KERR-NUT-ADS METRICS AND STRING THEORY

A Dissertation

by

WEI CHEN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2007

Major Subject: Physics

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Approved by:

Chair of Committee,	Christopher Pope
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	Stephen Fulling
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ABSTRACT

Kerr-NUT-AdS Metrics and String Theory. (December 2007)

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Chair of Advisory Committee: Dr. Christopher Pope

With the advent of supergravity and superstring theory, it is of great importance to study higher-dimensional solutions to the Einstein equations. In this dissertation, we study the higher dimensional Kerr-AdS metrics, and show how they admit further generalisations in which additional NUT-type parameters are introduced.

The choice of coordinates in four dimensions that leads to the natural inclusion of a NUT parameter in the Kerr-AdS solution is rather well known. An important feature of this coordinate system is that the radial variable and the latitude variable are placed on a very symmetrical footing. The NUT generalisations of the high-dimensional Kerr-AdS metrics obtained in this dissertation work in a very similar way. We first consider the Kerr-AdS metrics specialised to cohomogeneity 2 by appropriate restrictions on their rotation parameters. A latitude coordinate is introduced in such a way that it, and the radial variable, appeared in a very symmetrical way. The inclusion of a NUT charge is a natural result of this parametrisation. This procedure is then applied to the general D dimensional Kerr-AdS metrics with cohomogeneity $[D/2]$. The metrics depend on the radial coordinate r and $[D/2]$ latitude variables μ_i that are subject to the constraint $\sum_i \mu_i^2 = 1$. We find a coordinate reparameterisation in which the μ_i variables are replaced by $[D/2]-1$ unconstrained coordinates y_α, and put the coordinates r and y_α on a parallel footing in the metrics, leading to an immediate introduction of $([D/2]-1)$ NUT parameters. This gives the most general

Kerr-NUT-AdS metrics in D dimensions.

We discuss some remarkable properties of the new Kerr-NUT-AdS metrics. We show that the Hamilton-Jacobi and Klein-Gordon equations are separable in Kerr-NUT-AdS metrics with cohomogeneity 2. We also demonstrate that the general cohomogeneity-n Kerr-NUT-AdS metrics can be written in multi-Kerr-Schild form. Lastly, We study the BPS limits of the Kerr-NUT-AdS metrics. After Euclideanisation, we obtain new families of Einstein-Sassaki metrics in odd dimensions and Ricci-flat metrics in even dimensions. We also discuss their applications in String theory.

To my parents and my wife

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CHAPTER I

INTRODUCTION

A. A Brief History of General Relativity

It has been so far discovered that there exist four fundamental interactions in nature, i.e. the gravitational, electromagnetic, strong and weak nuclear forces. Although the gravitational force is much weaker than the other fundamental interactions, its long-range nature and additivity make it the dominative force in the large scale structures of the universe. Newton was the first who realized the universality of the gravitational interaction. He proposed that *any* two point masses are attracted to each other by a gravitational force whose magnitude is proportional the the product of their masses and inversely proportional to the square of the distance between them.

Newton's description of gravity has played an important rôle in understanding the planetary motion in our solar system. However, it was subsequently superseded by Einstein's general relativity. In this theory, the observed attraction between point masses is the manifestation of their motion along geodesics in a curved spacetime. The curvature is, in turn, generated by the energy and momentum content of matter according to Einstein equations. [1].

General relativity has been confirmed in many experimental tests. The first of these was verified by Einstein himself. He calculated the advance of the perihelion of the planet Mercury according to the laws of general relativity. The result perfectly matched the already well known measurement, namely 43" per century. Another proposal of Einstein, the bending of light by the sun, was confirmed by Eddington in his expedition to observe the solar eclipse of May 29, 1919.

This dissertation follows the style of Nuclear Physics B.

The experimental tests of general relativity can only be done at the weak-field level with our current technology. It is of great interest also to study the effects of general relativity in strong gravitational fields. A theoretical way to do this is to find exact solutions to the Einstein equations. This was first achieved in 1916 by Schwarzschild, by making a simplifying ansatz for a static metric with spherical symmetry. An interesting feature of this solution is that the metric has an "event horizon" which separates the spacetime into two regions. Nothing, even light, can escape from inside the horizon. This introduces a new object into theoretical physics, the black hole.

Schwarzschild's metric carries only one parameter, the mass. Two years after his discovery, Reissner [3] and Nordström [4] found a charged black hole solution in the Einstein-Maxwell theory.

The Schwarzschild and the Reissner-Nordström black hole are static, spherically symmetric and asymptotically flat. One can generalize further by relaxing these conditions. However, due to the complexity and non-linearity of the Einstein equations, this step took a long time, until in 1963, Kerr discovered a stationary and axial symmetric solution which describes a rotating black hole [5]. This important advance started a golden age of general relativity in which numerous solutions, techniques and ideas were developed. In 1965, Newman and his coworkers found the charged Kerr black hole [6]. In 1968, Carter generalized their solution further by including a cosmological constant and a NUT parameter [7]. An acceleration parameter was introduced into Kerr solution by Plebanski and Demianski in 1976 [8].

B. Quantum Theory of Gravity

As a classical theory of gravity, general relativity is well established and widely accepted. However, it is not sufficient to describe the phenomena in an extremely strong gravitational field, such as in the vicinity of a black hole curvature singularity or during the early evolution of the universe. Quantum effects, which dominate small-distance physics, come into play. With the advent of the modern cosmology, it is necessary to develop a quantum theory of gravity.

Demand for the quantum theory of gravity also has roots in the search for an underlying truth in physics. It is believed by most theoretical physicists that all the physical principles can be unified into one theory. The idea of unification has been a dominant theme in physics for well over a century, from Maxwell's unification of electric and magnetic phenomena in 19th century to the present Standard model which unifies the electromagnetic force with the strong and weak nuclear forces. The Standard Model is based on quantum field theory, in which forces are explained as the exchanges of force-carrying particles. The three underlying forces are then described as the exchanges of gauge bosons in an $SU(3) \times SU(2) \times U(1)$ symmetry group. The Standard Model is quite successful in explaining the experimental data, and in fact it agrees with all of our observations of the physical world. However, a major problem of the theory is that it contains about twenty free parameters, whose values can be determined only by experiment.

Now there is only one piece missing from the puzzle, namely gravity. However, the quantum field theoretic description of gravity has not been successful. In this approach, a massless, spin-2 particle called the "graviton" is introduced as the force-carrier. The distinguishing feature of gravitons is that they interact with each other, and thus they contribute to the energy-momentum sources which produce them. This recursion introduces infinite corrections to the theory which make it non-renormalisable.

C. Superstring Theory

A revolutionary idea, string theory, is employed to resolve the non-renormalisability problem. In this theory, strings take the place of particles as the building blocks of the physical world. The point-like interactions are replaced by the splitting and joining of strings, which smears out all the infinities associated with the non-renormalisability. After further studying, string theory has been deemed to be a very promising candidate for the unification theory of everything. The spectrum of bosonic particles may be explained as the various excitations of strings, and this spectrum automatically contains a massless spin-2 particle corresponding to the graviton. Moreover, the string theory has no adjustable parameter while the Standard Model has about twenty which can be determined only by experiment. In spite of these remarkable features, the bosonic string theory is not totally satisfactory. First, the ground state of string corresponds to a negative-mass particle, known as the tachyon. Second, this theory does not contain fermionic particles.

The key to all these problems is a symmetry principle, called supersymmetry. This describes the invariance of a theory under the exchange of bosons and fermions. The supersymmetric transformation acting twice turns the particle into itself but at a different location in spacetime. Therefore, supersymmetry is closely related to the spacetime translational and rotational symmetry of special relativity. Local supersymmetry, know as supergravity, is expected to contains the localized special relativity, i.e. general relativity. Knowledge of general relativity is therefore crucial to the understanding of supergravity.

The combination of string theory and supersymmetry leads to superstring theory, which contains both bosonic and fermionic particles but no tachyon. In superstring theory, the dimension of spacetime is predicted to be ten. Except for the observed $3 + 1$ dimensions, there are six extra dimensions which must be highly compactified. It is the geometrical and topological properties of these compact dimensions that determine the particle species and their couplings in the four dimensional world. Thus, the study of the geometry of spacetime is crucial to understanding the connection between the ten-dimensional superstring theory and a four-dimensional realistic model.

The low-energy effective theory of superstrings is ten-dimensional supergravity. Therefore, the classical solutions in supergravity, which are closely related to the higher dimensional solutions in general relativity, can reveal many important properties of superstring theory. In 1997, Maldacena found an implementation of this idea [9]. He studied the correspondence between the supergravity on an $AdS_5 \times S^5$ background and $SU(N)$ super-Yang-Mills gauge field theory in the large N limit in four dimensions, and made a bold conjecture that this so called "AdS/CFT correspondence" will continue to hold between the full quantum Type IIB superstring and $SU(N)$ super-Yang-Mills gauge field theory with arbitrary N in four dimensions. Then it is quite natural to ask:

Can we generalize this conjecture to the supergravity on other backgrounds?

Which background corresponds to the gauge field theory of our real world?

It is therefore important to find solutions in higher dimensional general relativity and extract the geometrical and topological information of various backgrounds.

D. Black Hole Solutions in Higher Dimensions

With the discovery of the superstring and supergravity, it becomes important to generalize solutions to the Einstein equations, and especially the "elementary" black hole solution, to higher dimensions. Schwarzschild's solution was generalized to higher dimensions by Tangherlini in 1963 [10]. In 1986, Myers and Perry [11] constructed an asymptotically flat, rotating black hole metric in arbitrary dimension D. This solution carries a mass parameter and $[(D-1)/2]$ independent rotation parameters, corresponding to independent rotations in the $[(D-1)/2]$ orthogonal spatial 2-planes. In 1998, Hawking, Hunter and Taylor-Robinson found another solution in 1998 [12] which describes the five-dimensional rotating black hole with a cosmological constant. They also obtained a special case of cosmological solution in all dimensions, in which there is a rotation in only one of the $[(D-1)/2]$ orthogonal spatial 2-planes. In 2004, Gibbons, Lü, Page and Pope constructed the general cosmological rotating black hole solution in arbitrary dimension, with $[(D-1)/2]$ independent rotation parameters [13, 14].

Since the four-dimensional cosmological Kerr solution can be further generalized to have a NUT charge, one might wonder whether this is possible in higher dimensions too.

E. General Kerr-NUT-AdS Metrics in All Dimensions

The main purpose of this dissertation is to obtain new NUT generalizations of the Kerr-AdS metrics which are of the most general possible type. It contains material from [15], [16], [17], [18] and [19].

The four dimensional Kerr-NUT-AdS metric is well known. It has cohomogeneity 2, and admits a coordinate system in which the NUT charge and the mass appear in a symmetrical way. In chapter II, we will first consider the higher dimensional Kerr-AdS metrics with cohomogeneity 2, try to cast them into a similar coordinate system as that in four dimensions, and then the inclusion of NUT charge will be quite natural [15].

In chapter III, this strategy will then be applied to the most general D dimensional Kerr-AdS metrics with cohomogeneity $[D/2]$ [16]. The metrics depend on the radial coordinate r and $[D/2]$ latitude variables μ_i that are subject to the constraint $\sum_i \mu_i^2 = 1$. We want to find a coordinate reparameterisation in which the μ_i variables are replaced by $[D/2]-1$ unconstrained coordinates y_α, and put the coordinates r and y_α on a parallel footing in the metric, leading to an immediate introduction of $([D/2]-1)$ NUT parameters. This gives the most general Kerr-NUT-AdS metrics in D dimensions.

In addition, we will discuss some remarkable properties of the new Kerr-NUT-AdS metrics. The separability of the Hamilton-Jacobi and Klein-Gordon equations in certain backgrounds played an important rôle in uncovering hidden symmetries associated with the existence of Killing tensors. In chapter IV, we will study whether the Hamilton-Jacobi and Klein-Gordon equations are separable in the new Kerr-NUT-AdS metrics with cohomogeneity 2, and manage to construct the associated irreducible rank-2 Killing tensor [17].

In chapter V, we will look for the linearly-independent, and mutually orthogonal null geodesic congruences in the Kerr-NUT-AdS solutions with cohomogeneity n, which enables us to write the metrics in multi-Kerr-Schild form [18].

In chapter VI, we will study the BPS limits of the Kerr-NUT-AdS metrics [19]. After Euclideanisation, they will give new families of Einstein-Sassaki metrics in odd dimensions and Ricci-flat metrics in even dimensions. The obtained Ricci-flat metric in six dimension has a specific application in the AdS/CFT correspondence. It can be interpreted as a cohomogeneity-three resolved Calabi-Yau cone over L^{abc} space with a blown up 2-cycle or 4-cycle. We will discuss D3-branes on this Calabi-Yau cone and their applications in the AdS/CFT correspondence. In terms of the dual quiver gauge theory, this corresponds to motion along the non-mesonic, or baryonic, directions in the moduli space of vacua. In particular, a dimension-two and/or dimension-six scalar operator gets a vacuum expectation value. These resolved cones support various harmonic $(2,1)$-forms which reduce the ranks of some of the gauge groups either by a Seiberg duality cascade or by Higgsing. We conclude this dissertation in chapter VII.

CHAPTER II

KERR-NUT-ADS METRICS WITH COHOMOGENEITY 2

A. Introduction

In view of the fact that the four-dimensional rotating black hole metrics admit generalisation where a NUT parameter is present, one might wonder whether such additional parameter could also be introduced in higher dimensions too. In fact, in a certain special class of higher-dimensional Kerr-de Sitter black holes, namely those in which there is just a rotation in a single 2-plane, a generalisation which includes a NUT parameter as well as the mass and the (single) rotation parameter has been obtained [20, 21]. It was shown in [21] that this generalisation, which is trivial in five dimensions but non-trivial in dimensions $D \geq 6$, still exhibits certain remarkable separability properties for the Hamilton-Jacobi and wave equations, which in fact played an important rôle in the original discovery of the generalised four-dimensional solutions.

The purpose of this chapter is to present a much wider classes of NUT generalisations of the Kerr-de Sitter metrics in which the rotation parameters a_i are divided into two sets, and all parameters within a set are equal. In odd dimensions, which we discuss in section B, we obtain generalised solutions for an arbitrary partition of the parameters into two such sets. In even dimensions, which we discuss in section C, the parameters are partitioned into one set with a non-vanishing value for the rotation, and the other set with vanishing rotation. In each of the odd and even dimensional cases, the net effect is to give a metric of cohomogeneity 2. In a manner that parallels rather closely the generalisations in $D = 4$, the two associated coordinates, on which the metric functions are intrinsically dependent, enter in a rather symmetrical way.

The metrics that we obtain are equivalent to the previously-known Kerr-de Sitter-Taub-NUT metrics in $D = 4$. In $D \geq 6$ the extra parameter that we introduce gives rise to non-trivial generalisations of the Kerr-de Sitter metrics. The new parameter is associated with characteristics that generalise those of Taub-NUT like metrics in four dimensions, and so we may think of it as being a higher-dimensional generalisation of the NUT parameter. In each of the odd and even-dimensional cases, we discuss also their supersymmetric limits. In odd dimensions, these yield, after Euclideanisation, new Einstein-Sasaki metrics. In even dimensions, the supersymmetric limit leads to new Rici-flat Kähler metrics.

In section D, we discuss some global aspects of the new Kerr-AdS-Taub-NUT metrics. In particular, in the case of even dimensions, the introduction of the NUT-type parameter implies that the time coordinate must be identified periodically, in the same way as happens in the previously-known four-dimensional solutions. By contrast, we find that in odd dimensions one can define a time coordinate that is not periodic.

In section E, we discuss the case of five dimensions in detail. We find that in this case, the new NUT-type parameter is actually bogus, in the sense that it can be removed by using a scaling symmetry that is specific to the five-dimensional metric. In the process of showing this, however, we uncover an intriguing and previously unnoticed property of the five-dimensional Kerr-AdS metric. We find that it has an "inversion symmetry," which implies that the metric with large values of its rotation parameters is equivalent, after a general coordinate transformation, to the metric with small values for the rotations. The fixed point of this symmetry occurs at the critical value of rotation that arises in the supersymmetric limit. This corresponds to the case where the rotation parameter is equal to the radius of the asymptotically AdS spacetime. The inversion symmetry is therefore a feature specifically of the five-dimensional Kerr black holes with a cosmological constant, and does not arise in the case of asymptotically flat black holes.

The chapter ends with conclusions in section F.

B. Kerr-de Sitter with NUT Parameter in $D = 2n + 1$

1. The Metric

We take as our starting point the general Kerr-de Sitter metric in $D = 2n + 1$ dimensions, which was constructed in [13, 14]. Specifically, we begin with the metrics written in an asymptotically non-rotating frame, as given in equation (E.3) of [13], specialised to the case of odd dimensions $D = 2n + 1$. We choose the cosmological constant to be negative, with the Ricci tensor given by $R_{\mu\nu} = -(D-1)g^2\, g_{\mu\nu}$. The constant g is the inverse of the AdS radius. The metric is described in terms of n "latitude" or direction cosine coordinates μ_i, subject to the constraint $\sum_{i=1}^n \mu_i^2 = 1$, n azimuthal coordinates ϕ_i, the radial coordinate r and time coordinate t. It has $(n+1)$ arbitrary parameters M and a_i, which can be thought of as characterising the mass and the n angular momenta in the n orthogonal spatial 2-planes.

$$\begin{aligned} ds^2 &= -W\,(1+g^2\,r^2)\,d\tau^2 + \frac{U\,dr^2}{V-2M} + \frac{2M}{U}\Big(W\,d\tau - \sum_{i=1}^n \frac{a_i\,\mu_i^2\,d\phi_i}{1-a_i^2\,g^2}\Big)^2 \\ &\quad + \sum_{i=1}^n \frac{r^2+a_i^2}{1-a_i^2\,g^2}\,[d\mu_i^2 + \mu_i^2\,d\phi_i^2] \\ &\quad - \frac{g^2}{W\,(1+g^2\,r^2)}\Big(\sum_{i=1}^n \frac{(r^2+a_i^2)\mu_i\,d\mu_i}{1-a_i^2\,g^2}\Big)^2\,, \end{aligned} \tag{2.1}$$

where U and V are defined by

$$U = \sum_{i=1}^{n} \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^{n} (r^2 + a_j^2) \,.$$

$$V = \frac{1}{r^2} \, (1 + g^2 \, r^2) \prod_{i=1}^{n} (r^2 + a_i^2) \,, \qquad (2.2)$$

W and F are given by

$$W = \sum_{i=1}^{n} \frac{\mu_i^2}{1 - a_i^2 \, g^2} \,, \qquad F = \frac{r^2}{1 + g^2 \, r^2} \sum_{i=1}^{n} \frac{\mu_i^2}{r^2 + a_i^2} \,. \qquad (2.3)$$

In order to find a generalisation that includes a NUT-type parameter, we first specialise the Kerr-AdS metrics by setting

$$a_1 = a_2 = \cdots = a_p = a \,, \qquad a_{p+1} = a_{p+2} = \cdots = a_n = b \,. \qquad (2.4)$$

We then reparameterise the latitude coordinates coordinates as

$$\mu_i = \nu_i \, \sin\theta \,, \qquad 1 \le i \le p \,, \qquad \sum_{i=1}^{p} \nu_i^2 = 1 \,,$$

$$\mu_{j+p} = \tilde\nu_j \, \cos\theta \,, \qquad 1 \le j \le q \,, \qquad \sum_{j=1}^{q} \tilde\nu_j^2 = 1 \,, \qquad (2.5)$$

where we have defined

$$n = p + q \,, \qquad (2.6)$$

and we also then introduce a coordinate v in place of θ, defined by

$$a^2 \cos^2\theta + b^2 \sin^2\theta = v^2 \,. \qquad (2.7)$$

It is convenient to divide the original n azimuthal coordinates ϕ_i into two sets, with p of them denoted by ϕ_i and the remaining q denoted by $\tilde\phi_j$.

Because of the specialisation of the rotation parameters in (2.4), the Kerr-AdS metric will now have cohomogeneity 2, rather than the cohomogeneity n of the general

$(2n+1)$-dimensional Kerr-AdS metrics. In fact, as we shall see explicitly below, the metric has homogeneous level sets $\mathbb{R} \times S^{2p-1} \times S^{2q-1}$, with the metric functions depending inhomogeneously on the coordinates r and v. Remarkably, the form in which the metric can now be written puts the radial coordinate r and the coordinate v on a parallel footing, and suggests a rather natural generalisation in which a NUT-type parameter L can be introduced. Rather than writing the metric first without the NUT contribution and then again with it added, we shall just directly present our final result with the NUT parameter included. The original Kerr-AdS, subject to the constraints on the rotation parameters specified in (2.4), corresponds to setting $L=0$. Our generalised metric including L is

$$\begin{aligned} ds^2 &= -\frac{(1+g^2r^2)(1-g^2v^2)}{\Xi_a\Xi_b}dt^2 + \frac{\rho^{2n-2}dr^2}{U} + \frac{\omega^{2n-2}dv^2}{V} \\ &\quad + \frac{2M}{\rho^{2n-2}}\Big(\frac{(1-g^2v^2)}{\Xi_a\Xi_b}\,dt - \mathcal{A}\Big)^2 + \frac{2L}{\omega^{2n-2}}\Big(\frac{(1+g^2r^2)}{\Xi_a\Xi_b}\,dt - \widetilde{\mathcal{A}}\Big)^2 \\ &\quad + \frac{(r^2+a^2)(a^2-v^2)}{\Xi_a(a^2-b^2)}\sum_{i=1}^p \Big(d\nu_i^2 + \nu_i^2 d\phi_i^2\Big) + \frac{(r^2+b^2)(b^2-v^2)}{\Xi_b(b^2-a^2)}\sum_{j=1}^q \Big(d\tilde\nu_j^2 + \tilde\nu_j^2 d\tilde\phi_j^2\Big)\,, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \mathcal{A} &= \frac{a(a^2-v^2)}{\Xi_a(a^2-b^2)}\sum_{i=1}^p \nu_i^2 d\phi_i + \frac{b(b^2-v^2)}{\Xi_b(b^2-a^2)}\sum_{j=1}^q \tilde\nu_j^2 d\tilde\phi_j\,, \\ \widetilde{\mathcal{A}} &= \frac{a(r^2+a^2)}{\Xi_a(a^2-b^2)}\sum_{i=1}^p \nu_i^2 d\phi_i + \frac{b(r^2+b^2)}{\Xi_b(b^2-a^2)}\sum_{j=1}^q \tilde\nu_j^2 d\tilde\phi_j\,, \\ U &= \frac{(1+g^2r^2)(r^2+a^2)^p\,(r^2+b^2)^q}{r^2} - 2M\,, \\ V &= -\frac{(1-g^2v^2)(a^2-v^2)^p\,(b^2-v^2)^q}{v^2} + 2L\,. \\ \rho^{2n-2} &= (r^2+v^2)(r^2+a^2)^{p-1}\,(r^2+b^2)^{q-1}\,, \qquad \Xi_a = 1-a^2g^2\,, \\ \omega^{2n-2} &= (r^2+v^2)(a^2-v^2)^{p-1}\,(b^2-v^2)^{q-1}\,, \qquad \Xi_b = 1-b^2g^2\,. \end{aligned} \tag{2.9}$$

It is straightforward (with the aid of a computer) to verify in a variety of low odd dimensions that the metric (2.8) does indeed solve the Einstein equations $R_{\mu\nu} = -(D-1)g^2 g_{\mu\nu}$, and since the construction does not exploit any special features of the low dimensions, one can be confident that the solution is valid in all odd dimensions. We have explicitly verified the solutions in $D \le 9$.

As we indicated above, the metric (2.8) can be re-expressed more elegantly in terms of two complex projective spaces $\mathbb{CP}^{p-1}$ and $\mathbb{CP}^{q-1}$. The proof is straightforward, following the same steps as were used in [13] when studying the Kerr-de Sitter metrics with equal angular momenta. The essential point is that one can write

$$\sum_{i=1}^{p}(d\nu_i^2 + \nu_i^2 d\phi_i^2) = d\Omega_{2p-1}^2 = (d\psi + A)^2 + d\Sigma_{p-1}^2\,, \qquad \sum_{i=1}^{p} \nu_i^2 d\phi_i = d\psi + A\,, \quad (2.10)$$

where $d\Sigma_{p-1}^2$ is the standard Fubini-Study metric on $\mathbb{CP}^{p-1}$ (with $R_{ab} = 2p g_{ab}$), and $\frac{1}{2}dA$ locally gives the Kähler form J. Note that $d\Omega_{2p-1}^2$ is the standard metric on the unit sphere S^{2p-1}, expressed here as the Hopf fibration over $\mathbb{CP}^{p-1}$.

With these results, and the analogous ones for the tilded coordinates $\tilde\nu_j$ and $\tilde\phi_j$, we find that (2.8) can be rewritten as

$$\begin{aligned} ds^2 \;=\; & -\frac{(1+g^2r^2)(1-g^2v^2)}{\Xi_a\Xi_b}dt^2 + \frac{\rho^{2n-2}dr^2}{U} + \frac{\omega^{2n-2}dv^2}{V} \\ & + \frac{2M}{\rho^{2n-2}}\Big(\frac{(1-g^2v^2)}{\Xi_a\Xi_b}\,dt - \mathcal{A}\Big)^2 + \frac{2L}{\omega^{2n-2}}\Big(\frac{(1+g^2r^2)}{\Xi_a\Xi_b}\,dt - \widetilde{\mathcal{A}}\Big)^2 \\ & + \frac{(r^2+a^2)(a^2-v^2)}{\Xi_a(a^2-b^2)}\Big((d\psi + A)^2 + d\Sigma_{p-1}^2\Big) \\ & + \frac{(r^2+b^2)(b^2-v^2)}{\Xi_b(b^2-a^2)}\Big((d\varphi + \widetilde{A})^2 + d\widetilde\Sigma_{q-1}^2\Big)\,, \end{aligned} \quad (2.11)$$

now with

$$\begin{aligned} \mathcal{A} &= \frac{a(a^2-v^2)}{\Xi_a(a^2-b^2)}(d\psi + A) + \frac{b(b^2-v^2)}{\Xi_b(b^2-a^2)}(d\varphi + \widetilde{A}) \\ \widetilde{\mathcal{A}} &= \frac{a(r^2+a^2)}{\Xi_a(a^2-b^2)}(d\psi + A) + \frac{b(r^2+b^2)}{\Xi_b(b^2-a^2)}(d\varphi + \widetilde{A})\,. \end{aligned} \quad (2.12)$$

Here A and $\widetilde{A}$ are potentials such that the Kähler forms of the complex projective spaces $\mathbb{CP}^{p-1}$ and $\mathbb{CP}^{q-1}$ are given locally by $J = \frac{1}{2}dA$ and $\widetilde{J} = \frac{1}{2}\widetilde{A}$ respectively. Another useful way of writing the metric is given in the appendix.

It can be seen from the form of (2.11) that the metrics have cohomogeneity 2, with principal orbits on the surfaces where r and v are constant that are the homogeneous spaces $\mathbb{R} \times S^{2p-1} \times S^{2q-1}$. The $\mathbb{R}$ factor is associated with the time direction, whilst the spheres S^{2p-1} and S^{2q-1} arise from the Hopf fibrations over $\mathbb{CP}^{p-1}$ and $\mathbb{CP}^{q-1}$ respectively. The sphere metrics on the principal orbits are squashed, and so the isometry group of (2.11) is $\mathbb{R} \times U(p) \times U(q)$.

We have presented the new solutions for the case of negative cosmological constant, but clearly these NUT generalisations of Kerr-AdS will also be valid if we send $g \to \mathrm{i}\, g$, yielding NUT generalisations of the Kerr-de Sitter metrics. It is also worth noting that even when the cosmological constant is set to zero, the solutions are still new, representing NUT generalisations of the asymptotically-flat rotating black holes of Myers and Perry [11].

Written in the form (2.8) or (2.11), the metric appears to be singular in the special case where one sets $a = b$. This is, however, an artefact of our introduction of the coordinate v, in place of θ. We did this in order to bring out the symmetrical relation between r and v, but clearly, as can be seen from (2.7), the coordinate v degenerates in the case $a = b$. This can be avoided by using θ as the coordinate instead, and performing appropriate rescalings.

Having written our new Kerr-AdS-Taub-NUT metrics in this form, it is clear that we could obtain more general Einstein metrics by replacing the Fubini-Study metrics $d\Sigma_{p-1}^2$ and $d\widetilde{\Sigma}_{q-1}^2$ on $\mathbb{CP}^{p-1}$ and $\mathbb{CP}^{q-1}$ by arbitrary Einstein-Kähler metrics of the same dimensions, and normalised to have the same cosmological constants as $d\Sigma_{p-1}^2$ and $d\widetilde{\Sigma}_{q-1}^2$. In the generalised metrics, A and $\widetilde{A}$ will now be potentials yielding the Kähler forms of the two Einstein-Kähler metrics, i.e. $J = \frac{1}{2}dA$ and $\widetilde{J} = \frac{1}{2}d\widetilde{A}$.

If we specialise to the case when $b = 0$, and define a new coordinate $\psi' = \psi - ag^2 t$, then the metric (2.12) reduces to

$$\begin{aligned} ds^2 &= \frac{r^2+v^2}{X}dr^2 + \frac{r^2+v^2}{Y}dv^2 - \frac{X}{r^2+v^2}\Big(dt - \frac{a^2-v^2}{a\,\Xi_a}(d\psi'+A)\Big)^2 \\ &\quad + \frac{Y}{r^2+v^2}\Big(dt - \frac{r^2+a^2}{a\,\Xi_a}(d\psi'+A)\Big)^2 + \frac{(r^2+a^2)(a^2-v^2)}{a^2\Xi_a}d\Sigma_{p-1}^2 \\ &\quad + \frac{r^2v^2}{a^2}d\Omega_{2q-1}^2\,, \end{aligned} \tag{2.13}$$

where $d\Omega_{2q-1}^2 = (d\varphi + \widetilde{A})^2 + d\widetilde{\Sigma}_{q-1}^2$ is the metric of a unit sphere S^{2q-1}, and

$$\begin{aligned} X &= (1+g^2r^2)(r^2+a^2) - \frac{2M}{(r^2+a^2)^{p-1}\,r^{2(q-1)}}\,, \\ Y &= (1-g^2v^2)(a^2-v^2) - \frac{2\widetilde{L}}{(a^2-v^2)^{p-1}\,v^{2(q-1)}}\,. \end{aligned} \tag{2.14}$$

The constant $\widetilde{L}$ is related to the original NUT parameter by $\widetilde{L} = (-1)^q\,L$. A special case of the metrics (2.13), namely when $p = 1$, was obtained in [20, 21].

2. The Supersymmetric Limit

Odd-dimensional Kerr-AdS black holes admit supersymmetric limits, which in Euclidean signature with positive cosmological constant become Einstein-Sasaki metrics [22, 23] (see also [24, 25] for discussions of how the supersymmetric limit arises in the Lorentzian regime, when a Bogomol'nyi inequality is saturated). We find that an analogous limit also exists for our new metrics where the NUT charge is introduced. We first set $g = \mathrm{i}$ so that the metric has a unit positive cosmological constant ($R_{\mu\nu} = (D-1)\,g_{\mu\nu}$). We then Euclideanise the metric by sending

$$t \to \mathrm{i}t\,, \qquad a \to \mathrm{i}a\,, \qquad b \to \mathrm{i}b\,, \tag{2.15}$$

define

$$1-a^2=\alpha\,\epsilon\,,\qquad 1-b^2=\beta\,\epsilon\,,\qquad M=-m\,\epsilon^{n+1}\,,\qquad L=\ell\,\epsilon^{n+1}\,,$$
$$1-r^2=\epsilon\,x\,,\qquad 1+v^2=\epsilon\,y\,,\tag{2.16}$$

and then take the limit $\epsilon\to0$. This leads to the metric

$$\begin{aligned}
ds^2 &= \Big(dt+\frac{(\alpha-x)(\alpha-y)}{\alpha(\alpha-\beta)}(d\psi+A)-\frac{(\beta-x)(\beta-y)}{\beta(\alpha-\beta)}(d\varphi+\widetilde{A})\Big)^2\\
&\quad+\frac{x-y}{4X}dx^2+\frac{x-y}{4Y}dy^2+\frac{(\alpha-x)(\alpha-y)}{\alpha(\alpha-\beta)}d\Sigma^2_{p-1}-\frac{(\beta-x)(\beta-y)}{\beta(\alpha-\beta)}d\widetilde{\Sigma}^2_{q-1}\\
&\quad+\frac{X}{x-y}\Big(\frac{(\alpha-y)}{\alpha(\alpha-\beta)}(d\psi+A)-\frac{(\beta-y)}{\beta(\alpha-\beta)}(d\varphi+\widetilde{A})\Big)^2\\
&\quad+\frac{Y}{x-y}\Big(\frac{(\alpha-x)}{\alpha(\alpha-\beta)}(d\psi+A)-\frac{(\beta-x)}{\beta(\alpha-\beta)}(d\varphi+\widetilde{A})\Big)^2\,,
\end{aligned}\tag{2.17}$$

where again $J=\frac12 dA$ and $\widetilde{J}=\frac12 d\widetilde{A}$ are the Kähler forms of the $\mathbb{CP}^{p-1}$ and $\mathbb{CP}^{q-1}$ complex projective spaces with metrics $d\Sigma^2_{p-1}$ and $d\widetilde{\Sigma}^2_{q-1}$ respectively, and

$$\begin{aligned}
X &= -\frac{2m}{(\alpha-x)^{p-1}(\beta-x)^{q-1}}-x(\alpha-x)(\beta-x)\,,\\
Y &= \frac{2\ell}{(\alpha-y)^{p-1}(\beta-y)^{q-1}}+y(\alpha-y)(\beta-y)\,.
\end{aligned}\tag{2.18}$$

It is straightforward to verify that the above metric (2.17) is an Einstein-Sasaki metric in $D=2n+1$ dimensions. Note that the metric has the form

$$ds^2_{2n+1}=(dt+2\mathcal{A})^2+ds^2_{2n}\,.\tag{2.19}$$

where ds^2_{2n} is an Einstein-Kähler metric and $\mathcal{A}$ is the corresponding Kähler potential, in the sense that the Kähler form for ds^2_{2n} can be written locally as $J=d\mathcal{A}$. As far as we know, these cohomogeneity-2 Einstein-Kähler metrics ds^2_{2n} have not been obtained explicitly before. Note that one can go to the Ricci-flat limit of ds^2_{2n} by performing a rescaling that amounts to dropping the x^3 term and y^3 term in (2.18).

