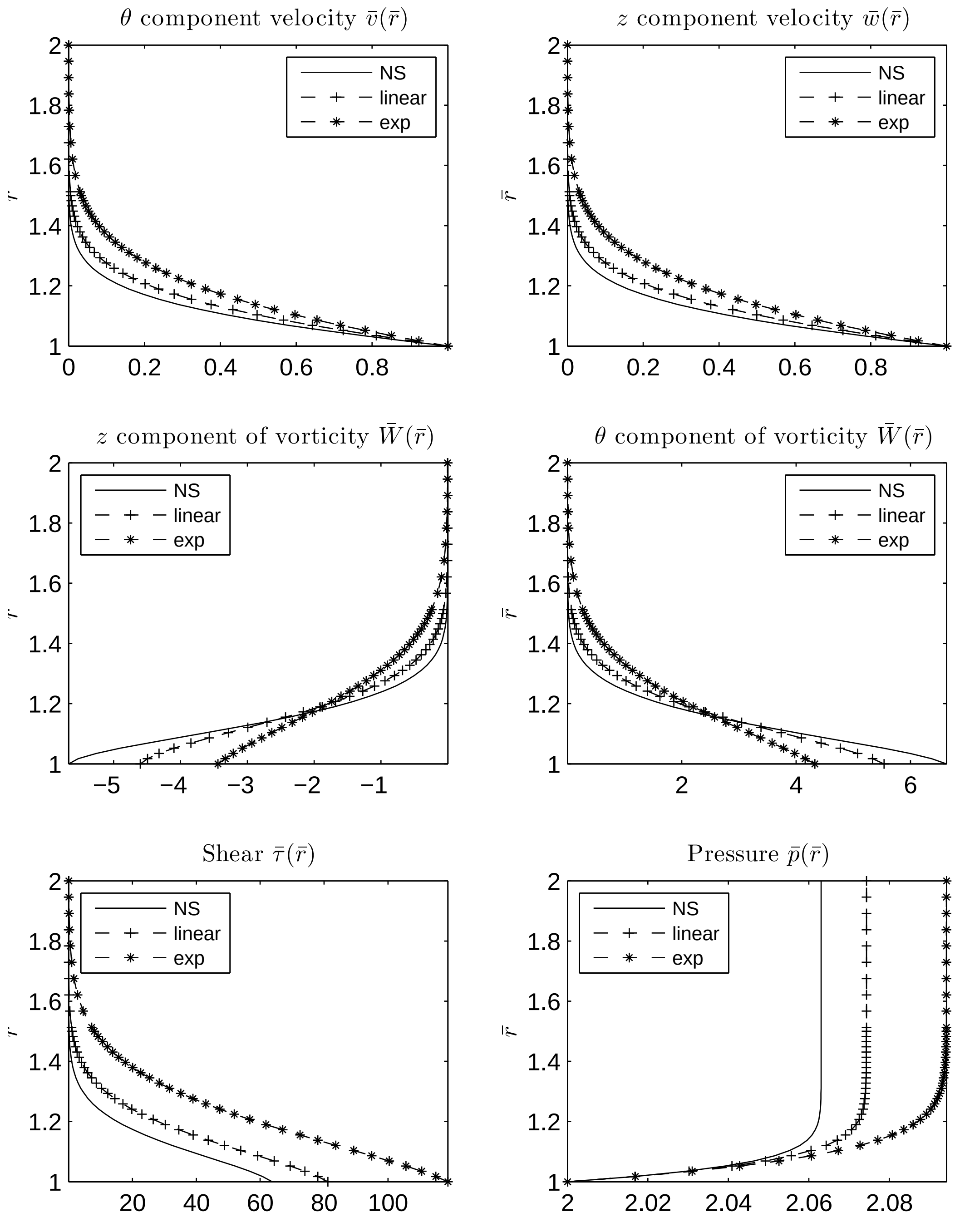

Fig. 20. Torsional and longitudinal oscillations ($Re = 100$, $C = 1$, $k = 2$) at $\bar{t} = \dfrac{\pi}{2}$

