PERFORMANCE ANALYSIS OF SIGNAL-TO-NOISE RATIO (SNR) ESTIMATES FOR ADDITIVE WHITE GAUSSIAN NOISE (AWGN) AND TIME-SELECTIVE FADING CHANNELS

A Thesis

by

HUSEYIN PEKSEN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

December 2008

Major Subject: Electrical Engineering

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ABSTRACT

Performance Analysis of Signal-to-Noise Ratio (SNR)
Estimates for Additive White Gaussian Noise (AWGN)
and Time-Selective Fading Channels. (December 2008)

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In this work, first the Cramer-Rao lower bound (CRLB) of the signal-to-noise ratio (SNR) estimate for binary phase shift keying (BPSK) modulated signals in additive white Gaussian noise (AWGN) channels is derived. All the steps and results of this CRLB derivation are shown in a detailed manner. Two major estimation scenarios are considered herein: the non-data-aided (NDA) and data-aided (DA) frameworks, respectively. The non-data-aided scenario does not assume the periodic transmission of known data symbols (pilots) to limit the system throughput, while the data-aided scenario assumes the transmission of known transmit data symbols or training sequences to estimate the channel parameters. The Cramer-Rao lower bounds for the non-data-aided and data-aided scenarios are derived. In addition, the modified Cramer-Rao lower bound (MCRLB) is also calculated and compared to the true CRLBs. It is shown that in the low SNR regime the true CRLB is tighter than the MCRLB in the non-data-aided estimation scenario.

Second, the Bayesian Cramer-Rao lower bound (BCRLB) for SNR estimate is considered for BPSK modulated signals in the presence of time-selective fading channels. Only the data-aided scenario is considered, and the time-selective fading channel is modeled by means of a polynomial function. A BCRLB on the variance of the SNR

estimate is found and the simulation results are presented.

To my parents and uncle Abdullah Cakmak

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CHAPTER I

INTRODUCTION

The main scope of this thesis is to analyze the performance of the signal-to-noise ratio (SNR) estimates in additive white Gaussian noise (AWGN) and time-selective fading channels through comparisons with theoretical performance benchmarks. First, the Cramer-Rao lower bound (CRLB) and modified CRLB will be used to analyze the performance of the SNR estimate in AWGN channels. Then the performance of the SNR estimate in time-selective fading channels will be analyzed by deriving the Bayesian-CRLB (BCRLB). The concepts and terminology necessary to conduct this analysis will be discussed in the following sections.

A brief description of SNR will help the reader to understand the answers of the following questions. What is SNR? What are the applications of the SNR? The simplest definition of the SNR is that of the ratio between the signal power to the noise power. Most of the times, this ratio is measured in decibels (dB). SNR is one of the most fundamental measures to characterize the performance of a communication system. For instance, bit error rate can be calculated from the knowledge of SNR at the receiver side. In general, the higher this ratio is the better the performance achieved by the communication system. The knowledge of the SNR can be utilized in many areas such as image and video transmission and processing, mobile communications, satellite communications and so on. Therefore, deriving efficient SNR estimators is an important task for any communication engineer.

After this brief description of SNR, the reasons to estimate SNR will be next explained. It is easier to compute the SNR if the signal power or noise power is

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known at the receiver. However, in most cases, such an information might not be available at the receiver site. Most of the times neither the signal power nor the noise power is known, so the estimation should be done for both of these parameters. Numerous papers can be found in literature about SNR estimation in different scenarios and many of them focus on finding the most accurate estimator. In general, an accurate estimate of SNR is required for saving energy in many applications. Thus far, a large number of different estimation techniques (Split-Symbol Moments Estimator (SSME), Maximum-Likelihood (ML) Estimator, Squared-Signal-to-Noise Variance (SNV) Estimator, Moments Method (MM) Estimator, Signal-to-Variation Ratio (SVR) Estimator) have been proposed to build SNR estimators. Reference [1] presents an excellent overview of the main SNR estimation algorithms proposed recently in the literature.

SNR estimators can be categorized into two classes: data-aided (DA) and non-data-aided (NDA). DA estimators make use of the knowledge of the transmitted data symbols while NDA estimators do not assume knowledge of data symbols. In general, DA estimators perform better than NDA or blind estimators. However, DA SNR estimators can be applied only if known data symbols are available. On the other hand, the NDA estimators present the advantage of not decreasing the throughput of the communication system, and therefore they are also known as "in-service" estimation techniques [1]. References [2]-[3] present a good overview of the DA and NDA estimation techniques. Both classes of estimators will be applied to our models in the next chapters of this thesis.

The performance analysis of the SNR estimates represents the main focus of this thesis. In general, the performance of an unbiased estimator can be found by placing a lower bound on its variance. By placing lower bounds on the variances of these estimators, the performance of these estimators can be compared with well defined performance benchmarks and a hierarchy in terms of performance could be established among different SNR estimators. Many bounds exist in the literature such as McAulay and Hofstetter 1971, Kendall and Stuart 1979, Seidman 1970, Ziv and Zakai 1969 (see e.g., [4]). In general, CRLB is easier to compute in terms of complexity. CRLB is a lower bound on the error variance of unbiased estimators. CRLB was first derived by Cramer and Rao, [5], [6].

In practice, it is highly desirable to design unbiased estimators with variances equal or very close to CRLB. In [7], the CRLBs for SNR estimators for BPSK and QPSK modulated signals were derived. This represents a fundamental reference but important details of the CRLB derivation were not fully disclosed. The technical details of the reference [7] will be presented in Chapter II, and additional novel results will be derived as well.

The modified Cramer-Rao lower bound (MCRLB) represents another lower bound which was introduced in [8]. The relationship between CRLB and MCRLB was also introduced in this reference. Some insights and details about MCRLB will be discussed in Chapter II. It is known that MCRLB is in general a looser bound than the CRLB, but it is easier to obtain. Being easier to compute makes MCRLB useful for numerous practical applications for which the derivation of the exact CRLB is either impossible or computationally prohibitive. Notice also that the vectorial form of MCRLB was presented in [9].

The true CRLB and the modified CRLB are not suited for assessing the performance of algorithms dealing with time-varying parameter estimation, since these bounds can only be applied to non-random parameters. The main reason for this is the fact that the statistical dependence between the amplitudes of channel gain at different instants is not taken into account by the true and the modified CRLBs.

In time-varying channels, the parameter vector to be estimated has to be considered random. Recently, deriving CRLBs suited for time-varying parameters has been addressed within the Bayesian statistics framework [10]. In [10], the authors presented a closed-form expression of the Bayesian CRLB (BCRLB) for estimating a dynamic phase offset. Also, in [11], the problem of adaptive parameter estimation was studied.

This thesis is organized as follows. In Chapter II, first we set the signal model. The CRLB for the SNR estimate for BPSK modulated signals in AWGN channels are derived for both non-data-aided and data-aided scenarios. Then the MCRLB is derived and the MCRB and CRB are compared. In Chapter III, the BCRLB is derived for BPSK modulated signals in time-selective fading channels assuming a data-aided scenario. Finally, computer simulation results are presented to corroborate the analytical results.

CHAPTER II

THE TRUE AND THE MODIFIED CRAMER-RAO BOUNDS FOR SNR ESTIMATE

A. The True Cramer-Rao Bound

1. Background

The CRLB of the SNR estimate for a binary phase shift keying (BPSK) modulated signal in an AWGN channel will be derived in this chapter. Even though CRLB was described briefly in the first chapter, the mathematical expressions needed to state the problem will be next revised. The CRLB for a scalar parameter is defined as follows. It is assumed that the probability density function $p(\mathbf{x};\theta)$ satisfies the regularity condition

$$E\left[-\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right] = 0 \quad \text{for all } \theta, \tag{2.1}$$

where $\mathbf{x}$ and θ denote the observed data and unknown parameter, respectively, and the expectation is taken with respect to $p(\mathbf{x};\theta)$. According to the CRLB, the variance of any unbiased estimator must satisfy the inequality:

$$\mathrm{var}(\hat{\theta}) \geq \frac{1}{-\mathrm{E}\left[\frac{\partial^2 \ln \mathrm{p}(\mathbf{x};\theta)}{\partial \theta^2}\right]} = \frac{1}{-\int \frac{\partial^2 \ln \mathrm{p}(\mathbf{x};\theta)}{\partial \theta^2} \mathrm{p}(\mathbf{x};\theta) \mathrm{d}\mathbf{x}}, \tag{2.2}$$

where the expectation is taken with respect to $p(\mathbf{x};\theta)$. The right-hand side term of (2.2) is referred to as the CRLB.

