# A NEW COMPUTATIONAL APPROACH TO THE SYNTHESIS OF FIXED ORDER CONTROLLERS

A Dissertation

by

WAQAR AHMAD MALIK

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2007

Major Subject: Mechanical Engineering

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# ABSTRACT


A New Computational Approach to

the Synthesis of Fixed Order Controllers. (December 2007)

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The research described in this dissertation deals with an *open* problem concerning the synthesis of controllers of fixed order and structure. This problem is encountered in a variety of applications. Simply put, the problem may be put as the determination of the set, $\mathcal{S}$ of controller parameter vectors, $K = (k_1, k_2, \ldots, k_l)$, that render Hurwitz a family (indexed by $\mathcal{F}$) of complex polynomials of the form $\{P_0(s,\alpha) + \sum_{i=1}^{l} P_i(s,\alpha)k_i,\ \alpha \in \mathcal{F}\}$, where the polynomials $P_j(s,\alpha),\ j = 0,\ldots,l$ are given data. They are specified by the plant to be controlled, the structure of the controller desired and the performance that the controllers are expected to achieve. Simple examples indicate that the set $\mathcal{S}$ can be non-convex and even be disconnected.

While the determination of the non-emptiness of $\mathcal{S}$ is decidable and amenable to methods such as the quantifier elimination scheme, such methods have not been computationally tractable and more importantly, do not provide a reasonable approximation for the set of controllers. Practical applications require the construction of a set of controllers that will enable a control engineer to check the satisfaction of performance criteria that may not be mathematically well characterized. The transient performance criteria often fall into this category.

From the practical viewpoint of the construction of approximations for $\mathcal{S}$, this dissertation is different from earlier work in the literature on this problem. A novel feature of the proposed algorithm is the exploitation of the interlacing property of Hurwitz polynomials to provide arbitrarily tight outer and inner approximation to $\mathcal{S}$. The approximation is given in terms of the union of polyhedral sets which are constructed systematically using the Hermite-Biehler theorem and the generalizations of the Descartes' rule of signs.

To my Mother and Father.

# TABLE OF CONTENTS

# LIST OF FIGURES

FIGURE Page

FIGURE Page

# CHAPTER I

## INTRODUCTION

The synthesis of fixed structure feedback controllers for dynamic systems is an open problem in control theory where even basic results are unavailable. Moreover this class of problems is very important in applications. The structure of a controller may be constrained in terms of its order (state space dimension) or by a requirement that a control input may only be a function of a certain specified subset of outputs.

The need for this research stems from the following: a) Practical applications require that controllers of fixed order and structure be designed from the empirical data of the plant owing to physical, informational and cost constraints. b) An important need in practical design problems is to have *sets* of feasible solutions meeting various objectives such as stability, robustness, time-delay tolerance etc. so that design tradeoffs can be compared. The present theory of fixed order controllers is far from answering such questions - indeed it cannot, in most cases, determine if even stabilization is possible at all. Furthermore, modern control theory provides optimal controllers of high order that cannot handle the constraint of fixed order or structure and can have severe structural and sensitivity problems.

Listed below are some illustrative problems that arise in different engineering applications and require controllers of fixed order or structure:

- In [1], the authors consider a collection of $N$ Unmanned Underwater Vehicles (UUVs) whose governing equations may be described as:

$$\dot{x}_i \quad = \quad A_i x_i + B_i u_i, \; i = 1, 2, \ldots, N,$$

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The journal model is *IEEE Transactions on Automatic Control.*

$$\dot{q} = \sum_{i=1}^{N} H_i x_i,$$

where $x_i$ is the state of the vehicle and $u_i$ is its control input. The term $q$ is the only information available to all UUVs due to constraints on communication. The problem is that of finding a controller, $u_i = K_i q$, so that the platoon is stabilized. This problem is clearly one of fixed-order (static output feedback) stabilization for a multi-input single-output LTI system.

- In the disk drive industry, the seek times are getting shorter and currently stand for desktop disk drives at 9-12 milliseconds; within this duration, operations such as data encoding and decoding, sensing and control must be completed. The signal processing and control computations associated with these operations are carried out on an inexpensive processor so as to minimize the cost as it is fast becoming a commodity industry. Moreover, only the position and voice coil motor current measurements are available for feedback when the arm is in the data zone. Recent trends towards miniaturization and making the seek times shorter only necessitate low complexity output feedback controllers that guarantee certain bandwidth while rejecting/attenuating repetitive and other disturbances.

- In [2, 3], the authors indicate the need for a bound on the order of the controller for a Hubble telescope based on the considerations of simplicity of implementation, hardware limitations and reliability.

- Applications requiring tuning of control parameters by a computer or human operator force a designer to minimize the number of controller parameters; some applications in this direction are in [4, 5, 6, 7].

In light of the pervasive use of fixed-order controllers in process control and the emergence of new applications such as formations of vehicles, it is imperative to understand whether fixed-order controllers that achieve a specified performance exist and if so, how one can find them and/or compute the set of all such stabilizing controllers that achieve a specified performance.

This dissertation focuses on the problem of determining the set of all controller parameters, $K = (k_1,\ k_2,\ \ldots,\ k_l)$ which render a set of real or complex polynomials Hurwitz, where each member of the set is of the form:

$$P(s, K) = P_o(s) + \sum_{i=1}^{l} k_i P_i(s). \tag{1.1}$$

Some important classes of problems which fall into this category, i.e. *where $K$ appears linearly*, are as follows:

- Consider the problem of stabilizing a single input single output (SISO) proper plant with a transfer function $\frac{N_p}{D_p}(s)$ with a proper controller $\frac{N_c}{D_c}(s)$. If $N_c(s) = n_0 + \ldots + n_m s^m$ and $D_c(s) = s^m + d_{m-1}s^{m-1} + \cdots + d_0$, then the characteristic polynomial for the closed loop system given below must be Hurwitz:

$$\begin{aligned}\Delta(s) &= \Delta(s, n_0, \cdots, n_m, d_0, \ldots, d_{m-1}) \\ &= \underbrace{s^m D_p(s)}_{P_0(s)} + n_0 \underbrace{N_p(s)}_{P_1(s)} + \cdots + n_m \underbrace{s^m N_p(s)}_{P_{m+1}(s)} + d_0 \underbrace{D_p(s)}_{P_{m+2}(s)} + \cdots \\ &\qquad \cdots + d_{m-1} \underbrace{s^{m-1} D_p(s)}_{P_{2m}(s)}.\end{aligned}$$

  The problem of synthesizing fixed order controllers for single input multiple output (SIMO) and multiple input single output (MISO) systems can be cast in a similar form.

- Consider the problem of guaranteeing a phase margin $\phi$ for a SISO proper plant with a transfer function $\frac{N_p}{D_p}(s)$ stabilized by a proper controller $\frac{N_c}{D_c}(s)$.

This requirement [8], may be converted to a specification that the family of polynomials $D_p(s)D_c(s)+e^{j\theta}N_p(s)N_c(s),\ \theta \in (-\phi,+\phi)$, be Hurwitz. If $N_c(s) = n_0 + \ldots + n_m s^m$ and $D_c(s) = s^m + d_{m-1}s^{m-1} + \cdots + d_0$, then each member of the family is of the form:

$$\begin{aligned}\Delta(s,\theta) &= \Delta(s, n_0, \cdots, n_m, d_0, \ldots, d_{m-1}) \\ &= \underbrace{s^m D_p(s)}_{P_0(s)} + n_0 \underbrace{N_p(s)e^{j\theta}}_{P_1(s)} + n_1 \underbrace{sN_p(s)e^{j\theta}}_{P_2(s)} + \cdots + n_m \underbrace{s^m N_p(s)e^{j\theta}}_{P_{m+1}(s)} \\ &\quad + d_0 \underbrace{D_p(s)}_{P_{m+2}(s)} + \cdots + d_{m-1} \underbrace{s^{m-1}D_p(s)}_{P_{2m}(s)}.\end{aligned}$$

- As described later in the dissertation, robust stability and performance specifications, such as upper bounding the $\mathcal{H}_\infty$ norm of a weighted sensitivity transfer function or requiring a closed loop transfer function to be strictly positive real (SPR), can be converted to the problem of ensuring a family of polynomials, of the form described by (1.1), is Hurwitz.

In the above problems, it is assumed that a complete description of a model of the plant is available for controller synthesis. It is widely recognized that an accurate analytical model of the plant may not be available to a control designer. However, it is reasonable in many applications that one will have an empirical model of the plant in terms of its frequency response data and from physical considerations or from the empirical time response data, one may have some coarse information about the plant such as the number of non-minimum phase zeros of the plant etc. In view of this, the problem of synthesizing sets of stabilizing controllers directly from the empirical data and such coarse information about the plant can have significant practical applications. There are many techniques for synthesizing controllers from empirical data of the plant; for example, the most notable are the PID controller design using

Ziegler-Nichols criteria [9], the rule-of-thumb designs for lead lag compensation [10] and loop-shaping. A systematic attempt to synthesize PID and first order controllers for delay-free Single Input Single Output (SISO) LTI plants using frequency response measurements was first presented in [11]. In [12], the author provides an initial attempt at synthesizing sets of stabilizing controllers of arbitrary order from the frequency response data and this dissertation provides further results in that direction.

In the following text, the organization of this dissertation, as well as the specific topics that are dealt with, will be discussed.

Chapter II introduces some preliminary concepts of control system design. It provides a description of a control system and discusses the requirements of a controller. Preliminary mathematical results are also provided which will be used in the other chapters.

Chapter III provides an approximation to the set of all controller parameters, $K$ that make a *real* polynomial $P(s, K)$, that is affinely dependent on $K$, Hurwitz. This chapter describes how the approximation can be accomplished though the use of the interlacing conditions for stability, as stated by the Hermite-Biehler Theorem, and the use of Descartes' rule of signs. The application of Hermite-Biehler Theorem and Descartes' rule of signs leads to the systematic construction of polyhedral sets, described by linear programs. The approximation is given in terms of the union of polyhedral sets and a systematic procedure is outlined that can capture the set.

Chapter IV and V deals with synthesis of the set of fixed order stabilizing controllers that achieve various performance specifications such as robust stability, phase margin, gain margin and $\mathcal{H}_\infty$ norm constraints, time response constraints (overshoot, undershoot, settling time) and simultaneous stabilization of a discrete set of plants or a continuum of plants. Chapter IV considers a large class of performance specification that can be reduced to the problem of determining a set of *stabilizing* controllers that render a set of complex or real polynomials Hurwitz [8]. This chapter provides a procedure analogous to the one provided in Chapter III, and uses the interlacing property of complex Hurwitz polynomials to systematically construct linear programs, the solution to which provides an approximation to the set of fixed structure stabilizing controllers satisfying the given performance criterion. Chapter V provides a procedure to construct an outer approximation (as a union of polyhedral sets) of the set of controllers $K$ so that a rational, proper transfer function, $\frac{N(s,K)}{D(s,K)}$ has a non-negative and decaying impulse response. It is assumed that the coefficients of the polynomials $N(s,K)$ and $D(s,K)$ are affine in $K$. A broad class of transient response control problems can be formulated in this way.

Chapter VI studies the structure of the set of minimal order stabilizing and performance attaining controllers for continuous time LTI plants in the controller parameter space. It shows that the minimal order of a controller that guarantee specified performance is $l$ if and only if (1) there is a controller of order $l$ guaranteeing the specified performance and (2) the set of strictly proper stabilizing controllers guaranteeing the performance is bounded. Moreover, if the order of the controller is increased, the set of higher order controllers which satisfies the specified performance, will necessarily be unbounded. A procedure is presented for controller order reduction through the construction of an under-determined system of linear equations. The system of linear equations is obtained by canceling the poles of the closed loop system obtained by a controller of higher order and replacing it with one less pole.

Chapter VII develops a theory for fixed order controller synthesis from the knowledge only of empirical frequency response data of the plant and from some coarse information about them. The coarse information that is required is the following: the number of non minimum phase zeros of the plant and the frequency range beyond which the phase response of the LTI plant does not change appreciably and the amplitude response goes to zero. The method also allows for measurement errors in the frequency response of the plant.

Chapter VIII develops a procedure for the synthesis of fixed order controllers for nonlinear systems with sector bounded nonlinearities. An inner and outer approximation of the set of absolutely stabilizing linear controllers is constructed by casting the closed loop system as a Lure-Postnikov system. The inner approximation is based on the well-known sufficient conditions that require Strict Positive Realness (SPR) of open loop transfer function (possibly with some multipliers) and a characterization of SPR transfer functions that require a family of complex polynomials to be Hurwitz. The outer approximation is based on the condition that the open loop transfer function must have infinite gain margin, which translates to a family of real polynomials being Hurwitz.

Finally, the contributions of this dissertation are summarized in the last chapter, and a few concluding remarks on possible directions for future research are presented.

# CHAPTER II

# PRELIMINARY CONCEPTS

In this chapter, a brief overview of control theory is provided. It develops the mathematical descriptions for the types of system with which this dissertation is concerned, namely, Linear Time Invariant (LTI) finite dimensional systems and discusses the objectives of a control action on such systems. It lists various classical and modern schemes to design the appropriate control action, and enumerates the difficulties which arises in applying these schemes to the problem discussed in this dissertation. Next, a few preliminary mathematical results are included which provides a background for the rest of the dissertation.

## A. Introduction to Control

Control theory deals with the analysis and design of dynamical systems which arise in various fields such as electrical engineering (motors, power systems), mechanical engineering (aircraft, mobile robots, lathes) and chemical engineering (oil refineries, distillation process). This dynamical system is usually referred to as the **Plant**. A systematic study of any such physical system (physical process) starts with the development of a model. A good model is desired to be detailed enough to accurately describe the phenomena of interest and yet be concise enough to allow a convenient mathematical representation. The development of a good model requires an engineer to be conversant with the working of the physical system and have a good physical insight. Once the model is developed, appropriate laws (e.g. Newton's law, Kirchhoff's law, conservation laws) are used to provide a mathematical description of the model. To represent the physical system accurately and in a tractable mathematical form is always a difficult task. Physical systems can be represented mathematically using modeling only in an approximate manner. In this dissertation, when I mention that we consider a system described by some given mathematical equations, I am considering the process described by the mathematical equations and not the actual system itself. Furthermore, I will not study the modeling procedure, but will assume that the mathematical description of the physical system is already provided.

This dissertation deals with dynamical systems which are continuous-time finite dimensional systems. In general, these systems can be modeled by a finite number of coupled first-order ordinary differential equations

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p, d_1, \ldots, d_d), \\ \dot{x_2} &= f_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p, d_1, \ldots, d_d), \\ &\vdots \\ \dot{x_n} &= f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p, d_1, \ldots, d_d), \end{aligned} \tag{2.1}$$

$$\begin{aligned} y_1 &= h_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p, d_1, \ldots, d_d), \\ y_2 &= h_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p, d_1, \ldots, d_d), \\ &\vdots \\ y_m &= h_m(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_p, d_1, \ldots, d_d), \end{aligned} \tag{2.2}$$

where $\dot{x}_i$ denotes the derivative of $x_i$ with respect to time variable $t$. $x_1, x_2, \ldots, x_n$ are called state variables and represent the memory that the dynamical system has of its past. $u_1, u_2, \ldots, u_p$ are the specified input variables. $d_1, d_2 \ldots, d_d$ are the unknown and unpredictable disturbance impacting the system. $y_1, y_2, \ldots, y_m$ are the output variables and comprises of variables which are of particular interest in the analysis of the dynamical system. These are variables that can generally be physically measured or variables that are required to behave in a specified manner. (2.1) is called the state equation and (2.2) is called the output equation. Together these equations constitute

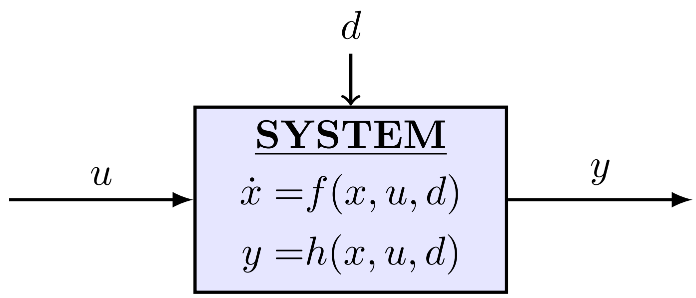



Fig. 1. A representation of the physical system.

a *state-space* description/representation of the model of the physical system.

These equations can be represented in a vector form as

$$\dot{x} = f(x, u, d),$$

$$y = h(x, u, d),$$

where $x$ is the state vector with components $x_i$, $u$ is the input vector with components $u_i$, $y$ is the output vector with components $y_i$, $d$ is the disturbance vector impacting the system and $f$ and $h$ are vector valued functions with components $f_i$ and $h_i$ respectively. The dimension of the state variable $x$ is referred to as the order of the system (plant). If $p = 1$ and $m = 1$ the plant is referred to as a Single Input Single Output (SISO) plant. If $p > 1$ and $m = 1$ it is referred to as the Multiple Input Single Output (MISO) plant. If $p = 1$ and $m > 1$ it is referred to as the Single Input Multiple Output (SIMO) plant. If $p > 1$ and $m > 1$ it is referred to as the Multiple Input Multiple Output (MIMO) plant.

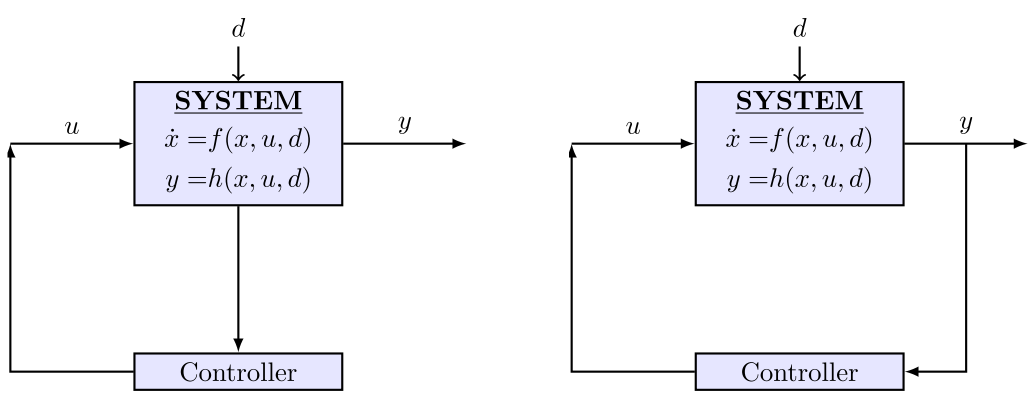
Fig. 1 shows a state-space representation of the physical system.

The purpose of designing the control action $u$ is to change the qualitative nature of the solutions of the dynamical system. The control action $u$ has to be designed so as to make certain physical variables of a system (plant) behave in a prescribed manner despite the presence of uncertainties in the plant model and disturbances acting on the plant.

Control is usually achieved by some form of feedback, in which the required control action, $u$, is generated by some device (controller) whose inputs are the measurements from the plant. If these measurements are the state variables, $x$, then the resulting control scheme is called a state feedback control scheme. In many cases, the state variables may not be directly available for measurement, or it may not be economically feasible to do so. In these cases the output of the plant are considered to be the measurements. If the device (controller) is itself modeled as a dynamical system, the resulting scheme is called a dynamic output feedback and can be represented by

$$\begin{aligned} \dot{\xi} =& g(\xi, y), \\ u =& \eta(\xi, y). \end{aligned}$$

The dimension of the vector $\xi$ is called the order of the controller. If the control input $u$ is not the output of a dynamical system, the resulting controller is called a static output feedback controller. Fig. 2 depicts the general feedback schemes.




(a) State feedback system. (b) Output feedback system.

Fig. 2. Feedback control systems.

The two fundamental issue in the design of a control system are *stability* and *performance* of the closed loop system. The controller must be designed such that it guarantees the stability of the closed loop system. *Stability of the closed loop system* is a loosely used term in control theory and it actually means that the zero state is asymptotically stable. The *performance* of a closed loop system refers to its ability to track reference signals closely and reject disturbances. The controller has to be designed such that the controlled outputs can be set to prescribed values (references) despite the presence of disturbances signals.

## B. Feedback Stabilization of Linear Systems

This dissertation further imposes restriction on the plant and the controller. It is assumed that the plant and controller are linear and time invariant. This is the only class of systems for which any *reasonable* theory for controller design has been developed. Under this assumption the plant and the controller can be represented with a set of linear ordinary differential equations with constant coefficients. Under these assumptions, the system can be written as,

Plant:
$$\dot{x_p} = A_p x_p + B_p u,$$
$$y = C_p x_p.$$

Controller:
$$\dot{x_c} = A_c x_c + B_c u,$$
$$y = C_c x_c.$$

$$\text{Closed loop:} \qquad \begin{bmatrix} \dot{x_p} \\ \dot{x_c} \end{bmatrix} = \underbrace{\begin{bmatrix} A_p & B_pC_c \\ B_cC_p & A_c \end{bmatrix}}_{A_{cl}} \begin{bmatrix} x_p \\ x_c \end{bmatrix}.$$

The above equation are written by setting the external input (reference) to the plant to be zero. Let $n_p$ be the size of the state vector $x_p$. It represents the order of the plant. Similarly, the size $n_c$ of $x_c$ represent the order of the controller. The order of the closed loop system is given by $n_p + n_c$.

The controller has to be designed such that it guarantees the stability of the closed loop system. By *stability of the closed loop system*, it is meant that the zero state is asymptotically stable. Basic results from linear system theory [13] indicate that the above system is asymptotically stable if and only if all eigenvalues of $A_{cl}$ have negative real parts (i.e. $\Re(\lambda_j) < 0, j = 1, \ldots, n_p + n_c$).

Since we are considering linear time invariant systems, they can be represented using Laplace transformations (Fig. 3). The plant and the feedback controller can be represented by the rational proper transfer function matrices $G(s)$ and $C(s)$ respectively. These can be written as

$$G(s) = D_p^{-1}(s)N_p(s) \qquad \text{and} \qquad C(s) = N_c(s)D_c^{-1}(s)$$

where $N_c(s), D_c(s), N_p(s)$ and $D_p(s)$ are polynomial matrices in the complex variable $s$. The characteristic polynomial of the closed loop system is

$$\delta(s) = \det\left[D_p(s)D_c(s) + N_p(s)N_c(s)\right].$$

The characteristic polynomial can also be represented as

$$\delta(s) = \det\left[sI - A_{cl}\right].$$

It can be easily shown that the eigenvalues of $A_{cl}$ coincide with the roots of the characteristic polynomial, $\delta(s)$. Hence, the closed loop system is stable if and only if $\delta(s)$ is Hurwitz, i.e., all its roots have negative real parts. Laplace transformation allows one to check the stability of a solution of the $n^{th}$ order differential equations with constant coefficients through an algebraic problem of determining whether the characteristic polynomial is Hurwitz.

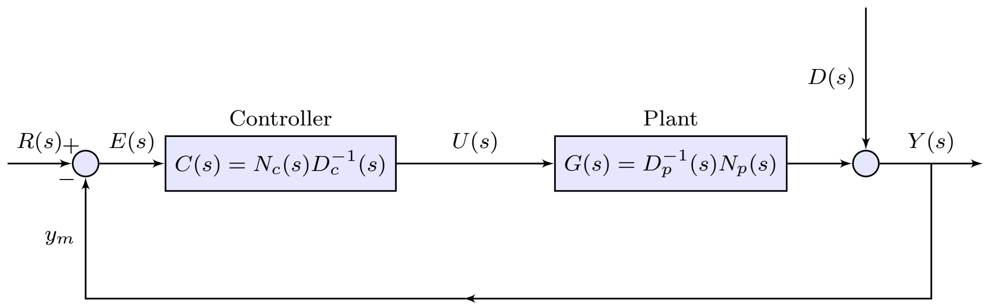



Fig. 3. Output feedback control system (Laplace Domain).

The design of stabilizing controllers can be done in many ways such as linear quadratic regulator, observer based state feedback and pole placement. These and other modern controller design schemes provide optimal controllers of high order that cannot handle the constraint of fixed order or structure and can have severe structural and sensitivity problems [10, 14]. If the desired controller required the determination of only one unknown parameter, then various classical methods, such as the root locus technique, the Nyquist stability criterion, and the Routh Hurwitz criterion could be effectively used. The root locus and the Nyquist stability criterion are graphical in nature, whereas the Routh-Hurwitz criteria provides an algebraic solution. Routh-Hurwitz criteria can be used to formulate the problem of fixed-order controllers, but it requires the simultaneous solution of a system of nonlinear equation in the controller parameters. The present state of art in control and optimization theory cannot provide a systematic and computationally tractable procedure to approximate the set of stabilizing controllers of fixed order.

## C. Mathematical Preliminaries

The problem of ascertaining the stability of a linear time invariant system reduces to the determination of the conditions under which a given real characteristic polynomial has all roots with negative real parts. This problem has been considered for over one hundred and fifty years. One of commonly used criteria is the Routh-Hurwitz criteria [15, 16]. There are other conditions for determining whether a polynomial is Hurwitz, and one of them, the Hermite-Biehler theorem predates the Routh-Hurwitz criteria. In 1856, Hermite [17] related the location of the roots of a polynomial with respect to a real line to the signature of a particular quadratic form. In this section, Hermite-Biehler theorem and its generalizations are provided and this theorem will be used extensively in this dissertation. Other root counting results, namely, the Descartes' rule of signs and its generalization are also provided.

### 1. Hermite-Biehler Theorem for Real Polynomials

Let $P(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$ be a *real* polynomial of degree $n$. Write $P(jw) := P_e(w^2) + jwP_o(w^2)$, where $P_e$ and $P_o$ are polynomials with real coefficients. The degrees of polynomials $P_e$ and $P_o$ are $n_e$ and $n_o$ respectively in $w^2$; specifically, if $n$ is odd, $n_e = n_o = \frac{n-1}{2}$ and if $n$ is even, $n_e = \frac{n}{2}$ and $n_o = n_e - 1$. Let $w_{e,i}$, $w_{o,i}$ denote the $i^{th}$ positive real roots of $P_e$ and $P_o$ respectively.

**Lemma 1.** *If $P(s)$ is Hurwitz, then all its coefficients are non-zero and have the same sign, either all positive or all negative.*

*Proof.* Let $s_1, s_2, \ldots, s_n$ be the roots of $P(s)$, and let $s_j^{'}$ be the real roots and let $s_k^{''}$

the complex roots. Then,

$$
\begin{aligned}
P(s) &= a_n \prod_j (s - s_j^{'}) \prod_k (s - s_k^{''}) \\
&= a_n \prod_j (s - s_j^{'}) \prod_k (s^2 - 2\Re(s_k^{''})s + |s_k^{''}|^2).
\end{aligned}
$$

Since, all $s_j^{'}$ and $\Re(s_k^{''})$ are negative, one can obtain only positive coefficients for the powers of $s$ when we compute the product of the monomials. Hence, all the coefficients of $P(s)$ must be of the same sign. □

**Lemma 2.** *If $P(s)$ is Hurwitz, then* $\arg[P(jw)]$*, also called the phase of $P(jw)$, is a continuous and strictly increasing function of $w$ on $(-\infty, \infty)$. Moreover the net change in phase from $-\infty$ to $\infty$ is*

$$\arg[P(j\infty)] - \arg[P(-j\infty)] = n\pi.$$

*Proof.* Let $s_1, s_2, \ldots, s_n$ be the roots of $P(s)$. Then,

$$P(s) = a_n \prod_{i=1}^{n} (s - s_i), \text{ with } s_i = a_i + jb_i \text{ and } a_i < 0.$$

Then,

$$
\begin{aligned}
\arg[P(jw)] &= arg\,[a_n] + \sum_{i=1}^{n} \arg[jw - a_i - jb_i], \\
&= arg\,[a_n] + \sum_{i=1}^{n} \arctan\left[\frac{w - b_i}{-a_i}\right].
\end{aligned}
$$

Since, for each root $s_i$, the term $\arctan\left[\frac{w - b_i}{-a_i}\right]$ is a continuous and strictly increasing function of $w$ on $(-\infty, \infty)$, it follows that $\arg[P(jw)]$ is also a continuous and strictly increasing function of $w$ on $(-\infty, \infty)$. The change in phase due to each root, as $w$

varies from $-\infty$ to $\infty$ is

$$\lim_{w\to\infty} \arctan\left[\frac{w-b_i}{-a_i}\right] - \lim_{w\to-\infty} \arctan\left[\frac{w-b_i}{-a_i}\right] = \frac{\pi}{2} - \frac{-\pi}{2} = \pi.$$

Hence the net change in phase of $P(jw)$ from $-\infty$ to $\infty$ is

$$\arg\left[P(j\infty)\right] - \arg\left[P(-j\infty)\right] = n\pi.$$

$\square$

**Lemma 3. Mikhailov stability criteria:** *The polynomial $P(s)$ is Hurwitz if and only if the frequency response plot (plot of $P(jw)$) starts on the real axis and passes through exactly $n$ quadrants in the counterclockwise direction as $w$ increases from $-\infty$ to $\infty$.*

The proof follows directly from Lemma 2.

The Hermite-Biehler theorem for real polynomials may be stated as follows:

**Theorem 1. Hermite-Biehler Theorem for real polynomials:** *A real polynomial $P(s)$ is Hurwitz iff*

1. *The constant coefficients of $P_e(w^2)$ and $P_o(w^2)$ are of the same sign,*
2. *All roots of $P_e(w^2)$ and $P_o(w^2)$ are real and distinct; the positive roots interlace according to the following:*
   - *if $n$ is even:*

     $$0 < w_{e,1} < w_{o,1} < \cdots < w_{o,n_e-1} < w_{e,n_e}$$

     ,
   - *if $n$ is odd:*

     $$0 < w_{e,1} < w_{o,1} < \cdots < w_{e,n_e} < w_{o,n_e}$$

     .

*Proof.* (Necessity) By Lemma 1 since all coefficients of $P(jw)$ are of the same sign, the constant coefficient of $P_e(w^2)$ and $P_o(w^2)$ will also be of the same sign. Now, wherever the frequency response plot intersects the imaginary axis, the value of $w$ is a root of $P_e(w^2)$ and where it intersects the real axis, the value of $w$ is a root of $P_o(w^2)$. Hence, by Mikhailov stability criteria, the interlacing condition holds (See Fig. 4).

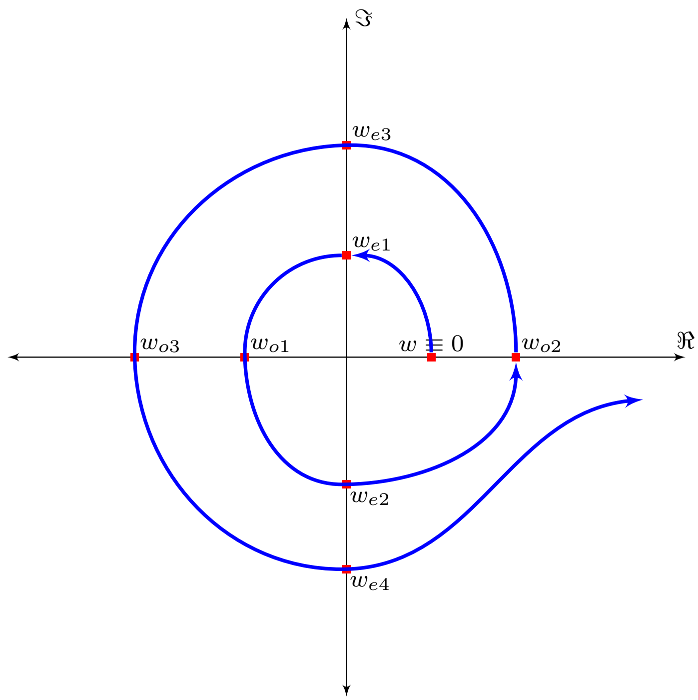



Fig. 4. Mikhailov plot.

(Sufficiency) Suppose there exists a polynomial $P(s) = P_e(s^2) + sP_o(s^2)$ which satisfies the conditions 1 and 2 stated above. Consider a one parameter family of polynomial, $P(s, \alpha) = P_e(s^2) + s\alpha P_o(s^2),\ \ \alpha > 0$. Since the roots of $P_e(s^2)$ and $P_o(s^2)$ interlace, the roots of the even and odd parts of $P(s, \alpha)$ also interlaces. Note, that since the constant coefficient of $P_e(s^2)$ and $P_o(s^2)$ are of the same sign (without loss of generality, we consider them to be positive), and since $\alpha$ is positive, the polynomial $P(s, \alpha)$ cannot have a root on the imaginary axis. Hence the root distribution (number of roots with positive and negative real parts) remains the same.

We will show that for all $\alpha$, $P(s, \alpha)$ is Hurwitz, and thus prove that $P(s)$ is Hurwitz.

Suppose $n$ is even. Consider the following root locus problem:

$$1 + \alpha \frac{sP_o(s)}{P_e(s)}.$$

The poles and zeros of $\frac{sP_o(s)}{P_e(s)}$ lie on the imaginary axis and they interlace according to,

$$-w_{e,n_e} < -w_{o,n_e-1} < \cdots < -w_{e,1} < 0 = w^* < w_{e,1} < w_{o,1} < \cdots < w_{o,n_e-1} < w_{e,n_e}.$$

Due to the zero at 0, and no other pole or zero on the real axis, a branch of the root locus exists on the real axis to the left of 0, i.e. in $[-\infty, 0]$ on the real axis. The angle of departure from any pole, $jw_{ej}$ will be

$$180^\circ + \sum_{i \neq j} (\angle w_{oi} - \angle w_{ei}) + \angle w_{oj} - \angle w^* = 180^\circ.$$

Similarly, the angle of arrival at any zero will be $0^\circ$. The branches of the root locus from the poles at $jw_{e1}$ and $-jw_{e1}$ goes to the zeros at $0 = w^*$ and $-\infty$. The other branches goes from $-w_{e,j}$ to $-w_{o,j-1}$ and $w_{e,j}$ to $w_{o,j-1}$, for $j = 2, \cdots, n_e$. The root locus is shown in Fig. 5 and it follows that for all $\alpha > 0$, $P(s, \alpha)$ is Hurwitz, and thus $P(s)$ is Hurwitz.

If $n$ is odd, then the root locus of $1 + \alpha^* \frac{P_e(s)}{sP_o(s)}$, $\alpha^* = \frac{1}{\alpha}$ shows that $P(s, \alpha)$ is Hurwitz.