If we consider the special case where $p = n - 1$ and $q = 1$, the Einstein-Sasaki metrics reduce to ones that were obtained recently in [26]. This may be seen by defining new parameters by the expressions

$$\hat{\alpha} = -4(\beta - 2\alpha)\,, \quad \hat{\beta} = \alpha(\alpha - \beta)\,, \quad m = \tfrac{1}{8}(-1)^N\,\mu\,, \quad \ell = \tfrac{1}{8}(-1)^N\,\nu\,, \qquad (2.20)$$

where $N \equiv p - 1$, and introducing new coordinates defined by

$$\hat{x} = x - \alpha\,, \quad \hat{y} = y - \alpha\,, \quad t = \tau + 2(\alpha - \beta)\chi\,, \quad \varphi = 2\beta\chi\,, \quad \psi = 2\hat{\psi} + 2\alpha\chi\,. \quad (2.21)$$

Defining also $\hat{X} = 4X$ and $\hat{Y} = 4Y$, we obtain, upon substitution into (2.17), the metric

$$\begin{aligned} ds^2 &= [d\tau - 2(\hat{x} + \hat{y})d\chi + \frac{2\hat{x}\hat{y}}{\hat{\beta}}\sigma]^2 + \frac{\hat{x} - \hat{y}}{\hat{X}}\, d\hat{x}^2 + \frac{\hat{x} - \hat{y}}{\hat{Y}}\, d\hat{y}^2 \\ &\quad + \frac{\hat{X}}{\hat{x} - \hat{y}}\,(d\chi - \frac{\hat{y}}{\hat{\beta}}\,\sigma)^2 + \frac{\hat{Y}}{\hat{x} - \hat{y}}\,(d\chi - \frac{\hat{x}}{\hat{\beta}}\,\sigma)^2 + \frac{\hat{x}\hat{y}}{\hat{\beta}}\, d\Sigma_N^2\,, \end{aligned} \qquad (2.22)$$

where $\sigma = d\hat{\psi} + \frac{1}{2}A$, and

$$\hat{X} = -4\hat{x}^3 - \hat{\alpha}\hat{x}^2 - 4\hat{\beta}\hat{x} - \frac{\mu}{\hat{x}^N}\,, \qquad \hat{Y} = 4\hat{y}^3 + \hat{\alpha}\hat{y}^2 + 4\hat{\beta}\hat{y} + \frac{\nu}{\hat{y}^N}\,. \qquad (2.23)$$

This is precisely of the form of the Einstein-Sasaki metrics that were obtained in section (4) of reference [26], in the case where the Einstein-Kähler base metric in that paper is taken to be $\mathbb{CP}^N$. A detailed discussion of the global structure of these metrics was given in [26], and new complete $D = 7$ Einstein-Sasaki spaces were obtained.

C. Kerr-de Sitter with NUT Parameter in $D = 2n$

The Kerr-de Sitter metrics in even spacetime dimensions take a slightly different form from those in odd dimensions. The reason for this is that now there are an odd number of spatial dimensions, and so there can be $(n-1)$ independent parameters characterising rotations in $(n-1)$ orthogonal 2-planes, with one additional spatial direction that is not associated with a rotation. Because of this feature, the $D = 2n$ dimensional Kerr-de Sitter black holes in general have cohomogeneity n, which can be reduced to cohomogeneity 2 if one sets all the $(n-1)$ rotation parameters equal. By contrast, in odd dimensions $D = 2n+1$ the general metrics have cohomogeneity n, reducing to cohomogeneity 1 if one sets all the rotation parameters equal.

It will be recalled that in section B, we were able to generalise the odd-dimensional Kerr-de Sitter to include a NUT parameter by dividing the angular momentum parameters a_i into two sets, equal within a set, thereby obtaining a metric of cohomogeneity 2. Our construction with the NUT parameter is intrinsically adapted to metrics of cohomogeneity 2, and so this means that in the present case, when we consider generalising the even-dimensional Kerr-de Sitter metrics, we shall first need to divide the rotation parameters a_i into two sets. In one set, the parameters will be equal and non-zero, while in the other set, the remaining rotation parameters will all be chosen to be zero.

Our starting point is the expression for the Kerr-de Sitter metrics given in equation (E.3) of reference [13], specialised to dimension $D = 2n$. We shall take the cosmological constant to be negative, with the resulting Kerr-AdS metrics satisfying $R_{\mu\nu} = -(D-1)g^2\, g_{\mu\nu}$.

$$\begin{aligned}
ds^2 &= -W\,(1+g^2r^2)\,d\tau^2 + \frac{U\,dr^2}{V-2M} + \frac{2M}{U}\Big(W\,d\tau - \sum_{i=1}^{n-1}\frac{a_i\,\mu_i^2\,d\phi_i}{1-a_i^2\,g^2}\Big)^2 \\
&\quad + \sum_{i=1}^{n}\frac{r^2+a_i^2}{1-a_i^2\,g^2}\,d\mu_i^2 + \sum_{i=1}^{n-1}\frac{r^2+a_i^2}{1-a_i^2\,g^2}\,\mu_i^2\,d\phi_i^2 \\
&\quad - \frac{g^2}{W\,(1+g^2r^2)}\Big(\sum_{i=1}^{n}\frac{(r^2+a_i^2)\mu_i\,d\mu_i}{1-a_i^2g^2}\Big)^2\,, \qquad (2.24)
\end{aligned}$$

where U and V are defined here by

$$\begin{aligned}
U &= r\sum_{i=1}^{n}\frac{\mu_i^2}{r^2+a_i^2}\prod_{j=1}^{n-1}(r^2+a_j^2)\,, \\
V &= \frac{1}{r}\,(1+g^2r^2)\prod_{i=1}^{n-1}(r^2+a_i^2) \qquad (2.25)
\end{aligned}$$

W and F are given in (2.3). We then set

$$a_1 = a_2 = \cdots = a_p = a\,, \qquad a_{p+1} = a_{p-2} = \cdots = a_{n-1} = 0\,. \qquad (2.26)$$

We then introduce new "latitude" coordinates ν_i, $\tilde{\nu}_j$ and θ, in place of the μ_i in [13],

$$\begin{aligned}
\mu_i &= \nu_i\,\sin\theta\,, \qquad 1 \le i \le p\,, \qquad \sum_{i=1}^{p}\nu_i^2 = 1\,, \\
\mu_{j+p} &= \tilde{\nu}_j\,\cos\theta\,, \qquad 1 \le j \le n-p\,, \qquad \sum_{j=1}^{n-p}\tilde{\nu}_j^2 = 1\,, \qquad (2.27)
\end{aligned}$$

In this case, because there are only $(n-1)$ azimuthal coordinates ϕ_i, we split them into two sets, which we shall denote by ϕ_i and $\tilde{\phi}_j$, defined for

$$\phi_i: \qquad 1 \le i \le p\,, \qquad\qquad \tilde{\phi}_j: \qquad 1 \le j \le q\,, \qquad (2.28)$$

where this time we have defined q such that

$$p + q = n - 1\,. \qquad (2.29)$$

We then introduce a new variable v, in place of θ, which this time is defined by

$$a^2 \cos^2\theta = v^2\,. \tag{2.30}$$

We can now write out the Kerr-AdS metric of [13, 14], subject to the restriction (2.26), in terms of the new variables defined above, and, as in the odd-dimensional case we discussed previously, this allows us to conjecture a generalisation that includes a NUT parameter L as well as the mass parameter M and angular momentum parameter a. Again, we shall just present our final result, having included the NUT parameter. Thus we obtain the new Kerr-AdS-Taub-NUT metric (which we have verified explicitly in $D \le 8$)

$$\begin{aligned} ds^2 &= -\frac{(1+g^2r^2)(1-g^2v^2)}{\Xi_a}dt^2 + \frac{\rho^{2n-3}dr^2}{U} + \frac{\omega^{2n-3}dv^2}{V} \\ &\quad + \frac{2M\,r}{\rho^{2n-3}}\Big(\frac{(1-g^2v^2)}{\Xi_a}\,dt - \mathcal{A}\Big)^2 - \frac{2L\,v}{\omega^{2n-3}}\Big(\frac{(1+g^2r^2)}{\Xi_a}dt - \widetilde{\mathcal{A}}\Big)^2 \\ &\quad + \frac{(r^2+a^2)(a^2-v^2)}{a^2\Xi_a}\sum_{i=1}^{p}(d\nu_i^2 + \nu_i^2\,d\phi_i^2) + \frac{r^2v^2}{a^2}\Big(d\tilde{\nu}_{q+1}^2 + \sum_{j=1}^{q}(d\tilde{\nu}_j^2 + \tilde{\nu}_j^2 d\tilde{\phi}_j^2)\Big)\,, \end{aligned} \tag{2.31}$$

where

$$\begin{aligned} \mathcal{A} &= \frac{a^2-v^2}{a\,\Xi_a}\sum_{i=1}^{p}\nu_i^2 d\phi_i\,, \qquad \widetilde{\mathcal{A}} = \frac{r^2+a^2}{a\,\Xi_a}\sum_{i=1}^{p}\nu_i^2 d\phi_i\,, \\ U &= (1+g^2r^2)(r^2+a^2)^p\,r^{2q} - 2M\,r\,, \\ V &= (1-g^2v^2)(a^2-v^2)^p\,v^{2q} - 2L\,v\,. \\ \rho^{2n-3} &= (r^2+v^2)(r^2+a^2)^{p-1}\,r^{2q}\,, \qquad \Xi_a = 1 - a^2g^2\,, \\ \omega^{2n-3} &= (r^2+v^2)(a^2-v^2)^{p-1}\,v^{2q}\,. \end{aligned} \tag{2.32}$$

The $(2n)$-dimensional Kerr-AdS-Taub-NUT metrics that we have constructed here can be seen to be quite similar in structure to the $(2n+1)$-dimensional examples that we constructed in section B, in the special case where we set the b parameter to zero. In fact we can re-express the metrics (2.31) in terms of a complex projective space and a sphere metric, in a manner that is closely analogous to (2.13). This is expressed most simply by making redefinitions as in (2.10), and then introducing a new Hopf fibre coordinate $\widetilde{\psi} = \psi - ag^2 t$ as we did in the odd-dimensional case. Having done this, we arrive at the metric

$$\begin{aligned} ds^2 &= \frac{r^2+v^2}{X}dr^2 + \frac{r^2+v^2}{Y}dv^2 - \frac{X}{r^2+v^2}\Big(dt - \frac{a^2-v^2}{a\,\Xi_a}(d\widetilde{\psi}+A)\Big)^2 \\ &\quad + \frac{Y}{r^2+v^2}\Big(dt - \frac{a^2+r^2}{a\,\Xi_a}(d\widetilde{\psi}+A)\Big)^2 + \frac{(a^2+r^2)(a^2-v^2)}{a^2\Xi_a}d\Sigma_{p-1}^2 + \frac{r^2v^2}{a^2}d\Omega_{2q}^2\,, \end{aligned} \tag{2.33}$$

where $d\Omega_{2q}^2$ is the metric on the unit sphere S^{2q},

$$\begin{aligned} X &= (1+g^2r^2)(r^2+a^2) - \frac{2M\,r}{(r^2+a^2)^{p-1}\,r^{2q}}\,, \\ Y &= (1-g^2v^2)(a^2-v^2) - \frac{2L\,v}{(a^2-v^2)^{p-1}\,v^{2q}}\,, \end{aligned} \tag{2.34}$$

and the Kähler form J for the $\mathbb{CP}^{p-1}$ metric $d\Sigma_{p-1}^2$ is given locally by $J = \frac{1}{2}dA$.

For the cases with $q=0$, there can also be a BPS limit of the solutions, giving rise to Ricci-flat Kähler metrics instead of Einstein-Kähler. To do this, we first Euclideanise the metric by setting $t \to \mathrm{i}\,t$, $a \to \mathrm{i}\,a$ and set $g = \mathrm{i}$. We then take the following limit

$$1-a^2 = \alpha\,\epsilon\,,\quad 1-r^2 = x\,\epsilon\,,\quad 1+v^2 = y\,\epsilon\,,\quad M = \mu\,(-\epsilon)^{p-1}\,,\quad L = \mathrm{i}\,\nu\,\epsilon^{p-1}\,, \tag{2.35}$$

with $\epsilon \to 0$. The metric becomes $ds^2 = \epsilon\,d\tilde{s}^2$, where $d\tilde{s}^2$ is a Ricci-flat Kähler metric, given by

$$\begin{aligned} d\tilde{s}^2 &= \frac{x-y}{4X}dx^2 + \frac{x-y}{4Y}dy^2 + \frac{(x-\alpha)(\alpha-y)}{\alpha}d\Sigma_{p-1}^2 \\ &\quad + \frac{X}{x-y}(dt + \frac{\alpha-y}{\alpha}(d\psi+A))^2 + \frac{Y}{x-y}(dt - \frac{x-\alpha}{\alpha}(d\psi+A))^2\,, \\ X &= x(x-\alpha) + \frac{2\mu}{(x-\alpha)^{p-1}}\,,\qquad Y = y(\alpha-y) - \frac{2\nu}{(\alpha-y)^{p-1}}\,. \end{aligned} \tag{2.36}$$

The Kähler 2-form is given locally by $J = dB$, where

$$B = \tfrac{1}{2}(x+y)dt + \frac{(x-\alpha)(\alpha-y)}{2\alpha}(d\psi + A)\,. \tag{2.37}$$

D. Global Analysis

The global analysis of Kerr-AdS black holes in general dimensions was given in [13, 14]. Here, we study the effect of introducing the NUT charge L. We shall consider the case where v is a compact coordinate, ranging over the interval $0 < v_1 \le v \le v_2$, where v_1 and v_2 are two adjacent roots of $V(v) = 0$, such that the function V is positive when v lies within the interval. In the case when $L = 0$, we would have $v_1 = a$ and $v_2 = b$. The coordinate r ranges from r_0 to infinity, where r_0 is the largest root of $U(r) = 0$. The discussion now divides into the cases of $D = 2n+1$ dimensions and $D = 2n$ dimensions.

1. $D = 2n+1$ Dimensions

The metric (2.11) is degenerate at $v = v_1$ and v_2, where $V(v_i) = 0$. The corresponding Killing vectors whose norms $\ell^2 = g_{\mu\nu}\,\ell^\mu\,\ell^\nu$ vanish at these surfaces have the form

$$\ell = \gamma_0 \frac{\partial}{\partial t} + \gamma_1 \frac{\partial}{\partial \phi} + \gamma_2 \frac{\partial}{\partial \psi}\,, \tag{2.38}$$

for constants γ_0, γ_1 and γ_2 to be determined. The associated "surface gravities" are of Euclidean type, in the sense that

$$\kappa_E^2 = \frac{g^{\mu\nu}\,(\partial_\mu \ell^2)\,(\partial_\nu \ell^2)}{4\ell^2}\Big|_{v=v_i} \tag{2.39}$$

is positive. Thus these degenerations are typical of an azimuthal coordinate at a spatial origin. We can scale the coefficients γ_i so that the Euclidean surface gravity is 1, implying that the Killing vector generates a closed translation with period 2π. One might conclude that the time coordinate is periodic, since γ_0 is non-vanishing. This is indeed the case for the solutions in even dimensions. However, in odd dimensions the $\partial/\partial t$ term can be removed by making the coordinate transformation

$$t = \tilde{t} + \frac{\Xi_b(a^2 - v_1^2)(b^2 - v_2^2)b\,\psi - \Xi_a(b^2 - v_1^2)(b^2 - v_2^2)a\,\varphi}{ab(a^2 - b^2)(1 - g^2 v_1^2)(1 - g^2 v_2^2)} \tag{2.40}$$

The two Killing vectors whose norms vanish at v_1 and v_2 are now given by

$$\ell_i = \frac{4L}{V'(v_i)}\Big(\frac{b}{b^2 - v_i^2}\frac{\partial}{\partial\varphi} + \frac{a}{a^2 - v_i^2}\frac{\partial}{\partial\psi}\Big)\,. \tag{2.41}$$

Both Killing vectors have unit Euclidean surface gravity, implying that they both generate closed 2π translations. Since it does not suffer a periodic identification, $\tilde{t}$ is perhaps a more natural choice than t for the time coordinate.

The metric also degenerates at $r = r_0$, and the corresponding null Killing vector has Lorentzian surface gravity κ, in the sense that $\kappa^2 = -\kappa_E^2$ is positive. Thus $r = r_0$ is an horizon. If we write the null Killing vector in terms of coordinate $\tilde{t}$, normalised to

$$\tilde{\ell}_0 = \frac{\partial}{\partial\tilde{t}} + \tilde{\gamma}_1\frac{\partial}{\partial\phi} + \tilde{\gamma}_2\frac{\partial}{\partial\psi}\,, \tag{2.42}$$

where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are determined from the condition that $\tilde{\ell}_0^2 = 0$ at $r = r_0$, we find that the surface gravity is given by

$$\kappa = \frac{(1 - g^2 v_1^2)(1 - g^2 v_2^2)(r_0^2 + a^2)(r_0^2 + b^2)(1 + g^2 r_0^2)U'(r_0)}{2\Xi_a\Xi_b(r_0^2 + v_1^2)(r_0^2 + v_2^2)[U(r_0) + 2M]}\,. \tag{2.43}$$

If instead we consider the null Killing vector in terms of the original coordinate t, and rescale it to give

$$\ell_0 = \frac{\partial}{\partial t} + \gamma_1\frac{\partial}{\partial\phi} + \gamma_2\frac{\partial}{\partial\psi}\,, \tag{2.44}$$

then the surface gravity is then given by

$$\kappa = \frac{(1+g^2 r_0^2)U'(r_0)}{2[U(r_0)+2M]}\,, \tag{2.45}$$

which is identical to the result for the Kerr-AdS black hole [13, 14] without the NUT parameter. It is not *a priori* obvious what the proper normalisation for the asymptotically timelike Killing vector should be, since the metrics with the non-vanishing NUT parameter are not asymptotic to AdS.

2. $D = 2n$ Dimensions

In even dimensions, the introduction of the NUT parameter implies that the time coordinate is necessarily periodic (as in four dimensions). To see this, we note from the metric (2.33) that, at the degenerate surfaces $v = v_1$ and v_2, the Killing vectors whose norms vanish are given by

$$\ell_i = \frac{2}{V'(v_i)}\Big((a^2 - v_i^2)\frac{\partial}{\partial t} + a\,\Xi_a\,\frac{\partial}{\partial \widetilde{\psi}}\Big)\,. \tag{2.46}$$

These Killing vectors are normalised to have unit Euclidean surface gravities, and hence they generate closed translations with period 2π. In the case when $L = 0$, then $v_1 = a$ and $v_2 = -a$, so the ℓ_i do not have $\partial/\partial t$ terms. However, when $L \neq 0$ there are necessarily $\partial/\partial t$ terms appearing in these Killing vectors that generate periodic translations, and so t must be identified periodically.

E. Inversion Symmetry of $D = 5$ Kerr-AdS Black Holes

In this section, we first demonstrate that the NUT parameter L introduced in our general rotating black holes is trivial in the special case of $D = 5$ dimensions. However, our demonstration also brings to light a rather remarkable property of the five-dimensional Kerr-AdS black hole metric, namely, that it admits a discrete symmetry transformation which shows that the metric with over-rotation (where the parameters a and b are such that $a^2g^2 > 1$ and/or $b^2g^2 > 1$) is equivalent to a Kerr-AdS metric with under-rotation.

We start with the five-dimensional Kerr-AdS metric written in the (2.11) with $p = 1$ and $q = 1$, and make the coordinate transformations

$$\psi \to ab^2\chi + ag^2t + a(1+b^2g^2)\phi\,, \qquad \varphi \to ba^2\chi + bg^2t + b(1+a^2g^2)\phi\,,$$
$$t \to t + a^2b^2\chi + (a^2+b^2)\phi\,, \tag{2.47}$$

and define $r^2 = x$ and $v^2 = y$. This leads to the five-dimensional metric

$$\begin{aligned} ds^2 &= (x+y)\Big(\frac{dx^2}{4X} + \frac{dy^2}{4Y}\Big) - \frac{X}{x(x+y)}(dt + y\,d\phi)^2 + \frac{Y}{y(x+y)}(dt - x\,d\phi)^2 \\ &\quad \frac{a^2b^2}{xy}\Big(dt - xyd\chi - (x-y)d\phi\Big)^2\,, \end{aligned} \tag{2.48}$$

where

$$\begin{aligned} X &= (1+g^2x)(x+a^2)(x+b^2) - 2Mx \\ &= g^2x^3 + (1+(a^2+b^2)g^2)x^2 + (a^2+b^2+a^2b^2g^2-2M)x + a^2b^2\,, \\ Y &= -(1-g^2y)(a^2-y)(b^2-y) + 2Ly \\ &= g^2y^3 - (1+(a^2+b^2)g^2)y^2 + (a^2+b^2+a^2b^2g^2+2L)y - a^2b^2\,. \end{aligned} \tag{2.49}$$

Although, the solution ostensibly has the four independent parameters (M, L, a, b), one can in fact scale away either M or L in this five-dimensional case. To do this, we set

$$\tilde{x} = \lambda^2 x\,, \quad \tilde{y} = \lambda^2 y\,, \quad \tilde{t} = \frac{t}{\lambda}\,, \quad \tilde{\chi} = \frac{\chi}{\lambda^5}\,, \quad \tilde{\phi} = \frac{\phi}{\lambda^3}\,. \tag{2.50}$$

The metric (2.48) is invariant under this transformation, if we simultaneously transform the parameters a, b, M and L. Thus we define $\widetilde{X} = \lambda^6\, X$ and $\widetilde{Y} = \lambda^6\, Y$, where $\widetilde{X}$ and $\widetilde{Y}$ are defined as in (2.49) except with tilded parameters $\tilde{a}$, $\tilde{b}$, $\widetilde{M}$ and $\widetilde{L}$. It follows that we shall have

$$\begin{aligned}
&\lambda^2 + \lambda^2(a^2+b^2)g^2 = 1 + (\tilde{a}^2+\tilde{b}^2)g^2\,, \qquad \lambda^6\, a^2\, b^2 = \tilde{a}^2\, \tilde{b}^2\,,\\
&\lambda^4(a^2+b^2+a^2b^2g^2+2L) = \tilde{a}^2+\tilde{b}^2+\tilde{a}^2\tilde{b}^2g^2+2\widetilde{L}\,,\\
&\lambda^4(a^2+b^2+a^2b^2g^2-2M) = \tilde{a}^2+\tilde{b}^2+\tilde{a}^2\tilde{b}^2g^2-2\widetilde{M}\,.
\end{aligned} \tag{2.51}$$

We can then choose, for example, to set $\widetilde{L} = 0$, and solve the four equations (2.51) for $\tilde{a}$, $\tilde{b}$, $\widetilde{M}$ and λ. Thus a solution with $L \ne 0$ is transformed into a tilded solution with $\widetilde{L} = 0$, and since this latter solution is just of the original five-dimensional Kerr-AdS form, it follows that the metric (2.48), even with $L \ne 0$, is also just the five-dimensional Kerr-AdS metric, but with changed values for the rotation and mass parameters. It is nevertheless interesting that the Kerr-de Sitter black hole in $D = 5$ can be put in such a symmetric form.

It should be stressed that the scaling symmetry that we used above in order to show that the parameter L in the five-dimensional metrics is "trivial" is very specific to five dimensions. In particular, it can be seen from (2.11) that in higher dimensions, when at least one of p or q exceeds 1, the associated metrics on the complex projective spaces will break the scaling symmetry. Thus, as in the case of the simpler NUT generalisations discussed [20, 21], five-dimensions is the exception in not admitting a non-trivial generalisation.

The transformation described above becomes particularly simple if we consider the case of an asymptotically flat five-dimensional rotating black hole, i.e. when $g = 0$. In this case, we have from (2.51) that $\lambda = 1$ and

$$\tilde{a}^2+\tilde{b}^2+2\widetilde{L} = a^2+b^2+2L\,, \quad \tilde{a}^2+\tilde{b}^2-2\widetilde{M} = a^2+b^2-2M\,, \quad \tilde{a}^2\tilde{b}^2 = a^2b^2\,. \tag{2.52}$$

Thus $\widetilde{L}+\widetilde{M}=L+M$, and so we can arrange to have $\widetilde{L}=0$ by taking $\widetilde{M}=L+M$, implying that $\tilde{a}^2+\tilde{b}^2=a^2+b^2+2L$, together with $\tilde{a}^2\tilde{b}^2=a^2b^2$. It is worth noting, however, that even though one can always map into a solution where $\widetilde{L}=0$, it may, depending upon the original values for a, b and L, correspond to having complex values for $\tilde{a}$ and $\tilde{b}$. Although the metric (2.48) would still be real, the metric written back in terms of the original ψ, ϕ and t coordinates would then be complex. Thus although the parameter L is really trivial in five dimensions, its inclusion can nevertheless allow one to parameterise the solutions in a wider class without the need for complex coordinate transformations. Similar remarks apply also to the case when $g\neq 0$.

There is another interesting consequence of the five-dimensional scaling symmetry discussed above, namely, that even with the parameter L omitted entirely, the five-dimensional rotating AdS black hole metrics have a symmetry that allows one to map an "over-rotating" black hole (i.e. where $a^2g^2>1$ or $b^2g^2>1$) into an under-rotating black hole. This can be understood by again considering the transformations in (2.51), where we now choose not only $\widetilde{L}=0$ but also $L=0$. The system of equations then admits a sextet of solutions for $(\tilde{a},\tilde{b},\widetilde{M},\lambda)$ (where we assume, without loss of generality, that the signs of the rotation parameters are unchanged):

$$\begin{aligned}
\tilde{a} &= a\,, \quad \tilde{b}=b\,, \quad \widetilde{M}=M\,, \quad \lambda=1\,,\\
\tilde{a} &= b\,, \quad \tilde{b}=a\,, \quad \widetilde{M}=M\,, \quad \lambda=1\,,\\
\tilde{a} &= \frac{1}{ag^2}\,, \quad \tilde{b}=\frac{b}{ag}\,, \quad \widetilde{M}=\frac{M}{a^4g^4}\,, \quad \lambda=\frac{1}{ag}\,,\\
\tilde{a} &= \frac{1}{bg^2}\,, \quad \tilde{b}=\frac{a}{bg}\,, \quad \widetilde{M}=\frac{M}{b^4g^4}\,, \quad \lambda=\frac{1}{bg}\,,\\
\tilde{a} &= \frac{a}{bg}\,, \quad \tilde{b}=\frac{1}{bg^2}\,, \quad \widetilde{M}=\frac{M}{b^4g^4}\,, \quad \lambda=\frac{1}{bg}\,,\\
\tilde{a} &= \frac{b}{ag}\,, \quad \tilde{b}=\frac{1}{ag^2}\,, \quad \widetilde{M}=\frac{M}{a^4g^4}\,, \quad \lambda=\frac{1}{ag}\,.
\end{aligned} \tag{2.53}$$

The first of these is the identity, the second is merely an exchange of the rôles of a and b, whilst the remaining four, modulo exchanges of the a's and the b's, are equivalent and non-trivial. Taking the third as an example, we see that if the metric is over-rotating by virtue of having $a^2 g^2 > 1$, then it can be re-expressed, by a change of variables, as a metric which is under-rotating. In fact any five-dimensional Kerr-AdS black hole with over-rotation is equivalent, after a change of coordinates, to one with under-rotation. Of course, after transforming back into the original coordinates in which the over-rotating black hole ostensibly exhibited singular behaviour, one would find that the coordinate ranges that actually reveal that it is well-behaved are not the "naive" ones that led to the original conclusion of singular behaviour.

It is instructive to rewrite the transformations (2.53) in terms of the original coordinates of the five-dimensional Kerr-AdS metric as given by Hawking, Hunter and Taylor-Robinson in [12]. The metric is given by

$$\begin{aligned} ds_5^2 &= -\frac{\Delta}{\rho^2}\Big[dt - \frac{a\,\sin^2\theta}{\Xi_a}\,d\phi - \frac{b\,\cos^2\theta}{\Xi_b}\,d\psi\Big]^2 + \frac{\Delta_\theta\,\sin^2\theta}{\rho^2}\Big[a\,dt - \frac{r^2+a^2}{\Xi_a}\,d\phi\Big]^2 \\ &\quad \frac{\Delta_\theta\,\cos^2\theta}{\rho^2}\Big[b\,dt - \frac{r^2+b^2}{\Xi_b}\,d\psi\Big]^2 + \frac{\rho^2\,dr^2}{\Delta} + \frac{\rho^2\,d\theta^2}{\Delta_\theta} \\ &\quad + \frac{(1+g^2r^2)}{r^2\,\rho^2}\Big[a\,b\,dt - \frac{b\,(r^2+a^2)\,\sin^2\theta}{\Xi_a}\,d\phi - \frac{a\,(r^2+b^2)\,\cos^2\theta}{\Xi_b}\,d\psi\Big]^2, \end{aligned} \tag{2.54}$$

where

$$\begin{aligned} \Delta &\equiv \frac{1}{r^2}\,(r^2+a^2)(r^2+b^2)(1+g^2r^2) - 2M\,, \\ \Delta_\theta &\equiv 1 - a^2\,g^2\,\cos^2\theta - b^2\,g^2\,\sin^2\theta\,, \\ \rho^2 &\equiv r^2 + a^2\,\cos^2\theta + b^2\,\sin^2\theta\,, \\ \Xi_a &\equiv 1 - a^2\,g^2\,, \qquad \Xi_b \equiv 1 - b^2\,g^2\,. \end{aligned} \tag{2.55}$$

It satisfies $R_{\mu\nu} = -4g^2\,g_{\mu\nu}$. Taking the transformation in the third line of (2.53) as an example, we find that after re-expressing our results back in terms of the quantities in (2.54), the symmetry transformation amounts to

$$\begin{aligned}
&a\rightarrow \frac{1}{ag^2}\,,\quad b\rightarrow \frac{b}{ag}\,,\quad M\rightarrow \frac{M}{a^4g^4}\,,\\
&\phi\rightarrow -\frac{1}{ag}\,\phi\,,\quad \psi\rightarrow \psi-\frac{b}{a}\,\phi\,,\quad t\rightarrow agt+\frac{1}{g}\,\phi\,,\\
&r\rightarrow \frac{1}{ag}\,r\,,\quad \cos\theta\rightarrow \Big(1-\frac{\Xi_a}{\Xi_b}\Big)^{1/2}\,\cos\theta\,.
\end{aligned}\tag{2.56}$$

It is straightforward to see that this transformation leaves the metric in (2.54) invariant, and that it therefore allows one to map an over-rotating Kerr-AdS metric into an under-rotating one. In other words, if we perform the transformation of parameters given in the first line in (2.56), then the metric is restored to its original form by making the general coordinate transformations given also in (2.56).

Another way of expressing this result is that for any given values of a and b, and provided one allows the coordinates to take complex values in general, then there exist real sections of the complex metric describing Kerr-AdS black holes with under-rotation, and also real sections of the same metric that describe Kerr-AdS black holes with over-rotation.

It is instructive also to re-express the coordinate transformations in (2.56) in terms of the coordinates y and $\hat\theta$ rather than r and θ, where y and $\hat\theta$ are the coordinates with respect to which the conformal boundary of the Kerr-AdS metric is precisely the standard $\mathbb{R}\times S^3$ Einstein universe, with a round S^3 factor. They are defined by [12]

$$\Xi_a\, y^2\,\sin^2\hat\theta=(r^2+a^2)\,\sin^2\theta\,,\qquad \Xi_b\, y^2\,\cos^2\hat\theta=(r^2+b^2)\,\cos^2\theta\,.\tag{2.57}$$

Applying the transformations in (2.56), we find that these imply the coordinate transformations

$$y^2\rightarrow -\frac{1}{g^2}-y^2\sin^2\hat\theta\,,\qquad \tan^2\hat\theta\rightarrow -\Big(1+\frac{1}{g^2y^2}\Big)\,\sec^2\hat\theta\,.\tag{2.58}$$

This result emphasises that the original y =constant boundary, which is the most natural choice from the AdS/CFT point of view [12, 27], is quite different from the y =constant boundary of the transformed metric.

A number of remarks are in order. First, we note that the symmetry we are discussing, which can be expressed in terms of dimensionless quantities as $ag \to 1/(ag)$, exists only in the case of the rotating black hole with a cosmological constant. In the case of asymptotically-flat black holes, for which $g = 0$, there is no inversion symmetry. The inversion symmetry for the five-dimensional Kerr-AdS black hole is reminiscent of a T-duality symmetry, in the sense that it implies there is a maximum allowed value for the rotation, namely $a^2g^2 = 1$. In fact, this value is associated with the supersymmetric limit. If one considers the case where a rotation parameter is becoming very large, i.e. $a^2g^2 >> 1$, then it can be seen from (2.56) that in the limiting case when a^2g^2 approaches infinity, the metric will actually approach the pure AdS metric.

It is interesting also to consider the effect on the canonical AdS metric of the transformations (2.58) taken in isolation. In other words, we start with the AdS metric

$$ds^2 = -(1 + g^2y^2)dt^2 + \frac{dy^2}{1 + g^2y^2} + y^2(d\hat{\theta}^2 + \sin^2\hat{\theta}^2\, d\phi^2 + \cos^2\hat{\theta}\, d\psi^2)\,, \qquad (2.59)$$

and impose just the coordinate transformations given in (2.58) (which are independent of the rotation parameters a and b). Upon doing so, we find that the AdS metric (2.59) transforms according to

$$ds^2 \to -\frac{1}{g^2}\,(1 + g^2y^2)d\phi^2 + \frac{dy^2}{1 + g^2y^2} + y^2(d\hat{\theta}^2 + \sin^2\hat{\theta}^2\, g^2dt^2 + \cos^2\hat{\theta}\, d\psi^2)\,. \quad (2.60)$$

This is identical in form to (2.59), with the rôles of ϕ and gt exchanged. It can easily be seen that in terms of the standard embeddding of AdS_5 in $\mathbb{R}^{4,2}$, the transformation (2.58) corresponds to exchanging the the rôles of the two timelike embedding coordinates with a pair of spacelike embedding coordinates.

F. Conclusions

In this chapter, we have constructed generalisations of certain Kerr-de Sitter and Kerr-AdS black holes in all dimensions $D \geq 6$, in which an additional NUT-type parameter is introduced. Specifically, the cases where we have obtained the more general solutions are where the rotation parameters are specialised so that the metrics have cohomogeneity 2. The nature of the generalisation is then analogous to the way in which a NUT parameter can be introduced in the four-dimensional Kerr-de Sitter metrics.

The same procedure can be followed also in five dimensions, but in this case we find that the additional NUT parameter is trivial, in the sense that it can be absorbed by a rescaling of parameters and coordinates. However, we also found that there exists a remarkable symmetry of the five-dimensional Kerr-AdS metrics, in which one can map a solution where one or both of the rotation parameters are large (the case of over-rotation, where $a^2g^2 > 1$ and/or $b^2g^2 > 1$) into a solution where the rotation parameters are small (i.e. under-rotation). This means that there is effectively a maximum rotation possible, corresponding to the supersymmetric case where $a^2g^2 = 1$ or $b^2g^2 = 1$.

We also studied the supersymmetric limits of the new Kerr-de Sitter-Taub-NUT metrics, showing that after Euclideanisation we can obtain new cohomogeneity-2 Einstein-Sasaki metrics in all odd dimensions $D \geq 7$, and new cohomogeneity-2 Ricci-flat Kähler metrics in all even dimensions $D \geq 6$.

CHAPTER III

GENERAL KERR-NUT-ADS METRICS IN ALL DIMENSIONS

A. Introduction

In chapter II, it was shown that one can introduce a NUT charge parameter in all the Kerr-AdS metrics in $D \geq 6$ if they are first specialised, by equating rotation parameters appropriately, to have cohomogeneity 2. The case of $D = 4$ had, of course, been obtained long ago, and the case $D = 5$ turns out to be rather degenerate, in that the NUT parameter is trivial and can be removed by a redefinition of the other parameters and the coordinates. (Cohomogeneity-one pure multi-nut solutions in higher dimensions were obtained in [28].)

