MIDDLE GRADES IN-SERVICE TEACHERS PEDAGOGICAL CONTENT KNOWLEDGE OF STUDENT INTERNAL REPRESENTATION OF EQUIVALENT FRACTIONS AND ALGEBRAIC EXPRESSIONS

A Dissertation

by

LESLIE WOODARD

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2008

Major Subject: Curriculum and Instruction

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Approved by:

Chair of Committee,	Gerald Kulm
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ABSTRACT

Middle Grades In-Service Teacher Pedagogical Content Knowledge of Student Representation of Equivalent Fractions and Algebraic Expressions. (December 2008)

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This study examined teacher pedagogical content knowledge changes through a Middle School Mathematics Program professional development workshop, development of noticing use of student representations, and teacher changes in hypothetical learning trajectories due to noticed aspects of student representation corresponding to the hypothetical learning trajectory model.

Using constant comparatives and repertory grid analysis, data was collected in two phases. Phase one, the teacher pre-test, occurred at the beginning of the summer of the 2003 professional development workshop. Phase two, the teacher post-test, occurred at the end of the workshop. Twenty-four teachers supplied data on pre- and post-tests during phases one and two. Eleven teachers were from Texas and 13 from Delaware. Six Texas and eight Delaware teachers worked with the algebraic expression concepts. Five Texas and five Delaware teachers worked with the equivalent fraction concepts. Four mathematics education researchers from Texas, three from Delaware, and two from the

American Association for the Advancement of Science participated in facilitating the professional development.

The results show that teacher pedagogical content knowledge changes with the help of a professional development partnership. The differences in knowledge can be measured with a hierarchal cluster analysis of the repertory grid by analyzing relationships between constructs and elements. Teacher hypothetical learning trajectories change depending on student representations of what they do and do not know about concepts.

The study encourages teachers to use knowledge of students' representation about a concept to determine what to teach next and how the concept should be taught. Teachers should use different types of representations including formal, imagistic, and action representations in teaching mathematical ideas. This will promote student development in all process standards including reasoning and proof, communication, problem solving, and connection.

The findings suggest that teacher pedagogical content knowledge can be redefined during professional development partnerships. Furthermore, teachers' knowledge of representation is varied and emphasis on the imagistic representation should be explored further. Finally, professional development models that facilitate how to extract what a student does and does not know based on representation, can be the basis for defining hypothetical learning trajectories.

DEDICATION

To my family

for their support, understanding, and sacrifice

ACKNOWLEDGEMENTS

I would like to thank my advisor Dr. Gerald Kulm and my committee members, Dr. Denton, Dr. Goldsby, and Dr. Stanley.

I am grateful for the support from my family. Your love and encouragement has brought me through to this point.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION

Background

The National Council of Teachers of Mathematics (NCTM) offers several content and process standards in its publication Principles and Standards for School Mathematics including number and operations, algebra, measurement, geometry, and probability and statistics as content standards and reasoning and proof, communication, problem solving, connection, and representation as process standards (NCTM, 2000). Reasonably, students' use of representation and teacher perception and cultivation of these uses can improve comprehension of all content standards and advance the understanding of the utilization of the remaining process standards (Goldin, 2003; NRC, 2001).

Teachers must have thorough understanding of pedagogy and content when analyzing student characteristics of representation for student success in reform mathematics curricula. Teachers make decisions every day in the classroom regarding the education of students, and the success of these decisions depends on professional reflection, adaptation to change, conceptualization, and application of principles relevant to student learners, pedagogy, content, and access to substantive, meaningful, and lifelong professional development to cultivate these principles (Tanner and Tanner, 2007). Undoubtedly, the latter has become a major tool used to achieve reform efforts

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This dissertation follows the style of *Journal of Research in Mathematics Education*.

that have emerged in mathematics education (NCTM, 1989, 2000). Professional development programs have become a means by which the mathematics education research community can bring current reform strategies and concepts to the classroom teacher so that the teacher can bring these models to life through classroom practice, hence improving student achievement.

Joyce and Showers (2002) stated that four conditions must be present in professional development to affect student achievement significantly. These conditions include (a) a collaborative community of professionals that practice, share results, and make revisions, (b) staff development content that evolves from curricular and instructional strategies that have a high probability of increasing students' ability to learn, (c) a magnitude of change in student learning that is evident and significant, and (d) professional development processes that emphasize skills to implement what the teacher is learning. Furthermore, the National Science Education Standards set by the National Research Council in 1997 state that professional developments must be coherent and integrated with collaboration among teachers and developers including university faculty and stakeholders. In essence, it is important teachers collaborate among themselves and with researchers to develop and enhance curriculum with the goals of student success in mind. By examining the dynamics of professional development partnerships, changes in teacher pedagogical content knowledge, and evaluating hypothetical learning trajectories, the applications of the outcomes should enhance progression of student success in mathematical problem areas.

Significance of the Study

The study is significant for three reasons. First, it examines the effectiveness of professional development partnerships between mathematics education researchers and teachers. Second, it examines changes in teacher pedagogical content knowledge of representation in two important areas in mathematics education in middle school s: number and algebra. Third, it offers a means to quantify qualitative data in education through the Repertory Grid Analysis method. The goals of this study align with the goals of the Middle School Mathematics Program (MSMP, 2002, p.4) focusing on the "study [of] classroom conditions that enable students to achieve the ambitious learning goals set forth in the new generation of reform curricula."

Professional Development Partnerships

Six strategy clusters for professional development for teachers of mathematics were introduced by Loucks-Horsley, Love, Stiles, Mundry, and Hewson (2003). These include (a) aligning and implementing curriculum, (b) adopting collaborative structures, (c) examining teaching and learning, (d) practicing immersion experiences, (e) practicing teaching, and (f) developing vehicles and mechanisms.

Collaborative structures have several underlying assumptions including (a) relating applicable content to teacher practices, (b) respecting teachers as adult learners, (c) collaborating for beneficial mathematics education, and (d) interacting with individuals with similar goals to promote a quality learning atmosphere. Three strategies that fall under the collaborative structure cluster are partnerships, professional networks, and study groups. Key elements for partnerships are (a) partners are equal; (b) roles for mathematics education researchers are clearly defined; (c) consistent values, goals, and objectives are shared by all partners; (d) specific benefits are given to the teachers; and (e) specific benefits are given to the mathematics education researchers.

Partnerships are a beneficial way for teachers to learn new content imperative to their practice, yet several challenges arise including inconsistencies with goals, objectives, and benefits for all partners. Goals, values, and objectives should respond to educational needs and not undermine curriculum. Benefits for the teacher as a partner include exposure to real world applications, different perspectives, and building a broader knowledge base by acquiring research from mathematics education researchers. The mathematics education researcher benefits by becoming familiar with the needs of the school system and by becoming more interactive in public education by stepping out of traditional university roles.

Loucks-Horsley et al. (2003) state conflict arises when the mathematics education researcher enters into the partnership trying to “fix” the problem with the belief that the teachers need more content. The researcher exerts this belief by taking control of the partnership. Teachers, on the other hand can feel intimidated by the mathematics education researchers who they view as the experts. Dewey’s “normal school problem” provides historic evidence of concerns between “university research mission and its role in the preparation of teachers” (Shulman, 1995, p. 511). More recently AAAS (2002), Joyce and Showers (2002), NCTM (2000), and NRC (2002) have pleaded for coherent, integrated, and collaborative participation among teachers and researchers to promote student success and achievement in the classroom.

This study seeks to investigate the effectiveness of professional development partnerships between university mathematics education researchers and middle school teachers by observing and analyzing teacher pedagogical content knowledge of student representation through a combination of action research, and by examining student work and thinking. The two aforementioned concepts fall under the examining teaching and learning strategy cluster for professional learning identified by Loucks-Horsley et al. (2003).

Middle School Problem Areas

Student achievement in fraction and algebra concepts can be increased by a teacher "having knowledge of student common conceptions and misconceptions about the subject matter" (Tirosh, 2000, p.5). Hence, the teacher's development of pedagogical content knowledge of representation regarding equivalent fractions and algebraic expressions is an important issue because both topics have proved to be monumental obstacles for mathematics students. Constructivist-based professional development concerning these obstacles offers an opportunity for obtaining empirical evidence about why teaching fractions has proved to be so difficult and why algebra teachers experience difficulty (Davis, Hunting, & Pearn, 1993; Kieran, 1992; Schmidt, 1994, 1996; Van Dooren, Verschaffel, & Onghena, 2002).

Student Representation

Representation is defined as a configuration of symbols, objects, and signs that represent mathematical ideas (Golding & Kaput, 1996). When looking to change or enhance teaching methodology in mathematics to promote student achievement, investigation of the concept of representation includes student representation of mathematical learning and teacher representation of pedagogical content knowledge.

Teacher pedagogical content knowledge consists of (a) synthesis of knowledge of mathematics; (b) knowledge of specific content within mathematics; and (c) knowledge of teaching, instruction, and classroom and organizational management (Shulman, 1986; NCTM, 2000). Teacher pedagogical content knowledge of student representations helps to broaden and deepen student mathematical understanding. By understanding student constructs, teachers are better able to structure hypothetical learning trajectories for individual students.

Personal Construct Psychology and Repertory Grids

In identifying teacher and student representation, one way to quantify data is through repertory grids. Repertory grids apply the personal construct psychology methodologies of Kelly (1955a). The grid offers a numerical way of comprehending an individual's psychological process through a matrix of elements and bipolar constructs. Students' use of representation and the ability of the teacher through pedagogical content knowledge to guide hypothetical learning trajectories can lead to generations of successful problem solvers.

Theoretical Model for the Study

Much research has been done on the teaching and learning of mathematics, pedagogical content knowledge, representation, and teacher professional development. Yet little has been done on the synthesis of these four components and their effects in changing hypothetical learning trajectories. Hypothetical learning trajectories were introduced by Simon and Tzur (1997) in *Explicating the Teachers Perspective from the Researchers' Perspective: Generating Accounts of Mathematics Teachers Practice.* The concept is defined as the path hypothesized by the teacher as instruction proceeds during development of pedagogical content knowledge. Simon and Tzur offered a methodological approach to understanding how a teacher develops by moving from traditional to reform teaching. The teachers in the present study were videotaped and interviewed and these videotapes of classroom teaching and personal interviews were used to develop rationales and plans for following lessons. These rationales and plans were founded on current educational research and the teachers were able to work simultaneously with education researchers to develop the next piece of the curriculum. The study found that teacher content and pedagogy should be reviewed initially in professional developments, and then processes to develop the teacher's practices should be implemented. Researchers can be integral parts of the teacher's classroom relationship by portraying the complex interrelationships among aspects of teacher knowledge of representation and its relationship to current and future instruction, curriculum development, and hypothetical learning trajectories.

Representations are dynamic vehicles used to solve problems and communicate the results of problem solving (Boaler, 2008). It is important to encourage students to represent their mathematical ideas in ways that make sense to them, even if those representations are not conventional. At the same time, students should learn conventional forms of representation in ways that facilitate their learning of mathematics and their communication with others about mathematical ideas (NCTM, 2000).

There are two types of representations: internal and external. Internal representation "gives us the knowledge for describing individual knowledge structures and problem-solving processes" while external representations "permit us to talk about mathematical relations and meaning apart from inferences concerning the individual learner or problem solver" (Goldin & Kaput, 1996, p. 406). Within these two types of representation, there are three dimensions: (a) display versus action, (b) imagistic/analogistic versus conventional character based/verbal, and (c) formal versus informal. Action representation is representation on paper that can be manipulated and represented in a different way. For example, action representation can be a linear equation that can be represented on a graph, a table, or a word problem. Display is a representation on paper that can only be represented one way and can not be changed. Formal representation is the representation of mathematics foundation and procedure. For example, formal representation can represent an arithmetic algorithm. Informal representation can become formal representation the more a student understands the content. Imagistic representation is non-verbal, non-formal representation with manipulative use such as use of the graphing calculator to find the solution to a system of equations while conventionally, students would be required to complete this by hand. Because the goals of the professional development were to focus on student knowledge structures by way of maturity of pedagogical content knowledge, the focus of this study was geared toward internal representation. The internal representation constructs were presented with the teacher-learned elements to determine what and/or how to teach the student to create representations that are more powerful. Teacher answers on the pre- and post- tests were coded as the representation constructs while the questions were coded as the elements. This research study approached the concepts of teacher changes in pedagogical content knowledge and hypothetical learning trajectories in a constructivist based professional development experience on representation of algebraic expressions and equivalent fractions at the middle school level.

To investigate teacher change in pedagogical content knowledge of representation, personal construct psychology (Kelly, 1955a) was used. By using personal construct psychology, the teacher elements are formed from classroom experiences and beliefs about student learning prior to professional development. After professional development, the teacher elements were re-evaluated for change. The repertory grid represents “personal constructs as a set of distinctions made about elements relevant to a problem domain” (Gaines & Shaw, 1986, p.317). This methodology can capture a coherent picture of researcher goals and teacher elements together before and after professional development experience on pedagogical content knowledge, eminently leading to a lens into student conceptions and misconceptions in both fractions and algebra. As teachers have more answers to questions after the professional development experience, knowledge students conceptions and misconceptions grow and show on the grid.

Figure 1 shows teacher pedagogical content knowledge as a synthesis of teacher beliefs, teacher content knowledge, and teacher pedagogical knowledge. Student representation is interpreted through teacher pedagogical content knowledge and the hypothetical learning trajectory for the student is defined.