For a vector parameter to be estimated $\theta = [\theta_1 \theta_2 \cdots \theta_p]^T$, the CRLB places a bound on the variance of each element assuming that the estimator $\hat{\theta}$ is unbiased [12]

$$\mathrm{var}(\hat{\theta}_{\mathrm{i}}) \geq \left[\mathbf{I^{-1}}(\theta)\right]_{\mathrm{ii}} \tag{2.3}$$

where $\mathbf{I}(\theta)$ is the $p \times p$ Fisher information matrix (FIM), which is defined generically in terms of its (i, j) entry as follows

$$[\mathbf{I}(\theta)]_{ij} = -E\left[\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial\theta_i\partial\theta_j}\right] \ , \tag{2.4}$$

for $i = 1, \ldots, p$ and $j = 1, \ldots, p$.

2. Problem Statement

Having revised the definition of CRLB, the problem now can be stated. In this section, the CRLB for the SNR estimate for a BPSK modulated signal in an AWGN channel will be derived. The derivation will be shown in a detailed manner. The Cramer-Rao lower bound for both NDA and DA scenarios will be presented.

In the absence of phase and frequency offsets, a BPSK signal at the output of the matched filter can be represented as

$$x_n = Sa_n + w_n, \text{ for } \ n = 1, \cdots, N$$

where S is a real scalar denoting the channel gain, and a_n is the corresponding transmitted symbol. In the non-data-aided scenario, a_n's are modeled as independent identically distributed (i.i.d) random symbols taking values from $\{+1, -1\}$ with equal probability. The additive terms w_n's are statistically independent zero-mean Gaussian random variables with a variance equal to σ^2. The *a prior* probabilities of a_n's, which are denoted as $\Pr(a_n = a^{(i)}), a^{(i)} \in \{+1, -1\}$, are assumed to be independent of S and w_n's. Suppose we have an observation window of N samples, within which S and σ^2 are constant.

In many applications, telecommunications engineers have to deal with intersymbol interference (ISI) in both wired and wireless communication systems due to the lack of perfect synchronization. The ISI effects are caused by the misalignment of data symbols at the specific sampling instants. In general, ISI causes a degradation in the performance of the digital communication system [13].

In [14], the Nyquist criterion for zero ISI is defined as: a necessary and sufficient condition for zero ISI at the receive filter output is that the folded Fourier transform $G_{fld}(w)$ of the equivalent shaping pulse is a constant for $|w| < \frac{\pi}{T}$ where $1/T$ is the symbol rate and

$$G_{fld}(\omega) = T \sum_{l=-\infty}^{\infty} g(-mT)\mathrm{exp(jm\omega T)}. \tag{2.5}$$

Having these definitions, we also assume Nyquist pulse shaping and ideal sampling at the receiver end so that the inter-symbol interference (ISI) at each sampling instance can be ignored. The signal-to-noise ratio (SNR) per symbol can therefore be defined as the ratio between signal power to noise power

$$\rho \triangleq \frac{E_s}{N_0} = \frac{S^2}{2\sigma^2}. \tag{2.6}$$

Notice that the SNR factor ρ needs to be estimated based on N observations of received samples x_n. Two parameters are involved in this estimate. In this case, the parameter vector can be defined as

$$\theta \triangleq \begin{bmatrix} S & \sigma^2 \end{bmatrix}^T.$$

The estimated SNR is generally represented in decibels since many engineering applications are interested in dB units in SNR, i.e., $\rho^{dB} = 10\log(\rho)$, so the following function is considered:

$$g(\theta) \triangleq 10\log\left(\frac{S^2}{2\sigma^2}\right). \tag{2.7}$$

As shown in [12], the following equation characterizes the CRLB of the SNR expressed in decibels:

$$CRLB(\rho) = \frac{\partial g(\theta)}{\partial \theta}\mathbf{I}^{-1}(\theta)\frac{\partial g(\theta)}{\partial \theta}^T, \tag{2.8}$$

where $\mathbf{I}(\theta)$ is Fisher information matrix (FIM)

$$\mathbf{I}(\theta) = \begin{bmatrix} -E\left\{\frac{\partial^2 lnp(x;\theta)}{\partial S^2}\right\} & -E\left\{\frac{\partial^2 \ln p(x;\theta)}{\partial S \partial \sigma^2}\right\} \\ -E\left\{\frac{\partial^2 \ln p(x;\theta)}{\partial \sigma^2 \partial S}\right\} & -E\left\{\frac{\partial^2 \ln p(x;\theta)}{\partial {\sigma^2}^2}\right\} \end{bmatrix}. \tag{2.9}$$

From (2.7), $\frac{\partial g(\theta)}{\partial \theta}$ can be found as

$$\frac{\partial g(\theta)}{\partial \theta} = \begin{bmatrix} \frac{20}{\ln(10)S} & \frac{-10}{\ln(10)\sigma^2} \end{bmatrix}. \tag{2.10}$$

a. Non-Data-Aided Case

For the NDA case, the transmitted symbol a_n is unknown to the receiver, so one can express the parametric probability density function (pdf) $p_{NDA}(x_n;\theta)$ as follows:

$$\begin{aligned} p_{NDA}(x_n;\theta) &= \sum_{i \in I} \Pr(a_n = a^{(i)}) p(x_n | a_n, \theta) \\ &= \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{x_n^2 + S^2}{2\sigma^2}\right]\exp\left[\frac{x_n S}{\sigma^2}\right] \\ &+ \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{x_n^2 + S^2}{2\sigma^2}\right]\exp\left[-\frac{x_n S}{\sigma^2}\right] \\ &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{x_n^2 + S^2}{2\sigma^2}\right]\left[\frac{1}{2}\left(\exp\left[\frac{x_n S}{\sigma^2}\right] + \exp\left[-\frac{x_n S}{\sigma^2}\right]\right)\right] \\ &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{x_n^2 + S^2}{2\sigma^2}\right]\cosh\left(\frac{x_n S}{\sigma^2}\right). \end{aligned} \tag{2.11}$$

Hence, the log-likelihood function for the N observed samples is given by

$$\begin{aligned}\ln p_{\mathrm{NDA}}(x;\theta) &= -\frac{N}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{n=1}^{N}\left(x_n^2 + S^2\right) \\ &+ \sum_{n=1}^{N}\ln\left(\cosh\left(\frac{x_n S}{\sigma^2}\right)\right). \qquad (2.12)\end{aligned}$$

Next, since the partial derivatives of the log-likelihood pdf are required to evaluate the elements of the Fischer information matrix, the following manipulations are necessary to be conducted:

$$\begin{aligned}\frac{\partial \ln p(x;\theta)}{\partial S} &= -\frac{NS}{\sigma^2} + \frac{1}{\sigma^2}\sum_{n=1}^{N}\tanh\left(\frac{x_n S}{\sigma^2}\right)x_n, \\ \frac{\partial^2 \ln p(x;\theta)}{\partial S^2} &= -\frac{N}{\sigma^2} + \frac{1}{\sigma^4}\sum_{n=1}^{N}\operatorname{sech}^2\left(\frac{x_n S}{\sigma^2}\right)x_n^2, \qquad (2.13) \\ \frac{\partial^2 \ln p(x;\theta)}{\partial S \partial\sigma^2} &= \frac{NS}{\sigma^4} - \frac{1}{\sigma^4}\sum_{n=1}^{N}\tanh\left(\frac{x_n S}{\sigma^2}\right)x_n \\ &- \frac{S}{\sigma^6}\sum_{n=1}^{N}\operatorname{sech}^2\left(\frac{x_n S}{\sigma^2}\right)x_n^2, \qquad (2.14) \\ \frac{\partial \ln p(x;\theta)}{\partial \sigma^2} &= -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{n=1}^{N}\left(x_n^2 + S^2\right) - \frac{S}{\sigma^4}\sum_{n=1}^{N}\tanh\left(\frac{x_n S}{\sigma^2}\right)x_n,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 \ln p(x;\theta)}{\partial {\sigma^2}^2} &= \frac{N}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{n=1}^{N}\left(x_n^2 + S^2\right) + \frac{2S}{\sigma^6}\sum_{n=1}^{N}\tanh\left(\frac{x_n S}{\sigma^2}\right)x_n \\ &+ \frac{S^2}{\sigma^8}\sum_{n=1}^{N}\operatorname{sech}^2\left(\frac{x_n S}{\sigma^2}\right)x_n^2. \qquad (2.15)\end{aligned}$$

Before proceeding to calculate the Fisher information matrix, the expected values of the following terms are required to to be evaluated.

$$
\begin{aligned}
E\left\{\operatorname{sech}^2\left(\frac{x_n S}{\sigma^2}\right)x_n^2\right\} &= \int_{-\infty}^{\infty} p(x_n;\theta)\operatorname{sech}^2\left(\frac{x_n S}{\sigma^2}\right)x_n^2 dx_n \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{S^2}{2\sigma^2}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{x_n^2}{2\sigma^2}\right)\frac{x_n^2}{\cosh\left(\frac{x_n S}{\sigma^2}\right)}dx_n
\end{aligned}
$$