The purpose of this chapter is to present NUT generalisations of the Kerr-AdS metrics which are of the most general possible type. We find that in D dimensions the general Kerr-AdS metric (with all rotation parameters allowed to be unequal) can be extended by the inclusion of $(D-5)/2$ independent NUT parameters when D is odd, and $(D-2)/2$ when D is even. We arrived at these solutions by first rewriting the Kerr-AdS metrics using a set of coordinate variables that make the introduction of the NUT parameters a very natural generalisation of the usual mass parameter.

The choice of coordinates in four dimensions that leads to the natural inclusion of a NUT parameter in the Kerr-AdS solution is rather well known. In the standard description of the Kerr-AdS solution one has angular coordinates (θ, ϕ) parameterising the 2-sphere spatial sections at constant radius r. If one defines $y = a \cos\theta$, where a is the rotation parameter, and makes appropriate linear redefinitions of the time and azimuthal coordinate ϕ, the metric can be written as

$$ds_4^2 = -\frac{\Delta_r}{r^2+u^2}\,(d\tau + y^2 d\psi)^2 + \frac{\Delta_y}{r^2+u^2}\,(d\tau - r^2 d\psi)^2 + \frac{(r^2+y^2)\,dr^2}{\Delta_r} + \frac{(r^2+y^2)\,dy^2}{\Delta_y}\,, \tag{3.1}$$

where

$$\Delta_r = (r^2+a^2)(1+g^2r^2) - 2Mr\,, \qquad \Delta_y = (a^2-y^2)(1-g^2y^2)\,. \tag{3.2}$$

This Kerr-AdS solution, satisfying $R_{\mu\nu} = -3g^2\, g_{\mu\nu}$, is generalised to include the NUT parameter L by replacing Δ_y by

$$\Delta_y = (a^2-y^2)(1-g^2y^2) + 2Ly\,. \tag{3.3}$$

An important feature of this parameterisation, which makes the inclusion of the NUT parameter very natural, is that the radial variable r and the "latitude" variable y are placed on a very symmetrical footing. The NUT generalisations of the higher-dimensional Kerr-AdS metrics that were obtained in chapter II worked in a very similar way. An essential part of the construction was that the rotation parameters had to be specialised in such a way that the cohomogeneity was reduced to 2, and so again a latitude-type coordinate y could be introduced in such a way that it, and the radial variable r, appeared in a very symmetrical way. The metric functions depended on r and y, with the $(D-2)$-dimensional hypersurfaces at constant r and y being homogeneous.

The key to finding the NUT generalisations that we obtain in the present chapter is to make a suitable reparameterisation of the multiple "latitude" coordinates that arise in the higher-dimensional Kerr-AdS metrics. In D dimensions one has the time and radial variables (t,r), $[(D-1)/2]$ azimuthal angles ϕ_i and $[D/2]$ latitude, or direction cosine, coordinates μ_i, which are subject to the constraint

$$\sum_{i=1}^{[D/2]} \mu_i^2 = 1\,. \tag{3.4}$$

The spatial sections at constant radius r have the geometry of deformed $(D-2)$-spheres. The unit S^{D-2} metric is given by

$$d\Omega^2 = \sum_{i=1}^{[D/2]} d\mu_i^2 + \sum_{i=1}^{[(D-1)/2]} \mu_i^2 d\phi_i^2 \tag{3.5}$$

in these variables. Associated with each azimuthal angle ϕ_i is a rotation parameter a_i.

We find that the appropriate reparameterisation of the μ_i coordinates is as follows. Taking $D = 2n+1$ in the odd-dimensional case, and $D = 2n$ in the even-dimensional case, we parameterise the n coordinates μ_i as

$$\mu_i^2 = \frac{\prod_{\alpha=1}^{n-1}(a_i^2 - y_\alpha^2)}{\prod_{k=1}^{\prime n}(a_i^2 - a_k^2)}\,, \tag{3.6}$$

where the prime on $\prod'$ indicates that the term that vanishes (i.e. when $k = i$) is omitted from the product.* Note that this parameterisation using just $(n-1)$ coordinates y_α explicitly solves the constraint (3.4). It also has the striking property that it diagonalises the metric (3.5) on the unit sphere, expressed in terms of the unconstrained latitude variables y_α:

$$d\Omega^2 = \sum_{\alpha=1}^{n-1} g_\alpha\, dy_\alpha^2 + \sum_{i=1}^{[(D-1)/2]} \mu_i^2\, d\phi_i^2\,, \tag{3.7}$$

where the notation $\prod'$ universally indicates that the vanishing factor is to be omitted

*Transformations of this type were first considered by Jacobi, in the context of constrained dynamical systems [29].

from the product, μ_i^2 is given by (3.6), and

$$g_\alpha = -\frac{y_\alpha^2 \prod'^{n-1}_{\beta=1}(y_\alpha^2 - y_\beta^2)}{\prod_{k=1}^n (a_k^2 - y_\alpha^2)}\,. \tag{3.8}$$

Note that the parameterisation (3.6) solves the constraint (3.4), and diagonalises the metric as in (3.7), for arbitrary choices of unequal constants a_i^2.

When we utilise (3.6) in the next section, we shall take the constants a_i to be the rotation parameters of the Kerr-AdS black holes. In the case of even dimensions $D = 2n$, there are only $(n-1)$ rotation parameters, and so a_n is taken to be zero. We shall see that with this choice of the parameters in the Jacobi transformations (3.6), the Kerr-AdS metrics obtained in [13, 14], which are non-diagonal in the latitude coordinate differentials $d\mu_i$, remarkably become diagonal with respect to the unconstrained coordinate differentials dy_α. Furthermore, we shall see that after writing the Kerr-AdS metrics in terms of the coordinates (t, r, y_α, ϕ_i), the radial variable r and the latitude variables y_α enter the metrics in a very symmetrical fashion, such that the generalisation to include a set of $(n-1)$ NUT parameters becomes very natural. It is explicitly verified [30] by a investigation of the Riemannian curvature that these generalisations of the Kerr-AdS metrics satisfy the Einstein equations in all dimensions.

After presenting the general Kerr-NUT-AdS metrics in section B, we then consider, in section C, some simpler expressions for the Kerr-NUT-AdS metrics. It turns out that the symmetrical appearance of the radial and latitude variables is further enhanced if one performs a "Wick rotation" of the radial coordinate r, and defines variables x_μ with $x_\alpha = y_\alpha$, $x_n = \mathrm{i}\, r$. This leads to a form for the metric in which all the coordinates x_μ enter on an exactly parallel footing. In a further simplification of the expressions for the metrics, we find that by defining appropriate linear combinations of the time and azimuthal coordinates, the Kerr-NUT-AdS metrics can be cast in a form that provides a natural generalisation of the four-dimensional metrics described in [31]. We also discuss certain scaling symmetries and discrete symmetries of the Kerr-NUT-AdS metrics. The scaling symmetries imply that there are $(n-2)$ non-trivial NUT parameters in odd dimensions $D = 2n+1$, and $(n-1)$ non-trivial NUT parameters in even dimensions $D = 2n$. The discrete symmetries imply that metrics with over-rotation, i.e. where one or more rotation parameters exceeds the AdS radius, are equivalent to metrics with under-rotation.

In section D, we focus on the particular cases of dimensions $D = 6$ and $D = 7$, since these are the lowest dimensions where our new results extend beyond those known previously. In section E we study the supersymmetric, or BPS, limits of the new metrics in odd and even dimensions. After performing a Euclideanisation, the odd-dimensional solutions give rise to new examples of Einstein-Sasaki metrics in $D \geq 7$. By writing these as circle fibrations over an Einstein-Kähler base, we thereby obtain new classes of Einstein-Kähler metrics in all even dimensions $D \geq 6$. The chapter ends with conclusions in section F.

B. The General Kerr-NUT-AdS Solutions

In this section, we shall present our general results for the Kerr-NUT-AdS metrics in D dimensions. These ostensibly have a total of $(D-1)$ independent parameters, comprising the mass M, the $[(D-1)/2]$ rotation parameters a_i, and $[(D-2)/2]$ NUT parameters L_α. As we shall discuss later, in odd dimensions there is a symmetry that allows one to eliminate one of the parameters, and so in odd dimensions there are actually in total $(D-2)$ non-trivial parameters in the solutions we obtain.

The first step is to rewrite the Kerr-AdS metrics, which were obtained in [13, 14], in terms of the new coordinate parameterisation introduced in (3.6). It is advantageous to separate the discussion into two cases, depending upon whether D is odd or even.

1. The Odd-dimensional Case: $D = 2n + 1$

As a preliminary, we make the following definitions:

$$\begin{aligned}
U &= \prod_{\alpha=1}^{n-1}(r^2 + y_\alpha^2)\,, \qquad U_\alpha = -(r^2 + y_\alpha^2){\prod}'^{\,n-1}_{\beta=1}(y_\beta^2 - y_\alpha^2)\,, \qquad 1 \le \alpha \le n-1\,,\\
W &= \prod_{\alpha=1}^{n-1}(1 - g^2 y_\alpha^2)\,, \qquad \gamma_i = \prod_{\alpha=1}^{n-1}(a_i^2 - y_\alpha^2)\,, \qquad 1 \le i \le n\,,\\
X &= \frac{1 + g^2 r^2}{r^2}\prod_{k=1}^{n}(r^2 + a_k^2) - 2M\,,\\
X_\alpha &= \frac{1 - g^2 y_\alpha^2}{y_\alpha^2}\prod_{k=1}^{n}(a_k^2 - y_\alpha^2) + 2L_\alpha \qquad 1 \le \alpha \le n-1\,.
\end{aligned} \tag{3.9}$$

We have actually already included the new NUT parameters L_α here; they appear just in the definitions of the functions X_α. Note that again the notation $\prod'$ indicates that the term in the full product that vanishes is to be omitted.

Using these functions, we find that the Kerr-NUT-AdS metrics in $D = 2n + 1$ dimensions are given by

$$\begin{aligned}
ds^2 &= \frac{U}{X}\,dr^2 + \sum_{\alpha=1}^{n-1}\frac{U_\alpha}{X_\alpha}\,dy_\alpha^2 - \frac{X}{U}\Big[W d\tilde{t} - \sum_{i=1}^{n} a_i^2\gamma_i d\tilde{\phi}_i\Big]^2\\
&\quad + \sum_{\alpha=1}^{n-1}\frac{X_\alpha}{U_\alpha}\Big[\frac{(1 + g^2 r^2)W}{1 - g^2 y_\alpha^2}\,d\tilde{t} - \sum_{i=1}^{n}\frac{a_i^2(r^2 + a_i^2)\,\gamma_i}{a_i^2 - y_\alpha^2}\,d\tilde{\phi}_i\Big]^2\\
&\quad + \frac{\prod_{k=1}^{n} a_k^2}{r^2\prod_{\alpha=1}^{n-1} y_\alpha^2}\Big[(1 + g^2 r^2)W\,d\tilde{t} - \sum_{i=1}^{n}(r^2 + a_i^2)\gamma_i\,d\tilde{\phi}_i\Big]^2\,.
\end{aligned} \tag{3.10}$$

With the parameters L_α set to zero, the metrics are just a rewriting of the Kerr-AdS metrics obtained in [13, 14], using the new coordinates y_α defined by (3.6).[†] They

[†]Note that the metric signature is just the usual $(-++\cdots+)$, for the appropriate

are written here in an asymptotically-static frame. We have also rescaled the time and azimuthal coordinates in order to simplify the expression. They are related to the original asymptotically static coordinates (t, ϕ_i) by

$$t = \tilde{t} \prod_{i=1}^{n} \Xi_i \,, \qquad \phi_i = a_i\, \Xi_i\, \tilde{\phi}_i \prod{}'^{\,n}_{k=1} (a_i^2 - a_k^2) \,, \tag{3.11}$$

where $\Xi_i = 1 - g^2\, a_i^2$. The coordinate t is canonically normalised, and the coordinates ϕ_i each have period 2π, in the Kerr-AdS metrics.

The new metrics that we have obtained, by including the $(n-1)$ parameters L_α in the definition of X_α in (3.9), describe the general Kerr-NUT-AdS metrics in dimension $D = 2n+1$. As we shall discuss in section C, in odd dimensions there is actually a redundancy among the $(n-1)$ NUT parameters, with one of them being trivial. Thus the total count of non-trivial parameters in the general Kerr-NUT-AdS metrics in dimension $D = 2n+1$ is $2n-1$, which can be thought of n rotation parameters, the mass, and $(n-2)$ NUT charges.

2. The Even-dimensional Case: $D = 2n$

In this case we begin by defining functions as follows:

$$\begin{aligned}
U &= \prod_{\alpha=1}^{n-1} (r^2 + y_\alpha^2) \,, \qquad U_\alpha = -(r^2 + y_\alpha^2) \prod{}'^{\,n-1}_{\beta=1} (y_\beta^2 - y_\alpha^2) \,, \qquad 1 \le \alpha \le n-1 \,, \\
W &= \prod_{\alpha=1}^{n-1} (1 - g^2 y_\alpha^2) \,, \qquad \gamma_i = \prod_{\alpha=1}^{n-1} (a_i^2 - y_\alpha^2) \,, \qquad 1 \le i \le n-1 \,, \\
X &= (1 + g^2 r^2) \prod_{k=1}^{n-1} (r^2 + a_k^2) - 2M\, r \,, \\
X_\alpha &= -(1 - g^2 y_\alpha^2) \prod_{k=1}^{n-1} (a_k^2 - y_\alpha^2) - 2L_\alpha\, y_\alpha \,, \qquad 1 \le \alpha \le n-1 \,.
\end{aligned} \tag{3.12}$$

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choices of the y_α coordinate intervals that correspond to the standard Kerr-AdS black hole solution.

We find that the Kerr-NUT-AdS metrics in $D = 2n$ dimensions are given by

$$\begin{aligned} ds^2 &= \frac{U}{X}\,dr^2 + \sum_{\alpha=1}^{n-1} \frac{U_\alpha}{X_\alpha}\,dy_\alpha^2 - \frac{X}{U}\Big[W d\tilde{t} - \sum_{i=1}^{n-1} \gamma_i d\tilde{\phi}_i\Big]^2 \\ &\quad + \sum_{\alpha=1}^{n-1} \frac{X_\alpha}{U_\alpha}\Big[\frac{(1+g^2r^2)W}{1-g^2y_\alpha^2}\,d\tilde{t} - \sum_{i=1}^{n-1}\frac{(r^2+a_i^2)\,\gamma_i}{a_i^2-y_\alpha^2}\,d\tilde{\phi}_i\Big]^2 . \end{aligned} \tag{3.13}$$

Again, the previously-known Kerr-AdS metrics correspond to setting the new NUT parameters L_α to zero in the definition of the functions X_α in (3.12). The coordinates $\tilde{t}$ and $\tilde{\phi}_i$ are related to the canonically-normalised coordinates t and ϕ_i of the $L_\alpha = 0$ Kerr-AdS metrics by

$$t = \tilde{t}\prod_{i=1}^{n} \Xi_i\,, \qquad \phi_i = a_i\,\Xi_i\,\tilde{\phi}_i {\prod_{k=1}^{n-1}}' (a_i^2 - a_k^2)\,, \tag{3.14}$$

When $L_\alpha = 0$, regularity of the Kerr-AdS metric dictates that the azimuthal angles ϕ_i should all have period 2π.

As we shall discuss in section C, all the NUT parameters are non-trivial in even dimensions, and so the general Kerr-NUT-AdS metrics in dimension $D = 2n$ have $2n-1$ independent parameters, comprising $(n-1)$ rotations, the mass, and $(n-1)$ NUT parameters.

C. A Simpler Form for the Kerr-NUT-AdS Metrics

We already saw in section B that the Kerr-NUT-AdS metrics assume a rather symmetrical form when the latitude coordinates μ_i are parameterised in terms of the coordinates y_α using (3.6). The parallel between the radial coordinate r and the latitude coordinates y_α becomes even more striking if we perform a Wick rotation of the radial variable, and define the n coordinates x_μ by

$$x_n = \mathrm{i}\,r\,, \qquad x_\alpha = y_\alpha\,, \qquad 1 \le \alpha \le n-1\,. \tag{3.15}$$

As we shall show, the Kerr-NUT-AdS metrics can now be written in a considerably simpler form. In fact, if we then perform further transformations on the time and azimuthal coordinates, we arrive at an even simpler way of presenting the Kerr-NUT-AdS metrics, which generalises the four-dimensional expressions obtained in [31]. As always, it is convenient to separate the discussion at this stage into the cases of odd and even dimensions.

1. $D = 2n+1$ Dimensions

We first define the functions

$$\begin{aligned} U_\mu &= {\prod}'^{n}_{\nu=1}(x_\nu^2 - x_\mu^2)\,, \qquad X_\mu = \frac{(1-g^2x_\mu^2)}{x_\mu^2}\prod_{k=1}^{n}(a_k^2 - x_\mu^2) + 2M_\mu\,, \\ \widetilde{W} &= \prod_{\nu=1}^{n}(1-g^2x_\nu^2)\,, \qquad \tilde{\gamma}_i = \prod_{\nu=1}^{n}(a_i^2 - x_\nu^2)\,. \end{aligned} \tag{3.16}$$

The odd-dimensional Kerr-NUT-AdS metric (3.10) can then be written as

$$\begin{aligned} ds^2 &= \sum_{\mu=1}^{n}\Big\{\frac{U_\mu}{X_\mu}\,dx_\mu^2 + \frac{X_\mu}{U_\mu}\Big[\frac{\widetilde{W}}{1-g^2x_\mu^2}\,d\tilde{t} - \sum_{i=1}^{n}\frac{a_i^2\,\tilde{\gamma}_i}{a_i^2 - x_\mu^2}\,d\tilde{\phi}_i\Big]^2\Big\} \\ &\quad -\frac{\prod_{k=1}^{n} a_k^2}{\prod_{\mu=1}^{n} x_\mu^2}\Big[\widetilde{W}\,d\tilde{t} - \sum_{i=1}^{n}\tilde{\gamma}_i\,d\tilde{\phi}_i\Big]^2\,. \end{aligned} \tag{3.17}$$

Note that M_n is just equal to the previous mass parameter M, while the remaining M_α are NUT parameters, previously denoted by L_α.

It is useful to give also the inverse of the metric (3.17). Defining

$$S_\mu = \prod_{k=1}^{n}(a_k^2 - x_\mu^2)^2\,, \qquad B_j = {\prod}'^{n}_{k=1}(a_j^2 - a_k^2)\,, \tag{3.18}$$

we find that the inverse metric is given by

$$\Big(\frac{\partial}{\partial s}\Big)^2 = \sum_{\mu=1}^n \Big\{ \frac{X_\mu}{U_\mu} \Big(\frac{\partial}{\partial x_\mu}\Big)^2 + \frac{S_\mu}{x_\mu^4\, U_\mu\, X_\mu} \Big[\frac{1}{(\prod_j \Xi_j)}\, \frac{\partial}{\partial \tilde t} + \sum_{k=1}^n \frac{(1-g^2 x_\mu^2)}{B_k\, \Xi_k\, (a_k^2 - x_\mu^2)}\, \frac{\partial}{\partial \tilde\phi_k} \Big]^2 \Big\}$$
$$- \frac{(\prod_{k=1}^n a_k^2)}{(\prod_{\nu=1}^n x_\nu^2)} \Big(\frac{1}{(\prod_j \Xi_j)}\, \frac{\partial}{\partial \tilde t} + \sum_{k=1}^n \frac{1}{a_k^2 B_k\, \Xi_k}\, \frac{\partial}{\partial \tilde\phi_k} \Big)^2 . \tag{3.19}$$

The inverse metric becomes somewhat simpler if expressed in terms of the original canonically normalised coordinates t and ϕ_k, whose relation to $\tilde t$ and $\tilde\phi_k$ is given in (3.11). The metric (3.19) then becomes

$$\Big(\frac{\partial}{\partial s}\Big)^2 = \sum_{\mu=1}^n \Big\{ \frac{X_\mu}{U_\mu} \Big(\frac{\partial}{\partial x_\mu}\Big)^2 + \frac{S_\mu}{x_\mu^4\, U_\mu\, X_\mu} \Big[\frac{\partial}{\partial t} + \sum_{k=1}^n \frac{a_k\,(1-g^2 x_\mu^2)}{(a_k^2 - x_\mu^2)}\, \frac{\partial}{\partial \phi_k} \Big]^2 \Big\}$$
$$- \frac{(\prod_{k=1}^n a_k^2)}{(\prod_{\nu=1}^n x_\nu^2)} \Big(\frac{\partial}{\partial t} + \sum_{k=1}^n \frac{1}{a_k}\, \frac{\partial}{\partial \phi_k} \Big)^2 . \tag{3.20}$$

It is straightforward to see that the Kerr-NUT-AdS metrics (3.10) and (3.17) have a set of discrete symmetries under which one of the rotation parameters a_i is inverted through the AdS radius $1/g$. Thus, choosing a_1 for this purpose as a representative example, we can see that (3.17) is invariant under the set of transformations

$$a_1\, g \to \frac{1}{a_1\, g}\,, \qquad a_j \to \frac{a_j}{a_1\, g}\,, \quad 2 \le j \le n\,,$$
$$M_\mu \to \frac{M_\mu}{(a_1\, g)^{2n}}\,, \qquad g t \to \phi_1\,, \qquad \phi_1 \to g t\,, \qquad x_\mu \to \frac{x_\mu}{a_1\, g}\,, \tag{3.21}$$

with ϕ_j for $2 \le j \le n$ left unchanged. This, and the other permutation-related inversion symmetries, can always map a metric with over-rotation (one or more parameters a_i satisfying $|a_i\, g| > 1$) into a metric with under-rotation (all parameters satisfying $|a_i\, g| < 1$).

A further simplification of the new Kerr-NUT-AdS metrics that we obtained in (3.10) and (3.17) in dimensions $D = 2n+1$ is possible, allowing them to be written in a manner that is a rather natural higher-dimensional analogue of the expression in [31] for the four-dimensional rotating black hole metrics. Again we begin by performing the "Wick rotation" of the radial variable, as in (3.15). We then find that after appropriate linear redefinitions of the time and azimuthal coordinates, the $D = 2n+1$ Kerr-NUT-AdS metrics can be written as

$$ds^2 = \sum_{\mu=1}^{n} \Big\{ \frac{dx_\mu^2}{Q_\mu} + Q_\mu \Big(\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k \Big)^2 \Big\} - \frac{c}{(\prod_{\nu=1}^{n} x_\nu^2)} \Big(\sum_{k=0}^{n} A^{(k)}\, d\psi_k \Big)^2 \,, \qquad (3.22)$$

where we define

$$\begin{aligned} Q_\mu &= \frac{X_\mu}{U_\mu} \,, \qquad U_\mu = {\prod}'^{n}_{\nu=1} (x_\nu^2 - x_\mu^2) \,, \qquad X_\mu = \sum_{k=1}^{n} c_k\, x_\mu^{2k} + \frac{c}{x_\mu^2} - 2b_\mu \,, \\ A_\mu^{(k)} &= \sideset{}{'}\sum_{\nu_1<\nu_2<\cdots<\nu_k} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 \,, \qquad A^{(k)} = \sum_{\nu_1<\nu_2\cdots<\nu_k} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 \,. \end{aligned} \qquad (3.23)$$

Here, the prime on the summation symbol in the definition of A_μ^k indicates that the index value μ is omitted in the summations of the ν indices over the range $[1, n]$. Note that ψ_0 plays the rôle of the time coordinate. It is worth remarking that $A^{(k)}$ and $A_\mu^{(k)}$ can be defined via the generating functions

$$\prod_{\nu=1}^{n} (1 + \lambda\, x_\nu^2) = \sum_{k=0}^{n} \lambda^k\, A^{(k)} \,, \qquad (1 + \lambda x_\mu^2)^{-1} \prod_{\nu=1}^{n} (1 + \lambda\, x_\nu^2) = \sum_{k=0}^{n-1} \lambda^k\, A_\mu^{(k)} \,. \qquad (3.24)$$

The constants c_k, c and b_μ are arbitrary, with $c_n = (-1)^{n+1}\, g^2$ determining the value of the cosmological constant, $R_{\mu\nu} = -2ng^2 g_{\mu\nu}$. The remaining $2n$ constants c_k, c and b_μ are related to the n rotation parameters a_i, the mass M and the $(n-1)$ NUT parameters L_α in the obvious way that follows by comparing X_μ in (3.23) with X_μ in (3.16). However, it should be noted that not all the parameters are non-trivial in the general solution. This can be seen from the fact that there is a scaling symmetry of the metric (3.22), under which we send

$$\begin{aligned} & x_\mu \to \lambda\, x_\mu\,, \qquad \psi_k \to \lambda^{-2k-1}\,\psi_k\,, \\ & c_k \to \lambda^{2n-2k}\,c_k\,, \qquad c \to \lambda^{2n+2}\,c\,, \qquad b_\mu \to \lambda^{2n}\,b_\mu\,. \end{aligned} \tag{3.25}$$

This scaling symmetry implies that there is one trivial parameter in the general Kerr-NUT-AdS solution, leaving a total of $2n-1$ non-trivial parameters in $D = 2n+1$ dimensions. In fact, in odd dimensions there is not necessarily a clear distinction between rotation parameters and NUT parameters, as can be seen by comparing the expressions for the functions X_μ in (3.23), and the expressions in terms of rotations, mass and NUT parameters in (3.16). One might for example find that for some values of the parameters, if the scaling symmetry (3.25) is used in order to remove a "redundant" NUT charge, then this leads to a rotation parameter becoming imaginary. In such a range of the parameters, it would be more natural to retain the redundant NUT parameter. This is quite different from the situation in even dimensions, where the mass and NUT parameters are distinguished by being the coefficients of linear powers of the coordinates, as can be seen in (3.12) and in (3.27) below.

We find that the inverse of the metric (3.22) is given by

$$\begin{aligned} \Big(\frac{\partial}{\partial s}\Big)^2 &= \sum_{\mu=1}^n \Big\{ Q_\mu \Big(\frac{\partial}{\partial x_\mu}\Big)^2 + \frac{1}{x_\mu^4\, Q_\mu\, U_\mu^2} \Big[\sum_{k=0}^n (-1)^k\, x_\mu^{2(n-k)}\, \frac{\partial}{\partial \psi_k}\Big]^2 \Big\} \\ &\quad - \frac{1}{c\,(\prod_{\nu=1}^n x_\nu^2)} \Big(\frac{\partial}{\partial \psi_n}\Big)^2 . \end{aligned} \tag{3.26}$$

The specific case of the Kerr-NUT-AdS metric in $D = 7$ dimensions is discussed in section D, including the explicit transformation of the time and azimuthal coordinates that brings the metric into the form (3.22). We also give a more extensive discussion of the counting of non-trivial parameters in this example.

2. $D = 2n$ Dimensions

In this case, in addition to performing the Wick rotation of the radial variable as in (3.15), one must additionally rescale the mass by a factor of i in order to obtain a real metric. We then define functions

$$\begin{aligned} U_\mu &= \prod\nolimits'^{n}_{\nu=1}(x_\nu^2 - x_\mu^2)\,, \qquad X_\mu = -(1-g^2 x_\mu^2)\prod_{k=1}^{n-1}(a_k^2 - x_\mu^2) - 2M_\mu\, x_\mu\,, \\ \widetilde{W} &= \prod_{\nu=1}^{n}(1-g^2x_\nu^2)\,, \qquad \tilde{\gamma}_i = \prod_{\nu=1}^{n}(a_i^2 - x_\nu^2)\,, \end{aligned} \tag{3.27}$$

where $M_n = \mathrm{i}\, M$ and $M_\alpha = L_\alpha$. The even-dimensional Kerr-NUT-AdS metrics (3.13) can then be written as

$$ds^2 = \sum_{\mu=1}^{n}\Big\{\frac{U_\mu}{X_\mu}\,dx_\mu^2 + \frac{X_\mu}{U_\mu}\Big[\frac{\widetilde{W}}{1-g^2x_\mu^2}\,d\tilde{t} - \sum_{i=1}^{n-1}\frac{\tilde{\gamma}_i}{a_i^2 - x_\mu^2}\,d\tilde{\phi}_i\Big]^2\Big\}\,. \tag{3.28}$$

We find that the inverse of the metric (3.28) is given by

$$\Big(\frac{\partial}{\partial s}\Big)^2 = \sum_{\mu=1}^{n}\Big\{\frac{X_\mu}{U_\mu}\Big(\frac{\partial}{\partial x_\mu}\Big)^2 + \frac{S_\mu}{U_\mu X_\mu}\Big[\frac{1}{(\prod_k \Xi_k)}\,\frac{\partial}{\partial \tilde{t}} + \sum_{k=1}^{n-1}\frac{1-g^2x_\mu^2}{\Xi_k\, B_k\,(a_k^2-x_\mu^2)}\,\frac{\partial}{\partial\tilde{\phi}_k}\Big]^2\Big\}\,, \tag{3.29}$$

where

$$S_\mu = \prod_{k=1}^{n-1}(a_k^2 - x_\mu^2)^2\,, \qquad B_j = \prod\nolimits'^{n-1}_{k=1}(a_j^2 - a_k^2)\,. \tag{3.30}$$

Note that in terms of the original canonically-normalised coordinates t and ϕ_i, the inverse metric (3.29) takes the slightly simpler form

$$\Big(\frac{\partial}{\partial s}\Big)^2 = \sum_{\mu=1}^{n}\Big\{\frac{X_\mu}{U_\mu}\Big(\frac{\partial}{\partial x_\mu}\Big)^2 + \frac{S_\mu}{U_\mu X_\mu}\Big[\frac{\partial}{\partial t} + \sum_{k=1}^{n-1}\frac{a_k(1-g^2x_\mu^2)}{(a_k^2-x_\mu^2)}\,\frac{\partial}{\partial\phi_k}\Big]^2\Big\}\,, \tag{3.31}$$

The even-dimensional Kerr-NUT-AdS metrics (3.13) and (3.28) also have a set of discrete symmetries under which any one of the rotation parameters a_i is inverted through the AdS radius $1/g$. Thus, for example, (3.28) is invariant under the set of transformations

$$a_1\, g \to \frac{1}{a_1\, g}\,, \qquad a_j \to \frac{a_j}{a_1\, g}\,, \quad 2 \le j \le n-1\,,$$
$$M_\mu \to \frac{M_\mu}{(a_1\, g)^{2n-1}}\,, \qquad gt \to \phi_1\,, \qquad \phi_1 \to gt\,, \qquad x_\mu \to \frac{x_\mu}{a_1\, g}\,, \tag{3.32}$$

with ϕ_j for $2 \le j \le n-1$ left unchanged. This, and the other permutation-related inversion symmetries, can always map a metric with over-rotation (one or more parameters a_i satisfying $|a_i\, g| > 1$) into a metric with under-rotation (all parameters satisfying $|a_i\, g| < 1$).

Again, we find that the new Kerr-NUT-AdS metrics in dimension $D = 2n$, which we have obtained in (3.13) and (3.28), can be further simplified and written elegantly in a form that is a natural higher-dimensional analogue of the four-dimensional metrics in [31]. After making the Wick rotation of the radial variable, as in (3.15), we then find that after appropriate linear redefinitions of the time and azimuthal coordinates, the $D = 2n$ Kerr-NUT-AdS metrics can be written as

$$ds^2 = \sum_{\mu=1}^{n} \Big\{ \frac{dx_\mu^2}{Q_\mu} + Q_\mu \Big(\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k \Big)^2 \Big\}\,, \tag{3.33}$$

where we define

$$Q_\mu = \frac{X_\mu}{U_\mu}\,, \qquad U_\mu = {\prod}'^{n}_{\nu=1} (x_\nu^2 - x_\mu^2)\,, \qquad X_\mu = \sum_{k=0}^{n} c_k\, x_\mu^{2k} + 2b_\mu\, x_\mu\,,$$
$$A_\mu^{(k)} = \sum_{\nu_1<\nu_2<\cdots<\nu_k}{}' x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2\,. \tag{3.34}$$

Again, the prime on the summation symbol in the definition of A_μ^k indicates that the index value μ is omitted in the summations of the ν indices over the range $[1, n]$. The constants c_k and b_μ are arbitrary, with $c_n = (-1)^{n+1}\, g^2$ determining the value of the cosmological constant, $R_{\mu\nu} = -(2n-1) g^2 g_{\mu\nu}$. The remaining constants c_k and b_μ are related to the rotation parameters, mass and NUT parameters in the obvious way that follows by comparing X_μ in (3.34) with X_μ in (3.27).

In this even-dimensional case there is ostensibly a mismatch between the total number of parameters in the metrics (3.13) or (3.28), namely $(n-1)$ rotation parameters a_i, the mass M and the $(n-1)$ NUT parameters L_α, and the number of parameters in the polynomials X_μ, namely n constants c_k for $0 \le k \le n-1$, and n constants b_μ. However, there is also a scaling symmetry that leaves the metric (3.33) invariant, namely

$$\begin{aligned} &x_\mu \longrightarrow \lambda\, x_\mu\,, \qquad \psi_k \longrightarrow \lambda^{-2k-1}\,\psi_k\,,\\ &c_k \longrightarrow \lambda^{2n-2k}\,c_k\,, \qquad b_\mu \longrightarrow \lambda^{2n}\,b_\mu\,. \end{aligned} \tag{3.35}$$

This implies one parameter in X_μ is trivial, leaving $2n-1$ non-trivial parameters in total in the general Kerr-NUT-AdS solution in dimension $D = 2n$.[‡]

It is useful also to record the inverse of the metric (3.33), which we find to be

$$\Big(\frac{\partial}{\partial s}\Big)^2 = \sum_{\mu=1}^{n} \Big\{ Q_\mu \Big(\frac{\partial}{\partial x_\mu}\Big)^2 + \frac{1}{Q_\mu\, U_\mu^2} \Big[\sum_{k=0}^{n-1} (-1)^k\, x_\mu^{2(n-1-k)}\, \frac{\partial}{\partial \psi_k}\Big]^2 \Big\}\,. \tag{3.36}$$

The specific case of the Kerr-NUT-AdS metric in $D = 6$ dimensions is discussed in section D, including the explicit transformation of the time and azimuthal coordinates that brings the metric into the form (3.33).

[‡]It should be emphasised that there is a significant difference therefore between even and odd dimensions, as regards the number of non-trivial NUT charges that can be introduced. In even dimensions $D = 2n$ the general Kerr-AdS metrics can be augmented with the introduction of $(n-1)$ non-trivial NUT parameters, while in odd dimensions $D = 2n+1$ the general Kerr-AdS metrics can be augmented with the introduction of $(n-2)$ non-trivial NUT parameters. Thus, in particular, there is a non-trivial NUT charge in $D = 4$, but there is no non-trivial NUT charge in $D = 5$. In odd dimensions, only in $D = 7$ and above does one have non-trivial NUT charges.