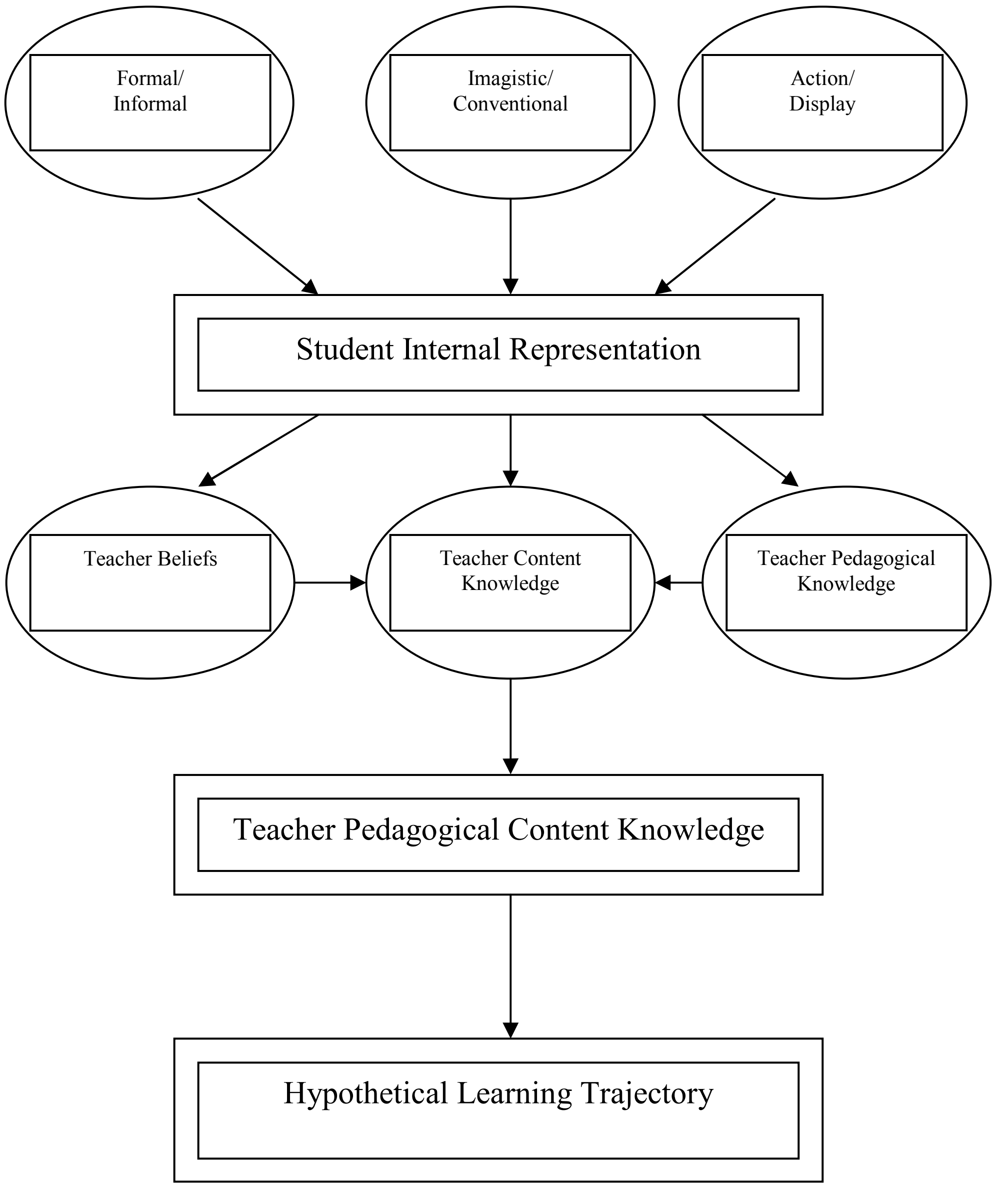

Figure 1. Model of teacher pedagogical content knowledge of student representation.

Teachers begin by interpreting student internal representations through all different aspects of pedagogical content knowledge in Figure 1. After the teachers interpret the representation they move to guiding the hypothetical learning trajectory determining what to teach next and how to teach it.

In Figure 2, a professional development partnership is imposed between the teacher pedagogical content knowledge of student representation and the hypothetical learning trajectory. This should allow teachers to redefine aspects of students' internal representation and modify hypothetical learning trajectories to promote improved student learning.

Teachers begin by interpreting student internal representations through all different aspects of pedagogical content knowledge in Figure 2. After the teachers interpret the representation they move to professional development partnerships with mathematics education researchers and constantly redefining the hypothetical learning trajectory determining what to teach next and how to teach it. This represents a cyclical process.

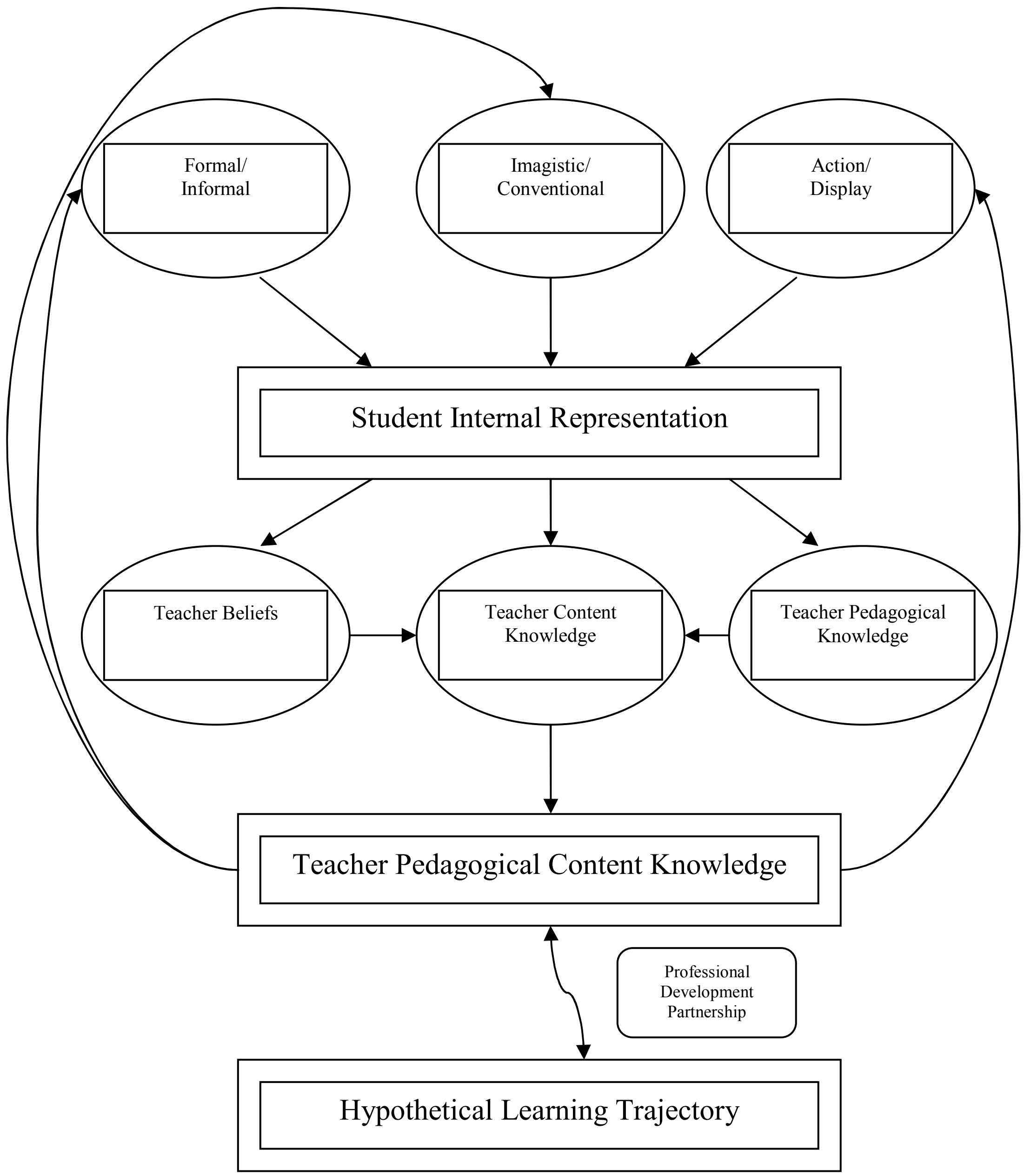

Figure 2. Model of teacher pedagogical content knowledge of student representation with professional development partnership.

Purpose of the Study

The purpose of this research is to investigate middle school teachers' development of pedagogical content knowledge and hypothetical learning trajectories of student representation of equivalent fractions and algebraic expressions during mathematics education reform. Specifically, the research addresses four questions.

1. What is teacher pedagogical content knowledge of student representation of equivalent fractions prior to and after a constructivist professional development experience based on pedagogical content knowledge?
2. What is teacher pedagogical content knowledge of student representation of algebraic expressions prior to and after a constructivist professional development experience based on pedagogical content knowledge?
3. Do the hypothetical learning trajectories of the teachers for equivalent fractions change after the professional development experience?
4. Do the hypothetical learning trajectories of the teachers for algebraic expressions change after the professional development experience?

The research questions align with the goals of MSMP(2001). The "findings will strengthen national policy decisions about the role of curriculum materials development, professional development, and ongoing support in promoting student achievement in mathematics. The project takes advantage of the variety of development and implementation efforts that currently exist in mathematics education and addresses key questions asked by educators and the public: Can the reform curricula really improve student learning? Under what conditions does such learning occur?" (MSMP Grant Proposal, 2002, p.4)

Delimitations

There are two delimitations to this study:

1. Participants were asked to participate voluntarily in the middle school mathematics program; therefore, data in the study only represents those who participated.
2. The video clip viewed by teachers was not in its entire context; therefore, there may be bias on the part of those analyzing the video.

Definitions

Terms used in this study are defined as follows:

1. *Content knowledge* is the knowledge of specific content in mathematics.
2. *Pedagogical knowledge* is the knowledge of teaching, instruction, and classroom and organizational management (NCTM, 2000).
3. *Pedagogical content knowledge* is the synthesis of the three knowledge basis, context, pedagogy, and subject matter (Shulman, 1986).
4. *Subject matter knowledge* is the knowledge of mathematics.
5. *Hypothetical learning trajectory* is the path hypothesized by the teacher and the researcher when proceeding in development of pedagogical content knowledge.

6. *Mathematical representation* is the configuration of symbols, objects, and signs that represent mathematical ideas (Goldin & Kaput, 1996).
7. *Teacher representation* is the knowledge of student representation to teach successfully (NCTM, 2000).
8. *Action Representation* is written representation that can be manipulated to be represented in several different ways (Goldin & Kaput, 1996).
9. *Display Representation* is written representation that can only be represented one way (Goldin & Kaput, 1996).
10. *Formal Representation* is representation of mathematical computation and procedure (Goldin & Kaput, 1996).
11. *Informal Representation* is representation of mathematical computation and procedure as a student interprets it in relation to personal congruence through non-mathematical terms (Goldin & Kaput, 1996).
12. *Imagistic Representation* is representation of manipulatives (Goldin & Kaput, 1996).
13. *Conventional Representation* is traditional representation (Goldin & Kaput, 1996).

CHAPTER II

REVIEW OF THE LITERATURE

Background

This study evaluates teacher pedagogical content knowledge changes based on student representation following a constructivist-based professional development that covered content strands on number and algebra within a reform mathematics curriculum. First, it is necessary to describe the reform curriculum and the reasons it exists. Second, pedagogical content knowledge and its importance in mathematics education will be addressed. This will be followed by a discussion of representation and algebra followed by fractions content and how they relate to the pedagogical content knowledge of teachers. The fourth area discussed will be professional development for the mathematics teacher and the constructivist basis for those professional developments. The final discussion covers the repertory grid method used to investigate teacher pedagogical content knowledge.

Current reforms in mathematics have called for a major change in the way mathematics is taught (NCTM, 2000). These changes have resulted in a focus on ongoing teacher development for teacher and student success in the classroom. One way this ongoing development has taken place in the mathematics education community is by "educating experienced teachers to transform their current practice to be more consistent with current reform principles" through "designing and implementing successful learning opportunities for teachers" (Simon & Tzur, 1999, p. 253).

Review of NCTM Principles and Standards and Reform

Mathematics education has brought about major reform in the past fifteen years. These major reform issues are outlined in *Principles and Standards for School Mathematics,* published in 2000 by the NCTM. Momentous changes in our world have occurred in both technology and diversity. Consequently, these changes have brought about redirection in mathematics education. It became necessary for mathematics students of all ages to be in a classroom where they are constructing and applying mathematical knowledge based on real-life experiences, cultural heritage, and the development of science and technology.

The NCTM previously published *Curriculum and Evaluation Standards for School Mathematics* in 1989. There were several changes from the 1989 document in the 2000 document, including the addition of principles, major technology reform, providing real life situations in an equitable environment, and constructivist learning. To address some of the aforementioned issues in the 2000 document, it is first necessary to introduce a timeline that explains variables that affect mathematics education before, during, and after the implementation of the 1989 Standards and how this led to the 2000 Principles and Standards (Reys, 2001; Schoen, Fey, Hirsch, & Coxford, 1999; Schoenfeld, 2004). Next, the six principles for school mathematics that describe integral parts of mathematics education and how the changes from 1989 to the 2000 document fall under the umbrella of those principles will then be addressed.

Changing research and epistemological bases regarding mathematical thinking and learning began in the 1970s, leading to the creation of the Curriculum and

Evaluation Standards for School Mathematics by the National Council of Teachers of Mathematics in 1989. Rosen (2000) stated there are three types of education: education for democratic equality, education for social efficiency, and education for social mobility. A major issue in mathematics teaching and learning is equity and inclusiveness. The questions of "Who gets to learn math?" and "What type of math gets to be learned?" had to be answered and be inclusive of socio-economically challenged and defy cultural barriers.