Remembering the definition of SNR in (2.6) and integration by substitution, one can easily obtain

$$
E\left\{\operatorname{sech}^2\left(\frac{x_n S}{\sigma^2}\right)x_n^2\right\} = \sigma^2 f(\rho), \tag{2.16}
$$

where $f(\rho)$ is a scalar function of ρ and is defined as

$$
f(\rho) = \frac{\exp(-\rho)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{u^2\exp(u^2/2)}{\cosh(u\sqrt{2\rho})}du.
$$

$$
\begin{aligned}
E\left\{\tanh\left(\frac{x_n S}{\sigma^2}\right)x_n\right\} &= \int_{-\infty}^{\infty} p(x_n;\theta)\tanh\left(\frac{x_n S}{\sigma^2}\right)x_n dx_n \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left(-\frac{x_n^2+S^2}{2\sigma^2}\right)\sinh\left(\frac{x_n S}{\sigma^2}\right)x_n dx_n \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x_n-S)^2}{2\sigma^2}\right]x_n dx_n \\
&- \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x_n+S)^2}{2\sigma^2}\right]x_n dx_n \\
&= S
\end{aligned} \tag{2.17}
$$

$$
\begin{aligned}
E\{x_n^2\} &= \int_{-\infty}^{\infty} p(x_n;\theta)x_n^2 dx_n \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left(-\frac{x_n^2+S^2}{2\sigma^2}\right)\cosh\left(\frac{x_n S}{\sigma^2}\right)x_n^2 dx_n \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x_n-S)^2}{2\sigma^2}\right]x_n^2 dx_n \\
&+ \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x_n+S)^2}{2\sigma^2}\right]x_n^2 dx_n \\
&= S^2+\sigma^2
\end{aligned} \tag{2.18}
$$

With (2.16)-(2.18) in mind, one can determine the expected values of the second order partial derivatives. The first element of the Fisher information matrix can be found as

$$\begin{aligned}
E\left\{\frac{\partial^2 \ln p(x;\theta)}{\partial S^2}\right\} &= -\frac{N}{\sigma^2} + \frac{1}{\sigma^4}\sum_{n=1}^{N} E\left\{\operatorname{sech}^2\left(\frac{x_n S}{\sigma^2}\right) x_n^2\right\} \\
&= -\frac{N}{\sigma^2} + \frac{N}{\sigma^2} f(\rho) \\
&= -\frac{N}{\sigma^4}(\sigma^2 - \sigma^2 f(\rho)). \qquad (2.19)
\end{aligned}$$

The second and the third elements of the Fisher information matrix can be determined by the following manipulations

$$\begin{aligned}
E\left\{\frac{\partial^2 \ln p(x;\theta)}{\partial S \partial \sigma^2}\right\} &= -\frac{NS}{\sigma^4} - \frac{1}{\sigma^4}\sum_{n=1}^{N} E\left\{\tanh\left(\frac{x_n S}{\sigma^2}\right) x_n\right\} \\
&- \frac{S}{\sigma^6}\sum_{n=1}^{N} E\left\{\operatorname{sech}^2\left(\frac{x_n S}{\sigma^2}\right) x_n^2\right\} \\
&= \frac{NS}{\sigma^4} - \frac{NS}{\sigma^4} - \frac{S}{\sigma^6} N\sigma^2 f(\rho) \\
&= -\frac{N}{\sigma^4} S f(\rho). \qquad (2.20)
\end{aligned}$$

And the last element is given by

$$\begin{aligned}
E\left\{\frac{\partial^2 \ln p(x;\theta)}{\partial {\sigma^2}^2}\right\} &= \frac{N}{2\sigma^4} - \frac{NS^2}{\sigma^6} - \frac{1}{\sigma^6}\sum_{n=1}^{N} E\left\{x_n^2\right\} \\
&+ \frac{2S}{\sigma^6}\sum_{n=1}^{N} E\left\{\tanh\left(\frac{x_n S}{\sigma^2}\right) x_n\right\} + \frac{S^2}{\sigma^8}\sum_{n=1}^{N} E\left\{\operatorname{sech}^2\left(\frac{x_n S}{\sigma^2}\right) x_n^2\right\} \\
&= \frac{N}{2\sigma^4} - \frac{NS^2}{\sigma^6} - \frac{1}{\sigma^6} N(S^2 + \sigma^2) + \frac{2S}{\sigma^6} NS + \frac{S^2}{\sigma^8} N\sigma^2 f(\rho) \\
&= -\frac{N}{\sigma^4}\left(\frac{1}{2} - \frac{S^2 f(\rho)}{\sigma^2}\right) \qquad (2.21)
\end{aligned}$$

With the expected values (2.19)-(2.21) ready, the expression for the Fisher information matrix is given by

$$\mathbf{I}_{NDA}(\theta) = \frac{N}{\sigma^4}\begin{bmatrix} \sigma^2 - \sigma^2 f(\rho) & Sf(\rho) \\ Sf(\rho) & \frac{1}{2} - \frac{S^2 f(\rho)}{\sigma^2} \end{bmatrix}.$$

The inverse of the $\mathbf{I}_{NDA}(\theta)$ matrix should be found to calculate the CRLB. It's inverse is expressed as

$$\mathbf{I}_{NDA}^{-1}(\theta) \quad = \quad \frac{\sigma^4}{N\left[\frac{\sigma^2}{2} - S^2 f(\rho) - \frac{\sigma^2}{2} f(\rho)\right]} \cdot \begin{bmatrix} \frac{1}{2} - \frac{S^2 f(\rho)}{\sigma^2} & -Sf(\rho) \\ -Sf(\rho) & \sigma^2 - \sigma^2 f(\rho) \end{bmatrix} \qquad (2.22)$$

Plugging (2.10) and (2.22) into (2.8), one can readily obtain the CRLB

$$CRLB_{NDA}(\rho) = \frac{200\left(\frac{1}{\rho} - f(\rho) + 1\right)}{N(\ln 10)^2[1 - f(\rho) - 4\rho f(\rho)]} \quad (\mathrm{dB}^2). \qquad (2.23)$$

b. Data-Aided Case

For the data-aided (DA) case, the transmitted symbol a_n is known to the receiver, the modulation can be removed perfectly and the resulting signal model can be expressed as

$$x_n = S + v_n, \text{ for } n = 1, \cdots, N.$$

The parametric probability density function (pdf) $p_{DA}(x_n; \theta)$ can be expressed as

$$p_{DA}(x_n; \theta) \quad = \quad \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_n - S)^2}{2\sigma^2}\right]. \qquad (2.24)$$

Notice that the Fisher information matrix was already defined in (2.9). The same procedure will be conducted as in the NDA case. The log-likelihood function of the probability density function given N samples takes the form:

$$\ln p_{\mathrm{DA}}(x;\theta) \quad = \quad -\frac{N}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n - S)^2 . \tag{2.25}$$

So, the terms $\frac{\partial^2 \ln p(x;\theta)}{\partial S^2}$, $\frac{\partial^2 \ln p(x;\theta)}{\partial S \partial \sigma^2}$, and $\frac{\partial^2 \ln p(x;\theta)}{\partial {\sigma^2}^2}$ can be obtained through the following calculations:

$$\begin{aligned}
\frac{\partial \ln p(x;\theta)}{\partial S} &= \frac{1}{\sigma^2}\sum_{n=1}^{N}(x_n - S), \\
\frac{\partial^2 \ln p(x;\theta)}{\partial S^2} &= -\frac{N}{\sigma^2}, \\
\frac{\partial^2 \ln p(x;\theta)}{\partial S \partial \sigma^2} &= -\frac{1}{\sigma^4}\sum_{n=1}^{N}(x_n - S), \\
\frac{\partial \ln p(x;\theta)}{\partial \sigma^2} &= -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{n=1}^{N}(x_n - S)^2 ,
\end{aligned} \tag{2.26}$$

and

$$\frac{\partial^2 \ln p(x;\theta)}{\partial {\sigma^2}^2} \quad = \quad \frac{N}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{n=1}^{N}(x_n - S)^2 \tag{2.27}$$

After taking the negative expectations, one can easily find the Fisher information matrix as

$$\mathbf{I}_{DA}(\theta) = \begin{bmatrix} \frac{N}{\sigma^2} & 0 \\ 0 & \frac{N}{2\sigma^4} \end{bmatrix}.$$