D. Kerr-NUT-AdS Metrics in $D=6$ and $D=7$

1. Seven-dimensional Kerr-NUT-AdS

Here we present the specific example of $D=7$, with rotation parameters $a_i=\{a,b,c\}$, mass M and two NUT parameters L_1 and L_2. The Kerr-NUT-AdS metric is given by

$$\begin{aligned}
ds^2 \;=\;& \frac{(r^2+y^2)(r^2+z^2)\,dr^2}{X}+\frac{(r^2+y^2)(y^2-z^2)\,dy^2}{Y}+\frac{(r^2+z^2)(z^2-y^2)\,dz^2}{Z}\\
&-\frac{X}{(r^2+y^2)(r^2+z^2)}\Big[(1-g^2y^2)(1-g^2z^2)\,d\tilde{t}-a^2(a^2-y^2)(a^2-z^2)\,d\tilde{\phi}_1\\
&\qquad -b^2(b^2-y^2)(b^2-z^2)\,d\tilde{\phi}_2-c^2(c^2-y^2)(c^2-z^2)d\tilde{\phi}_3\Big]^2\\
&+\frac{Y}{(r^2+y^2)(y^2-z^2)}\Big[(1+g^2r^2)(1-g^2z^2)\,d\tilde{t}-a^2(a^2+r^2)(a^2-z^2)\,d\tilde{\phi}_1\\
&\qquad -b^2(b^2+r^2)(b^2-z^2)\,d\tilde{\phi}_2-c^2(c^2+r^2)(c^2-z^2)\,d\tilde{\phi}_3\Big]^2\\
&+\frac{Z}{(r^2+z^2)(z^2-y^2)}\Big[(1+g^2r^2)(1-g^2y^2)\,d\tilde{t}-a^2(a^2+r^2)(a^2-y^2)\,d\tilde{\phi}_1\\
&\qquad -b^2(b^2+r^2)(b^2-y^2)\,d\tilde{\phi}_2-c^2(c^2+r^2)(c^2-y^2)\,d\tilde{\phi}_3\Big]^2\\
&+\frac{a^2b^2c^2}{r^2y^2z^2}\Big[(1+g^2r^2)(1-g^2y^2)(1-g^2z^2)\,d\tilde{t}-(a^2+r^2)(a^2-y^2)(a^2-z^2)\,d\tilde{\phi}_1\\
&\qquad -(b^2+r^2)(b^2-y^2)(b^2-z^2)\,d\tilde{\phi}_2-(c^2+r^2)(c^2-y^2)(c^2-z^2)\,d\tilde{\phi}_3\Big]^2
\end{aligned} \tag{3.37}$$

where

$$\tilde{t} = \frac{t}{\Xi_a \Xi_b \Xi_c}\,, \qquad \tilde{\phi}_1 = \frac{\phi_1}{a\,\Xi_a (b^2-a^2)(c^2-a^2)}\,,$$
$$\tilde{\phi}_2 = \frac{\phi_2}{b\,\Xi_b (a^2-b^2)(c^2-b^2)}\,, \qquad \tilde{\phi}_3 = \frac{\phi_3}{c\,\Xi_c (a^2-c^2)(b^2-c^2)}\,,$$
$$\Xi_a = 1 - a^2 g^2\,, \qquad \Xi_b = 1 - b^2 g^2\,, \qquad \Xi_c = 1 - c^2 g^2\,,$$
$$X = \frac{1}{r^2}(1+g^2 r^2)(a^2+r^2)(b^2+r^2)(c^2+r^2) - 2M\,,$$
$$Y = \frac{1}{y^2}(1-g^2 y^2)(a^2-y^2)(b^2-y^2)(c^2-y^2) + 2L_1\,,$$
$$Z = \frac{1}{z^2}(1-g^2 z^2)(a^2-z^2)(b^2-z^2)(c^2-z^2) + 2L_2\,. \qquad (3.38)$$

Note that regularity of the metric dictates that the coordinates ϕ_i each have period 2π when the NUT parameters L_1 and L_2 are set to zero.

The metric has six parameters, (a, b, c, M, L_1, L_2), but one of them is redundant. To show this, we first rewrite the metric after making the coordinate transformations

$$\begin{aligned}
t &= t' + (a^2+b^2+c^2)\psi_1 + (a^2b^2+b^2c^2+c^2a^2)\psi_2 + a^2b^2c^2\psi_3\,,\\
\frac{\phi_1}{a} &= \psi_1 + (b^2+c^2)\psi_2 + b^2c^2\psi_3 + g^2(t' + (b^2+c^2)\psi_1 + b^2c^2\psi_2)\,,\\
\frac{\phi_2}{b} &= \psi_1 + (a^2+c^2)\psi_2 + a^2c^2\psi_3 + g^2(t' + (a^2+c^2)\psi_1 + a^2c^2\psi_2)\,,\\
\frac{\phi_3}{c} &= \psi_1 + (a^2+b^2)\psi_2 + a^2b^2\psi_3 + g^2(t' + (a^2+b^2)\psi_1 + a^2b^2\psi_2)\,, \qquad (3.39)
\end{aligned}$$

which leads to

$$\begin{aligned} ds^2 \;=\; & \frac{(r^2+y^2)(r^2+z^2)\,dr^2}{X} + \frac{(r^2+y^2)(y^2-z^2)\,dy^2}{Y} + \frac{(r^2+z^2)(z^2-y^2)\,dz^2}{Z} \\ & - \frac{X}{(r^2+y^2)(r^2+z^2)}\Big(dt' + (y^2+z^2)d\psi_1 + y^2z^2d\psi_2\Big)^2 \\ & + \frac{Y}{(r^2+y^2)(z^2-y^2)}\Big(dt' + (z^2-r^2)d\psi_1 - r^2z^2d\psi_2\Big)^2 \\ & + \frac{Z}{(r^2+z^2)(y^2-z^2)}\Big(dt' + (y^2-r^2)d\psi_1 - r^2y^2d\psi_2\Big)^2 \qquad (3.40) \\ & + \frac{C_3}{r^2y^2z^2}\Big(dt' + (y^2+z^2-r^2)d\psi_1 + (y^2z^2-r^2y^2-r^2z^2)d\psi_2 - r^2y^2z^2d\psi_3\Big)^2 . \end{aligned}$$

The functions X, Y and Z can be expressed as

$$\begin{aligned} X &= g^2r^6 + C_0r^4 + C_1r^2 + C_2 - 2M + \frac{C_3}{r^2}\,, \\ Y &= g^2y^6 - C_0y^4 + C_1y^2 - C_2 + 2L_1 + \frac{C_3}{y^2}\,, \\ Z &= g^2z^6 - C_0z^4 + C_1z^2 - C_2 + 2L_2 + \frac{C_3}{z^2}\,, \qquad (3.41) \end{aligned}$$

where

$$\begin{aligned} C_0 &= 1 + g^2(a^2+b^2+c^2)\,, \qquad C_1 = a^2+b^2+c^2 + g^2(a^2b^2+b^2c^2+c^2a^2)\,, \\ C_2 &= a^2b^2+b^2c^2+c^2a^2 + g^2a^2b^2c^2\,, \qquad C_3 = a^2b^2c^2\,. \qquad (3.42) \end{aligned}$$

We can now view the solution as being parameterised by (C_0, C_1, C_3), together with $X_0 = C_2 - 2M$, $Y_0 = 2L_1 - C_2$, $Z_0 = 2L_2 - C_2$. The solution has a scaling symmetry, namely

$$\begin{aligned} & r \to \lambda\, r\,, \qquad y \to \lambda\, y\,, \qquad z \to \lambda\, z\,, \\ & C_0 \to \lambda^2 C_0\,, \qquad C_1 \to \lambda^4 C_1\,, \qquad C_3 \to \lambda^8 C_3\,, \\ & X_0 \to \lambda^6 X_0\,, \qquad Y_0 \to \lambda^6 Y_0\,, \qquad Z_0 \to \lambda^6 Z_0\,, \\ & \tilde{t} \to \lambda^{-1}\tilde{t}\,, \qquad \psi_1 \to \lambda^{-3}\psi_1\,, \qquad \psi_2 \to \lambda^{-5}\psi_2\,, \qquad \psi_3 \to \lambda^{-7}\psi_3\,, \qquad (3.43) \end{aligned}$$

This implies that one of the parameters in (3.43) can be set to 1 without loss of generality. In turn, this allows us to set one of the original parameters, say L_2 to zero. Thus there are actually five non-trivial parameters in the solution.

For any fixed gauged choice of M, L_1, L_2, the metric still has discrete residual symmetry, namely

$$a \to \frac{1}{a\,g^2}\,, \qquad b \to \frac{b}{a\,g}\,, \qquad c \to \frac{c}{a\,g}\,, \qquad \{M, L_1, L_2\} \to \lambda^6\,\{M, L_1, L_2\}\,, \tag{3.44}$$

with $\lambda = 1/(a\,g)$.

2. Six-dimensional Kerr-NUT-AdS

Here we present the explicit $D = 6$ metric, given by

$$\begin{aligned} ds^2 \;=\;& \frac{(r^2+y^2)(r^2+z^2)\,dr^2}{X} + \frac{(r^2+y^2)(y^2-z^2)\,dy^2}{Y} + \frac{(r^2+z^2)(z^2-y^2)\,dz^2}{Z} \\ & -\frac{X}{(r^2+y^2)(r^2+z^2)}\Big((1-g^2y^2)(1-g^2z^2)\,d\tilde{t} - (a^2-y^2)(a^2-z^2)\,d\tilde{\phi}_1 \\ & \qquad -(b^2-y^2)(b^2-z^2)\,d\tilde{\phi}_2\Big)^2 \\ & +\frac{Y}{(r^2+y^2)(y^2-z^2)}\Big((1+g^2r^2)(1-g^2z^2)\,d\tilde{t} - (a^2+r^2)(a^2-z^2)\,d\tilde{\phi}_1 \\ & \qquad -(b^2+r^2)(b^2-z^2)\,d\tilde{\phi}_2\Big)^2 \\ & +\frac{Z}{(r^2+z^2)(z^2-y^2)}\Big((1+g^2r^2)(1-g^2y^2)\,d\tilde{t} - (a^2+r^2)(a^2-y^2)\,d\tilde{\phi}_1 \\ & \qquad -(b^2+r^2)(b^2-y^2)\,d\tilde{\phi}_2\Big)^2\,. \end{aligned} \tag{3.45}$$

where X, Y and Z are given by

$$X = (1+g^2r^2)(r^2+a^2)(r^2+b^2) - 2M\,r\,, \quad Y = -(1-g^2y^2)(a^2-y^2)(b^2-y^2) - 2L_1\,y\,,$$
$$Z = -(1-g^2z^2)(a^2-z^2)(b^2-z^2) - 2L_2\,z\,. \tag{3.46}$$

The coordinate $\tilde{t}$, $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are related to the canonically defined t, ϕ_1 and ϕ_2 by (3.14). We can then make the coordinate transformation

$$\begin{aligned} t &= t' + (a^2+b^2)\,\psi_1 + a^2b^2\,\psi_2\,, \qquad \frac{\phi_1}{a} = \psi_1 + b^2\,\psi_2 + g^2(d\tilde{t} + b^2\,\psi_1)\,, \\ \frac{\phi_2}{b} &= \psi_1 + a^2\,\psi_2 + g^2(d\tilde{t} + a^2\,\psi_1)\,, \end{aligned} \tag{3.47}$$

which leads to the metric

$$\begin{aligned} ds^2 &= \frac{(r^2+y^2)(r^2+z^2)\,dr^2}{X} + \frac{(r^2+y^2)(y^2-z^2)\,dy^2}{Y} + \frac{(r^2+z^2)(z^2-y^2)\,dz^2}{Z} \\ &-\frac{X}{(r^2+y^2)(r^2+z^2)}\Big(dt' + (y^2+z^2)\,d\psi_1 + y^2z^2\,d\psi_2\Big)^2 \\ &+\frac{Y}{(r^2+y^2)(y^2-z^2)}\Big(dt' + (z^2-r^2)\,d\psi_1 - r^2z^2\,d\psi_2\Big)^2 \\ &+\frac{Z}{(r^2+z^2)(z^2-y^2)}\Big(dt' + (y^2-r^2)\,d\psi_1 - r^2y^2\,d\psi_2\Big)^2\,. \end{aligned} \tag{3.48}$$

The functions X, Y and Z given in (3.46) can now be written as

$$\begin{aligned} X &= g^6r^6 + C_0\,r^4 + C_1r^2 - 2M\,r + C_2\,, \\ Y &= g^6y^6 - C_0\,y^4 + C_1y^2 - 2L_1\,y - C_2\,, \\ Z &= g^6z^6 - C_0\,z^4 + C_1z^2 - 2L_1\,z - C_2\,, \end{aligned}$$

where C_i are constants, expressed in terms two constants a and b, given by

$$C_0 = 1 + g^2(a^2+b^2)\,, \qquad C_1 = a^2+b^2+g^2a^2b^2\,, \qquad C_2 = a^2b^2\,. \tag{3.49}$$

In fact, the constants C_i can be arbitrary, since the form of the metric has the following symmetry:

$$\begin{aligned}
&r \to \lambda\, r\,, \qquad y \to \lambda\, y\,, \qquad z \to \lambda\, z\,,\\
&C_0 \to \lambda^2\, C_0\,, \qquad C_1 \to \lambda^4\, C_1\,, \qquad C_2 \to \lambda^6\, C_2\,,\\
&M \to \lambda^5\, M\,, \qquad L_1 \to \lambda^5\, L_1\,, \qquad L_2 \to \lambda^5\, L_2\,,\\
&\tilde{t} \to \lambda^{-1}\, \tilde{t}\,, \qquad \psi_1 \to \lambda^{-3}\, \psi_1\,, \qquad \psi_2 \to \lambda^{-5}\, \psi_2\,.
\end{aligned} \tag{3.50}$$

Thus to fix C_i as given by (3.49) is to have fixed the symmetry. It follows that unlike in the case of odd dimensions, the NUT parameters here are all non-trivial. For the above fixed parameter gauge, the metric has residual discrete symmetry, namely

$$a \to \frac{1}{a\, g^2}\,, \qquad b \to \frac{b}{a\, g} \tag{3.51}$$

with $\lambda = a\, g$.

The form in which the six-dimensional Kerr-NUT-AdS metric is written in equation (3.48) is closely analogous to the form of the four-dimensional Plebanski metrics [31].

E. BPS Limits

1. BPS Limit for $D = 2n + 1$

In this section we shall investigate the BPS limit of the odd-dimensional Kerr-NUT-AdS metrics. In this limit the metrics admit Killing spinors, and if one furthermore performs a Euclideanisation to positive-definite metric signature, and sets the cosmological constant to be positive (by taking g^2 to be negative) one will obtain Einstein-Sasaki metrics.

For convenience, we shall scale the metrics in this limit so that their Ricci tensor is the same as that of a unit sphere of the same dimension. This is achieved by setting $g = \mathrm{i}$. It is convenient also to write the metrics in a specific asymptotically-rotating frame, by sending $\phi_i \to \phi_i - g\, dt$.

We shall first consider the 7-dimensional metric discussed in the previous section. The Euclideanisation is achieved by sending

$$t \to \mathrm{i}\,\tau\,, \qquad a \to \mathrm{i}\,a\,, \qquad b \to \mathrm{i}\,b\,, \qquad c \to \mathrm{i}\,c\,. \tag{3.52}$$

To take the BPS limit we define

$$\begin{aligned} &1 - a^2 = \alpha\,\epsilon\,, \qquad 1 - b^2 = \beta\,\epsilon\,, \qquad 1 - c^2 = \gamma\,\epsilon\,,\\ &1 - r^2 = x\,\epsilon\,, \qquad 1 + y^2 \to y\,\epsilon\,, \qquad 1 + z^2 \to z\,\epsilon\,,\\ &M = m\,\epsilon^4\,, \qquad L_1 = \ell_1\,\epsilon^4\,, \qquad L_2 = \ell_2\,\epsilon^4\,, \end{aligned} \tag{3.53}$$

and then send $\epsilon \to 0$. This leads to the metric

$$ds_7^2 = (d\tau + \mathcal{A})^2 + ds_6^2\,, \tag{3.54}$$

where

$$
\begin{aligned}
ds_6^2 &= \frac{(x-y)(x-z)\,dx^2}{4X} + \frac{(y-x)(y-z)\,dy^2}{4Y} + \frac{(z-x)(z-y)}{4Z}dz^2 \\
&\quad + \frac{X}{(x-y)(x-z)}\Big((\alpha-y)(\alpha-z)d\tilde{\phi}_1 + (\beta-y)(\beta-z)d\tilde{\phi}_2 \\
&\qquad\qquad + (\gamma-y)(\gamma-z)d\tilde{\phi}_3\Big)^2 \\
&\quad + \frac{Y}{(y-x)(y-z)}\Big((\alpha-x)(\alpha-z)d\tilde{\phi}_1 + (\beta-x)(\beta-z)d\tilde{\phi}_2 \\
&\qquad\qquad + (\gamma-x)(\gamma-z)d\tilde{\phi}_3\Big)^2 \\
&\quad + \frac{Z}{(z-x)(z-y)}\Big((\alpha-x)(\alpha-y)d\tilde{\phi}_1 + (\beta-x)(\beta-y)d\tilde{\phi}_2 \\
&\qquad\qquad + (\gamma-x)(\gamma-y)d\tilde{\phi}_3\Big)^2 \\
\mathcal{A} &= (\alpha-x)(\alpha-y)(\alpha-z)d\tilde{\phi}_1 + (\beta-x)(\beta-y)(\beta-z)d\tilde{\phi}_1 \\
&\quad + (\gamma-x)(\gamma-y)(\gamma-z)d\tilde{\phi}_1 \\
X &= x(\alpha-x)(\beta-x)(\gamma-x) - 2m\,, \qquad Y = y(\alpha-y)(\beta-y)(\gamma-y) - 2\ell_1\,, \\
Z &= z(\alpha-z)(\beta-z)(\gamma-z) - 2\ell_2\,. \qquad (3.55)
\end{aligned}
$$

The $\tilde{\phi}_i$ are related to the original ϕ_i by the constant scalings

$$
\begin{aligned}
\phi_1 &= \alpha(\alpha-\beta)(\alpha-\gamma)\tilde{\phi}_1\,, \qquad \phi_2 = \beta(\beta-\alpha)(\beta-\gamma)\tilde{\phi}_2\,, \\
\phi_3 &= \gamma(\gamma-\alpha)(\gamma-\beta)\tilde{\phi}_3\,. \qquad (3.56)
\end{aligned}
$$

For the general case of $D = 2n+1$ dimensions, we find after performing analogous computations that the Einstein-Sasaki metric is given by

$$ds_{2n+1}^2 = (d\tau + \mathcal{A})^2 + ds_{2n}^2\,, \qquad (3.57)$$

where

$$\begin{aligned}
ds^2_{2n} &= \sum_{\mu=1}^n \frac{U_\mu\, dx_\mu^2}{4X_\mu} + \sum_{\mu=1}^n \frac{X_\mu}{U_\mu}\Big(\sum_{i=1}^n \frac{W_i\, d\tilde\phi_i}{\alpha_i - x_\mu}\Big)^2\,, \\
\mathcal{A} &= \sum_{i=1}^n W_i d\tilde\phi_i\,, \qquad U_\mu = \prod\nolimits'^{n}_{\nu=1}(x_\nu - x_\mu)\,, \\
X_\mu &= x_\mu \prod_{i=1}^n (\alpha_i - x_\mu) - 2\ell_\mu\,, \qquad W_i = \prod_{\nu=1}^n (\alpha_i - x_\nu)
\end{aligned} \tag{3.58}$$

Thus we obtain a large class of local Einstein-Sasaki metrics in arbitrary $(2n+1)$ dimensions. These metrics extend the results obtained in [22, 23], where there were no NUT charges, and those in [26, 15], where metrics of cohomogeneity two were considered. We expect that (3.57,3.58) is the most general metric for Einstein-Sasaki spaces with $U(1)^{n+1}$ isometry in $(2n+1)$ dimensions.

It is of considerable interest to study the global structure of these Einstein-Sasaki metrics, and thereby to obtain the conditions on the parameters under which they extend onto smooth manifolds. This was done for $D=5$ in [22, 23], where complete metrics for the Einstein-Sasaki manifolds L^{pqr} were obtained. Those results extended previous results for the Y^{pq} [32] manifolds, which corresponded to the specialisation where the two angular momentum parameters were set equal. For seven dimensions, the global structure has been previously discussed for various special cases. When Y and Z in (3.55) both have a double root, the solution reduces to that obtained in [33], where the global structure was analysed in detail. If two angular momenta are set equal, the solution reduces to that obtained in [26] where the global structure was also discussed. Aside from these special cases, our general results in $D=7$ that we have obtained in this paper are new. Similarly, our results in $D \geq 9$ extend those obtained previously.

2. BPS Limit for $D = 2n$

The BPS limit in this case can give rise to Ricci-flat Kähler metrics. Consider first the example of the six-dimensional Kerr-NUT-AdS metric. We perform a Euclideanisation and take an analogous BPS limit to the one we discussed above for the seven-dimensional case, by setting

$$M = m\epsilon^3, \qquad L_1 = i\ell_1\epsilon^3, \qquad L_2 = i\ell_2\epsilon^3 \,. \tag{3.59}$$

In the BPS limit, when ϵ goes to zero, we obtain the Ricci flat metric

$$\begin{aligned} ds_6^2 &= \frac{(y-x)(z-x)\,dx^2}{4X} + \frac{(x-y)(z-y)\,dy^2}{4Y} + \frac{(x-z)(y-z)}{4Z}dz^2 \\ &\quad + \frac{X}{(y-x)(z-x)}\Big(y\,z\,d\tilde\tau - (\alpha-y)(\alpha-z)d\tilde\phi_1 - (\beta-y)(\beta-z)d\tilde\phi_2\Big)^2 \\ &\quad + \frac{Y}{(x-y)(z-y)}\Big(x\,z\,d\tilde\tau - (\alpha-x)(\alpha-z)d\tilde\phi_1 - (\beta-x)(\beta-z)d\tilde\phi_2\Big)^2 \\ &\quad + \frac{Z}{(x-z)(y-z)}\Big(x\,y\,d\tilde\tau - (\alpha-x)(\alpha-y)d\tilde\phi_1 - (\beta-x)(\beta-y)d\tilde\phi_2\Big)^2 \\ X &= x(\alpha-x)(\beta-x) - 2m\,, \qquad Y = y(\alpha-y)(\beta-y) - 2\ell_1\,, \\ Z &= z(\alpha-z)(\beta-z) - 2\ell_2\,. \end{aligned} \tag{3.60}$$

The coordinates $\tilde\tau$ and $\tilde\phi_i$ are related to the original τ and ϕ_i by the constant scalings

$$\phi_1 = \alpha(\alpha-\beta)\tilde\phi_1\,, \qquad \phi_2 = \beta(\beta-\alpha)\tilde\phi_2\,, \qquad \tau = \alpha\beta\,\tilde\tau\,. \tag{3.61}$$

Note that this metric can in fact be viewed as the zero cosmological constant limit of the six-dimensional Einstein-Kähler metric (3.55) that we obtained above.

For the general case of $D = 2n$ dimensions, we find that the BPS limit of the

Euclideanised Kerr-NUT-AdS metrics yields the Ricci-flat metrics

$$\begin{aligned} ds^2_{2n} &= \sum_{\mu=1}^{n} \frac{U_\mu\, dx_\mu^2}{4X_\mu} + \sum_{\mu=1}^{n} \frac{X_\mu}{U_\mu}\Big(\frac{\gamma}{x_\mu} d\tilde{\tau} - \sum_{i=1}^{n-1} \frac{W_i\, d\tilde{\phi}_i}{\alpha_i - x_\mu}\Big)^2\,, \\ X_\mu &= x_\mu \prod_{i=1}^{n-1} (\alpha_i - x_\mu) - 2\ell_\mu\,, \qquad U_\mu = {\prod}'^{n}_{\nu=1} (x_\nu - x_\mu)\,, \\ W_i &= \prod_{\nu=1}^{n} (\alpha_i - x_\nu)\,, \qquad \gamma = \prod_{\nu=1}^{n} x_\nu\,. \end{aligned} \tag{3.62}$$

Again, these metrics can be obtained also as limiting cases of the metrics (3.58), in which the cosmological constant is sent to zero.

F. Conclusions

In this chapter, we have obtained new results for the inclusion of NUT parameters in the Kerr-AdS metrics that were constructed in [13, 14]. Our strategy for doing this involved first making a judicious choice of coordinates parameterising the latitude variables in the Kerr-AdS metrics. By making a change of variables analogous to one considered long ago by Jacobi in the theory of constrained dynamical systems, we were able to rewrite the Kerr-AdS solutions of [13, 14] in such a way that the metrics become diagonal in a set of unconstrained latitude coordinates y_α. These coordinates then appear in a manner that closely parallels that of the radial variable r, and this immediately suggests a natural generalisation of the Kerr-AdS metrics to include NUT charges. It is explicitly verified [30] by a investigation of the Riemannian curvature that these generalisations of the Kerr-AdS metrics satisfy the Einstein equations in all dimensions. After further changes of variable, we arrived at the very simple expressions (3.22) and (3.33) for the general Kerr-NUT-AdS metrics in all odd and even dimensions. These expressions can be thought of as natural generalisations of the four-dimensional results obtained in [31].

The general Kerr-NUT-AdS metrics that we have obtained in this chapter have a total of $(2n-1)$ non-trivial parameters, where the spacetime dimension is $D = 2n+1$ in the odd-dimensional case, and $D = 2n$ in the even-dimensional case. In odd dimensions these parameters can be viewed as comprising n rotations, a mass, and $(n-2)$ NUT charges. In even dimensions they instead comprise $(n-1)$ rotations, a mass, and $(n-1)$ NUT charges. In odd dimensions, but not in even dimensions, there is some measure of arbitrariness in the interpretation of parameters as rotations or NUT charges.

An interesting feature of the Kerr-AdS and Kerr-NUT-AdS metrics that is uncovered by our work is that in all dimensions there exist discrete symmetries of the metrics in which one of the rotation parameters is inverted through the AdS radius $1/g$, together with appropriate scalings of the other rotation parameters, the mass and the NUT charges. An implication of these symmetries is that any metric with over-rotation, i.e. where one or more of the rotation parameters exceeds the AdS radius, is identical, up to coordinate transformations, to a metric with only under-rotation. This was observed in chapter II for the Kerr-AdS metric in $D = 5$. The inversion symmetry was apparently not previously noticed in the four-dimensional Kerr-AdS metric, and we have presented it explicitly, in the standard coordinate system, in appendix C.

We also considered the BPS, or supersymmetric, limits of the Kerr-NUT-AdS metrics. In odd dimensions these yield, after Euclideanisation, new examples of Einstein-Sasaki metrics. We expect that by making appropriate choices for the various parameters in the solutions, one can obtain new examples of complete Einstein-Sasaki spaces defined on non-singular compact manifolds.

CHAPTER IV

SEPARABILITY IN COHOMOGENEITY-2 KERR-NUT-ADS METRICS

A. Introduction

In order to obtain explicit solutions to the Einstein equations, or coupled Einstein/matter equations, it is generally necessary to make simplifying symmetry assumptions about the form the metric. In some cases, where a high degree of symmetry is assumed, this alone can be sufficient to render the reduced system of equations solvable. A typical example is when one considers an ansatz for cohomogeneity-1 metrics, meaning that the remaining metric functions depend non-trivially on only a single coordinate, and hence the Einstein equations reduce to a system of ordinary differential equations.

In more complicated circumstances, it may be that symmetries of a less manifest nature can play an important rôle in allowing one to construct an explicit solution to the Einstein equations. A nice example of this kind is provided by the Kerr solution for a four-dimensional rotating black hole [5]. This is a metric of cohomogeneity 2, with non-trivial coordinate dependence on both a radial and an angular variable. It was observed, after the original discovery of the solution, that it exhibits the remarkable property, associated with a "hidden symmetry," of allowing the separability of the Hamilton-Jacobi equation and the Klein-Gordon equation. In fact, it can be shown that the separability is related to the existence of a 2-index Killing tensor $K_{\mu\nu}$ in the Kerr geometry, satisfying $\nabla_{(\mu} K_{\nu\rho)} = 0$. By exploiting this property, and conjecturing that it would continue to hold for the more general situation with a cosmological constant and a NUT charge, Carter was able to construct the solution for a four-dimensional Kerr-NUT-AdS black hole [7].

It is of considerable interest to investigate the issue of separability in other gravitational solutions, including in particular solutions describing black holes in higher dimensions. Not only can this shed light on the existence of hidden symmetries, associated with the existence of Killing tensors, in the known black hole metrics; it can also point the way to constructing more general solutions with additional parameters.

In chapter II, we reviewed the general D-dimensional Kerr-AdS black hole constructed in [13, 14]. In this solution, there are $[(D-1)/2]$ independent rotation parameters a_i, characterising angular momenta in orthogonal spatial 2-planes. The general metrics are of cohomogeneity $[D/2]$, with principal orbits $\mathbb{R} \times U(1)^{[(D-1)/2]}$. It was shown in [34] that the Hamilton-Jacobi and Klein-Gordon equations are separable in all odd dimensions D, if one make the specialisation that all $(D-1)/2$ rotation parameters a_i are set equal. This has the effect of enhancing the symmetry of the principal orbits from $\mathbb{R} \times U(1)^{(D-1)/2}$ to $\mathbb{R} \times U((D-1)/2)$, and reducing the cohomogeneity from $(D-1)/2$ to 1. In fact, the enhanced manifest symmetry in this case is already sufficient to permit the separability, without the need for any additional hidden symmetry. Indeed, it was shown in [34] that the Killing tensor in this case is reducible, being a linear combination of direct products of Killing vectors. A non-trivial, irreducible, Killing tensor was found to exist in the case where all rotation parameters except one are vanishing [21]. Irreducible Killing tensors were also shown to exist in the special case of the five-dimensional asymptotically flat metric of [11], for arbitrary values of the two rotation parameters [35]. In the case of rotating AdS black holes in five dimensions, the separability, and associated irreducible Killing tensor, were found in [36].

A feature common to all the known cases exhibiting the phenomenon of separability is that the metric in question is of cohomogeneity ≤ 2. A natural next step, in the investigation of separability, is therefore to examine all the D-dimensional rotating black holes under the appropriate specialisation of parameters that reduces their cohomogeneity from $[D/2]$ to 2. In fact, the D-dimensional rotating AdS black holes with this specialisation were studied in chapter II, and it was shown that they admit a generalisation in which a NUT parameter is introduced. Specifically, the specialisation that reduces the cohomogeneity to 2 is achieved by taking sets of rotation parameters to be equal in an appropriate way.

In odd dimensions, $D = 2n + 1$, cohomogeneity 2 is achieved by dividing the $n = p + q$ rotation parameters a_i into two sets, with p of them equal to a, and the remaining q parameters equal to b. At the same time, the isometry group enlarges from $\mathbb{R} \times U(1)^{p+q}$ to $\mathbb{R} \times U(p) \times U(q)$.

In even dimensions $D = 2n$, cohomogeneity 2 is achieved by instead dividing the $n - 1 = p + q$ rotation parameters into a set of p that are taken to equal a, with the remaining q parameters taken to be zero. In this case, the isometry group enlarges from $\mathbb{R} \times U(1)^{p+q}$ to $\mathbb{R} \times U(p) \times SO(2q + 1)$.

In this chapter, we shall show that all these cohomogeneity-2 Kerr-AdS metrics have the property that the Hamilton-Jacobi equation and the Klein-Gordon equation are separable. Furthermore, we show that this property persists when the NUT parameter introduced in chapter II is included. We also obtain the 2-index Killing tensor $K_{\mu\nu}$ that is associated with the hidden symmetry responsible for allowing the equations to separate. Unlike the case of the further specialisation to cohomogeneity 1 in odd dimensions that was studied in [34], in these cohomogeneity-2 cases the Killing tensor is irreducible.

We also study some further properties of the cohomogeneity-2 Kerr-NUT-AdS metrics that were obtained in chapter II. In particular, we examine the case where one adjusts the NUT parameter so that the two adjacent roots of the metric function whose vanishing defines the endpoints of the range of one of the inhomogeneous coordinates become coincident. After appropriate scalings, this limit yields NUT-type metrics of cohomogeneity 1, which in some special cases coincide with NUT generalisations obtained previously in [28].

B. Separability in $D = 2n$ Dimensions

We copy the cohomogeneity-2 Kerr-NUT-AdS metric in $D = 2n$ dimensions here which was given in (2.33)

$$\begin{aligned} ds^2 &= \frac{r^2+v^2}{X}dr^2 + \frac{r^2+v^2}{Y}dv^2 - \frac{X}{r^2+v^2}\Big(dt - \frac{a^2-v^2}{a\,\Xi_a}(d\psi + A)\Big)^2 \\ &\quad + \frac{Y}{r^2+v^2}\Big(dt - \frac{a^2+r^2}{a\,\Xi_a}(d\psi + A)\Big)^2 + \frac{(a^2+r^2)(a^2-v^2)}{a^2\Xi_a}d\Sigma_{p-1}^2 + \frac{r^2v^2}{a^2}d\Omega_{2q}^2\,. \end{aligned} \tag{4.1}$$

Here, $d\Omega_{2q}^2$ is the metric on the unit sphere S^{2q}, $d\Sigma_{p-1}^2$ is the standard Fubini-Study metric on the "unit" complex projective space $\mathbb{CP}^{p-1}$ with Kähler form $J = \frac{1}{2}dA$, and the metric functions X and Y are given by

$$\begin{aligned} X &= (1+g^2r^2)(r^2+a^2) - \frac{2M\,r}{(r^2+a^2)^{p-1}\,r^{2q}}\,, \\ Y &= (1-g^2v^2)(a^2-v^2) - \frac{2L\,v}{(a^2-v^2)^{p-1}\,v^{2q}}\,. \end{aligned} \tag{4.2}$$

It should be noted that one can replace the unit-sphere metric $d\Omega_{2q}^2 = \gamma_{ij}\,dx^i dx^j$ by any $(2q)$-dimensional Einstein metric normalised to $R_{ij} = (2q-1)\,\gamma_{ij}$, and the Fubini-Study metric $d\Sigma_{p-1}^2 = h_{mn}\,dx^m dx^n$ on $\mathbb{CP}^{p-1}$ can be replaced by any Einstein-Kähler $(2p-2)$-metric normalised to $R_{mn} = 2p h_{mn}$, and one again has a local solution of the Einstein equations.