$\square$

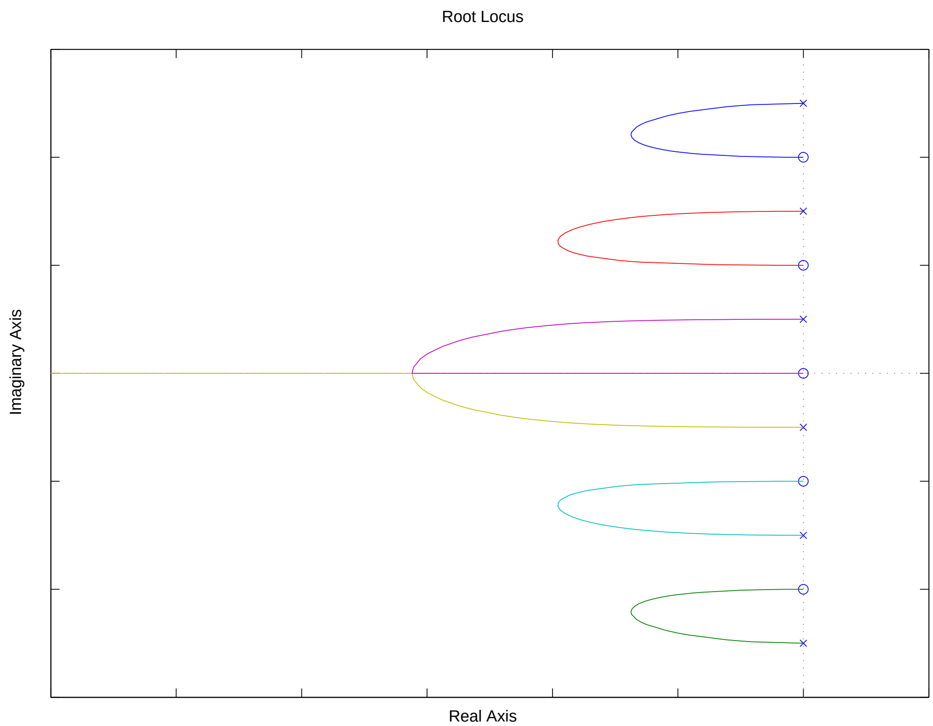



Fig. 5. Root locus.

## 2. Hermite-Biehler Theorem for Complex Polynomials

Let $P(s)$ be a polynomial with complex coefficients; the number of its roots in the left half plane is the same as the number of roots of the polynomial $P(jw)$ in the upper half plane, (i.e., number of roots with positive imaginary parts). Hermite considered exactly the same problem of counting the roots of a complex polynomial in the upper half plane [18]; the Hermite-Biehler theorem for real polynomials stated in the previous subsection is a special case.

Let $P(jw) = P_r(w) + jP_i(w)$. where $P_r(s)$ and $P_i(s)$ are polynomials with real coefficients. Without loss of generality, one may assume that $P_r(w)$ and $P_i(w)$ to be of degree $n$. Let $w_{r1}, w_{r2}, \ldots, w_{rn}$ be the roots of $P_r(w)$ and $w_{i1}, w_{i2}, \ldots, w_{in}$ be the roots of $P_i(w)$. The Hermite Biehler theorem for complex polynomials may then be stated as:

**Theorem 2.** ***Hermite-Biehler Theorem for complex polynomials:*** *The polynomial $P(s)$ is Hurwitz if and only if all roots $P_r(w)$ and $P_i(w)$ are real and interlace according to the following:*

- *If the leading coefficient of $P_r(w)$ and $P_i(w)$ are of the same sign, then*

$$-\infty < w_{r1} < w_{i1} < w_{r2} < w_{i2} < \ldots < w_{rn} < w_{in} < \infty.$$

- *If the leading coefficient of $P_r(w)$ and $P_i(w)$ are of the same sign, then*

$$-\infty < w_{i1} < w_{r1} < w_{i2} < w_{r2} < \ldots < w_{in} < w_{rn} < \infty.$$

The proof of the Hermite-Biehler Theorem for complex polynomials is a straightforward extension of that of the real case. The essential idea is that the Mikhailov plot ($P(jw)$) of a complex Hurwitz polynomial $P(s)$ must go through $2n$ quadrants in the counterclockwise direction as $w$ increases from $-\infty$ to $\infty$. The first condition given above corresponds to the Mikhailov plot starting in the first or third quadrant at $w = -\infty$. The second condition given above corresponds to the Mikhailov plot starting in the second or fourth quadrant at $w = -\infty$.

## 3. Descartes' Rule of Signs and Its Generalizations

Descartes' rule of signs, first described by Rene Descartes in 1637 in his work La Geometrie, is a technique for determining the number of positive real roots of a polynomial. The Descartes' rule of signs [19, 20] may be stated as follows:

**Theorem 3. Descartes' Rule of Signs:** *Let*

$$P(s) = p_0 + p_1 s + \ldots + p_n s^n,$$

*be a $n^{th}$ degree real polynomial ($p_n \neq 0$). Then:*

1. *The number of positive, real roots of $P(s)$ is at most equal to the number of the variations in sign of the sequence of its coefficients $p_0, p_1, \ldots, p_n$; moreover, if the number of positive roots is less than the number of variations, the difference is an even number.*

2. *If $P(s)$ has all real roots, then the number of variations in sign of the sequence of coefficients equals the number of real, positive roots of $P(s)$.*

*Proof.* There are different ways to prove the first part Descartes' Rule of Signs, see the subsections on Descartes' Rule of Signs in Chapter V of [20] and section 1 of [19].

Without loss of generality one can delete the roots of $P(s)$ at zero, hence we assume that $p_0 \neq 0$. Also, assume that the leading coefficient $p_n$ of $P(s)$ is unity, since, division of $P(s)$ by a nonzero real number neither influences the location or number of the variations in sign of the sequence of its coefficients, nor the location of its roots. If $p_0 > 0$, then the variations in sign of the sequence of its coefficients must be even, since the first and last coefficient of $P(s)$ are both positive. Moreover, the number of real positive roots (counted with multiplicity) must also be even, since $P(0) > 0$ and $P(s)$ is also positive for very large $s$. Similar arguments show that if $p_0 < 0$, then the number of variations in sign of the sequence of its coefficients is odd and the number of positive roots is odd. Hence, the number of the variations in sign of the sequence of its coefficients and number of roots have the same parity.

Let $a$ be a positive root of $P(s)$, and $P(s) = (s-a)Q(s)$, with

$$Q(s) = s^{n-1} + q_{n-2}s^{n-2} + ... + q_1 s + q_0.$$

Here, $q_{n-1} = 1$, since we've assumed $p_n$ is 1. Also,

$$-aq_{n-k} + q_{n-k-1} = p_{n-k} \text{ for } k = 1, 2, \ldots, n-1,$$

and

$$-aq_0 = p_0.$$

Now, as we work down from the highest-degree terms, we find that at every variation in the signs of $q_j$ and $q_{j-1}$, $p_j$ has the same sign as $q_{j-1}$. Now suppose there are $m$ variations in signs of $q_j, j = 0, \ldots, n-1$, then there will at least $m$ sign changes in the sequence $p_j,\ j = 1, \ldots, n$. Since, $p_0$ is the opposite sign of $q_0$, it follows that $P(s)$ has at least $m+1$ variations in the sign of the sequence of its coefficients. Moreover, due to the parity condition, any additional sign changes has to come in pairs.

Hence, if $P(s)$ has exactly $l$ real positive roots, and let

$$P(s) = \left( \prod_{i=1}^{l} (s - a_i) \right) Q_{n-l}(s),$$

then $P(s)$ has at least $l$ variations in the sign of the sequence of its coefficients. Note that as $Q_{n-l}(s)$ has no real positive roots and thus, due to the parity condition, has an even number of variations in the sign of the sequence of its coefficients.

*Proof of second part:* Suppose $P(s)$ has all real roots; then $P(s)$ may be expressed as $p_n(s+a_1)\ldots(s+a_r)\hat{Q}_{n-r}(s)$, where $a_1, \ldots, a_r > 0$ and $\hat{Q}_{n-r}(s)$ is a monic polynomial that has all real, positive roots. (Without any loss of generality, one can delete the roots of $P(x)$ at zero). It will be shown that, if $V_k$ is the number of variations (in the sign of the coefficients) of $\hat{P}_k(s) := p_n(s+a_1)(s+a_2)\ldots(s+a_k)\hat{Q}_{n-r}(s)$, then $V_k = n - r$ for $k = 1, \ldots, r$. The following will be established:

1. $V_k \geq V_{k+1} \geq n - r,\ k = 1, \ldots, r$, and
2. $V_1 = n - r$ to complete the proof.

Consider $\hat{P}_{k+1}(s) = (s + a_{k+1})\hat{P}_k(s)$; the number of variations of $\hat{P}_k(s)$ is the same as the number of variations of $\hat{P}_k(a_{k+1}s)$ and therefore, the number of variations of

$\hat{P}_{k+1}(a_{k+1}s)$ is the same as the number of variations of $a_{k+1}(s+1)\hat{P}_k(a_{k+1}s)$. Suppose $\hat{P}_k(a_{k+1}s) = q_0 + q_1 s + \ldots + q_{n+k-r}s^{n+k-r}$. Then, $(s+1)\hat{P}_k(a_{k+1}s) = q_0 + (q_1+q_0)s + (q_2+q_1)s^2 + \ldots + (q_{n+k-r} + q_{n+k-r-1})s^{n+k-r} + q_{n+k-r}s^{n+k-r}$. Since $q_i + q_{i-1}$ is of the same sign as either $q_i$ or $q_{i-1}$, adding $q_i + q_{i-1}$ in between $q_i$ and $q_{i-1}$ in a sequence $\{q_0,\ q_1,\ \cdots,\ q_{i-1},\ q_i,\ \cdots,\ q_{n+k-r}\}$ will not change the number of variations in the sign as one moves from the left to the right. In other words, $V_k$ is the same as the variations of the following sequences:

$$\{q_0,\ q_1,\ \cdots,\ q_{i-1},\ q_{i-1}+q_i,\ q_i,\ \cdots,\ q_{n+k-r}\},$$

and hence,

$$\{q_0,\ q_1+q_0,\ q_1,\ q_1+q_2,\ \cdots,\ q_{i-1},\ q_i+q_{i-1},\ q_i,\ \cdots, \\ \cdots q_{n+k-r-1},\ q_{n+k-r}+q_{n+k-r-1},\ q_{n+k-r}\}.$$

Since dropping terms in the sequence does not increase the number of variations in the sign of the terms, $V_k$ is greater than or equal to the number of changes in sign of

$$\{q_0,\ q_1+q_0,\ q_1+q_2,\ \cdots,\ q_i+q_{i-1}\ \cdots,\ q_{n+k-r}+q_{n+k-r-1},\ q_{n+k-r}\},$$

which equals $V_{k+1}$. Therefore, $V_k \geq V_{k+1}$. By the first part $V_k \geq n-r$ for every $k = 1, \ldots, r$.

In particular, $V_1 \geq n-r$ and $V_1 - n + r$ is a positive, even number. However, $\hat{P}_1$ is a polynomial of degree $n+1-r$ and hence, $V_1 \leq n+1-r$. Therefore, $V_1$, being an integer, can only equal $n-r$. $\square$

**Theorem 4. Poincaré's generalization:** *The number of sign changes in the coefficients of $(s+1)^k P(s)$ is a non-increasing function of $k$; for a sufficiently large $k$, the number of sign changes in the coefficients exactly equals the number of real, positive* *roots of* $P(s)$.

*Proof.* The proof of the generalization due to Poincaré is given in [20]. The essential idea of the proof provided in [20] is that, for a sufficiently large $k$, the number of sign changes of the coefficients of $(s+1)^k P(s)$ is the same as the number of sign changes of the sequence of values an associated polynomial $R(u)$ takes at $u = \frac{l}{k},\ l = 1, 2, \ldots, k-1$; the associated polynomial $R(u)$ is defined as follows:

$$R(u) = (1-u)^n P(\frac{u}{1-u})$$

In other words, Poincaré's scheme samples the polynomial $R(u)$ *uniformly* with a grid width of $\frac{1}{k}$ and examines the variations in the value of the polynomial at these points. Clearly, for a sufficiently fine grid, the number of sign variations in the sampled values of $R(u)$ exactly equals the number of roots of $R(u)$ in $(0, 1)$, and this is equal to the number of real, positive roots of $P(s)$. $\square$

# CHAPTER III

## STABILIZING CONTROLLERS FOR LINEAR-TIME INVARIANT PLANTS

The problem of fixed-order stabilization of a Linear-Time Invariant (LTI) dynamical system is one of the most important open problems in control theory [21]. It has attracted significant attention over the last four decades [22, 23]. This problem may be simply stated as follows: Given a finite-dimensional LTI dynamical system, is there a set of stabilizing proper, rational controllers of a given order? This set is the *basic* set in which all design must be carried out. Despite many results concerning this problem, there is no systematic procedure for synthesizing a fixed-order controller.

This chapter describes a new approach to the synthesis of fixed structure (including fixed order) controllers which are required in many practical applications. A broad class of fixed structure controller synthesis problems can be reduced to the determination of a real controller parameter vector (or simply, a controller), $K = (k_1, \quad k_2, \quad \ldots, k_l)$, which render a set of real polynomials Hurwitz, where each member of the set is of the form:

$$P(s, K) \quad = \quad P_o(s) + \sum_{l=1}^{N} k_l P_l(s) \tag{3.1}$$

The assumption of linear dependence on $K$, albeit restrictive, applies at least to all compensator design problem for single-input/single-output, single-input/multi-output or multi-input/single-output plants.

### A. Relation to Current State of Knowledge

The problem of fixed structure controller synthesis can be posed as the feasibility of a set of polynomial inequalities in the controller parameters through the Routh-Hurwitz criterion and is shown to be decidable by Anderson et. al [24] using the Quantifier

Elimination (QE) technique of Tarski and Seidenberg. However, this method is not computationally tractable. A good survey of the attempts to solve this problem and a related problem of Static Output Feedback (SOF) stabilization is given in [22, 21, 23] and the references therein. Recent work on control system design using QE technique is in [25]. The associated problem of pole placement is presented in [26, 27, 28].

The set of all fixed order/structure stabilizing controllers is non-convex in general and can even be disconnected in the space of controller parameters [29, 8]. This is a source of difficulty in its computation. In [30], the Hermite-Biehler theorem is used in getting an approximation of the set of all stabilizing PID controllers for SISO plants. The basic idea is to make the Mikhailov plot [31, 8] of the characteristic polynomial go through an appropriate number of quadrants in the counterclockwise direction. For a fixed proportional gain, the frequencies at which the Mikhailov plot cuts the real axis is fixed; requiring the imaginary part to take appropriate signs at these frequencies is tantamount to making the Mikhailov plot go through the required number of quadrants; however, this is equivalent to solving a linear program in the other two parameters. The approximation is completed by sweeping through the allowable values of the proportional gain. In [32], the D-decomposition technique (see [31]) is used for the synthesis of first order stabilizing controllers for discrete-time systems.

In [33], another interesting route to approximating the set of stabilizing controllers is presented. This approach combines ideas from Strict Positive Realness, positive polynomials written as sum of squares (SOS) and LMIs. This approach also considers characteristic polynomials that are linear in the parameters of the controller. This approximation is an inner approximation of the set as it is based on the following sufficient condition for stabilization: Given a (central) polynomial $Q(s)$, a characteristic polynomial, $P(s)$, is Hurwitz *if* the rational transfer function $\frac{P(s)}{Q(s)}$ is

SPR; using the condition of [34], the SPR condition is translated to the condition that an associated polynomial, $P(jw)Q^*(jw) + Q(jw)P^*(jw) - 2\gamma Q^*(jw)Q(jw) \geq 0$ for some $\gamma > 0$. By using techniques from [35], the associated polynomial is written as sum-of-squares, which then is expressed as an LMI in the controller parameters. The feasible set of the LMI is convex and is an inner approximation to the set of stabilizing controllers.

The parametrization of all stabilizing controllers of fixed order via Quadratic Lyapunov Functions is presented in [36]. It is accomplished through the use of two coupled Riccatti equations, one for $P$ and the other for $P^{-1}$, where $P$ is a symmetric matrix. However, this parametrization requires the determination of a fixed order controller a priori. In [37], the synthesis of a low order stabilizing controller is posed as the feasibility of a pair of LMIs with a coupling rank constraint. This constraint is convex for the full order controller synthesis problem but is *not convex* for the low order controller synthesis problem. Alternating projections are then used to synthesize a low order controller; the convergence of alternating projections is local owing to the non-convex nature of the coupling constraint.

The LMI approach for synthesizing SOF controller is also adopted in [38, 39, 40]. In [38], a cone complementary linearization algorithm is used to obtain a SOF stabilizing controller, whose order is guaranteed to be less than or equal to $n - \max\{n_u, n_y\}$, where $n$ is the order of the plant, $n_u$ and $n_y$ are the number of inputs and outputs respectively of the plant, a generic stabilizability result of [28]. The authors of [38] indicate that often they find controller order less than or equal to $n - n_u - n_y + 1$ matching the generic stabilizability result of [26].

Linear Quadratic Regulator (LQR) theory is employed in [41, 42, 43, 44, 45] to develop necessary and sufficient conditions for the existence of a stabilizing static output feedback for a given LTI system. In [46], an iterative method is provided for the computation of a stabilizing controller; however, convergence of the iteration is not guaranteed. Output feedback controller synthesis based on a specified degree of sub-optimality is presented in [47, 48].

Gradient based techniques for the synthesis of stabilizing SOF controllers is presented in [49], where a characteristic polynomial, whose coefficients are affine in controller parameters, is considered. A gradient update scheme for the controller parameters is proposed based on the minimization of the spectral radius of the characteristic polynomial. Since the spectral radius is not necessarily a convex function of the controller parameters, the gradient scheme may yield local optima.

A necessary condition for a polynomial to be Hurwitz is that all the coefficients of the polynomial be of the same sign. In particular, if the coefficients of the closed loop polynomial are linear functions of the controller parameters, this necessary condition can be equivalently expressed as the feasibility of two linear programs, since the coefficients can all be negative or positive. One may view the union of the feasible sets of the two linear programs as an outer approximation of the set of stabilizing controllers. In [50], this approach was taken to arrive at a lower bound on the minimal order of stabilization. A further refinement of this approach was taken up in [51], where a sufficient condition for the existence of a fixed order controller was established by requiring that the distance of a given Hurwitz polynomial, $\delta$ from the achievable closed loop characteristic polynomials be less than the stability radius of the polynomial $\delta$.

In [52], the interior of the monotone increasing (convex) non-negative cone of $n$ frequencies is bijectively mapped into the set of all Hurwitz polynomials of degree $n$. This is a convex parametrization. However, all Hurwitz polynomials of order $n$ may not be achievable closed loop polynomials, especially with controllers of fixed order. Nevertheless, by working in the space of frequencies, one can produce approximations to the stabilizing set which are unions of convex sets. This approach to approximating the set of stabilizing controllers of a fixed order using linear programming techniques was initiated in [53].

The given approach differs from the contributions in the literature on this problem in its exploitation of the Interlacing Property of Hurwitz polynomials to approximate the stabilizing sets. This leads to a systematic approach to the construction of the sets of stabilizing controllers of a fixed order using polyhedral sets. The set of all stabilizing controllers is approximated by the union of the feasible sets of systematically constructed (finite) linear programs. The approximation can be made arbitrarily accurate by increasing the number of feasible sets of linear programs. The main tools that are used in the construction of the sets of linear inequalities are the Hermite-Biehler theorem, Descartes' Rule of Signs and its generalization due to Poincaré.

## B. Synthesis of Sets of Stabilizing Controllers

In this section, the Interlacing Property (IP) of Hurwitz polynomials is used to systematically generate sets of controllers in the parameter space and contained in $\mathcal{S}$. This approach leads to sets of Linear Programs (LPs). The procedure proposed here for generating the set of all fixed order controllers using the feasible sets of LPs can be applied to discrete-time LTI plants also.

### 1. On Characterizing the Set of Stabilizing Controllers via Linear Programming

Let $P(s,K)$ be a *real* closed loop characteristic polynomial whose coefficients are affinely dependent on the design parameters $K$. Write $P(jw,K) := P_e(w^2,K) + jwP_o(w^2,K)$, where $P_e$ and $P_o$ are polynomials with real coefficients. The degrees of polynomials $P_e$ and $P_o$ are $n_e$ and $n_o$ respectively in $w^2$; specifically, if $n$ is odd,

$n_e = n_o = \frac{n-1}{2}$ and if $n$ is even, $n_e = \frac{n}{2}$ and $n_o = n_e - 1$. Let $w_{e,i}$, $w_{o,i}$ denote the $i^{th}$ positive real roots of $P_e$ and $P_o$ respectively.

The Hermite-Biehler theorem for real polynomials was stated in the previous chapter (Theorem 1). The set $\mathcal{S} = \{K : P(s,K) \text{ is Hurwitz}\}$ is, therefore, the set of all controllers, $K$, that simultaneously satisfy conditions (1) and (2) of the Hermite-Biehler theorem. The following version of the Hermite-Biehler theorem poses the problem of rendering $P(s,K)$ Hurwitz through a choice of $n-1$ frequencies. By way of notation, the polynomials $P_e$ and $P_o$ are represented compactly in the following form, owing to the affine dependence of their coefficients on the controller parameter vector $K$:

$$P_e(w^2, K) = \begin{bmatrix} 1 & w^2 & \cdots & w^{2n_e} \end{bmatrix} \Delta_e \begin{bmatrix} 1 \\ K \end{bmatrix}, \tag{3.2}$$

$$P_o(w^2, K) = \begin{bmatrix} 1 & w^2 & \cdots & w^{2n_o} \end{bmatrix} \Delta_o \begin{bmatrix} 1 \\ K \end{bmatrix}. \tag{3.3}$$

In (3.2) and (3.3), $\Delta_e$ and $\Delta_o$ are real constant matrices that depend on the plant data and the structure of the controller sought; they are respectively of dimensions $(n_e+1)\times(l+1)$ and $(n_o+1)\times(l+1)$, where, for $n$ odd, we have $n_e = n_o = \frac{n-1}{2}$, and for even $n$, we have $n_e = \frac{n}{2}$, $n_o = n_e - 1$; $l$ is the size of the controller parameter vector. For $i = 1, 2, 3, 4$, let $C_i$ and $S_i$ be diagonal matrices of size $n$; the $(m+1)^{st}$ diagonal entry of $C_i$ is $\cos\left(\frac{(2i-1)\pi}{4} + \frac{m\pi}{2}\right)$ and the corresponding entry for $S_i$ is $\sin\left(\frac{(2i-1)\pi}{4} + \frac{m\pi}{2}\right)$. For any given set of $n$ distinct frequencies, $w_0 < w_1 < \cdots < w_{n-1}$, and for any integer

$m$ define a Vandermonde-like matrix, $V(w_0, w_1, \ldots, w_{n-1}, m)$, as:

$$V(w_0, w_1, \ldots, w_{n-1}, m) := \begin{bmatrix} 1 & w_0^2 & \ldots & w_0^{2m} \\ 1 & w_1^2 & \ldots & w_1^{2m} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & w_{n-1}^2 & \ldots & w_{n-1}^{2m} \end{bmatrix}.$$

The following theorem characterizes the set of stabilizing controllers can be characterized in terms of $(n-1)$ frequencies:

**Theorem 5.** *There exists a real control parameter vector $K = (k_1,\ k_2,\ \cdots,\ k_l)$ so that the real polynomial*

$$\begin{aligned} P(s, K) &:= P_0(s) + k_1 P_1(s) + \ldots + k_l P_l(s) \\ &= p_n(K)s^n + p_{n-1}(K)s^{n-1} + \cdots + p_0(K) \end{aligned}$$

*is Hurwitz iff there exists a set of $n-1$ frequencies, $0 = w_0 < w_1 < w_2 < w_3 < \cdots < w_{n-1}$, so that one of the following two Linear Programs (LPs) is feasible:*

$$\begin{aligned} C_i V(w_0, w_1, \ldots, w_{n-1}, n_e)\Delta_e \begin{bmatrix} 1 \\ K \end{bmatrix} &> 0, \\ S_i V(w_0, w_1, \ldots, w_{n-1}, n_o)\Delta_o \begin{bmatrix} 1 \\ K \end{bmatrix} &> 0. \end{aligned} \tag{3.4}$$

*for $i = 1, 3$.*

*Moreover, the union of the feasible sets of the above LPs corresponding to all such sets of frequencies $(0 < w_1 < w_2 < \ldots < w_{n-1})$ is the set of all stabilizing controllers.*

*Proof.* The first condition of the Hermite-Biehler theorem requires that the constant coefficients of $P_e$ and $P_o$ be of the same sign. This condition implies that

$$P_e(0,K) > 0, P_o(0,K) > 0 \text{ or}$$

$$P_e(0,K) < 0, P_o(0,K) < 0.$$

For $w_0 = 0$, the condition $P_e(0,K) > 0, P_o(0,K) > 0$ is equivalent to

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} V(w_0, \ldots, w_{n-1}, n_e)\Delta_e \begin{bmatrix} 1 \\ K \end{bmatrix} > 0,$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} V(w_0, \ldots, w_{n-1}, n_o)\Delta_o \begin{bmatrix} 1 \\ K \end{bmatrix} > 0.$$

The second condition of the Hermite-Biehler theorem is equivalent to the existence of $n-1$ frequencies, $0 < w_1 < w_2 < \cdots < w_{n-1}$ such that the roots of the even polynomial, $P_e$, lie in $(0, w_1), (w_2, w_3), (w_4, w_5), \ldots,$ while the roots of the odd polynomial, $P_o$, lie in $(w_1, w_2), (w_3, w_4), \ldots.$
If $P_e(0,K) > 0, P_o(0,K) > 0$, then the placement of roots will require $P_e(w_1^2, K) < 0, P_e(w_2^2, K) < 0, P_e(w_3^2, K) > 0, P_e(w_4^2, K) > 0, \ldots$ and $P_o(w_1^2, K) > 0, P_o(w_2^2, K) < 0, P_o(w_3^2, K) < 0, P_o(w_4^2, K) > 0, \ldots$.

In other words, the signs of $P_e(w_j^2, K)$ and $P_o(w_j^2, K)$ are the same as that of $\cos(\frac{\pi}{4} + j\frac{\pi}{2})$ and $\sin\left(\frac{\pi}{4} + j\frac{\pi}{2}\right)$ respectively. Therefore, for the case when $P_e(0,K) > 0, P_o(0,K) > 0$,

$$\cos\left(\frac{\pi}{4} + j\frac{\pi}{2}\right) P_e(w_j^2, K) > 0 \text{ and } \sin\left(\frac{\pi}{4} + j\frac{\pi}{2}\right) P_o(w_j^2, K) > 0.$$

Thus, for $P_e(0,K) > 0, P_o(0,K) > 0$, by putting the inequality conditions together, there exists a stabilizing controller $K$ iff there exists a set of $(n-1)$ frequencies $0 < w_1 < \ldots < w_{n-1}$ such that the Linear Program (LP) given by (3.4) is feasible for $i = 1$.

Similarly $P_e(0, K) < 0, P_o(0, K) < 0$ corresponds to the case $i = 3$ and the Linear Program (LP) given by equation (3.4) is feasible for a stabilizing controller $K$. □

*Remark* 1. If for some $(n-1)$ tuples of frequencies, $0 = w_0 < w_1^* < w_2^* < w_3^* < \cdots < w_{n-1}^*$, one of the LPs in (3.4) is feasible, then the solution to the LP provides a *convex subset* of the set of stabilizing controllers.

*Remark* 2. The problem of determining the set of all stabilizing controllers can be posed as the search for all possible $n$ tuples ($(n-1)$ tuples of frequencies and the binary number indicating the sign of the coefficients of the characteristic polynomial), whose corresponding LP is feasible.

As can be seen from the LPs given by (3.4), one can associate with every linear program an $n-1$ tuple of frequencies and a binary number which indicates the sign of the coefficients of the characteristic polynomial. The frequency information is used in constructing the Vandermonde matrix $V$ and the sign information is used in the choice of $C_1, S_1$ or $C_3, S_3$.

*Remark* 3. If the characteristic polynomial is monic, then only the LP corresponding to $i = 1$ needs to be considered for checking the feasibility; the LP corresponding to $i = 3$ will not be feasible since all the coefficients of a characteristic polynomial must be of the same sign - in particular, if the characteristic polynomial is monic, the coefficients $P_e(0, K)$ and $P_o(0, K)$ must also be positive.

*Remark* 4. Let $w_{e,i}^*$, $w_{o,i}^*$ denote the $i^{th}$ positive real roots of $P_e$ and $P_o$ respectively for a stabilizing controller $K^*$. Then for any $(n-1)$ tuples of frequencies, $w_1, w_2 \ldots, w_{n-1}$ such that $w_1 \in (w_{e,1}^*, w_{o,1}^*), w_2 \in (w_{e,2}^*, w_{o,2}^*) \ldots$, one of the LPs in (3.4) will be feasible and the solution will provide a set of controllers which contains $K^*$.

*Remark* 5. Let $\mathcal{S}_1, \mathcal{S}_2$ be the feasible sets of LPs described by equation (3.4) and corresponding to the cases $i = 1$ and $i = 3$ respectively. Then,

- $\mathcal{S}_1$ is exactly the set of all controllers $K$ which place the positive roots of $P_e(w^2, K)$ and $P_o(w^2, K)$ alternately in the intervals

$$(0, w_1), (w_1, w_2), \cdots, (w_{n-2}, w_{n-1})$$

and contained in the intersection of half spaces $P_e(0, K) > 0, P_o(0, K) > 0$.

- $\mathcal{S}_2$ is exactly the set of all controllers $K$ which place the positive roots of $P_e(w^2, K)$ and $P_o(w^2, K)$ alternately in the intervals

$$(0, w_1), (w_1, w_2), \cdots, (w_{n-2}, w_{n-1})$$

and contained in the intersection of half spaces $P_e(0, K) < 0, P_o(0, K) < 0$.

- $\bigcup_{i=1}^{2} \mathcal{S}_i$ is exactly the set of all controllers $K$ which place the positive roots of $P_e(w^2, K)$ and $P_o(w^2, K)$ alternately in the intervals

$$(0, w_1), (w_1, w_2), \ldots, (w_{n-2}, w_{n-1}).$$

*Remark* 6. The problem of simultaneous stabilization of two or more SIMO/MISO plants can be naturally accommodated with the proposed procedure; it is equivalent to finding the intersection of the set of stabilizing controllers of fixed order for each plant. This can be accomplished using linear programming, since one is now intersecting a finite number of sets, each of which is approximated using a union of polyhedral sets.

We now proceed to develop a computational procedure for Theorem 5. The idea for the systematic search for the $n-1$ tuples of frequencies came from the proofs of the Poincaré's generalization of Descartes' rule of signs. Descartes' rule of signs and its generalization also form a basis for formulation of outer approximation.

The essential idea of the proof of the generalization by Poincaré to Descartes' rule of signs provided in [20] is that, for a sufficiently large $k$, the number of sign changes of the coefficients of $(s+1)^k P(s)$ is the same as the number of sign changes of the sequence of values an associated polynomial $R(u)$ takes at $u = \frac{l}{k},\ l = 1, 2, \ldots, k-1$; the associated polynomial $R(u)$ is defined as follows:

$$R(u) = (1-u)^n P(\frac{u}{1-u}).$$

In other words, Poincaré's scheme samples the polynomial $R(u)$ *uniformly* with a grid width of $\frac{1}{k}$ and examines the variations in the value of the polynomial at these points. Clearly, for a sufficiently fine grid, the number of sign variations in the sampled values of $R(u)$ exactly equals the number of roots of $R(u)$ in $(0, 1)$, and this is equal to the number of real, positive roots of $P(s)$.

This idea will be exploited in finding the sets of fixed structure stabilizing controllers and the bounds on the set of all fixed structure stabilizing controllers.

## 2. Constructing the Set of Stabilizing Controllers

Motivated by Poincaré's generalization of the rule of signs, we introduce the transformation,

$$w^2 = \frac{u}{1-u},$$

and for convenience call $u$ as the generalized frequency. This mapping allows us to compactify the frequency space from $(0, \infty)$ to $(0, 1)$. We define the polynomials,

$$\begin{aligned} R_e(u, K) =& (1-u)^{n_e} P_e(\frac{u}{1-u}, K) \text{ and} \\ R_o(u, K) =& (1-u)^{n_o} P_o(\frac{u}{1-u}, K). \end{aligned}$$

Using Theorem 5 and the above mapping, the problem of finding stabilizing controllers may be posed as the problem of searching for all sets of $n-1$ points,

$u_1, \ldots, u_{n-1} \in (0,1)$ such that at least one of the following two LPs is feasible:

$$C_i D_e V(\frac{u_0}{1-u_0}, \ldots, \frac{u_{n-1}}{1-u_{n-1}}, n_e)\Delta_e \begin{bmatrix} 1 \\ K \end{bmatrix} > 0,$$
$$S_i D_o V(\frac{u_0}{1-u_0}, \ldots, \frac{u_{n-1}}{1-u_{n-1}}, n_o)\Delta_o \begin{bmatrix} 1 \\ K \end{bmatrix} > 0, \tag{3.5}$$

for $i = 1, 3.$

where $D_e$ and $D_o$ be diagonal matrices whose $i^{th}$ diagonal entries are $(1-u_i)^{2n_e}$ and $(1-u_i)^{2n_o}$ respectively.

The union of feasible sets of the above LPs corresponding to all possible $(n-1)$ tuples, $0 < u_1 < u_2 < \cdots < u_{n-1} < 1$, is the set of all stabilizing controllers, $\mathcal{S}$. Computationally, only finite $(n-1)$ tuples can be considered and hence the computed set will be an inner approximation of the set of stabilizing controllers. The determination of the $(n-1)$ tuples of generalized frequencies, corresponding to stabilizing controllers, for the general case is a difficult and open problem. The $(n-1)$ tuples, leading to feasible sets, depend on the plant and the controller structure. We provide an algorithm which systematically searches through the generalized frequency partitions. The inputs are: (i) desired number of partitions, and (ii) plant data and the controller structure from which LPs are constructed. One can get the inner approximation of the set of stabilizing controllers as follows:

- **Step 1:** Enter plant data and controller structure. Form the matrices $\Delta_e$ and $\Delta_o$. Let *ntup* be the number of tuples of generalized frequencies required.

- **Step 2:** Enter the number of partitions $p$.

- **Step 3:** Form the set of generalized frequencies $0 < u_1 < u_2 < \ldots < u_p < 1$.