The inverse of $\mathbf{I}_{DA}(\theta)$ matrix is

$$\mathbf{I}_{DA}^{-1}(\theta) \quad = \quad \frac{2\sigma^6}{N^2} \cdot \begin{bmatrix} \frac{N}{2\sigma^4} & 0 \\ 0 & \frac{N}{\sigma^2} \end{bmatrix} \tag{2.28}$$

Again, by using (2.10) and (2.28) into (2.8), the true CRLB for the DA case can be expressed as

$$CRLB_{DA}(\rho) = \frac{200\left(\frac{1}{\rho}+1\right)}{N(\ln 10)^2} \quad (\mathrm{dB}^2). \tag{2.29}$$

B. The Modified Cramer-Rao Bound

1. Background

As we have seen in (2.8), the evaluation of the CRLB is mathematically quite tedious when the observed signal contains, besides the parameter to be estimated some nuisance parameters (denoted symbolically by the notation $\mathbf{u}$ and in whose values we are not interested in and in the same whose values might be hard or impossible to estimate). The joint probability density function can be expressed as

$$p(\mathbf{x};\theta) = \int_{-\infty}^{\infty} p(\mathbf{x}|\mathbf{u};\theta)p(\mathbf{u})d\mathbf{u} = E_{\mathbf{u}}\left[p(\mathbf{x}|\mathbf{u};\theta)\right]. \tag{2.30}$$

It is clear now that the integration in (2.30) is difficult. As shown in [8], by changing the logarithm operators and the order of the expectation operators, MCRLB can be expressed as

$$MCRLB(\theta) = \frac{1}{E_{\mathbf{u}}\left[E_{\mathbf{x}|\mathbf{u}}\left[\frac{\partial^2 \ln p(\mathbf{x}|\mathbf{u};\theta)}{\partial\theta^2}\right]\right]} \tag{2.31}$$

where $E_{\mathbf{x}|\mathbf{u}}[\cdot]$ denotes the expectation with respect to $p(\mathbf{x}|\mathbf{u};\theta)$.

The CRLB in (2.2) can also be further expressed as

$$CRLB(\theta) = \frac{1}{E\left[\left[\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right]^2\right]} = \frac{1}{\int\left[\frac{\partial \ln p(\mathbf{x};\theta)}{\partial\theta}\right]^2 p(\mathbf{x};\theta)d\mathbf{x}}. \tag{2.32}$$

So the MCRLB can be found as

$$MCRLB(\theta) = \frac{1}{E_{\mathbf{u}}\left[E_{\mathbf{x}|\mathbf{u}}\left[\frac{\partial \ln p(\mathbf{x}|\theta)}{\partial \theta}\right]^2\right]}. \tag{2.33}$$

2. Problem Statement

In this section, the modified CRLB will be derived for the SNR estimate for BPSK modulated signals in AWGN channels. The signal model is given by (2.5). For vector parameters, the modified CRLB is given by the following formula:

$$MCRLB(\rho) = \frac{\partial g(\theta)}{\partial \theta}\mathbf{I}'^{-1}(\theta)\frac{\partial g(\theta)}{\partial \theta}^T,$$

where $\mathbf{I}'^{-1}(\theta)$ is the inverse modified Fisher information matrix (MFIM)

$$\mathbf{I}'(\theta) = \begin{bmatrix} -E_{x,a}\left\{\frac{\partial^2 lnp(x|a,\theta)}{\partial S^2}\right\} & -E_{x,a}\left\{\frac{\partial^2 \ln p(x|a,\theta)}{\partial S \partial \sigma^2}\right\} \\ -E_{x,a}\left\{\frac{\partial^2 \ln p(x|a,\theta)}{\partial \sigma^2 \partial S}\right\} & -E_{x,a}\left\{\frac{\partial^2 \ln p(x|a,\theta)}{\partial {\sigma^2}^2}\right\} \end{bmatrix}. \tag{2.34}$$

The log-likelihood function given the transmitted symbols $\ln p(x|a,\theta)$ can be represented as

$$\ln p(x|a,\theta) = -\frac{N}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n - Sa_n)^2.$$

To find the modified Fisher information matrix, we will need to find the partial derivatives of the log-likelihood function just like we did before.

$$\frac{\partial \ln p(x|a,\theta)}{\partial S} = \frac{1}{\sigma^2}\sum_{n=1}^{N}(x_n - Sa_n)a_n,$$

$$\frac{\partial^2 \ln p(x|a,\theta)}{\partial S^2} = -\frac{1}{\sigma^2}\sum_{n=1}^{N}a_n^2, \tag{2.35}$$

$$\frac{\partial^2 \ln p(x|a,\theta)}{\partial S \partial \sigma^2} = -\frac{1}{\sigma^4}\sum_{n=1}^{N}a_n(x_n - Sa_n), \tag{2.36}$$

$$\frac{\partial \ln p(x|a,\theta)}{\partial \sigma^2} = -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{n=1}^{N}(x_n - Sa_n)^2,$$

and

$$\frac{\partial^2 \ln p(x|a,\theta)}{\partial {\sigma^2}^2} = \frac{N}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{n=1}^{N}(x_n - Sa_n)^2. \tag{2.37}$$

Thus, the expected values of the elements for the modified Fisher information matrix can be determined as follows:

$$\begin{aligned}
E_{x,a}\left\{\frac{\partial^2 \ln p(x|a,\theta)}{\partial S^2}\right\} &= -\frac{1}{\sigma^2}\sum_{n=1}^{N} E_{x_n,a_n}\{a_n^2\} \\
&= -\frac{N}{\sigma^2} && (2.38) \\
E_{x,a}\left\{\frac{\partial^2 \ln p(x|a,\theta)}{\partial S \partial \sigma^2}\right\} &= -\frac{1}{\sigma^4}\sum_{n=1}^{N} E_{x_n,a_n}\{a_n x_n\} + \frac{S}{\sigma^4}\sum_{n=1}^{N} E_{x_n,a_n}\{a_n^2\} \\
&= \frac{NS}{\sigma^4} - \frac{1}{\sigma^4}\sum_{n=1}^{N} E_{a_n}\left\{a_n E_{x_n|a_n}\{x_n\}\right\} \\
&= \frac{NS}{\sigma^4} - \frac{S}{\sigma^4}\sum_{n=1}^{N} E_{a_n}\{a_n^2\} \\
&= 0 && (2.39) \\
E_{x,a}\left\{\frac{\partial^2 \ln p(x|a,\theta)}{\partial {\sigma^2}^2}\right\} &= \frac{N}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{n=1}^{N} E_{x_n,a_n}\left\{(x_n - Sa_n)^2\right\} \\
&= \frac{N}{2\sigma^4} - \frac{1}{\sigma^6}N\sigma^2 \\
&= -\frac{N}{2\sigma^4} && (2.40)
\end{aligned}$$

Therefore, by plugging (2.38)-(2.40) into (2.34), it is found that the modified Fisher information matrix takes the diagonal matrix form

$$\mathbf{I}'(\theta) = \begin{bmatrix} \frac{N}{\sigma^2} & 0 \\ 0 & \frac{N}{2\sigma^4} \end{bmatrix}.$$

The inverse of the modified Fisher information matrix can be found as

$$\mathbf{I}'^{-1}(\theta) = \frac{2\sigma^6}{N^2} \cdot \begin{bmatrix} \frac{N}{2\sigma^4} & 0 \\ 0 & \frac{N}{\sigma^2} \end{bmatrix}, \tag{2.41}$$

and the modified CRLB can be easily found by plugging (2.41) into (2.34) as

$$MCRLB(\rho) = \frac{200\left(\frac{1}{\rho} + 1\right)}{N(\ln 10)^2} \quad (\mathrm{dB}^2). \tag{2.42}$$

C. Numerical Results

Figures 1, 2 and 3 compare the CRLB (2.23) and MCRLB (2.42) for the non-data-aided case assuming different observation lengths. The observation lengths $N = 100$, $N = 200$, and $N = 400$ are assumed for Figures 1, 2 and 3, respectively. Since the MCRLB (2.42) takes the same form as the CRLB (2.29) for the data-aided estimation case, the dotted lines also represent the performance bounds when the training symbols are transmitted. One can compare the CRLBs for non-data aided and data-aided cases by replacing MCRLB with CRLB for the data-aided case.

Notice that for large SNR values, $f(\rho)$ approaches zero and, therefore, the two bounds get close. As one can see, for SNR values greater than 6 dB, the two bounds become identical. While in the low SNR regime, the difference between the two bounds is significant. In other words, the MCRLB is quite loose at low SNR levels.