It is not hard to see that the inverse of the metric (4.1), which we can write as

$(\partial/\partial s)^2 \equiv g^{\mu\nu}\,\partial_\mu\partial_\nu$, is given by

$$\begin{aligned}(r^2+v^2)\Big(\frac{\partial}{\partial s}\Big)^2 \;=\;& X\Big(\frac{\partial}{\partial r}\Big)^2 + Y\Big(\frac{\partial}{\partial v}\Big)^2 - \frac{1}{X}\Big((r^2+a^2)\frac{\partial}{\partial t} + a\Xi_a\frac{\partial}{\partial\psi}\Big)^2 \\ & + \frac{1}{Y}\Big((a^2-v^2)\frac{\partial}{\partial t} + a\Xi_a\frac{\partial}{\partial\psi}\Big)^2 + \; a^2\big(\frac{1}{r^2}+\frac{1}{v^2}\big)\Big(\frac{\partial}{\partial\Omega}\Big)^2 \\ & - a^2\Xi_a\Big(\frac{1}{r^2+a^2} - \frac{1}{a^2-v^2}\Big)h^{mn}\Big(\partial_m - A_m\frac{\partial}{\partial\psi}\Big)\Big(\partial_n - A_n\frac{\partial}{\partial\psi}\Big)\,,\end{aligned} \tag{4.3}$$

where A_m are the components of the 1-form A, $(\frac{\partial}{\partial\Omega})^2 = \gamma^{ij}\partial_i\partial_j$ is the inverse of the metric on the unit $(2q)$-sphere, and h^{mn} are the components of the inverse of the Fubini-Study metric on $\mathbb{CP}^{p-1}$.

1. Separability of the Hamilton-Jacobi Equation

The covariant Hamiltonian function on the cotangent bundle of the metric (4.1) is given by

$$\mathcal{H}(\mathcal{P}_\mu, x^\mu) \equiv \tfrac{1}{2}g^{\mu\nu}\,\mathcal{P}_\mu\,\mathcal{P}_\nu\,, \tag{4.4}$$

where $\mathcal{P}_\mu$ are the canonical momenta conjugate to the coordinates x^μ. In terms of Hamilton's principle function S, one has $\mathcal{P}_\mu = \partial_\mu S$, and the Hamilton-Jacobi equation is given by

$$\mathcal{H}(\partial_\mu S, x^\mu) = -\tfrac{1}{2}\mu^2\,. \tag{4.5}$$

It is evident from (4.3) that the Hamilton-Jacobi equation admits separable solutions of the form

$$S = -Et + J_\psi\,\psi + F(r) + G(v) + P + Q\,, \tag{4.6}$$

where P is a function of the $\mathbb{CP}^{p-1}$ coordinates only, and Q is a function of the S^{2p} coordinates only. We introduce separation constants K_Σ and K_Ω for the functions on these spaces, so that

$$\Big(\frac{\partial Q}{\partial\Omega}\Big)^2 = K_\Omega^2\,,\qquad h^{mn}\,(\partial_m P - J_\psi\, A_m P)(\partial_n P - J_\psi\, A_n P) = K_\Sigma^2\,. \tag{4.7}$$

From the above, we can read off the remaining non-trivial equations for the functions $F(r)$ and $G(v)$ in (4.6), finding

$$\begin{aligned} -2\kappa &= X F'^2 - \frac{1}{X}\Big(E(r^2+a^2) - a\Xi_a J_\psi\Big)^2 + \frac{a^2 K_\Omega^2}{r^2} - \frac{a^2\Xi_a K_\Sigma^2}{r^2+a^2} + \mu^2 r^2\,,\\ 2\kappa &= Y\dot G^2 + \frac{1}{Y}\Big(E(a^2-v^2) - a\Xi_a J_\psi\Big)^2 + \frac{a^2 K_\Omega^2}{v^2} + \frac{a^2\Xi_a K_\Sigma^2}{a^2-v^2} + \mu^2 v^2\,, \end{aligned} \tag{4.8}$$

where F' denotes dF/dr, $\dot G$ denotes dG/dv, and κ is the separation constant associated with the non-trivial hidden symmetry that permits the separation of the Hamilton-Jacobi equation.

Note that the separation demonstrated thus far works equally well if $d\Sigma_{p-1}^2$ is any $(2p-2)$-dimensional Einstein-Kähler metric and $d\Omega_{2q}^2$ is any Einstein metric with the same scalar curvatures as the $\mathbb{CP}^{p-1}$ and S^{2q} metrics respectively. A complete separability, in which the functions P and Q are themselves fully separated, depends upon the complete separability of the Hamilton-Jacobi equations in these two spaces. In particular, this is possible whenever they are homogeneous spaces, as is the case for $\mathbb{CP}^{p-1}$ and S^{2q}. Note that in the case of the Einstein-Kähler space, the relevant Hamilton-Jacobi equation is the one describing a particle of charge J_ψ in geodesic motion, with minimal coupling to the potential A whose field strength is $2J$, where J is the Kähler form.

Following the discussion in [21], we note that associated with the separation constant κ is a Poisson function $\mathcal{K}$, which Poisson commutes with the Hamiltonian $\mathcal{H}$. The function $\mathcal{K}$ is equal to the separation constant κ if the Hamilton-Jacobi equations are satisfied, and so we can simply read it off from either of the equations in (4.8), or any linear combination thereof. Thus, for example, from the first equation in (4.8) we may read off

$$\begin{aligned}\mathcal{K} &= -\tfrac{1}{2}X\mathcal{P}_r^2 + \frac{1}{2X}\Big((r^2+a^2)\mathcal{P}_t + a\Xi_a\mathcal{P}_\psi\Big)^2 - \frac{a^2\mathcal{P}_\Omega^2}{2r^2} + \tfrac{1}{2}r^2 g^{\mu\nu}\,\mathcal{P}_\mu\mathcal{P}_\nu \\ &\quad + \frac{a^2\Xi_a}{2(r^2+a^2)}\,h^{mn}(\mathcal{P}_m - A_m\mathcal{P}_\psi)(\mathcal{P}_n - A_n\mathcal{P}_\psi)\,, \end{aligned} \tag{4.9}$$

where $\mathcal{P}_\Omega^2 \equiv \gamma^{ij}\mathcal{P}_i\mathcal{P}_j$ and γ^{ij} is the inverse metric on the unit sphere S^{2q}. An alternative way of writing $\mathcal{K}$, which puts the r and v coordinates on an equivalent footing, is to take the linear combination of the two equations in (4.8) that eliminates μ^2, yielding

$$\begin{aligned}\mathcal{K} &= \frac{1}{2(r^2+v^2)}\Big[\frac{v^2}{X}\,[(r^2+a^2)\mathcal{P}_t + a\Xi_a\mathcal{P}_\psi]^2 + \frac{r^2}{Y}\,[(a^2-v^2)\mathcal{P}_t + a\Xi_a\mathcal{P}_\psi]^2 \\ &\qquad -v^2X\mathcal{P}_r^2 + r^2Y\mathcal{P}_v^2\Big] + \frac{a^2}{2}\Big(\frac{1}{v^2} - \frac{1}{r^2}\Big)\mathcal{P}_\Omega^2 \\ &\quad + \frac{a^2+r^2-v^2}{2(r^2+a^2)(a^2-v^2)}\,h^{mn}(\mathcal{P}_m - A_m\mathcal{P}_\psi)(\mathcal{P}_n - A_n\mathcal{P}_\psi)\,, \end{aligned} \tag{4.10}$$

The function $\mathcal{K}$ defines a Stäckel-Killing tensor with components $K^{\mu\nu}$, given by

$$\mathcal{K} = \tfrac{1}{2}K^{\mu\nu}\,\mathcal{P}_\mu\mathcal{P}_\nu \tag{4.11}$$

Thus the components $K^{\mu\nu}$ can be read off trivially from (4.9) or (4.10) by inspection. The Stäckel-Killing tensor satisfies

$$\nabla_{(\mu}\,K_{\nu\rho)} = 0\,, \tag{4.12}$$

by virtue of the fact that $\mathcal{K}$ Poisson commutes with $\mathcal{H}$.

2. Separability of the Klein-Gordon Equation

The separability of the Klein-Gordon equation is closely related to that of the Hamilton-Jacobi equation. A key observation, which can easily be seen from (4.1), is that

$$\sqrt{-g} = a^{1-2p-2q}\, \Xi_a^{-p}\, (r^2+a^2)^{p-1}\, (a^2-v^2)^{p-1}\, r^{2q}\, v^{2q}\, \sqrt{h}\, \sqrt{\gamma}\, (r^2+v^2)\,. \qquad (4.13)$$

Aside from the factor (r^2+v^2), the coordinate dependence of $\sqrt{-g}$ therefore factorises into a product of a function of r, a function of v, a function of the S^{2q} coordinates and a function of the $\mathbb{CP}^{p-1}$ coordinates. Since the Laplacian is given by

$$\Box = \frac{1}{\sqrt{-g}}\, \partial_\mu \Big(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu \Big)\,, \qquad (4.14)$$

it follows from (4.3) that the Klein-Gordon equation $\Box f = \lambda f$ becomes

$$\begin{aligned}
&\frac{1}{(r^2+a^2)^{p-1}\, r^{2q}}\, \frac{\partial}{\partial r} \Big((r^2+a^2)^{p-1}\, r^{2q}\, X\, \frac{\partial f}{\partial r} \Big) \\
&+ \frac{1}{(a^2-v^2)^{p-1}\, v^{2q}}\, \frac{\partial}{\partial v} \Big((a^2-v^2)^{p-1}\, v^{2q}\, Y\, \frac{\partial f}{\partial v} \Big) \\
&- \frac{1}{X} \Big((r^2+a^2) \frac{\partial}{\partial t} + a\Xi_a\, \frac{\partial}{\partial \psi} \Big)^2 f + \frac{1}{Y} \Big((a^2-v^2) \frac{\partial}{\partial t} + a\Xi_a\, \frac{\partial}{\partial \psi} \Big)^2 f \qquad (4.15) \\
&+ a^2 \Big(\frac{1}{r^2} + \frac{1}{v^2} \Big)\, \frac{1}{\sqrt{\gamma}}\, \partial_i (\sqrt{\gamma}\, \gamma^{ij}\, \partial_j f) \\
&- a^2 \Xi_a \Big(\frac{1}{r^2+a^2} - \frac{1}{a^2-v^2} \Big)\, \frac{1}{\sqrt{h}}\, D_m (\sqrt{h}\, h^{mn}\, D_n f) = \lambda\, (r^2+v^2)\, f\,,
\end{aligned}$$

where $D_m \equiv \partial_m - A_m\, \partial/\partial\psi$. It is manifest that the equation can be separated by writing f as a product of functions of r, v, ψ, the S^{2q} coordinates and the $\mathbb{CP}^{p-1}$ coordinates. Of course the complete separability of the equation depends upon the fact that one can fully separate the Klein-Gordon equations on S^{2q} and $\mathbb{CP}^{p-1}$, by virtue of the homogeneity of these spaces.

C. Separability in $D = 2n + 1$ Dimensions

The cohomogeneity-2 Kerr-NUT-AdS metric in $D = 2n+1$ dimensions can be written in a "vielbein basis" as in (A.1),

$$\begin{aligned} ds^2 \;=\; & \frac{r^2+v^2}{X}\,dr^2 - \frac{X}{r^2+v^2}\Big[dt + v^2 d\phi - \frac{a(a^2-v^2)}{\Xi_a(a^2-b^2)}\,A - \frac{b(b^2-v^2)}{\Xi_b(b^2-a^2)}\,B\Big]^2 \\ & + \frac{r^2+v^2}{Y}\,dv^2 + \frac{Y}{r^2+v^2}\Big[dt - r^2 d\phi - \frac{a(r^2+a^2)}{\Xi_a(a^2-b^2)}\,A - \frac{b(r^2+b^2)}{\Xi_b(b^2-a^2)}\,B\Big]^2 \\ & + \frac{(r^2+a^2)(a^2-v^2)}{\Xi_a(a^2-b^2)}\,d\Sigma_{p-1}^2 + \frac{(r^2+b^2)(b^2-v^2)}{\Xi_b(b^2-a^2)}\,d\widetilde{\Sigma}_{q-1}^2 \\ & + \frac{a^2b^2}{r^2v^2}\Big[dt - (r^2-v^2)d\phi - r^2v^2 d\psi \\ & \qquad - \frac{(r^2+a^2)(a^2-v^2)}{a\Xi_a(a^2-b^2)}A - \frac{(r^2+b^2)(b^2-v^2)}{b\Xi_b(b^2-a^2)}B\Big]^2 \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} X &= \frac{(1+g^2r^2)(r^2+a^2)(r^2+b^2)}{r^2} - \frac{2M}{(r^2+a^2)^{p-1}\,(r^2+b^2)^{q-1}}\,, \\ Y &= \frac{-(1-g^2v^2)(a^2-v^2)(b^2-v^2)}{v^2} + \frac{2L}{(a^2-v^2)^{p-1}\,(b^2-v^2)^{q-1}}\,. \end{aligned} \tag{4.17}$$

Here, $d\Sigma_{p-1}^2$ and $d\widetilde{\Sigma}_{q-1}^2$ are the standard "unit" metrics on two complex projective spaces $\mathbb{CP}^{p-1}$ and $\mathbb{CP}^{q-1}$, with Kähler forms given locally by $J = \frac{1}{2}dA$ and $\widetilde{J} = \frac{1}{2}dB$. One can also obtain more general solutions by replacing the complex projective spaces with their Fubini-Study metrics by any other Einstein-Kähler metrics with the same Ricci scalars.

One can straightforwardly show that the inverse $(\partial/\partial s)^2$ of the metric (4.16) is given by

$$
\begin{aligned}
(r^2+v^2)\Big(\frac{\partial}{\partial s}\Big)^2 &= X\Big(\frac{\partial}{\partial r}\Big)^2 + Y\Big(\frac{\partial}{\partial v}\Big)^2 + \frac{1}{a^2b^2}\Big(\frac{1}{r^2}+\frac{1}{v^2}\Big)\Big(\frac{\partial}{\partial\psi}\Big)^2 \\
&\quad -\frac{1}{X}\left(r^2\frac{\partial}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial\psi}+\frac{\partial}{\partial\phi}\right)^2 + \frac{1}{Y}\left(v^2\frac{\partial}{\partial t}+\frac{1}{v^2}\frac{\partial}{\partial\psi}-\frac{\partial}{\partial\phi}\right)^2 \\
&\quad -(a^2-b^2)\,\Xi_a\left(\frac{1}{r^2+a^2}-\frac{1}{a^2-v^2}\right)h^{mn}\,D_m\,D_n \\
&\quad -(b^2-a^2)\,\Xi_b\left(\frac{1}{r^2+b^2}-\frac{1}{b^2-v^2}\right)\tilde h^{k\ell}\,\widetilde D_k\,\widetilde D_\ell\,, \qquad (4.18)
\end{aligned}
$$

where

$$
\begin{aligned}
D_m &\equiv \partial_m - \frac{a\,A_m}{(a^2-b^2)\Xi_a}\left(\frac{\partial}{\partial\phi}-a^2\frac{\partial}{\partial t}-\frac{1}{a^2}\frac{\partial}{\partial\psi}\right), \\
\widetilde D_k &\equiv \partial_k - \frac{b\,B_k}{(b^2-a^2)\Xi_b}\left(\frac{\partial}{\partial\phi}-b^2\frac{\partial}{\partial t}-\frac{1}{b^2}\frac{\partial}{\partial\psi}\right), \qquad (4.19)
\end{aligned}
$$

and h^{mn} and $\tilde h^{k\ell}$ are the inverses of the Fubini-Study metrics $d\Sigma^2_{p-1}$ and $d\widetilde\Sigma^2_{q-1}$ on the complex projective spaces $\mathbb{CP}^{p-1}$ and $\mathbb{CP}^{q-1}$.

1. Separability of the Hamilton-Jacobi Equation

Following analogous steps to those we described in section B, it can be seen that the Hamilton-Jacobi equation is separable, if we write the Hamilton principle function as

$$
S = -Et + J_\psi\,\psi + J_\phi\,\phi + F(r) + G(v) + P + \widetilde P\,, \qquad (4.20)
$$

where P depends only on the coordinates of $\mathbb{CP}^{p-1}$, and $\widetilde P$ depends only on the coordinates of $\mathbb{CP}^{q-1}$. We have separation constants K_Σ and $K_{\widetilde\Sigma}$ associated with the two complex projective space factors. The Hamilton-Jacobi equations in these two subspaces themselves describe particles of charges q and $\tilde q$ minimally coupled to the vector potentials A and B respectively, where

$$
q = \frac{a}{(a^2-b^2)\Xi_a}\,(J_\phi + a^2E - \frac{1}{a^2}J_\psi)\,, \qquad \tilde q = \frac{b}{(b^2-a^2)\Xi_b}\,(J_\phi + b^2E - \frac{1}{b^2}J_\psi)\,, \quad (4.21)
$$

and

$$h^{mn}\,(\partial_m P - qA_m)(\partial_n P - qA_n) = K_\Sigma^2\,, \qquad \tilde{h}^{k\ell}\,(\partial_k \widetilde{P} - \tilde{q}A_k)(\partial_\ell \widetilde{P} - \tilde{q}A_\ell) = K_{\widetilde{\Sigma}}^2\,. \quad (4.22)$$

From (4.18), it then follows that there is a further non-trivial separation constant κ, leading to the equations

$$\begin{aligned} -2\kappa &= XF'^2 + \frac{J_\psi^2}{a^2b^2r^2} - \frac{1}{X}\,(Er^2 - \frac{1}{r^2}J_\psi - J_\phi)^2 \\ &\quad -\frac{(a^2-b^2)\Xi_a\,K_\Sigma^2}{r^2+a^2} - \frac{(b^2-a^2)\Xi_b\,K_{\widetilde{\Sigma}}^2}{r^2+b^2} + \mu^2\,r^2\,, \\ 2\kappa &= Y\dot{G}^2 + \frac{J_\psi^2}{a^2b^2v^2} + \frac{1}{Y}\,(Ev^2 - \frac{1}{v^2}J_\psi + J_\phi)^2 \\ &\quad +\frac{(a^2-b^2)\Xi_a\,K_\Sigma^2}{a^2-v^2} + \frac{(b^2-a^2)\Xi_b\,K_{\widetilde{\Sigma}}^2}{b^2-v^2} + \mu^2\,v^2\,. \end{aligned} \quad (4.23)$$

We can then read off the associated Poisson function $\mathcal{K}$ that commutes with the Hamiltonian $\mathcal{H}$, and which takes the constant value κ upon use of the Hamilton-Jacobi equations. As in section B, one can organise the expression for $\mathcal{K}$ in different ways, depending on the choice of linear combination of the two expressions in (4.23) that one makes. Thus, for example, from the first expression we can write $\mathcal{K}$ as

$$\begin{aligned} \mathcal{K} &= -\tfrac12 X\mathcal{P}_r^2 - \frac{1}{2a^2b^2r^2}\,\mathcal{P}_\psi^2 + \frac{1}{2X}\,(r^2\mathcal{P}_t + \frac{1}{r^2}\mathcal{P}_\psi + \mathcal{P}_\phi)^2 \\ &\quad +\frac{(a^2-b^2)\Xi_a}{2(r^2+a^2)}\,\mathcal{P}_\Sigma^2 + \frac{(b^2-a^2)\Xi_b}{2(r^2+b^2)}\,\mathcal{P}_{\widetilde{\Sigma}}^2 + \tfrac12\,r^2\,g^{\mu\nu}\,\mathcal{P}_\mu\mathcal{P}_\nu\,, \end{aligned} \quad (4.24)$$

where

$$\begin{aligned} \mathcal{P}_\Sigma^2 &\equiv h^{mn}\,[\mathcal{P}_m - \frac{aA_m}{(a^2-b^2)\Xi_a}\,(\mathcal{P}_\phi - a^2\mathcal{P}_t - \frac{1}{a^2}\mathcal{P}_\psi)][\mathcal{P}_n - \frac{aA_n}{(a^2-b^2)\Xi_a}\,(\mathcal{P}_\phi - a^2\mathcal{P}_t - \frac{1}{a^2}\mathcal{P}_\psi)]\,, \\ \mathcal{P}_{\widetilde{\Sigma}}^2 &\equiv \tilde{h}^{k\ell}\,[\mathcal{P}_k - \frac{bB_k}{(b^2-a^2)\Xi_b}\,(\mathcal{P}_\phi - b^2\mathcal{P}_t - \frac{1}{b^2}\mathcal{P}_\psi)][\mathcal{P}_\ell - \frac{bB_\ell}{(b^2-a^2)\Xi_a}\,(\mathcal{P}_\phi - b^2\mathcal{P}_t - \frac{1}{b^2}\mathcal{P}_\psi)]. \end{aligned} \quad (4.25)$$

The components of the associated Killing tensor $K_{\mu\nu}$ can be read off directly from (4.24), via $\mathcal{K} = \frac12 K^{\mu\nu}\mathcal{P}_\mu\mathcal{P}_\nu$. Again, as in the even-dimensional case discussed in section B, one can equivalently express $\mathcal{K}$ in a more symmetrical fashion by taking the linear combination of the two equations in (4.23) that eliminates μ^2.

2. Separability of the Klein-Gordon Equation

As in the case of even dimensions, here too the separability of the Klein-Gordon equation is closely related to the separability of the Hamilton-Jacobi equation. Again, the key point is that $\sqrt{-g}$ has a simple form, being proportional to (r^2+v^2) times a product of functions of r, v and the coordinates on the two complex projective spaces:

$$\sqrt{-g} = \frac{ab\,r\,v\,\sqrt{h}\,\sqrt{\tilde{h}}}{|a^2-b^2|^{n-2}\,\Xi_a^{p-1}\,\Xi_b^{q-1}}\,(r^2{+}a^2)^{p-1}\,(r^2{+}b^2)^{q-1}\,(a^2{-}v^2)^{p-1}\,(b^2{-}v^2)^{q-1}\,(r^2{+}v^2)\,. \tag{4.26}$$

Together with the the expression (4.18) for the inverse metric, we see that the Klein-Gordon equation $\Box f = \lambda f$ assumes the manifestly separable form

$$\begin{aligned}
&\frac{1}{r(r^2+a^2)^{p-1}(r^2+b^2)^{q-1}}\,\frac{\partial}{\partial r}\Big(r(r^2+a^2)^{p-1}(r^2+b^2)^{q-1}\,X\,\frac{\partial f}{\partial r}\Big)\\
&\frac{1}{v(a^2-v^2)^{p-1}(b^2-v^2)^{q-1}}\,\frac{\partial}{\partial v}\Big(v(a^2-v^2)^{p-1}(b^2-v^2)^{q-1}\,Y\,\frac{\partial f}{\partial v}\Big)\\
&-\frac{1}{X}\Big(r^2\frac{\partial}{\partial t}+\frac{1}{r^2}\,\frac{\partial}{\partial\psi}+\frac{\partial}{\partial\phi}\Big)^2 f+\frac{1}{Y}\Big(v^2\frac{\partial}{\partial t}+\frac{1}{v^2}\,\frac{\partial}{\partial\psi}-\frac{\partial}{\partial\phi}\Big)^2 f+\frac{1}{a^2b^2}\Big(\frac{1}{r^2}+\frac{1}{v^2}\Big)\frac{\partial^2 f}{\partial\psi^2}\\
&-(a^2-b^2)\Xi_a\Big(\frac{1}{r^2+a^2}-\frac{1}{a^2-v^2}\Big)\,\frac{1}{\sqrt{h}}\,D_m(\sqrt{h}\,h^{mn}D_n f)\\
&-(b^2-a^2)\Xi_b\Big(\frac{1}{r^2+b^2}-\frac{1}{b^2-v^2}\Big)\,\frac{1}{\sqrt{\tilde{h}}}\,\widetilde{D}_k(\sqrt{\tilde{h}}\,\tilde{h}^{k\ell}\widetilde{D}_\ell f)=\lambda(r^2+v^2)f\,.
\end{aligned} \tag{4.27}$$

Note that D_m and $\widetilde{D}_k$, defined in (4.19), yield gauge covariant derivatives acting on charged wavefunctions in the two complex projective spaces, once one separates variables by writing f as a product of functions of the coordinates. As in the previous discussions, the complete separability of the system depends upon the separability of the Klein-Gordon equations in the complex projective spaces.

D. Specialisation to NUT Metrics of Cohomogeneity 1

The NUT generalisations of the Kerr-AdS metrics that were found in chapter II all have cohomogeneity 2, and, as we have shown in this chapter, they all share the feature that the Hamilton-Jacobi equation and the Klein-Gordon equation are separable in these backgrounds. It is also of interest to see how these cohomogeneity-2 Kerr-NUT-AdS metrics reduce to certain previously-known solutions under specialisations of the parameters. In particular, we shall show that if one applies a limiting procedure in which the cohomogeneity is reduced from 2 to 1, then the resulting metrics include some higher-dimensional NUT metrics that were obtained in [28]. As usual, the discussion divides into the cases of even-dimensional metrics and odd-dimensional metrics.

1. $D = 2n$

Our starting point is the class of even-dimensional cohomogeneity-2 Kerr-NUT-AdS metrics that were given in (4.1). The cohomogeneity can be reduced from 2 to 1 by specialising the parameters in such a way that the two adjacent roots of the function $Y(v)$ that define the range of the v coordinate become coincident. Provided the v coordinate is rescaled appropriately as the limit is taken, one obtains a non-singular metric that now no longer has any dependence on the rescaled v coordinate.

The function $Y(v)$ acquires a double root, at $v = v_0$, if the parameters a and L are chosen to satisfy

$$\begin{aligned} L &= L_0 \equiv \frac{(a^2 - v_0^2)^{p+1} v_0^{2q-1}}{(2q+1)a^2 - (2p+2q+1)v_0^2} \,, \\ g^2 &= \frac{(2q-1)a^2 - (2p+2q-1)v_0^2}{v_0^2((2q+1)a^2 - (2p+2q+1)v_0^2)} \,. \end{aligned} \tag{4.28}$$

In order to approach this limit with an appropriately rescaled v coordinate, we define

$$v = v_0 + \epsilon \cos\chi \,, \qquad L = L_0(1 + \epsilon^2 c) \,, \tag{4.29}$$

with the constant c given by

$$c = \frac{a^4(1-4q^2) + 2a^2(2q-1)(2p+2q+1)v_0^2 + (1-4(p+q)^2)v_0^4}{2(a^2-v_0^2)^2 v_0^2} \,, \tag{4.30}$$

where ϵ will shortly be sent to zero. The function Y under this limit becomes

$$Y = \epsilon^2 Y_0 \sin^2\chi \,, \tag{4.31}$$

where

$$Y_0 = \frac{2(a-v_0)^2 c}{(2p+2q+1)v_0^2 - (2q+1)a^2} \,. \tag{4.32}$$

In order for the metric (4.1) to be nonsingular in the limit, we must also make the coordinate transformations

$$\psi \to \frac{a\,\Xi_a}{\epsilon\, Y_0}\tilde{\psi} \,, \qquad t \to t + \frac{a^2 - v_0^2}{\epsilon\, Y_0}\,\psi \,. \tag{4.33}$$

Sending ϵ to zero, the metric (4.1) then becomes

$$\begin{aligned} ds^2 &= -\frac{X}{r^2+v_0^2}\Big(dt + \frac{2v_0}{Y_0}\cos\chi\, d\psi - \frac{a^2-v_0^2}{a\,\Xi_a}A\Big)^2 + \frac{r^2+v_0^2}{X}dr^2 \\ &\quad + \frac{(r^2+v_0^2)}{Y_0}(d\chi^2 + \sin^2\chi\, d\psi^2) + \frac{r^2+a^2}{a^2\,\Xi_a}d\Sigma_{p-1}^2 + \frac{r^2 v_0^2}{a^2}d\Omega_{2q}^2 \,. \end{aligned} \tag{4.34}$$

The metrics (4.34) are contained within a rather general class of cohomogeneity-1 NUT metrics that were obtained in [28]. The case $p = 1$ and $q = 0$ reduces to the standard Taub-NUT-AdS metric in four dimensions.

2. $D = 2n + 1$

In odd dimensions, our starting point is the cohomogeneity-2 Kerr-NUT-AdS metrics presented in equation (4.16).

Proceeding in an analogous fashion to the discussion we gave in even dimensions, we first consider the conditions under which Y has a double root, at $v = v_0$. This happens when the constants L, a and b are chosen such that

$$\begin{aligned} L &= L_0 \equiv \frac{(a^2 - v_0^2)^p (b^2 - v_0^2)^q (1 - g^2 v_0^2)}{2 v_0^2} \,, \\ g^2 &= \frac{a^2 b^2 + (a^2(q-1) + b^2(p-1)) v_0^2 - (p+q-1) v_0^4}{(a^2 q + b^2 p - (p+q) v_0^2) v_0^4} \,. \end{aligned} \tag{4.35}$$

Next, we deform away slightly from the double root, and introduce a new coordinate χ in place of v:

$$v = v_0 + \epsilon \cos\chi \,, \qquad L = L_0 (1 + \epsilon^2 c) \,, \tag{4.36}$$

where the constant c is given by

$$\begin{aligned} c &= \frac{2}{(a^2 - v_0^2)^2 (b^2 - v_0^2)^2} \Big(-2a^2 b^2 (a^2 q + b^2 p) - (a^4 q(q-1) + b^4 p(p-1) \\ &\quad + 2a^2 b^2 (pq - 2p - 2q)) v_0^2 + 2(p+q)(a^2(q-1) + b^2(p-1)) v_0^4 \\ &\quad - (p+q)(p+q-1) v_0^6 \Big) \,. \end{aligned} \tag{4.37}$$

The function Y in this limit becomes

$$Y = \epsilon^2 Y_0 \sin^2\chi \,, \tag{4.38}$$

where

$$Y_0 = \frac{(a^2 - v_0^2)^2 (b^2 - v_0^2)^2 \, c}{v_0^4 ((p+q) v_0^2 - a^2 q - b^2 p)} \,. \tag{4.39}$$

Making the further coordinate transformation

$$t \to t - \frac{v_0^2\phi}{\epsilon\, Y_0}\,, \qquad \phi \to \frac{\phi}{\epsilon\, Y_0}\,, \qquad \psi \to -\frac{\phi}{v_0^2\epsilon\, Y_0} - \frac{\psi}{v_0^4}\,, \tag{4.40}$$

we can now obtain a smooth limit in which ϵ is sent to zero, for which the metric (4.16) becomes

$$\begin{aligned} ds^2 &= -\frac{X}{r^2+v_0^2}\Big[dt + \frac{2v_0}{Y_0}\cos\chi\, d\phi - \frac{a(a^2-v_0^2)}{\Xi_a(a^2-b^2)}A - -\frac{b(b^2-v_0^2)}{\Xi_b(b^2-a^2)}B\Big]^2 \\ &\quad + \frac{a^2b^2}{r^2v_0^2}\Big[dt + \frac{2v_0}{Y_0}\cos\chi\, d\phi + \frac{r^2}{v_0^2}(d\psi + \frac{2v_0}{Y_0}\cos\chi\, d\phi) - \frac{(r^2+a^2)(a^2-v_0^2)}{a\Xi_a(a^2-b^2)}A \\ &\quad - \frac{(r^2+b^2)(b^2-v_0^2)}{b\Xi_b(b^2-a^2)}B\Big]^2 + \frac{r^2+v_0^2}{X}dr^2 + \frac{r^2+v_0^2}{Y_0}(d\chi^2 + \sin^2\chi\, d\phi^2) \\ &\quad + \frac{(r^2+a^2)(a^2-v_0^2)}{\Xi_a(a^2-b^2)}d\Sigma_{p-1}^2 + \frac{(r^2+b^2)(b^2-v_0^2)}{\Xi_b(b^2-a^2)}d\widetilde{\Sigma}_{q-1}^2\,. \end{aligned} \tag{4.41}$$

This metric is contained within the class of cohomogeneity-1 NUT generalisations that were considered in [28].

E. Conclusions

The separability of the Hamilton-Jacobi and Klein-Gordon equations in the background of a rotating four-dimensional black hole played an important rôle in the construction of generalisations of the Kerr metric, and in the uncovering of hidden symmetries associated with the existence of Killing tensors. In this chapter, we have shown that the Hamilton-Jacobi and Klein-Gordon equations are separable in Kerr-AdS backgrounds in all dimensions, if one specialises the rotation parameters so that the metrics have cohomogeneity 2. Furthermore, we have shown that this property of separability extends to the NUT generalisations of these cohomogeneity-2 black holes that we obtained in chapter II. In all these cases, we also constructed the associated irreducible rank-2 Killing tensor whose existence reflects the hidden symmetry that leads to the separability. We also considered some cohomogeneity-1 specialisations of the new Kerr-NUT-AdS metrics, and showed how they relate to previous results in the literature [28].

The results on separability that we have obtained in this chapter raise the interesting question of whether it might extend to the higher-dimensional rotating black holes with more general choices for the rotation parameters, and thus having cohomogeneity larger than 2. This question was finally answered by [37], in which Frolov, Krtous and Kubiznak proved the separability of the Hamilton-Jacobi and Klein-Gordon equations in the general ($D \geq 4$) cohomogeneity-n Kerr-NUT-AdS spacetimes presented in chapter III.

CHAPTER V

KERR-SCHILD STRUCTURE AND HARMONIC 2-FORMS ON KERR-NUT-ADS METRICS

A. Introduction

One intriguing feature of General Relativity is that, despite its high degree of non-linearity, many exact solutions can be cast into a Kerr-Schild form [38] where non-trivial parameters such as mass, charge, or cosmological constant enter the metrics as a linear perturbation of flat spacetime. A simple example is the (A)dS metric, which can be written as

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_n^2 + \Lambda\, r^2 (dt - dr)^2\,, \tag{5.1}$$

where the first three terms describe the $(n+2)$-dimensional Minkowski spacetime and the cosmological constant enters the last term linearly. More complicated examples include the Plebanski metric [31]; in (2,2) signature, the Plebanski metric can have a double Kerr-Schild form where both the mass and the NUT charge enter the metric linearly [21].

The general cohomogeneity-n Kerr-NUT-AdS solutions in chapter III can be viewed as higher-dimensional generalisations of the Plebanski metric. The solutions are parameterised by the mass, multiple NUT charges and arbitrary orthogonal rotations. In this chapter, we demonstrate in section B that the D-dimensional Kerr-NUT-AdS solution admits $[D/2]$ linearly-independent, mutually-orthogonal and affinely parameterised null geodesic congruences upon Wick-rotation of the metric to $([D/2], [(D+1)/2])$ signature. This enables us to cast the metric into the multi-Kerr-Schild form, where the mass and all of the NUT parameters enter the metric linearly. In section C, we obtain n harmonic 2-forms on the Kerr-NUT-AdS metrics in $D = 2n$ dimensions. In the BPS limit, these n harmonic 2-forms becomes linearly dependent, and the number of linearly-independent ones becomes $n-1$. However, a Kähler 2-form emerges under the BPS limit, and hence the total number of harmonic 2-forms remains n. We conclude the chapter in section D.