- **Step 4:** Call function SOLVER$(start = 1, loop = 1, \mathbf{U} = [\ 0\ ], set = [\ ])$.

- **Step 5:** If inner approximation from **step 4** is satisfactory, then **EXIT** else goto **step 2** and increase number of partitions.

In **step 1**, the plant and the controller structure is entered. The characteristic polynomial, affinely dependent on the controller parameter, is calculated. Then the compact representation, i.e. $\Delta_e$ and $\Delta_o$, of the even and odd polynomials are calculated.

In **Step 2** the number of partitions, $p$, of the interval $(0, 1)$ is entered. The minimum separation between the roots of even and odd polynomials of $P(s, K)$ is a measure of the coprimeness of the two polynomials, and hence, a measure of the distance of the polynomial $P(s, K)$ to a polynomial with imaginary axis roots. For this reason, a lower bound on the minimum width of the partition (and thus the number of partitions) is reasonable, from both a computational as well as a closed loop robust stability point of view.

In **Step 3** we partition $(0, 1)$ using $p$ points. Instead of choosing equally spaced points for the partition, it is better to choose points that are bunched towards the right end of the interval $(0, 1)$. Hence, to obtain better results, a partition using the positive roots of an appropriate Chebyshev polynomial is used.

In **Step 4** the function SOLVER is called. This function can be described through the following pseudo-code:

**Function :** $\textsc{Solver}(start, loop, \mathbf{U}, set)$

**comment:** Recursive algorithm for calculating the inner approximation

**if** $loop > ntup$

  **then return** $(set)$

$loop \leftarrow loop + 1;$

**for** $i \leftarrow start$ **to** $p$

$$\textbf{do} \begin{cases} \mathbf{U} \leftarrow [\mathbf{U}\ \ u_i]; \\ S1 \leftarrow \text{Solve a subset of the } \textbf{LPs} \text{ in (3.5)} \\ \textbf{if } \text{LP is feasible} \\ \quad \textbf{then} \begin{cases} \text{SOLVER}(i+1, loop, \mathbf{U}, set) \\ \textbf{if } loop > ntup \\ \quad \textbf{then } set \leftarrow [set\ \ S1] \end{cases} \end{cases}$$

**return** $(set)$

The function SOLVER implements a recursive algorithm for finding an inner approximation. The inputs to this function are (i) *loop* which is a count for the tuples of generalized frequencies considered in each iteration. It increases from 1 to *ntup*. (ii) *start* which is the index of the generalized frequency from which the search for the next possible frequency in the tuple should start. For each *loop*, this index has to be greater than the generalized frequency being considered in the previous *loop*. (iii) $\mathbf{U}$ are the tuples of generalized frequencies. It starts with [0] and in $loop = 1$ it becomes $[0, u_1]$. Finally, when $loop = ntup$, it represents an $(n-1)$ tuple of generalized frequencies. (iv) *set*, the set of polyhedrons corresponding to all possible $(n-1)$ generalized frequencies which were found. The function SOLVER uses a pruning technique to find the feasible $n-1$ tuples of frequencies. Suppose a subset of the LPs in (3.5) is infeasible for $0 < u_1^* < u_2^*$, then all sets of $n-1$ tuples

$0 < u_1^* < u_2^* < u_3 < \ldots < u_{n-1} < 1$ will be infeasible and can be discarded. The outer approximation also aids in the search for all $n-1$ tuples of generalized frequencies. The union of the feasible sets (each of which is polyhedral) corresponding to all possible $n-1$ tuples, therefore, provides an approximation of the set of stabilizing controllers. It is an *inner approximation* - every element of the approximate set *is* a stabilizing controller.

**Step 5** lists the stopping criteria and also deals with refining the partition. The approximation may be made more accurate by refining the partition of $(0, 1)$, because if $K$ is a stabilizing controller, a partition fine enough to separate the roots of $R_e(u, K)$ and $R_o(u, K)$ will always capture $K$. This refinement is done by increasing the number of partitions (**Step 5**). However, if one must increase the number of partitions (and thus decrease the grid width) arbitrarily in order to find a stabilizing controller of fixed order, it is quite likely that the resulting controller may not be robust to plant or controller variations.

### 3. Outer Approximation

In the previous subsection, a procedure to construct LPs whose feasible set is contained in $\mathcal{S}$ is outlined. Their union $\mathcal{S}_i$ is an inner approximation to $\mathcal{S}$. In this subsection, a procedure is developed to generate a countable union of polyhedral sets, $\mathcal{S}_o$, which contains all the fixed structure stabilizing controllers. As an example of an outer approximation, consider the scheme presented in [54]. A necessary condition for a polynomial to be Hurwitz is that its coefficients must be of the same sign; using this fact, Bhattacharyya and Keel [54] construct two LPs that require the coefficients of the characteristic polynomial to be of the same sign; hence, the union of the feasible sets, say $\bigcup_{i=1}^{2} \mathcal{S}_{outer,i}$, of the two LPs is an *outer approximation* of the set of all stabilizing controllers, i.e., it contains the set of all stabilizing controllers of the desired structure. One may ask the following question: Exactly how does the requirement, that the coefficients of the characteristic polynomial be of the same sign, relate to the conditions of Hermite-Biehler theorem? An answer to this question can provide the gap between the set of the stabilizing controllers, $\mathcal{S}$ and $\bigcup_{i=1}^{2} \mathcal{S}_{outer,i}$. It can also provide directions for tightening the outer approximation.

For the sake of a discussion on outer approximation the polynomials, $P_e(w^2, K)$ and $P_o(w^2, K)$, are treated as polynomials in $w^2$. Let $\lambda = w^2$ and let the $i^{th}$ roots of $P_e(\lambda, K)$ and $P_o(\lambda, K)$ be represented as $\lambda_{e,i}$ and $\lambda_{o,i}$ respectively. The Hermite-Biehler theorem is applied to this polynomial. The condition, that the coefficients of characteristic polynomial are of the same sign, ensures that the polynomials, $P_e(\lambda, K)$ and $P_o(\lambda, K)$ have the maximum possible number of sign changes in its coefficients, ($n_e$ and $n_o$ respectively). This is necessary, by Descartes' rule of signs, for the polynomials $P_e$ and $P_o$ to have all real roots and hence, satisfy a part of the second necessary condition of the Hermite-Biehler theorem. An easy way to tighten the requirement that all roots of $P_e$ and $P_o$ are positive is through the use of Poincaré's generalization and is stated below:

**Lemma 4.** *If $K$ is a stabilizing control vector, then $(\lambda + 1)^{k-1} P_e(\lambda, K)$ and $(\lambda + 1)^{k-1} P_o(\lambda, K)$ have exactly $n_e$ and $n_o$ sign changes in their coefficients respectively for every $k \geq 1$.*

Exactly how can one use this lemma to construct an outer approximation? For any given $k \geq 1$, the polynomial $(\lambda + 1)^{k-1} P_e(\lambda, K)$ is of degree $n_e + k - 1$; requiring $n_e$ sign changes in its coefficients is tantamount to choosing $n_e + 1$ coefficients in the increasing powers of $\lambda$ and enforcing $n_e$ sign changes in the chosen coefficients. Enforcing sign changes in the coefficients is equivalent to constructing two linear programs for every choice of $n_e + 1$ coefficients. One can now check the feasibility of the constructed LPs and let $\mathcal{S}_{e,k}$ be the union of their feasible sets. A similar procedure can be applied to the polynomial $P_o(\lambda, K)$ and let $\mathcal{S}_{o,k}$ be the corresponding set for the polynomial $P_o$. One can construct the intersection of $\mathcal{S}_{e,k}$ and $\mathcal{S}_{o,k}$ and let us call the intersection $\mathcal{S}_{outer,k}$. Clearly, the set $\mathcal{S}_{outer,k}$ contains $\mathcal{S}$, since every stabilizing $K \in \mathcal{S}$ satisfies the second requirement of Hermite-Biehler theorem and thus belongs to $\mathcal{S}_{outer,k}$.

It must be remarked that $\mathcal{S}_{outer,k+1} \subset \mathcal{S}_{outer,k}$, indicating the tightening of the approximation (with increasing $k$) using Lemma 4. To see this, suppose $K \notin \mathcal{S}_{outer,k}$, indicating that either $(\lambda+1)^{k-1}P_e(\lambda, K)$ has fewer than $n_e$ sign changes in its coefficients or $(\lambda+1)^{k-1}P_o(\lambda, K)$ has fewer than $n_o$ sign changes in its coefficients. By Poincaré's generalization, since the number of sign changes in the coefficients of $(\lambda+1)^k P_e(\lambda, K)$ and $(\lambda+1)^k P_o(\lambda, K)$ is a non-increasing function of $k$, it follows that $K \notin \mathcal{S}_{outer,k+1}$.

Interlacing of roots has not been accounted for in the construction of an outer approximation using Lemma 4. The following lemma converts the requirement of interlacing of roots to checking the signs of the coefficients of a family of polynomials.

**Lemma 5.** *Suppose,*

$$\begin{aligned}
Q(\lambda) =& P_e(\lambda, K) = q_0 + q_1\lambda + \ldots + q_m\lambda^m \text{ and} \\
R(\lambda) =& \lambda P_o(\lambda, K) = r_1\lambda + \ldots + r_m\lambda^m + r_{m+1}\lambda^{m+1}.
\end{aligned}$$

*Let $q_m$ and $r_m$ be of the same sign. Let $\mu_1, \ldots \mu_m$ be the roots of $Q(\lambda)$ and $0, \xi_1, \ldots, \xi_m$ be the roots of $R(\lambda)$. Consider a one-parameter family of polynomials:*

$$\tilde{Q}(\lambda, \eta) = Q(\lambda) - \eta R(\lambda)$$

*. Then, the following two statements are equivalent:*

1. *The roots of $Q(\lambda)$ and $R(\lambda)$ are real and interlacing, i.e.,* $0 < \mu_1 < \xi_1 < \mu_2 < \xi_2 < \ldots < \mu_m < \xi_m$.

2. *The number of real positive roots of $\tilde{Q}(\lambda, \eta)$ is exactly $m+1$ for $\eta > 0$ and exactly $m$ for $\eta < 0$.*

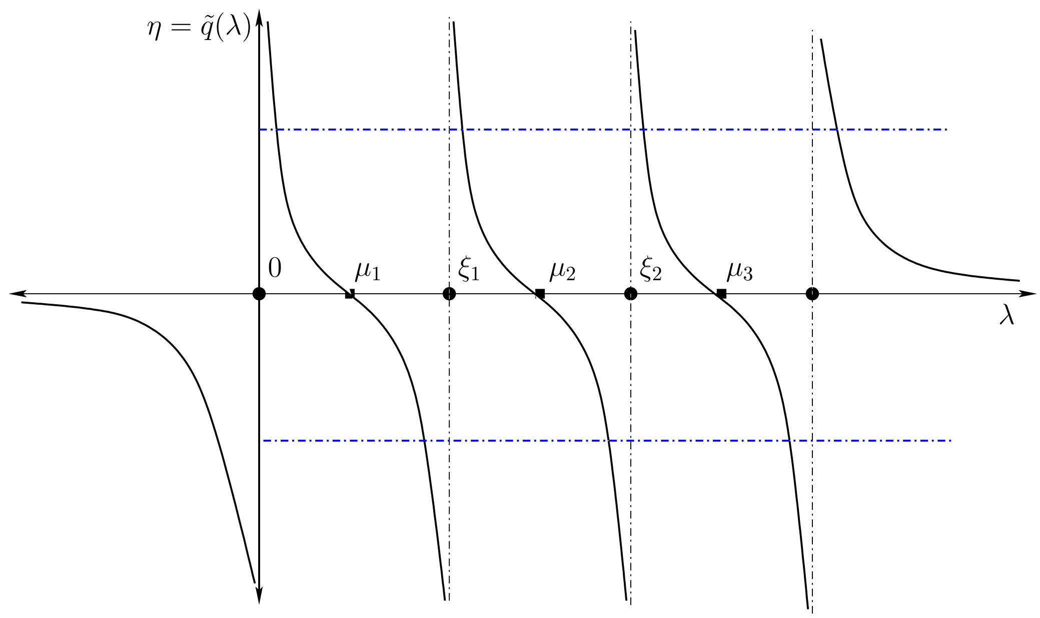



Fig. 6. Plot of $\tilde{q}(\lambda) = \frac{Q(\lambda)}{R(\lambda)}$.

*Proof.* (Necessity) Consider the plot of $\tilde{q}(\lambda) = \frac{Q(\lambda)}{R(\lambda)}$ (see Fig. 6). We first observe that $\lim_{|\lambda|\to\infty} \tilde{q} = 0$. At the roots of $R(\lambda)$, the function $\tilde{q}(\lambda)$ has asymptotes. Since there are $m+1$ asymptotes and since there is exactly one zero of $\tilde{q}(\lambda)$ in between any two successive asymptotes, there are exactly $m$ real zeros between $(0, \xi_m)$ for the equation $\tilde{q}(\lambda) = \eta$ for any $\eta \in \Re$. Consider the interval $(-\infty, 0)$. The function $\tilde{q}(\lambda)$ is monotonic on $(-\infty, 0)$; otherwise, for some $\eta \in \Re$, the equation $\tilde{q}(\lambda) = \eta$ has two zeros in $(-\infty, 0)$ implying that the equation $\tilde{q}(\lambda) = \eta$ (and hence, the $m + 1^{th}$ degree polynomial $\tilde{Q}(\lambda, \eta)$) has $m+1$ real roots in $(-\infty, \xi_m)$, which is clearly impossible.

Since $\lim_{\lambda \to 0^-} \tilde{q}(\lambda) < 0$ and $\lim_{\lambda \to \xi_m^+} \tilde{q}(\lambda) > 0$ are of opposite signs, the function $\tilde{q}$ is monotonically decreasing on $(-\infty, \xi_1)$ and $(\xi_m, \infty)$. Hence for any $\eta \in \Re$, there is exactly one real zero in the interval $(-\infty, \xi_1) \cup (\xi_m, \infty)$. Therefore, the equation $\tilde{q}(\lambda) = \eta$ (and hence, the polynomial $\tilde{Q}(\lambda, \eta)$) has exactly $m+1$ real zeros for any $\eta > 0$ and exactly $m$ for $\eta < 0$.

(Sufficiency) If the second condition holds, then for $\eta = 0$, one must have real roots, i.e., $Q(\lambda)$ has all real roots. For $\eta \to \infty$, one must have all real roots, implying that $R(\lambda)$ also has all real roots. Suppose the roots of $Q(\lambda)$ and $R(\lambda)$ do not interlace. Consider the root locus problem,

$$1 - \eta \frac{Q(\lambda)}{R(\lambda)} = 0.$$

Since roots do not interlace, either two successive zeros of $Q(\lambda)$ or $R(\lambda)$ do not have the zeros of the other polynomial between them. Therefore, for either positive or negative values, $\eta^*$ of $\eta$, there is a break-in or a breakaway point between the roots and for some perturbation of $\eta^*$, there will be two complex conjugate roots for the equation $1 - \eta \frac{Q(\lambda)}{R(\lambda)} = 0$. In other words, the $m^{th}$ degree polynomial $\tilde{Q}(\lambda, \eta)$ will have two complex conjugate roots, implying that it cannot have all real roots, which is a contradiction. $\square$

Lemma 5 can be used to convert the problem of checking whether roots of $P_e(\lambda, K)$ and $P_o(\lambda, K)$ interlace into the problem of counting of the number of real roots for every member of a one-parameter family of polynomials. Without any loss of generality, we will assume that $P(s, K)$ is of odd degree for this discussion; otherwise, one can consider the polynomial $(s+1)P(s, K)$ which is Hurwitz *iff* $P(s, K)$ is Hurwitz and is of odd degree. If $P(s, K)$ is of odd degree, then the polynomials $P_e(\lambda, K)$ and $P_o(\lambda, K)$ are of the same degree and the leading coefficients of $P_e$ and

$P_o$ are of the same sign. If the roots of $P_e(\lambda, K)$ and $P_o(\lambda, K)$ were to be positive, distinct and interlacing, then the number of real positive roots of

$$P_e(\lambda, K) - \eta\lambda P_o(\lambda, K)$$

is exactly $n_o + 1$ for $\eta > 0$ and exactly $n_o$ for $\eta < 0$.

Lemmas 4 and 5 can be put together to show that an arbitrarily tight outer approximation can be constructed.

**Proposition 1.** Let $P(s, K)$ be of odd degree. Then $K$ is such that $P(s, K)$ is not Hurwitz *iff* one of the following holds:

1. All coefficients of $P(s, K)$ are not of the same sign.
2. For some $l > 1$, the number of sign changes in the coefficients of $(\lambda + 1)^{l-1} P_e(\lambda, K)$ is fewer than $n_e$.
3. For some $l > 1$, the number of sign changes in the coefficients of $(\lambda + 1)^{l-1} P_o(\lambda, K)$ is fewer than $n_o$.
4. For some $l > 1$ and for some $\eta > 0$, the number of sign changes in the coefficients of $(\lambda + 1)^{l-1}(\lambda P_o(\lambda, K) - \eta P_e(\lambda, K))$ is fewer than $n_o + 1$.
5. For some $l > 1$ and for some $\eta < 0$, the number of sign changes in the coefficients of $(\lambda + 1)^{l-1}(\lambda P_o(\lambda, K) - \eta P_e(\lambda, K))$ is fewer than $n_o$.

One can get an outer approximation of the set of stabilizing controllers as follows:

- **Step 1:** Construct and check the feasibility of the two LPs corresponding to the cases when all coefficients of $P(s, K)$ are positive and when they are negative.
- **Step 2:** Choose $l > 1$. Construct and check the feasibility of LPs corresponding to the case that the coefficients of the polynomial $(\lambda+1)^{l-1} P_e(\lambda, K)$ have exactly $n_e$ sign changes. Suppose $p_e$ of them are feasible.

- **Step 3:** Similarly, construct and check the feasibility of LPs corresponding to the case that the coefficients of the polynomial $(\lambda+1)^{l-1}P_o(\lambda, K)$ have exactly $n_o$ sign changes. Suppose $p_o$ of them are feasible.

- **Step 4:** By picking an LP each from the steps 1, 2 and 3, check the simultaneous feasibility. There will be at most $2p_ep_o$ of such augmented LPs; of these, let $p_{eo}$ be feasible.

- **Step 5:** For this step, consider the degree of $P(s, K)$. If it is even, consider $(s+1)P(s, K)$ and construct its odd and even parts, $P_e(\lambda, K)$ and $P_o(\lambda, K)$. Pick some values of $\eta \in \Re$. For each $\eta \in \Re$, construct and check the feasibility of LPs corresponding to the case that the number of sign changes in the coefficients of $(\lambda+1)^{l-1}(\lambda P_o(\lambda, K, \eta) - \eta P_e(\lambda, K))$ is equal to $n_o + 1$ if $\eta > 0$ and to $n_o$ if $\eta < 0$. Let $p^*$ of them be feasible.

- **Step 6:** Check the simultaneous feasibility of LPs constructed by taking one LP from Step 4 and one from Step 5. There will be at most $p^*p_{eo}$ LPs to be checked.

- **Step 7:** Update $l$ to any number greater than $l$ for refinement and go to Step 2.

4. Using the Outer Approximation to Restrict the Search for $n-1$ Frequencies

The outer approximation outlined above involves determining a number of LPs; the union of the feasibility sets of the LPs contains the set of all stabilizing controllers. This set (outer approximation) is relatively easier to find and can help in the systematic calculation of the inner approximation in the following way:

For any $m(< n-1)$, the $m-$tuples, $(w_1, w_2, \ldots, w_m)$, can be used to solve a subset of the LPs in Theorem 5 to obtain a polyhedron $\mathcal{S}_{partial}$. If the intersection of $\mathcal{S}_{partial}$ with $\mathcal{S}_o$ is empty, then there is no need to search for the other $n-m-1$ tuples. This can considerably simplify the computation of the inner approximation by assisting in the pruning of the search for feasible $(n-1)$ tuples of frequencies in the function SOLVER.

The modified function SOLVER, which takes advantage of the above remark is given below:

**Function :** $\textsc{Solver}(start, loop, \mathbf{U}, set, \mathcal{S}_o)$

**comment:** Recursive algorithm for calculating the inner approximation

**if** $loop > ntup$

  **then return** $(set)$

$loop \leftarrow loop + 1;$

**for** $i \leftarrow start$ **to** $p$

**do**
- $\mathbf{U} \leftarrow [\mathbf{U} \;\; u_i];$
- $\mathcal{S}_{partial} \leftarrow$ Solve a subset of the **LPs** in (3.5)
- **if** $\mathcal{S}_{partial} \cap \mathcal{S}_o$ is non empty
  - **then**
    - $\text{Solver}(i+1, loop, \mathbf{U}, set, \mathcal{S}_o)$
    - **if** $loop > ntup$
      - **then** $set \leftarrow [set \;\; \mathcal{S}_{partial}]$

**return** $(set)$

One can also use the outer approximation in determining an inner approximation of $\mathcal{S}$ by restricting the range of $(n-1)$ tuple frequencies that must be searched for. Using the method described in the previous subsection, one usually gets a set of polyhedra whose union contains $\mathcal{S}$. One can then ask the following question: Given a polyhedron, what is the range of $(n-1)$ tuple frequencies corresponding to the stabilizing controllers in the polyhedron? Once the range is found, partitioning of the $(n-1)$ tuples can be carried over the narrower range of frequencies in order to determine an inner approximation of the stabilizing controllers. The following lemmas work towards such a restriction of the $(n-1)$ tuples of frequencies:

**Lemma 6.** *Let $\mathcal{K}$ be a polyhedron and for every $K \in \mathcal{K}$, let $Q(\lambda, K)$ be a real polynomial of degree $(r+1)$ with coefficients affine in $K$. Let $R(\lambda, K)$ be its derivative with respect to $\lambda$. For all polynomials $R(\lambda, K),\ K \in \mathcal{K}$ that have all real and distinct roots, suppose their roots lie in the intervals $(a_i, b_i),\ i = 1, \ldots, r$ with $a_{i+1} \geq a_i$ and $b_{i+1} \geq b_i,\ i = 1, \ldots, r-1$. Then, all polynomials, $Q(\lambda, K),\ K \in \mathcal{K}$ that have all real and distinct roots have their roots in the intervals, $(-\infty, b_1)$, $(a_1, b_2)$, …, $(a_{r-1}, b_r)$ and $(a_r, \infty)$.*

*Proof.* Let $Q(\lambda, K)$ have all real and distinct roots for some $K \in \mathcal{K}$. Then, $R(\lambda, K)$ has all real and distinct real roots and let them be $\eta_1, \ldots, \eta_r$. Since the roots of $Q(\lambda, K)$ and $R(\lambda, K)$ interlace, the roots of $Q(\lambda, K)$ must lie in $(-\infty, \eta_1)$, $(\eta_1, \eta_2)$, …, $(\eta_{r-1}, \eta_r)$ and $(\eta_r, \infty)$. But $\eta_i \in (a_i, b_i)$. Therefore, the result follows. $\square$

**Lemma 7.** *Consider a linear polynomial $a(K)s + b(K)$, where the coefficients $a, b$ are affine functions of $K$ and $K \in \mathcal{K}$, a polyhedral set. If $a(K) > 0$ for $K \in \mathcal{K}$, the root of the linear polynomial, for every $K \in \mathcal{K}$ lies in $[-\lambda_{low}, \lambda_{high}]$, where $\lambda_{low}$ and $\lambda_{high}$ can be determined by the following linear fractional programs:*

$$\lambda_{low} = \min -\frac{b(K)}{a(K)}, \qquad \lambda_{high} = \max -\frac{b(K)}{a(K)},$$
$$a(K) > 0,\ K \in \mathcal{K}. \qquad a(K) > 0,\ K \in \mathcal{K}.$$

The above linear fractional program can be cast as linear programs, see Boyd and Vandenberghe [55]. Using Lemmas 6 and 7 and the following lemma, one can construct bounds for the roots of a real polynomial with all real roots.

**Lemma 8.** *Let $\mathcal{K} = \{K : AK \leq b\}$ be a polyhedron and corresponds to a set of polynomials*

$$\{R(\lambda, K),\ K \in \mathcal{K}\}$$

*of degree $r$ and let the leading coefficient of every $R(\lambda, K),\ K \in \mathcal{K}$ be positive. Then, the roots of the polynomials $R(\lambda, K),\ K \in \mathcal{K}$ with all real and distinct roots can be bounded recursively using Lemmas 6 and 7.*

*Proof.* Let $\mathcal{K}_r \subset \mathcal{K}$ be defined such that the polynomial $R(\lambda, K)$ will have all real roots if $K \in \mathcal{K}_r$. If $\mathcal{K}_r = \emptyset$, then any bound on the roots will suffice.

Let $R^{(k)}(\lambda, K)$ denote the $k^{th}$ derivative of $R(\lambda, K)$ with respect to $\lambda$. It is clear that $R^{(r-1)}$ is a linear polynomial, the roots of which can be bounded using Lemma 7. Since, for every $K \in \mathcal{K}_r$, the roots of $R(\lambda, K)$ are real, it follows that the roots of $R^{(k)}(\lambda, K)$ are also real for every $K \in \mathcal{K}_r$. Further, for every $K \in \mathcal{K}_r$, the roots of $R^{(r-2)}$ must lie in $(-\infty, \lambda_{high})$ and $(\lambda_{low}, \infty)$. These bounds can be tightened further by solving the following programs:

$$\text{Let } b_1 := \max \lambda \text{ such that}$$

$$\lambda < \lambda_{high},$$
$$R^{(r-2)}(\lambda, K) > 0, \text{ and}$$
$$R^{(r-2)}(\lambda_{high}, K) < 0,\ K \in \mathcal{K}$$

is feasible. If the linear program is not feasible for any $\lambda < \lambda_{high}$, it implies that the polynomial $R^{(r-2)}(\lambda, K)$ cannot have two real roots and hence, the polynomial, $R(\lambda, K)$ cannot have all real roots. This in turn will imply that $\mathcal{K}_r = \emptyset$. If the linear program is feasible for some $\lambda < \lambda_{high}$, the quantity $b_1$ may be computed using a bisection technique as the program can be written as a linear program for any given

$\lambda$. Similarly,

$$\text{Let } a_2 := \min \lambda \text{ such that}$$
$$\lambda > \lambda_{low},$$
$$R^{(r-2)}(\lambda, K) < 0 \text{ and}$$
$$R^{(r-2)}(\lambda_{low}, K) > 0,\ K \in \mathcal{K}$$

is feasible. It is clear that the roots of $R^{(r-2)}(\lambda, K)$ lie in $(-\infty, b_1)$ and $(a_2, \infty)$. We will show, by induction, that such bounds can be computed for every $R^{(k)}(\lambda, K)$. Since the base case $k = r-1$ has already been considered, it is sufficient to show that the bounds for the $i^{th}$ root of $R^{(k-1)}(\lambda, K)$ can be computed when the bounds for $R^{(k)}$ have been computed. Let the $i^{th}$ root of $R^{(k)}(\lambda, K)$ lie in $(a_i^{(k)}, b_i^{(k)})$. Then, it is clear that the roots of $R^{(k-1)}(\lambda, K)$ for $K \in \mathcal{K}_r$ must lie in

$$(-\infty, b_1^{(k)}), (a_1^{(k)}, b_2^{(k)}), \ldots, (a_{r-k-1}^{(k)}, b_{r-k}^{(k)}) \text{ and } (a_{r-k}^{(k)}, \infty).$$

These bounds can be further tightened using linear programming technique illustrated for the case $k = r-1$. Define $a_1^{(k-1)} = -\infty$, and further define $b_{r-k+1}^{(k-1)} = \infty$. For $j \leq r-k$, define

$$b_j^{(k-1)} = \max \lambda \text{ such that}$$
$$\lambda \leq b_j^{(k)},$$
$$(-1)^{r-k+j} R^{(k-1)}(\lambda, K) > 0 \text{ and}$$
$$(-1)^{r-k+j} R^{(k-1)}(b_j^{(k)}, K) < 0,\ K \in \mathcal{K}$$

is feasible. (Again, if the inequalities are not feasible, then $\mathcal{K}_r = \emptyset$).

Similarly, for $j \geq 2$, define

$$\begin{aligned} a_j^{(k-1)} &= \min\ \lambda \text{ such that} \\ &\lambda \geq a_{j-1}^{(k)}, \\ &(-1)^{r-k+j} R^{(k-1)}(\lambda, K) < 0 \text{ and} \\ &(-1)^{r-k+j} R^{(k-1)}(a_{j-1}^{(k)}, K) > 0,\ K \in \mathcal{K} \end{aligned}$$

is feasible. It is clear that the $j^{th}$ root of $R^{(k-1)}(\lambda, K),\ K \in \mathcal{K}$ lies in $(a_j^{(k-1)}, b_j^{(k-1)})$. By recursion, one can get the bounds for the roots of $R^{(0)}(\lambda, K)$. □

*Remark* 7. If $R(\lambda, K)$ corresponds to either $P_e(\lambda, K)$ or $P_o(\lambda, K)$ and the set $\mathcal{K}_r = \emptyset$ corresponding to any polyhedron of an outer approximation of $\mathcal{S}$, then such a polyhedron can be removed from further consideration in the construction of inner approximation as it cannot contain any stabilizing controllers.

*Remark* 8. By the above lemma, the computation of bounds for the roots of $P_e(\lambda, K)$ and $P_o(\lambda, K)$ corresponding to a polyhedron representing the outer approximation of stabilizing controllers is independent of the calculations of the vertices of the polyhedron. It only requires the computation of linear programs and hence, is computationally tractable.

*Remark* 9. In the inner approximation discussed previously, since we have to calculate the *positive* real roots of $P_e(\lambda, K)$ and $P_o(\lambda, K)$, we can restrict the lower bound on the first root to be 0. This bound can be further tightened using linear programming technique described in Lemma 5. Define

$$\begin{aligned} a_1^{(k-1)} &= \min \lambda \text{ such that} \\ &\lambda \geq 0, \\ &(-1)^{r-k+j} R^{(k-1)}(\lambda, K) < 0 \text{ and} \end{aligned}$$

$$(-1)^{r-k+j} R^{(k-1)}(0, K) > 0,\ K \in \mathcal{K}$$

is feasible. For numerical implementation, one can assume that that the roots of the polynomials always lie within the interval $[0, \lambda_{max}]$, where the bound $\lambda_{max}$ can be computed using the Cauchy bound [56] and linear fractional programming. One can similarly, compute a tighter upper bound for $b_{r-k+1}^{(k-1)}$.

## C. Examples

To illustrate the proposed approach, consider the following examples:

*Example* 1. This example is from [24]. Consider a LTI plant described by the following equation:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 5 & -1 \\ -1 & -1 & 0 \end{bmatrix} x. \end{aligned}$$

The aim is to find the set of all static stabilizing controllers, i.e., $u = Ky$ where $K = [K_1 \ \ K_2]$, for this system using the method proposed here.

The characteristic polynomial of the closed loop system is

$$P(s, K) = \begin{bmatrix} 1 & s & s^2 & s^3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -13 & -5 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix}.$$

For this example, $n_e = 1$ and $n_o = 1$. The real and imaginary parts of the characteristic polynomial, when evaluated at $jw$ are given by:

$$P_e(w^2, K) = \begin{bmatrix} 1 & w^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix},$$

$$P_o(w^2, K) = \begin{bmatrix} 1 & w^2 \end{bmatrix} \begin{bmatrix} -13 & -5 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix}.$$

The polynomials $R_e(u, K)$ and $R_o(u, K)$ are given by:

$$R_e(u, K) = \begin{bmatrix} (1-u) & u \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix},$$

$$R_o(u, K) = \begin{bmatrix} (1-u) & u \end{bmatrix} \begin{bmatrix} -13 & -5 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix}.$$

**Construction of the sets of fixed order stabilizing controllers for this example problem:** For the closed loop system to be stable, there must exist a set of generalized frequencies $0 = u_0 < u_1 < u_2 < 1$ such that:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1-u_0 & u_0 \\ 1-u_1 & u_1 \\ 1-u_2 & u_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix} > 0,$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1-u_0 & u_0 \\ 1-u_1 & u_1 \\ 1-u_2 & u_2 \end{bmatrix} \begin{bmatrix} -13 & -5 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix} > 0.$$

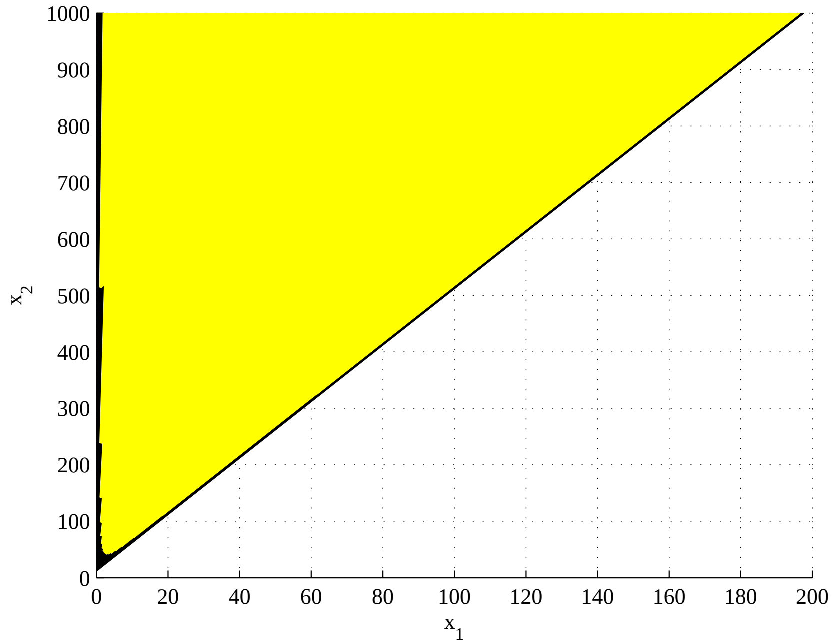



Fig. 7. Inner and outer approximation of the set of stabilizing controllers for example 1.

Fig. 7 shows the results of inner and outer approximation of the set of stabilizing controllers. The lightly shaded area shows the inner approximation and the black region in Fig. 7 is the difference in the outer and inner approximation. This shows that we achieve a tight bound on the set of controllers.

*Example* 2. Consider the system given by

$$P(s) = \frac{1}{s(s^3+1)}.$$

This plant is *not stabilizable* by a first order controller. We consider second order controllers of the form

$$C(s) = \frac{k_1 s^2 + k_2 s + k_3}{s^2 + k_4 s + k_5}.$$

The method developed in this paper is able to find an inner approximation efficiently.