The National Research Council in its articles *Everybody Counts* (1989) and *A Challenge of Numbers* (Madison & Hart, 1990) showed great concern with the troubling attrition rates associated with the African American and Latino communities. In the 1890s, only the elite were educated and only 7% of 14 year olds attended high school, with only 3.5% achieving graduation. Elementary school at this time was considered education for all and only minimal mathematics skills were taught. The high school curriculum was considered very rigorous in mathematics. By 1949, 75% of children ages 14-17 attended high school, with 49% graduating. However, the rigorous mathematics curriculum did not change, and fewer students took algebra and geometry. This was considered a major pitfall as many who tried to enlist in the army lacked the basic mathematics skills necessary for an officer. Because of the U.S.S.R.'s launching of Sputnik in 1957, the National Science Foundation supported a "new math" that included set theory, modular arithmetic, and symbolic logic. One producer of the new math curriculum was the School Mathematics Study Group.

Teachers and parents felt uncomfortable with the new math and were ill prepared to teach it; within about 13 years, the "new math" was no more and the focus returned to basic skills and procedures. During this time, around the 1970s, enrollment in high school increased. However, the standard mathematics track remained traditional and it was intended primarily for college-bound students. For those students not intending to go to college, courses such as Math of Money, Business Math, and Shop Math were added to the curriculum. By 1980, international studies such as the *Second International Mathematics Study* showed that American students had little problem solving ability and that improvement in basic skills had been minimal. In 1983, the national crisis of the deficit caused the United States to re-evaluate the curriculum once again. The National Science Foundation funded the project *Man: A Course of Study* (MACOS). The project was failed due to political backlash. Hence, the National Council for Teachers of Mathematics had to take on a leadership role in mathematics education but had to be very careful because the MACOS situation proved how politics affect curriculum.

In addition to examining the history behind the United States re-developing mathematics curricula during and directly after national crises, it is also necessary to take an in-depth look at how textbooks evolved during this time to understand the development of the NCTM's Principles and Standards. Reys (2001) introduced several factors that contributed to improvements in mathematics education during this difficult period. These include:

1. There is no national curriculum.

2. Every state has its own framework that influence what mathematical content is taught and when.
3. About half the states are "adoption states" in which state committees review and approve textbooks.
4. The rest are "open states" in which each district, or sometimes the school, chooses its own textbooks.
5. Most districts adopt new mathematics books within a five to seven year cycle but there is no single period where all schools are adopting textbooks.
6. The availability of technology, including calculators and computers, varies greatly, so assuming the existence of a basic core of technology across all schools is risky.
7. A serious shortage of certified mathematics teachers and a lack of deep mathematical knowledge among many who do teach limit the types of mathematics curricula that can be developed. (Reys, 2001, p. 256)

Reys further adds that these factors cause textbook publishers to develop materials that are marketable with many different frameworks and that can be used by teachers that have a wide range of different knowledge bases. Schoenfeld (2004) further adds that the textbooks are bundled in packages for elementary and middle school levels for coherence.

With this history in mind, the National Council of Teachers of Mathematics developed *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). The goals of the Standards were to create a vision of what it means to be mathematically literate and competent and to create a set of standards to guide the redevelopment and revision of the mathematics curriculum. According to Schoenfeld (2004), mathematical competence depended on a number of factors including: (a) metacognition; (b) beliefs about self and mathematics; (c) knowledge base; and (d) access to problem solving strategies. The social goals of the curriculum were to create mathematically literate workers, lifelong learning, an opportunity for all, and an informed electorate. Furthermore, the curriculum was also supposed to produce confident problem solvers who valued mathematics and could reason and communicate the concepts effectively. Not only did NCTM want to produce different learners, but for this mission to take place teachers would also have to be different. Teachers would have to learn to be active facilitators and guide students through group discussions, mathematics models, and projects.

The NCTM standards presented a framework of guidelines for curriculum development and assessment, but it was not a scope, sequence, or curriculum. They encompassed four sections in three different grade bands. The grade bands were K-4, 5-8, and 9-12, and began with math as problem solving, math as communication, math as reasoning, and math as connections. Three of the sections were content and process standards while the other one was to define standards for student and program evaluation. The standards could be applied to all mathematics teachers, but they were considered vague. Furthermore, they were seen as a threat to social order by making the traditional curriculum easier by calling for mathematics for all.

The NCTM Standards were used in many research proposals, and in 1990-1991, the NSF called for proposals of curricula that were consistent with the Standards. In 1995, the NRC followed by developing the National Science Education Standards. In 1992, the California Department of Education designed a framework for K-12 in mathematics. This was followed by the publishing of various reform texts in 1993-1994.

These reform texts once again looked different to parents and teachers. Therefore, new challenges were faced including teachers not having adequate pedagogical content knowledge to facilitate this type of instruction. The California Learning Assessment System and websites against reform were started by parents and caused a review of the framework developed by the California Department of Education. Assembly Bill 170 (after hearings in 1995 and 1996) called for a general education curriculum based on fundamental skills, but not limited to basic computation skills in mathematics. Furthermore, in 1997, mathematics research professors devised an anti-reform mathematics curriculum. Carnine (1997) reviewed the research on reform standards based curricula, but was an advocate of direct instruction. In 1998, Richard Riley, who was then the United States Secretary of Education, called for a cease-fire between the traditionalists with classical mathematical values and the reformers that were process oriented.

In 2000, NCTM published the *Principles and Standards for School Mathematics*. These principles include equity, curriculum, teaching, learning, assessment, and technology. Equity involves support for every type of student. The curricula are focused, coherent, and aligned across grades and sometimes content. Teaching not only requires knowing what students know and need to know, it also means knowing with understanding and building new knowledge from prior knowledge, using assessment that supports learning and benefits student and teacher, and using technology that influences and enhances mathematics learning.

Increased use of technology, constructs of algorithms, revision of student goals, and addition of discrete mathematics across the entire curriculum all fall under categories in the principles. The principles were an addition to the standards developed by the NCTM. They describe the features of an efficient and effective mathematics classroom, whereas the standards describe the content constructed within the classroom. The principles are described below.

1. *Technology Principle.* Technology is not a replacement for foundational understanding but helps foster it. Changes in technology and our society have brought about a need for all students to have calculators available to them, to have computers in every classroom, to have computer access for all students, and the use of calculators and computers to solve problems.
2. *Teaching and Leaning Principle.* Discrete mathematics was a major change; spread discrete mathematics across the entire curricula at all grade levels instead of just for grades 9-12.
3. *Curriculum and Equity Principle.* Reform goals were re-examined and mathematical literacy became a major goal to create mathematically literate workers, lifelong learning, opportunities for all, and an informed electorate.

4. *All Principles.* Student goals were redefined to include valuing mathematics, having confidence in mathematics, becoming problem solvers, and communicating and reasoning mathematics.
5. *Teaching, Learning, and Assessment Principle.* Students were to be allowed to construct algorithms on their own.

With the historical background of the reform of mathematics curriculum in mind, professional workshops were to be geared toward the development of teacher pedagogical content knowledge and student problem solving through representation was to be multiplied.

Pedagogical Content Knowledge

Long and Coldren (2006) state that effective instruction to promote student learning involves a confluence of at least three fundamental processes on the part of the instructor: (a) adequate knowledge of the material and content, (b) knowledge of how to present material known as pedagogical knowledge, and (c) the ability to create an interpersonal context in which the material is to be learned. The first two processes that Long and Coldren (2006) present were developed in Shulman's *Pedagogical Content Knowledge* (1987) and state that effective teachers have particular knowledge of teaching instruction, the curriculum, and also comprehension of the background and culture of the students and the contexts that they learn. Pedagogical content knowledge is described as the knowledge formed by the synthesis of three knowledge bases: subject matter, pedagogy, and context. It is considered the set of attributes that help the teacher transfer the knowledge of content to students and guide the students to understand the content in a manner that is personally meaningful. Shulman's *Knowledge Growth in Teaching* (1995) project was a model for understanding these aspects of teaching and learning. Pedagogical content knowledge epitomizes the belief that the knowledge base of teaching lies at the intersection of "content and pedagogy" and in the teacher's capacity to "transform the content knowledge" into "pedagogically powerful" and adaptive forms that apply to each student. Therefore, the teacher must know the content, know how to present it, and make it relevant to the ever-changing student.

Much of the research on pedagogical content knowledge is based on elementary education, although the concept can be highly evolved to address middle and secondary education, teacher researcher relationships, and curriculum development. The studies look intensely at teacher knowledge of mathematics, general pedagogical practice, and pre-service teachers. However, they do not closely examine aspects such as knowledge of the development of students' mathematical understanding and how this knowledge can be used to develop and revise curricula. Bright and Vacc (1994) studied the changes in pre-service elementary school teachers' beliefs about teaching and learning mathematics by investigating their ability to provide mathematics instruction based on student thinking. The Cognitively Guided Instruction Belief Scale was used to assess teacher beliefs about instruction. Teachers worked through problems and then watched students work through problems on video. The teachers' work and the students' work was compared and contrasted. The teachers and researchers failed to hypothesize what should be done next based on the comparison and contrast of data.

Tirosh (2000) investigated the development of pre-service elementary school teachers' pedagogical content knowledge of division of fractions. The teachers were asked to work through two fraction problems and pedagogical content knowledge and subject matter content knowledge were assessed. Teachers were asked to list mistakes they believed students would make on these problems. It was found that the teachers could divide fractions but could not thoroughly explain the procedure and were further unaware of the sources of students' incorrect responses.

Kinach (2002) found that teachers teach adding and subtracting integers that were not mathematically meaningful to students, and that they continually teach and attempt to build on this process, even though developing this aspect of the curriculum was not successful for students. Van Dooren, Verschaffel, and Onghena (2003) found pre-service teachers use their content knowledge to evaluate students and build curriculum in this manner, but fail to use pedagogy. Therefore, it is important to realize teachers with no experience in the classroom have been studied based on their beliefs about mathematics content. Unfortunately, the union of pedagogy and content knowledge is not expanded on sufficiently. Furthermore, the relationship between the teacher and researcher during professional development has failed to be addressed.

Cognitive theories including those of Piaget (1983) and Vygotsky (1962) also approach the three previously mentioned processes in different ways. Piaget places an emphasis on the student and how knowledge is constructed and reconstructed to accommodate new information. Vygotsky (1962, 1978), through the concept of the Zone of Proximal Development, focuses on the third factor, context, by interaction of the dyad of teacher and student leading to the teacher scaffolding and the student reaching higher comprehension levels. Scaffolding is the support offered by the joint participation of a more expert person and a student in a task that has a level of complexity just beyond the level that the student could perform independently.

Pedagogical content knowledge is knowledge of teaching, knowledge of content, and knowledge of curriculum. An, Kulm, and Wu (2004) further define this knowledge base as having specific mathematics knowledge for the grade level being taught, knowledge of selection of appropriate materials as well as the goals and key ideas of the curricula, knowledge of student thinking, and mastery of instruction delivery. Many studies have been done on pedagogical content knowledge, the three knowledge bases that form the concept, and the professional development experiences that help develop the concept in teachers (DeBoer et al., 2004; Koency & Swanson, 2000; Phillips, 1992). Further studies have been done on the changes in pre-service teachers' and elementary school teachers' development of pedagogical content knowledge in professional development but little has been done in regards to middle school in-service teachers' growth of pedagogical content knowledge in pedagogically constructivist based professional development. Consequently, research using the personal constructs psychology methodology (Kelly, 1955b) or any methodology to analyze the pedagogical content knowledge of teachers during professional development, is minimal.

Content Knowledge

Studies based primarily on the content knowledge of teachers offer strong support for teachers having a deep and broad knowledge of mathematical content, holistically and on the classroom level that they teach (Ball, 1990; Ball, 2004; Even & Tirosh, 1995; Franke & Lehrer, 1992; Lloyd & Wilson, 1998; Ma, 1999; Van Doreen, Vershaffel, & Onghena, 2002). Yet this content, as seen in *Network of Pedagogical Content Knowledge* (An, Kulm, & Wu, 2004), shows that if teachers connect this knowledge with the knowledge of curriculum and teaching, student learning and achievement can be enhanced.

Teacher Beliefs

Loucks-Horsley et al. (2003) maintain that the knowledge base encompasses two different aspects, knowledge, and beliefs. Knowledge is considered information that depends on research and is solid, whereas beliefs are what we think we know or what we may come to know based on new experiences and information (Ball, 1996). The *Network of Pedagogical Content Knowledge* (An, Kulm, & Wu, 2004) includes a component of teacher beliefs. Research concerning the importance of teachers' beliefs in the knowledge of effective teachings has been compiled with aspects to pedagogical content knowledge (Cooney, Shealy, & Arvold, 1998; Fennema & Franke, 1992; Pligge, Kent, & Spence, 2000; Simon et al., 2000). It was found that teachers' belief systems play a major role in pedagogical content knowledge and the effectiveness of teaching (An, Kulm, & Wu, 2004; Showder, Phillip, Amstrong, & Schappelle, 1998).

Constructivism

According to Loucks-Horsley et al. (2003), learners evolve from a current state of knowledge when new ideas fit naturally and are added, when learners create a new idea out of what exists, when new ideas challenge current knowledge and lead to minor modifications, and when new ideas challenge current knowledge so powerfully that current knowledge is rejected. This is known as cognitive dissonance.