B. Multi-Kerr-Schild Structure

Let us first consider the case of $D = 2n+1$ dimensions, for which the metric was given in (3.22). In order to put the metric in a Kerr-Schild form, it is necessary to Wick rotate to $(n, n+1)$ signature. This can be easily achieved by Wick rotating all the spatial $U(1)$ coordinates. The corresponding metric is then given by

$$ds^2 = \sum_{\mu=1}^{n} \Big\{ \frac{dx_\mu^2}{Q_\mu} - Q_\mu \Big(\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k \Big)^2 \Big\} + \frac{c}{(\prod_{\nu=1}^{n} x_\nu^2)} \Big(\sum_{k=0}^{n} A^{(k)}\, d\psi_k \Big)^2 , \qquad (5.2)$$

where

$$Q_\mu = \frac{X_\mu}{U_\mu} , \qquad U_\mu = {\prod}'^{n}_{\nu=1} (x_\nu^2 - x_\mu^2) , \qquad X_\mu = \sum_{k=1}^{n} c_k\, x_\mu^{2k} + \frac{c}{x_\mu^2} - 2b_\mu ,$$
$$A_\mu^{(k)} = \sum'_{\nu_1<\nu_2<\cdots<\nu_k} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 , \qquad A^{(k)} = \sum_{\nu_1<\nu_2\cdots<\nu_k} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2 , \qquad (5.3)$$

The prime on the summation and product symbols in the definition of $A_\mu^{(k)}$ and U_μ indicates that the index value μ is omitted in the summations of the ν indices over the range $[1, n]$. Note that ψ_0 was denoted as t in chapter III, playing the rôle of the time like coordinate in the $(1, 2n)$ spacetime signature. In this way of writing the metric, all of the integration constants of the solution enter only in the functions X_μ. The constant $c_n = (-1)^{n+1} g^2$ is fixed by the value of the cosmological constant, with $R_{\mu\nu} = -2ng^2\, g_{\mu\nu}$. The other $2n$ constants c_k, c and b_μ are arbitrary. These are related to the n rotation parameters, the mass and the $(n-1)$ NUT parameters, with one parameter being trivial and removable through a scaling symmetry, as shown in chapter III. Note that in $(n, n+1)$ signature, the NUT charges are really masses with respect to different time-like Killing vectors. However, we shall continue to refer them as NUT charges.

We now re-arrange the metric (5.2) into the form

$$\begin{aligned} ds^2 &= -\sum_{\mu=1}^{n} \frac{X_\mu}{U_\mu}\Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k + \frac{U_\mu}{X_\mu} dx_\mu\Big]\Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k - \frac{U_\mu}{X_\mu} dx_\mu\Big] \\ &\quad + \frac{c}{(\prod_{\nu=1}^{n} x_\nu^2)}\Big(\sum_{k=0}^{n} A^{(k)}\, d\psi_k\Big)^2 . \end{aligned} \tag{5.4}$$

If we perform the following coordinate transformation,

$$d\hat{\psi}_k = d\psi_k + \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-k-1}}{X_\mu} dx_\mu\,, \qquad k = 0\,, \cdots, n\,, \tag{5.5}$$

the metric can then be cast into the n-Kerr-Schild form, namely

$$ds^2 = d\bar{s}^2 + \sum_{\mu=1}^{n} \frac{2b_\mu}{U_\mu}\Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\hat{\psi}_k\Big]^2 , \tag{5.6}$$

where

$$\begin{aligned} d\bar{s}^2 &= -\sum_{\mu=1}^{n} \Big\{\frac{\bar{X}_\mu}{U_\mu}\Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\hat{\psi}_k\Big]^2 - 2\Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\hat{\psi}_k\Big] dx_\mu\Big\} \\ &\quad + \frac{c}{(\prod_{\nu=1}^{n} x_\nu^2)}\Big(\sum_{k=0}^{n} A^{(k)}\, d\hat{\psi}_k\Big)^2 , \\ \bar{X}_\mu &= \sum_{k=1}^{n} c_k\, x_\mu^{2k} + \frac{c}{x_\mu^2} . \end{aligned} \tag{5.7}$$

It is straightforward to verify that the metric $d\bar{s}^2$ is that of pure AdS spacetime. The mass and NUT parameters b_μ appear linearly in the metric ds^2. It should be emphasised that although the constants c and c_k with $k < n$ are trivial in the metric

$d\bar{s}^2$, they provide non-trivial angular momentum parameters in the metric ds^2. It is interesting to note that all of the constants c_k, including c_n that is related to the cosmological constant, appear linearly in the metric, and can all be extracted from $d\bar{s}^2$ and grouped in the second term of (5.6). This implies that all the parameters, the mass, NUTs and angular momenta and cosmological constant can enter the metric linearly as a perturbation of flat spacetime. In this chapter, we shall consider in detail only the Kerr-Schild form where the mass and NUT parameters enter the metric linearly as a perturbation of pure AdS spacetime.

The AdS metric (5.7) can be diagonalised, in a way that the second term of (5.6) remains simple. To do so, let us first rewrite the $\bar{X}_\mu$ as follows

$$\bar{X}_\mu = \frac{(1 - g^2\, x_\mu^2)}{x_\mu^2} \prod_{k-1}^{n} (a_k^2 - x_\mu^2)\,. \tag{5.8}$$

Then we complete the square in $d\bar{s}^2$:

$$d\bar{s}^2 = \sum_{\mu=1}^{n} \Big\{ \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \frac{\bar{X}_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\hat{\psi}_k - \frac{U_\mu}{\bar{X}_\mu} dx_\mu \Big]^2 \Big\} + \frac{c}{(\prod_{\nu=1}^{n} x_\nu^2)} \Big(\sum_{k=0}^{n} A^{(k)}\, d\hat{\psi}_k \Big)^2 , \tag{5.9}$$

and make the coordinate transformation,

$$d\tilde{\psi}_k = -d\hat{\psi}_k + \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-k-1}}{\bar{X}_\mu} dx_\mu\,, \qquad k \;=\; 0\,, \cdots, n\,. \tag{5.10}$$

The metric can be put into a new form,

$$ds^2 = d\bar{s}^2 + \sum_{\mu=1}^{n} \frac{2b_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\tilde{\psi}_k - \frac{U_\mu}{\bar{X}_\mu} dx_\mu \Big]^2 , \tag{5.11}$$

where

$$d\bar{s}^2 = \sum_{\mu=1}^{n} \Big\{ \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \frac{\bar{X}_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\tilde{\psi}_k \Big]^2 \Big\} + \frac{c}{(\prod_{\nu=1}^{n} x_\nu^2)} \Big(\sum_{k=0}^{n} A^{(k)}\, d\tilde{\psi}_k \Big)^2 . \tag{5.12}$$

Performing a recombination of the $U(1)$ coordinates, namely

$$\tau=\sum_{k=0}^{n}B^{(k)}\,d\tilde{\psi}_k\,,\qquad \frac{\varphi_i}{a_i}=\sum_{k=1}^{n}B_i^{(k-1)}d\tilde{\psi}_k+g^2\sum_{k=0}^{n-1}B_i^{(k)}\,d\tilde{\psi}_k\,,\qquad i\,=\,1\,,\,\cdots\,,n\,, \tag{5.13}$$

where

$$B_i^{(k)}=\sideset{}{'}\sum_{j_1<j_2<\cdots<j_k}a_{j_1}^2a_{j_2}^2\cdots a_{j_k}^2\,,\qquad B^{(k)}=\sum_{j_1<j_2\cdots<j_k}a_{j_1}^2a_{j_2}^2\cdots a_{j_k}^2\,, \tag{5.14}$$

the odd dimensional Kerr-NUT-AdS metrics can be expressed as

$$ds^2=d\bar{s}^2+\sum_{\mu=1}^{n}\frac{2b_\mu}{U_\mu}(k_{(\mu)\alpha}dx^\alpha)^2\,, \tag{5.15}$$

$$d\bar{s}^2 = \frac{W}{\prod_{i=1}^n\Xi_i}d\tau^2+\sum_{\mu=1}^{n}\frac{U_\mu}{\bar{X}_\mu}\,dx_\mu^2-\sum_{i=1}^{n}\frac{\gamma_i}{\Xi_i{\prod'}_{k=1}^{n}(a_i^2-a_k^2)}d\varphi_i^2\,, \tag{5.16}$$

$$k_{(\mu)\alpha}dx^\alpha = \frac{W}{1-g^2\,x_\mu^2}\frac{d\tau}{\prod_{i=1}^n\Xi_i}-\frac{U_\mu\,dx_\mu}{\bar{X}_\mu}-\sum_{i=1}^{n}\frac{a_i\,\gamma_i d\varphi_i}{(a_i^2-x_\mu^2)\Xi_i{\prod'}_{k=1}^{n}(a_i^2-a_k^2)}, \tag{5.17}$$

where

$$\Xi_i=1-g^2\,a_i^2\,,\qquad \gamma_i=\prod_{\nu=1}^{n}(a_i^2-x_\nu^2)\,,\qquad W=\prod_{\nu=1}^{n}(1-g^2\,x_\nu^2)\,. \tag{5.18}$$

If we set all but one of the b_μ to zero, the result reduces to the Kerr-Schild form for rotating AdS black holes obtained previously in [13].

We now turn our attention to the the case of $D=2n$ dimensions. The corresponding Kerr-NUT-AdS metrics were obtained in (3.33). After performing Wick rotations, the metric with (n,n) signature is given by

$$ds^2=\sum_{\mu=1}^{n}\Big\{\frac{dx_\mu^2}{Q_\mu}-Q_\mu\Big(\sum_{k=0}^{n-1}A_\mu^{(k)}\,d\psi_k\Big)^2\Big\}\,, \tag{5.19}$$

where we Q_μ, U_μ and $A_\mu^{(k)}$ have the same form as those in the odd dimensions, given in (5.3). The functions X_μ are given by

$$X_\mu = \sum_{k=0}^{n} c_k\, x_\mu^{2k} + 2b_\mu\, x_\mu\,. \tag{5.20}$$

The constants c_k and b_μ are arbitrary, except for $c_n = (-1)^{n+1} g^2$, which is fixed by the value of the cosmological constant, $R_{\mu\nu} = -(2n-1)g^2\, g_{\mu\nu}$. The metric can be re-arranged into the form

$$ds^2 = -\sum_{\mu=1}^{n} \frac{X_\mu}{U_\mu}\Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k + \frac{U_\mu}{X_\mu} dx_\mu\Big]\,\Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k - \frac{U_\mu}{X_\mu} dx_\mu\Big]\,. \tag{5.21}$$

After performing the coordinate transformation

$$d\hat{\psi}_k = d\psi_k + \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-k-1}}{X_\mu} dx_\mu\,, \qquad k\ =\ 0\,,\ \cdots\ , n-1\,, \tag{5.22}$$

the metric can be cast into the n-Kerr-Schild form,

$$ds^2 = d\bar{s}^2 - \sum_{\mu=1}^{n} \frac{2b_\mu x_\mu}{U_\mu}\Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\hat{\psi}_k\Big]^2 \tag{5.23}$$

where

$$\begin{aligned} d\bar{s}^2 &= -\sum_{\mu=1}^{n}\Big\{\frac{\bar{X}_\mu}{U_\mu}\Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\hat{\psi}_k\Big]^2 - 2\Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\hat{\psi}_k\Big] dx_\mu\Big\}\,, \\ \bar{X}_\mu &= \sum_{k=0}^{n} c_k\, x_\mu^{2k}\,. \end{aligned} \tag{5.24}$$

It is straightforward to verify that $d\bar{s}^2$ is the metric for pure AdS spacetime. As in the odd dimensions, this metric can be put into a diagonal form, while keeping the second term of (5.23) simple. To do that, we first reparameterise X_μ as

$$\bar{X}_\mu = -(1-g^2 x_\mu^2)\prod_{k=1}^{n-1}(a_k^2 - x_\mu^2)\,. \tag{5.25}$$

We then complete the square in $d\bar{s}^2$, *i.e.*

$$d\bar{s}^2 = \sum_{\mu=1}^{n} \Big\{ \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \frac{\bar{X}_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\hat{\psi}_k - \frac{U_\mu}{\bar{X}_\mu} dx_\mu \Big]^2 \Big\} \tag{5.26}$$

and make the coordinate transformation

$$d\tilde{\psi}_k = -d\hat{\psi}_k + \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-k-1}}{\bar{X}_\mu} dx_\mu \,, \qquad k \;=\; 0\,,\; \cdots\,, n-1\,. \tag{5.27}$$

The metric (5.23) can then be put into a new form:

$$ds^2 = d\bar{s}^2 - \sum_{\mu=1}^{n} \frac{2b_\mu x_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\tilde{\psi}_k - \frac{U_\mu}{\bar{X}_\mu} dx_\mu \Big]^2 \,, \tag{5.28}$$

where

$$d\bar{s}^2 = \sum_{\mu=1}^{n} \Big\{ \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \frac{\bar{X}_\mu}{U_\mu} \Big[\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\tilde{\psi}_k \Big]^2 \Big\} \,. \tag{5.29}$$

The $d\bar{s}^2$ metric can now straightforwardly be diagonalised by means of the coordinate transformation

$$\tau = \sum_{k=0}^{n-1} B^{(k)}\, d\tilde{\psi}_k \,, \qquad \frac{\varphi_i}{a_i} = \sum_{k=1}^{n-1} B_i^{(k-1)} d\tilde{\psi}_k + g^2 \sum_{k=0}^{n-2} B_i^{(k)}\, d\tilde{\psi}_k \qquad i \;=\; 1\,,\; \cdots\,, n-1\,, \tag{5.30}$$

where

$$B_i^{(k)} = \sum_{j_1<j_2<\cdots<j_k}{}^{'} a_{j_1}^2 a_{j_2}^2 \cdots a_{j_k}^2 \,, \qquad B^{(k)} = \sum_{j_1<j_2\cdots<j_k} a_{j_1}^2 a_{j_2}^2 \cdots a_{j_k}^2 \,. \tag{5.31}$$

The even dimensional Kerr-NUT-AdS metrics can now be expressed as

$$ds^2 = d\bar{s}^2 - \sum_{\mu=1}^{n} \frac{2b_\mu x_\mu}{U_\mu} (k_{(\mu)\alpha} dx^\alpha)^2 \,, \tag{5.32}$$

where

$$d\bar{s}^2 = \frac{W}{\prod_{i=1}^{n-1}\Xi_i}\,d\tau^2 + \sum_{\mu=1}^{n}\frac{U_\mu}{\bar{X}_\mu}\,dx_\mu^2 - \sum_{i=1}^{n-1}\frac{\gamma_i}{a_i^2\Xi_i\prod'^{\,n-1}_{k=1}(a_i^2-a_k^2)}d\varphi_i^2\,, \tag{5.33}$$

$$k_{(\mu)\alpha}dx^\alpha = \frac{W}{1-g^2x_\mu^2}\frac{d\tau}{\prod_{i=1}^{n-1}\Xi_i} - \frac{U_\mu dx_\mu}{\bar{X}_\mu} - \sum_{i=1}^{n-1}\frac{\gamma_i d\varphi_i}{(a_i^2-x_\mu^2)a_i\Xi_i\prod'^{\,n-1}_{k=1}(a_i^2-a_k^2)}\,, \tag{5.34}$$

where Ξ_i, γ_i and W have the same structure as that in the odd dimensions, given by (5.18). When all but one of the b_μ vanishes, the metric reduces to the Kerr-Schild form of the rotating AdS black hole obtained in [13].

To summarise, we find that in both even and odd dimensions, the Kerr-NUT-AdS solution can be cast into the following multi-Kerr-Schild form:

$$ds^2 = d\bar{s}^2 + \sum_{\mu=1}^{n}\frac{2b_\mu\,f(x_\mu)}{U_\mu}(k_{(\mu)\alpha}dx^\alpha)^2\,, \tag{5.35}$$

where $f(x_\mu) = 1$ for odd dimensions and $f(x_\mu) = x_\mu$ for even dimensions. The vectors $k_{(\mu)\alpha}$ are n linearly-independent, mutually-orthogonal and affinely-parameterised null geodesic congruences, satisfying

$$k_{(\mu)\alpha}k^\alpha_{(\nu)} = 0\,, \qquad k^\alpha_{(\mu)}\bar{\nabla}_\alpha k_{(\mu)\beta} = 0\,. \tag{5.36}$$

Note that the index α in $k_{\alpha(\mu)}$ can be raised with either $g^{\alpha\beta}$ or $\bar{g}^{\alpha\beta}$ for the above conditions to be satisfied.

C. Harmonic 2-forms in $D = 2n$ Dimensions

In this section, we find n harmonic 2-forms $G^{(\mu)}_{(2)} = dB^{(\mu)}_{(1)}$ on the $2n$-dimensional Kerr-NUT-AdS metric (5.19), where we use the index $\mu = 1, 2, \ldots n$ to label the harmonic 2-forms. The potentials have a rather simple form, given by

$$B^{(\mu)}_{(1)} = \frac{x_\mu}{U_\mu}\Big(\sum_{k=0}^{n-1}A^{(k)}_\mu\,d\psi_k\Big)\,. \tag{5.37}$$

The metric (5.19) admits a natural vielbein basis, namely

$$e^\mu = \frac{dx_\mu}{\sqrt{Q_\mu}}\,, \qquad \tilde{e}^\mu = \sqrt{Q_\mu}\Big(\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k\Big)\,. \tag{5.38}$$

In this vielbein basis, the harmonic 2-forms $G_{(2)}^{(\mu)}$ are given by

$$G_{(2)}^{(\mu)} = \sum f_\nu^{(\mu)}\, e^\nu \wedge \tilde{e}^\nu\,, \tag{5.39}$$

where the coefficients are

$$\begin{aligned} f_\mu^{(\mu)} &= \frac{1}{U_\mu^2}\Big[A^{(n-1)} + \sum_{k=1}^{n-2}(-1)^k(2k+1)x_\mu^{2(k+1)}A_\mu^{(n-k-2)}\Big]\,,\\ f_\nu^{(\mu)} &= -\frac{2x_\mu x_\nu}{U_\mu^2}\prod_{\rho\neq\mu,\nu}(x_\rho^2 - x_\mu^2)\,, \qquad \text{with } \mu\neq\nu\,. \end{aligned} \tag{5.40}$$

We verify with low-lying examples that all of the $G_{(2)}^{(\mu)}$ are harmonic, *i.e.* $dG_{(2)}^{(\mu)} = 0 = d * G_{(2)}^{(\mu)}$. It is worth observing that these 2-forms are harmonic regardless of the detailed structure of the functions X_μ.

It was shown in chapter III that the BPS limit of the metric (5.19) gives rise to the non-compact Calabi-Yau metric that can provide a resolutions of the cone over the Einstein-Sasaki spaces. Under suitable coordinate transformation, the metric is given by

$$ds^2 = \sum_{\mu=1}^{n}\Big\{\frac{dx_\mu^2}{Q_\mu} + Q_\mu\,\Big(\sum_{k=0}^{n-1}A_\mu^{(k)}\,d\psi_k\Big)^2\Big\}\,, \tag{5.41}$$

where we define

$$\begin{aligned} Q_\mu &= \frac{4X_\mu}{U_\mu}\,, \qquad U_\mu = {\prod_{\nu=1}^{n}}'(x_\nu - x_\mu)\,, \qquad X_\mu = x_\mu\prod_{k=1}^{n-1}(x_\mu+\alpha_k) + 2b_\mu\,,\\ A_\mu^{(k)} &= \sum_{\nu_1<\nu_2<\cdots<\nu_k}{}' \, x_{\nu_1}x_{\nu_2}\cdots x_{\nu_k}\,. \end{aligned} \tag{5.42}$$

Note that we have Wick rotated the metric to have Euclidean signature. We can choose the same form of the vielbein basis as in (5.38). The Kähler 2-form is then given by

$$J_{(2)} = \sum_{\mu=1}^{n} e^\mu \wedge \tilde{e}^\mu \,. \tag{5.43}$$

The 1-form potentials for the harmonic 2-forms are given by

$$B_{(1)}^{(\mu)} = \frac{1}{U_\mu}\Big(\sum_{k=0}^{n-1} A_\mu^{(k)}\, d\psi_k\Big) \,. \tag{5.44}$$

The corresponding harmonic 2-forms $G_{(2)}^{(\mu)}$ have the same form as in (5.39), with the functions $f_\nu^{(\mu)}$ are given by

$$f_\nu^{(\mu)} = \frac{2}{U_\mu^2} \prod_{\rho\neq\mu,\nu} (x_\rho - x_\mu)\,, \text{with } \mu \neq \nu\,, \qquad f_\mu^{(\mu)} = -\sum_{\nu\neq\mu} f_\nu^{(\mu)}\,. \tag{5.45}$$

Note that $G_{(2)}^{(\mu)}$ satisfy the linear relation $\sum_{\mu=1}^n G_{(2)}^{(\mu)} = 0$. Thus, in the BPS limit, there are $(n-1)$ linearly independent such harmonic 2-forms. Together with the Kähler 2-form, the total number of harmonic 2-forms is n again.

D. Conclusion

In this chapter, we explicitly express the general Kerr-NUT-AdS metrics in Kerr-Schild form for both even and odd dimensions. We demonstrate that, in a suitable coordinate system the mass, NUT and angular momentum parameters enter linearly in the metric, and hence they can be viewed as a linear perturbation of pure AdS spacetime.

We also obtain n harmonic 2-forms on the $2n$-dimensional Kerr-NUT-AdS metrics. An interesting property of these harmonic 2-forms is that the closure and co-closure do not depend on the detailed structure of the functions X_μ. This provides a potential ansatz for charged Kerr-NUT-AdS solutions for pure Einstein-Maxwell theories in higher dimensions, whose explicit analytical solutions remain elusive. In the case of four dimensions, the back-reaction of the gauge field to the Einstein equations gives precisely the charged Plebanski metric [31], where only the functions X_μ in the metric have extra contributions from the electric and magnetic charges. However, the same phenomenon does not occur in higher dimensions; nevertheless, the harmonic 2-forms we constructed can be viewed as charged Kerr-NUT-AdS solutions at the linear level for small-charge expansion. Together with the charged slowly-rotating black holes obtained in [39, 40], our results may lead to the general charged Kerr-NUT-AdS solutions.

CHAPTER VI

RESOLVED CALABI-YAU CONES AND FLOWS FROM L^{ABC} SUPERCONFORMAL FIELD THEORIES

A. Introduction

The AdS/CFT correspondence relates type IIB string theory on $\mathrm{AdS}_5 \times S^5$ to four-dimensional $\mathcal{N} = 4$ $U(N)$ superconformal Yang-Mills theory [9, 41, 42]. More generally, type IIB string theory on $\mathrm{AdS}_5 \times X^5$, where X^5 is an Einstein-Sasaki space such as $T^{1,1}$, Y^{pq} [43, 32] or L^{abc} [22, 23], corresponds to an $\mathcal{N} = 1$ superconformal quiver gauge theory. The dual gauge theories have been identified in [44] for $T^{1,1}$, in [45, 46] for Y^{pq} and in [47, 48, 49] for L^{abc}.

There is a prescription for mapping perturbations of the supergravity background to operators in the dual gauge theory [41, 42]. In particular, motion in the Kähler moduli space of the Calabi-Yau cone over the Einstein-Sasaki space corresponds to giving vacuum expectation values (vevs) to the fundamental fields, such that only non-mesonic operators get vevs. This is because the mesonic directions of the full moduli space correspond to the motion of the D3-branes in the Calabi-Yau space whereas the non-mesonic, or baryonic, directions are associated with either deformations of the geometry or turning on B-fields. This has been studied for a blown-up 2-cycle in the resolved conifold in [50], as well as for a blown-up 4-cycle in the resolved conifold [51], Y^{pq} cones [52], L^{abc} cones [53] and general Calabi-Yau cones [54]. All of these resolved Calabi-Yau cones with blown-up 4-cycles follow the general construction given in [55, 56].

In this chapter, we shall apply the state/operator correspondence to a general class of resolved Calabi-Yau cones over L^{abc} with a blown-up 2-cycle or 4-cycle. These metrics are obtained from the Euclideanization of the BPS limit of the six-dimensional Kerr-NUT-AdS solutions constructed in chapter II, III.* In particular, blowing up a 2-cycle or 4-cycle corresponds to giving a vev to a real dimension-two and/or six scalar operator. Although cycles are being blown up, in all but two cases there remain singularities [57, 58]. However, there is a countably infinite subset of cases where there is an ALE singularity, on which perturbative string dynamics is well-defined. Some of these cases were studied in [54]. While adding a large number of D3-branes that are uniformly distributed, or "smeared", on the blown-up cycle ends up inducing a power-law singularity at short distance†, the resulting backgrounds can nevertheless be reliably used to describe perturbations around the UV conformal fixed point of the quiver gauge theories. Close to the UV fixed point, blowing up a 2-cycle on the L^{abc} cone corresponds to giving a vev to an operator that is analogous to the case of the resolved conifold. Therefore, we shall refer to these spaces as resolved cones, though it should be understood that there are still orbifold-type singularities.

The supergravity background can also be perturbed by adding a harmonic 3-form which lives on the Calabi-Yau metrics. If this is a pure $(2,1)$-form then supersymmetry will be preserved. Furthermore, if this form carries nontrivial flux then it corresponds to D5-branes wrapped on a 2-cycle in the Calabi-Yau space. The introduction of these fractional D3-branes eliminates the conformal fixed point in the UV limit of the quiver gauge theory. The theory undergoes a Seiberg duality cascade and the ranks of some of the gauge groups are reduced with decreasing energy scale. The supergravity solutions corresponding to fractional branes have been constructed

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*This is the even-dimensional analog of the relation between the Einstein-Sasaki spaces constructed in [26] and odd-dimensional BPS Kerr-NUT-AdS solutions.

†This singularity is due to the smearing of the D3-brane charge on the blown-up cycle. A completely non-singular solution with D3-branes stacked at a single point on the resolved conifold has been constructed [59].

for the cones over T^{11} [60, 61], Y^{pq} [62] and L^{abc} spaces [63, 64]. Fractional branes have also been considered for Calabi-Yau spaces with blown-up cycles, such as the deformed conifold [65], resolved conifold [66] and regularized conifold [51], as well as the resolved Y^{pq} cones with blown-up 4-cycles [54]. We shall also consider continuous families of 3-forms that do not have nontrivial flux. In this case, there remains a conformal fixed point in the UV limit of the field theory. It has been proposed that the ranks of some of the gauge groups are reduced with decreasing energy scale via the Higgs mechanism [67].

Since the L^{abc} spaces have cohomogeneity two, the form fields constructed on the corresponding Calabi-Yau spaces will generally have nontrivial dependence on the radial direction as well as the two non-azimuthal coordinates of L^{abc}. In addition, these forms generally break the $U(1)_R \times U(1) \times U(1)$ global symmetry group of the theory down to a $U(1) \times U(1)$ symmetry group which, in particular, breaks the R-symmetry. However, this is done in such a way that the theory preserves $\mathcal{N} = 1$ supersymmetry.

The various perturbations of the $\mathrm{AdS}_5 \times L^{abc}$ supergravity background that will be discussed are shown in Figure 1. These perturbations, which can be superimposed with one another, correspond to continuous families of Renormalization Group (RG) flows from the UV superconformal fixed point of the quiver gauge theory.

The chapter is organized as follows. In section B, we discuss the geometry of the resolved Calabi-Yau cones over the L^{abc} spaces. A subset of these are the resolved cones over Y^{pq} and their various limits. We find various harmonic $(2,1)$-forms on these metrics, some of which carry nontrivial flux and some of which do not. In section C, we apply some of our results to the AdS/CFT correspondence. In particular, we relate the perturbations of the $\mathrm{AdS}_5 \times L^{abc}$ background to various flows from the UV conformal fixed point of the dual quiver gauge theory. In section D, we

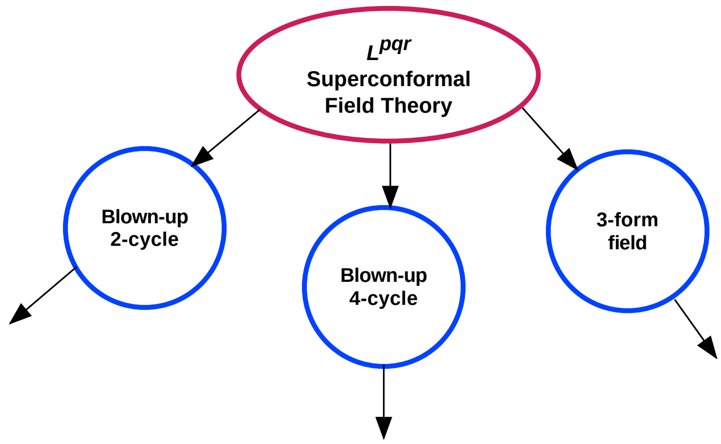

Fig. 1. RG flows from the superconformal fixed point of the L^{abc} quiver gauge theory correspond to various deformations of the supergravity background.

consider eight-dimensional resolved cones over L^{pqrs} and the various harmonic forms that live on them. In section E, we carry out the corresponding analysis for the higher-dimensional resolved cones. Lastly, conclusions are presented in section F.

B. Six-dimensional Resolved Calabi-Yau Cones

Although the L^{abc} spaces themselves are non-singular for appropriately chosen integers p, q, r [22, 23], the cones over these spaces have a power-law singularity at their apex. In the case of the cone over $T^{1,1}$, this singularity can be smoothed out in two different ways [68]. Firstly, one can blow up a 3-cycle, which corresponds to a complex deformation. The resulting deformed conifold has been crucial for the construction of a well-behaved supergravity dual of the IR region of the gauge theory, providing a geometrical description of confinement [65].

One might hope that a similar resolution procedure could be performed on other L^{abc} cones. Although a first-order deformation of the complex structure of Y^{pq} cones has been found in [69], there exists an obstruction to finding the complex deformations beyond first order [70, 71]. There is also evidence from the field theory side that such deformations will break supersymmetry for the Y^{pq} cones [72, 73, 74, 75] as well as for a large class of L^{abc} cones [75]. Nevertheless, there are L^{abc} cones which allow for complex structure deformations [70, 71], which can be understood from the corresponding toric diagrams [48].‡ However, the explicit metrics for these deformed L^{abc} cones, let alone the solutions for D3-branes on these cones, are not known.

Fig. 2. A 4-cycle within the base space of a cone over L^{abc} can be blown up. Within this 4-cycle lies a 2-cycle. The volumes of these two cycles correspond to two independent Kähler moduli.

The second way in which the $T^{1,1}$ cone can be rendered regular is by blowing up a 2-cycle [68]. Also, for the case of a cone over $T^{1,1}/\mathbb{Z}_2$, the singularity can be resolved by blowing up a 4-cycle. Both of these resolutions are examples of Kähler deformations which, as we shall see shortly, can also be performed on the L^{abc} cones $C(L^{abc})$. Moreover, the 2-cycle actually lives within the 4-cycle, as illustrated in Figure 2. This means that there are two Kähler moduli associated with the 4-cycle. For certain parameter choices, we can have the 4-cycle corresponds to the Einstein-Kähler base space of L^{abc}, whose metric can be obtained by taking a certain scaling limit of a Euclideanized form of the Plebanski-Demianski metric [63]. It is also possible to have

‡We thank Angel Uranga for discussions on this point.

the volume of the 4-cycle vanishes, whilst keeping a 2-cycle blown up.

It has been found that the cone over $Y^{2,1}$ can be rendered completely regular by blowing up an appropriate 4-cycle [57]. However this, together with the resolved cones over $T^{1,1}$ and $T^{1,1}/\mathbb{Z}_2$, constitute the only examples of non-singular resolved cones over the L^{abc} spaces [58]. Although we shall refer to these spaces as "resolved" L^{abc} cones, there are generally orbifold-type singularities remaining. In the limit of a vanishing 2-cycle, this can be seen simply because at short distance the geometry becomes a direct product of $\mathbb{R}^2$ and the four-dimensional Einstein-Kähler base space of L^{abc}, which is itself an orbifold. Nevertheless, the resolved cones over L^{abc} can be embedded in ten dimensions to give Ricci-flat backgrounds $\text{Mink}_4 \times C(L^{abc})$, on which perturbative string dynamics is well-defined. However, as we shall see in section C, the back-reaction of D3-branes leads to a power-law singularity at short distance. This singularity is due to the fact that we are smearing the D3-branes on the blown-up cycle. For the case of the resolved conifold, it has been shown that if the D3-branes are stacked at a single point then the supergravity solution is completely regular [59].

1. Resolved Cones over Y^{pq}

Before turning to resolved cones over the general cohomogeneity-two L^{abc} spaces, it is instructive first to consider the subset involving the cohomogeneity-one Y^{pq} spaces. The metric of the resolved cone over Y^{pq} is given in chapter II,

$$ds_6^2 = \frac{x+y}{4X}dx^2 + \frac{X}{x+y}(d\tau + \frac{y}{2\alpha}\,\sigma_3)^2 + \frac{x+y}{4Y}dy^2 + \frac{Y}{x+y}(d\tau - \frac{x}{2\alpha}\,\sigma_3)^2 + \frac{x\,y}{4\alpha}\,(\sigma_1^2 + \sigma_2^2)\,. \tag{6.1}$$

where

$$X = x(x+\alpha) - \frac{2\mu}{x}\,, \qquad Y = y(\alpha - y) + \frac{2\nu}{y}\,, \tag{6.2}$$

and that

$$\sigma_3 = d\psi + \cos\theta\, d\phi\,, \qquad \sigma_1^2 + \sigma_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2\,. \tag{6.3}$$

It has been shown that the only completely regular examples are the resolved cones over $T^{1,1}$, $T^{1,1}/\mathbb{Z}_2$ and $Y^{2,1}$ [57, 58]. We shall now consider various limits of the metric (6.1).

Resolved conifold

In order to reduce to a resolved cone over T^{11} (or $T^{11}/\mathbb{Z}_2$), we need to select ν such that $Y(y)$ has a double root. This happens when $\nu = -\frac{2}{27}\alpha^3$. Making the coordinate redefinition

$$y = \tfrac{2}{3}\alpha + \epsilon\, \cos\tilde{\theta}\,, \qquad \nu = -\tfrac{2}{27}\alpha^3 + \tfrac{1}{2}\alpha\epsilon^2\,, \qquad \tau = -\frac{2}{9\epsilon}\tilde{\phi}\,, \qquad \sigma_3 \to \sigma_3 + \frac{2\alpha}{3\epsilon}d\tau\,, \tag{6.4}$$

and setting the parameter ϵ to zero, we find that the metric becomes

$$ds_6^2 = \frac{x+\frac{2}{3}\alpha}{4X}dx^2 + \frac{X}{9(x+\frac{2}{3}\alpha)}(\sigma_3 + \cos\tilde{\theta}\, d\tilde{\phi})^2 + \tfrac{1}{6}(x+\tfrac{2}{3}\alpha)(d\tilde{\theta}^2 + \sin^2\tilde{\theta}\, d\tilde{\phi}^2) + \tfrac{1}{6}x\,(\sigma_1^2 + \sigma_2^2)\,. \tag{6.5}$$

If $\mu = 0$, there is a blown-up S^2 and the solution describes the resolved conifold [68]. If, on the other hand, $\alpha = 0$, then there is a blown-up $S^2 \times S^2$ and the solution describes the regularized conifold [51]. In fact, it has been shown that one can always blow up a 4-cycle on any cone over an Einstein-Sasaki space [55, 56]. We shall now take a look at the analogous limits for the resolved cones over the Y^{pq} spaces.

The $\alpha = 0$ limit

If we let $y \to \alpha y$, $\nu \to \alpha^3 \nu$ and then take $\alpha \to 0$, we obtain the limit

$$ds^2 = \frac{x}{4X}dx^2 + \frac{X}{x}(d\tau + \tfrac{1}{2}y\sigma_3)^2 + x\Big[\frac{dy^2}{4Y} + Y\,\sigma_3^2 + \tfrac{1}{4}y(\sigma_1^2 + \sigma_2^2)\Big]\,, \tag{6.6}$$

where

$$X = x^2 - \frac{2\mu}{x}\,, \qquad Y = y(1-y) + \frac{2\nu}{y}\,. \tag{6.7}$$

There is a single Kähler modulus, which corresponds to a blown-up 4-cycle with a volume parameterized by μ. This is the analog of the resolved cone for general Y^{pq} spaces. However, unlike the $T^{1,1}/\mathbb{Z}_2$ case, this metric has an orbifold-type singularity at its apex, since the geometry reduces to the direct product of $\mathbb{R}^2$ and an Einstein-Kähler orbifold.