The characteristic equation is given by

$$s^6 + k_4 s^5 + k_5 s^4 + s^3 + (k_1 + k_4)s^2 + (k_5 + k_2)s + k_3.$$

Using roots of a $20^{th}$ degree Chebyshev polynomial of the first kind for partitioning, we get 2380 polyhedrons. Every point contained in these polyhedrons stabilizes the plant. Each of these polyhedrons corresponds to a set of 5-tuples of points: $0 < u_1 < u_2 < u_3 < u_4 < u_5 < 1$. There are 2380 such set of frequencies. One such set of generalized frequency is $(0,\ 0.12054,\ 0.20003,\ 0.3546,\ 0.5,\ 0.5806)$. Corresponding to this set of generalized frequency, we can find the corresponding polyhedron. This polyhedron was found to have 32 vertices. All points inside this polyhedron provides a value of $K$ which stabilizes the plant. For any point inside this polyhedron, the roots of the even and odd polynomials should interlace and $u_{1e} \in (0, 0.12054)$, $u_{1o} \in (0.12054, 0.20003)$, $u_{2e} \in (0.20003, 0.3546)$, $u_{2o} \in (0.3546, 0.5)$, $u_{3e} \in (0.5, 0.5806)$. To show this, consider a point in the polyhedron, $K = (-0.2235, -1.6020, 0.0339, 0.8879, 1.7594)$. For this value of $K$, we find that $u_e = (0.056884, 0.31025, 0.55542)$ and $u_o = (0.15912, 0.48373)$ satisfy the interlacing property and hence is a feasible point. For this controller, the roots of the closed loop are at $(-0.088 \pm 1.027i, -0.2091 \pm 0.5626i, -0.147 \pm 0.2888i)$. The computer program was written in MATLAB and used the functions provided in [57] for solving LPs and for polyhedron visualization.

*Example* 3. The following example is from [58]. The example is that of a Saturn V booster and its model can be described by a single input, two output seventh order model:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & -0.65 & -0.002 & 2.6 & 0 \\ -0.014 & 1 & -0.041 & 0.0002 & -0.015 & -0.033 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -45 & -0.13 & 255 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -50 & -10 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}',$$

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x. \end{aligned}$$

The construction of an inner and outer approximation of the set of stabilizing controllers of the form $u = [K_1 \ \ K_2]y$ is provided.

The characteristic polynomial of the closed loop system with a static output feedback stabilizing controller is:

$$P(s, K) = \begin{bmatrix} 1 & s & s^2 & s^3 & s^4 & s^5 & s^6 & s^7 \end{bmatrix} \begin{bmatrix} 6.3 & 2.2856 & 0 \\ -448.7218 & 49.5229 & 2.2856 \\ 1.2196 & .072 & 49.5229 \\ 2249.5 & -2.6 & .072 \\ 458.4251 & 0 & -2.6 \\ 96.5153 & 0 & 0 \\ 10.171 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix}.$$

Since the characteristic polynomial is monic, all its coefficients must be positive. The real and imaginary parts of the characteristic polynomial, when evaluated at $s = jw$, are given by:

$$P_e(w^2,K) \;=\; \begin{bmatrix} 1 & w^2 & w^4 & w^6 \end{bmatrix} \begin{bmatrix} 6.3 & 2.2856 & 0 \\ -1.2196 & -0.072 & -49.5229 \\ 458.4251 & 0 & -2.6 \\ -10.171 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix},$$

$$P_o(w^2,K) \;=\; \begin{bmatrix} 1 & w^2 & w^4 & w^6 \end{bmatrix} \begin{bmatrix} -448.7218 & 49.5229 & 2.2856 \\ -2249.5 & 2.6 & -0.072 \\ 96.5153 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix}.$$

After using the transformation $w^2 = \frac{u}{1-u}$, we get

$$R_e(u) = \begin{bmatrix} (1-u)^3 & u(1-u)^2 & u^2(1-u) & u^3 \end{bmatrix} \begin{bmatrix} 6.3 & 2.286 & 0 \\ -1.220 & -0.072 & -49.523 \\ 458.425 & 0 & -2.6 \\ -10.171 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix},$$

$$R_o(u) = \begin{bmatrix} (1-u)^3 & u(1-u)^2 & u^2(1-u) & u^3 \end{bmatrix} \begin{bmatrix} -448.722 & 49.523 & 2.286 \\ -2249.5 & 2.6 & -0.072 \\ 96.515 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix}.$$

Let

$$U = U(u_0, u_1, u_2, u_3, u_4, u_5, u_6, 3) = \begin{bmatrix} (1-u_0)^3 & (1-u_0)^2 u_0 & (1-u_0)u_0^2 & u_0^3 \\ (1-u_1)^3 & (1-u_1)^2 u_1 & (1-u_1)u_1^2 & u_1^3 \\ (1-u_2)^3 & (1-u_2)^2 u_2 & (1-u_2)u_2^2 & u_2^3 \\ (1-u_3)^3 & (1-u_3)^2 u_3 & (1-u_3)u_3^2 & u_3^3 \\ (1-u_4)^3 & (1-u_4)^2 u_4 & (1-u_4)u_4^2 & u_4^3 \\ (1-u_5)^3 & (1-u_5)^2 u_5 & (1-u_5)u_5^2 & u_5^3 \\ (1-u_6)^3 & (1-u_6)^2 u_6 & (1-u_6)u_6^2 & u_6^3 \end{bmatrix}.$$

For the system to be stable, there must exist 6-tuples, $0(= u_0) < u_1 < u_2 < u_3 < u_4 < u_5 < u_6 < 1$, such that,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} U \begin{bmatrix} 6.3 & 2.2856 & 0 \\ -1.2196 & -0.072 & -49.5229 \\ 458.4251 & 0 & -2.6 \\ -10.171 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix} > 0,$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} U[ \begin{bmatrix} -448.7218 & 49.5229 & 2.2856 \\ -2249.5 & 2.6 & -0.072 \\ 96.5153 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix} > 0.$$

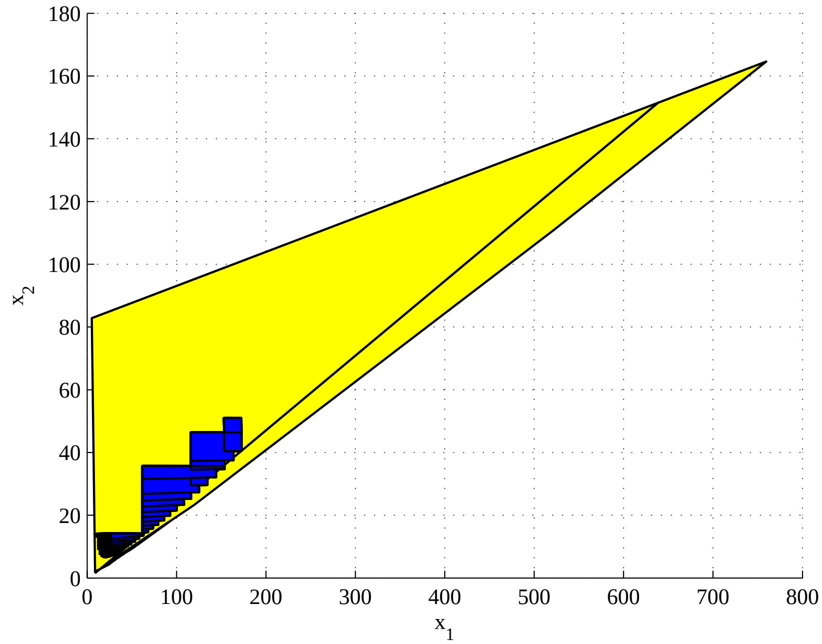



Fig. 8. Outer and inner approximation of the set of static stabilizing controllers for the seventh order system described in example 3.

Fig. 8 illustrates the outer and inner approximation of the set of stabilizing controllers. Initially, only the outer approximation was found as a union of polyhedrons. The obtained set can be represented as $\mathcal{K} = \{K : AK \leq b\}$. Using Lemma 8, the bounds for the 6-tuples are found. The even and odd parts of the characteristic polynomial, $P_e(\lambda, K)$ and $P_o(\lambda, K)$, $\lambda = w^2$, are of order 3. The range of the root for the second derivative of $P_e(\lambda, K)$ is $(1, 14.8886)$ and that of $P_o(\lambda, K)$ is found to have a constant value of 32.1718. The ranges of the first and second roots of the first derivative of $P_e(\lambda, K)$ are $(0.088913, 5.9406)$ and $(26.2933, 29.6883)$ respectively. The ranges of the first and second roots of the first derivative of $P_o(\lambda, K)$ are $(1.5204, 15.2135)$ and $(49.1301, 62.8232)$ respectively.

The range of the frequencies calculated for the $P_e(\lambda, K)$ and $P_o(\lambda, K)$ are:

$$\lambda_{e1} = (0.0042604, 0.2394) \qquad \lambda_{o1} = (0, 7.6114)$$

$$\lambda_{e2} = (0.683, 6.0015) \qquad \lambda_{o2} = (29.4393, 32.9093)$$

$$\lambda_{e3} = (26.2933, 44.4891) \qquad \lambda_{o3} = (57.5571, 97.5243)$$

The values reported above do not take the interlacing into consideration. These bounds can be made tighter by simply observing the ranges and ensuring that interlacing occurs. This availability of these bounds simplifies the calculation of the inner approximation.

*Example* 4. An example of approximating a disconnected set of stabilizing controllers is the following [30]:

The plant to be controlled has the transfer function:

$$H(s) = \frac{s^3 + 3s^2 + 9}{s^4 + 2s^3 + 3s^2 + 7s + 14}.$$

The controller considered is a PID controller,

$$C(s) = K_p + \frac{K_I}{s} + K_D s.$$

The closed loop characteristic polynomial, $P(s, K_P, K_I, K_D)$ may be written as:

$$P = \begin{bmatrix} 1 & s & s^2 & s^3 & s^4 & s^5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 9 \\ 14 & 0 & 9 & 0 \\ 7 & 9 & 0 & 3 \\ 3 & 0 & 3 & 1 \\ 2 & 3 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_D \\ K_P \\ K_I \end{bmatrix}.$$

and

$$P_e = \begin{bmatrix} 1 & w^2 & w^4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 9 \\ -7 & -9 & 0 & -3 \\ 2 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_D \\ K_P \\ K_I \end{bmatrix},$$

$$P_o = \begin{bmatrix} 1 & w^2 & w^4 \end{bmatrix} \begin{bmatrix} 14 & 0 & 9 & 0 \\ -3 & 0 & -3 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_D \\ K_P \\ K_I \end{bmatrix}.$$

For the polynomial $P(s, K_P, K_I, K_D)$ to be Hurwitz, there must exist frequencies 4-tuples of frequencies. As the characteristic polynomial is not monic, it requires

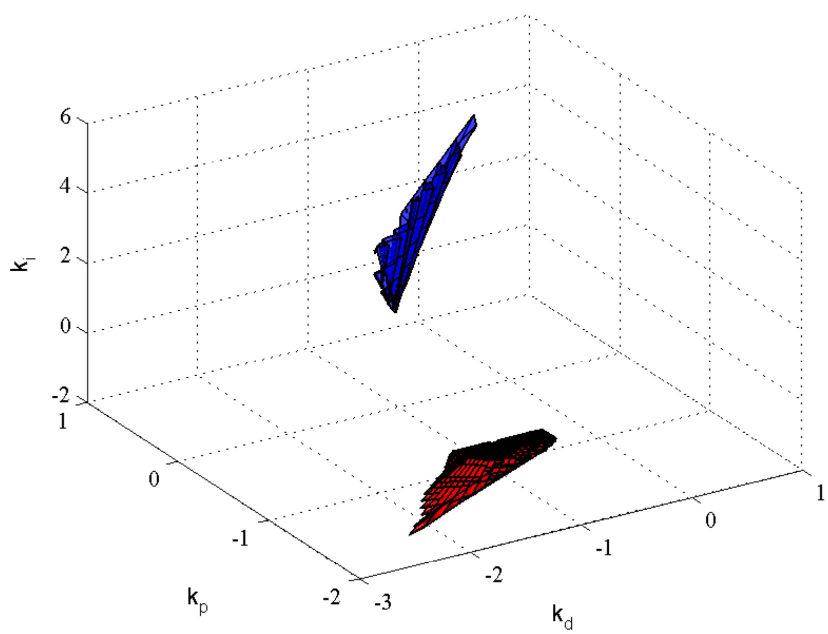



Fig. 9. Set of stabilizing PID controllers - An inner approximation.

the solution of (3.5) for $i = 1, 3$. Fig. 9 illustrates the inner approximation of disconnected set of stabilizing controllers. There are two discontinuous sets. A partition with 25 positive roots of Chebyshev polynomial of first kind was considered. Equation (3.5) was solved and 28 feasible polyhedrons were found for $i = 1$ and 461 for $i = 3$. The approximate ranges of $(u_1, u_2, u_3)$ for $i = 1$ are found to be $[(0.7614, 0.8381)\ ,\ (0.8014, 0.9269)\ ,\ (0.8381, 0.9491)]$. If a refinement is needed for superior results, then the number of partitions can be increased and the search for $(n-1)$ tuples can be approximately restricted to the range found above.

*Example* 5. The following example is from [59]. The example is that of stabilizing two plants simultaneously by a fixed order controller. The book [59] considers a second order controller to stabilize these plants. We approximate the set of all first order controllers

$$C(s) = \frac{(K_1 s + K_2)}{s + K_3}$$

that simultaneously stabilize two plants with the following transfer functions:

$$P_1(s) = \frac{2(2-s)}{(s+1)(s+2)} \text{ and } P_2(s) = \frac{(2-s)}{(s-1)(s+2)}$$

**Plant 1:**

The closed loop characteristic polynomial of the first plant is,

$$P_1(s,K) = \begin{bmatrix} 1 & s & s^2 & s^3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 4 & 2 \\ 2 & 4 & -2 & 3 \\ 3 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix}.$$

The real and imaginary parts of the characteristic polynomial, evaluated at $s = jw$, are given by:

$$P_e^1(w^2, K) = \begin{bmatrix} 1 & w^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 4 & 2 \\ -3 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix},$$

$$P_o^1(w^2, K) = \begin{bmatrix} 1 & w^2 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 3 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix}.$$

After using the transformation $w^2 = \frac{u}{1-u}$,

$$R_e^1(u, K) = \begin{bmatrix} (1-u) & u \end{bmatrix} \begin{bmatrix} 0 & 0 & 4 & 2 \\ -3 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix},$$

$$R_o^1(u, K) = \begin{bmatrix} (1-u) & u \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 3 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix}.$$

**Plant 2:**

The closed loop characteristic polynomial of the second plant is:

$$P_2(s, K) = \begin{bmatrix} 1 & s & s^2 & s^3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & -2 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix}.$$

The real and imaginary parts of the characteristic polynomial are given by:

$$P_e^2(w^2, K) = \begin{bmatrix} 1 & w^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & -2 \\ -1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix},$$

$$P_o^2(w^2, K) = \begin{bmatrix} 1 & w^2 \end{bmatrix} \begin{bmatrix} -2 & 2 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix}.$$

After using the transformation $w^2 = \frac{u}{1-u}$,

$$R_e^2(u, K) = \begin{bmatrix} (1-u) & u \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & -2 \\ -1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix},$$

$$R_o^2(u, K) = \begin{bmatrix} (1-u) & u \end{bmatrix} \begin{bmatrix} -2 & 2 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix}.$$

For the system to be stable, there must exist two sets of generalized frequencies $0(= u_0) < u_1 < u_2 < 1$ and $0 = u_3 < u_4 < u_5 < 1$ such that:

$$C_1 \begin{bmatrix} (1-u_0) & u_0 \\ (1-u_1) & u_1 \\ (1-u_2) & u_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 4 & 2 \\ -3 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} > 0,$$

$$S_1 \begin{bmatrix} (1-u_0) & u_0 \\ (1-u_1) & u_1 \\ (1-u_2) & u_2 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 3 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} > 0,$$

$$C_1 \begin{bmatrix} (1-u_3) & u_3 \\ (1-u_4) & u_4 \\ (1-u_5) & u_5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & -2 \\ -1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} > 0,$$

$$S_1 \begin{bmatrix} (1-u_3) & u_3 \\ (1-u_4) & u_4 \\ (1-u_5) & u_5 \end{bmatrix} \begin{bmatrix} -2 & 2 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \\ K_3 \end{bmatrix} > 0.$$

Each plant is separately stabilized using the computer program. Intersection of the two stabilizing sets provides the solution for the simultaneous stabilization of the two plants. Fig. 10 and Fig. 11 show the results of this approximation; a partition with 20 positive roots of Chebyshev polynomial of first kind is used.

*Example* 6. Consider a plant $P_1(s) = \frac{1}{(s^2-0.1s+1)}$; it is clear that there is a proper stabilizing controller of first order. We want to find strictly proper output feedback stabilizing controllers of first order, i.e. controllers of the form: $C(s) = \frac{K_1}{s+K_2}$.

The closed loop characteristic polynomial of the plant is:

$$P(s,K) = \begin{bmatrix} 1 & s & s^2 & s^3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -0.1 \\ -0.1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix}.$$

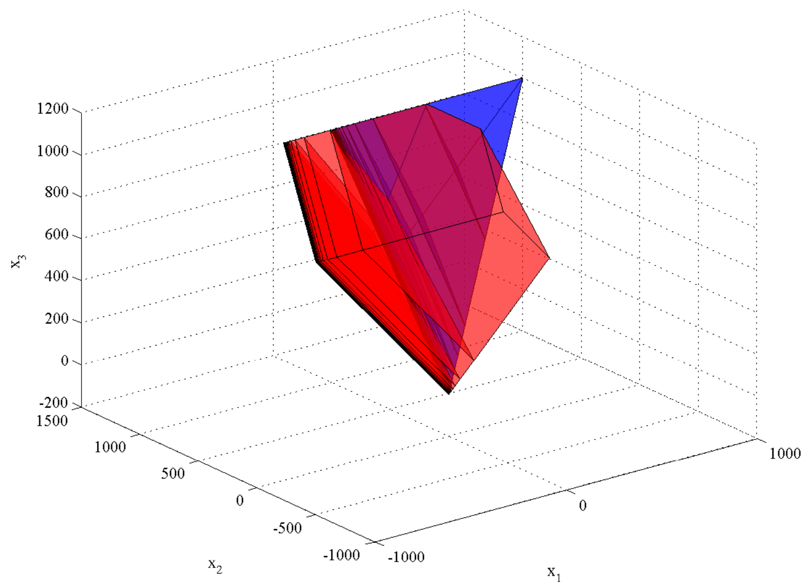



Fig. 10. Inner approximation of both the plants (example 5).

The real and imaginary parts of the characteristic polynomial are given by,

$$P_e(\lambda, K) = \begin{bmatrix} 1 & \lambda \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0.1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix},$$

$$P_o(\lambda, K) = \begin{bmatrix} 1 & \lambda \end{bmatrix} \begin{bmatrix} 1 & 0 & -0.1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ K_1 \\ K_2 \end{bmatrix}.$$

Enforcing the necessary condition that the coefficients of $P(s, K)$ be of the same sign and the condition that $\lambda P_o(\lambda) - \eta P_e(\lambda)$ has exactly two sign changes, one gets an outer approximation of the set of stabilizing controllers.

The characteristic polynomial is monic, hence (3.5) has to be solved for only $i = 1$. 2-tuples of frequencies have to be searched. Partitions using positive roots of Chebyshev polynomial of first kind is used. For 20 partitions, 5 polyhedrons are found

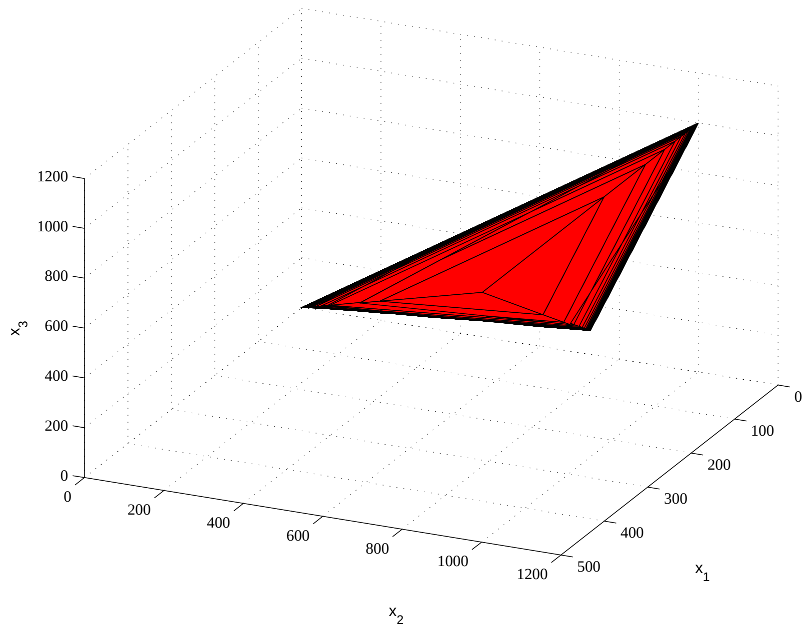



Fig. 11. Inner approximation for simultaneous stabilization (example 5).

and approximate range of $u_1$ is $(0.1205, 0.4287)$. On refining the number of partitions to 40, 12 polyhedrons are found and approximate range of $u_1$ is $(0.0596, 0.4769)$. On increasing the number of partitions to 100, the inner approximation consists of 32 polyhedrons and the approximate range of $u_1$ is $(0.0237, 0.4909)$. Fig. 12 shows the inner and outer approximation. The lighter shaded region is the inner approximation with 100 partitions. The black region is the difference in the outer and inner approximations.

## D. Summary

This chapter considers the problem of the synthesis of fixed order and structure controllers, where the coefficients of the closed loop polynomial are linear in the parameters of the controller. A novel feature of this paper is the systematic exploitation

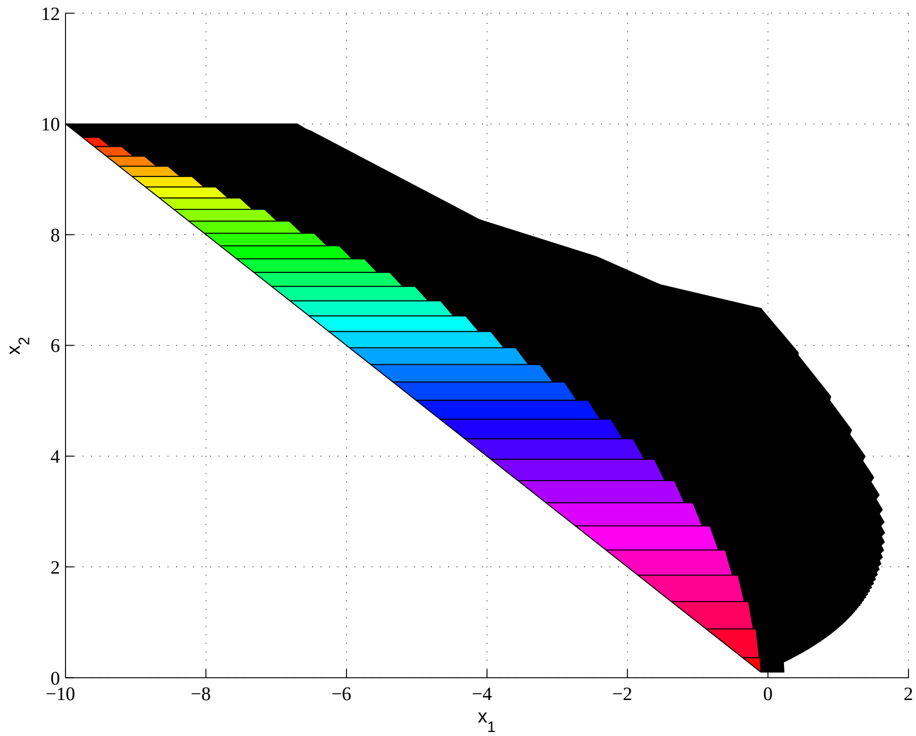



Fig. 12. Inner and outer approximation of example 6 showing a bounded approximation for first order controllers.

of the interlacing property of Hurwitz polynomials and the use of Descartes' rule of signs to generate LPs in the parameters of a fixed order controller. The problem of inner approximation of the set of stabilizing controllers was posed as the search for all sets of ordered $n-1$-tuples of frequencies for which the associated LP is feasible; the union of all feasible LPs is an inner approximation for the set of all stabilizing controllers. For constructing the pouter approximation, we use the fact that a necessary condition for a polynomial to be Hurwitz is that the roots of even and odd parts of the polynomial have all real, positive and interlacing roots. The Descartes' rule of signs and its generalization due to Poincaré were used to construct the LPs for the outer approximation. A significant advantage of the presented methodology is that robust stability and performance specifications such as gain and phase margins can be naturally accommodated by imposing further linear inequality constraints. These will be discussed in the next chapter.

# CHAPTER IV

## CONTROLLERS SATISFYING GIVEN PERFORMANCE CRITERIA

The previous chapter dealt with the synthesis of sets of fixed order and fixed structure stabilizing feedback controllers. It is often required that these feedback controllers satisfy some specified performance criterion. This class of fixed structure controller synthesis problems can be reduced to the determination of a real controller parameter vector (or simply, a controller), $K$, so that a family of real or complex polynomials, linear in the parameters of the controllers, is Hurwitz. An algorithm is provided which exploits the Interlacing Property of complex Hurwitz polynomials to systematically construct an arbitrarily large number of sets of linear inequalities in $K$. The simultaneous stabilization of the family of real or complex polynomials provides an approximation of the set of controllers satisfying the given performance criteria.

### A. Introduction

One can impose additional restrictions on the set of stabilizing controllers, $\mathcal{S}$, and check if a certain level of performance can be achieved. Some performance criteria are desired gain margin, phase margin, desired upper bound on the $\mathcal{H}_\infty$ norm of a weighted sensitivity transfer function, or a requirement that a certain closed loop transfer function be SPR etc. A large class of performance problems such as those listed here can be reduced to the problem of determining a set of *stabilizing* controllers that render a set of complex or real polynomials Hurwitz [8].

#### a. Performances That Require a Family of Real Polynomials to be Rendered Hurwitz

The solution to the set of controllers satisfying certain performance specifications, such as the gain margin, can be reduced to finding set of controllers that simultaneously stabilize a one parameter compact family of real polynomials.

- The criterion for guaranteeing a gain margin of $k^*$ for a SISO plant with a transfer function $\frac{N_p(s)}{D_p(s)}$ stabilized by a fixed order compensator, $\frac{N_c(s)}{D_c(s)}$, is that, for every $k \in [1, k^*]$, the real polynomial

  $$D_p(s)D_c(s) + kN_p(s)N_c(s)$$

  must be Hurwitz.

- The solution for the set of controllers for the robust stability of an interval plant can be similarly posed as requiring a finite family of real polynomials to be rendered Hurwitz through the choice of the controller parameters. An interval plant is a class of interval systems where the uncertain parameters lie in intervals and appear linearly in the numerator and denominator coefficients of the transfer functions. Consider an unity feedback system with the controller $C(s) = \frac{N_c(s)}{D_c s)}$ and the interval plant $G(s) = \frac{N(s)}{D(s)}$. The interval polynomial sets is defined as

  $$\mathbf{D}(s) = D(s) : a_o + \cdots + a_n s^n,\ a_k \in [{a_k}^-, {a_k}^+],\ k = 1 \ldots n,$$
  $$\mathbf{N}(s) = N(s) : b_o + \cdots + b_m s^m,\ b_k \in [{b_k}^-, {b_k}^+],\ k = 1 \ldots m,$$

  and the corresponding set of interval systems:

  $$\mathbf{G}(s) = \left[ \frac{N(s)}{D(s)} : (N(s), D(s)) \in (\mathbf{N}(s) \times \mathbf{D}(s)) \right].$$

  The set of system characteristic polynomials can be written as

  $$\mathbf{\Delta}(s) = N_c(s)\mathbf{N}(s) + D_c(s)\mathbf{D}(s).$$

  This control system is robustly stable if each polynomial in $\mathbf{\Delta}(s)$ is of the same degree and is Hurwitz. Using the Generalized Kharitonov Theorem [8], the Hurwitz stability of the control system over the set $\mathbf{G}(s)$ can be reduced to testing over the much smaller extremal set of system $\mathbf{G_E}(s)$. Moreover, if the polynomials $N_c(s)$ and $D_c(s)$ are of the form $s^{t_i}(a_i s + b_i)U_i(s)Q_i(s)$, where $t_i \geq 0$ is an arbitrary integer, $a_i$ and $b_i$ are arbitrary real numbers, $U_i(s)$ is an anti-Hurwitz polynomial, and $Q_i(s)$ is an even or odd polynomial, then it is sufficient that the controller *simultaneously* stabilizes the finite set of Kharitonov vertex polynomials $\mathbf{\Delta_K}(s)$.

b. Performances That Require a Family of Complex Polynomials to be Rendered Hurwitz

A large class of performance problems such as those listed here can be reduced to the problem of determining a set of *stabilizing* controllers that render a set of complex polynomials Hurwitz [8]. For example,

- the criterion for guaranteeing a phase margin of $\phi$ for a SISO plant with a transfer function $\frac{N_p(s)}{D_p(s)}$ stabilized by a fixed order compensator, $\frac{N_c(s)}{D_c(s)}$ is that, for every $\theta \in (-\phi, \phi)$, the polynomial

$$D_p(s)D_c(s) + e^{j\theta}N_p(s)N_c(s)$$

must be Hurwitz.

- For the same controller to achieve a $\mathcal{H}_\infty$ norm of the sensitivity transfer function less than $\gamma$ is equivalent to having the following family of complex polynomials

$$\gamma D_p(s)D_c(s) + e^{j\theta}N_p(s)N_c(s),$$

Hurwitz, for every $\theta \in [0, 2\pi]$.

- Requiring a transfer function $\frac{N(s,K)}{D(s,K)}$ to be Strictly Positive Real (SPR) is equivalent to requiring the family of polynomials

$$D(s,K) + j\alpha N(s,K),\ \alpha \in \Re,$$

Hurwitz. In fact, this problem arises in guaranteeing *absolute stability*, that is, robust stability to sector bounded nonlinearities, which are common in mechanical systems.

In the following subsection, a method is provided to construct a subset $\mathcal{P} \subset \mathcal{K}$ that make a complex polynomial, $P(s,K)$, of the form described by equation (3.1), Hurwitz. Controllers that achieve a specified performance can be synthesized in the following way: (i) discretize the relevant set of polynomials to yield a finite number of polynomials to be made Hurwitz, (ii) construct set of controllers that simultaneously make the finite number of polynomials Hurwitz.

## B. On Characterizing Stabilizing Sets of Controllers for Complex Polynomials via Linear Programming

Let $P(s)$ be a complex polynomial; the number of its roots in the left half plane is the same as the number of roots of the polynomial $P(jw)$ in the upper half plane, (i.e., number of roots with positive imaginary parts). Hermite considered exactly the same problem of counting the roots of a complex polynomial in the upper half plane [18]. For a complex polynomial to be Hurwitz, the Hermite-Biehler theorem requires the separation between the roots of the real and imaginary parts of $P(jw)$ (see Theorem 2. As can be seen from our earlier treatment for real stabilization, such a treatment would lend itself for the construction of linear programs in the controller parameters.

Let $P(jw) = P_r(w) + jP_i(w)$. If its degree is $n$, then for a sufficiently large $w^*$, the Mikhailov plot of $P(jw)$ will lie entirely in one quadrant for all $w < -w^*$; we will say that $p_{r,n} + jp_{i,n}$ defined through $\lim_{w\to-\infty} \frac{P(jw)}{|w|^n}$ belongs to the same quadrant. One can assume without any loss of generality that $p_{r,n}p_{i,n} \neq 0$; in fact, if $p_{r,n}p_{i,n} = 0$, one can consider $(1 + j\tau)P(jw)$; for this polynomial, the leading coefficients of the corresponding real and imaginary polynomials are different from zero whenever $\tau \neq 0$ and the location of the roots of this polynomial being the same as that of $P(jw)$ . Let $C_k, S_k,\ k = 1, 2, 3, 4$ be diagonal matrices of dimension $2n$; the $(m+1)^{st}$ diagonal elements of these matrices are respectively the signs of $\cos\left((2k-1)\frac{\pi}{4} + m\frac{\pi}{2}\right)$ and $\sin\left((2k-1)\frac{\pi}{4} + m\frac{\pi}{2}\right)$ .

In the following theorem, the Hermite Biehler theorem for complex polynomials is restated in a form, which allows the construction of linear programs.