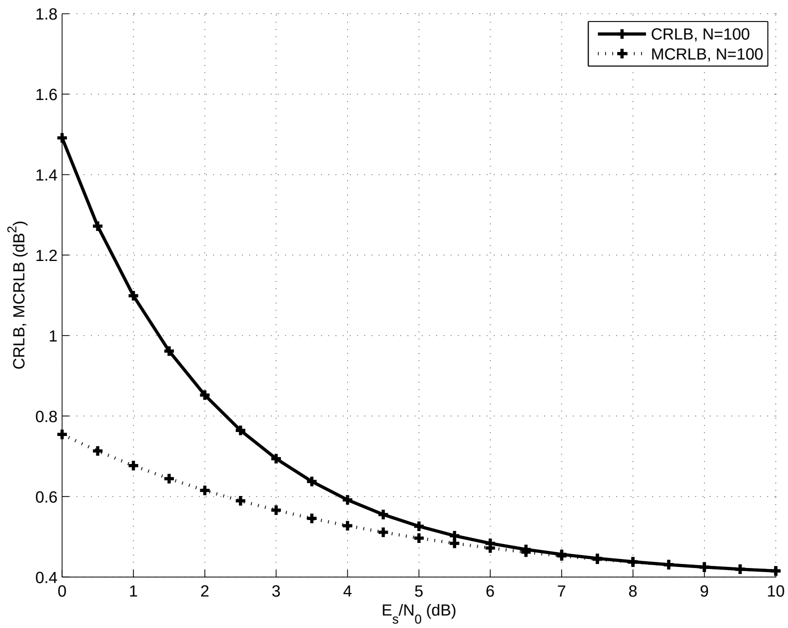

Fig. 1. CRLB and MCRLB for SNR estimators for a BPSK signal with N=100.

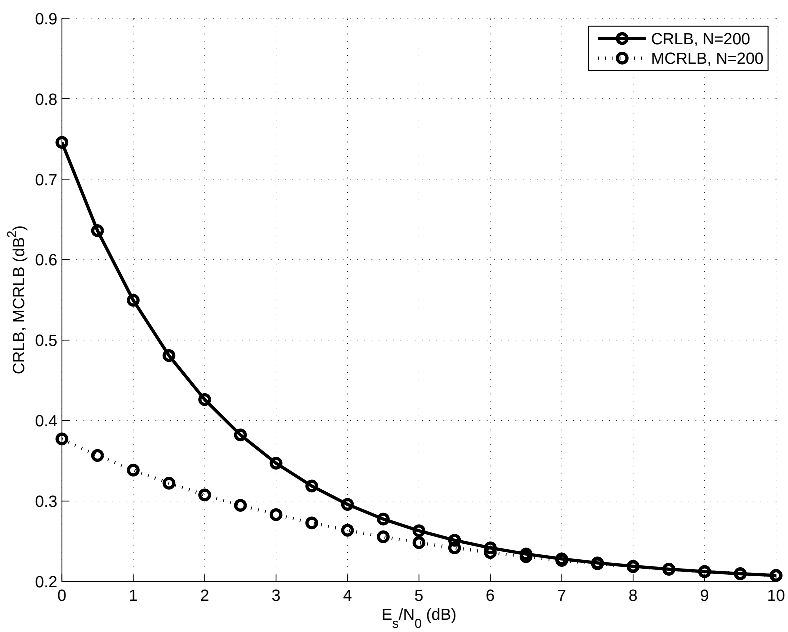

Fig. 2. CRLB and MCRLB for SNR estimators for a BPSK signal with N=200.

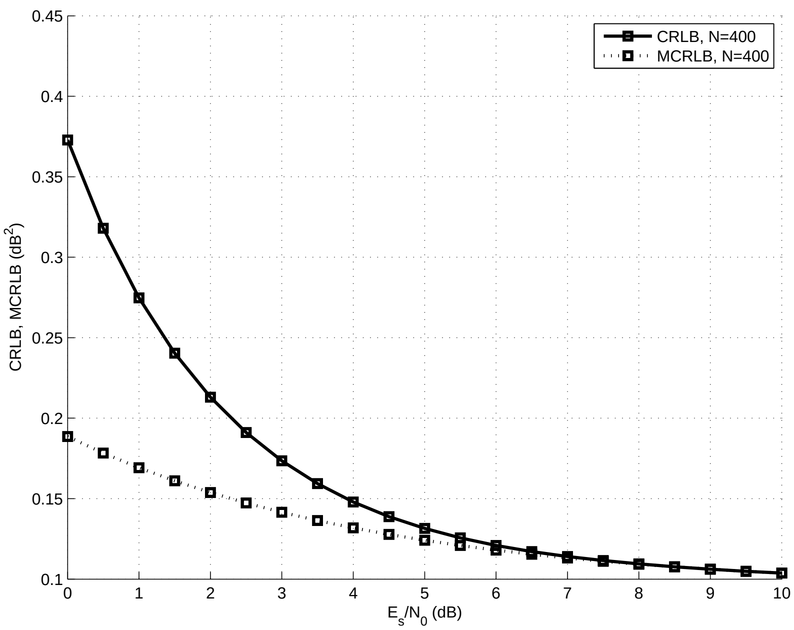

Fig. 3. CRLB and MCRLB for SNR estimators for a BPSK signal with N=400.

CHAPTER III

THE BAYESIAN CRAMER-RAO BOUNDS FOR SNR ESTIMATE

A. Background

There exists two basic models for the static parameter estimation. The parameters to be estimated are assumed to be non-random unknown variables in the first model. The parameter estimation problem with this type of model is called Fisher or non-Bayesian estimation. For instance, S and σ^2 were assumed to be unknown non-random parameters in Chapter II, and the resulting estimation framework could be referred to as non-Bayesian. The second model assumes that the parameters to be estimated are random variables with a priori probability densities. This type of estimation is called Bayesian estimation. Bayesian estimation will represent our main interest in this chapter since the parameters will be considered random variables with a priori information. Notice also that the selected model always depends on the specific application.

It is important to develop lower bounds on the performance of any Bayesian estimator. There are already several Bayesian bounds proposed in the literature such as the Bayesian Cramer-Rao lower bound, Bayesian Bhattacharyya bound, Bobrovsky-Zakai bound and Weiss-Weinstein bound.

The Bayesian Cramer-Rao lower bound (BCRLB) will be derived in this chapter. BCRLB was first derived by Van Trees in 1968 [15]. Van Trees defined a $D \times D$

Bayesian information matrix $\mathbf{J}_B$ in [4], whose elements are

$$\begin{aligned}[\mathbf{J}_B]_{i,j} &= E_{\mathbf{y},\boldsymbol{\theta}}\left\{\frac{\partial \ln p(\mathbf{y},\boldsymbol{\theta})}{\partial \theta_i}\cdot\frac{\partial \ln p(\mathbf{y},\boldsymbol{\theta})}{\partial \theta_j}\right\} \\ &= -E_{\mathbf{y},\boldsymbol{\theta}}\left\{\frac{\partial^2 \ln p(\mathbf{y},\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}\right\}. \qquad (3.1)\end{aligned}$$

$\mathbf{J}_B$ can be expressed as

$$\mathbf{J}_B = \mathbf{J}_D + \mathbf{J}_P, \qquad (3.2)$$

where

$$\begin{aligned}[\mathbf{J}_D]_{i,j} &= E_{\mathbf{y},\boldsymbol{\theta}}\left\{\frac{\partial \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial \theta_i}\cdot\frac{\partial \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial \theta_j}\right\} \\ &= -E_{\mathbf{y},\boldsymbol{\theta}}\left\{\frac{\partial^2 \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}\right\} \\ &= E_{\boldsymbol{\theta}}\left\{-E_{\mathbf{y}|\boldsymbol{\theta}}\left[\frac{\partial^2 \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}\right]\right\} \\ &= E_{\boldsymbol{\theta}}\left\{[\mathbf{I}(\boldsymbol{\theta})]_{i,j}\right\}, \qquad (3.3)\end{aligned}$$

and

$$\begin{aligned}[\mathbf{J}_P]_{i,j} &= E_{\boldsymbol{\theta}}\left\{\frac{\partial \ln p(\boldsymbol{\theta})}{\partial \theta_i}\cdot\frac{\partial \ln p(\boldsymbol{\theta})}{\partial \theta_j}\right\} \\ &= -E_{\boldsymbol{\theta}}\left\{\frac{\partial^2 \ln p(\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}\right\}. \qquad (3.4)\end{aligned}$$

In (3.3), $\mathbf{I}(\boldsymbol{\theta})$ is the Fisher information matrix which was defined in (2.4). The contribution of the data is represented by the $\mathbf{J}_D$ term in (3.3) and the $\mathbf{J}_P$ term in (3.4) represents the contribution of the prior information.

$$\mathbf{R}(\hat{\theta}) \geq \mathbf{J}_{\mathrm{B}}{}^{-1} \triangleq \mathrm{BCRLB}, \qquad (3.5)$$

and

$$E_{\mathbf{y},\boldsymbol{\theta}}\left\{\left(\hat{\theta}_i(\mathbf{y}) - \theta_i\right)^2\right\} \geq [\mathbf{J}_B]_{ii}\,, \qquad (3.6)$$

with equality if and only if the following linear relationship holds:

$$\hat{\theta}_i(\mathbf{y}) - \theta_i = \sum_{j=1}^{D} k_{ij} \frac{\partial \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial \theta_j}, \tag{3.7}$$

for some constants k_{ij}. The reader should note that k_{ij} is not a function of $\boldsymbol{\theta}$.