The $\mu = 0$ limit: blowing up 2-cycles

One can also consider the limit in which μ vanishes, in which case x runs from 0 to asymptotic ∞. Near $x = 0$, we can express the metric as

$$ds^2 = y\Big(dr^2 + \tfrac{1}{4}r^2(\sigma_3 + \frac{2}{y}d\tau)^2 + \tfrac{1}{4}r^2(\sigma_1^2 + \sigma_2^2) + \frac{dy^2}{4Y}\Big) + Y(d\tau - \tfrac{1}{2}r^2\sigma_3)^2\,, \tag{6.8}$$

where $x = r^2$. At $r = 0$ there is a collapsing 3-sphere, instead of a circle as in the previous limit. There is a single Kähler modulus corresponding to the volume of a blown-up 2-cycle, which is parameterized by α. However, unlike the analogous resolved conifold for which there is a smooth 2-sphere, in general this 2-cycle is a "tear-drop" with a conical singularity.

Calabi-Yau structure

The Calabi-Yau structure on the metric (6.1) is given by a Kähler form J and a holomorphic (3,0)-form $G_{(3)}$. These can be expressed in the complex vielbein basis

$$\epsilon^1 = e^1 + \mathrm{i}\, e^2\,, \qquad \epsilon^2 = e^3 + \mathrm{i}\, e^4\,, \qquad \epsilon^3 = e^5 + \mathrm{i}\, e^6\,, \tag{6.9}$$

where the vielbein is conveniently chosen to be

$$e^1=\sqrt{\frac{x+y}{4X}}\,dx\,,\quad e^2=\sqrt{\frac{X}{x+y}}\,\Big(d\tau+\frac{y}{2\alpha}\,\sigma_3\Big)\,,\quad e^3=\sqrt{\frac{x+y}{4Y}}\,dy\,,$$
$$e^4=\sqrt{\frac{Y}{x+y}}\,\Big(d\tau-\frac{x}{2\alpha}\,\sigma_3\Big)\,,\quad e^5=\sqrt{\frac{xy}{4\alpha}}\,\sigma_1\,,\quad e^6=\sqrt{\frac{xy}{4\alpha}}\,\sigma_2\,.\qquad(6.10)$$

The Kähler 2-form is then given by

$$J=\tfrac{\mathrm{i}}{2}\,\epsilon^i\wedge\bar\epsilon^i\,,\qquad(6.11)$$

and the complex self-dual harmonic $(3,0)$-form is given by

$$G_{(3)}=e^{-3\mathrm{i}\tau}\,\epsilon^1\wedge\epsilon^2\wedge\epsilon^3\equiv W_{(3)}+\mathrm{i}\,{*W_{(3)}}\,.\qquad(6.12)$$

Harmonic $(2,1)$-forms

We are interested in harmonic $(2,1)$-forms that live on the resolved Y^{pq} cones, since their presence preserves the minimal supersymmetry of the theory. We find there exist the following five such $(2,1)$-forms:

$$\begin{aligned}\Phi_1&=\frac{e^{-3\mathrm{i}\tau}}{x\,X}\,\bar\epsilon_1\wedge\epsilon_2\wedge\epsilon_3\,,\quad \Phi_2=\frac{e^{-3\mathrm{i}\tau}}{y\,Y}\,\bar\epsilon_2\wedge\epsilon_1\wedge\epsilon_3\,,\quad \Phi_3=\frac{e^{3\mathrm{i}\tau}}{x\,y\,X\,Y}\bar\epsilon_3\wedge\epsilon_1\wedge\epsilon_2\,,\\ \Phi_4&=\frac{1}{x\,y\,\sqrt{x+y}}\Big(\frac{1}{x\sqrt{Y}}\epsilon_2\wedge(\bar\epsilon_3\wedge\epsilon_3-\bar\epsilon_1\wedge\epsilon_1)-\frac{1}{y\sqrt{X}}\epsilon_1\wedge(\bar\epsilon_3\wedge\epsilon_3-\bar\epsilon_2\wedge\epsilon_2)\Big)\,,\\ \Phi_5&=\frac{1}{\sqrt{x+y}}\Big(\frac{1}{x^2\sqrt{Y}}\epsilon_2\wedge(\bar\epsilon_3\wedge\epsilon_3-\bar\epsilon_1\wedge\epsilon_1)+\frac{1}{y^2\sqrt{X}}\epsilon_1\wedge(\bar\epsilon_3\wedge\epsilon_3-\bar\epsilon_2\wedge\epsilon_2)\Big)\,.\end{aligned}\qquad(6.13)$$

All of these forms have singularities at all distances x, for certain values of y, except for Φ_1, which has a singularity only at small distance. Φ_1 has a rapid fall off at large distance, such that it does not support nontrivial flux. On the other hand, in the large-x limit the last harmonic form behaves like

$$\begin{aligned}\Phi_5 &= \tfrac{1}{4}\sigma_1\wedge\sigma_2\wedge(\sigma_3+\frac{2}{y}\,d\tau)+\frac{1}{2y^2}\sigma_3\wedge d\tau\wedge dy+\frac{1}{4x}\Big(-2\sigma_1\wedge\sigma_2\wedge d\tau \\ &\quad +\mathrm{i}\Big[(\frac{1}{y}\sigma_1\wedge\sigma_2-\frac{1}{y^2}\sigma_3\wedge dy)\wedge dx+\frac{y}{Y}\sigma_1\wedge\sigma_2\wedge dy\Big]\Big)+\mathcal{O}(\frac{1}{x^2}). \qquad (6.14)\end{aligned}$$

This indicates that this form does support nontrivial flux. In the $\alpha = 0$ limit, in which we have first rescaled $y \to \alpha y$, Φ_4 and Φ_5 reduce to the same form. This form has a singularity that is confined to small distance.

2. Resolved Cones over L^{abc}

We now turn to the resolved cones over the general cohomogeneity-two L^{abc} spaces. The metric is given in chapter III,

$$\begin{aligned}ds^2 &= \tfrac{1}{4}(u^2dx^2+v^2dy^2+w^2dz^2)+\frac{1}{u^2}(d\tau+(y+z)d\phi+yz\,d\psi)^2 \\ &\quad +\frac{1}{v^2}(d\tau+(x+z)d\phi+xz\,d\psi)^2+\frac{1}{w^2}(d\tau+(x+y)d\phi+xy\,d\psi)^2, \qquad (6.15)\end{aligned}$$

where the functions u, v, w are given by

$$\begin{aligned}&u^2=\frac{(y-x)(z-x)}{X}\,, \qquad v^2=\frac{(x-y)(z-y)}{Y}\,, \qquad w^2=\frac{(x-z)(y-z)}{Z}\,, \\ &X = x(\alpha-x)(\beta-x)-2M\,, \qquad Y = y(\alpha-y)(\beta-y)-2L_1\,, \\ &Z = z(\alpha-z)(\beta-z)-2L_2\,. \qquad (6.16)\end{aligned}$$

Notice that the coordinates x, y and z appear in the metric on a symmetrical footing. We shall choose x to be the radial direction, and y and z to be the non-azimuthal coordinates on the L^{abc} level sets. This reduces to the Y^{pq} subset when $a = p-q$, $b = p+q$ and $c = d = p$.

Calabi-Yau structure

The complex vielbein can be written as

$$\epsilon^1 = e^1 + \mathrm{i}\, e^2\,, \qquad \epsilon^2 = e^3 + \mathrm{i}\, e^4\,, \qquad \epsilon^3 = e^5 + \mathrm{i}\, e^6\,, \tag{6.17}$$

in the vielbein basis

$$\begin{aligned} e^1 &= \tfrac{1}{2} u\, dx\,, \qquad e^2 = \frac{1}{u}(d\tau + (y+z)d\phi + yz\, d\psi)\,,\\ e^3 &= \tfrac{1}{2} v\, dy\,, \qquad e^4 = \frac{1}{v}(d\tau + (x+z)d\phi + xz\, d\psi)\,,\\ e^5 &= \tfrac{1}{2} w\, dz\,, \qquad e^6 = \frac{1}{w}(d\tau + (x+y)d\phi + xy\, d\psi)\,. \end{aligned} \tag{6.18}$$

Then the Kähler 2-form and complex self-dual harmonic $(3,0)$-form are given by

$$J = \frac{\mathrm{i}}{2}\bar{\epsilon}_i \wedge \epsilon_i\,, \qquad G_{(3)} = e^{\mathrm{i}\,\nu}\, \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3\,, \tag{6.19}$$

where

$$\nu = 3\tau + 2(\alpha+\beta)\phi + \alpha\beta\psi\,. \tag{6.20}$$

Harmonic $(2,1)$-forms

There is a harmonic $(2,1)$-form given by

$$\Psi_1 = \frac{e^{\mathrm{i}\,\nu}}{X}\, \bar{\epsilon}^1 \wedge \epsilon^2 \wedge \epsilon^3\,. \tag{6.21}$$

Using this, one can then construct a general class of harmonic $(2,1)$-forms

$$\Phi_1 = f(\gamma)\, \Psi_1\,, \tag{6.22}$$

for any function f so long as $d\gamma \wedge \Psi_1 = 0$. This orthogonality condition is obeyed by

$$\gamma = \frac{YZ}{X}\, e^{\mathrm{i}\,2\nu}\,, \tag{6.23}$$

as can be seen by calculating its exterior derivative:

$$d\gamma = \frac{2\gamma}{(x-y)(y-z)(z-x)} \Big(u(y-z)X'\,\bar{\epsilon}^1 - v(z-x)Y'\,\epsilon^2 - w(x-y)Z'\,\epsilon^3 \Big)\,. \tag{6.24}$$

We can consider the special case for which

$$\Phi_1 = \frac{(YZ)^\delta}{X^{\delta+1}} e^{\mathrm{i}(2\delta+1)\nu}\,\bar{\epsilon}^1\wedge\epsilon^2\wedge\epsilon^3\,, \tag{6.25}$$

where δ is a continuous parameter. Due to the ν dependence, this field only preserves $U(1)^2$ of the $U(1)^3$ isometry of the six-dimensional space. Although the full $U(1)^3$ is preserved for $\delta = -1/2$, the form field would blow up at the degeneracies of X, Y and Z, which would lead to a singular surface in the ten-dimensional geometry. In order for the singularity to be confined to $X = 0$, so that we have a reasonable gravity description near the UV region of the dual field theory, we require that $\delta \geq 0$.

We find there exist the following $(2,1)$-forms:

$$\begin{aligned}
\Phi_1 &= f\Big(\frac{YZ}{X}\,e^{\mathrm{i}\,2\nu}\Big)\,\frac{e^{\mathrm{i}\,\nu}}{X}\,\bar{\epsilon}^1\wedge\epsilon^2\wedge\epsilon^3\,,\\
\Phi_2 &= f\Big(\frac{XZ}{Y}\,e^{\mathrm{i}\,2\nu}\Big)\,\frac{e^{\mathrm{i}\,\nu}}{Y}\,\epsilon^1\wedge\bar{\epsilon}^2\wedge\epsilon^3\,,\\
\Phi_3 &= f\Big(\frac{XY}{Z}\,e^{\mathrm{i}\,2\nu}\Big)\,\frac{e^{\mathrm{i}\,\nu}}{Z}\,\epsilon^1\wedge\epsilon^2\wedge\bar{\epsilon}^3\,,\\
\Phi_4 &= a_1 A\,\epsilon^1\wedge(\bar{\epsilon}^2\wedge\epsilon^2 - \bar{\epsilon}^3\wedge\epsilon^3) + a_2 B\,\epsilon^2\wedge(\bar{\epsilon}^3\wedge\epsilon^3 - \bar{\epsilon}^1\wedge\epsilon^1)\\
&\quad + a_3 C\,\epsilon^3\wedge(\bar{\epsilon}^1\wedge\epsilon^1 - \bar{\epsilon}^2\wedge\epsilon^2)\,,\\
\Phi_5 &= b_1 Ax\,\epsilon^1\wedge(\bar{\epsilon}^2\wedge\epsilon^2 - \bar{\epsilon}^3\wedge\epsilon^3) + b_2 By\,\epsilon^2\wedge(\bar{\epsilon}^3\wedge\epsilon^3 - \bar{\epsilon}^1\wedge\epsilon^1)\\
&\quad + b_3 Cz\,\epsilon^3\wedge(\bar{\epsilon}^1\wedge\epsilon^1 - \bar{\epsilon}^2\wedge\epsilon^2)\,,
\end{aligned} \tag{6.26}$$

where

$$\begin{aligned}
A^{-1} &= (y-z)^2\sqrt{(y-x)(z-x)X}\,, \qquad B^{-1} = (x-z)^2\sqrt{(x-y)(z-y)Y}\,,\\
C^{-1} &= (x-y)^2\sqrt{(x-z)(y-z)Z}\,,
\end{aligned} \tag{6.27}$$

and a_i and b_i are constants which satisfy the conditions $a_1+a_2+a_3 = 0$ and $b_1+b_2+b_3 = 0$. Notice that the first three forms in (6.27) are related to each other by interchanging the x, y and z coordinates, while the last two forms remain invariant. This reflects the fact that the x, y and z coordinates appear in a completely symmetric manner in the metric of the resolved cone over L^{abc}. Φ_1 has a singularity that is confined to small distance, as do Φ_4 and Φ_5 if one performs the rescaling $y \to \alpha y$, $z \to \alpha z$ and then takes the limit $\alpha \to 0$. Φ_4 and Φ_5 have nontrivial flux, while Φ_1 does not.

In the cohomogeneity-two limit, the resolved L^{abc} cones reduce to the resolved Y^{pq} cones. In this limit, Φ_4 and Φ_5 reduce to the corresponding forms given in (6.13), while the first three forms generalize those in (6.13) to include an arbitrary function f. In particular, taking $f = 1$ reproduces the Φ_1 and Φ_2 in (6.13), whilst taking f to be the inverse of its argument reproduces Φ_3.

C. D3-branes and the AdS/CFT Correspondence

A supersymmetric D3-brane solution of the type IIB theory with six-dimensional Calabi-Yau transverse space is given by

$$
\begin{aligned}
ds^2 &= H^{-\frac{1}{2}}(-dt^2+dx_1^2+dx_2^2+dx_3^2)+H^{\frac{1}{2}}ds_6^2\,, \\
F_5 &= G_{(5)}+*G_{(5)}\,, \qquad G_{(5)} = dt\wedge dx_1\wedge dx_2\wedge dx_3\wedge dH^{-1}\,, \\
F_{(3)} &= F^{\rm RR}_{(3)}+{\rm i}\,F^{\rm NS}_{(3)} = m\,\omega_{(3)}\,,
\end{aligned}
\tag{6.28}
$$

with

$$\Box_6 H = m^2|\omega_{(3)}|^2\,. \tag{6.29}$$

Here the $\Box_6$ is a Laplacian of the Calabi-Yau metric ds_6^2 and $\omega_{(3)}$ is a harmonic $(2,1)$-form in ds_6^2. We shall refer to this as a modified D3-brane solution, owing to the inclusion of the additional 3-form. If this 3-form carries nontrivial flux, then it corresponds to fractional a D3-brane.

We shall take the six-dimensional metric ds_6^2 of the transverse space to be the resolved cone over L^{abc}. We first consider the case of vanishing m. It was shown in [76, 37, 77] that the Klein-Gordon equation for the general Kerr-NUT-AdS solutions constructed in [16] is separable. Since our metrics arise as the Euclideanization of the supersymmetric limit of Kerr-NUT-AdS solutions, the corresponding equation for H is hence also separable. To see this, we consider a real superposition of the ansatz

$$H = H_1(x)\, H_2(y)\, H_3(z)\, e^{2\mathrm{i}(a_0\psi - a_1\phi + a_2\,\tau)}\,. \tag{6.30}$$

In general, this ansatz breaks the $U(1)^3$ global symmetry.

The Laplace equation is then given by

$$\begin{aligned} 0 \;=\;& \frac{1}{(y-x)(z-x)}\Big(\frac{(X\,H_1')'}{H_1} - \frac{(a_0+a_1x+a_2x^2)^2}{X}\Big) \\ &+\frac{1}{(x-y)(z-y)}\Big(\frac{(Y\,H_2')'}{H_2} - \frac{(a_0+a_1y+a_2y^2)^2}{Y}\Big) \\ &+\frac{1}{(x-z)(y-z)}\Big(\frac{(Z\,H_3')'}{H_3} - \frac{(a_0+a_1z+a_2z^2)^2}{Z}\Big)\,, \end{aligned} \tag{6.31}$$

where a prime denotes a derivative with respect to the separated variable associated with the function H_i. This equation can be expressed as three separate equations in x, y and z:

$$\begin{aligned} (X\,H_1')' - \Big(\frac{(a_0+a_1x+a_2x^2)^2}{X} + b_0 + b_1x\Big)H_1 &= 0\,, \\ (Y\,H_2')' - \Big(\frac{(a_0+a_1y+a_2y^2)^2}{Y} + b_0 + b_1y\Big)H_2 &= 0\,, \\ (Z\,H_3')' - \Big(\frac{(a_0+a_1z+a_2z^2)^2}{Z} + b_0 + b_1z\Big)H_3 &= 0\,, \end{aligned} \tag{6.32}$$

where b_0 and b_1 are separation constants. These equations do not have explicit closed-form solutions for general a_i and b_i. We shall consider the simplest solution obtained by setting all of the a_i and b_i to zero and letting H depend on x only. The solution is given by

$$H = c_0 - \frac{c_1 \log(x-x_1)}{(x_1-x_2)(x_1-x_3)} + \frac{c_1 \log(x-x_2)}{(x_2-x_1)(x_2-x_3)} + \frac{c_1 \log(x-x_3)}{(x_3-x_1)(x_3-x_2)} \,. \qquad (6.33)$$

where x_1, x_2 and x_3 are the three roots of X, satisfying

$$x_1+x_2+x_3 = \alpha+\beta\,, \qquad x_1x_2+x_1x_3+x_2x_3 = \alpha\,\beta\,, \qquad x_1x_2x_3 = 2M\,. \qquad (6.34)$$

Consider the radial coordinate x with $x_1 \le x \le \infty$. Then the function H has a logarithmic divergence at x_1.

We now consider solutions for which the 3-form $\omega_{(3)}$ is turned on. A simple solution can be obtained by rescaling $y \to \alpha y$, $z \to \beta z$ and then taking the limit $\alpha = \beta = 0$. The general construction of [55, 56] is recovered for the case of this class of resolved L^{abc} cones [53]. We can then take $\omega_{(3)}$ to be the harmonic $(2,1)$-form Ψ_1 given by (6.21). Then, for a certain choice of integration constants, the resulting H is given by

$$H = \frac{x}{18M(x^3-2M)}\,, \qquad (6.35)$$

which diverges at $x^3 = 2M$.

The divergence of the H function in both (6.33) and (6.35) corresponds to a naked singularity in the short-distance region of the geometry. This singularity of the D3-brane solution arises even in the case of the resolved cone over $Y^{2,1}$, which itself is completely regular [57]. This singularity is due to the fact that the D3-branes have been smeared over the blown-up 4-cycle. A shell of uniformly distributed branes tends to be singular at its surface. For the case of the resolved conifold, in which there is a blown-up 2-cycle, a completely regular solution has been found for which the D3-branes are stacked at a single point [59]. This involves solving the equations (6.32) for the case of $T^{1,1}$ for which there is a delta function source. The solution is expressed as an expansion in terms of the angular harmonics. It would be interesting to explore than analogous construction for the resolved L^{abc} cones. All of these other examples, with the sole exception of $Y^{2,1}$, will still have orbifold singularities.

Another possible way in which regular solutions can be obtained is to blow up a 3-cycle instead of a 4-cycle. Then an appropriate 3-form would prevent the 3-cycle from collapsing, as in the case of the deformed conifold [65]. As already discussed in the previous section, while there exists an obstruction to complex deformations of Y^{pq} cones there are other subsets of the L^{abc} cones which do allow for complex structure deformations [70, 71, 48]. However, the explicit metrics for these deformed L^{abc} cones are not known.

Although the solution describing D3-branes on a resolved L^{abc} cone becomes singular at short distance, we can still use this background at large distance to study various flows of the quiver gauge theory in the region of the UV conformal fixed point. At large x, (6.29) becomes

$$\frac{4}{x^2}\partial_x\Big(X\partial_x H\Big) = m^2|\omega_{(3)}|^2\,, \tag{6.36}$$

where X is given by (6.16). Note that this equation applies for arbitrary α and β, since for large x we can consistently neglect the dependence of H on the non-azimuthal "angular" coordinates y and z. Again considering the case of the self-dual harmonic $(2,1)$-form Ψ_1 given by (6.21), the resulting asymptotic expansion of H is

$$H = \frac{Q}{x^2}\Big(1+\frac{c_2}{x}+\frac{c_4}{x^2}+\frac{c_6}{x^3}+\cdots\Big)\,, \tag{6.37}$$

where

$$\begin{aligned} c_2 &= \frac{2}{3}(\alpha+\beta)\,, \\ c_4 &= \frac{1}{2}(\alpha^2+\alpha\beta+\beta^2)\,, \\ c_6 &= \frac{1}{30}\Big(\frac{m^2}{Q}+12(\alpha^2+\beta^2)(\alpha+\beta)+2M\Big)\,. \end{aligned} \qquad (6.38)$$

We have set an additive constant to zero so that the geometry is asymptotically $\mathrm{AdS}_5\times L^{abc}$. This can be seen from the leading $x^{-2}\sim r^{-4}$ term in H (since x has dimension two, we can take $x\sim r^2$ for large x). The transformation properties and dimensions of the operators being turned on in the dual field theory can be read off from the linearized form of the supergravity solution (6.28). The metric perturbations due to H have the same form as those within the metric ds_6^2 itself. Therefore, from the asymptotic expansion of H given in (6.37), we can read off that there are scalar operators of dimension two, four and six with expectation values that go as c_2, c_4 and c_6, respectively. This is consistent with the perturbations of the 2-form and 4-form potentials. We shall now discuss the gauge theory interpretation of the blown-up 2-cycles, as well as the 3-form, in more detail.

Blown-up 2-cycle

First, we consider the case with vanishing M, for which the six-dimensional space is the L^{abc} analog of the resolved conifold, in the sense that there is a blown-up 2-cycle. The volume of the 2-cycle is characterized by the parameters α and β. This is a global deformation, in that it changes the position of the branes at infinity [54].

The parameters α and β specify the expectation values of dimension n non-mesonic scalar operators in the dual gauge theory. For the case $\beta=-\alpha$, c_2 and c_6 vanish, while c_4 can only vanish for $\alpha=\beta=0$. To identify the specific dimension-two operator whose expectation value goes as c_2, it is helpful to consider the description of the resolved cone over L^{abc} in terms of four complex numbers z_i which satisfy the constraint

$$\sum_{i=1}^{4} Q_i\,|z_i|^2 = t\,, \tag{6.39}$$

where one then takes the quotient by a $U(1)$ action [45]. The parameter t is the area of the blown-up CP^1 and corresponds to the coefficient of the Fayet-Iliopoulos term in the Lagrangian of the field theory. The z_i correspond to the lowest components of chiral superfields. This can be described as a gauged linear sigma model with a $U(1)$ gauge group and 4 fields with charges Q_i. Then the above constraint corresponds to setting the D-terms of the gauged linear sigma model to zero to give the vacuum. For the L^{abc} spaces, the Q_i are given by $Q_i = (a, -c, b, -d)$ where $d = a+b-c$ [47]. The requirement $\sum_{i=1}^{4} Q_i = 0$ guarantees that the 1-loop β-function vanishes, so that the sigma model is Calabi-Yau.

Since t acts as a natural order-parameter in the gauge theory, from (6.39) it is reasonable to suppose that blowing up the 2-cycle corresponds to giving an expectation value that goes as c_2 to the dimension-two scalar operator[§]

$$\mathcal{K} = a\,A_\alpha \bar{A}^{\dot\alpha} - c\,B_{\dot\alpha}\bar{B}^{\dot\alpha} + b\,C_\alpha \bar{C}^{\alpha} - d\,D_{\dot\alpha}\bar{D}^{\dot\alpha}\,. \tag{6.40}$$

This operator lies within the $U(1)$ baryonic current multiplet. Since this conserved current has no anomalous dimension, the dimension of $\mathcal{K}$ is protected. $\mathcal{K}$ reduces to the operator discussed in [54] for the case of a resolved cone over $T^{11}/\mathbb{Z}_2$, for which $a = b = c = d = 1$.

Blown-up 4-cycle

For nonvanishing M in the function X, one generically blows up a 4-cycle. Unlike

[§]We thank Amihay Hanany and Igor Klebanov for correspondence on this point.

the case of a blown-up 2-cycle, this is a local deformation since it does not change the position of the branes at infinity [54]. In the limit of vanishing α and β, one recovers the general construction obtained in [55, 56] that has been recently discussed in [52, 53, 54]. Also note that c_6 vanishes for the appropriate values of M, α and β.

It has been shown that the number of formal Fayet-Iliopoulos parameters can be matched with the possible deformations, which is suggestive that the dimension-six operator that is turned on is associated with the gauge groups in the quiver. Although the specific operator has not been identified, it has been proposed that they are of the schematic form [54]

$$\mathcal{O}_i = \sum_g c_{i,g} \mathcal{W}_g \bar{\mathcal{W}}_g \,, \tag{6.41}$$

where the gauge groups in the quiver have been summed over, $\mathcal{W}_g$ is an operator associated with the field strength for the gauge group g, and $c_{i,g}$ are constants. The dimension-six operator might also have contributions from the bifundamental fields of the form

$$a_1 A_\alpha \bar{A}^\alpha B_{\dot\alpha} \bar{B}^{\dot\alpha} C_\beta \bar{C}^\beta + a_2 A_\alpha \bar{A}^\alpha B_{\dot\alpha} \bar{B}^{\dot\alpha} D_{\dot\beta} \bar{D}^{\dot\beta} + a_3 A_\alpha \bar{A}^\alpha C_\beta \bar{C}^\beta D_{\dot\beta} \bar{D}^{\dot\beta} + a_4 B_{\dot\alpha} \bar{B}^{\dot\alpha} C_\beta \bar{C}^\beta D_{\dot\beta} \bar{D}^{\dot\beta}, \tag{6.42}$$

where the a_i are constants. It is proposed that a particular combination of all of these terms in (6.41) and (6.42) correspond to the blown-up 4-cycle¶. One possibility is that the contributions from the bifundamental fields in (6.42) are present only when α and β are nonvanishing.

Turning on the 3-form

Turning on a 3-form results in the ranks of some of the gauge groups of the dual quiver gauge theory being reduced with decreasing energy scale. For the case in

¶We thank Sergio Benvenuti for correspondence on this point.

which the 3-form has nontrivial flux, the theory undergoes a Seiberg duality cascade [60, 61, 65]. On the other hand, the 3-form Ψ_1 given by (6.21) does not have nontrivial flux. For a case such as this, it has been proposed that the reduction in ranks of gauge groups is due to Higgsing [67]. In particular, from (6.37), we see that the parameter m associated with the 3-form also contributes to the expectation value c_6 of a dimension-six scalar operator. An additional effect of this 3-form is that the $U(1)$ R-symmetry is broken. The theory still preserves $\mathcal{N}=1$ supersymmetry.

D. Eight-dimensional Resolved Calabi-Yau Cones

1. Cohomogeneity-two Metrics

We now turn to eight-dimensional Calabi-Yau spaces, which can be used to construct M2-brane solutions of eleven-dimensional supergravity. Before considering the general cohomogeneity-four resolved cones over L^{pqrs}, we shall first look at the cohomogeneity-two metrics, which can be built over an $S^2\times S^2$ base space. These metrics are given in chapter II, III,

$$\begin{aligned}
ds_8^2 &= \tfrac14 u^2 dx^2+\tfrac14 v^2 dy^2+\frac{1}{u^2}\Big[d\tau+\frac{y}{3\alpha}(\sigma_3+\tilde\sigma_3)\Big]^2\\
&\quad+\frac{1}{v^2}\Big[d\tau-\frac{x}{3\alpha}(\sigma_{(3)}+\tilde\sigma_3)\Big]^2+c^2(\sigma_1^2+\sigma_2^2+\tilde\sigma_1^2+\tilde\sigma_2^2)\\
u^2 &= \frac{x+y}{X}\,,\qquad v^2=\frac{x+y}{Y}\,,\qquad c^2=\frac{xy}{6\alpha}\,,\\
X &= x(x+\alpha)-\frac{2\mu}{x^2}\,,\quad Y=y(\alpha-y)+\frac{2\nu}{y^2}\,,
\end{aligned}\tag{6.43}$$

Completely regular examples were discussed in [58].

Calabi-Yau structure

We can define the vielbein basis

$$\begin{aligned} e^1 &= \tfrac{1}{2}u\,dx\,, \qquad e^2 = -\frac{1}{u}(d\tau + \frac{y}{3\alpha}(\sigma_3+\tilde\sigma_3))\,, \qquad e^3 = \tfrac{1}{2}v\,dy\,, \\ e^4 &= \frac{1}{v}(d\tau - \frac{x}{3\alpha}(\sigma_3+\tilde\sigma_3))\,, \quad e^5 = c\,\sigma_1\,, \quad e^6 = c\,\sigma_2\,, \qquad e^7 = c\,\tilde\sigma_1\,, \quad e^8 = c\,\tilde\sigma_2\,, \end{aligned} \tag{6.44}$$

and then the complex vielbein

$$\epsilon_1 = e^1 + \mathrm{i}\,e^2\,, \qquad \epsilon_2 = e^3 + \mathrm{i}\,e^4\,, \qquad \epsilon_3 = e^5 + \mathrm{i}\,e^6\,, \qquad \epsilon_4 = e^7 + \mathrm{i}\,e^8\,. \tag{6.45}$$

The Kähler 2-form and holomorphic $(4,0)$-form are given by

$$J = \tfrac{\mathrm{i}}{2}\epsilon^i \wedge \bar\epsilon^i\,, \tag{6.46}$$

and

$$G_{(4)} = e^{-4\mathrm{i}\tau}\epsilon^1\wedge\epsilon^2\wedge\epsilon^3\wedge\epsilon^4\,. \tag{6.47}$$

Harmonic $(2,2)$-forms

We find four self-dual $(2,2)$-forms; they are given by

$$\begin{aligned} \Phi_1 &= \frac{(\bar\epsilon_1\wedge\epsilon_1+\bar\epsilon_2\wedge\epsilon_2)\wedge(\bar\epsilon_3\wedge\epsilon_3+\bar\epsilon_4\wedge\epsilon_4)-2(\bar\epsilon_1\wedge\epsilon_1\wedge\bar\epsilon_2\wedge\epsilon_2+\bar\epsilon_3\wedge\epsilon_3\wedge\bar\epsilon_4\wedge\epsilon_4)}{x^3y^3} \\ \Phi_2 &= \frac{(\bar\epsilon_1\wedge\epsilon_1-\bar\epsilon_2\wedge\epsilon_2)\wedge(\bar\epsilon_3\wedge\epsilon_3-\bar\epsilon_4\wedge\epsilon_4)}{xy(x+y)^2}\,, \\ \Phi_3 &= \frac{e^{-4\mathrm{i}\tau}\,(\bar\epsilon_1\wedge\bar\epsilon_2\wedge\epsilon_3\wedge\epsilon_4+\epsilon_1\wedge\epsilon_2\wedge\bar\epsilon_3\wedge\bar\epsilon_4)}{x^2y^2\,XY}\,, \\ \Phi_4 &= \frac{(\bar\epsilon_1\wedge\epsilon_2-\epsilon_1\wedge\bar\epsilon_2)\wedge(\bar\epsilon_3\wedge\epsilon_3+\bar\epsilon_4\wedge\epsilon_4)}{xy\sqrt{XY}}\,. \end{aligned} \tag{6.48}$$

Notice that Φ_1 and Φ_2 are square integrable, in that they are well behaved at both small and large asymptotic distance. For the cases in which the eight-dimensional Calabi-Yau spaces are regular [58], these harmonic forms can be used to construct completely non-singular M2-brane solutions to eleven-dimensional supergravity.

2. Cohomogeneity-four Metrics on Resolved Cones over L^{pqrs}

We now turn to the general cohomogeneity-four metrics on resolved Calabi-Yau cones over the seven-dimensional Einstein-Sasaki spaces L^{pqrs}, which can be written as in chapter III

$$\begin{aligned} ds_8^2 &= \tfrac{1}{4}(u_1^2\,dx_1^2+u_2^2\,dx_2^2+u_3^2\,dx_3^2+u_4^2\,dx_4^2) \\ &+\frac{1}{u_1^2}\,[d\tau+(x_2+x_3+x_4)d\phi+(x_2x_3+x_2x_4+x_3x_4)d\psi+x_2x_3x_4d\chi]^2 \\ &+\frac{1}{u_2^2}\,[d\tau+(x_1+x_3+x_4)d\phi+(x_1x_3+x_1x_4+x_3x_4)d\psi+x_1x_3x_4d\chi]^2 \\ &+\frac{1}{u_3^2}\,[d\tau+(x_1+x_2+x_4)d\phi+(x_1x_2+x_1x_4+x_2x_4)d\psi+x_1x_2x_4d\chi]^2 \\ &+\frac{1}{u_4^2}\,[d\tau+(x_1+x_2+x_3)d\phi+(x_1x_2+x_1x_3+x_2x_3)d\psi+x_1x_2x_3d\chi]^2\,, \end{aligned} \tag{6.49}$$

where

$$\begin{aligned} u_1^2 &= \frac{(x_2-x_1)(x_3-x_1)(x_4-x_1)}{X_1}\,, \qquad u_2^2 = \frac{(x_1-x_2)(x_3-x_2)(x_4-x_2)}{X_2}\,, \\ u_3^2 &= \frac{(x_1-x_3)(x_2-x_3)(x_4-x_3)}{X_3}\,, \qquad u_4^2 = \frac{(x_1-x_4)(x_2-x_4)(x_3-x_4)}{X_4}\,, \\ X_1 &= x_1(a-x_1)(b-x_1)(c-x_1)-2M_1\,, \\ X_2 &= x_2(a-x_2)(b-x_2)(c-x_2)-2M_2\,, \\ X_3 &= x_3(a-x_3)(b-x_3)(c-x_3)-2M_3\,, \\ X_4 &= x_4(a-x_4)(b-x_4)(c-x_4)-2M_4\,. \end{aligned} \tag{6.50}$$

Calabi-Yau structure

We shall choose the vielbein basis

$$\begin{aligned}
e^1 &= \tfrac{1}{2}u_1\,dx_1\,, \quad u_3 = \tfrac{1}{2}u_2\,dx_2\,, \quad e^5 = \tfrac{1}{2}u_3\,dx_3\,, \quad e^7 = \tfrac{1}{2}u_4\,dx_4\,, \\
e^2 &= \frac{1}{u_1}\,[d\tau+(x_2+x_3+x_4)d\phi+(x_2x_3+x_2x_4+x_3x_4)d\psi+x_2x_3x_4d\chi]\,, \\
e^4 &= \frac{1}{u_2}\,[d\tau+(x_1+x_3+x_4)d\phi+(x_1x_3+x_1x_4+x_3x_4)d\psi+x_1x_3x_4d\chi]\,, \\
e^6 &= \frac{1}{u_3}\,[d\tau+(x_1+x_2+x_4)d\phi+(x_1x_2+x_1x_4+x_2x_4)d\psi+x_1x_2x_4d\chi]\,, \\
e^8 &= \frac{1}{u_4}\,[d\tau+(x_1+x_2+x_3)d\phi+(x_1x_2+x_1x_3+x_2x_3)d\psi+x_1x_2x_3d\chi]\,. \quad (6.51)
\end{aligned}$$

The holomorphic vielbein are then given by

$$\epsilon^1 = e^1+\mathrm{i}\,e^2\,, \quad \epsilon^2 = e^3+\mathrm{i}\,e^4\,, \quad \epsilon^3 = e^5+\mathrm{i}\,e^6\,, \quad \epsilon^4 = e^7+\mathrm{i}\,e^8\,. \qquad (6.52)$$

Defining

$$\begin{aligned}
J &= \frac{\mathrm{i}}{2}\,(\bar\epsilon^1\wedge\epsilon^1+\bar\epsilon^2\wedge\epsilon^2+\bar\epsilon^3\wedge\epsilon^3+\bar\epsilon^4\wedge\epsilon^4)\,, \\
\Omega &= e^{\mathrm{i}\,\nu}\,\epsilon^1\wedge\epsilon^2\wedge\epsilon^3\wedge\epsilon^4\,, \qquad (6.53)
\end{aligned}$$

where

$$\nu = 4\tau+3(a+b+c)\,\phi+2(ab+bc+ca)\,\psi+abc\,\chi\,, \qquad (6.54)$$

it is straightforward to verify that

$$dJ = 0\,, \qquad d\Omega = 0\,, \qquad (6.55)$$

and hence that the metric is indeed Ricci-flat Kähler, with J being the Kähler form and Ω the holomorphic $(4,0)$-form.