**Theorem 6.** *Let $P(s)$ be a complex polynomial of degree $n$ with $p_{r,n}p_{i,n} \neq 0$; the following statements are equivalent:*

1. *$P(s)$ is Hurwitz.*
2. *All roots of the polynomials $P_r(w)$ and $P_i(w)$ are real and interlace; specifically, there exists a set of $(2n-1)$ real frequencies satisfying $-\infty < w_1 < w_2 < \ldots < w_{2n-1} < \infty$ that separates the roots of the real polynomials in such a way that for exactly one of $k = 1, 2, 3, 4$, the following conditions hold:*

$$C_k \begin{bmatrix} P_{r,n} \\ P_r(w_1) \\ P_r(w_2) \\ \vdots \\ P_r(w_{2n-1}) \end{bmatrix} > 0, \qquad S_k \begin{bmatrix} P_{i,n} \\ P_i(w_1) \\ P_i(w_2) \\ \vdots \\ P_i(w_{2n-1}) \end{bmatrix} > 0.$$

*Proof.* The Mikhailov plot $(P(jw))$ of a complex Hurwitz polynomial $P(s)$ must go through $2n$ quadrants in the counterclockwise direction as $w$ increases from $-\infty$ to $\infty$. The conditions given above correspond to the Mikhailov plot starting in the $k^{th}$ quadrant at $w = -\infty$. □

One can now convert the problem of checking interlacing to that of checking the feasibility of LPs with the knowledge of a set of $2n-1$ frequencies; the following theorem is akin to Theorem 5. Let $\alpha_n(K)\beta_n(K) \neq 0$ and let $P(s,K) = (\alpha_n(K)+j\beta_n(K))s^n + (\alpha_{n-1}(K)+j\beta_{n-1}(K))s^{n-1} + \cdots + (\alpha_0(K)+j\beta_0(K))$. Further let $P(jw,K) = P_r(w,K)+jP_i(w,K)$ with $\bar{\alpha}(K)$ and $\bar{\beta}(K)$ as the leading coefficients of $P_r(w,K)$ and $P_i(w,K)$ respectively. If $n$ is even, then $\bar{\alpha} = (-1)^{\frac{n}{2}}\alpha_n(K)$ and $\bar{\beta} = (-1)^{\frac{n}{2}}\beta_n(K)$; if $n$ is odd, then $\bar{\alpha} = (-1)^{\frac{n+1}{2}}\beta_n(K)$ and $\bar{\beta} = (-1)^{\frac{n-1}{2}}\alpha_n(K)$. If $K \in \Re^l$, following the notation used earlier for real stabilization, the polynomials $P_r(w,K)$ and $P_i(w,K)$ may be expressed in terms of some real matrices $\Delta_r, \Delta_i \in \Re^{(n+1)\times(l+1)}$ as follows:

$$\begin{aligned} P_r(w,K) &= \begin{bmatrix} 1 & w & \cdots & w^n \end{bmatrix} \Delta_r \begin{bmatrix} 1 \\ K \end{bmatrix}, \\ P_i(w,K) &= \begin{bmatrix} 1 & w & \cdots & w^n \end{bmatrix} \Delta_i \begin{bmatrix} 1 \\ K \end{bmatrix}. \end{aligned}$$

Define a Vandermonde type matrix, $\bar{V}_c$ for any set of $(2n-1)$ distinct real frequencies arranged in increasing order, $w_1, w_2, \ldots, w_{2n-1}$ as follows:

$$V_c(w_1, w_2, \cdots, w_{2n-1}) = \begin{bmatrix} 0 & 0 & 0 & \cdots & (-1)^n \\ 1 & w_1 & w_1 & \cdots & w_1^n \\ 1 & w_2 & w_2 & \cdots & w_2^n \\ 1 & \vdots & \vdots & \cdots & \vdots \\ 1 & w_{2n-1} & w_{2n-1} & \cdots & w_{2n-1}^n \end{bmatrix}.$$

An application of Hermite-Biehler theorem for the complex polynomial $P(s, K)$ yields the following theorem:

**Theorem 7.** *There exists a controller $K$ such that the complex polynomial $P(s, K)$, is Hurwitz iff there exist $2n-1$ frequencies, $w_1 < w_2 < \cdots < w_{2n-1}$, such that at least one of the following four Linear Programs is feasible:*

$$C_k V_c(w_0, w_1, \ldots, w_{2n-1}) \Delta_r \begin{bmatrix} 1 \\ K \end{bmatrix} > 0,$$

$$S_k V_c(w_0, w_1, \ldots, w_{2n-1}) \Delta_i \begin{bmatrix} 1 \\ K \end{bmatrix} > 0,$$

*for* $k = 1, 2, 3, 4.$

The set of controllers $K$ that render a complex polynomial, $P(s, K)$, Hurwitz can, therefore, be reduced to the search for the set of $2n-1$ frequencies so that at least one of the corresponding 4 LPs is feasible. If one were to index the LPs corresponding to a set of $2n-1$ frequencies with a two bit binary number, then the set of all controllers $K$ which render a complex polynomial Hurwitz is the union of the feasible sets of all LPs corresponding to $(2n-1)$ frequencies and the two bit binary numbers. In order to facilitate the search, compactify $\Re$ using the following transformation: $w = \frac{u}{1-u^2}$. The rationale is as follows: if $u = \tan\theta$, then $\tan 2\theta = 2w$; in other words, if

$\theta \in (-\frac{\pi}{4}, +\frac{\pi}{4})$, then $u \in (-1, 1)$ and $w \in (-\infty, \infty)$ and the relation is bijective. With this transformation, if one defines $R_e(u, K) = (1-u^2)^n P_e(\frac{u}{1-u^2}, K)$ and $R_o(u, K) = (1-u^2)^n P_o(\frac{u}{1-u^2}, K)$, then the polynomials $P_e(w, K)$ and $P_o(w, K)$ have exactly the same number of real roots on $\Re$ as that of the polynomials $R_e(u, K)$ and $R_o(u, K)$ respectively in the interval $(-1, 1)$; since the transformation $w = \frac{u}{1-u^2},\ |u| < 1$ is bijective, the real roots of $P_e$ and $P_o$ interlace in a specific pattern *iff* the real roots of $R_e$ and $R_o$ in the interval $(-1, 1)$ interlace in the same pattern. Using Theorem 7, there exists a controller $K$ such that the complex polynomial $P(s, K)$ is Hurwitz *iff* there exist $2n-1$ numbers, $-1 < u_1 < u_2 < \cdots < u_{2n-1} < 1$ such that one of the following four sets of linear inequalities is feasible:

$$\begin{aligned} C_k U_c(u_1, u_2, \ldots, u_{2n-1}) \Delta_r \begin{bmatrix} 1 \\ K \end{bmatrix} > 0, \\ S_k U_c(u_1, u_2, \ldots, u_{2n-1}) \Delta_i \begin{bmatrix} 1 \\ K \end{bmatrix} > 0, \end{aligned} \tag{4.1}$$

for $k = 1, 2, 3, 4$.

where $U_c(u_1, u_2, \ldots, u_{2n-1})$ is as follows:

$$U_c(u_1, u_2, \ldots, u_{2n-1}) = \begin{bmatrix} 0 & 0 & \cdots & (-1)^n \\ (1-u_1^2)^n & u_1(1-u_1^2)^{n-1} & \cdots & u_1^n \\ (1-u_2^2)^n & u_2(1-u_2^2)^{n-1} & \cdots & u_2^n \\ \vdots & \vdots & \cdots & \vdots \\ (1-u_{2n-1})^n & u_{2n-1}(1-u_{2n-1}^2)^{n-1} & \cdots & u_{2n-1}^n \end{bmatrix}.$$

As in the case of real stabilization, the procedure is to partition the interval $(-1, 1)$ using more than $(2n-1)$ points and systematically searching for the feasibility of the above set of linear inequalities (described in (4.1)). Every feasible point, $K$, makes the polynomial $P(s, K)$ Hurwitz. The union of all the feasible sets of the LPs described in (4.1) for all possible sets of $(2n-1)$ points in $(-1, 1)$ is the set of all stabilizing controllers of the given structure. Partitioning $(-1, 1)$ enables one to capture only finitely many of the possible sets of $(2n-1)$ points, $u_1, \ldots, u_{2n-1}$. The feasible sets of the LPs corresponding to these finitely many possible sets will provide an *inner approximation* of the set of all stabilizing controllers. This approximation can be improved by refining the partition - i.e., if $K$ is a stabilizing controller not in the approximate set, then there is refinement (which will separate the roots of $R_e(u, K)$ and $R_o(u, K)$) of the partition from which one can pick $2n-1$ points so that one of the four LPs corresponding to these points is feasible.

## C. Outer Approximation for Complex Polynomials

The previous section provides a procedure to construct an inner approximation of the set of controllers that achieve a certain level of performance. In this subsection, Lemma 5, Descartes' rule of signs and it's generalization due to Poincaré is used to construct an outer approximation of the stabilizing, performance achieving set of controllers.

**Lemma 9.** *Let $P(jw, K) = P_r(w, K) + jP_i(w, K)$ be a complex polynomial of order $n$, $(p_{r,n}p_{i,n} \neq 0)$. Let $\mu_1, \ldots \mu_n$ be the roots of $P_r(w, K)$ and $\xi_1, \ldots, \xi_n$ be the roots of $P_i(w, K)$. Let $\tilde{Q}(w, K, \eta) = P_r(w, K) - \eta P_i(w, K)$, where $\eta \in \Re$. Then the following statements are equivalent:*

1. *$P(jw, K)$ is Hurwitz.*
2. *All the roots of the polynomials $P_r(w, K)$ and $P_i(w, K)$ are real and interlace according to the following:*
   - *if $p_{r,n}p_{i,n} > 0$, then $-\infty < \mu_1 < \xi_1 < \mu_2 < \xi_2 < \ldots < \mu_n < \xi_n < \infty$.*

- *if* $p_{r,n}p_{i,n} < 0$, *then* $-\infty < \xi_1 < \mu_1 < \xi_2 < \mu_2 < \ldots < \xi_n < \mu_n < \infty$.

3. *The sum of positive roots of* $\tilde{Q}(w, K, \eta)$ *and* $\tilde{Q}(-w, K, \eta)$ *should be* $n$*, for any* $\eta \in \Re$ *and there exists a real* $\lambda$ *such that,*
   - *if* $p_{r,n}p_{i,n} > 0$, *then* $P_r(w - \lambda, K)$ *and* $P_i(w - \lambda, K)$ *have exactly* $n - 1$ *and* $n$ *sign changes respectively.*
   - *if* $p_{r,n}p_{i,n} < 0$, *then* $P_r(w - \lambda, K)$ *and* $P_i(w - \lambda, K)$ *have exactly* $n$ *and* $n - 1$ *sign changes respectively.*

*Proof.* The first two conditions are from the Hermite-Biehler theorem for complex polynomials. $P(jw, K)$ is Hurwitz if its Mikhailov plot goes though $2n$ quadrants in the *counterclockwise* direction as $w$ increases from $-\infty$ to $\infty$. This is ensured by the interlacing pattern of the second condition. The third condition follows from Lemma 5, i.e., for any $\eta \in \Re$, the roots of $\tilde{Q}(w, K, \eta)$ are all real. Suppose $m$ of them are positive. Then the polynomial $\tilde{Q}(-w, K, \eta)$ has $n - m$ positive roots. □

As in the case of real stabilization, Poincaré's generalization can be used to tighten the outer approximation.

**Lemma 10.** *If* $K$ *is a stabilizing, performance achieving control vector, then* $(w + 1)^{k-1}\tilde{Q}(w, K, \eta)$ *and* $(w + 1)^{k-1}\tilde{Q}(-w, K, \eta)$ *have exactly* $m$ *and* $n - m$ *sign changes in their coefficients respectively for every* $k \geq 1$ *and for all* $\eta \in \Re$.

The above lemma is used for generating the outer approximation. This is done by choosing some $\eta$ and $k$, and constructing linear programs for every choice of $m$.

## D. Examples

*Example* 7. Consider a plant $P_1(s) = \frac{1}{s^2 - 0.1s + 1}$; it is clear that there is a stabilizing controller of first order. Let us assume that the controller of the form: $C(s) = \frac{K_1}{s + K_2}$.

Furthermore, suppose we are required to guarantee a 20° phase margin.

The complex polynomial obtained from $(s^2-0.1s+1)(s+K_2)+e^{j\theta}K_1$, by using the Euler's identity ($e^{j\theta} = cos\theta + jsin\theta$), should be Hurwitz for every $\theta \in (-20°, 20°)$. The set of stabilizing controllers is found by discretizing $\theta$ and finding the simultaneous solution of the family of complex polynomials thus found.

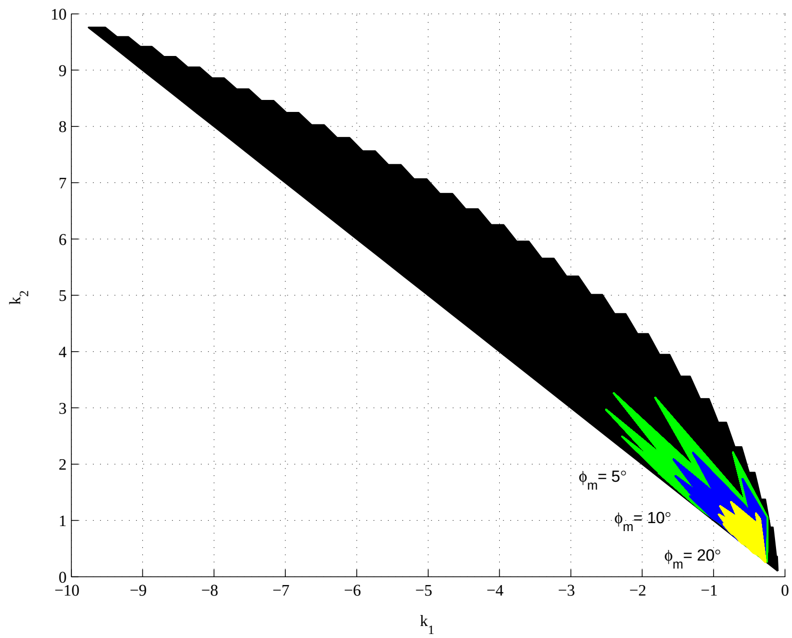



Fig. 13. Inner approximation of the set of stabilizing controllers guaranteeing a given phase margin (example 7).

The inner approximation is shown in Fig. 13. The outer region is the set stabilizing controllers with no performance specification, the regions with phase margins of 5°, 10° and 20° are shown in the figure.

*Example* 8. In this example, we find an approximation to the set of stabilizing PI controllers that ensures that the transfer function has $\mathcal{H}_\infty$ norm less than a prescribed value. The plant is chosen to be:

$$G(s) = \frac{N_p}{D_p} = \frac{s-1}{s^2+0.8s-0.2}.$$

The controller is a PI controller, i.e.,

$$C(s) = \frac{N_c}{D_c} = \frac{k_p s + k_i}{s}.$$

The constraint is to achieve a $H_\infty$ norm of the complementary sensitivity transfer function to be less than $\gamma = 1$. Consider the weight $W(s)$ as a high pass transfer function:

$$W(s) = \frac{W_n}{W_d} = \frac{s+0.1}{s+1}.$$

The given performance ($\mathcal{H}_\infty$) specification is expressed as:

$$\left\| \frac{W_n}{W_d} \frac{N_p N_c}{(D_p D_c + N_p N_c)} \right\|_\infty \leq \gamma.$$

Finding the set of controllers which satisfies this condition is equivalent to the set of controllers that simultaneously stabilizes the family of plants given by:

$$\gamma W_d (D_p D_c + N_p N_c) + e^{j\theta} W_n N_p N_c,$$

for every $\theta \in (0, 2\pi)$.

An inner approximation of the set of PI gains is shown in Fig. 14. The black region is the inner approximation of the complete set of the PI gains, and the lighter colored region shows the set of PI gains satisfying the given $\mathcal{H}_\infty$ norm specification.

*Example* 9. Consider the interval plant,

$$P(s) = \frac{P_1(s)}{P_2(s)} = \frac{a_1 s + a_0}{b_2 s^2 + b_1 s + b_0},$$

where the plant parameters vary as follows: $a_1 \in [0.1,\ 0.2]$, $a_0 \in [0.9,\ 1.1]$, $b_2 \in$

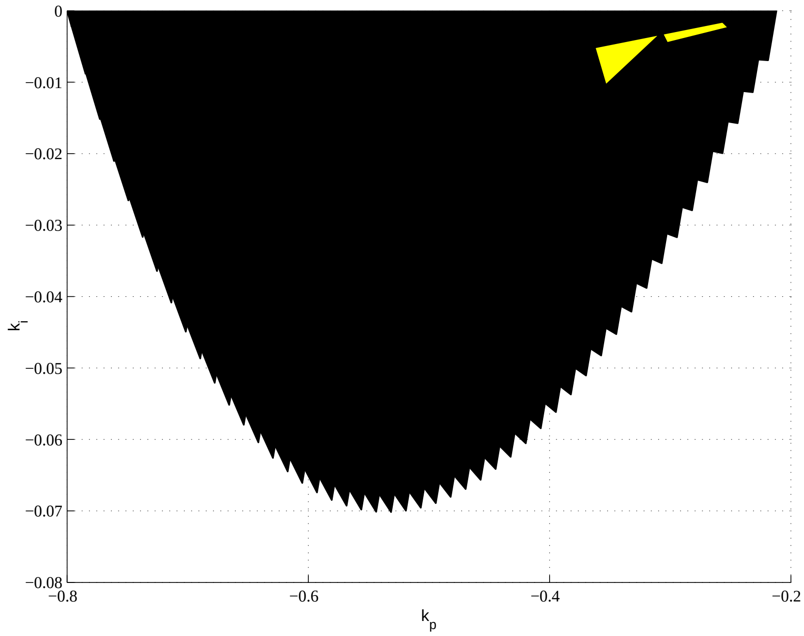



Fig. 14. Inner approximation of the set of stabilizing controllers guaranteeing a given $\mathcal{H}_\infty$ specification (example 8).

$[0.9,\ 1.0]$, $b_1 \in [1.8,\ 2.0]$ and $b_0 \in [1.9,\ 2.1]$. The controller is of the form $\dfrac{k_1 s + k_2}{s + k_3}$. The Kharitonov polynomials of the interval polynomials $P_1(s)$ and $P_2(s)$ are respectively,

$$k_1{}^1(s) = 0.9 + 0.1s, \qquad k_1{}^2(s) = 0.9 + 0.2s,$$
$$k_1{}^3(s) = 1 + 0.1s, \qquad k_1{}^4(s) = 1 + 0.2s,$$

$$k_2{}^1(s) = 1.9 + 1.8s + s^2, \qquad k_2{}^2(s) = 1.9 + 2s + s^2,$$
$$k_2{}^3(s) = 2.1 + 1.8s + s^2, \qquad k_2{}^4(s) = 2.1 + 2s + 0.9s^2.$$

It is sufficient to find the set of controllers that simultaneously stabilize the plants corresponding to the sixteen Kharitonov vertices. The inner approximation of the set of controllers is shown in Fig. 15.

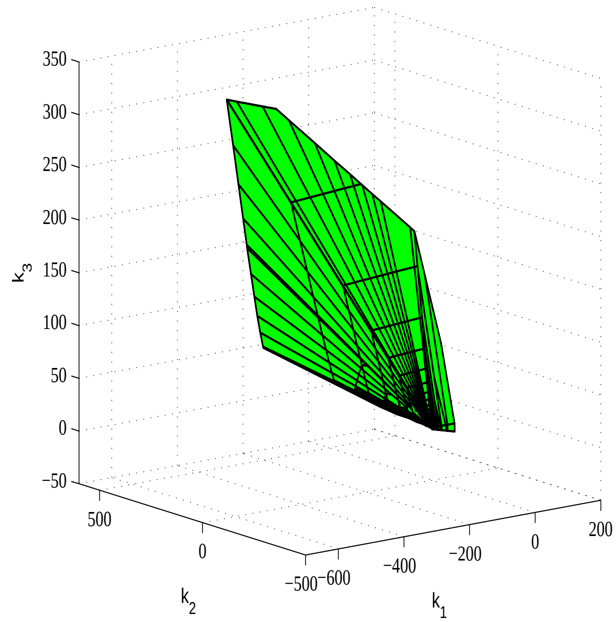



Fig. 15. Inner approximation of the set of stabilizing controllers for example 9.

## E. Summary

This chapter, provides an algorithm which systematically used the interlacing property of complex Hurwitz polynomials to construct LPs in the parameters of a fixed order controller. For complex stabilization, the feasible set of any LP constructed for an inner approximation of the set of stabilizing controllers, can be indexed by a set of $2n-1$ increasing frequencies, $-\infty = w_0 < w_1 < w_2 < \cdots < w_{2n-1} < \infty$. The problem of constructing an inner approximation of the set of stabilizing controllers for the complex polynomial was then posed as the search for all sets of ordered $2n-1$-tuples of frequencies for which the associated LP was feasible; the union of all feasible LPs provided an inner approximation for the set of all stabilizing controllers. Robust stability and performance specifications such as gain margin, phase margin and $\mathcal{H}_\infty$ performance specification, were solved by converting the performance criteria to the condition of simultaneous stabilization of a family of real or complex polynomial.

# CHAPTER V

# CONTROLLING THE TRANSIENT RESPONSE OF LINEAR TIME INVARIANT (LTI) SYSTEMS WITH FIXED STRUCTURE CONTROLLERS

The main topic of investigation of this chapter is to find a bound for the set of control parameters, $K$, so that a rational, proper transfer function, $\frac{N(s,K)}{D(s,K)}$ has a decaying, non-negative impulse response. It is assumed that the coefficients of the polynomials $N(s,K)$ and $D(s,K)$ are affine in $K$. A broad class of transient response control problems can be formulated in this way. The results of chapter III enables one to constructively find the set of controllers $K$ that render $D(s,K)$ Hurwitz as a countable union of polyhedral sets. [60] provides two necessary conditions for the transfer function $\frac{N(s,K)}{D(s,K)}$ to have a non-negative impulse response: the dominant root of $D(s,K)$ be real and no real root of $N(s,K)$ be greater than the dominant root of $D(s,K)$. Using these two results, and the Descartes' rule of signs, a procedure is outlined to construct an outer approximation (as a union of polyhedral sets) of the set of controllers $K$ so that $\frac{N(s,K)}{D(s,K)}$ has a non-negative and decaying impulse response. An extension of the developed procedure to the discrete time linear time invariant systems is also presented at the end of this chapter.

## A. Introduction

The problem of controlling the transient response is important in the design of controllers for practical applications. Despite its importance, very little is known in terms of a systematic solution technique, even for Single Input Single Output (SISO) Linear Time Invariant (LTI) systems.

The following problem involving the control of transient response of a SISO LTI system is considered:

*Problem* 1. Given a proper, rational transfer function

$$H(s,K) = \frac{N(s,K)}{D(s,K)},$$

where the coefficients of $N(s,K)$ and $D(s,K)$ are affine in the controller parameter vector $K$, determine the set of $K$'s such that the impulse response, $h(t)$, of $H(s,K)$ is non-negative.

A large class of problems involving transient response can be posed in this form:

1. Consider the problem of synthesizing a first order controller, $C(s) = \dfrac{as+b}{s+c}$, for a plant whose transfer function is

$$H_p(s) = \frac{N_p(s)}{D_p(s)}.$$

The feedback system corresponding to this controller is shown in Fig. 16.

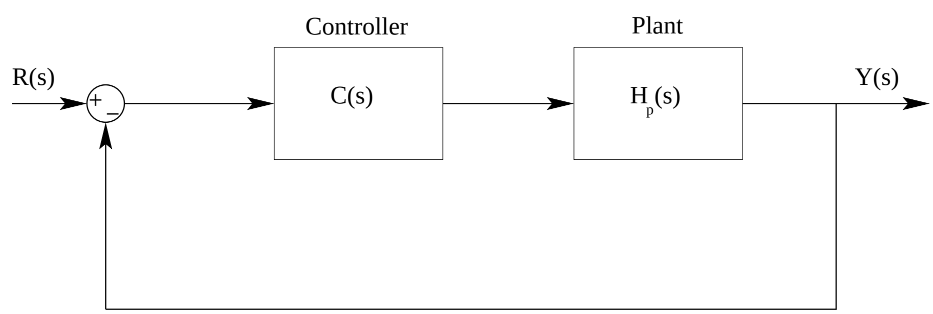



Fig. 16. Feedback control system.

Suppose $s_l(t)$ and $s_u(t)$ be the lower and upper bounds for the step response of the closed loop system and let $\frac{N_l}{D_l}(s)$ and $\frac{N_u}{D_u}(s)$ be the Laplace transforms of $s_l(t)$ and $s_u(t)$. Then this problem may be posed as requiring the impulse response of the following two transfer functions be non-negative:

$$H_1(s) = \frac{N_u D_p s^2 + a(N_u N_p s^2 - D_u N_p s) + b(N_u N_p s - D_u N_p) + c N_u D_p s}{s D_u D_p + a s^2 D_u N_p + b s D_u N_p + c D_u D_p},$$

and

$$H_2(s) = \frac{-N_l D_p s^2 - a(N_l N_p s^2 - D_l N_p s) - b(N_l N_p s - D_l N_p) - c N_l D_p s}{s D_l D_p + a s^2 D_l N_p + b s D_l N_p + c D_u D_p}.$$

The problem of synthesizing a system with a desired undershoot and overshoot can be solved by specifying $s_l(t) = -\epsilon_u$ and $s_u(t) = (1 + \epsilon_o)$, where $\epsilon_u$ and $\epsilon_o$ is the tolerable undershoot and overshoot respectively for the step response of the closed loop system.

2. The problem of synthesizing a monotonic step response is equivalent to synthesizing the impulse response of the closed loop transfer function to be non-negative. This problem arises in the synthesis of automatic vehicle following control algorithms for Automated Highway Systems(AHS) [61]. In such applications, an error propagation function, $H(s)$, indicates how the error in maintaining a safe distance from the vehicle ahead is propagated upstream in a string of automatically controlled vehicles. The coefficients of the numerator and denominator polynomial of $H(s)$ are dependent linearly on the controller gains. In some practically useful information architectures, the error propagation transfer function has a zero frequency gain ($H(0)$) of unity and the requirement is that the $\mathcal{L}_\infty$ norm of the error propagation transfer function must be less than or equal to unity. In such cases, $\|H(s)\| = 1$ if its impulse response is non-negative. In fact, if the impulse response of a transfer function $H(s)$ is non-negative, all its induced $\mathcal{L}_p$ norms equal $H(0)$.

The step response of a plant with a rational, proper stable transfer function $H(s)$ is monotonic iff its impulse response is non-negative. If the step response is monotonic, then there is no undershoot or overshoot. The problem of synthesizing a controller so that a closed loop system has non-overshooting step response was considered in [62] for discrete-time LTI systems. It is shown in [62] that there always exists a stabilizing two-parameter compensator that achieves a non-overshooting step response. However, there may not exist a controller that can eliminate the undershoot in the step response.

Some important qualitative results concerning undershoot and overshoot in the step response of a LTI system have, as their basis, Descartes' rule of signs and its generalization. A generalization of Descartes' rule of signs, due to Laguerre, is especially useful in the synthesis of non-undershooting step response for LTI systems [60, 63] and may be stated as follows [19]:

**Lemma 11.** *Let $F(s)$ be a power series ordered by decreasing powers of $s$. If it is convergent for all values of $s$ greater than a given positive number $a$ but does not converge for $s = a$, then*

1. *the number of positive values of $s$ for which $F(s)$ is convergent and takes the value zero is at most equal to the number of variations (in the sign of the coefficients) of the power series, and*
2. *if the number of values of $s$ which have that property is less than the number of variations of the series, then the difference is an even number.*

A result related to the above generalization is the following: If $H(s) = \frac{N(s)}{D(s)}$ is a proper, rational transfer function and if $N(s)$ has $r$ real, positive zeros greater than the root(s) of $H(s)$ with maximal real part, then the impulse response of $H(s)$ changes sign at least $r$ times.

Descartes' rule of signs was effectively employed in [60, 63] to constructively demonstrate that the step response of a system can be controlled to be monotonic (no undershoot and overshoot) with a two parameter compensator *iff* there are no real, non-minimum phase zeros for a continuous LTI plant (*iff* there are no real, positive non-minimum phase zeros for a discrete LTI plant). Using elementary properties of Laplace transforms, it is also shown in [64] that there exists a proper, rational two parameter compensator that achieves non-overshooting step response for any continuous LTI system which does not have a zero at the origin.

However, the schemes proposed in [62, 60, 63, 64] use compensators of sufficiently high order and it is not clear whether there exists controllers of fixed order that can accomplish the same objectives concerning the transient response. A controller of lower order is practically appealing, since a majority of industrial controllers are simple controllers such as the PID controllers. Since the primary objective of a controller is stabilization, one must first construct the set of all stabilizing fixed order controllers as this is the *basic set* in which design must be carried out. The solution for stabilizing an LTI plant with a fixed order controller was discussed in Chapter III. The set of stabilizing fixed order controllers was constructed in parameter space using Hermite-Biehler theorem and the generalization of Descartes' rule of signs due to Poincare [20]. The underlying idea is to search for all $n-1$ tuples of increasing frequencies $0 < w_1 < w_2 < \cdots < w_{n-1}$ so that the roots of the even and odd polynomials associated with the characteristic polynomial have roots that alternatively lie in the intervals $(0, w_1), (w_1, w_2), (w_2, w_3), \cdots, (w_{n-1}, w_n), (w_n, \infty)$ and hence, satisfy the interlacing requirement of Hermite-Biehler theorem. Each set of $n-1$ tuples of increasing frequencies can potentially provide two LPs in the controller coefficients(gains), with each element in the feasible set of the LPs being a stabilizing controller. Hence, one can parametrize the set of all stabilizing controllers of fixed order as the union of feasible sets of LPs associated with all such $n-1$ tuples of increasing frequencies.

In this chapter, a bound (an outer approximation) is constructed for set of $K$'s such that

1. $D(s, K)$ is Hurwitz, and

2. $H(s, K)$ has non-negative impulse response.

The following sections deal with such a construction and presentation of numerical results.

## B. Approximation of the Set of Controllers with Non-Negative Impulse Response

In this section, the necessary conditions for a transfer function $H(s)$ to have non-negative impulse response is first provided. The associated linear programs are then constructed to approximate the set the controllers with non-negative impulse response.

The following lemma is from [60]:

**Lemma 12.** *Suppose $D(s)$ is Hurwitz and suppose $H(s) = \frac{N}{D}(s)$ is a rational, strictly proper transfer function with non-negative impulse response. Then, $N(s)$ does not have real, positive zeros.*

*Proof.* If $N(s)$ has a root $\alpha > 0$, then $D(s+\alpha)$ is Hurwitz and $N(s+\alpha) = sN_1(s)$. The impulse response, $h_\alpha(t)$, of $H(s+\alpha) = \frac{sN_1}{D(s+\alpha)}$ has the same number of sign changes as the impulse response, $h(t)$, of $H(s)$. Since $H_1(s) = \frac{N_1(s)}{D(s+\alpha)}$ is stable and has relative degree two; therefore, its impulse response, $h_1(t)$, satisfies the following properties: $h_1(0) = 0$ by the initial value theorem and $\lim_{t\to\infty} h_1(t) = 0$, since $D(s+\alpha)$ is Hurwitz. Also, $h_\alpha(t) = \frac{d}{dt}h_1(t)$, since $H(s+\alpha) = sH_1(s) - h_1(0) = sH_1(s)$. Therefore, by mean value theorem, it follows that $h_\alpha(t)$ must change sign at least once and therefore, $h(t)$ must also change sign. □

Poincare's generalization of Descartes' rule of signs (see Chapter III) is used to construct a bound for the set of stabilizing controllers that ensure the impulse response $H(s, K)$ is non-negative.

The set of $K$'s such that $D(s, K)$ is Hurwitz can be computed as the union of feasible sets of linear programs as shown in Chapter III. Let $\mathcal{F}_i$ be one such feasible (polyhedral) set that may be expressed as the feasible set of the linear program $A_i K \leq b_i$.

**Lemma 13.** *Suppose $\mathcal{F}_i$ is a set of stabilizing controllers, i.e., for all $K \in \mathcal{F}_i$, the polynomial $D(s, K)$ is Hurwitz. Let $\frac{N(s,K)}{D(s,K)}$ be strictly proper. Let $2r$ be the largest positive integer less than the degree of $N(s, K)$. Let $\mathcal{G}_{l_k,k}$ be the union of the feasible sets of the linear programs constructed by requiring that the number of variations of $(s+1)^k N(s, K)$ be $2l_k$. If $K \in \mathcal{F}_i$ renders the impulse response of $\frac{N(s,K)}{D(s,K)}$ non-negative, then, for some non-increasing sequence $l_k \in \{0, 1, \ldots, r\}$ with limit zero, $K \in \mathcal{G}_{l_k,k}$ for all $k \geq 0$.*

*Proof.* Suppose the impulse response of $\frac{N(s,K)}{D(s,K)}$ is non-negative for some $K \in \mathcal{F}_i$. Then, $N(s, K)$ cannot have any real positive zeros. By Descartes' rule of signs, $N(s, K)$ can at most have even number of variations. Suppose $N(s, K)$ has $2l$ variations, where $2l$ is less than the degree of $N(s, K)$. Then, let $2l_k$ be the number of variations of $(s + 1)^k N(s, K)$. By Poincare's result, $l \geq l_1 \geq l_2 \geq \cdots \geq l_k \to 0$. Therefore, $K \in \mathcal{G}_{l_k,k}$ for all $k$. $\square$

A real root, $\lambda$, of $D(s)$ is dominant if

- $D(s)$ has roots with real parts less than or equal to $\lambda$, and
- Among the roots of $D(s)$ with a real part $\lambda$, the real root has the maximum multiplicity.

**Lemma 14.** *If $\frac{N(s,K)}{D(s,K)}$ has non-negative impulse response, then $D(s, K)$ must have a dominant real root.*

*Proof.* The proof is in [60]. In [60], it is shown that if this condition does not hold, the impulse response of $\frac{N(s,K)}{D(s,K)}$ changes sign an infinite number of times. $\square$

Since $K$ is a parameter of choice, generically, a real root of $\lambda$ is dominant if it is greater than the real part of all other roots of $D(s, K)$. For this reason, we seek an answer to the following question: What is the set of $K \in \mathcal{F}_i$ for which a real root is generically dominant? The following lemma answers this question:

**Lemma 15.** *If a real root of a Hurwitz polynomial $D(s)$ is dominant, there exists an $\alpha < 0$ such that all coefficients except the constant coefficient of $D_\alpha(s) := D(s - \alpha)$ are of the same sign and the constant coefficient is of a different sign.*

*Proof.* Let $\lambda$ be a (generically) dominant real root of $D(s)$. Consider $\alpha = \lambda - \epsilon$, where $\epsilon > 0$ and can be chosen. Clearly, when $\epsilon = 0$, all coefficients except the constant coefficient are positive and the constant coefficient is zero. When $\epsilon > 0$, there is one root in the RHP plane and it must be positive since the polynomial is real. Since the coefficients are continuous functions of the roots, for sufficiently small $\epsilon > 0$, all but the constant coefficient will be positive; since the product of roots is negative, this implies that the constant coefficient is of opposite sign. $\square$

One can now pose the problem of determining the set of controllers in $\mathcal{F}_i$ that have a dominant real pole as a search for $\alpha$ so that a linear program parametrized in $\alpha$ is feasible. The following lemma deals with this problem:

**Lemma 16.** *Let $\mathcal{F}_i$ be a set of controllers $K$ such that $D(s, K)$ is Hurwitz. If $\mathcal{L} \subset \mathcal{F}_i$ is the set of controllers such that $D(s, K)$ has a generically dominant real root, then there exists an $\alpha < 0$ such that any $K \in \mathcal{L}$ must satisfy one of the following additional set of linear constraints.*

*Let $D_\alpha(s, K) := D(s + \alpha, K) := d_0(\alpha, K) + d_1(\alpha, K)s + \cdots + d_n(\alpha, K)s^n$.*

- **LP1:**

$$d_0(\alpha, K) < 0, d_1(\alpha, K) > 0, d_2(\alpha, K) > 0, \cdots, d_n(\alpha, K) > 0.$$

- **LP2:**

$$d_0(\alpha, K) > 0, d_1(\alpha, K) < 0, d_2(\alpha, K) < 0, \cdots, d_n(\alpha, K) < 0.$$

Combining the results of Lemmas 13 and 16, one can find an outer approximation for the set of stabilizing controllers $K$ that render the impulse response of $\frac{N(s,K)}{D(s,K)}$ non-negative.