Constructivism is defined as having a new experience and internalizing it through previous experiences and constructs. Piaget (1983) believed that people adapt their thinking to include new experiences; therefore, student interest, previous experiences, and cognition are considered important when designing curriculum and even more important when developing and training teachers in pedagogical content knowledge (Berger, 1978; Crowther, 1997). Vygotsky (1967) believed cognitive activities form in a matrix of socio-historical development. Hence, cognitive skills are not factors that are innate but are products of activities of the society and culture in which we grow up. Cognitive constructivism focus is the view that all knowledge is constructed and instruments of construction are either innate (Chomsky, 1971), or are products of developmental construction (Piaget, 1971).

Piaget draws on the philosophy of Kant, who “described the structures that by which any competent subject acquires or generates knowledge” (Noddings, 1997, p.8). Kant, along with Chomsky, believed that instruments of construction are innate. Piaget further draws on the concepts of reflective abstraction with respect to cognitive constructivism and the development of mathematical structures (Chomsky, 1971).

Noddings (1997) writes that in mathematics education, cognitive constructivism is considered pedagogical constructivism, which means, “acceptance of constructivist premise about knowledge and knowers implies a way of teaching that acknowledges the learners as active knowers” (p. 10). When using pedagogical constructivism, tools that uncover ways of thinking, errors, and misconceptions are necessary. Constructivism should be used in the mathematics classroom when teaching students but should further be applied to teacher training. This guides teachers to construct knowledge as students do so that they can teach more effectively. Research shows that experiencing constructivist learning does not take place through a lecture, but through learning continuously in ways that are constructivist (Little, 1993; Loucks-Horsley et al., 1990). Schifter (1996) further supports these claims by stating that constructivist principles and practices do not allow teachers a stopping point in growth but lead to continuous development and change.

Professional development in mathematics education has followed the claims of research on constructivism to create constructivist-based methodologies for further teacher development (Herbel-Eisenmann & Phillips, 2005; Simon, 1995; Simon & Schifter, 1991; Simon & Tzur, 1999). Several researches also studied teacher retention and growth after the constructivist based professional development experience (Cobb & Steffe, 1983; Farmer, Gerretson, & Lassak, 2003; Schifter & Lester, 2002, Simon & Tzur, 1999). The Middle School Mathematics Project (MSMP, 2001) provides an example of constructivist based professional development supported by not only providing learning experiences with researchers based on curriculum, pedagogical content knowledge, and student achievement, but also by providing ongoing support that allows teachers to reflect effectively and optimize learning during professional development opportunities (DeBoer et al., 2004; Stein, Smith, & Silver, 1999; Wu, 2004).

In conclusion, there are four specific points that can be agreed upon regarding constructivism. Noddings (1997) notes:

> All knowledge is constructed and mathematical knowledge is constructed through reflective abstraction, Cognitive structures are activated during the process of construction and explain the result of cognitive activity, Cognitive structures continually develop, Acknowledgement of constructivism as a cognitive position leads to methodological constructivism (identify various cognitive structures at all phases of construction. (p. 10)

Student learning in mathematics is a constructivist process. Students use many different types of representations to display this process. Hence, the lesson based on representation incorporates a synthesis of the formal, imagistic, and action type of representations that help students look at a problem and answer from a variety of constructed viewpoints. Annenberg Media (2008) states:

> The act of representing a concept or relationship may result in the use of manipulative materials, the construction of graphs or diagrams, the writing of number sentences, or the presentation of a written or oral explanation. When using representations to solve a problem or make sense of a new concept, students are likely to go back and forth, using the representation to help clarify the problem and using the problem to extend their understanding of the representation.

The NCTM representation standard states mathematics instructional programs should build the capacity of all students. They should be taught to (a) create and use representations to organize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems, and (c) use representations to model and interpret physical, social, and mathematical phenomena. Hence, “representations are useful in all areas of mathematics because they help us develop, share, and preserve our mathematical thoughts… to portray, clarify, or extend a mathematical idea by focusing on its essential features" (NCTM, 2000, p. 206).

Mathematical representation is considered a way to capture abstract mathematical relationships. They can be internal or external and can be represented by way of imagistic representation, action representation, and formal representation. Whether internal, external, imagistic, action, or formal, representations can not only improve students’ communication of mathematical ideas but also improve the other content standards of problem solving and reasoning.

Summary

The goal of effective professional development is to develop teachers to promote change in the classroom and ultimately promote student success. By focusing on the development of teacher pedagogical content knowledge, teacher hypothetical learning trajectories can be redefined by noticing different aspects of student representations. Student representations can help meet the representation standard set forth by NCTM (2000) but can also assist in the remaining process standards including comprehension, problem solving, and communication.

CHAPTER III

METHODOLOGY

Background of the Study

This study was a part of a five-year longitudinal study investigating teaching and learning of mathematics at the middle school level through the Middle School Mathematics Project (MSMP, 2001) professional development opportunities. The study examines (a) the effectiveness of professional development partnerships between mathematics education researchers and teachers, (b) changes of teacher pedagogical content knowledge of representation in two important areas in mathematics education in middle schools, number and algebra, and (c) offers a means to quantify qualitative data in education through the Repertory Grid Analysis method.

The MSMP, the context for this study, focuses on providing professional development support that is needed to improve student achievement in mathematics by investigating and promoting the growth and development of curriculums, teacher pedagogical content knowledge, and student achievement. During the second summer of the program, the focus during the professional development session was on three instructional criteria that used representations effectively: probing student understanding, guiding student interpretation, and reasoning (Wu, 2004).

In this study, data was collected from 24 middle school teachers over a professional development period of 2 weeks during the second summer of the program.

Videotapes, video observation instruments, and questionnaires served as the main source for the study.

Participants and Setting

Eleven middle school mathematics teachers and three mathematics education researchers in Texas, 13 middle school mathematics teachers and one mathematics education researcher in Delaware, along with two AAAS researchers, participated in this study. Criteria used for the teacher subjects in the study were:

1. Volunteered to participate in MSMP.
2. Attendance of the constructivist- based professional development workshop during the summer of 2003 with completion of pre and post video observation analysis assessment instrument.

For a detailed analysis of the teacher pedagogical content knowledge of student representation, pre- and post- tests were completed to determine teacher constructs of pedagogical content knowledge. The repertory grid not only analyzes the teacher constructs as elements, but also teacher development and growth in the workshop, agreement of the type of student representations teacher noticed, and the direction teachers plan to go next in the classroom.

Procedure

MSMP Workshop Background

The overall goal of the MSMP project, a five year longitudinal study, is to provide professional development for middle school teachers that helps to improve student achievement by way of investigating curricula, textbook usage, and teacher knowledge supported by reform efforts through the National Council of Teachers of Mathematics (NCTM, 2000) and the American Association for the Advancement of Science (AAAS, 1993). The workshop conducted during the summer of 2003 comprised a review and discussion of learning goals including NCTM standards (2000) and Project 2061's benchmarks (AAAS, 1993). This approach helped teachers to comprehend the learning goals so that they could better investigate the alignment of these goals alongside their classroom practice. The professional development experience was grounded in development of pedagogical content knowledge using constructivist theory.

Conceptual Framework for the Research

The professional development workshop that was the foundation for the study took place in the summer of 2003 and the three criteria that guided this particular workshop were using representations, probing student understanding, and guiding student interpretation and reasoning. It was found through previous videotape analysis that teachers building student conceptual understanding should be a major focus.

The research goals were clearly defined in the *Middle School Mathematics Program Grant Proposal:*

> By examining the connections among the structure of instructional materials, teacher knowledge, classroom activities, professional development, ongoing support, and student learning, our research will shed light and provide valid statistical evidence on how these elements work together to improve student learning in mathematics. We have designed, and plan to execute, a rigorous longitudinal study. We will create and adapt research-based instruments to collect valid teacher and student learning data that can provide information on the conditions under which students learn mathematics well.

Professional Development Background

The teachers watched a video clip presentation of a lesson on equivalent fractions and a lesson on algebraic expressions. After viewing the presentation, they were given a pre-test designed by the professional development team based on the theorhetical framework of Shulman (1986) that addressed the teacher pedagogical content knowledge of student learning before the professional development workshop. The video clip on the algebraic expression lesson lasted approximately 4 minutes and began 22 minutes into the daily lesson. Students begin to working on paper on an algebraic expression to represent cost as a function of number of bikes rented and camp & can costs. One student comes to the overhead projector to display his expression of cost as the number of bikes rented. The video clip on the equivalent fractions lesson lasted approximately 4 minutes and began approximately 20 minutes into the daily lesson. Students are attempting to find a fraction that is equivalent to the fraction 6/9. Two students come to the board and show there work.

The pre-test questions were as followed:

Algebraic Expressions

1. Describe what the student knows about algebraic expressions.
2. Describe how you know what the student knows.
3. Describe what the student seems NOT to know about algebraic expressions.
4. What evidence from the video helps you infer what the student does not know?
5. What would you do next with the student?

Equivalent Fractions

1. Describe what the student knows about equivalent fractions.
2. Describe how you know what the student knows.
3. Describe what the student seems NOT to know about equivalent fractions.
4. What evidence from the video helps you infer what the student does not know?
5. What would you do next with the student?

During the previous school year, the teachers were videotaped on three to five classroom lessons based on equivalent fractions and algebraic expressions according to grade level. The summer of 2003 workshop allowed the teachers to use these videos as data to improve pedagogical content knowledge when analyzing student conceptualization and misconceptions. In the workshop, they watched a video of themselves and answered pedagogical content knowledge questions as they worked through the workshop with the researchers. The workshop questions from the individual teacher video file are listed below:

1. Describe what the student knows.
2. Describe how you know what the student knows.
3. Describe how you know what the student does not know.
4. What do you not know?
5. How can you figure out what you do not know?

Consequently, the teachers engaged in a data driven professional development experience that was enhanced because they were involved in analyzing their curriculum, teaching, and student performance data. They received guidance from the researcher, who did not simply offer a lecture on content and pedagogy, but also provided empirical evidence that can be readily applied to the current teacher practice and classroom as viewed on the video.

In this professional development workshop, after teacher practice was documented and analyzed, the researcher offered models based on the teacher practice that allowed the teachers to transfer these models to their classrooms and apply them directly. A post-test was given to the teachers at the end of the second summer professional development workshop, and they viewed the same videos from the pre-test and were asked the same questions.

Instrumentation

To examine the relationship between the pedagogical content knowledge of student representation during the workshop, constant comparisons, and an adaptation of repertory grids analysis of teacher answers on the pre and post test were used. This study used data that existed from the summer of 2003 workshop.

Videotapes

There were three types of videotapes used during the MSMP professional development workshop. Teachers' voluntary participation in the MSMP program required that they allow themselves to be videotaped three to five times during classroom teaching. The second type of video teachers viewed was a student-centered video on the teaching of an algebraic expressions lesson. The third video was a student-centered video on the teaching of an equivalent fractions lesson.

Sixth grade teachers viewed the equivalent fractions video while seventh grade teachers viewed the algebraic expressions. They first viewed the videos and answered the pre-test questions. During the workshop, the teachers viewed their own videos and answered the workshop questionnaire. Lastly, the teachers reviewed the initial videos and answered the post-test questions.

Video Observation Analysis Instrument

The video observation analysis instrument allowed the teacher to observe a segment of a teacher's lesson. The video clip focused on the interaction of the teacher with two students concerning equivalent fractions. They were asked to review the segment as many times as they wished before completing the instrument so that they would have a complete understanding of the students' thinking from the video. The teachers were also given a transcript of the segment so that they could better answer the pre- and post-test questions for equivalent fractions. The algebraic expressions video focused on a teacher's interaction with a student concerning algebraic expressions. The teachers were asked to follow the same process as the equivalent fractions video observation analysis.

Data Collection

Data collection took place in two phases. Phase one was the teacher data phase pre-test and phase two was the teacher data phase post-test. Phase one occurred at the beginning of the summer of the 2003 professional development workshop and phase two occurred at the end of the workshop. A total of 24 teachers during phases one and two supplied data on pre- and post-tests. Eleven teachers were from Texas and 13 from Delaware. Six Texas teachers and eight Delaware teachers worked with the algebraic expression concepts. Five Texas teachers and five Delaware teachers worked with the equivalent fraction concepts. Four mathematics education researchers from Texas, three from Delaware, and two from AAAS participated in facilitating the professional development.

Data Analysis

Constant Comparative Method

The constant comparative qualitative method (Lincoln & Guba, 1985) was used to analyze teachers' answers to the pre- and post-test questions. The constant comparison method provides an approach to recording and classifying phenomena that can be categorized descriptively and explanatorily. After looking at the data, a list is made of categories and the data is then put in those categories and further refined. The data that emerged from this method were used as the bipolar constructs including action vs. display, formal vs. informal, and imagistic vs. conventional representing teacher pedagogical content knowledge of the types of student internal representations while the questions on the pre- and post-tests represent the elements.

Repertory Grids

The repertory grid method was used because it supports the idea that beliefs about teaching and learning affect the classroom practice and change thereof (Kelly, 1955a; Lester & Onore, 1990). The repertory grid is designed by organizing researchers' knowledge of student learning as the constructs for the professional development and using pre- and post-test questions as the elements. The teachers' answers on the pre- and post-tests are then categorized by the type of student representation recognized by the teachers, and also by how the teachers proceeded with hypothetical learning trajectories based on these representations. The five questions on the algebra and fraction pre- and post-tests were used as horizontal elements on repertory grids and teachers' knowledge of student representation and type of representation were listed as the bipolar constructs.