Table VI. Drag force per unit length for oscillating cylinder ($Re = 100$, $C = 1$, $k = 2$)

Drag on cylinder	$\mu(p) = \alpha$	$\mu(p) = \alpha(1 + \beta p)$	$\mu(p) = \alpha e^{\beta p}$
$\bar{F}_D$	63.72	81.22	1.18×10^2

3. Torsional oscillations of inner cylinder

When the inner cylinder performs pure torsional oscillation, the governing equations can be obtained just by setting $w = 0$ in (3.16) and (3.18) and solving as before.

Since the pressure p and velocity v are independent of the longitudinal velocity w, the solutions for them will be identical to the ones obtained in Figures 18 - 19. The remarks made earlier regarding the nature of the profiles for velocity, vorticity and shear are applicable here as well.

4. Longitudinal oscillations of inner cylinder

The governing equations for the case when the inner cylinder executes pure longitudinal oscillations are found in turn by setting $v = 0$ in (3.16) and (3.18).

Thus we obtain $\frac{\partial p}{\partial r} = 0$, i.e., p is a constant. The viscosity hence remains constant everywhere and we only need to solve for w subject to the boundary conditions $w(R_{in}, t) = w_0 \sin \omega t$ and $w(R_{out}, t) = 0$.

Therefore the solutions depend only on the Reynolds number Re and are independent of C and k and will be identical to ones obtained for a Navier-Stokes fluid under the same conditions.

CHAPTER IV

CONCLUSION

In this thesis, we investigated two flow domains where we considered the flow of an incompressible fluid with pressure dependent viscosity.

The first involved flow between infinite parallel plates separated by a finite distance where the lower plate was impulsively started or harmonically oscillated. Gravity was taken into account and that was the cause of the high pressures involved in the problem.

In the second case, we considered the flow in the annulus of infinitely long coaxial cylinders where the inner cylinder was free to rotate or oscillate. The centripetal forces developed due to the rotation gave rise to high pressures.

In both cases the high pressure caused appreciable departure from the conventional Navier-Stokes solutions and justified the need to take pressure dependence of viscosity into account. Thus, in flows of incompressible fluids involving high pressures, it is prudent to consider the variation of viscosity with pressure to obtain more realistic solutions.

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APPENDIX A

FINITE DIFFERENCE SCHEME USED

The following system of coupled equations are to be solved using a suitable finite difference scheme.(see (3.20) and (3.19))

$$
\begin{aligned}
&\frac{\partial \bar{v}}{\partial \bar{t}} = \frac{\bar{\mu}(\bar{p})}{Re}\left[\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\frac{\partial \bar{v}}{\partial \bar{r}}\right) + \frac{1}{\bar{r}}\frac{\partial \bar{v}}{\partial \bar{r}} - \frac{\bar{v}}{\bar{r}^2}\right] + \frac{\bar{\mu}'(\bar{p})}{Re}\left[\frac{\partial \bar{p}}{\partial \bar{r}}\left(\frac{\partial \bar{v}}{\partial \bar{r}} - \frac{\bar{v}}{\bar{r}}\right)\right],\\
&\frac{\partial \bar{p}}{\partial \bar{r}} - \frac{\bar{v}^2}{\bar{r}} = 0,\\
&\frac{\partial \bar{w}}{\partial \bar{t}} = \frac{\bar{\mu}(\bar{p})}{Re}\left[\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\frac{\partial \bar{w}}{\partial \bar{r}}\right) + \frac{1}{\bar{r}}\frac{\partial \bar{w}}{\partial \bar{r}}\right] + \frac{\bar{\mu}'(\bar{p})}{Re}\left[\frac{\partial \bar{p}}{\partial \bar{r}}\left(\frac{\partial \bar{w}}{\partial \bar{r}}\right)\right]
\end{aligned}
$$

satisfying the following boundary conditions

$$
\begin{aligned}
&\bar{v}(1,\bar{t}) = \sin \bar{t}, \quad \bar{v}(\eta,\bar{t}) = 0,\\
&\bar{w}(1,\bar{t}) = \gamma \sin \alpha_0 \bar{t}, \quad \bar{w}(\eta,\bar{t}) = 0,\\
&\bar{p}(1,\bar{t}) = k.
\end{aligned}
$$