### 1. Extension to Discrete Time LTI Systems

The problem statement for the discrete time counterpart is as follows:

*Problem* 2. Given a proper, rational transfer function

$$H(z, K) = \frac{N(z, K)}{D(z, K)},$$

where the coefficients of $N(z, K)$ and $D(z, K)$ are affine in the controller parameter vector $K$, determine the set of $K$'s such that the impulse response, $h(k)$, of $H(z, K)$ is non-negative and decaying.

Through a bilinear transformation, the problem of rendering a polynomial Schur can be converted to some other polynomial being made Hurwitz. The results of [65] can be brought to bear to find an approximate set of stabilizing controllers; recently, using Chebyshev polynomials, a direct method of approximating the set of controllers, $K$, that render $D(z, K)$ Schur is presented in [66]. With this as a starting point, we can focus on how to make the impulse response of $h(k)$ non-negative.

The following counterparts of Lemmas 12 and 14 are of relevance and are proved in [63]:

**Lemma 17.** *Suppose $D(z, K)$ is Schur. If $N(z, K)$ has a real positive root on or outside the unit disk, then the impulse response $h(k)$ will change sign at least once.*

**Lemma 18.** *If $h(k)$ does not change sign, then there is a real positive root $z_0$ of $D(z, K)$ of maximum modulus. Moreover, if there are more than one root that is of maximum modulus, the multiplicity of the real positive root is maximum.*

The counterpart of Lemma 13 may be stated as follows:

**Lemma 19.** *Suppose $\mathcal{F}_i$ is a set of stabilizing controllers, i.e., for all $K \in \mathcal{F}_i$, the polynomial $D(z, K)$ is Schur. Let $\frac{N(z,K)}{D(z,K)}$ be strictly proper. Let $2r$ be the largest positive integer less than the degree of $N(z, K)$. Let $\mathcal{G}_{l_k,k}$ be the union of the feasible sets of the linear programs constructed by requiring that the number of variations of $(z+1)^k N(z, K)$ be $2l_k$. If $K \in \mathcal{F}_i$ renders the impulse response of $\frac{N(z,K)}{D(z,K)}$ non-negative, then, for some non-increasing sequence $l_k \in \{0, 1, \ldots, r\}$ with limit zero, $K \in \mathcal{G}_{l_k,k}$ for all $k \geq 0$.*

The proof follows along the same lines as Lemma 13.

To use Lemma 18, one can construct a polynomial $\tilde{D}(s, K) = (1 - s)^n D(\frac{s+1}{1-s}, K)$ and claim the following:

**Lemma 20.** *Let $D(z, K)$ be Schur and of degree $n$. If a real, positive root, $\rho$, of $D(z, K)$ is of maximum modulus then the polynomial $\tilde{D}(s, K)$ has roots with real parts less than or equal to $\frac{\rho-1}{\rho+1}$.*

*Proof.* Let $re^{j\theta}$ be a root of $D(z, K)$. The roots of $D(z, K)$ and $\tilde{D}(s, K)$ are related through $r(\cos\theta + j\sin\theta) = \frac{s+1}{1-s}$. Then $\Re(s) = \frac{r^2-1}{r^2+2r\cos\theta+1} \leq \frac{r-1}{r+1}$. Since $r \leq \rho < 1$, the real part of $s$ is a maximum, when $\theta = 0$. Moreover, for $\theta = 0$, the real part of $s$ is an increasing function of $r$. Therefore, a real positive root of $D(z, K)$ is of maximum modulus if and only if $\Re(s) \leq \frac{\rho-1}{\rho+1}$. $\square$

To get an outer approximation, one therefore uses Lemma 16, where $\mathcal{F}_i$ will correspond to a set of controllers that render $D(z, K)$ Schur.

## C. Examples

*Example* 10. Consider the plant $\dfrac{s+2}{s^2+0.8s-0.2}$ and a controller of the form $\dfrac{s+k_1}{s+k_2}$. The closed loop dynamics are given by the transfer function

$$\frac{N(s,K)}{D(s,K)} = \frac{s^2+(2+k_1)s+2k_1}{s^3+(1.8+k_2)s^2+(0.8k_2+1.8+k_1)s-0.2k_2+2k_1}.$$

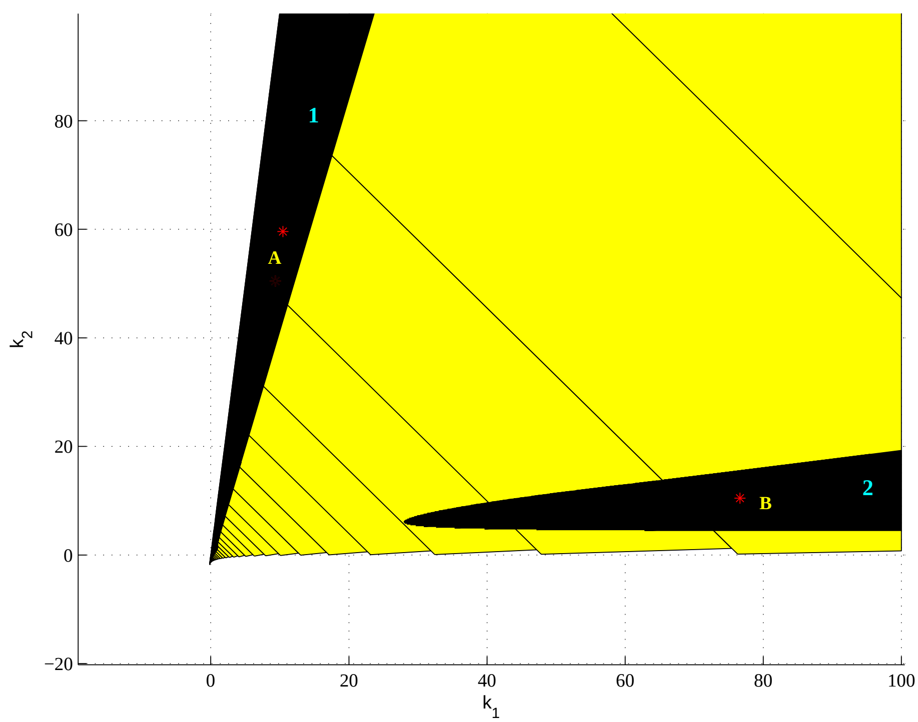



Fig. 17. Set of controllers for example 10.

Using Lemmas 12 and 13, the condition for $N(s, K)$ to have no real, positive zeros gives the **LP** $k_1 > 0$.

Applying Lemma 16, $D_\alpha(s, K) = s^3 + (1.8 - 3\alpha + k_2)s^2 + (k_1 - 3.6\alpha + 0.8k_2 + 3\alpha^2 - 2k_2\alpha + 1.8)s - 1.8\alpha + k_2\alpha^2 - k_1\alpha - \alpha^3 + 2k_1 + 1.8\alpha^2 - 0.8k_2\alpha - 0.2k_2$ and solving **LP1** and **LP2** gives us an approximation of the region where the closed loop has non-negative impulse response. The complete set of stabilizing controllers is found using the method described in Chapter III. Figure 17 shows the set of

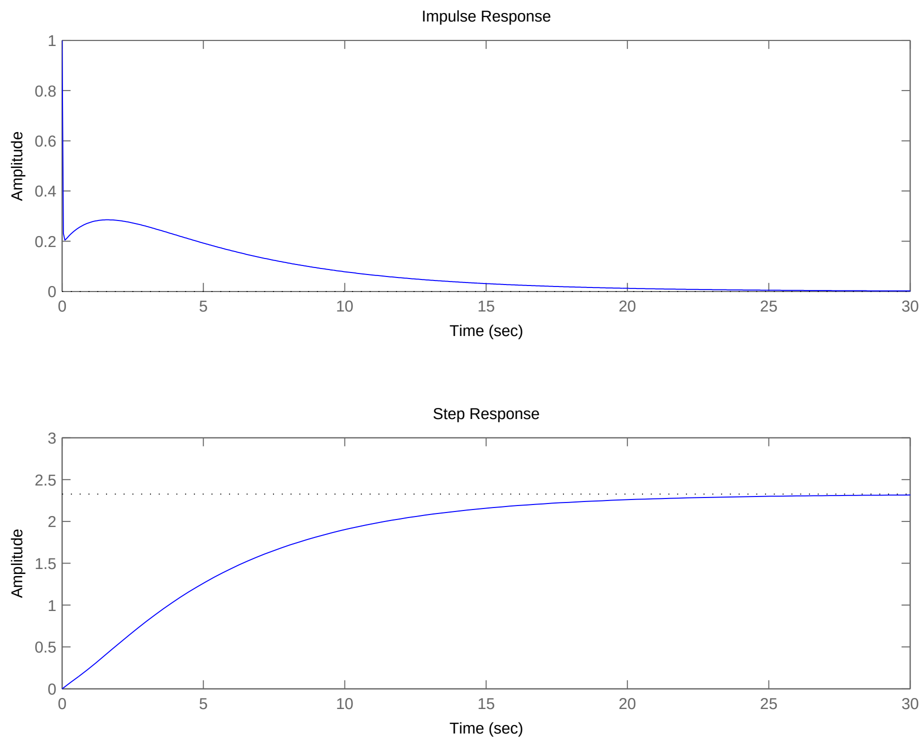



Fig. 18. Impulse and step response for point A in Fig. 17.

controllers. The light colored region is the inner approximation of the set of stabilizing controllers. The black regions shows the outer approximation for the set of controllers with non-negative impulse response. Figure 18 shows the impulse and step response for point A in region 1. Figure 19 shows the impulse and step response for point B in region 2. Corresponding to Point A, the controller is $\dfrac{s + 10.4501}{s + 59.581}$. The zeros of the close loop are at $(-10.4501, -2)$. The poles of the closed loop

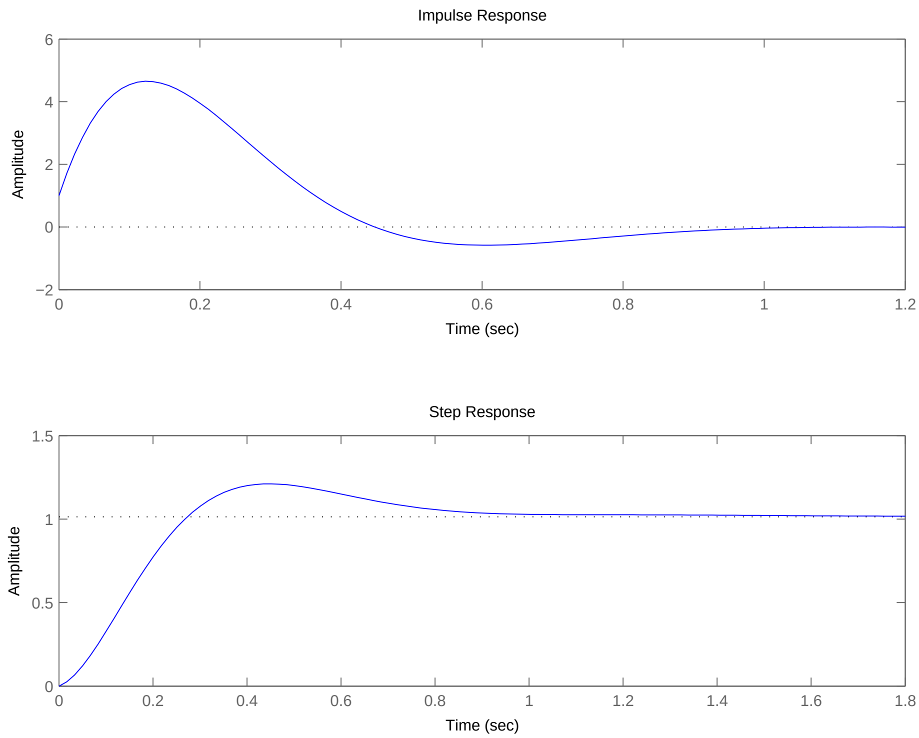



Fig. 19. Impulse and step response for point B in Fig. 17.

are at $(-60.3913, -0.804805, -0.184842)$. Corresponding to point B, the controller is $\dfrac{s + 76.6311}{s + 10.4821}$. The zeros of the close loop are at $(-76.6311, -2)$. The poles of the closed loop are at $(-4.9441 \pm 6.221i, -2.3939)$. The proximity of the closed loop dominant pole to the zero of the closed loop causes the impulse response of the system to become negative, thus causing the step response to overshoot.

## D. Summary

In this chapter, the problem of controlling the transient response is posed as one of rendering the impulse response of a rational, proper transfer function, $\frac{N(s,K)}{D(s,K)}$, to be decaying and non-negative. The coefficients of the numerator and denominator polynomials are assumed to be affine in $K$. Using earlier results about the construction of fixed order stabilizing controllers, the non-negativeness of the impulse response and the Descartes' rule of signs, a method is presented to construct an outer approximation of the set of $K$'s that corresponding to the transfer function $\frac{N(s,K)}{D(s,K)}$ having a non-negative and decaying impulse response. The procedure is also extended to discrete time linear time invariant systems.

# CHAPTER VI

## ON THE BOUNDEDNESS OF THE SET OF STABILIZING CONTROLLERS

In Chapter III, algorithms were proposed for generating approximations (inner and outer) to the set of fixed structure/stabilizing controllers for Linear Time Invariant (LTI) plants. This chapter shows that the set of rational, strictly proper robustly stabilizing controllers for Single Input Single Output (SISO) Linear Time Invariant (LTI) plants will form a bounded (can even be empty) set in the controller parameter space if and only if the order of the stabilizing controller can not be reduced any further; if the set of proper stabilizing controllers of order $r$ is not empty and the set of strictly proper controllers of order $r$ is bounded, then $r$ is the minimal order of stabilization.

This result result is further extended to characterize the set of controllers that guarantee some pre-specified performance specifications. Procedure to generate such sets were provided in Chapter IV. In particular, it is shown here that the minimal order of a controller that guarantees specified performance is $l$ iff (1) there is a controller of order $l$ guaranteeing the specified performance and (2) the set of strictly proper robustly stabilizing controllers of order $l$ and guaranteeing the performance is bounded. Moreover, if the order of the controller is increased, the set of higher order controllers which satisfies the specified performance, will necessarily be unbounded. This characterization is provided for performance specifications, such as gain margin and robust stability, which require a one-parameter family of real polynomials to be Hurwitz, where the parameter is in a closed interval. Other performance specifications, such as phase margin and $\mathcal{H}_\infty$ norm, can be reduced to the problem of determining a set of stabilizing controllers that renders a family of complex polynomials Hurwitz.

## A. Properties of the Set of Stabilizing Controllers

The following lemmas are simple observations which provide key basis for the proposed characterization of stabilizing controllers.

**Lemma 21.** *If $C_r(s) = \frac{N_r(s)}{D_r(s)}$ is a $r^{th}$ order rational, proper controller that stabilizes $P(s) = \frac{N_p(s)}{D_p(s)}$, then given any polynomials $\tilde{N}_r(s)$ and $\tilde{D}_r(s)$ of degree $r$, there is a $\tau^* > 0$ such that the $(r+1)^{st}$ order strictly proper, rational controller,*

$$C_{r+1}(s) = \frac{N_r(s) + \tau \tilde{N}_r(s)}{D_r(s) + \tau(s^{r+1} + \tilde{D}_r(s))},$$

*also stabilizes $\dfrac{N_p(s)}{D_p(s)}$ for every $0 < \tau \leq \tau^*$.*

*Proof.* Let $\Delta(s) := N_p(s)N_r(s) + D_p(s)D_r(s)$.
The characteristic polynomial, $\Delta_{pert}(s,\tau)$, associated with the perturbed controller is $\Delta(s) + \tau(s^{r+1}D_p(s) + (\tilde{N}_r(s)N_p(s) + \tilde{D}_r(s)D_p(s)))$. If $\tau$ is treated as a variable in the following root locus problem,

$$1 + \frac{1}{\tau}\frac{\Delta(s)}{s^{r+1}D_p(s) + \tilde{N}_r(s)N_p(s) + \tilde{D}_r(s)D_p(s)} = 0,$$

and if we notice that the relative degree of the rational proper transfer function in the above equation is one, it follows that there is a $\tau^* > 0$ such that for all $0 < \tau \leq \tau^*$, the polynomial, $\Delta_{pert}(s,\tau)$, is Hurwitz. □

The following are the consequences of Lemma 21:

1. If there is a $r^{th}$ order stabilizing controller, then there is a (strictly proper) stabilizing controller of order $r+1$. Therefore, there is no gap in the order of stabilization. Hence, minimal order compensators can be synthesized by recursively reducing the order of stabilizing controller by one. A scheme for recursive order reduction is provided later in this chapter.

2. One can associate a vector,

$$K = (k_0, k_1, \ldots, k_r, k_{r+1}, \ldots, k_{2r}),$$

with a rational, proper controller, $C_r(s)$, where

$$C_r(s) = \frac{k_0 + k_1 s + \ldots + k_r s^r}{k_{r+1} + k_{r+2} s + \ldots + k_{2r} s^{r-1} + s^r}.$$

Clearly, there is a one-to-one correspondence with $K \in \Re^{2r+1}$ and a rational, proper $r^{th}$ order controller $C_r(s)$. The set of all $K$, with $k_r = 0$, will constitute the parameter space of the $r^{th}$ order strictly proper stabilizing controller. Without any loss of generality, $K$ and $C_r(s)$ can be used interchangeably.

Let $\tilde{N}_r(s) = \tilde{k}_0 + \tilde{k}_1 s + \ldots + \tilde{k}_r s^r$, and $\tilde{D}_r(s) = \tilde{k}_{r+1} + \tilde{k}_{r+2} s + \ldots + \tilde{k}_{2r} s^{r-1} + \tilde{k}_{2r+1} s^r$, so that, by Lemma 21, there is a $\tau^*$ such that for all $0 < \tau \leq \tau^*$, the following $(r+1)^{st}$ order controller, $\tilde{C}_{r+1}(s)$, is also stabilizing:

$$\tilde{C}_{r+1}(s) = \frac{(k_0 + \tau\tilde{k}_0) + (k_1 + \tau\tilde{k}_1)s + \ldots + (k_r + \tau\tilde{k}_r)s^r}{(k_{r+1} + \tau\tilde{k}_{r+1}) + \ldots + (k_{2r} + \tau\tilde{k}_{2r})s^{r-1} + (1 + \tau\tilde{k}_{2r+1})s^r + \tau s^{r+1}}.$$

In order to find an associated vector, $\tilde{K} \in \Re^{2r+2}$, divide the numerator and denominator by $\tau$, so that

$$\tilde{C}_{r+1}(s) = \frac{\left(\frac{k_0}{\tau} + \tilde{k}_0\right) + \left(\frac{k_1}{\tau} + \tilde{k}_1\right)s + \ldots\ldots + \left(\frac{k_r}{\tau} + \tilde{k}_r\right)s^r}{\left(\frac{k_{r+1}}{\tau} + \tilde{k}_{r+1}\right) + \ldots + \left(\frac{k_{2r}}{\tau} + \tilde{k}_{2r}\right)s^{r-1} + \left(\frac{1}{\tau} + \tilde{k}_{2r+1}\right)s^r + s^{r+1}}.$$

From here, one can get a $\tilde{K}(\tau) \in \Re^{2r+2}$:

$$\tilde{K}(\tau) = \left(\frac{k_0}{\tau} + \tilde{k}_0, \frac{k_1}{\tau} + \tilde{k}_1, \ldots, \frac{k_r}{\tau} + \tilde{k}_r, 0, \frac{k_{r+1}}{\tau} + \tilde{k}_{r+1}, \right.$$
$$\left. \frac{k_{r+2}}{\tau} + \tilde{k}_{r+2}, \ldots, \frac{k_{2r}}{\tau} + \tilde{k}_{2r}, \frac{1}{\tau} + \tilde{k}_{2r+1}\right).$$

Define $K_0 := \tilde{K}(\tau^*), \lambda := \frac{1}{\tau} - \frac{1}{\tau^*}$, and let $K_1$ be

$$K_1 := (k_0, k_1, \ldots, k_r, 0, k_{r+1}, k_{r+2}, \ldots, k_{2r}, 1).$$

Then, one can express $\tilde{K}$ as $K_0 + \lambda K_1$ and is stabilizing for every $\lambda \geq 0$, by Lemma 21. Thus, $\tilde{K}$ is a ray originating at $K_0$ and is in the direction of $K_1$ in the space of parameters of $(r+1)^{st}$ order strictly proper stabilizing controllers. Two things can be inferred from above:

(a) If an $r^{th}$ order stabilizing compensator exists, the set of $(r+1)^{st}$ order strictly proper stabilizing controller parameters is unbounded. In particular, the set of $(r+1)^{st}$ order strictly proper stabilizing controllers contains a ray of the form $K_0 + \lambda K_1$ in $\Re^{2r+2}$ that is stabilizing for every $\lambda \geq 0$.

(b) If, by some means, one were to find a ray, $\{K_0 + \lambda K_1,\ \lambda \geq 0\}$, of strictly proper $(r+1)^{st}$ order stabilizing controllers, with $K_1$ having the $(r+2)^{nd}$ entry to be zero and the last entry to be unity, then it seems likely to recover a lower order controller from $K_1$ considering the correspondence between $K_1$ and $C(s)$.

**Lemma 22.** *If $C_r(s) = \frac{N_r(s)}{D_r(s)}$ is a strictly proper stabilizing controller of order $r$ for the plant $P(s) = \frac{N_p(s)}{D_p(s)}$ of order $n$, then there also exists a biproper stabilizing controller of order $r$ for $P(s)$.*

*Proof.* Without any loss of generality, one consider a $C_r(s)$ of relative degree one to prove this lemma. Let $\Delta(s) = N_p(s)N_r(s) + D_p(s)D_r(s)$. Consider a biproper controller to be $(\varepsilon s+1)C_r(s)$. The corresponding closed loop characteristic polynomial is $\Delta(s)+\varepsilon s N_p(s)N_r(s)$. In the standard form for the root locus, since the characteristic equation for the perturbed case can be expressed as,

$$1 + \varepsilon \frac{sN_p(s)N_r(s)}{\Delta(s)} = 0,$$

it follows that the closed loop system with the perturbed controller becomes stable by a standard root locus argument if $\varepsilon$ is sufficiently small. $\square$

Lemma 22 indicates that if ever a reduction in the order of stabilizing controller from $r$ to $r-1$ is possible there always exists a biproper stabilizing controller of order $r-1$.

**Lemma 23.** *If a stabilizing controller of order $r$ exists for the given plant of order $n(> r)$, then stabilizing controllers of order between $r$ and $n$ exist.*

*Proof.* It is easily proved by applying Lemma 21 and Lemma 22 recursively. $\square$

Lemma 21 shows that if $C(s)$ is an $r^{th}$ order stabilizing controller for $P(s)$, the set of $(r+1)^{st}$ order strictly proper stabilizing controllers is unbounded. However, the converse is in general not true. The following example serves to illustrate that point.

Consider the plant $P(s) = \frac{1}{s^2+1}$. Clearly, no static output feedback controllers exist. In fact, if $K$ is the static feedback gain, the best one can achieve is marginal stability when $K > 0$. Now consider first order strictly proper stabilizing controllers. Let $C(s) = \frac{K_1}{s+K_2}$ be a representative controller and the corresponding characteristic polynomial is:

$$s^3 + K_2 s^2 + s + K_1 + K_2.$$

From the Routh-Hurwitz criterion, the stability criterion requires that $K_2 > 0,\ K_1 + K_2 > 0$ and $K_1 < 0$. Clearly, the set of strictly proper stabilizing controllers is unbounded; however, there is no stabilizing controller of lower order.

To counter such an example, one requires that the ray of strictly proper stabilizing controllers place the closed loop poles to the left of $\Re(s) = -\epsilon$.

Let $P(s)$ denote a proper transfer function of an LTI system and let $P(s)$ be given by:

$$P(s) = \frac{a_0 + a_1 s + \ldots + a_m s^m}{s^n + b_{n-1}s^{n-1} + \ldots + b_0}.$$

The following theorem provides the conditions for the existence of a lower order controller from the unboundedness of the set of higher order controllers:

**Theorem 8.** *A proper controller of order $r$ stabilizing $P(s)$ exists iff there exists an $\epsilon > 0$ and a ray of strictly proper stabilizing controllers of order $r+1$, namely $\{K_0 + \lambda K_1,\ \lambda > 0\}$, that place the closed loop poles to the left of $\Re(s) = -\epsilon$.*

*Proof.* A controller of order $n-1$ always exists for a SISO plant of order $n$. Hence, we will assume that $r \leq n-2$.
(Necessity) Suppose an $r^{th}$ order proper controller, $C(s)$ stabilizes $P(s)$ and let $z_i$ be the roots of the closed loop polynomial. Define $\epsilon_0 := \max(\Re(z_i))$, (i.e. the maximum real part of the roots), and define $\epsilon := -\frac{\epsilon_0}{2}$. It is clear that the controller $C(s-\epsilon)$ will stabilize the plant $P(s-\epsilon)$. By Lemma 21, there exists a $\tau^*$ such that the controller $\frac{1}{\tau s+1}C(s-\epsilon)$ stabilizes $P(s-\epsilon)$ for every $\tau \in (0, \tau^*]$. Again, by shifting the imaginary axis of the complex plane, the controller, $\tilde{C}(s) = \frac{1}{\tau(s+\epsilon)+1}C(s)$ stabilizes $P(s)$ and places the poles of the closed loop to the left of $Re(s) \leq -\epsilon$.
If $C(s)$ is of the form

$$C(s) = \frac{c_0 + c_1 s + \ldots + c_r s^r}{s^r + d_{r-1}s^{r-1} + \ldots + d_0},$$

then $\frac{C(s)}{\tau s + 1 + \tau\epsilon}$ is of the form,

$$\frac{\frac{1}{\tau}(c_0 + c_1 s + \ldots + c_r s^r)}{\Delta_p(s)},$$

where

$$\Delta_p(s) := s^{r+1} + (\epsilon + d_{r-1})s^r + (\epsilon d_{r-1} + d_{r-2})s^{r-1} + \dots$$
$$\dots + (\epsilon d_1 + d_0)s + \epsilon d_0 + \frac{1}{\tau}(s^r + d_{r-1}s^{r-1} + \dots + d_0).$$

In the parameter space of $(r+1)^{st}$ order controller, it is of the form, $K_0 + \lambda K_1$, where

$$\lambda := \frac{1}{\tau} - \frac{1}{\tau^*},$$
$$K_1 = (c_0,\ c_1,\ \dots, c_r, 0,\ d_0,\ d_1,\ \dots, d_{r-1},\ 1),$$
$$K_0 = \frac{1}{\tau^*}K_1 + (\underbrace{0, \dots, 0}_{r+1\ zeros},\ \epsilon d_0,\ (\epsilon d_1 + d_0),\ \dots, (\epsilon + d_{r-1})),$$

and this ray of controllers, $\{K_0 + \lambda K_1,\ \lambda > 0\}$ stabilize the plant $P(s)$ and place the closed loop poles to the left of $\Re(s) = -\epsilon$.

(Sufficiency) Consider a ray of strictly proper controllers of order $r$ as given below:

$$C(s, \lambda) = \frac{N_c(s) + \lambda N_c^*(s)}{D_c(s) + \lambda D_c^*(s)},$$

where,

$$N_c(s) = c_0 + c_1 s + c_2 s^2 + \dots + c_{r-1}s^r$$
$$N_c^*(s) = e_0 + e_1 s + e_2 s^2 + \dots + e_{r-1}s^r$$
$$D_c(s) = d_0 + d_1 s + d_2 s^2 + \dots + d_r s^r + s^{r+1}$$
$$D_c^*(s) = f_0 + f_1 s + f_2 s^2 + \dots + f_r s^r$$

Suppose this ray of strictly proper controllers $C(s, \lambda)$ of order $r+1$ stabilize a plant $P(s)$ and place the closed loop poles to the left of $\Re(s) = -\epsilon$ for some $\epsilon > 0$. If $P(s) = \frac{N_p(s)}{D_p(s)}$, then, the closed loop characteristic polynomial for the plant $P(s)$ with a controller from the ray (identified by $\lambda$) may be written as:

$$\Delta(P(s), \lambda) = \Delta_0 (P(s)) + \lambda \Delta_1 (P(s)),$$

where $\Delta_0 = N_p(s)N_c(s) + D_p(s)D_c(s)$ and $\Delta_1 = N_p(s)N_c^*(s) + D_p(s)D_c^*(s)$. Notice that the degree of $\Delta_1(s)$ is less than that of $\Delta_0(s)$, since we are considering a ray of strictly proper controllers. Since $\Delta(P(s), \lambda)$ is Hurwitz for all $\lambda > 0$ and has roots with real part less than $-\epsilon$, from a root locus argument, it must be true that the roots of $\Delta_1(P(s))$ must lie in the open left half plane.

Since $\Delta_1(P(s))$ is Hurwitz, one of the following two cases can occur:

1. If $N_c^*(s)$ and $D_c^*(s)$ are co-prime, $\frac{N_c^*(s)}{D_c^*(s)}$ is a $r^{th}$ order proper stabilizing controller.
2. If $N_c^*(s)$ and $D_c^*(s)$ are not co-prime, then $\frac{N_c^*(s)}{D_c^*(s)}$ in its reduced form stabilizes $P(s)$.

There is a possibility that $C(s) = \frac{N_c^*(s)}{D_c^*(s)}$ may not be proper; since $C(s)$ stabilizes $P(s)$, and since $\frac{C(s)}{\tau s+1}$ stabilizes $P(s)$ for a sufficiently small $\tau > 0$, and since the degree of $N_c^*$ is no more than $r$, one can always synthesize a proper controller of order $r$ that stabilizes $P(s)$. □

*Example* 11. An arbitrarily tight outer approximation can be constructed as a union of finite number of feasible sets of linear programs (LPs) [65]. If this outer approximation is bounded, which can be checked by checking the boundedness of the polyhedral sets generated by associated LPs, then, by using the results in this manuscript, one can establish the minimal order of stabilization computationally.

Consider a plant $P_1(s) = \dfrac{1}{(s^2 - 0.1s + 1)}$. We consider the set of strictly proper output feedback stabilizing controllers of first order, i.e. controllers of the form: $C(s) = \dfrac{K_1}{s + K_2}$. Fig. 20 shows the inner and outer approximation (Chapter III). The black region is the outer approximation and the colored region is the inner approximation. Since the outer approximation is bounded, the minimal order of

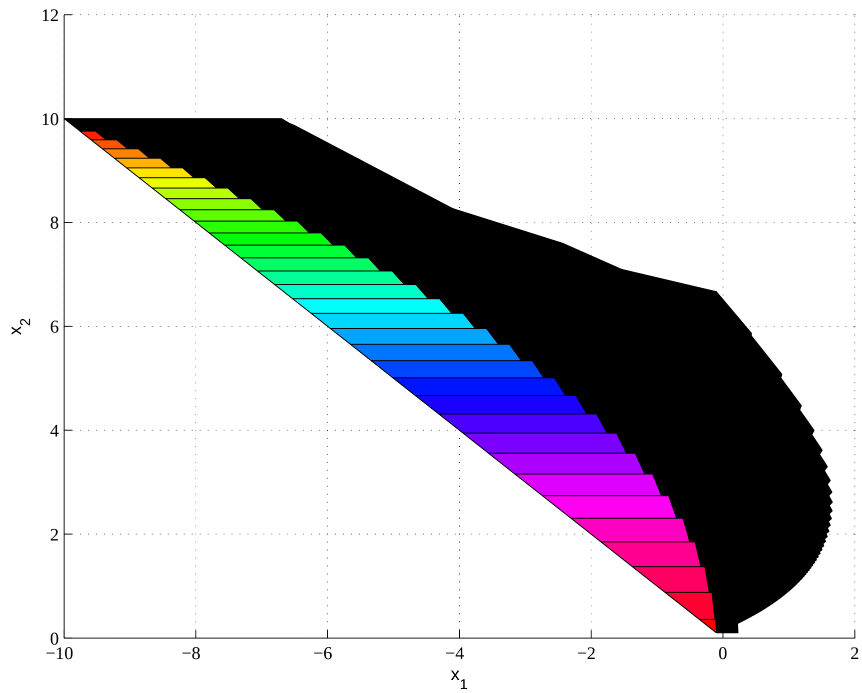



Fig. 20. Inner and outer approximation showing a bounded approximation for first order controllers.

stabilization for this plant is one.

## B. Boundedness of Strictly Proper Stabilizing Controllers of Minimal Order that Guarantee Performance.

Following the results of Chapter IV, one can refer to controllers guaranteeing performance as those controllers that render a one-parameter family of possibly complex polynomials of the form $\{D_p(s)D_c(s) + f(\alpha)N_p(s)N_c(s),\ \alpha \in \mathcal{K}\}$ to be Hurwitz for some pre-specified continuous function $f$ and a pre-specified compact interval $\mathcal{K}$. It is easy to associate the function $f$ and the compact interval $\mathcal{K}$ associated with the performance specifications discussed in Chapter IV.

The following lemmas enable the characterization of the set of controllers which satisfy these performance specifications.