By rating the constructs on the bipolar matrix or repertory grid, teacher pedagogical content knowledge of student representation before and after the workshop was investigated. It can be determined if there is a change not only by evaluating answers to each but by closely paying attention to the hypothetical learning trajectory from the pre- and post-tests displayed in question five. One repertory grid was completed for each test: one for the pre-test for algebraic expressions, one for the post-test for algebraic expressions, one for the pre-test for equivalent fractions, and one for the post-test for equivalent fractions.

To analyze the grid, correlation was used to analyze the relationship between constructs and elements. The correlation between the elements can be seen by watching the elements reorder on the grid. Elements that move to the left have a low rating while elements further to the right have a high rating. Constructs that move down have a higher rating while constructs that move up have a lower rating. Hierarchical cluster analysis was used to investigate the nature of the cluster correlations of the elements and constructs. In hierarchal cluster analysis the relationship of percentage similarity of the elements and constructs and examined. In conclusion by viewing how much information the teacher interprets about student representations in the elements and the type of representations the teachers interprets in the constructs we can view relationships on the pre and post test using the grid.

CHAPTER IV

ANALYSIS

Several different methods were used to evaluate the extent to which teacher pedagogical content knowledge of student representations changed after professional development. During the summer of 2003, teachers watched a video presentation on fraction or algebra concepts and completed a pre-test using a video observation instrument. The teachers then took part in professional development exercises and analyzed videos of them teaching. Afterwards, teachers viewed the initial video presentation on fraction or algebra concepts and completed the same video observation instrument. Constant comparison and repertory grid analysis techniques were used to review the teacher constructs of pedagogical content knowledge of student representation.

As previously stated the data was first sorted using constant comparisons. Teacher answers during the pre- and post-tests were documented and then sorted according to similarity. Next, all answers were sorted according to type of internal representation and placed on the repertory grid on the vertical axis. The bipolar constructs for the vertical axis were the internal representation types, which included action vs. display, imagistic/analogistic vs. conventional, and formal vs. informal. The definition of action representation is a written representation that can be manipulated while display representation is a written representation that can not be manipulated. An imagistic/analogistic representation is a non-verbal representation using pictures and/or manipulatives while conventional representations use traditional methods to represent the mathematical idea. Lastly, formal representations are mathematical systems that are consciously constructed for specific goals such as mathematical computations while informal representations can become formal as the mathematics learning occurs. The elements were identified as the questions and placed on the horizontal axis.

The theoretical model (see Figures 1 and 2 in Chapter I) was the foundation used for this study. Fit of the model based on *Explicating the Teachers Perspective from the Researchers' Perspective: Generating Accounts of Mathematics Teachers Practice* (Simon & Tzur, 1995) in conjunction with the Middle School Mathematics Project (MSMP, 2001) professional development model was explored. The variables of bipolar constructs of student internal representation, teacher pedagogical content knowledge, professional development partnerships, and hypothetical learning trajectories were extracted using the video observation analysis instrument.

The first step was to identify teacher pedagogical content knowledge of student representation on the video observation analysis instrument pre-test; this is shown in Tables 1 and 2 for content strands algebra and number as guided by MSMP mathematics education researchers. The second step was to tally the teacher pedagogical content knowledge of student internal representations three times according to the bipolar constructs of imagistic vs. convention, formal vs. informal, and action vs. display as noted in column four of Tables 1 and 2. Column two represents the answers to the questions on the video observation analysis instrument. Column three represents how many teachers gave the same answer in column two

Table 1

Categories and Frequencies of Answers Based on Teacher Interpretation of Student Internal Representation for Algebra Pretest

Question 1 – Describe what the student knows about algebraic expressions.			
	Category	*N*	*Type of Representation*
A	The student knows how to construct equations with variables	5	Formal
B	The student knows how to connect variables to unknowns in the problem	12	Action
C	The student knows that multiplication must take place	4	Formal
D	The student knows that when setting up equation there is an = sign	2	Formal
E	The student knows algebraic expressions represent a process	1	Action
F	The student knows how to substitute different values for variables / represent unknown quantities	3	Action
G	The student knows numbers together mean multiplication	4	Formal
H	The student knows 30, B, and N are a part of the algebraic expression	1	Formal
I	The student knows different letters represent different variables	2	Formal
Question 2 – Describe how you know what the student knows.			
	Category	*N*	*Type of Representation*
A	The student is writing	7	Action
B	The student is verbalizing	7	Formal
C	The student connects variable B to real life situation and problem	2	Imagistic
D	The student multiplies people times cost	2	Action
E	The student sets up the equation	6	Action
F	The student understands what variables and constants stand for	4	Formal
G	The student recognizes and corrects mistake	4	Action
H	The student receives prompts by teacher	1	Formal

Table 1, continued.

Question 3 – Describe what the student seems not to know about algebraic expressions.			
	Category	*N*	*Type of Representation*
A	The student does not know the correct order for writing the specific expressions	5	Action
B	The student does not know which variable to multiply the cost by	3	Action
C	The student does not know what each variable represents	6	Imagistic
D	The student does not know what they are solving for	4	Action
E	The student is unsure about what each constant represents	2	Imagistic
F	The student cannot translate logical thoughts into expression	3	Action
G	The student does not know total cost should be the variable C not B	1	Imagistic
H	The student does not know 30 x 100 = 100 x 30	1	Formal
Question 4 – What evidence from the video helps you infer what the student does not know?			
	Category	*N*	*Type of Representation*
A	The student is confused when making verbal explanation	4	Action
B	The student writes the equation incorrectly	4	Action
C	The teacher helps by making connection to real life	1	Imagistic
D	The student does not know which constant to use	3	Action
E	The variables are being used interchangeably	3	Formal
F	The teacher prompts and ask questions	4	Formal
G	The variables are not defined	2	Formal
H	An incorrect expression is written N = 30B	3	Action
I	The student does not know what he is solving for	1	Action

Table 1, continued.

Question 5 – What would you do next with this student?			
	Category	*N*	*Type of Representation*
A	I would discuss how to identify and define variables	4	Formal
B	I would show how to check	2	Formal
C	I would verbalize the next problem	2	Formal
D	I would walk through problem again / verbalize	3	Formal
E	I would show how to substitute different quantity for variables – customers	2	Action
F	I would focus on what 30 means	1	Imagistic
G	I would give a similar problem	1	Formal
H	I would do the next problem w/o prompts	4	Formal
I	I would walk through problem again / written	1	Formal
J	I would do an exercise on translating words to math symbols	1	Action

Table 2

Categories and Frequencies of Answers Based on Teacher Interpretation of Student Internal Representation for Algebra Pretest

Question 1 – Describe what these students know about equivalent fractions.			
	Category	*N*	*Type of Representation*
A	The student knows equivalent fractions represent a pattern	7	Imagistic
B	The student knows to find an equivalent fraction you must multiply numerator and denominator by the same number	2	Action
C	The student knows how to find equivalent fractions (method unidentified)	4	Action
D	The student knows that equivalent fractions are equivalent	1	Formal
E	The student knows there is a relationship	2	Imagistic
F	The student knows changes have to be the same in equivalent fractions	1	Formal
G	The student knows the terms numerator and denominator	1	Formal
H	The student knows equivalent fractions in this problem represent a pattern of adding 2s and 3s	1	Imagistic
I	Others	1	NA
Question 2 – Describe how you know what the student knows.			
	Category	*N*	*Type of Representation*
A	The student Looks for patterns	4	Imagistic
B	The student multiply the numerator and denominator by the same number	2	Action
C	The student explains the pattern	4	Action
D	The student know how to find more than one equivalent fraction	1	Action
E	The student uses mathematical symbols	1	Formal
G	The student is willing to go to the board	1	Action
H	Response to the teacher questions	1	Formal
I	Demonstrates knowledge	1	Action

Table 2, continued.

Question 3 – Describe what the students seem not to know about equivalent fractions.			
	Category	*N*	*Type of Representation*
A	The student does not know why the method works	4	Formal
B	The student does not know why and how they should multiply	3	Formal
C	The student does not know multiplying by a/a is the same as multiplying by 1	2	Formal
D	The student does not know the relationship between the numerator and denominator	1	Formal
E	The student does not know the relationship of base fractions to equivalent fractions	2	Formal
F	The student does not know how to generate fractions with a list or pattern of fractions	1	Action
G	The student does not know the meaning of equivalence or showing equivalence	1	Formal
Question 4 – What evidence from the video helps you infer what the students do not know?			
	Category	*N*	*Type of Representation*
A	The teacher prompts the students for answers	1	Formal
B	The students could not explain the reason for their processes	2	Formal
C	The student made the statement "Counting by 2s and 3s" and "Go by 2s"	4	Action
D	The student representations on board show a disconnect between numerator and denominator	3	Formal
E	The student does not understand they are multiplying by 1 when multiplying by a/a	1	Formal
F	The student never sees a math connection	1	Imagistic
G	There is no discussion on why the student's method works	1	Action
K	There is no relationship to strips	1	Imagistic

Table 2, continued.

Question 5 – What would you do next with these students?			
	Category	*N*	*Type of Representation*
A	I would show multiplication by 1 whole	4	Formal
B	I would check another fraction with student patterns	2	Action
C	I would draw a picture	3	Imagistic
D	I would have them show why the pattern works	2	Action
E	I would use manipulatives to show patterns represent the same quantity	1	Imagistic
F	Re-teach using manipulatives	1	Imagistic
G	Use concrete representations and models	1	Imagistic

The tallied bipolar constructs in columns three and four of Tables 1 and 2 were placed into rating bands ranging from 1-5, as shown in Table 3 and Table 4 for algebra and number, respectively. The bands are formed by dividing the total number of answers to the question by five. The type of representations are separated and put into the bands they fall in: 1st fifth, second fifth, 3rd fifth, 4th fifth, or 5th fifth. This process is done so that the ratings can be placed on the repertory grid for analysis. The rating and the number of answers for the type of representation are listed in the last three columns.

Table 3

Repertory Grid Coding of Teacher Pedagogical Content Knowledge of Student Representation for Algebra Pre-test

Question Number	Total number of Answers	Answer Bands	Formal Answer Band, Number of Responses	Action Answer Band, Number of Responses	Imagistic Answer Band, Number of Responses
1	37	1 0-7.4 2 >7.4-14.8 3 >14.8-22.2 4 >22.2-29.6 5 >29.6-37	3, 21	3, 16	1, 0
2	33	1 0-6.6 2 >6.6-13.2 3 >13.2-19.8 4 >19.8-26.4 5 >26.4-33	2, 12	3, 19	1, 2
3	25	1 0-5 2 >5-10 3 >10-15 4 >15-20 5 >20-5	1, 1	3, 15	2, 9
4	25	1 0-5 2 >5-10 3 >10-15 4 >15-20 5 >20-25	2, 9	3, 15	1, 1
5	21	1 0-4.2 2 >4.2-8.4 3 >8.4-12.6 4 >12.6-16.8 5 >16.8-21	5, 17	1, 3	1, 1

The bands are formed by dividing the total number of answers to the question by 5 as shown in column 3. For example Question 3 had 25 answers, therefore the 5 bands are Band 1 0-5, Band 2 >5-10, Band 3 >10-15, Band 4 >15-20, Band 5 >20-25. Because 15 of the answers were interpreted as action representation, then the band coding is 3.

Table 4

Repertory Grid Coding of Teacher Pedagogical Content Knowledge of Student Representation for Number Pre-test

Question Number	**Total number of Answers**	**Answer Bands**	**Formal** **Answer Band, Number of Responses**	**Action** **Answer Band, Number of Responses**	**Imagistic** **Answer Band, Number of Responses**
1	19	1 0-3.8 2 >3.8-7.6 3 >7.6-11.4 4 >11.4-15.2 5 >15.2-19	1, 3	2, 6	3, 10
2	15	1 0-3 2 >3-6 3 >6-9 4 >9-12 5 >12-15	1, 2	3, 9	2, 4
3	14	1 0-2.8 2 >2.8-5.6 3 >5.6-8.4 4 >8.4-11.2 5 >11.2-14	5, 13	1, 1	1, 0
4	14	1 0-2.8 2 >2.8-5.6 3 >5.6-8.4 4 >8.4-11.2 5 >11.2-14	3, 7	2, 5	1, 2
5	14	1 0-2.8 2 >2.8-5.6 3 >5.6-8.4 4 >8.4-11.2 5 >11.2-14	2, 4	2, 4	3, 6

The rating bands in Table 3 and Table 4 were entered into the repertory grids using the Rep IV 1.1 repertory grid program and displayed in Figures 3 and 4. Gaines and Shaw (2005) notes:

> *Rep IV* is a suite of tools supporting research into, and applications of, a range of conversational constructivist methodologies based on George Kelly's (1955b) *Personal Construct Psychology.* In its various versions, it aims to provide personal, professional, and research support for promoting understanding of individual and communal psychological and social processes. It provides conversational tools for constructing and analyzing *grids* (*RepGrid*)*,* and *nets* (*RepNet*). Grids are a generalization of Kelly's *repertory grids* for eliciting construct networks through examples of their application, and nets are a generalization of visual syntactic structures used for representing construct networks directly in visual languages. (p. 3)

The hierarchal cluster analysis or focus was found for algebra and fractions, respectively (see Figures 3, 4, 5, and 6, and Tables 5 and 6). Hierarchal cluster analysis was described by Jankowicz (2004).