B. Problem Statement

In this section, a closed-form expression of a BCRLB for SNR estimate will be presented. The transmission of a BPSK modulated signal over a time-selective fading channel is considered. Assuming an ideal receiver with perfect synchronization, the output of the receiver's matched filter can be expressed as

$$y_n = a_n h_n + w_n, \qquad n = 1, ..., N \tag{3.8}$$

where y_n is the received signal, a_n stands for the transmitted data symbol ($a_n = \pm 1$), h_n denotes the time-varying positive channel gain and w_n represents the realization of a zero-mean, white, complex Gaussian noise with known variance σ_w^2 at the time index n.

It is straightforward to extend the analysis to vector case. For example, the received samples could be represented in terms of the $N \times 1$ vector

$$\mathbf{y} = \mathbf{A}\mathbf{h} + \mathbf{w}, \tag{3.9}$$

where $\mathbf{y} \equiv [y_1 \cdots y_N]^T$, $\mathbf{h} \equiv [h_1 \cdots h_N]^T$, $\mathbf{w} \equiv [w_1 \cdots w_N]^T$ and A is the $N \times N$ diagonal matrix with $[A]_{nn} = a_n$.

Notice also that the communication channel can be modeled in several ways. There exit various papers in this regard. Clarke's model [16] is widely used, i.e., where h_n is a zero mean complex normal random process with the following correlation function

$$E[h_n h_{n-l}^*] = \sigma_h^2 J_0\left(2\pi \frac{f_d}{f_s} l\right), \tag{3.10}$$

where $[\cdot]^*$ denotes the conjugate operator, σ_h^2 is the channel variance, f_d is the maximum Doppler frequency, f_s is the sampling rate, and $J_0(\cdot)$ is the Bessel function of zero zero. This model is proven to be very useful from the viewpoint of performance evaluation criteria [2]. The polynomial model is more appropriate for capturing the changes in time selective channels and was used in [17], [18]. By assumption, the slowly time-varying process h_n is bandlimited. So, by using Taylor's series expansion theorem, the time-varying channel gain can be expanded as a M-order polynomial

$$h_n = \sum_{m=0}^{M-1} c_m {t_n}^m + R_M(n), \tag{3.11}$$

where c_m is the unknown complex random channel gain, $\boldsymbol{\tau_n} = [t_n^0 \cdots t_n^{M-1}]^T$ stands for the index of the nth sample, and $R_M(n)$ denotes the remainder of the Taylor series expansion. The $R_M \to 0$ as $M \to \infty$ approximation was already presented in [17]. Therefore, for M sufficiently high, the channel can be approximated as $\mathbf{h} \approx \mathbf{Tc}$ where

$$\mathbf{T} = \begin{bmatrix} 1 & t_1 & \cdots & {t_1}^{M-1} \\ 1 & t_2 & \cdots & {t_2}^{M-1} \\ \vdots & \vdots & & \vdots \\ 1 & t_N & \cdots & {t_N}^{M-1} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{M-1} \end{bmatrix}. \tag{3.12}$$

In this chapter, just the data-aided estimation case will be considered. So, the transmitted symbol a_n is assumed to be known to the receiver. The channel gains are assumed to be unknown complex Gaussian random parameters. Hence, the probability density functions of each sample, in the case of BPSK transmission, are given by

$$p_{DA}(y_n|\boldsymbol{\theta}) \;=\; \frac{1}{\pi\sigma_w^2}\exp\left[-\frac{|y_n - \boldsymbol{\tau_n}^T\mathbf{c}|^2}{\sigma_w^2}\right]. \tag{3.13}$$

SNR was defined as the ratio of the signal power to the noise power. So, SNR over a channel realization can be represented as

$$\rho = \frac{\mathbf{c}^H\mathbf{T}^T\mathbf{T}\mathbf{c}}{N\sigma_w^2}, \tag{3.14}$$

where SNR ρ is a function of a vector of unknown parameters $\boldsymbol{\theta} = [\mathbf{c}^T, \sigma_w^2]^T$ in complex channels. SNR will be estimated with the given known symbols a_{DA} and $\mathbf{y}$ with the assumed statistical model. Many engineering applications use the quantities decibels (dB). Using dB has many advantages such as representing very small and large numbers, logarithmic scaling possibility and ability to carry out multiplication of ratios by simple addition and subtraction. In our case, the SNR can be expressed in dB as follows: $\rho^{(dB)} = 10\log(\rho)$. Note that the superscript $[\cdot]^{(dB)}$ denotes quantities in dB.

The term J_B was defined in equation (3.2). J_B can be further expanded as follows

$$\mathbf{J}_B = E_{\boldsymbol{\theta}}\left[\mathbf{I}(\boldsymbol{\theta})\right] + E_{\boldsymbol{\theta}}\left[-\Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}}\ln p(\boldsymbol{\theta})\right], \tag{3.15}$$

where $\Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}}$ represents the second-order partial derivative and $\mathbf{I}(\boldsymbol{\theta})$ is the Fisher information matrix which was expressed in (2.4). The Fisher information matrix can also be expressed as

$$\mathbf{I}(\boldsymbol{\theta}) \;=\; E_{y_n|\boldsymbol{\theta}}\left[-\Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}}\ln p(y_n|\boldsymbol{\theta})\right]. \tag{3.16}$$

Now, the BCRLB can be derived as follows. First, we will compute $E_{\boldsymbol{\theta}}\left[\mathbf{I}(\boldsymbol{\theta})\right]$ which corresponds to the first term in the righthand side of equation (3.15) in the subsection Step 1. Then, $E_{\boldsymbol{\theta}}\left[-\Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}}\ln p(\boldsymbol{\theta})\right]$, which corresponds to the second term in (3.15) will be achieved in Step 2. Finally, BCRLB will be found by calculating the Bayesian information matrix. The inverse of the addition of two matrices found in Step 1 and Step 2 respectively will give us the bound on the unknown parameters. SNR is a function of these unknown parameters. Thus, BCRLB will be achieved by making use of the unknown parameters.

1. Step 1: Calculation of $E_{\boldsymbol{\theta}}\left[\mathbf{I}(\boldsymbol{\theta})\right]$

In this subsection, the term $E_{\boldsymbol{\theta}}\left[\mathbf{I}(\boldsymbol{\theta})\right]$ will be evaluated. The Fisher information matrix $I(\boldsymbol{\theta})$ should be computed first. The following manipulations should be done by using equation (3.16). The Fisher information matrix $\mathbf{I}(\boldsymbol{\theta})$ can be found using the identity given in [12] as

$$E\left[\frac{\partial \ln p(y_n|\boldsymbol{\theta})}{\partial \theta_i}\cdot\frac{\partial \ln p(y_n|\boldsymbol{\theta})}{\partial \theta_j}\right] = -E\left[\frac{\partial^2 \ln p(y_n|\boldsymbol{\theta})}{\partial \theta_i \partial \theta_j}\right]. \tag{3.17}$$

So, the second order derivative matrix can be obtained directly from (3.13) by expanding the squared magnitude into a product of a complex number with its conjugate as

$$\frac{\ln p_{DA}(y_n|\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^*}\cdot\frac{\ln p_{DA}(y_n|\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^*}^H = \frac{|y_n - \boldsymbol{\tau_n}^T c|^2}{\sigma_w^4}\begin{bmatrix} 1 & t_n & \cdots & t_n^{M-1} \\ t_n & t_n^2 & \cdots & t_n^M \\ \vdots & \vdots & & \vdots \\ t_n^{M-1} & t_n^M & \cdots & t_n^{2(M-1)} \end{bmatrix}. \tag{3.18}$$

Therefore, we can obtain the FIM as

$$E_{\boldsymbol{\theta}}\left[\mathbf{I}(\boldsymbol{\theta})\right] = E\left[\frac{\ln p_{DA}(y_n|\boldsymbol{\theta})}{\partial\boldsymbol{\theta}^*}\frac{\ln p_{DA}(y_n|\boldsymbol{\theta})}{\partial\boldsymbol{\theta}^*}^H\right] = \frac{1}{\sigma_w^2}\begin{bmatrix} 1 & t_n & \cdots & t_n^{M-1} & 0 \\ t_n & t_n^2 & \cdots & t_n^M & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ t_n^{M-1} & t_n^M & \cdots & t_n^{2(M-1)} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{2\sigma_w^2} \end{bmatrix}. \tag{3.19}$$