**Lemma 24.** *Let $\Delta(s)$ be a complex Hurwitz polynomial of the form $\Pi_{i=1}^{n+r}(s+\alpha_i)$. Then for any real monic polynomial $D_p(s)$ of degree $n$, there exists a $\tau^* > 0$ such that the perturbed polynomial*

$$\Delta_p(s) := \tau s^{r+1} D_p(s) + \Delta(s)$$

*is Hurwitz for $0 < \tau \leq \tau^*$.*

*Proof.* One can use a regular perturbation technique for determining the locus of roots, $\alpha_i$ for small $\tau$, and use the relationship between the coefficients of a polynomial and the sum of the roots to find the locus of the $(n+r+1)^{st}$ root of $\Delta_p(s)$.
Let $-\alpha_i$ be a root of $\Delta(s)$ of multiplicity $k$. Define $\epsilon$ through $\tau = \epsilon^k$. Let $(s+\alpha_i) = \epsilon z + O(\epsilon^2)$, where $z$ is a complex number and indicates the direction of the root locus and is to be determined. In terms of $z$ and $\epsilon$, the perturbed polynomial may be expressed as:

$$\Delta_p(z,\epsilon) = \epsilon^k(-\alpha_i + \epsilon z + O(\epsilon^2))^{r+1} D_p(-\alpha_i + \epsilon z + O(\epsilon^2)) + \epsilon^k \Pi_{j:\alpha_i \neq \alpha_j}(\alpha_j - \alpha_i + O(\epsilon^2)),$$

which upon collecting terms of the smallest order (in this case $O(\epsilon^k)$) yields the following equation for the perturbation of the root $-\alpha_i$:

$$(-\alpha_i)^{r+1} D_p(-\alpha_i) + z^k \Pi_{i:j\neq i}(\alpha_j - \alpha_i).$$

It is ready then that $z$ is the $k^{th}$ (complex) root (say $z_k$) of

$$-\frac{(-\alpha_i)^{r+1} D_p(-\alpha_i)}{\Pi_{j\neq i}(\alpha_j - \alpha_i)}.$$

Therefore, for small $\tau$, the finite root of $\Delta_p(z,\epsilon)$ vary as $-\alpha_i + \tau^{\frac{1}{k}} z_k + O(\tau^{\frac{2}{k}})$.

Since the sum of the roots of $\Delta_p(s)$ is $-\dfrac{1 + d_{n-1}\tau}{\tau}$, where $d_{n-1}$ is the coefficient of $s^{n-1}$ in $D_p(s)$, it is clear that the sum of the roots goes to $-\infty$ as $\tau \to 0$. But the other $n+r$ roots are finite as we have shown through the regular perturbation technique. Therefore, one root escapes to infinity. For the asymptotic behavior of this root, it is sufficient to consider the two highest order terms of $\Delta_p(s)$. Asymptotically, the $(n+r+1)^{st}$ root satisfies:

$$\tau s^{n+r+1} + (1 + d_{n-1}\tau)s^{n+r} = 0,$$

implying that $s \approx -\frac{(1+d_{n-1}\tau)}{\tau}$. Hence, for sufficiently small $\tau > 0$, all the $(n+r+1)$ roots of $\Delta_p(s)$ will be in the open left half plane, implying that it is Hurwitz. $\square$

**Lemma 25.** *Let $C(s) = \frac{N_c(s)}{D_c(s)}$ be an $r^{th}$ order stabilizing controller for a plant $P(s) = \frac{N_p(s)}{D_p(s)}$ and guarantees performance. Then, there is a $\tau^* > 0$ such that for all $\tau \in (0, \tau^*]$ and for $\alpha \in \mathcal{K}$, the $(r+1)^{st}$ order strictly proper controller, $\tilde{C}(s) = \frac{N_c(s)}{\tau s^{r+1} + D_c(s)}$ guarantees the specified performance. Consequently, the set of $(r+1)^{st}$ order controllers guaranteeing the specified performance is unbounded in the parameter space of the controller.*

*Proof.* Let $j^{r+1}D_p(jw)$ be expressed as $\bar{D}_{pr}(w) + j\bar{D}_{pi}(w)$. Let $\Delta(s,\alpha) = D_p(s)D_c(s) + f(\alpha)N_p(s)N_c(s)$ and $\Delta_p(s,\alpha) = D_p(\tau s^{r+1} + D_c(s)) + f(\alpha)N_p(s)N_c(s)$. The specified performance is given in terms of the continuous function $f$ and the compact interval $\mathcal{K}$ and is fixed throughout the proof. Let $\Delta(s,\alpha)|_{s=jw} = \Delta_r(w,\alpha) + j\Delta_i(w,\alpha)$. Consider the values of $\tau$ for which $\Delta_p(s,\alpha)|_{s=jw} = 0$. This amounts to the determination of the values of $\tau$ that solve the following system of nonlinear equations:

$$\tau w^{r+1}\bar{D}_{pr}(w) + \Delta_r(w,\alpha) = 0,$$
$$\tau w^{r+1}\bar{D}_{pi}(w) + \Delta_i(w,\alpha) = 0.$$

Observe that 0 is not a root of the above equation as it would imply that the polynomial $\Delta(s,\alpha)$ is not Hurwitz. The values of (real) $w$ that satisfy the above equations form a subset of the roots of

$$\tilde{\Delta}(w) := \Delta_r(w,\alpha)\bar{D}_{pi}(w) - \Delta_i(w,\alpha)\bar{D}_{pr}(w).$$

However, the roots of $\tilde{\Delta}(w)$ vary continuously with $\alpha$. Consider any real root of $\tilde{\Delta}(w)$. At such a $w$, one cannot have $\bar{D}_{pr}(w) = 0$ and $\bar{D}_{pi}(w) = 0$ as it would imply that $\Delta(jw, \alpha) = 0$ and contradict the Hurwitzness of $\Delta(s, \alpha)$. Without any loss of generality, one can consider the case that $\bar{D}_{pr}(w) \neq 0$. Let $w_1, \ldots, w_k$ be the distinct real roots of $\tilde{\Delta}(w)$. The corresponding value of $\tau_k$ satisfying the system of equations is given by:

$$\tau_k = -\frac{\Delta_r(w, \alpha)}{w^{r+1}\bar{D}_{pr}(w)},$$

and hence, $\tau_k$ is a continuous function of $\alpha$ as it is a continuous function of continuous functions of $\alpha$. Define $\tau(\alpha) = \min_k \tau_k(\alpha)$ and is clearly a continuous function of $\alpha$, since $\tau_k(\alpha)$ is continuous in $\alpha$ for every $k$. By Lemma 24, $\Delta_p(s)$ is Hurwitz for every $\tau \in [0, \tau(\alpha)]$. Since $\alpha$ belongs to a compact interval, by Weierstrass's theorem, the argument, $\alpha^*$ minimizing $\tau(\alpha)$ lies in the interval $[-\phi^*, \phi^*]$. Define $\tau^* := \tau(\alpha^*)$. Since $\tau(\alpha) \geq \tau^*$, it follows that for every $\tau \in (0, \tau^*]$ and for every $\alpha \in [-\phi^*, \phi^*]$, the polynomial $\Delta_p(s, \alpha)$ is Hurwitz.

To show the unboundedness of the set of $(r+1)^{st}$ order strictly proper stabilizing controllers that guarantee performance, let $C(s)$ be of the form

$$C(s) = \frac{c_0 + c_1 s + \ldots + c_r s^r}{s^r + d_{r-1}s^{r-1} + \ldots + d_0},$$

Consider the $(r+1)^{th}$ order controller, $\hat{C}(s) = \dfrac{N_c(s)}{\tau s^{r+1} + D_c(s)}$ and can be written in the form,

$$\frac{\frac{1}{\tau}(c_0 + c_1 s + \ldots + c_r s^r)}{s^{r+1} + \frac{1}{\tau}(s^r + d_{r-1}s^{r-1} + \ldots + d_0)}.$$

Suppose, $C(s)$ stabilizes a plant $P(s) = \frac{N_p(s)}{D_p(s)}$ and satisfies the given performance criterion, then the family of polynomials

$$\Delta(s, \alpha) = D_p(s)D_c(s) + f(\alpha)N_p(s)N_c(s)$$

must be Hurwitz. Here, $f$ is some continuous function, depending on the performance specification and $\alpha$ is a compact interval.

If the higher order controller $\hat{C}(s)$ also has to satisfy the same performance criteria, then the following family of polynomials must be Hurwitz:

$$\begin{aligned}&D_p(s)(\tau s^{r+1} + D_c(s)) + f(\alpha)N_p(s)N_c(s)\\=&\tau s^{r+1}D_p(s) + D_p(s)D_c(s) + f(\alpha)N_p(s)N_c(s)\\=&\tau s^{r+1}D_p(s) + \Delta(s,\alpha)\\=&\Delta_p(s,\alpha).\end{aligned}$$

Lemma 24 states that there is a $\tau^* > 0$ such that for all $\tau \in (0, \tau^*]$ the family of polynomial $\Delta_p(s,\alpha)$ is Hurwitz.

In parameter vector form, $\hat{C}(s)$ is of the form, $K_0 + \lambda K_1$, where

$$\begin{aligned}\lambda &:= \frac{1}{\tau} - \frac{1}{\tau^*},\\K_1 &= (c_0,\ c_1,\ \ldots, c_r, 0,\ d_0,\ d_1,\ \ldots, d_{r-1},\ 1),\\K_0 &= \frac{1}{\tau^*}K_1,\end{aligned}$$

and this ray of controllers, $\{K_0 + \lambda K_1,\ \lambda > 0\}$ guarantees the specified performance criteria. $\square$

A performance guaranteeing minimal order stabilizing controller is a controller of the least order that not only stabilizes the plant but also guarantees the specified performance. Analogous to the problem of real stabilizing minimal order controllers, one can state the following:

**Theorem 9.** *A performance guaranteeing controller of order $l$ exists if and only if there is an $\epsilon > 0$ and a ray $\{K_0 + \lambda K_1,\ \lambda \geq 0\}$ of strictly proper, performance* *guaranteeing $(l+1)^{st}$ order stabilizing controllers that place the roots of*

$$D_p(s)D_c(s) + f(\alpha)N_p(s)N_c(s)$$

*to the left of $Re(s) \leq -\epsilon$ for every $\alpha \in \mathcal{K}$.*

*Proof.* The necessity proof follows along the lines of the necessity proof of Theorem 8 but uses Lemma 25 (required for complex stabilization).

Let $C(s) = \frac{N_c(s)}{D_c(s)}$ be a $l^{th}$ order performance guaranteeing controller for $P(s) = \frac{N_p(s)}{D_p(s)}$. Then, for some specified continuous $f$ and a compact interval $\mathcal{K}$, the polynomial

$$\Delta(s,\alpha) = D_p(s)D_c(s) + f(\alpha)N_p(s)N_c(s)$$

is Hurwitz for every $\alpha \in \mathcal{K}$. For every $\alpha$, let $\epsilon(\alpha) := \max_i\{Re\left(\lambda_i\left(\Delta(s,\alpha)\right)\right) < 0\}$, where $\lambda_i\left(\Delta(s,\alpha)\right)$ is the $i^{th}$ root of the characteristic polynomial. Then, $\epsilon(\alpha)$ is a continuous function of $\alpha$. Since $\alpha \in \mathcal{K}$, a compact interval, there exists an $\epsilon$ such that $2\epsilon = -\epsilon(\alpha^*) \geq \epsilon(\alpha)$ for every $\alpha \in \mathcal{K}$. It is clear that the shifted polynomial $\Delta(s-\epsilon,\alpha)$ is Hurwitz for every $\alpha \in \mathcal{K}$. In other words, the controller $C(s-\epsilon)$ guarantees the specified performance when controlling the plant $P(s-\epsilon)$. By Lemma 25, it follows that there exists a $\tau^*$ such that for all $\tau \in (0,\tau^*]$, the controller $\dfrac{N_c(s-\epsilon)}{\tau s^{l+1} + D_c(s-\epsilon)}$ guarantees the specified performance on the plant $P(s-\epsilon)$; this is equivalent to saying that the controller $\dfrac{N_c(s)}{\tau(s+\epsilon)^{l+1} + D_c(s)}$ stabilizes the plant $P(s)$ while placing the poles of the polynomial $(\tau(s+\epsilon)^{l+1} + D_c(s))D_p(s) + f(\alpha)N_p(s)N_c(s)$ to the left of $Re(s) \leq -\epsilon$. The set of controllers corresponding to $\tau \in (0,\tau^*]$, as in the discussion until now, can be expressed as a ray in the parameter space as $\{K_0 + \lambda K_1,\ \lambda \geq 0\}$, where $\lambda = \frac{1}{\tau} - \frac{1}{\tau^*}$ and $K_1 = (c_0,\ c_1,\ \ldots, c_r, 0,\ d_0,\ d_1,\ \ldots, d_{r-1},\ 1)$, $K_0 = \frac{1}{\tau^*}K_1 + (\underbrace{0,\ \ldots, 0}_{r+1 zeros}\ \epsilon^{l+1},\ (l+1)\epsilon^l,\ \ldots,\ (l+1)\epsilon)$.

The sufficiency proof follows along the lines of the sufficiency proof of Theorem 8.

Let $C(s, \lambda) = \frac{N_c(s)+\lambda N_c^*(s)}{D_c(s)+\lambda D_c^*(s)}$ be a performance guaranteeing controller and be of the same form as in the sufficiency part of Theorem 8. For some $\epsilon > 0$, since the roots of

$$(D_p(s)D_c(s) + f(\alpha)N_p(s)N_c(s)) + \lambda(D_p(s)D_c^*(s) + f(\alpha)N_p(s)N_c^*(s)$$

have real parts less than $-\epsilon$ for every $\alpha \in \mathcal{K}$ and for every $\lambda \geq 0$, it follows that

$$D_p(s)D_c^*(s) + f(\alpha)N_p(s)N_c^*(s)$$

must be Hurwitz for every $\alpha \in \mathcal{K}$. This implies that the controller $C^*(s) := \frac{N_c^*(s)}{D_c^*(s)}$ in its reduced form is a performance guaranteeing controller for $P(s)$. The degree of the numerator and denominator of $C^*(s)$ is no more than $l$. In the event that the controller $C^*(s)$ is not proper, one can resort to a reasoning along the lines of Lemma 25 to show that there is a proper performance guaranteeing controller of order $l$ for the given plant $P(s)$. $\square$

The upshot of the last theorem is that the performance guaranteeing strictly proper stabilizing controllers of minimal order are necessarily bounded. If one can construct a tight outer approximation for the set of performance guaranteeing controllers of a given order, this result can be used to check if it is of minimal order.

The unboundedness property of the set of higher order stabilizing and performance attaining controllers, as discussed above, can be used to develop and recursive procedure for reducing the order of the controller. Motivation for developing such an algorithm and its details are provided in the next section.

## C. Order Reduction

In this section, a procedure is presented for controller order reduction through the construction of an under-determined system of linear equations. The system of linear equations is obtained by canceling the poles of the closed loop system obtained by a controller of higher order and replacing it with one less pole. The free parameter in the solution of the under-determined system is then used to search for stability and performance.

In the last two decades, there have been numerous results in the field of modern control theory, [67], [68], [69], which provide a precise formulation and elegant solutions to the problem of synthesizing a controller which minimizes the generalized sensitivity of a given transfer function. Many robust stability and performance problems can be formulated as similar problems of optimization.

The order of the optimal controller obtained through these traditional techniques is almost always very high, being equal to that of the generalized plant. The generalized plant is of higher order than the original plant due to the inclusion of frequency dependent weights needed to achieve performance. The difficulty involved in implementing a high order controller for practical applications has been a deterrent to the use of these controllers. The need for low order controllers arises when simplicity, hardware limitations or reliability in the implementation of a controller dictates low order of stabilization.

There are in general three basic approaches to obtain a low order controller [70]. The first method is to directly obtain a low order controller from the given plant data. The second method is to find a simpler lower order representation of the plant which captures the essential features of the plant and then construct a possible lower order controller using the lower order representation of the plant. The third approach is to compute a controller directly from the higher order plant. This controller can be of high order. Controller order reduction schemes are then applied to synthesize a lower order controller.

The direct synthesis of low order controllers involve the problem of fixed-order stabilization. The problem of fixed-order stabilization of a Linear-Time Invariant (LTI) dynamical system and a systematic procedure for synthesizing a fixed-order controller were discussed in Chapter III.

Another approach to achieve a low order controller is to approximate the original system, and then obtain a controller based on the approximated plant. The model of the system is approximated by various existing methods (see [71]) which are all based on minimization of some error. A method based on truncating the balanced realization was proposed by Moore [72]. In many applications, the interest is in approximating the higher order plant only in a specific frequency interval. The use of weighted-frequency improves the model reduction by trying to reduce the error only over a specified frequency range [73]. Comparison of different model reduction techniques is given in [74]. The main drawback of this method is that the errors due to model approximation will cause problems in subsequent controller design synthesis [75].

The procedure of direct controller order reduction can be categorized in two parts, the open-loop and closed-loop methods. In open loop methods, it is required that the reduced controller, $C_r(s)$ be a good approximation of the original controller $C(s)$. Requiring $C_r(s)$ to be a good approximation to $C(s)$ does not guarantee the desired closed-loop performance. The controller reduction requires taking the plant dynamics into account and hence closed-loop methods are used. This is generally achieved through frequency weighting (see [70],[71]). In frequency-weighted controller reduction, the aim is to find a lower order controller $C_r(s)$ that minimizes the weighted error $||W_o(s)\,(C(s) - C_r(s))\,W_i(s)||_\infty$, where $W_i(s)$ and $W_o(s)$ are appropriate frequency weighting functions. These weights can be chosen to satisfy closed loop stability and performance [76].

The next subsection provides a procedure to recursively reduce the order of the high order controller and can be applied to high order controllers obtained through classical control synthesis techniques. Since all achievable closed loop maps are affine in the $\mathcal{Q}$ (Youla) parameter, the procedure searches for $\mathcal{Q}$ parameter of a certain form to induce a pole zero cancellation in the closed loop map and obtain a lower order controller. The initial work on this structure of controller parameter was discussed in [53] and preliminary results regarding stabilization were provided in [77]. This work differs from [53, 77] in the admission of a free parameter that can be used to search for stabilizing controller that satisfy the desired performance specifications. The simplicity of the scheme as well as the limitation of the scheme stems from the search for $\mathcal{Q}$ parameter of a certain form. A recursive order reduction procedure based on pole-zero cancellation, which guarantees a specified performance specification is provided. This procedure is *sufficient* for order reduction and may not be necessary.

## 1. A Parametrization of Stabilizing Controllers

Consider a rational, proper transfer function, $P(s) = \frac{N_p(s)}{D_p(s)}$ of order $n$, where $N_p(s)$ and $D_p(s)$ are co-prime polynomials and a rational, proper stabilizing transfer function, $C_1(s) = \frac{N_{c1}(s)}{D_{c1}(s)}$ of order $r(<= n)$, where $N_c(s)$ and $D_c(s)$ are co-prime polynomials. The problem is to find a low order controller, $C_2(s)$ which stabilizes the plant, $P(s)$ and meets some specified $\mathcal{H}_\infty$-norm performance specification.

In order to obtain a controller order reduction, we consider Youla parameters of the form

$$Q = \frac{k_m s^m + k_{m-1} s^{m-1} + \ldots + k_1 s + k_0}{q_{n-r+m}(s)}, \tag{6.1}$$

where the order of the polynomial $q_{n-r+m}(s)$ is $n - r + m$. Then, the stabilizing controller associated with $Q$ can be expressed as

$$C_s(s) = \frac{N_{c1}(s) \cdot q_{n-r+m}(s) + (k_m s^m + k_{m-1} s^{m-1} + \ldots + k_1 s + k_0) \cdot D_p(s)}{D_{c1}(s) \cdot q_{n-r+m}(s) - (k_m s^m + k_{m-1} s^{m-1} + \ldots + k_1 s + k_0) \cdot N_p(s)}. \tag{6.2}$$

The modified closed loop characteristic polynomial is given by

$$
\begin{aligned}
N_p(s)[N_{c1}(s)\cdot q_{n-r+m}(s) + (k_m s^m + k_{m-1}s^{m-1} + \ldots + k_1 s + k_0)\cdot D_p(s)] \\
+ D_p(s)[D_{c1}(s)\cdot q_{n-r+m}(s) - (k_m s^m + k_{m-1}s^{m-1} + \ldots + k_1 s + k_0)\cdot N_p(s)] \\
= \Delta(s)\cdot q_{n-r+m}(s). \quad (6.3)
\end{aligned}
$$

It is clear from the right hand side of the equation that we add $n-r+m$ poles to the closed loop system through the $Q$ parameter. For an order reduction of the controller, the polynomials $N_{c1}(s)\cdot q_{n-r+m}(s) + (k_m s^m + k_{m-1}s^{m-1} + \ldots + k_1 s + k_0)\cdot D_p(s)$ and $D_{c1}(s)\cdot q_{n-r+m}(s) - (k_m s^m + k_{m-1}s^{m-1} + \ldots + k_1 s + k_0)\cdot N_p(s)$ must have at least $n-r+m+1$ factors in common; otherwise, the resulting controller will not be of reduced order. If they have a polynomial factor, $\bar{q}_{n-r+m+1}(s)$ of order $n-r+m+1$ in common, this factor must divide $\Delta(s)\cdot q_{n-r+m}(s)$. This indicates that $n-r+m+1$ poles of the closed loop system corresponding to $C_1(s)$ have been canceled to obtain one reduced order controller. That is, at $\lambda_i$, $i = 1, ..., n-r+m+1$ with $\Delta(\lambda_i) = 0$, $i = 1, ..., n+r$, we must have

$$
\begin{aligned}
N_{c1}(s)\cdot q_{n-r+m}(s) + (k_m s^m + k_{m-1}s^{m-1} + \ldots + k_1 s + k_0)\cdot D_p(s)|_{s=\lambda_i} = 0, \\
i = 1, ..., n-r+m+1. \quad (6.4)
\end{aligned}
$$

and therefore, have the following dependent set of equations:

$$
\begin{aligned}
D_{c1}(s)\cdot q_{n-r+m}(s) - (k_m s^m + k_{m-1}s^{m-1} + \ldots + k_1 s + k_0)\cdot N_p(s)|_{s=\lambda_i} = 0, \\
i = 1, ..., n-r+m+1. \quad (6.5)
\end{aligned}
$$

The construction of $q_{n-r+m}(s)$ does not care which of the $(n-r+m+1)$ roots of $\Delta(s)$ are picked, as long as complex conjugates are chosen together. Hence, to obtain a controller order reduction by one with a proper controller, we solve (6.4) for $q_{n-r+m}(s)$

and $(k_m s^m + k_{m-1} s^{m-1} + \ldots + k_1 s + k_0)$. Without any loss of generality, $q_{n-r+m}(s)$ may be chosen to be a monic polynomial. Therefore, there are $(n - r + 2m + 1)$ unknowns and $(n-r+m+1)$ linear equations. The formulation of the problem and the procedure for the solution in terms of these variables is provided in the following sections. If a Hurwitz polynomial $q_{n-r+m}(s)$ satisfying (6.4) is found, then a stabilizing controller, whose order is reduced by one, is obtained by (6.2).

## 2. Problem Formulation

Equation (6.4) can be expressed as follows:

$$\mathbb{A} \underbrace{\begin{bmatrix} \tilde{q}_0 \\ q_1 \\ \vdots \\ \tilde{q}_{n-r+m-2} \\ \tilde{q}_{n-r+m-1} \\ k_0 \\ k_1 \\ \vdots \\ k_m \end{bmatrix}}_{\alpha} = \mathbb{B},$$

where, $\mathbb{A}$ is a numeric matrix of size $(n - r + m + 1) \times (n - r + 2m + 1)$, $\mathbb{B}$ is a numeric matrix of size $(n - r + m + 1) \times (1)$. $\tilde{q}_0, \tilde{q}_1, \cdots, \tilde{q}_{n-r+m-1}$ are the coefficients of the monic polynomial $q_{n-r+m}(s)$. The variable parameters in this procedure are given by the vector $\alpha = [\ \tilde{q}_0 \quad \tilde{q}_1 \quad \cdots \quad \tilde{q}_{n-r+m-1} \quad k_0 \quad \cdots \quad k_m\ ]'$. An appropriate solution to $\alpha$, which makes $q_{n-r+m}(s)$ Hurwitz, will yield the desired low order controller.

Let the desired solution be

$$\alpha = \alpha^{\dagger} + \lambda \alpha_i^{\mathcal{N}},$$

where $\alpha^{\dagger}$ is the minimum norm solution and $\alpha^{\mathcal{N}}$ is the null space of the above system of equations. $\alpha_i^{\mathcal{N}}$ represents $i^{th}$ vector in a basis of the null space. Hence, the solution can be represented in terms of one parameter, $\lambda$. The span of the null space can be controlled by choosing $m$, i.e. it depends on the form of the $\mathcal{Q}$ parametrization.

The closed loop controller (one order lower) can be expressed in the form :

$$C_s(s) = \frac{\tilde{N}_c(s)}{\tilde{D}_c(s)} = \frac{N_{c0}(s) + \lambda N_{c1}(s)}{D_{c0}(s) + \lambda D_{c1}(s)}. \tag{6.6}$$

This equation is obtained by substituting $\alpha$ into (6.2) and removing the $(n-r+m+1)$ roots of $\triangle_1$, which we picked earlier, from both the numerator and the denominator.

At this stage, the search for a lower order *stabilizing* controller reduces to a root locus problem in $\lambda$. Over the range of stabilizing values of $\lambda$, one can then compute the prescribed *performance specifications* and determine if the specifications have been met. In case, either a stabilizing set has not been found or the performance specifications have not been met, one may try a different vector in the null space. While there is no guarantee that the procedure will find a stabilizing controller of lower order in the first place, the procedure seemed to find lower order controller whenever possible in the numerical examples.

a. Lower Order Stabilizing Controller

The closed loop characteristic equation, for the reduced order controller, is given by

$$\begin{aligned}\tilde{\Delta}(s,\lambda) &= \tilde{N}_c(s)N_p(s) + \tilde{D}_c(s)D_p(s) \\ &= \Delta_0(s) + \lambda\Delta_1(s).\end{aligned} \tag{6.7}$$

*Problem* 3. Find $\lambda$ such that $\tilde{\Delta}(s,\lambda)$ is Hurwitz.

*Procedure.* The controller can be reduced to a one order lower stabilizing controller if there exists a Hurwitz $q_{n-r+m}(s)$. Hence, one needs to find $\lambda$ such that the monic polynomial $q_{n-r+m}(s)$ is Hurwitz. This problem reduces to a root locus problem for the range of $\lambda$ such that the following polynomial is Hurwitz.

$$s^{n-r+m} + \tilde{q}^{\dagger}_{n-r+m-1} s^{n-r+m-1} + \ldots + \tilde{q}^{\dagger}_0 + \lambda \left[ \tilde{q}^{\mathcal{N}}_{n-r+m-1} s^{n-r+m-1} + \ldots + \tilde{q}^{\mathcal{N}}_0 \right].$$

b. Lower Order Controller Satisfying Given Performance Specification

Various performance specification which can be expressed as a complex stabilization problem (see Chapter IV) can be considered. For example, consider the performance specification to be a desired upper bound on the $\mathcal{H}_\infty$ norm of a weighted sensitivity transfer function. The given performance ($\mathcal{H}_\infty$) specification is expressed as:

$$\left\| \frac{N_w(s)}{D_w(s)} \frac{N_p(s)(N_{c0}(s) + \lambda N_{c1}(s))}{(D_p(s)(D_{c0}(s) + \lambda D_{c1}(s)) + N_p(s)(N_{c0}(s) + \lambda N_{c1}(s)))} \right\|_\infty \leq \gamma.$$

This can be expressed as,

$$\left\| \frac{N_0(s) + \lambda N_1(s)}{D_0(s) + \lambda D_1(s)} \right\|_\infty \leq \gamma.$$

The above $\mathcal{H}_\infty$ specification can be expressed in the form of a complex stabilization problem, i.e.

$$P(s,\lambda,\theta) = \gamma(D_0(s) + \lambda D_1(s)) + e^{j\theta}(N_0(s) + \lambda N_1(s))$$

should be Hurwitz $\forall \theta \in [0, 2\pi]$. Using Euler's formula ($e^{j\theta} = cos\theta + jsin\theta$), this can be converted into a problem of simultaneous stabilization of family of complex polynomials, $P(s,\lambda,\theta)$.

*Problem* 4. Find $\lambda$ such that the family of complex polynomials $P(s, \lambda, \theta)$ is Hurwitz $\forall \theta \in [0, 2\pi]$.

*Procedure.* The solution presented is based on determining those values of $\lambda$ for which the complex polynomial $P(s, \lambda, \theta)$ has a root on the imaginary axis. For a fixed $\theta$, these values of $\lambda$ partition the $\lambda$-axis into root invariant regions. One can then find those invariant regions for which the polynomial $P(s, \lambda, \theta)$ is Hurwitz. By intersection of such regions, as $\theta$ varies in its domain, one obtains the desired ranges of $\lambda$ for which the family of complex polynomials is Hurwitz. This range of values, when non-empty, provides the desired performance. Procedurally, the steps involved are:

- Discretize $\theta$. For each $\theta_j$, let $P(s, \lambda, \theta_j)$ be the corresponding complex polynomial.

- To find root invariant regions, one has to calculate $\lambda$, such that a root of the complex polynomial $P(s, \lambda, \theta_j)$ lies on the imaginary axis. Let $s = jw$ be a root. Then,

  - $P(jw, \lambda, \theta_j) = P_r(w, \lambda, \theta_j) + jP_i(w, \lambda, \theta_j) = 0$
  - The above equation reduces to

$$P_r(w, \lambda, \theta_j) = P_{r0}(w, \theta_j) + \lambda P_{r1}(w, \theta_j) = 0 \tag{6.8}$$

$$P_i(w, \lambda, \theta_j) = P_{i0}(w, \theta_j) + \lambda P_{i1}(w, \theta_j) = 0 \tag{6.9}$$

  - Eliminating $\lambda$ from (6.8) and (6.9) gives a polynomial,

$$P_{r0}(w, \theta_j)P_{i1}(w, \theta_j) - P_{i0}(w, \theta_j)P_{r1}(w, \theta_j) = 0 \tag{6.10}$$

  - Find the real roots, $w^*$, of the polynomial (6.10).

  - For each $w^*$, find the corresponding value of $\lambda$ from (6.8) or (6.9). This provides the separation of the parameter $\lambda$ into root invariant regions. Pick one value of $\lambda$ from each of the root invariant regions and find the root distribution of the polynomial $P(s, \lambda, \theta_j)$. Let $\lambda_j$ denote the range of $\lambda$'s for which the polynomial $P(s, \lambda, \theta_j)$ is Hurwitz.

- The intersection of stabilizing ranges ($\lambda_j$'s) for all polynomials (different values of $\theta_j$'s) in the family of polynomial provide a range of values for $\lambda$, which can be used to obtain a one order lower controller which satisfies a pre-specified performance criterion.

### D. Examples

*Example* 12. Consider a fourth order plant,

$$P(s) = \frac{s^2 + 3s + 2}{s^4 - 10s^3 + 35s^2 - 50s + 24}.$$

The initial controller is

$$C(s) = \frac{1000s^3 + 13000s^2 + 54000s + 72000}{s^3 + 42s^2 + 395s + 1050}.$$

The weighting function is considered to be $W = 1$. The $\mathcal{H}_\infty$ norm of the complementary sensitivity function is 3.2704. The aim is to recursively find reduced order controllers with $\mathcal{H}_\infty \leq 2$. This particular controller is chosen to illustrate the point that the initial closed loop system can have an $\mathcal{H}_\infty$ norm greater than the desired value.

Recursive order reduction results using the two procedures in the previous section are presented.

**Without performance criterion:** Choosing $m = 1$, the form of the Youla Parameter is

$$Q(s) = \frac{k_1 s + k_0}{s^2 + q_1 s + q_0}.$$

The closed loop poles are $-6.123 \pm j24.195$, $-6.691 \pm j2.834$, $-0.592 \pm j0.800481$ and $-5.187$. We choose to remove $-6.123 \pm j24.195$, $-5.187$. The range of $\lambda$ is found to be $-2365.9523 \longrightarrow \infty$.

In the procedure given above, the choice of only one free parameter $\lambda$ provides an advantage. Over the range of stabilizing values of $\lambda$, one can compute the prescribed *performance specifications* and determine if the specifications have been met. A plot showing the $\mathcal{H}_\infty$ norm of the complementary sensitivity function and the phase margin for different values of $\lambda$ is shown in Fig. 21. This provides one with a tool to choose a value of $\lambda$ which allows multiple performance specifications to be met.

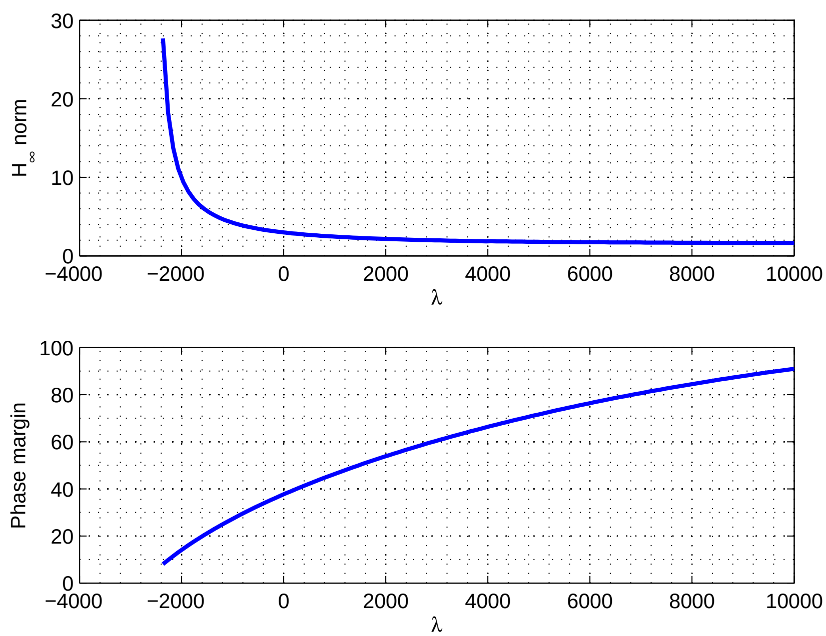



Fig. 21. $\mathcal{H}_\infty$ norm of the complementary sensitivity function and the phase margin for different values of $\lambda$.

Since in this case there is no performance specifications, $\lambda = 2000$ is chosen. The reduced controller obtained is :

$$C_1(s) = \frac{1450s^2 + 10290s + 18160}{s^2 + 49.74s + 263.5}.$$

The $\mathcal{H}_\infty$ norm of reduced system is 2.1536.

The poles of the closed loop of the reduced system are $-12.588 \pm j25.613$, $-6.691 \pm j2.833$ and $-0.592 \pm j0.801$. Choosing $m = 1$, the form of the Youla Parameter is $Q(s) = \frac{k_1 s + k_0}{s^3 + q_2 s^2 q_1 s + q_0}$. Choosing to remove $-12.588 \pm j25.613$, $-6.691 \pm j2.833$, the range of $\lambda$ is $-\infty \longrightarrow -8765.0404$. Picking $\lambda = -9000$, the reduced order controller obtained is,

$$C_2(s) = \frac{618.3s + 1525}{s + 22.66}.$$

The $\mathcal{H}_\infty$ norm of reduced system is 12.9481 .