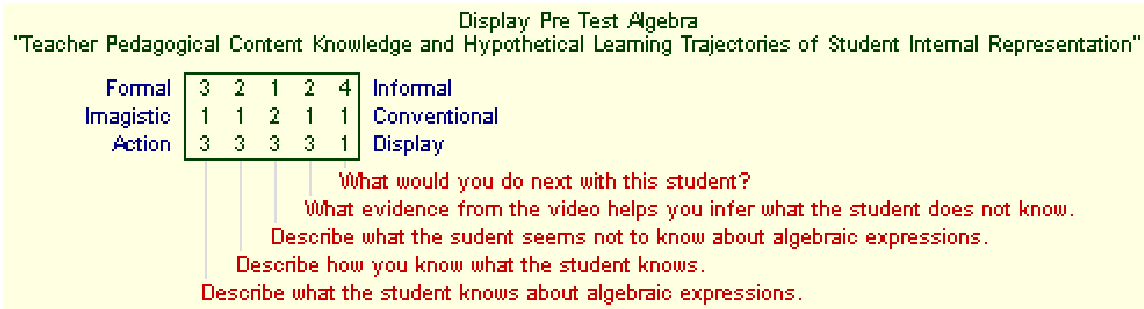

Figure 3. Repertory grid pre-test for algebra.

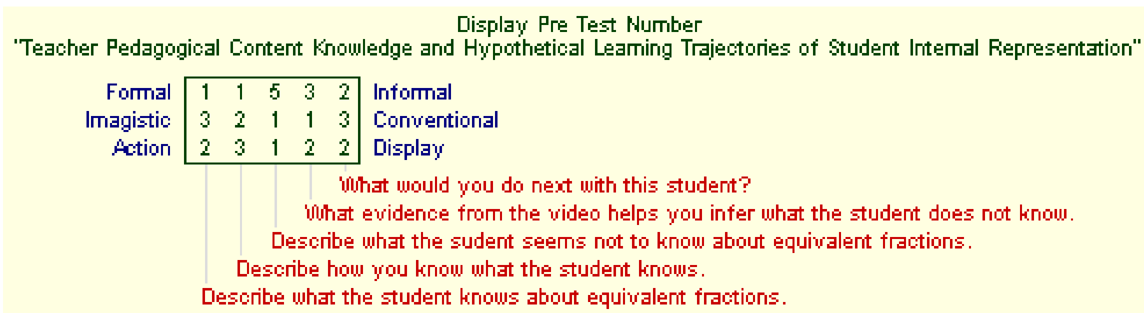

Figure 4. Repertory grid pre-test for number.

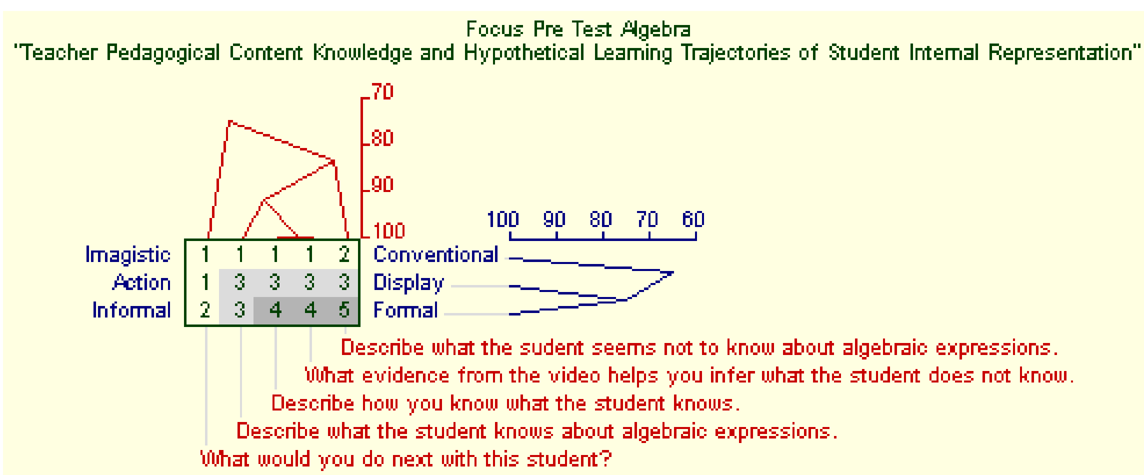

Figure 5. Hierarchal cluster analysis pre-test for algebra.

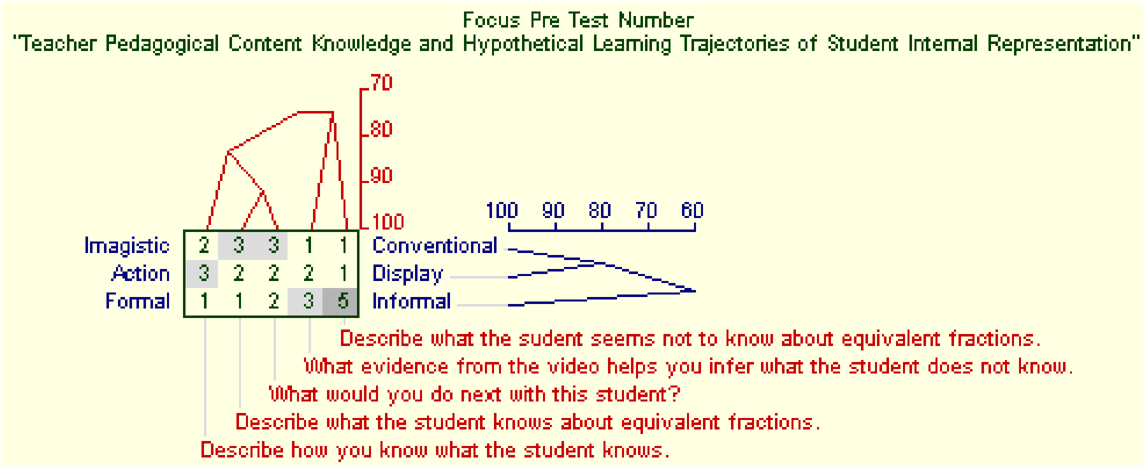

Figure 6. Hierarchal cluster analysis pre-test for number.

Table 5

Similarity Percentages for Hierarchal Cluster Analysis for Algebra Pre-test

Constructs (Internal Representations)	**Similarity Percentages**
Imagistic vs. Conventional Action vs. Display	65%
Imagistic vs. Conventional Informal vs. Formal	65%
Action vs. Display Informal vs. Formal	85%
Element (Questions)	**Similarity Percentages**
Question 5 vs. Question 3	75%
Question 1 vs. Question 3	85%
Question 1 vs. Question 2	90%
Question 1 vs. Question 4	90%
Question 2 vs. Question 4	100%

Table 6

Similarity Percentages for Hierarchal Cluster Analysis for Number Pre-test

Constructs (Internal Representations)	**Similarity Percentages**
Imagistic vs. Conventional Action vs. Display	80%
Action vs. Display Formal vs. Informal	60%
Imagistic vs. Conventional Formal vs. Informal	60%
Element (Questions)	**Similarity Percentages**
Question 2 vs. Question 3	75%
Question 2 vs. Question 4	75%
Question 4 vs. Question 3	75%
Question 2 vs. Question 1	80%
Question 1 vs. Question 5	90%
Question 2 vs. Question 5	90%

<u>Elements</u>

1. Examine the elements and note which elements have been reordered and are now next to each other.
2. Examine the shape of the element dendrogram. How many major branches does it have; in other words, how many distinct clusters of elements exist?

3. Identify construct similarities and differences. For each cluster, follow the lines to the left and up to the relevant set of adjacent columns in the main grid. On which constructs do these elements receive similar ratings, and on which do they differ?
4. What does this mean in terms of the way in which your [teacher] is thinking?
5. Find the highest percentage similarity score. Look at the element dendrogram again. You will see that there is a percentage scale above it, which allows you to read off the percentage similarity scores between any two adjacent elements. Each element has a line to its right, which meets with its neighbor in a sideways V-shape. If you draw a perpendicular line from the apex of that V-shape to the percentage scale, you can read off the percentage similarity score between those two adjacent elements. Next, find the two adjacent elements, which have the highest percentage similarity, score and note its value. Then note the next pair, their percentage similarity scores, and if the pair forms a separate cluster from the other pairs you identified, or whether they belong to that cluster.
6. Examine the remaining scores. Continue this procedure.

<u>Constructs</u>

1. Examine the constructs and note how they have been reordered.
2. Look at the shape of the construct dendrogram, and decide what this might suggest about the similarities and differences in your [teacher's] construing.

3. Identify element similarities and differences. For each cluster, follow left to the relevant rows of ratings. Which elements have received similar ratings on these constructs, and which received different ones?
4. What does this mean? Discuss the implications with your interviewee.
5. Find the highest percentage similarity score. Working with the separate construct percentage similarity scale, note the two adjacent constructs that have the highest percentage similarity score and follow the lines to the right until they meet at an apex of the V-shape, and draw a perpendicular line to the percentage similarity scale to read off the value. Find the next pair, note their score, and see whether they are a distinct cluster or form part of the same cluster as the previous pair.
6. Examine the remaining scores. Continue this procedure (pp. 122-124).

Questions 1-5 are ordered from left to right on the above grids. Constructs are ordered randomly order on the grid above grid vertically. The answer bands are the numbers located on the grid.

The elements have been reordered from left to right with the lowest ratings on the left and the higher ratings on the right. The element dendrogram is on the top of the grid. The dendrogram for the elements show at least 4 distinct clusters of relationships among the elements. The dendrogram for the constructs are shown on the right side of the grid. The dendrogram for the constructs show at least 2 distinct clusters of relationships among the constructs. These relationships are show in similarity percentages in Table 5.

The dendrogram for the elements show at least 3 distinct clusters of relationships among the elements. The dendrogram for the constructs are shown on the right side of the grid. The dendrogram for the constructs show at least 2 distinct clusters of relationships among the constructs. These relationships are show in similarity percentages in Table 6.

After the teachers completed the professional development workshop, they watched the video presentation again and took the same assessment given in the beginning. The steps taken with the post-test were similar to the pre-test. Teacher pedagogical content knowledge of student representation on the video observation analysis instrument post-test is displayed in Tables 7 and 8 in content strands for algebra and number, respectively. Next, the teacher pedagogical content knowledge of student internal representations was tallied according to the bipolar constructs of imagistic vs. convention, formal vs. informal, and action vs. display as noted in column four of Tables 9 and 10. Column two represents the answers to the questions on the video observation analysis instrument. Finally, column three represents how many teachers gave the same answer in column two.

Table 7

Categories and Frequencies of Answers Based on Teacher Interpretation of Student Internal Representation for Algebra Post-test

Question 1 – Describe what the student knows about algebraic expressions.			
	Category	*N*	*Type of Representation*
A	The student knows how construct equations with variables	2	Formal
B	The student knows how to connect variables to unknowns in the problem	7	Action
C	The student knows multiplication must take place	5	Formal
D	The student knows when setting up equation there is an =	3	Formal
E	The student knows how to substitute different values for variables / represent unknown quantities	2	Action
F	The student knows numbers together mean multiplication	1	Formal
G	The student knows 30, B, and N are a part of the algebraic expression	3	Formal
H	The student knows different letters represent different variables	1	Formal
I	The student knows algebraic expressions represents a pattern	2	Imagistic
J	The student knows 30N = N30	2	Formal
K	The student knows how to find the cost	2	Action
L	The student knows how to find the number of customers	1	Action
M	The student knows y = mx + b	1	Formal
N	The student knows what an equation is and how to write one	1	Formal

Table 7, continued.

Question 2 – Describe how you know what the student knows.			
	Category	*N*	*Type of Representation*
A	The student is writing	8	Action
B	The student is verbalizing	9	Formal
C	The student connects variable B to real life situation and problem	2	Imagistic
D	The student multiplies people times cost	4	Action
E	The student sets up the equation	3	Action
F	The student understands what variables and constants stand for	1	Formal
G	The student recognizes and corrects mistake	1	Action
H	The student writes =	1	Formal
I	The student connects variable N to problem	3	Imagistic
Question 3 – Describe what the student seems not to know about algebraic expressions.			
	Category	*Post*	
A	The student does not know the correct order for writing expression	6	Action
B	The student cannot define what variable represents	8	Imagistic
C	The students do not know what they are solving for	1	Action
D	The student is unsure about what the constants represent	8	Imagistic
E	The student cannot translate logical thoughts into expression	4	Action
F	The student does not know 30x100=100x30	1	Formal
G	The student needs guidance from teacher	1	Formal
H	The student does not know what 30 represents	1	Imagistic

Table 7, continued.

Question 4 – What evidence from the video helps you infer what the student does not know?			
	Category	*N*	*Type of Representation*
A	The student is confused when making verbal explanation	4	Action
B	The equation is written incorrectly	3	Action
C	The student does not know which constant to use	3	Action
D	The teacher prompts and ask questions	6	Imagistic
E	The variables are not defined	1	Formal
F	The incorrect expression is written N=30B	4	Action
G	The student realizes and corrects mistake	2	Action
H	The student does not know where to plug in values for variables, i.e. "Do you want me to use 20 or 30?"	7	Action
Question 5 – What would you do next with this student?			
	Category	*Post*	
A	I would discuss how to identify and define variables	4	Formal
B	I would show how to check	2	Action
C	I would walk through problem again / verbalize	3	Formal
D	I would show how to substitute different quantity for variables – customers	4	Action
E	I would give a similar problem	2	Action
F	I would do the next problem w/o prompts	5	Action
G	I would walk through problem again	1	Action
H	I would work 1 on 1	1	Formal
I	I would discuss constants	2	Formal
J	I would see if students can for equations using table	2	Action

Table 8

Categories and Frequencies of Answers Based on Teacher Interpretation of Student Internal Representation for Number Post-test

Question 1 – Describe what these students know about equivalent fractions.			
	Category	*N*	*Type of Representation*
A	The student knows equivalent fractions represent Patterns	3	Imagistic
B	The student knows to Multiply numerator and denominator by the same number	6	Action
C	The student knows to Find equivalent fractions (method unidentified)	3	Action
D	The student knows Fractions are equivalent	2	Formal
E	The student knows this equivalent fractions represent Patterns of adding 2s and 3s	5	Imagistic
F	The student knows this equivalent fractions represent patterns of Counting – 2s and 3s	1	Imagistic
Question 2 – Describe how you know what the student knows.			
	Category	*N*	*Type of Representation*
A	The student knows this equivalent fractions represent there is a pattern	1	Imagistic
B	The student knows to Multiply the numerator and denominator by the same number	2	Action
C	The student knows how Explain the pattern	4	Formal
D	The student Adds – 2s and 3s	4	Action
E	The student Willing to go to the board	4	Action
F	The student knows this equivalent fractions represent Uses a variety of strategies	1	Action

Table 8, continued.