We choose to use an implicit backward Euler scheme, i.e, to approximate the time derivative as a backward difference and the space derivatives as a central difference. Sufficient number of time levels and grid points are selected for accuracy. In the discussion that follows, it will be understood that for any quantity $\bar{\theta}$, $\bar{\theta}_i^n$ denotes the value of $\bar{\theta}(\bar{r}_i)$ at time level n.

Thus,

$$
\begin{aligned}
\frac{\bar{v}_i^n - \bar{v}_i^{n-1}}{\Delta \bar{t}} =& \frac{\bar{\mu}(\bar{p}_i^n)}{Re}\left[\frac{1}{\bar{r}_i}\frac{\bar{v}_{i+1}^n - 2\bar{v}_i^n + \bar{v}_{i-1}^n}{\Delta \bar{r}^2} + \frac{1}{\bar{r}_i}\frac{\bar{v}_{i+1}^n - \bar{v}_{i-1}^n}{2\Delta \bar{r}} - \frac{\bar{v}_i^n}{\bar{r}_i^2}\right]\\
&+ \frac{\bar{\mu}'(\bar{p}_i^n)}{Re}\left[\frac{\bar{p}_{i+1}^n - \bar{p}_{i-1}^n}{2\Delta \bar{r}}\right]\left[\frac{\bar{v}_{i+1}^n - \bar{v}_{i-1}^n}{2\Delta \bar{r}} - \frac{\bar{v}_i^n}{\bar{r}_i}\right]
\end{aligned}
$$

This step is unconditionally stable and second order accurate in space. In our computations we use a time step $\Delta\bar{t} = 10^{-4}$ while $\Delta\bar{r} = 10^{-2}$.

Using the values of $\bar{v}_i^n$, $\bar{p}_i^n$ is found by using a forward difference approximation,

$$\bar{p}_i^n = \bar{p}_{i-1}^n + \Delta\bar{r}\left[\frac{\left(\bar{v}_{i-1}^n\right)^2}{\bar{r}_{i-1}}\right]$$

This is an explicit scheme which is first order accurate in space, making the numerical scheme first order accurate over all.

At each time step, values of $\bar{v}_i^n$ and $\bar{p}_i^n$ are iteratively computed till convergence is reached. Following that, $\bar{w}_i^n$ is computed for that time step in a manner similar to $\bar{v}_i^n$ as

$$\begin{aligned}\frac{\bar{w}_i^n - \bar{w}_i^{n-1}}{\Delta\bar{t}} = & \frac{\bar{\mu}(\bar{p}_i^n)}{Re}\left[\frac{1}{\bar{r}_i}\frac{\bar{w}_{i+1}^n - 2\bar{w}_i^n + \bar{w}_{i-1}^n}{\Delta\bar{r}^2} + \frac{1}{\bar{r}_i}\frac{\bar{w}_{i+1}^n - \bar{w}_{i-1}^n}{2\Delta\bar{r}}\right] \\ & + \frac{\bar{\mu}'(\bar{p}_i^n)}{Re}\left[\frac{\bar{p}_{i+1}^n - \bar{p}_{i-1}^n}{2\Delta\bar{r}}\right]\left[\frac{\bar{w}_{i+1}^n - \bar{w}_{i-1}^n}{2\Delta\bar{r}}\right]\end{aligned}$$

Thus by time marching, the evolution of pressure and velocity is traced.

VITA

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