Harmonic $(3,1)$-forms

We find that harmonic $(3,1)$-forms can be constructed as follows. First, it can be verified that

$$G_{(3,1)} = \frac{1}{X_1}\, e^{\mathrm{i}\,\nu}\, \bar\epsilon^1\wedge\epsilon^2\wedge\epsilon^3\wedge\epsilon^4 \qquad (6.56)$$

is closed, and hence harmonic. Next, we define the function

$$\gamma = \sqrt{\frac{X_2X_3X_4}{X_1}}\, e^{\mathrm{i}\,\nu}\,, \qquad (6.57)$$

which can be shown to satisfy the relation

$$\begin{aligned} d\gamma \;=\; & \frac{u_1\, e^{\mathrm{i}\,\nu}}{u_2u_3u_4\,(x_1-x_2)(x_1-x_3)(x_1-x_4)}\Big(u_1\,(x_2-x_3)(x_2-x_4)(x_4-x_3)X_1'\,\bar\epsilon^1 \\ & -u_2\,(x_3-x_1)(x_3-x_4)(x_4-x_1)\,X_2'\,\epsilon^2+u_3\,(x_1-x_2)(x_4-x_1)(x_4-x_2)\,X_3'\,\epsilon^3 \\ & +u_4\,(x_1-x_2)(x_3-x_1)(x_2-x_3)\,X_4'\,\epsilon^4\Big)\,, \end{aligned} \qquad (6.58)$$

where X_i' denotes the derivative of X_i with respect to its argument x_i. It therefore follows that $d\gamma\wedge G_{(3,1)} = 0$, and so

$$\Phi_{(3,1)} = f(\gamma)\, G_{(3,1)} \qquad (6.59)$$

is a harmonic $(3,1)$-form for any function f. In particular, we have a family of harmonic $(3,1)$-forms given by

$$\Psi_{(3,1)} = \frac{X_2^\delta\, X_3^\delta\, X_4^\delta}{X_1^{\delta+1}}\, e^{(2\delta+1)\,\mathrm{i}\,\nu}\, \bar\epsilon^1\wedge\epsilon^2\wedge\epsilon^3\wedge\epsilon^4 \qquad (6.60)$$

for any constant δ. For nonzero δ, these forms preserve only a $U(1)^3$ subgroup of the $U(1)^4$ isometry of the eight-dimensional space. Note that $\Psi_{(3,1)}$ has a singularity only at short distance if $\delta \geq 0$, where we have taken x_1 to be the radial direction. Additional harmonic $(3,1)$-forms can be constructed by permuting the x_i directions, but these forms have singularities for all x_1. They are analogous to the $(2,1)$-forms Φ_1,

Φ_2 and Φ_3 in (6.27) for a six-dimensional space, and they do not support nontrivial flux.

Harmonic (2, 2)-forms

We can also construct harmonic (2, 2)-forms as follows. We define (2, 2)-forms

$$\begin{aligned} G_{(2,2)} &= f\,(\bar\epsilon^1\wedge\epsilon^1\wedge\bar\epsilon^2\wedge\epsilon^2+\bar\epsilon^3\wedge\epsilon^3\wedge\bar\epsilon^4\wedge\epsilon^4) \\ &\quad +g\,(\bar\epsilon^1\wedge\epsilon^1\wedge\bar\epsilon^3\wedge\epsilon^3+\bar\epsilon^2\wedge\epsilon^2\wedge\bar\epsilon^4\wedge\epsilon^4) \\ &\quad +h\,(\bar\epsilon^1\wedge\epsilon^1\wedge\bar\epsilon^4\wedge\epsilon^4+\bar\epsilon^2\wedge\epsilon^2\wedge\bar\epsilon^3\wedge\epsilon^3)\,, \end{aligned} \tag{6.61}$$

where f, g and h are functions of (x_1, x_2, x_3, x_4). Imposing the closure of $G_{(2,2)}$ leads to three independent solutions for f, g and h, namely

$$f = g = h = 1\,, \tag{6.62}$$

$$\begin{aligned} f &= \frac{1}{(x_1-x_2)^2(x_1-x_3)(x_2-x_4)(x_3-x_4)^2}\,, \\ g &= \frac{x_1\,(2x_4-x_2-x_3)+x_2\,(2x_3-x_4)-x_3\,x_4}{(x_1-x_2)^2(x_1-x_3)^2(x_2-x_4)^2(x_3-x_4)^2}\,, \\ h &= \frac{1}{(x_1-x_2)(x_1-x_3)^2(x_2-x_4)^2(x_3-x_4)}\,, \end{aligned} \tag{6.63}$$

and

$$\begin{aligned} f &= \frac{1}{(x_1-x_3)(x_2-x_3)^2(x_1-x_4)^2(x_2-x_4)}\,, \\ g &= \frac{1}{(x_1-x_3)^2(x_2-x_3)(x_1-x_4)(x_2-x_4)^2}\,, \\ h &= \frac{x_1\,(x_3+x_4-2x_2)+x_2\,(x_3+x_4)-2x_3\,x_4}{(x_1-x_3)^2(x_2-x_3)^2(x_1-x_4)^2(x_2-x_4)^2}\,. \end{aligned} \tag{6.64}$$

These forms are somewhat analogous to the (2, 1)-forms Φ_4 and Φ_5 given in (6.27) for a six-dimensional space. The first solution, (6.62), is just the harmonic (2, 2)-form

$J\wedge J$. It follows from (6.61) that $J\wedge G_{(2,2)}$ is proportional to $(f+g+h)$, and so $J\wedge G_{(2,2)}$ is non-zero for (6.62). However, each of the solutions (6.63) and (6.64) satisfies $f+g+h = 0$, and so these two harmonic $(2,2)$-forms satisfy the supersymmetric condition

$$J\wedge G_{(2,2)} = 0\,. \tag{6.65}$$

Notice also that these harmonic $(2,2)$-forms are square integrable. These can be used to construct modified M2-brane solutions, which have only orbifold-type singularities. Note that none of these cohomogeneity-four Calabi-Yau spaces are completely regular [58].

3. M2-brane Solutions

We can use these eight-dimensional spaces, and the harmonic 4-forms which they support, to construct a modified M2-brane solution to eleven-dimensional supergravity, given by

$$\begin{aligned} ds_{11}^2 &= H^{-2/3}(-dt^2+dx_1^2+dx_2^2)+H^{1/3}ds_8^2\,,\\ F_{(4)} &= dt\wedge dx_1\wedge dx_2\wedge dH^{-1}+m\,L_{(4)}\,, \end{aligned} \tag{6.66}$$

where

$$\Box H = -\frac{1}{48}m^2 L_{(4)}^2\,, \tag{6.67}$$

and $L_{(4)}$ is an (anti)self-dual harmonic 4-form on the eight-dimensional space with the metric ds_8^2.

Let us first consider the case with $m = 0$, for which the Laplace equation on the Calabi-Yau metric is separable. The solution for general dimensionality is presented in the appendix E. Here we just give a solution for the eight-dimensional case that depends only on the radial variable x_1; it is given by

$$H = \int^{x_1} \frac{3Q}{X(x_1')} dx_1' \,. \tag{6.68}$$

Thus in the asymptotic region at large x_1, the function H has the behavior

$$H = \frac{Q}{x_1^3}\Big(1 + \frac{c_2}{x_1} + \cdots\Big)\,, \qquad \text{where} \;\; c_2 = \frac{3}{4}(\alpha+\beta+\gamma)\,. \tag{6.69}$$

We have taken an arbitrary additive constant to zero, so that the geometry is asymptotically $\mathrm{AdS}_4 \times L^{pqrs}$. Since x_1 has dimension two, we see that there is a non-mesonic dimension-two scalar operator being turned on with expectation value c_2.

It is especially interesting to construct M2-brane solutions using one of the square-integrable harmonic $(2,2)$-forms that we found previously, since this guarantees that with the appropriate integration constants the only singularities are of orbifold type. This is because the 4-form prevents the blown-up 4-cycle from collapsing. Moreover, examples of regular eight-dimensional Calabi-Yau spaces that have been discussed in [58] can be used to construct completely non-singular M2-brane solutions. The resulting geometry smoothly interpolates between $\mathrm{AdS}_4 \times L^{pqrs}$ asymptotically, and a direct product of $\mathrm{Minkowski}_3$ and a compact space at short distance. Many examples of cohomogeneity-one solutions of this type were constructed in [81, 82, 83]. Although not much is known even about the UV conformal fixed point of the dual three-dimensional $\mathcal{N} = 2$ super Yang-Mills field theory, based on the geometrical properties of the supergravity background it flows to a confining phase in the IR region.

E. Harmonic Forms on Higher-dimensional Resolved Cones

In this section, we extend some of the constructions of harmonic middle-dimension forms to the case of higher-dimensional metrics on the resolutions of cones over Einstein-Sasaki spaces. We take as our starting point the local Ricci-flat Kähler metrics in dimension $D = 2n+4$ that were considered in [58]:

$$\begin{aligned} d\tilde{s}^2 &= \frac{x+y}{4X}dx^2 + \frac{x+y}{4Y}dy^2 + \frac{X}{x+y}\Big[d\tau + \frac{y}{\alpha}\sigma\Big]^2 + \frac{Y}{x+y}\Big[d\tau - \frac{x}{\alpha}\sigma\Big]^2 + \frac{xy}{\alpha}d\Sigma_n^2 \\ \sigma &= d\psi + A\,, \qquad X = x(x+\alpha) - \frac{2\mu}{x^n}\,, \qquad Y = y(\alpha - y) + \frac{2\nu}{y^n}\,, \end{aligned} \tag{6.70}$$

where $d\Sigma_n^2$ is a metric on a $2n$-dimensional Einstein-Kähler space Z, satisfying $R_{ab} = 2(n+1)\,g_{ab}$, with Kähler form $J = \frac{1}{2}dA$. (We have made some minor changes of coordinates compared to the metric presented in [58].) For convenience, we shall set the constant α to unity. This can always be done, when $\alpha \neq 0$, by means of coordinate scalings together with an overall rescaling of the Ricci-flat metric. The special case $\alpha = 0$ can be recovered via a limiting procedure.

Next, we define the 2-forms

$$\omega_x = \tfrac{1}{2}dx\wedge(d\tau + y\,\sigma)\,, \qquad \omega_y = \tfrac{1}{2}dy\wedge(d\tau - x\,\sigma)\,, \qquad \omega = xy\,J\,. \tag{6.71}$$

It can easily be verified that $\hat{J} \equiv \omega_x - \omega_y + \omega$ is closed and, in fact, this is the Kähler form of the Ricci-flat Kähler metric (6.70). In the case that n is even ($n = 2m$), we find that the middle-degree form

$$G_{(2m+2)} = \frac{1}{(xy)^{2m+1}}\Big[\omega_x\wedge\omega_y\wedge\omega^{m-1} + \frac{1}{m+1}\,(\omega_x - \omega_y)\wedge\omega^m - \frac{1}{m(m+1)}\,\omega^{m+1}\Big] \tag{6.72}$$

is closed. Since it is also self-dual, it follows that it is a harmonic form. This generalises the harmonic $(2,2)$-form Φ_1 in eight dimensions given in (6.48) and is somewhat analogous to the $(2,1)$-forms Φ_4 and Φ_5 given in (6.27) for a six dimensions.

Further harmonic forms can be obtained if one takes the Einstein-Kähler base metric $d\Sigma_n^2$ to be a product of Einstein-Kähler metrics. For example, if we choose it to be the product of metrics on two copies of $\mathbb{CP}^m$ (recall that we are considering the case where $n = 2m$ is even), with Kähler forms J_1 and J_2 respectively (so $J = J_1 + J_2$), then defining

$$\omega_1 = xy\, J_1\,, \qquad \omega_2 = xy\, J_2\,, \tag{6.73}$$

we find that

$$\widetilde{G}_{(2m+2)} = \frac{1}{(x+y)^2\,(xy)^m}\,(\omega_x + \omega_y)\wedge\sum_{p=0}^{m}(-1)^p\,\omega_1^{m-p}\wedge\omega_2^p \tag{6.74}$$

is closed and self-dual, and therefore it is harmonic.

F. Conclusions

We have investigated the Kähler moduli associated with blowing up a 2-cycle or 4-cycle on Calabi-Yau cones over the L^{abc} spaces. This yields a countably infinite number of backgrounds with ALE singularities on which perturbative string dynamics is well-defined. Although adding D3-branes induces a power-law type singularity at short distance, one can still use the AdS/CFT dictionary to relate the blown-up cycles to deformations of the dual quiver gauge theory close to the UV conformal fixed point. In particular, we identify the non-mesonic dimension-two real scalar operator that acquires a vev, thereby generalizing the state/operator correspondence for the resolved conifold over T^{11} [50] and $T^{11}/\mathbb{Z}_2$ [54] to resolved cones over the L^{abc} spaces. On the other hand, blowing up a 4-cycle corresponds to a dimension-six non-mesonic scalar operator getting a vev.

The resolved cones over the cohomogeneity-two L^{abc} spaces support various harmonic $(2,1)$-forms, some of which depend nontrivially on three non-azimuthal coordinate directions. These forms can be further generalized by a multiplicative function, so long as the exterior derivative of this function satisfies a certain orthogonality condition. In particular, there are harmonic $(2,1)$-forms which depend on continuous parameters. 3-forms carrying nontrivial flux correspond to fractional D3-branes, while those which do not correspond to giving a vev to a dimension-six operator.

For the D3-brane solutions constructed with resolved cones over L^{abc}, we have restricted ourselves to the case in which the D3-branes are smeared over the blown-up cycle. As we already mentioned, this yields to a power-law singularity at short distance. For solutions involving a 3-form field, one may be able to smooth out this singularity by a complex deformation of the Calabi-Yau space that results in a blown-up 3-cycle. Although it has been shown that there are obstructions to the existence of complex deformations of cones over Y^{pq} spaces, there are other subsets of the L^{abc} cones which do allow for complex structure deformations [70, 71, 48]. It would be useful to construct the explicit metrics describing these deformed L^{abc} cones, as well as the non-singular supergravity solutions that describe fractional D3-branes on these spaces.

Alternatively, one can consider stacking the D3-branes at a single point. For the case of the resolved conifold, this has been shown to yield a completely regular solution [59]. Perhaps there are analogous constructions with the resolved cones over L^{abc}. With the exceptions of $T^{1,1}$, $T^{1,1}/\mathbb{Z}_2$ and $Y^{2,1}$, the resolved L^{abc} cones have orbifold singularities. Although these singularities will remain there when D3-branes are stacked at a single point, perturbative string dynamics is well-defined on such backgrounds.

One can also consider fibering a D3-brane worldvolume direction (which need not be compact) over a resolved L^{abc} cone in such a way that the resulting geometry only has orbifold-type singularities. For the case of the resolved conifold, such a

D3-brane solution has already been constructed and is completely regular, and it is also supersymmetric [78]. The corresponding D3-brane solutions for the resolved L^{abc} cones are currently being investigated [79].

We also discussed the geometry of higher-dimensional Calabi-Yau spaces with blown-up cycles, as well as the various harmonic forms which live on them. In particular, we have found that eight-dimensional resolved cones over the L^{pqrs} spaces support harmonic 4-forms that are square integrable. They can be used to construct M2-brane solutions of eleven-dimensional supergravity which have only orbifold-type singularities. Unfortunately, not much is known about the dual three-dimensional $\mathcal{N} = 2$ gauge theories, other than that they flow from a UV conformal fixed point to a confining phase in the IR region.

Lastly, the type IIB supergravity backgrounds dual to certain marginal deformations (β deformations) of the conformal fixed point of the Y^{pq} and L^{abc} quiver gauge theories were obtained in [80, 84]. The solution-generating method works for any gravity solution with $U(1)\times U(1)$ global symmetry. It might be interesting to see if these deformations can be applied to the gravity solutions discussed in this paper, since they possess the necessary global symmetry.

CHAPTER VII

CONCLUSION

In this dissertation, we have studied the higher dimensional Kerr-AdS black hole solutions, and showed how they admit further generalisation in which NUT-type parameters are introduced.

we first constructed Kerr-NUT-AdS metrics in all dimensions where the rotation parameters are specialised so that the metrics have cohomogeneity 2. The nature of the generalisation is then analogous to the way in which a NUT parameter can be introduced in the four-dimensional Kerr-AdS metrics.

This strategy was then applied to the general D dimensional Kerr-AdS metrics with cohomogeneity $[D/2]$. By making a choice of coordinates parameterising the latitude variables in the Kerr-AdS metrics, we were able to rewrite the Kerr-AdS solutions in such a way that the metrics become diagonal in a set of unconstrained latitude coordinates y_α. These coordinates then appear in a manner that closely parallels that of the radial variable r, and this immediately suggests a natural generalisation of the Kerr-AdS metrics to include $([D/2]-1)$ NUT charges. After further changes of variable, we arrived at the very simple expressions (3.22) and (3.33) for the general Kerr-NUT-AdS metrics in all odd and even dimensions. These expressions can be thought of as natural generalisations of the four-dimensional results obtained in [31].

The new Kerr-NUT-AdS metrics have some remarkable properties. We showed that the Hamilton-Jacobi and Klein-Gordon equations are separable in the Kerr-NUT-AdS background with cohomogeneity 2 and constructed the associated irreducible rank-2 Killing tensor whose existence reflects the hidden symmetry that leads to the separability. We also demonstrated that the general cohomogeneity-n Kerr-NUT-AdS solutions in D dimensions admit $[D/2]$ linearly-independant and mutually orthogonal null geodesic congruences, which enables us to write the metrics in multi-Kerr-Schild form.

We also studied the BPS limits of the Kerr-NUT-AdS metrics. These yield, after Euclideanisation, new examples of Einstein-Sasaki metrics in odd dimensions, and Ricci-flat Kähler cones in even dimensions. In six dimension, this gives a resolved Calabi-Yau cone over L^{abc} spaces with a blow up 2-cycle or 4-cycle. We discussed D3-branes on this Calabi-Yau cone and their applications in AdS/CFT correspondence.

To conclude, our results not only contribute to the classification of solutions in general relativity, but also provide interesting non-trivial backgrounds for string theory.

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APPENDIX A

ANOTHER FORM FOR THE ODD-DIMENSIONAL KERR-NUT-ADS METRICS WITH COHOMOGENEITY 2

If we perform the same angular redefinitions (2.47) in the odd-dimensional Kerr-NUT-AdS metrics with cohomogeneity 2 (2.11), they may be re-expressed as

$$\begin{aligned} ds^2 \;=\;& \frac{r^2+v^2}{X}\,dr^2+\frac{r^2+v^2}{Y}\,dv^2+\frac{(r^2+a^2)(a^2-v^2)}{\Xi_a(a^2-b^2)}\,d\Sigma^2_{p-1}+\frac{(r^2+b^2)(b^2-v^2)}{\Xi_b(b^2-a^2)}\,d\widetilde{\Sigma}^2_{q-1} \\ &+\frac{a^2b^2}{r^2v^2}\left[dt-(r^2-v^2)d\phi-r^2v^2d\chi-\frac{(r^2+a^2)(a^2-v^2)}{a\Xi_a(a^2-b^2)}\,A-\frac{(r^2+b^2)(b^2-v^2)}{b\Xi_b(b^2-a^2)}\,B\right]^2 \\ &-\frac{X}{r^2+v^2}\left[dt+v^2d\phi-\frac{a(a^2-v^2)}{\Xi_a(a^2-b^2)}\,A-\frac{b(b^2-v^2)}{\Xi_b(b^2-a^2)}\,B\right]^2 \\ &+\frac{Y}{r^2+v^2}\left[dt-r^2d\phi-\frac{a(r^2+a^2)}{\Xi_a(a^2-b^2)}\,A-\frac{b(r^2+b^2)}{\Xi_b(b^2-a^2)}\,B\right]^2, \end{aligned} \tag{A.1}$$

where we have defined X and Y as

$$\begin{aligned} X \;\equiv\;& \frac{U}{(r^2+a^2)^{p-1}\,(r^2+b^2)^{q-1}} \\ =\;& \frac{(1+g^2r^2)(r^2+a^2)(r^2+b^2)}{r^2}-\frac{2M}{(r^2+a^2)^{p-1}\,(r^2+b^2)^{q-1}}\,, \\ Y \;\equiv\;& \frac{V}{(a^2-v^2)^{p-1}\,(b^2-v^2)^{q-1}} \\ =\;& \frac{-(1-g^2v^2)(a^2-v^2)(b^2-v^2)}{v^2}+\frac{2L}{(a^2-v^2)^{p-1}\,(b^2-v^2)^{q-1}}\,. \end{aligned} \tag{A.2}$$

APPENDIX B

A SYMMETRY BETWEEN THE TIME AND AZIMUTHAL COORDINATES

It can be observed from the expressions for the Kerr-NUT-AdS metrics that we obtained in chapter III, section C that the time coordinate and the azimuthal angular coordinates appear on a very parallel footing. It is possible, therefore, to present further simplifications of the expressions (3.17) and (3.28) in odd and even dimensions that exploit this observation.

For the Kerr-NUT-AdS metrics in odd dimensions $D = 2n+1$, we make the definitions

$$\begin{aligned} a_0 &= \frac{1}{g}\,, \qquad \Gamma_I = \prod_{\nu=1}^{n}(a_I^2 - x_\nu^2)\,, \quad 0 \le I \le n\,, \\ \tilde{\phi}_0 &= -g^{2n}\,\tilde{t}\,, \qquad X_\mu = \frac{g^2}{x_\mu^2}\prod_{I=0}^{n}(a_I^2 - x_\mu^2) + 2M_\mu\,. \end{aligned} \tag{B.1}$$

The metric (3.17) can then be written as

$$ds^2 = \sum_{\mu=1}^{n}\Big\{\frac{U_\mu}{X_\mu}\,dx_\mu^2 + \frac{X_\mu}{U_\mu}\Big(\sum_{I=0}^{n}\frac{a_I^2\,\Gamma_I\,d\tilde{\phi}_I}{a_I^2 - x_\mu^2}\Big)^2\Big\} - \frac{(\prod_{k=1}^{n} a_k^2)}{(\prod_{\mu=1}^{n} x_\mu^2)}\Big(\sum_{I=0}^{n}\Gamma_I\,d\tilde{\phi}_I\Big)^2\,. \tag{B.2}$$

For the Kerr-NUT-AdS metrics in even dimensions $D = 2n$, we make the definitions

$$\begin{aligned} a_0 &= \frac{1}{g}\,, \qquad \Gamma_I = \prod_{\nu=1}^{n}(a_I^2 - x_\nu^2)\,, \quad 0 \le I \le n-1\,, \\ \tilde{\phi}_0 &= -g^{2n-2}\,\tilde{t}\,, \qquad X_\mu = -g^2\prod_{I=0}^{n-1}(a_I^2 - x_\mu^2) - 2M_\mu\,x_\mu\,. \end{aligned} \tag{B.3}$$

The metric (3.28) can then be written as

$$ds^2 = \sum_{\mu=1}^{n}\Big\{\frac{U_\mu}{X_\mu}\,dx_\mu^2 + \frac{X_\mu}{U_\mu}\Big(\sum_{I=0}^{n-1}\frac{\Gamma_I\,d\tilde{\phi}_I}{a_I^2 - x_\mu^2}\Big)^2\Big\}\,. \tag{B.4}$$

APPENDIX C

INVERSION SYMMETRY OF THE $D=4$ ROTATING BLACK HOLE

We made the observation in chapter III, section C that there exists an inversion symmetry in all the Kerr-NUT-AdS metrics, in which one of the rotation parameters is inverted through the AdS radius, together with corresponding scalings of the other parameters. A case of particular interest is in four dimensions. The four-dimensional Kerr-AdS metric can be written as

$$\begin{aligned} ds^2 &= \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta}\, d\theta^2 - \frac{\Delta_r}{\rho^2}(dt - \frac{a}{\Xi}\sin^2\theta\, d\phi^2)^2 + \frac{\Delta_\theta\, \sin^2\theta}{\rho^2}(a\, dt - \frac{r^2+a^2}{\Xi} d\phi)^2\,, \\ \rho^2 &= r^2 + a^2\cos^2\theta\,, \Delta_r = (1+g^2r^2)(r^2+a^2) - 2M\, r\,, \Delta_\theta = 1 - a^2g^2\cos^2\theta\,, \end{aligned} \tag{C.1}$$

where $\Xi = 1 - a^2g^2$. It is straightforward to verify that the metric is invariant under the transformation

$$\begin{aligned} & a \rightarrow \frac{1}{a\, g^2}\,, \qquad M \rightarrow \frac{M}{a^3g^3}\,, \\ & r \rightarrow \frac{r}{a\, g}\,, \quad \cos\theta \rightarrow ag\, \cos\theta\,, \quad \phi \rightarrow -\frac{\phi}{a\, g}\,, \quad t \rightarrow a\, g\, t + \frac{\phi}{g}\,. \end{aligned} \tag{C.2}$$

Note that the metric (C.1) is written in a frame that is asymptotically rotating at infinity. As a consequence the required ignorable coordinate transformations in (C.2) that bring the transformed metric back to its original form do not, unlike those given in (3.32) for an asymptotically-static frame, simply involve an exchange of the azimuthal coordinate and g times the time coordinate. If we define an asymptotically-static frame by replacing the azimuthal coordinate with $\hat{\phi} = \phi + ag^2t$, then the last two transformations in (C.2) become simply

$$\hat{\phi} \rightarrow gt\,, \qquad gt \rightarrow \hat{\phi}\,. \tag{C.3}$$

APPENDIX D

COMPLEX STRUCTURE AND FIRST-ORDER EQUATIONS

In this appendix, we construct Ricci-flat Kähler spaces in dimension $D = 2n+4$, built over an Einstein-Kähler base space of real dimension $2n$ with metric $d\Sigma_n^2$. We normalise this metric so that it satisfies $R_{ij} = 2(n+1)g_{ij}$. Its Kähler form will be written as $J = \frac{1}{2}dA$. We may also assume that it admits a holomorphic $(n,0)$-form Ω, satisfying (see, for example, section 4 of [26])

$$d\Omega = \mathrm{i}\,(n+1)A\wedge\Omega\,. \tag{D.1}$$

The ansatz for the $(3n+4)$-dimensional Ricci-flat Kähler metrics will be

$$d\hat{s}^2 = u^2dx^2+v^2dy^2+a^2(d\tau+f_1\sigma)^2+b^2(d\tau+f_2\sigma)^2+c^2d\Sigma_n^2\,, \tag{D.2}$$

where a, b, c, u, v, f_1 and f_2 are functions of x and y, and

$$\sigma = d\psi+A\,. \tag{D.3}$$

We define the vielbein

$$\hat{e}^1 = udx\,,\qquad \hat{e}^2 = a(d\tau+f_1\sigma)\,,\qquad \hat{e}^3 = vdy\,,\qquad \hat{e}^4 = b(d\tau+f_2\sigma)\,,\qquad \hat{e}^i = ce^i\,, \tag{D.4}$$

where e^i is a vielbein for the Einstein-Kähler base metric $d\Sigma_n^2$.

We make the ansatz

$$\hat{J} = e^1\wedge e^2+e^3\wedge e^4+c^2J \tag{D.5}$$

for the Kähler form. It is then natural to define a complex vielbein by

$$\hat{\epsilon}^1 = \hat{e}^1+\mathrm{i}\,\hat{e}^2\,,\qquad \hat{\epsilon}^2 = \hat{e}^3+\mathrm{i}\,\hat{e}^4\,,\qquad \hat{\epsilon}^i = c\,\epsilon^i\,, \tag{D.6}$$

where ϵ^i is a complex vielbein for the base metric $d\Sigma_n^2$. We also make the ansatz

$$\hat{\Omega} = e^{\mathrm{i}\alpha\tau+\mathrm{i}\beta\psi}\, c^n\, \hat{\epsilon}^1\wedge\hat{\epsilon}^2\wedge\Omega \tag{D.7}$$

for the holomorphic $(n+2,0)$-form. The conditions for $d\hat{s}^2$ to be Ricci flat and Kähler are then given by

$$d\hat{J} = 0\,, \qquad d\hat{\Omega} = 0\,. \tag{D.8}$$

One immediately finds that the constant β should be chosen to be

$$\beta = n+1\,. \tag{D.9}$$

However, the constant α can be left arbitrary.

We now obtain the first-order equations:

$$\begin{aligned}
d\hat{J} = 0: &\quad (bv)' - (au)^{\boldsymbol{\cdot}} = 0\,, \qquad (c^2)' - 2auf_1 = 0\,, \qquad (c^2)^{\boldsymbol{\cdot}} - 2bvf_2 = 0\,,\\
d\hat{\Omega} = 0: &\quad \alpha\, uvc^n - (avc^n)' - (buc^n)^{\boldsymbol{\cdot}} = 0\,,\\
&\quad \alpha buc^n f_2 - (n+1)buc^n + [abc^n(f_1-f_2)]' = 0\,,\\
&\quad \alpha avc^n f_1 - (n+1)avc^n - [abc^n(f_1-f_2)]^{\boldsymbol{\cdot}} = 0\,.
\end{aligned} \tag{D.10}$$

The constant α appearing in the first-order equations (D.10) is always trivial, in the sense that it can be set to any chosen non-zero value without loss of generality. To see this, we perform the following rescaling of coordinates and functions:

$$\begin{aligned}
&x \to \lambda\, x\,, \qquad y \to \lambda\, y\,, \qquad \tau \to \lambda\,\tau\,,\\
&c \to \lambda\, c\,, \qquad f_1 \to \lambda\, f_1 \qquad f_2 \to \lambda\, f_2\,,
\end{aligned} \tag{D.11}$$

whilst leaving the functions a, b, u and v unscaled. It can be seen that the effect of these rescalings is to scale the metric $d\hat{s}^2$ in (D.2) according to

$$d\hat{s}^2 \longrightarrow \lambda^2\, d\hat{s}^2\,. \tag{D.12}$$

The rescalings have the effect of replacing α by $\lambda\alpha$ in the first-order equations (D.10), thus giving

$$\begin{aligned}
d\hat{J} = 0: &\qquad (bv)' - (au)^{\cdot} = 0\,, \qquad (c^2)' - 2au f_1 = 0\,, \qquad (c^2)^{\cdot} - 2bv f_2 = 0\,,\\
d\hat{\Omega} = 0: &\qquad \lambda\alpha\, uvc^n - (avc^n)' - (buc^n)^{\cdot} = 0\,,\\
&\qquad \lambda\alpha\, buc^n f_2 - (n+1)buc^n + [abc^n(f_1 - f_2)]' = 0\,,\\
&\qquad \lambda\alpha\, avc^n f_1 - (n+1)avc^n - [abc^n(f_1 - f_2)]^{\cdot} = 0\,.
\end{aligned} \tag{D.13}$$

Since a rescaling of a Ricci-flat metric by a non-zero constant leaves it Ricci-flat, it follows that the constant λ can be chosen at will, and so no generality is lost by setting α to any desired finite and non-zero value.

APPENDIX E

SEPARABILITY OF LAPLACIAN ON CALABI-YAU METRICS

We consider the Calabi-Yau metrics obtained in chapter II, III. The metric can be expressed as

$$\begin{aligned} ds^2 &= \sum_{\mu=1}^{n} \Big[\frac{U_\mu\, dx_\mu^2}{4X_\mu} + \frac{X_\mu}{U_\mu} (\sum_{i=0}^{n-1} W_i d\phi_i)^2\Big]\,, \\ X_\mu &= x_\mu \prod_{i=1}^{n-1} (\alpha_i - x_\mu) - 2\ell_\mu\,, \qquad U_\mu = {\prod_{\nu=1}^{n}}' \,(x_\nu - x_\mu)\,, \end{aligned} \tag{E.1}$$

where W_i is defined by

$$\prod_{\mu=1}^{n} (1+qx_\mu) \equiv \sum_{i=0}^{n-1} W_i\, q^{i+1}\,. \tag{E.2}$$

It turns out that the equation $\Box H = 0$ is separable in the x_μ coordinates, where $\Box$ is the Laplacian taken on the above metric. (The separability for the more general non-extremal Kerr-NUT-AdS metrics was shown explicitly in [76, 37, 77]. Making the ansatz

$$H = \Big(\prod_{\mu=1}^{n} H_\mu(x_\mu)\Big)\, \exp\Big(2\mathrm{i} \sum_{i=0}^{n-1} (-1)^i a_i \phi_{n-1-i}\Big)\,, \tag{E.3}$$

for the harmonic function, we find that the $H_\mu(x_\mu)$ satisfy

$$(X_\mu H_\mu')' - \Big(\frac{(\sum_{i=0}^{n-1} a_i\, x_\mu^i)^2}{X_\mu} + \sum_{i=1}^{n-2} b_i x_\mu^i\Big)\, H_\mu = 0\,, \tag{E.4}$$

where a prime on H_μ or X_μ denotes a derivative with respect to its argument x_μ. The system thus has $2n-1$ independent separation constants $a_0, a_1, \ldots a_{n-1}$ and $b_0, b_1, \ldots, b_{n-2}$.

VITA

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