Question 3 – Describe what the students seem not to know about equivalent fractions.			
	Category	*N*	*Type of Representation*
A	The student does not know why the student's method works	3	Formal
B	The student does not know why and how they should multiply	2	Formal
C	The student does not know the relationship between numerator and denominator	1	Formal
D	The student does not know how to generate fractions with a list or pattern of fractions	1	Action
E	The student does not know the meaning of equivalence, showing equivalence	9	Formal
F	The student does not know why they are adding	1	Imagistic
Question 4 – What evidence from the video helps you infer what the students do not know?			
	Category	*N*	*Type of Representation*
A	The student made the statement "Counting by 2s and 3s" and "Go by 2s"	2	Action
B	The student representations on board show a disconnect between numerator and denominator	1	Imagistic
C	The student does not understand they are multiplying by 1 when multiplying by a/a	1	Formal
D	There is no discussion on why the student's method works	1	Action
E	The student does not know the meaning/definition of equivalence	2	Formal
F	The student does not know the relationship of base fraction to equivalent fractions	3	Formal
G	The student does not use pictures or diagrams	1	Imagistic
H	The student uses rote memorization	1	Action
I	Student procedure	1	Formal

Table 8, continued.

Question 5 – What would you do next with these students?			
	Category	*Post*	
A	I would show multiplication by 1 whole	2	Formal
B	I would check another fraction with student patterns	1	Action
C	I would draw a picture	4	Imagistic
D	I would have them show why the pattern works	1	Imagistic
E	I would use manipulatives to show patterns represent the same quantity	1	Imagistic
G	I would use concrete representations and models	5	Imagistic
H	I would show uniqueness of multiplying numerator and denominator by the same number	1	Formal

The rating bands in Tables 7 and 8 were entered into the repertory grids using the Rep IV 1.1 repertory grid program and displayed in Figures 7 and 8. The hierarchal cluster analysis or focus was found for algebra and fractions respectively, and is shown in Figures 9 and 10.

Table 9

Repertory Grid Coding of Teacher Pedagogical Content Knowledge of Student Representation on Algebra Post-test

Question Number	**Total number of Answers**	**Answer Bands**	**Formal** Answer Band, Number of Responses	**Action** Answer Band, Number of Responses	**Imagistic** Answer Band, Number of Responses
1	47	1 0-9.4 2 >9.4-18.8 3 >18.8-28.2 4 >28.2-37.6 5 >37.6-47	4, 33	2, 12	1, 2
2	32	1 0-6.4 2 >6.4-12.8 3 >12.8-19.2 4 >19.2-25.6 5 >25.6-32	2, 11	3, 16	1, 5
3	30	1 0-6 2 >6-12 3 >12-18 4 >18-24 5 >24-30	1, 2	2, 11	3, 17
4	30	1 0-6 2 >6-12 3 >12-18 4 >18-24 5 >24-30	1, 1	4, 23	1, 6
5	26	1 0-5.2 2 >5.2-10.4 3 >10.4-15.6 4 >15.6-20.8 5 >20.8-26	2, 10	4, 16	1, 0

The bands are formed by dividing the total number of answers to the question by 5 as shown in column 3. For example Question 3 had 30 answers, therefore the 5 bands are Band 1 0-6, Band 2 >6-12, Band 3 >12-18, Band 4 >18-24 and Band 5 >24-30. Since 17 of the answers were interpreted as imagistic representation then the band coding is 3.

Table 10

Repertory Grid Coding of Teacher Pedagogical Content Knowledge of Student Representation on Number Post-test

Question Number	**Total number of Answers**	**Answer Bands**	**Formal** Answer Band, Number of Responses	**Action** Answer Band, Number of Responses	**Imagistic** Answer Band, Number of Responses
1	20	1 0-4 2 >4-8 3 >8-12 4 >12-16 5 >16-20	1, 2	3, 9	3, 9
2	16	1 0-3.2 2 >3.2-6.4 3 >6.4-9.6 4 >9.6-12.8 5 >12.8-16	2, 4	4, 11	1, 1
3	17	1 0-3.4 2 >3.4-6.8 3 >6.8-10.2 4 >10.2-13.6 5 >13.6-17	5, 15	1, 1	1, 1
4	13	1 0-2.6 2 >2.6-5.2 3 >5.2-7.8 4 >7.8-10.4 5 >10.4-13	3, 7	2, 4	1, 2
5	15	1 0-3 2 >3-6 3 >6-9 4 >9-12 5 >12-15	1, 3	1, 1	4, 11

The rating bands in Tables 9 and 10 were entered into the repertory grids using the Rep IV 1.1 repertory grid program and displayed in Figures 7 and 8. The hierarchal cluster analysis or focus was found for algebra and fractions, respectively, and is shown in Figures 9 and 10.

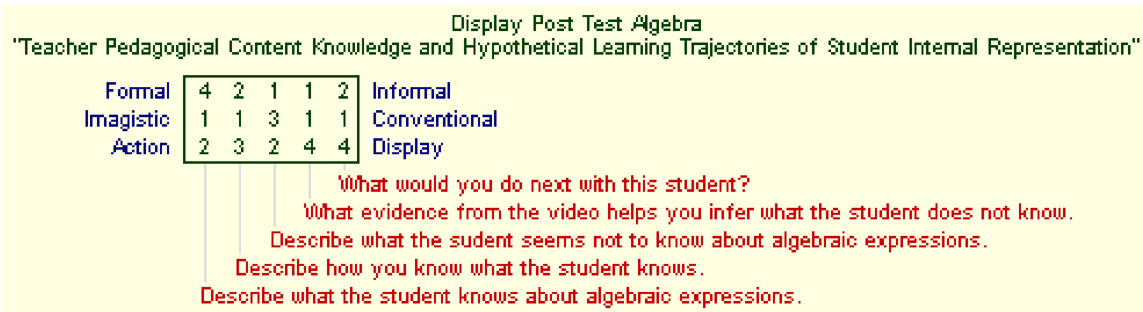

Figure 7. Repertory grid for algebra post-test.

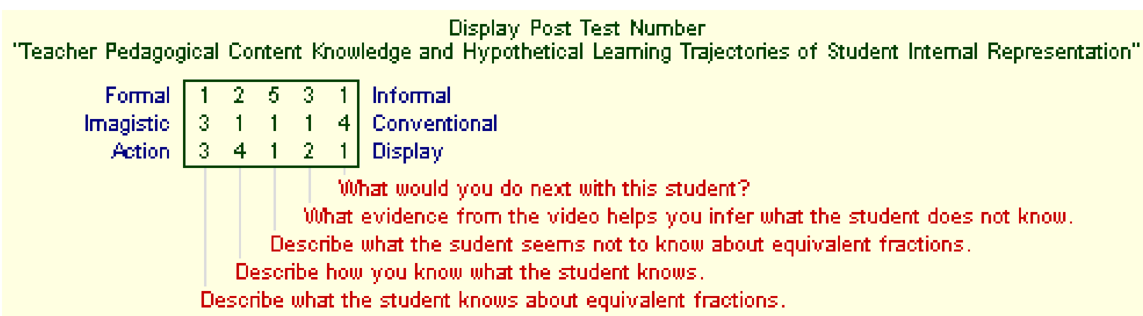

Figure 8. Repertory grid for number post-test.

Questions 1-5 are ordered from left to right on the above grids. Constructs are ordered randomly order on the grid above grid vertically. The answer bands are the numbers located on the grid.

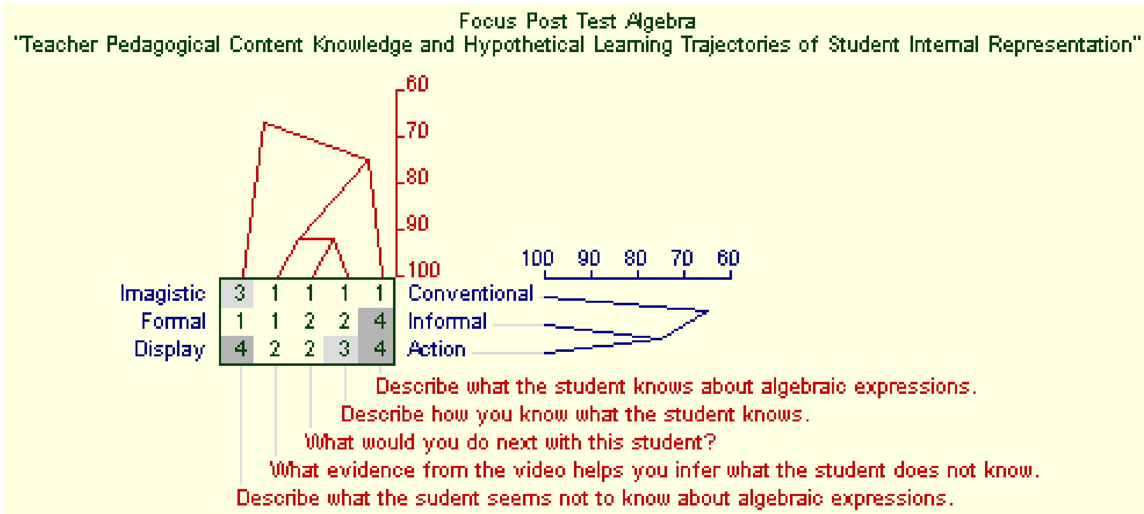

Figure 9. Hierarchal cluster analysis for algebra post-test.

The elements again have been reordered from left to right with the lowest ratings on the left and the higher ratings on the right. The element dendrogram is on the top of the grid. The dendrogram for the elements show at least 4 distinct clusters of relationships among the elements. The dendrogram for the constructs are shown on the right side of the grid. The dendrogram for the constructs show at least 2 distinct clusters of relationships among the constructs. These relationships are show in similarity percentages in Table 11.

Table 11

Similarity Percentages for Hierarchal Cluster Analysis for Algebra Post-test

Constructs (Internal Representations)	**Similarity Percentages**
Imagistic vs. Conventional Formal vs. Informal	65%
Formal vs. Informal Display vs. Action	75%
Imagistic vs. Conventional Display vs. Action	65%
Element (Questions)	**Similarity Percentages**
Question 3 vs. Question 1	65%
Question 4 vs. Question 1	75%
Question 4 vs. Question 5	90%
Question 4 vs. Question 2	90%
Question 5 vs. Question 2	90%

Element Analysis for Algebra

When determining the relationship between the elements, the question order shifted from 5, 1, 2, 4, and 3 on the pre-test to 3, 4, 5, 2, and 1 on the post-test. On the pre-test, questions 5 and 3 have the least in common while questions 1, 2, and 4 have the most in common. This result indicates that the teachers' hypothetical learning trajectory based on the student representation was the most unrelated to what the student knew. Yet in the post-test, what the student did or did not know based on the representation was more closely related to the hypothetical learning trajectory.

Construct Analysis for Algebra

When determining the relationship between the constructs, the original construct order was formal vs. informal, imagistic vs. conventional, and action vs. display. In the pre-test for algebra, the order changed to conventional vs. imagistic, display vs. action, and formal vs. informal. The ratings on constructs increased in order from left to right and top to bottom. Hence, the least pronounced construct of student representation that teachers noticed in the pre-test was conventional, while the most pronounced construct was informal for student representation. In the post-test, the order was changed to imagistic vs. conventional, formal vs. informal, and display vs. action. Hence, the least pronounced construct of student representation that teachers noticed in the post-test was imagistic, while the most pronounced construct noticed for student representation was action.

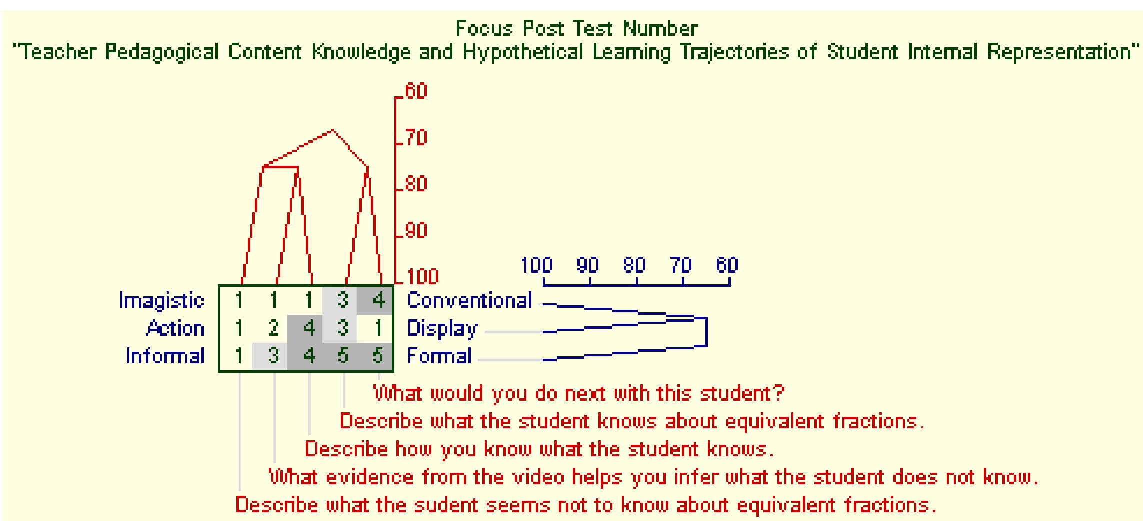

Figure 10. Hierarchal cluster analysis for number post-test.