2. Step 2: Calculation of $E_{\boldsymbol{\theta}}\left[-\Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}}\ln p(\boldsymbol{\theta})\right]$

The term $E_{\boldsymbol{\theta}}\left[-\Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}}\ln p(\boldsymbol{\theta})\right]$ will be evaluated in this subsection. It represents the contribution of the prior information. The parameter vector $\boldsymbol{\theta} = [\mathbf{c}^T, \sigma_w^2]^T$ contains both $\mathbf{c}$ and σ_w^2 terms. So we need to have knowledge about these terms. The channel gain is assumed to be complex normal, so the probability density function of the parameter vector can be expressed as

$$p(c_m) = \frac{1}{\pi\sigma_c^2}\exp\left(-\frac{|c_m|^2}{\sigma_c^2}\right). \tag{3.20}$$

And we will model the channel noise as a uniform distribution

$$p(\sigma_w^2) = \frac{1}{b-a} \quad \text{for } \; 0 < a < \sigma_w^2 < b, \tag{3.21}$$

where a is the minimum value and b is the maximum value. The joint probability density function of these two models is

$$p(\boldsymbol{\theta}) = \frac{1}{b-a}\prod_{m=0}^{M-1}\frac{1}{\pi\sigma_c^2}\exp\left(-\frac{|c_m|^2}{\sigma_c^2}\right). \tag{3.22}$$

The log-likelihood joint probability density function can be found as

$$\ln p(\boldsymbol{\theta}) = -\ln(b-a) - M\ln(\pi\sigma_c^2) - \sum_{m=0}^{M-1}\frac{|c_m|^2}{\sigma_c^2}. \tag{3.23}$$

The first-order partial derivative of the log-likelihood joint pdf with respect to the complex conjugate of the parameter vector can be obtained as

$$\frac{\partial \ln p(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^*} = -\begin{bmatrix} \frac{c_0}{\sigma_c^2} \\ \frac{c_1}{\sigma_c^2} \\ \vdots \\ \frac{c_{M-1}}{\sigma_c^2} \\ 0 \end{bmatrix}. \tag{3.24}$$

Again, by using the identity in equation (3.17) given in [12], one can find the contribution of the prior information as

$$E_{\boldsymbol{\theta}}\left[-\Delta_{\boldsymbol{\theta}}^{\boldsymbol{\theta}} \ln p(\boldsymbol{\theta})\right] = E_{\boldsymbol{\theta}}\left\{\frac{\partial \ln p(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^*} \cdot \frac{\partial \ln p(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^*}^H\right\} = \begin{bmatrix} \frac{1}{\sigma_c^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\sigma_c^2} & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \frac{1}{\sigma_c^2} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{3.25}$$

3. Step 3: Calculation of BCRLB

By plugging (3.19) and (3.25) into (3.15), we can readily obtain the Bayesian information matrix as

$$\mathbf{J_B} = \frac{1}{\sigma_w^2}\mathbf{X} + \mathbf{Y}, \tag{3.26}$$

where

$$\mathbf{X} = \begin{bmatrix} 1 & t_n & \cdots & t_n^{M-1} & 0 \\ t_n & t_n^2 & \cdots & t_n^M & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ t_n^{M-1} & t_n^M & \cdots & t_n^{2(M-1)} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{2\sigma_w^2} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} \frac{1}{\sigma_c^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\sigma_c^2} & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \frac{1}{\sigma_c^2} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{3.27}$$

$\mathbf{X}$ is the matrix part of the equation (3.19) and $\mathbf{Y}$ is the matrix found in equation (3.25). Both of these matrix's dimensions are $(M+1) \times (M+1)$.

From the definition of the BCRLB given in (3.5), we can obtain the bound on the variances of unknown parameters $\boldsymbol{\theta}$. However, we want to find the Bayesian bound on the variance of SNR which is a function of $\boldsymbol{\theta} = [c_0, c_1, \ldots, c_{M-1}, \sigma_w^2]^T$. We define the function $\gamma(\boldsymbol{\theta})$ to be estimated which was also presented in (3.14) as

$$\gamma(\boldsymbol{\theta}) = \rho = \frac{\mathbf{c}^H \mathbf{T}^T \mathbf{T} \mathbf{c}}{N\sigma_w^2}. \tag{3.28}$$

Then, the BCRLB is defined in [4] as

$$E_{\mathbf{y},\boldsymbol{\theta}} \left\{ (\hat{\gamma}(\boldsymbol{\theta}) - \gamma(\boldsymbol{\theta}))^2 \right\} \geq \boldsymbol{\Gamma} \mathbf{J}_B^{-1} \boldsymbol{\Gamma}^{\mathbf{T}}, \tag{3.29}$$

where $\boldsymbol{\Gamma}$ is

$$\begin{aligned} \boldsymbol{\Gamma} &= E_{y_n,\boldsymbol{\theta}} \left\{ \gamma(\boldsymbol{\theta}) \frac{\partial \ln p(y_n, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right\} \\ &= \int \int \gamma(\boldsymbol{\theta}) \frac{\partial \ln p(y_n, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} p(y_n, \boldsymbol{\theta}) dy_n d\boldsymbol{\theta} \\ &= \int \left(\int \gamma(\boldsymbol{\theta}) \frac{\partial \ln p(y_n, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} d\boldsymbol{\theta} \right) dy_n \\ &= \int p(y_n) \left(\int \gamma(\boldsymbol{\theta}) \frac{\partial \ln p(\boldsymbol{\theta}|y_n)}{\partial \boldsymbol{\theta}} d\boldsymbol{\theta} \right) dy_n. \end{aligned} \tag{3.30}$$

Evaluating the inner integral gives

$$\int \gamma(\boldsymbol{\theta}) \frac{\partial \ln p(\boldsymbol{\theta}|y_n)}{\partial \boldsymbol{\theta}} d\boldsymbol{\theta} = \gamma(\boldsymbol{\theta}) p(\boldsymbol{\theta}|y_n) \mid_{-\infty}^{\infty} - \int \frac{\partial \gamma(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} p(\boldsymbol{\theta}|y_n) d\boldsymbol{\theta}. \tag{3.31}$$

Assuming

$$\lim_{\boldsymbol{\theta} \to \pm\infty} \gamma(\boldsymbol{\theta}) p(\boldsymbol{\theta}|y_n) = 0 \text{ for all } y_n, \tag{3.32}$$

we have

$$\begin{aligned} \boldsymbol{\Gamma} &= -\int \int \frac{\partial \gamma(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} p(y_n, \boldsymbol{\theta}) dy_n d\boldsymbol{\theta} \\ &= -E_{\boldsymbol{\theta}} \left\{ \frac{\partial \gamma(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right\}. \end{aligned} \tag{3.33}$$

After some straightforward calculations, one can find the first-order partial derivative of the $\gamma(\boldsymbol{\theta})$ as

$$\frac{\partial \gamma(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \frac{2\mathbf{T}^T\mathbf{T}\mathbf{c}}{N\sigma_w^2} \\ -\frac{\mathbf{c}^{\mathbf{H}}\mathbf{T}^{\mathbf{T}}\mathbf{T}\mathbf{c}}{N\sigma_w^4} \end{bmatrix}^T. \tag{3.34}$$

Applying the expectation operator to the first-order partial derivative of the $\gamma(\boldsymbol{\theta})$ function leads to

$$\boldsymbol{\Gamma} = -E_{\boldsymbol{\theta}} \left\{ \frac{\partial \gamma(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right\} = \begin{bmatrix} \mathbf{0} \\ \frac{\mathrm{tr}(\mathbf{T}^T\mathbf{T})\sigma_c^2}{Nab} \end{bmatrix}^T, \tag{3.35}$$

where the notation $\mathrm{tr}(\cdot)$ denotes the trace of a matrix and is defined to be the sum of the elements on the main diagonal (the diagonal from the upper left to the lower right) of that matrix.