The elements again have been reordered from left to right with the lowest ratings on the left and the higher ratings on the right. The element dendrogram is on the top of the grid. The dendrogram for the elements show at least 4 distinct clusters of relationships among the elements. The dendrogram for the constructs are shown on the right side of the grid. The dendrogram for the constructs show at least 2 distinct clusters of relationships among the constructs. These relationships are show in similarity percentages in Table 12.

Table 12

Similarity Percentages for Hierarchal Cluster Analysis for Number Post-test

Constructs (Internal Representations)	**Similarity Percentages**
Imagistic vs. Conventional Action vs. Display	65%
Imagistic vs. Conventional Informal vs. Formal	65%
Action vs. Display Informal vs. Formal	65%
Element (Questions)	**Similarity Percentages**
Question 3 vs. Question 5	65%
Question 3 vs. Question 1	65%
Question 3 vs. Question 4	75%
Question 3 vs. Question 2	75%
Question 1 vs. Question 5	75%

Element Analysis for Number

When determining the relationship between the elements, the question order shifted from 2, 1, 5, 4, and 3 on the pre-test to 3, 4, 2, 1, and 5 on the post-test. On the pre-test, questions 2 and 3 had the least in common while questions 1, 5, and 4 had the most in common. This indicated that the teachers' hypothetical learning trajectory based on the student representation was most related to what the student knew and evidence in the video of what the student did not know. Yet in the post-test, the hypothetical learning trajectory was least related to what the representations of what student did not know.

Construct Analysis for Number

When determining the relationship between the constructs, the original construct order was formal vs. informal, imagistic vs. conventional, and action vs. display. In the pre-test for algebra, the order changed to imagistic vs. conventional, action vs. display, and informal vs. formal. The ratings on constructs increased in order from left to right and top to bottom. Hence, the least pronounced construct of student representation that teachers noticed in the pre-test was imagistic, while the most pronounced construct noticed for student representation was formal. In the post-test, the order was changed to imagistic vs. conventional, action vs. display, and formal vs. informal. Hence, the least pronounced construct of student representation teachers noticed in the post-test was imagistic while the most pronounced construct was informal for student representation.

In summary, the Middle School Mathematics Project professional development workshop on pedagogy and content in student representations was analyzed through pre-and post-test video observation analysis instruments. The hypothetical learning trajectories in question 5 were also analyzed for changes from pre to post. These findings will be discussed in the following chapter.

CHAPTER V

CONCLUSION

In this study, it is imperative to understand that the intent was not to compare teachers to each other. Instead, the goal was to investigate how the answers to the pre-test and post-test compared and contrasted to determine if the professional development collaboration with the researcher helped to create a more clearly defined hypothetical learning trajectory. Such a trajectory can be used to guide curriculum development through knowledge of student representation.

For curriculum to be developed properly, teachers must have input about the effective use of textbooks and other curriculum material and about their own teaching. However, input from other teachers or professional development leaders is not sufficient; there must be a relationship between educational researchers and teachers. This relationship allows the teacher to learn practical skills from colleagues and researchers. Teachers can also learn how to implement these skills successfully in the classroom and adapt them to the learning style for each student. By examining the teachers' hypothetical learning trajectories before and after development with the researchers, one can clearly see that the teachers can hypothesize more effectively about how to guide the students through the curriculum based on the students' current level of knowledge because in the hierarchal cluster analysis grids the similarity relationships either have higher percentages between the trajectories and the student misconceptions and conceptions.

The theoretical model in Figure 1 was used for analyzing the pre-tests while the theoretical model in Figure 2 was used for analyzing the post-tests. Overall, Figure 1 showed that teacher pedagogical content knowledge is a synthesis of teacher beliefs, teacher content knowledge, and teacher pedagogical knowledge. Student representation was interpreted through teacher pedagogical content knowledge and the hypothetical learning trajectory for the student was defined. In Figure 2, a professional development partnership was imposed between the teacher pedagogical content knowledge of student representation and the hypothetical learning trajectory. This should have allowed teachers to redefine aspects of students' internal representation and modify hypothetical learning trajectories to promote improved student learning.

The idea the second model represented was that there should be a change in the hypothetical learning trajectories of the teacher after a professional development partnership. Evidence of change in number and algebra took place. In the following sections, a discussion is provided on the results for each of the research questions in the study.

Teacher Pedagogical Content Knowledge of Student Representation of Number

Research Question 1. What is teacher pedagogical content knowledge of student representation of equivalent fractions prior to and after a constructivist professional development experience based on pedagogical content knowledge?

On the pre-test, informal representation was the representation teachers noticed most about what the student did or did not know. There was little distinction between the action vs. display and the imagistic vs. conventional constructs as they had a high similarity rating. Constructs that move to the right and down because of higher ratings are considered favorable while constructs that move to the left and up are considered unfavorable and have lower teacher ratings. Imagistic and actions are considered the most favorable constructs among what teachers notice about student representation misconceptions, which means they have a higher rating and are on the positive end of bipolar constructs, while display and conventional representations are the most unfavorable, which means they have a lower rating and are on the negative end of bipolar constructs. Imagistic representation is the most unfavorable construct on what the teacher would do next yet NCTM (2000) states multiple representations should be used in the mathematics classroom to promote student achievement in all process standards.

On the post-test, action representation was the representation that teachers noticed most about what the student does and does not know. This is favorable because the formation of algebraic expressions from word problems is an action representation. There is little distinction between the formal vs. informal and the display vs. action constructs because they have a high similarity rating. Informal and actions are considered the most favorable among what teachers notice, while display and formal representations are the most unfavorable. Imagistic representation is again the most unfavorable construct on what the teacher would do next with the student. This may have taken place because the goal of this professional development experience was geared toward teachers interpreting student representations and not teaching teachers different representations (MSMP, 2001). Furthermore, research shows that teachers that teach currently were taught traditionally and will use more formal and conventional representation as opposed to imagistic representation (Stocks & Schoenfeld, 1997).

The ranking of constructs changed with number from informal, formal, display, action, conventional, and imagistic, to formal, informal, display, action, conventional, and imagistic. Although the ranking did not change much, the switch with informal to formal as the highest rating is extremely important. This draws on the research of Goldin (2003), in that number and the representation of computing the algorithms thereof is a formal representation. Informal representations can change to formal representations through student cognitive development in the classroom. Formal representation is considered the representation taught most by traditional teachers, hence lending itself to the research trends noticed here. Per Wu (2000), and Stocks and Schoenfeld (1997), teachers in this era were taught traditionally, and hence follow the traditional education and put it into their practice.

Teacher Pedagogical Content Knowledge of Student Representation of Algebra

Research Question 2. What is teacher pedagogical content knowledge of student representation of algebraic expressions prior to and after a constructivist professional development based on pedagogical content knowledge?

On the pre-test, formal representation was the representation that teachers noticed most about what the student does or does not know. There was little distinction between the action vs. display and the informal vs. formal constructs and had high similarity percentage ratings. Informal and actions are considered the most unfavorable constructs among what teachers notice about student representation conceptions and misconceptions , while display and formal representations are the most favorable and have the highest rating by the teachers.

On the post-test, action representation was the representation that teachers noticed most about what the student does and does not know. This is favorable because the formation of algebraic expressions from word problems is an action representation. There is little distinction between the informal vs. formal and the action vs. display constructs as they have high similarity ratings. Informal and action are considered the most favorable among what teachers notice as student conceptions and misconceptions, while display and formal representations are the most unfavorable. Imagistic representation is the most unfavorable construct based on what the teacher would do next. Multiple of representations should be used in the mathematics classroom by the teacher and student to promote student achievement in all process standards.

Ranking of constructs for number changed from formal, informal, display, action, conventional, and imagistic to action, display, informal, formal, conventional, and imagistic. The action constructs switched with the formal constructs to become the highest teacher ratings. The bipolar ends of action and informal switched with display and formal for a higher rating. NCTM (2000) called for changes in teaching mathematics. Furthermore, this change of instruction was to be streamlined with reform curriculum including the use of manipulatives and connection to real world applications. The teachers were better able to recognize student informal representations after the professional development. The teachers also realized the strength of student action representation in the post-test. Per Goldin (2003), informal representations should be noticed and developed to become formal in the classroom. Furthermore, algebraic expressions and the several representations thereof are action representations. Although teachers noticed that most of what students did not know was imagistic, they still failed to rate imagistic higher in what and how to teach the concept next. This corresponds to Wu's research (2000) in noting that teacher knowledge of the use of representations factors into use of that representation in the classroom.

Hypothetical Learning Trajectory for Number

Research Question 3. Do the hypothetical learning trajectories of the teachers for equivalent fractions change after the professional development?

On the pre-test, the hypothetical learning trajectory addressed all representations. Question 5 on the video observation analysis instrument moved to column 3 and was closest to what the student did or did not know as represented in the video clip. This is favorable because there should be a closer relationship between what the student does not know and what the teacher should teach next and how he or she goes about it.

On the post-test, question 5 on the video observation analysis instrument moved from column 3 to and stayed in column 3. This move stayed close to what the student did or did not know as represented in the video clip. This is favorable because there should be a close relationship between what the student does not know and what the teacher should teach next, as well as how the teacher teaches. In addition, the teacher rated imagistic representation high on what the student did not know, but it was rated low on what the teacher would teach next.

Teachers notice student representation as action, imagistic, formal, or the bipolar opposites and use this pedagogical content knowledge to determine hypothetical learning trajectories based on what the students know and do not know. Evidence from the data suggests that teachers were not better able to hypothesize the learning trajectory in number, but they did move from using what the students know and did not know as a tool for the trajectory. The teachers initially showed a relationship with just the hypothetical learning trajectory and what the students knew with 90% similarity. Although this similarity percentage dropped in the post-test to 75%, teachers replied with a 65% similarity percentage with a relationship to the hypothetical learning trajectory and what the students did not know. The model fits in number, this data lends itself to the study of Tirosh (2000), where it was found that teachers could not explain a fraction procedure but also could not explain the student error yet the changes in the pedagogical content knowledge of the teachers are attributed not only to knowledge of content and teaching but also to teachers' previous beliefs (Fennema & Franke, 1992; Lehrer, R. & Franke, M. L., 1992). In this study, after the professional development, teachers were able to identify student error and use this information as a foundation for what and how to teach the next concept.

Hypothetical Learning Trajectory for Algebra

Research Question 4. Do the hypothetical learning trajectories of the teachers for algebraic expressions change after the professional development?

On the pre-test, not only did the hypothetical learning trajectory fail to address imagistic representation, but also question 5 on the video observation analysis instrument moved to column 1 and was furthest from what the student did not know. This is unfavorable because there should be a closer relationship between what the student does not know and what the teacher should teach next.

On the post-test, Question 5 on the video observation analysis instrument moved from column 1 to column 3. This move was closer to what the student did not know as represented in the video clip and what the student knew. This is favorable because there should be a closer relationship between what the student does not know and between what the teacher does next and how he or she teaches the next concept. In addition, the teacher rated imagistic representation high on what the student did not know, but it was rated low on what the teacher would teach next.

The data shows that teachers were better able to hypothesize the learning trajectory in the post-test for number. The trajectory moved from having 75% similarity with what the student did not know and no relationship to what the student did know to a 95% similarity relationship with what the student does and does not know. Clearly, hypothetical learning trajectories are redefined after the professional development partnership, and they relate to the guidelines for effective collaborative partnerships recommended by Joyce and Showers (2002) and Loucks-Horsley et al. (2003).

Conclusions

The MSMP professional development was designed to train teachers to notice aspects of student representation and redefine scope and sequence of curriculum based on this knowledge. Although it appears in the data that this took place, it is also obvious that teacher training in uses of specific representations in number and algebra is necessary. This supports Wu's research (2000), which indicated that teachers "gained content knowledge of representation for fractions and algebraic patterns of change" (p. 132) but that their pedagogical knowledge toward implementing the use of representations into teaching did not advance.

Imagistic representation is representation that uses manipulatives or other materials that are meaningful to students. This representation was severely under-represented in all tests. Teachers should use different types of representations including formal, imagistic, and action representations in teaching students mathematical ideas. This will promote student learning in all process standards including reasoning and proof, communication, problem solving, and connection.

Defining hypothetical learning trajectories based on what the student does or does not know is extremely important in student success. Hence, professional development that continues to develop and redefine these trajectories can promote teacher pedagogical content knowledge.

Professional development that concentrates on strengthening teacher pedagogical content knowledge of student representation offers a reasonable starting point for observing student problem areas and promoting achievement in those areas through development of hypothetical learning trajectories. Focusing on reform goals in professional development can become a monumental task. Therefore, the professional development partnership must be goal oriented to promote change. The partnership between the mathematics education researcher and the teacher contributes to teacher growth by giving perspectives based on empirical knowledge in the content and pedagogical areas.

It is expected that correlations between constructs and elements will be redefined after professional development to represent a relationship between the professional development partnership and teacher growth. The changes in the correlations and hierarchal cluster analysis imply teacher knowledge growth and retention, and also increase the likelihood of the practice and transfer to the classroom.

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