The $\boldsymbol{\Gamma}$ term in equation (3.29) has already been determined. Now we need to find the inverse of the Bayesian information matrix. The inverse of the Bayesian matrix can be found by the following manipulations

$$\begin{aligned}
\mathbf{J}_B^{-1} &= \left(\frac{1}{\sigma_w^2}\mathbf{X} + \mathbf{Y}\right)^{-1} \\
&= \left(\frac{1}{\sigma_w^2} \begin{bmatrix} 1 & t_n & \cdots & t_n^{M-1} & 0 \\ t_n & t_n^2 & \cdots & t_n^M & 0 \\ \vdots & & & & \\ t_n^{M-1} & t_n^M & \cdots & t_n^{2(M-1)} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{2\sigma_w^2} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sigma_c^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\sigma_c^2} & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \frac{1}{\sigma_c^2} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \right)^{-1}
\end{aligned} \tag{3.36}$$

$$= \left(\frac{1}{\sigma_w^2} \underbrace{\begin{bmatrix} 1 & t_n & \cdots & t_n^{M-1} & 0 \\ t_n & t_n^2 & \cdots & t_n^M & 0 \\ \vdots & & & & \\ t_n^{M-1} & t_n^M & \cdots & t_n^{2(M-1)} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}}_{\boldsymbol{\omega}\boldsymbol{\omega}^T} + \underbrace{\begin{bmatrix} \frac{1}{\sigma_c^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\sigma_c^2} & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \frac{1}{\sigma_c^2} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{2\sigma_w^4} \end{bmatrix}}_{\mathbf{Z}} \right)^{-1} , \tag{3.37}$$

where

$$\boldsymbol{\omega} = \begin{bmatrix} \boldsymbol{\tau_n} \\ 0 \end{bmatrix}. \tag{3.38}$$

There exists a known matrix inversion lemma, which is called Woodbury identity, e.g., [19]:

$$(\mathbf{A} + b\mathbf{u}\mathbf{v}^H)^{-1} = \mathbf{A}^{-1} - \frac{b}{1 + b\mathbf{v}^H\mathbf{A}^{-1}\mathbf{u}}\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^H\mathbf{A}^{-1}. \tag{3.39}$$

Equation (3.37) looks familiar to the form of Woodbury identity, so

$$\begin{aligned}\mathbf{J}_B^{-1} &= \left(\mathbf{Z} + \frac{1}{\sigma_w^2}\boldsymbol{\omega}\boldsymbol{\omega}^T\right)^{-1} \\ &= \mathbf{Z}^{-1} - \frac{1}{\sigma_w^2 + \boldsymbol{\omega}^T\mathbf{Z}^{-1}\boldsymbol{\omega}}\mathbf{Z}^{-1}\boldsymbol{\omega}\boldsymbol{\omega}^T\mathbf{Z}^{\mathbf{-1}}, \end{aligned} \tag{3.40}$$

where

$$\mathbf{Z}^{-1} = \begin{bmatrix} \sigma_c^2\mathbf{I}_M & \\ & 2\sigma_w^4 \end{bmatrix}. \tag{3.41}$$

By plugging (3.41) in (3.40), we can find the inverse of the Bayesian information matrix as

$$\begin{aligned}\mathbf{J}_B^{-1} &= \begin{bmatrix} \sigma_c^2\mathbf{I}_M & \\ & 2\sigma_w^4 \end{bmatrix} - \frac{1}{\sigma_w^2 + \boldsymbol{\omega}^T \begin{bmatrix} \sigma_c^2\mathbf{I}_M & \\ & 2\sigma_w^4 \end{bmatrix} \boldsymbol{\omega}} \begin{bmatrix} \sigma_c^2\mathbf{I}_M & \\ & 2\sigma_w^4 \end{bmatrix} \boldsymbol{\omega}\boldsymbol{\omega}^T \begin{bmatrix} \sigma_c^2\mathbf{I}_M & \\ & 2\sigma_w^4 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_c^2\mathbf{I}_M - \frac{\sigma_c^4\boldsymbol{\tau_n}\boldsymbol{\tau_n}^T}{\sigma_w^2+\sigma_c^2\boldsymbol{\tau_n}^T\boldsymbol{\tau_n}} & \\ & 2\sigma_w^4 \end{bmatrix}. \end{aligned} \tag{3.42}$$

By plugging (3.35) and (3.42) into (3.29), one can get the Bayesian CRLB on the variance of the SNR as

$$\begin{aligned}\text{BCRLB} &= \begin{bmatrix} \mathbf{0}^T & \frac{\operatorname{tr}(\mathbf{T}^T\mathbf{T})\sigma_c^2}{Nab} \end{bmatrix} \begin{bmatrix} \sigma_c^2\mathbf{I}_M - \frac{\sigma_c^4\boldsymbol{\tau_n}\boldsymbol{\tau_n}^T}{\sigma_w^2+\sigma_c^2\boldsymbol{\tau_n}^T\boldsymbol{\tau_n}} & \\ & 2\sigma_w^4 \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \frac{\operatorname{tr}(\mathbf{T}^T\mathbf{T})\sigma_c^2}{Nab} \end{bmatrix} \\ &= \left[\frac{\sqrt{2}\sigma_w^2\sigma_c^2}{Nab}\operatorname{tr}(\mathbf{T}^T\mathbf{T})\right]^2. \end{aligned} \tag{3.43}$$

C. Numerical Results

Figure 4 shows the Bayesian CRLB for the SNR estimate for the BPSK modulated signal in time-selective fading channels. BCRLB for data-aided case is considered. For simplicity, $\{\tau_1 \ldots \tau_N\}$ has been chosen such that $\mathbf{T}^T\mathbf{T} = N\mathbf{I}$. The same simplification is also done in our work. We assumed $\mathrm{tr}(\mathbf{T}^T\mathbf{T}) = N$ so that the observation length term N is canceled in equation (3.43). It is observed that the BCRLB is not sensitive to the polynomial order M. We obtain the same plots for different values of M.

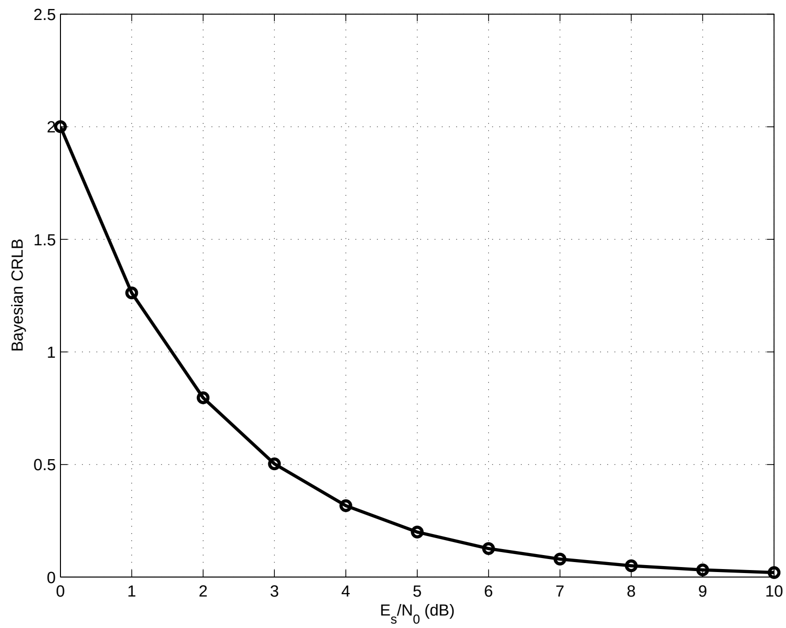

Fig. 4. BCRLB for SNR estimators for BPSK signal (DA scenario).

CHAPTER IV

CONCLUSIONS

We summarize this work and suggest possible future research topics in this chapter.

A. Summary of the Thesis

To summarize, this thesis has considered the problem of finding lower bounds of SNR estimates.

First, background information about CRLB has been presented. CRLB for SNR estimate for BPSK modulated signals in AWGN channels has been shown in a detailed way. CRLBs for both non-data-aided and data-aided cases have been investigated. Moreover, the MCRLB has been calculated and compared to the true CRLBs. It is found that all the lower bounds are inversely proportional to the observation length. It is observed that the MCRLB is the same as the CRLB for the data-aided case. At low SNR levels, the MCRLB is loose for the non-data-aided estimation scenario.

Next, background information about BCRLB has been presented. The BCRLB has been derived for the SNR estimate of a BPSK modulated signal in time-selective fading channels. The time-selective fading channel is modeled using a polynomial-in-time model. BCRLB for data-aided case is considered in this work. Simulation results have been also presented for BCRLB.

B. Future Works

There are numerous directions for future research work. First, different types of lower bounds can be applied to both estimation scenarios such as the Barankin and Bayesian Bhattacharyya lower bounds. Performance of the lower bounds for SNR estimate could be calculated in different directions such as frequency flat Rician fading channels. Additionally, in our current work, the noise was assumed to be an uncorrelated normal process. However, in general, the noise is caused by interference. So, finding performance bounds of signal-to-noise-and-interference ratio (SINR) can be considered as a interesting future work. Furthermore, determining which bounds predict most accurately the performance of SNR estimators at low SNR values represents another challenging and very important research problem.

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