No further reduction of the controller occurs. Failure to calculate does not guarantee that we have achieved the minimal order controller. However for this system, the minimal order of stabilization is indeed first order.

**With performance criterion:** The form of the Youla Parameter considered is the same as before. $\theta$ is discretized at intervals of $45°$. The roots which are sought to be removed are $-6.123 \pm j24.195,\ -5.187$.

For $\theta = 0\,\mathrm{deg}$, the value of $\lambda$ where one root of the polynomial $P(s, \lambda, \theta = 0)$ is on the imaginary axis is found to be $-7340.372$.

The number of roots are checked for two intervals of $\lambda$, $[-\infty,\ -7340.372]$ and $[-7340.372,\ \infty]$. It is found that for $\lambda \in (-7340.372,\ \infty)$, all roots of $P(jw, \lambda)$ are in the Left Half Plane. For other values of $\theta$, the ranges of $\lambda's$ are given below.

$$\begin{array}{ll|ll}
45^\circ & \Rightarrow (-954.659,\ \infty) & 90^\circ & \Rightarrow (1432.454,\ \infty) \\
135^\circ & \Rightarrow (2842.117,\ \infty) & 180^\circ & \Rightarrow (-7368.890,\ \infty) \\
225^\circ & \Rightarrow (2842.117,\ \infty) & 270^\circ & \Rightarrow (1432.454,\ \infty) \\
315^\circ & \Rightarrow (-954.659,\ \infty) & &
\end{array}$$

Choosing $\lambda = 10000$, a reduced order controller which provides a system with $\mathcal{H}_\infty$ norm of 1.6365 is obtained. The reduced order controller is given by,

$$C_1(s) = \frac{3013s^2 + 19500s + 33750}{s^2 + 94.03s + 485.2}.$$

The same procedure is repeated to recursively obtain a lower order controller $C_2(s)$ which yields a system with $\mathcal{H}_\infty$ norm of 1.6049. The controller is given by

$$C_2(s) = \frac{3054s + 3013}{s + 89.23}.$$

*Example* 13. In this example, the order reduction algorithm is applied to the popular four-disk system. This system was first studied by [73] and comparison of various controller order reduction techniques applied to this problem have been done in [70], [78]. The problem is to control the angle of a disk that is mounted with three other disks on a shaft with torsion flexibility. The actuation is on the third disk and the angle of concern is the angle of the first disk. The disks have unit rotational inertia, and the springs have unit torsional stiffness. The system has one rigid-body mode. The three vibration modes are assumed to be lightly damped.

The system (plant) to be controlled is represented as linear, time-invariant, single input and single output, unstable, non-minimum phase and of eighth order. The minimal realization of $A, B, C$ is given by:

$$A = \begin{bmatrix} -0.161 & -6.004 & -0.58215 & -9.9835 & -0.40727 & -3.982 & 0.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix},$$

$$B' = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.0 & 0.0 & 0.0064432 & 0.0023196 & 0.071252 & 1.0002 & 0.10455 & 0.99551 \end{bmatrix}.$$

The transfer function of this plant is:

$$P(s) = \frac{0.0064432s^5 + 0.0023196s^4 + 0.071252s^3 + 1.0002s^2 + 0.10455\,s + 0.99551}{s^8 + 0.161s^7 + 6.004s^6 + 0.5822s^5 + 9.983s^4 + 0.4073s^3 + 3.982s^2}.$$

The initial controller is:

$$C_8(s) = \frac{0.191s^7 + 0.039s^6 + 1.1475s^5 + 0.1603s^4 + 1.913s^3 + 0.1596s^2 + 0.768s + 0.0327}{s^8 + 1.298s^7 + 6.824s^6 + 7.235s^5 + 13.91s^4 + 10.29s^3 + 9.59s^2 + 3.351s + 1.382}.$$

The $\mathcal{H}_\infty$ norm of the complementary transfer function with this controller is 1.2683. The proposed algorithm is applied to the higher order controller $C_8(s)$ and the aim is to obtain the lowest possible order controller with $\mathcal{H}_\infty$ norm less than 1.5.

The following provides a result of the algorithm developed in this chapter

$$C_7(s) = \frac{.001(2.304s^7 + .527s^6 + 13.72s^5 + 2.331s^4 + 22.74s^3 + 2.317s^2 + 9.068\,s + .00857)}{s^7 + 1.233s^6 + 3.056s^5 + 2.493s^4 + 2.486s^3 + 0.9420s^2 + 0.3795\,s + 0.05178}$$

$$\Downarrow$$

$$C_6(s) = \frac{0.2943s^6 + 0.2047s^5 + 0.7812s^4 + 0.4902s^3 + 0.3791s^2 + 0.2104\,s + 0.008968}{s^6 + 1.241s^5 + 3.104s^4 + 2.529s^3 + 2.451s^2 + 0.9070\,s + 0.2938}$$

$$\Downarrow$$

$$C_5(s) = \frac{0.3030s^5 + 2.829s^4 + 0.3622s^3 + 1.658s^2 + 0.05818\,s + 0.002284}{s^5 + 1.615s^4 + 1.687s^3 + 0.9683s^2 + 0.3614\,s - 0.2888}$$

$$\Downarrow$$

$$C_4(s) = \frac{-0.09514s^4 + 0.2188s^3 - 0.03941s^2 + 0.1309\,s + 0.005588}{s^4 + 1.177s^3 + 1.399s^2 + 0.5578\,s + 0.2499}$$

$$\Downarrow$$

$$C_3(s) = \frac{0.02110s^3 + 0.001600s^2 + 0.01338\,s + 0.0002998}{s^3 + 0.4461s^2 + 0.2646\,s + 0.04818}$$

$$\Downarrow$$

$$C_2(s) = \frac{0.01580s^2 + 0.1426\,s + 0.006308}{s^2 + 0.4597\,s + 0.2365}$$

$$\Downarrow$$

$$C_1(s) = \frac{0.03304\,s + 0.000003404}{s + 0.1764}$$

The $\mathcal{H}_\infty$ norm of the closed loop with $C_1(s)$ is calculated to be 1.004.

## E. Summary

In this chapter, the structure of the set of minimal order stabilizing and performance attaining controllers for continuous time LTI plants in the controller parameter space is studied. The minimal order of a controller that guarantee specified performance is $l$ if and only if (1) there is a controller of order $l$ guaranteeing the specified performance and (2) the set of strictly proper stabilizing controllers guaranteeing the performance is bounded. Moreover, if the order of the controller is increased, the set of higher order controllers which satisfies the specified performance, will necessarily be unbounded. These characterization are provided for performance specifications, such as gain margin and robust stability, which can be posed as the simultaneous stabilization of a one-parameter family of plants. Other performance specifications, such as phase margin and $\mathcal{H}_\infty$ norm, is reduced to the problem of determining a set of stabilizing controllers that renders a family of complex polynomials Hurwitz. The characterization of the set of controllers for the stabilization of complex polynomials is provided and is used to show the boundedness properties for the set of controllers that guarantee a given phase margin or an upper bound on the $\mathcal{H}_\infty$ norm. Also, if the set of proper stabilizing controllers of order $r$ is not empty and the set of strictly proper robustly stabilizing controllers of order $r$ is bounded *iff* $r$ is the minimal order of stabilization for the plant.

A procedure was presented for controller order reduction through the construction of an under-determined system of linear equations. The system of linear equations was obtained by canceling the poles of the closed loop system obtained by a controller of higher order and replacing it with one less pole. The free parameter in the solution of the under-determined system was then used to search for stability and performance.

# CHAPTER VII

# SYNTHESIS OF FIXED ORDER STABILIZING CONTROLLERS USING FREQUENCY RESPONSE MEASUREMENTS

## A. Introduction

The synthesis of fixed order/structure controllers for LTI plants is an important open problem with a wide variety of practical applications [21, 59]. Given an analytical model of the plant, a procedure to synthesize an approximation to the set of fixed order/structure controllers was provided in Chapter III. It is also widely recognized that an accurate analytical model of the plant may not be available to a control designer. However, it is reasonable in many applications that one will have an empirical model of the plant in terms of its frequency response data and from physical considerations or from the empirical time response data, one may have some coarse information about the plant such as the number of non-minimum phase zeros of the plant. In view of this, the problem of synthesizing sets of stabilizing controllers directly from the empirical data and such coarse information about the plant is considered.

There are many techniques for synthesizing controllers from empirical data of the plant; for example, the most notable are the PID controller design using Ziegler-Nichols criteria [9], the rule-of-thumb designs for lead lag compensation [10] and loop-shaping. A systematic attempt to synthesize PID and first order controllers for delay-free Single Input Single Output (SISO) LTI plants using frequency response measurements was first presented in [11].

By way of notation, the transfer function of the plant is denoted by $H_p(s)$. The following are the standing assumptions about the plant:

1. The transfer function $H_p(s)$ of the plant is rational and strictly proper, i.e.,

$H_p(s) = \frac{N_p}{D_p}(s)$, for some co-prime polynomials, $N_p(s)$ and $D_p(s)$, with the degree, $n$, of $D_p(s)$ greater than the degree $m$ of $N_p(s)$. It is not required to know either $m$ or $n$.

2. There are no poles and zeros of the plant on the imaginary axis, i.e., $D_p(jw) \neq 0$, $N_p(jw) \neq 0$ for every $w \in \Re$.

3. There is a frequency $w_b$ beyond which the phase of the plant does not change appreciably and the amplitude response of the plant is negligible. To quantify this statement, let $H_p(jw)$ be expressed as $H_r(w) + jwH_i(w)$, where $H_r$ and $H_i$ are real, rational functions of $w$. For some known $\alpha > 0, \epsilon > 0$, assume that $|H_p(jw)| \leq \epsilon$ and $|\frac{H_i}{H_r}(w)| \leq \alpha$ for all $w \geq w_b$. This is a reasonable assumption since the plant is strictly proper.

4. The relative degree $n - m$ is known. This can be inferred from the amplitude response of the plant at sufficiently high frequencies.

5. It is assumed that the functions $|H_p(jw)|^2$, $H_r(w)$, $H_i(w)$ have been approximated using polynomials $P_0(w)$, $P_1(w)$, $P_2(w)$ respectively and the maximum estimation errors are bounded by $\mu_0$, $\mu_1$, $\mu_2$ and the maximum derivatives of the estimation errors are bounded by $\eta_0$, $\eta_1$, $\eta_2$ respectively. Mathematically, for all $w \in [0, w_b]$,

$$\begin{aligned}
\big||H_p(jw)|^2 - P_0(w)\big| &\leq \mu_0, \\
|H_r(w) - P_1(w)| &\leq \mu_1, \\
|H_i(w) - P_2(w)| &\leq \mu_2, \\
\left|\frac{d(|H_p(jw)|^2 - P_0(w))}{dw}\right| &\leq \eta_0, \\
\left|\frac{d(H_r(w) - P_1(w))}{dw}\right| &\leq \eta_1,
\end{aligned}$$

$$\left| \frac{d(H_i(w) - P_2(w))}{dw} \right| \leq \eta_2.$$

It is assumed that $\mu_i,\ \eta_i,\ i = 0, 1, 2$ and the polynomials $P_0(w),\ P_1(w),\ P_2(w)$ are known.

6. The number of non-minimum phase zeros, $u$ of the plant are known. This can be found, in some cases, from frequency response data.

This chapter deals with synthesizing a rational, proper stabilizing controller $C(s)$, i.e., for some monic polynomial $D_c(s)$ of degree $r$ and a polynomial $N_c(s)$ of degree at most $r$, $C(s) = \frac{N_c}{D_c}(s)$. Let $N_c(s) = n_0 + n_1 s + \ldots + n_r s^r$ and $D_c(s) = d_0 + d_1 s + \ldots + d_{r-1} s^{r-1} + s^r$. Let $K$ be the vector of controller coefficients:

$$K = \begin{bmatrix} n_0 & n_1 & \ldots & n_r & d_0 & d_1 & \ldots & d_{r-1} \end{bmatrix}^T$$

. The determination of the vector $K$ is equivalent to the determination of the stabilizing controller $C(s)$.

The basic ideas used in the construction of stabilizing sets are as follows:

- Construct a rational function

$$\delta(s) = H_p(s)H_p(-s)N_c(s) + H_p(-s)D_c(s).$$

  In fact, if $\Delta(s) := N_p(s)N_c(s) + D_p(s)D_c(s)$ is the characteristic polynomial of the closed loop system, then it is easy to see

$$\delta(s) = \Delta(s)\frac{N_p(-s)}{D_p(s)D_p(-s)}.$$

  If $\Delta(s)$ has coefficients that are affine in the controller coefficients, then the rational function, $\delta(s)$, is also affine in the controller coefficients.

- All controllers, $C(s)$, that stabilize $H_p(s)$, are such that the total phase accumulation of $\delta(jw)$ as $w$ varies from 0 to $\infty$ is the same and equals $(n-m+r+2u)\frac{\pi}{2}$. Since $n-m$, $r$ and $u$ are known, the total desired accumulation of phase is known.

- Let $\delta(jw) = \delta_r(w) + jw\delta_i(w)$, where $\delta_r(w)$ and $\delta_i(w)$ are real, rational functions. Lemma 1 provides an expression for how the total accumulation of phase is related to the roots of $\delta_i(w)$ and the sign of $\delta_r(w)$ at those roots.

  Essentially, the numerator of $\delta(s)$ must have a certain number of roots with negative real parts. This can happen only if the Nyquist plot of $\delta(s)$ is one of finitely many patterns, where each pattern is identified with the signs of the real part of the Nyquist plot when the imaginary part is zero. The set of such patterns can be characterized using the generalized phase formula developed in [30, 79].

- The existence of a stabilizing controller for the plant can be expressed in terms of the existence of an appropriate set of frequency intervals which admit exactly one or zero roots of the imaginary part of the Nyquist plot and no roots of the real part. This is shown in Theorem 10. For every set of frequency intervals, these conditions can be translated into linear inequality constraints or linear matrix inequality (LMI) constraints involving the controller parameters. This step involves the Chebyshev approximation of the frequency response in the frequency band $[0, w_b]$. It subsequently involves the use of Markov-Lucaks theorem [80] to convert the conditions into an LMI form. Lemmas 27, 28 and Theorem 11 deal with the synthesis of stabilizing controllers using LMIs.

## B. Inner Approximation

Following the outline of the main ideas of the chapter presented in the earlier section, this section begins with a generalization of Hermite-Biehler theorem for rational functions in Lemma 26.

**Lemma 26.** *Consider* $\delta(s) = \frac{\Delta(s)N_p(-s)}{D_p(s)D_p(-s)}$. *Let the positive roots of* $\delta_i(w)$ *be* $w_1 < w_2 < \ldots < w_l$ *and let* $w_0 = 0$ *and* $w_{l+1} = \infty$. *Let the sign of* $\delta_r(w)$ *at these frequencies be correspondingly* $i_0, \ldots, i_{l+1}$. *Then* $\Delta(s)$ *is Hurwitz if and only if* $n - m + r + 2u =$

$$= \begin{cases} sgn[\delta_i(0)]\{i_0 - 2i_1 + \ldots + 2(-1)^l i_l + (-1)^{l+1} i_{l+1}\} & \text{when } n-m+r \text{ is even,} \\ sgn[\delta_i(0)]\{i_0 - 2i_1 + 2i_2 + \ldots + 2(-1)^l i_l\} & \text{when } n-m+r \text{ is odd.} \end{cases} \tag{7.1}$$

*Proof.* Note that the degree of the polynomial $\Delta(s)N_p(-s)$ is $n + r + m$. Hence, the parity of the degree of the polynomial $\Delta(s)N_p(-s)$ is the same as that of $n - m + r$.

Let the sign of $\frac{d\delta_i(w)}{dw}$ at $w = w_k$ be $I_k$. The change in the phase of $\delta(jw)$ from $w_l$ to $w_{l+1}$ is given by: $I_l(i_l - i_{l+1})\frac{\pi}{2}$. Since $I_{l+1} = -I_l$, the phase change in $\delta(jw)$ from $w = w_0$ to $w = w_l$ can be expressed as:

$$I_0((i_0 - i_1) - (i_1 - i_2) + (i_2 - i_3) + \ldots + (-1)^{l-1}(i_{l-1} - i_l))\frac{\pi}{2}.$$

The phase change in $\delta(jw)$ from $w = w_l$ to $\infty$ will depend on the degree of the polynomial $\Delta(s)N_p(-s)$; if the degree is odd, it will be $I_l\frac{\pi}{2}i_l$, and if the degree is even, it will be $I_l(i_l - i_{l+1})\frac{\pi}{2}$. Since $I_0 = sign(\delta_i(0))$ and $I_l = (-1)^l I_0$, the change in the phase of $\delta(jw)$ as $w$ changes from 0 to $\infty$ is:

$$\begin{aligned} sgn[\delta_i(0)]\{i_0 - 2i_1 + \ldots + 2(-1)^l i_l + (-1)^{l+1} i_{l+1}\}\frac{\pi}{2} \qquad & \text{when } n-m+r \text{ is even,} \\ sgn[\delta_i(0)]\{i_0 - 2i_1 + 2i_2 + \ldots + 2(-1)^l i_l\}\frac{\pi}{2} \qquad & \text{when } n-m+r \text{ is odd.} \end{aligned}$$

Since $D_p(s)$ does not have any zeros on the imaginary axis, the phase change in $\delta(jw)$ as $w$ changes from 0 to $\infty$ is the same as that of $\Delta(jw)N_p(-jw)$ as $w$ changes from 0 to $\infty$. The accumulation or change of phase of $\Delta(jw)N_p(-jw)$ is $(n - m + r + 2u)\frac{\pi}{2}$ if and only if $\Delta(s)$ is Hurwitz. With this observation $(n - m + r + 2u)$ equals the quantity expressed in equation (7.1). □

The following theorem will use Lemma 26 to characterize a stabilizing controller of a fixed order in terms of frequency response of the plant.

**Theorem 10.** *A controller $C(s)$ stabilizes the plant if and only if*

- *There exists a sequence $i_1, \ldots, i_l$ satisfying equation (7.1), and*
- *For the sequence of integers $i_1, \ldots, i_l$, there exists correspondingly $l$ disjoint frequency bands, $[w_{p,1}, w_{p,2}]$, $p = 1, \ldots, l$ such that*
    1. *there exists exactly one root of $\delta_i(w)$ in $(w_{p,1}, w_{p,2})$,*
    2. *the sign of $\delta_r(w)$ in $[w_{p,1}, w_{p,2}]$ is the same as that of $i_p$, and*
    3. *there is no sign change of $\delta_i(w)$ in the disjoint intervals $[0, w_{1,1}]$, $[w_{l,2}, \infty]$ and $[w_{p,2}, w_{p+1,1}]$, $p = 1, \ldots, l-1$.*

*Proof.* Let the root of $\delta_i(w)$ in $(w_{p,1}, w_{p,2})$ be $w_p$. Since the sign of $\delta_r(w)$ at $w_p$ is $i_p$, the change in phase of $\delta(jw)$ as $w$ varies from 0 to $\infty$ is $(n-m+r+2u)\frac{\pi}{2}$, indicating that $\Delta(s)N_p(-s)$ has $m-u$ roots with positive real part. However, this is the case if and only if $\Delta(s)$ is Hurwitz. □

*Remark* 10. 1. $\delta(s)$ may be expressed as $\delta_0(s) + \sum_{p=1}^{2r+1} \delta_p(s)k_p$, where $k_p$ is the $p^{th}$ component of the controller vector, $K$, and $\delta_0, \delta_1, \ldots, \delta_{2r+1}$ are rational functions, which can be determined once the expression for $H_p(s)$ is known. Similarly, $\delta_r$ and $\delta_i$ are affinely dependent on the controller parameter vector, $K$. To emphasize the dependence on $K$, the notation $\delta_r(w, K)$ and $\delta_i(w, K)$ is used as appropriate. One may express the affine dependence of $\delta_r(w, K)$ and $\delta_i(w, K)$ as:

$$\delta_r(w, K) = \Delta_r(w, |H_p(jw)|^2, H_r(w), H_i(w)) \begin{bmatrix} K \\ 1 \end{bmatrix},$$

$$\delta_i(w,K) = \Delta_i(w, |H_p(jw)|^2, H_r(w), H_i(w)) \begin{bmatrix} K \\ 1 \end{bmatrix}.$$

for some vectors $\Delta_r$ and $\Delta_i$ that depend affinely on $|H_p(jw)|^2$, $H_r(w)$ and $H_i(w)$.

2. The conditions in Theorem 10 may be replaced as follows:

   (a) The first condition may be replaced by: $\delta_i(w_{p_1}, K)\delta_i(w_{p,2}, K) < 0$ and $\dfrac{d\delta_i(w,K)}{dw}$ has the sign $I_0(-1)^p$ in $[w_{p,1}, w_{p,2}]$. This ensures that $\delta_i(w,K)$ has exactly one root in the interval of interest. If the frequency response at frequencies, $w_{p,1}$ and $w_{p,2}$ are known, then the first condition,

   $$\delta_i(w_{p,1}, K)\delta_i(w_{p,2}, K) < 0,$$

   can be written as two sets of linear inequalities.

   (b) The second and third conditions may similarly be replaced as:

   $$i_p\delta_r(w) \geq 0, \ \forall w \in [w_{p,1}, w_{p,2}],$$
   $$I_0\delta_i(w) \geq 0, \ \forall w \in [0, w_{1,1}],$$
   $$(-1)^l I_0\delta_i(w) \geq 0, \ \forall w \in [w_{l,2}, \infty),$$
   $$(-1)^q I_0\delta_i(w) \geq 0, \ \forall w \in [w_{q,2}, w_{q+1,1}],$$

   where $p = 1, 2, \ldots, l$, $q = 1, \ldots, l-1$ and dependence on $K$ is suppressed.

If $H_p(jw)$ is exactly known, the condition that $I_0(-1)^p \frac{d\delta_i(w,K)}{dw}$ be non-negative in $[w_{p,1}, w_{p,2}]$ can be posed as a robust Semi-Definite Program (SDP) using Markov-Lucaks theorem as is done in [81] or in [82]. If $H_p(jw)$ is approximately known as is typically the case when fitting a rational function approximation to the given data contaminated with noise, the non-negativity condition can be posed as a robust SDP.

In the pursuit of posing non-negativity conditions of the polynomial approximations of rational functions, Lemma 27 is required. To prepare for lemma 27, let $\tilde{P}_0 := |H_p(jw)|^2 - P_0(w)$, $\tilde{P}_1 := H_r(w) - P_1(w)$, $\tilde{P}_2 := H_i(w) - P_2(w)$ and let $\tilde{Q}_i := \frac{d\tilde{P}_i}{dw}$, $i = 0, 1, 2$. Let $B_\mu$ be the box, $|\tilde{P}_i| \leq \mu_i$, $i = 0, 1, 2$ and $B_\eta$ be the box, $|\tilde{Q}_i| \leq \eta_i$, $i = 0, 1, 2$. Define $w_{0,1} = w_{0,2} = 0$ and $w_{l+1,1} = \infty$.

The following lemma provides a sufficient condition for checking the nonnegativity of a rational function through its polynomial approximation and the approximation error bounds. Let,

$$\Delta_r^*(w, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) := \Delta_r(w, P_0(w) + \mu_{0,e}, P_1(w) + \mu_{1,e}, P_2(w) + \mu_{2,e}),$$

where $\mu_{i,e}$, $i = 0, 1, 2$ are the vertices of the box $B_\mu$.

**Lemma 27.** *Let $[w_{low}, w_{high}] \subset [0, w_b]$. Let $K$ be such that for all vertices of the box $B_\mu$,*

$$\Delta_r^*(w, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} \geq 0, \quad \forall w \in [w_{low}, w_{high}]. \tag{7.2}$$

*Then, $K$ satisfies*

$$\delta_r(w, K) = \Delta_r(w, |H_p(jw)|^2, H_r(w), H_i(w)) \begin{bmatrix} K \\ 1 \end{bmatrix} \geq 0$$

*for all $w \in [w_{low}, w_{high}]$.*

*Proof.* The proof is by contraposition.

Suppose $\delta_r(\bar{w}, K) < 0$ for some $\bar{w} \in [w_{low}, w_{high}]$.

Set $\tilde{\mu}_i = \tilde{P}_i(\bar{w})$, $i = 0, 1, 2$. Therefore,

$$\Delta_r^*(\bar{w}, \tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2) \begin{bmatrix} K \\ 1 \end{bmatrix} < 0.$$

Since $|\tilde{\mu}_i| \leq \mu_i$, and since $\Delta_r$ depends affinely on $\tilde{\mu}_i$, $i = 0, 1, 2$, it must be that at some vertex $(\mu_{0,e}, \mu_{1,e}, \mu_{2,e})$ of the box $|\tilde{P}_i| \leq \mu_i$,

$$\Delta_r^*(\bar{w}, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} < 0.$$

□

*Remark* 11. The *polynomial* condition given by equation (7.2) is a sufficient condition for the the rational function $\delta_r(w, K)$ to be non-negative on the interval $[w_{low}, w_{high}]$ for the given value of $K$. In particular, the set of $K$'s that satisfy the polynomial condition at every vertex of the box also render the rational function $\delta_r(w, K)$ to be non-negative on $[w_{low}, w_{high}]$. The set of $K$'s satisfying the polynomial condition at a vertex of the box can be written as a SDP; for example, one may use the recent formulation of [81] or that of [33]. Since there are only eight vertices for the box $|\tilde{P}_i| \leq \mu_0$, this means that the set of $K$'s that simultaneously satisfy eight SDP's (which can be cast as a bigger SDP) also renders the rational function $\delta_r(w, K)$ to be non-negative on $[w_{low}, w_{high}]$.

Similar conditions can be derived for the non-negativity of rational functions $\delta_i(w, K)$, $\dfrac{d\delta_i(w, K)}{dw}$.

The following lemma deals with the non negativity of $\delta_i(w, K)$ on $[w_b, \infty)$, where the polynomial approximation does not hold.

**Lemma 28.** *Let* $B_\epsilon := \{(\epsilon_0, \epsilon_1, \epsilon_2) : 0 \leq \epsilon_0 < \epsilon^2, \ \max\{|\epsilon_1|, |\epsilon_2|\} < \epsilon\}$. *Let* $(\epsilon_{0,e}, \epsilon_{1,e}, \epsilon_{2,e})$, $e = 1, \ldots, 8$ *be the vertices of the box* $B_\epsilon$. *If, for some* $K$ *and* $l$ *and for* $e = 1, \ldots, 8$,

$$(-1)^l \Delta_i(w, \epsilon_{0,e}, \epsilon_{1,e}, \epsilon_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} \geq 0, \ \ \forall w \in [w_b, \infty),$$

*then*

$$(-1)^l \Delta_i(w, |H_p(jw)|^2, H_r(w), H_i(w)) \begin{bmatrix} K \\ 1 \end{bmatrix} \geq 0, \quad \forall w \in [w_b, \infty).$$

The proof for this lemma is similar to that of Lemma 27 and for reasons of space, it is omitted.

The following lemma is required before stating the main result. Let

$$\Delta_i^*(w_{p,1}, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) = \\ \Delta_i(w_{p,1}, P_0(w_{p,1}) + \mu_{0,e}, P_1(w_{p,1}) + \mu_{1,e}, P_2(w_{p,1}) + \mu_{2,e})).$$

**Lemma 29.** *Let* $[w_{p,1}, w_{p,2}] \subset [0, w_b]$. *Let* $I_0 \in \{-1, +1\}$. *If, for some* $K$, *and for all* $e = 1, \ldots, 8$,

$$I_0(-1)^{p-1} \Delta_i^*(w_{p,1}, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} \geq 0,$$

$$I_0(-1)^{p} \Delta_i^*(w_{p,2}, \mu_{0,e}, \mu_{1,e}, \mu_{2,e}) \begin{bmatrix} K \\ 1 \end{bmatrix} \geq 0$$

*then*

$$I_0(-1)^{p-1} \delta_i(w_{p,1}) > 0, \quad I_0(-1)^{p} \delta_i(w_{p,2}) > 0.$$

*Proof.* The proof is by contraposition. Suppose $I_0(-1)^{p-1}\delta_i(w_{p,1}, K) < 0$. Let $\tilde{\mu}_i := \tilde{P}_i(w_{p,1})$. Then,

$$I_0(-1)^{p-1} \Delta_i^*(w_{p,1}, \tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2) \begin{bmatrix} K \\ 1 \end{bmatrix} > 0.$$

However, this cannot happen unless at some vertex $e$,

$$I_0(-1)^{p-1}\Delta_i^*(w_{p,1},\mu_{0,e},\mu_{1,e},\mu_{2,e})\begin{bmatrix} K \\ 1 \end{bmatrix} > 0.$$

A similar reasoning can be applied to the second condition in the lemma to complete the proof. □

*Remark* 12. If $I_0, p, w_{p,1}, w_{p,2}$ are known in the above lemma, the sufficient conditions are linear inequalities in $K$. In particular, every $K$ that satisfies the system of linear inequalities at the vertices of the box $B_\mu$, also satisfies the linear inequalities $(-1)^{p-1}I_0\delta_i(w_{p,1},K) > 0$ and $(-1)^p I_0\delta_i(w_{p,2},K) > 0$. Note that $\delta_i(w,K)$ is not known exactly, but only, a polynomial approximation of $\delta_i(w,K)$ is available for every $K$.

Since the condition in Remark 10(2a) requires the non-negativity of $\frac{d\delta_i}{dw}$, $\frac{d\delta_i}{dw}$ is expressed as:

$$\Delta_{d,i}(w,|H_p(jw)|^2,H_r(w),H_i(w),\frac{d|H_p(jw)|^2}{dw},\frac{dH_r}{dw},\frac{dH_i}{dw})\begin{bmatrix} K \\ 1 \end{bmatrix},$$

for some array $\Delta_{d,i}$ that is polynomial in $w$ and is dependent affinely on $|H_p(jw)|^2$, $H_r(w)$, $H_i(w)$ and their derivatives.

The following is the main result and provides a sufficient condition for the direct synthesis of sets of stabilizing controllers from the frequency response data:

**Theorem 11.** *Let $i_1,\ldots,i_l$ be a sequence of integers from the set $\{-1,1\}$ satisfying equation (7.1). Let $\mu = (\mu_{0,e},\mu_{1,e},\mu_{2,e})$, $e = 1,\ldots,8$ be the vertices of box $B_\mu$ and $\eta = (\eta_{0,f},\eta_{1,f},\eta_{2,f})$, $f = 1,\ldots,8$ be the vertices of the box $B_\eta$. Let $K$ satisfy every constraint in the following set of constraints for $I_0 = -1$ or for $I_0 = +1$ and for every*

*$e = 1, \ldots, 8$ and $f = 1, \ldots, 8$:*

$$I_0(-1)^{p-1}\Delta_i^*(w_{p,1}, \mu)\begin{bmatrix} K \\ 1 \end{bmatrix} > 0, \qquad p = 1, \ldots, l, \tag{7.3}$$

$$I_0(-1)^{p}\Delta_i^*(w_{p,2}, \mu)\begin{bmatrix} K \\ 1 \end{bmatrix} > 0, \qquad p = 1, \ldots, l, \tag{7.4}$$

$$I_0(-1)^{p}\Delta_{d,i}^*(w, \mu, \eta)\begin{bmatrix} K \\ 1 \end{bmatrix} > 0, \qquad \forall w \in [w_{p,1}, w_{p,2}],\ p = 1, \ldots, l, \tag{7.5}$$

$$I_0(-1)^{p}\Delta_i^*(w, \mu)\begin{bmatrix} K \\ 1 \end{bmatrix} > 0, \qquad \forall w \in [w_{p,2}, w_{p+1,1}],\ p = 0, 1, \ldots, l, \tag{7.6}$$

$$i_p\Delta_r^*(w, \mu)\begin{bmatrix} K \\ 1 \end{bmatrix} > 0, \qquad \forall w \in [w_{p,1}, w_{p,2}],\ p = 1, \ldots, l, \tag{7.7}$$

$$(-1)^l\Delta_i(w, \epsilon_{0,e}, \epsilon_{1,e}, \epsilon_{2,e})\begin{bmatrix} K \\ 1 \end{bmatrix} > 0, \qquad \forall w \in [w_b, \infty). \tag{7.8}$$

*Then, $K$ is a stabilizing controller for the plant.*

This theorem covers all the cases discussed in this section and provides a sufficient condition for the synthesis of sets of stabilizing controllers.

- Equations (7.3) and (7.4) together ensure that $\delta_i(w_{p,1}, K)\delta_i(w_{p,2}, K) < 0$. This follows from Lemma 29.

- Equation (7.5) guarantees that $\frac{d\delta_i(w,K)}{dw}$ has the sign $I_0(-1)^p$ in $[w_{p,1}, w_{p,2}]$. This is an application of Lemma 27 to $\frac{d\delta_i(w,K)}{dw}$.

- Equations (7.3), (7.4) and (7.5) provide the condition for $\delta_i(w, K)$ to have only one real root in the interval $[w_{p,1}, w_{p,2}]$.

- Equations (7.6) and (7.8) provides the condition for the real roots of $\delta_i(w, K)$ to not lie outside the intervals $[w_{p,1}, w_{p,2}]$. This is necessary for the correct application of Lemma 26. This condition is satisfied by ensuring that the polynomial is either positive or negative in the complete range of $[w_{p,2}, w_{p+1,1}]$.
- Equation (7.7) ensures that at the real roots of $\delta_i(w, K)$, the sign of $\delta_r(w, K)$ is correct and is given by the sequence of integers satisfying equation (7.1).

The next section provides an example using some of the conditions discussed in this section.

## C. Numerical Example

Consider a plant:

$$P(s) = \frac{s^4 + 4s^3 + 23s^2 + 46s - 12}{s^5 + s^4 + 20s^3 + 36s^2 + 99s + 100}.$$

The frequency response measurements of this plant are shown in Fig. 22. In the simulations it is assumed that the plant structure is not known. Frequency response measurements are gathered from this 'unknown' plant. From the amplitude response at high frequencies, it is determined that $n - m = 1$. For frequencies greater than $w_b = 10$, it is observed that the phase of the plant response does not change appreciably and the amplitude response of the plant is negligible. Frequency response information of the plant at 30 discrete frequency points, corresponding to the Chebyshev nodes, are considered for the synthesis of the stabilizing controllers.

The aim is to find a first order controller which stabilizes the closed loop. The controller is given by:

$$C(s) = \frac{k_1 s + k_2}{s + k_3}.$$

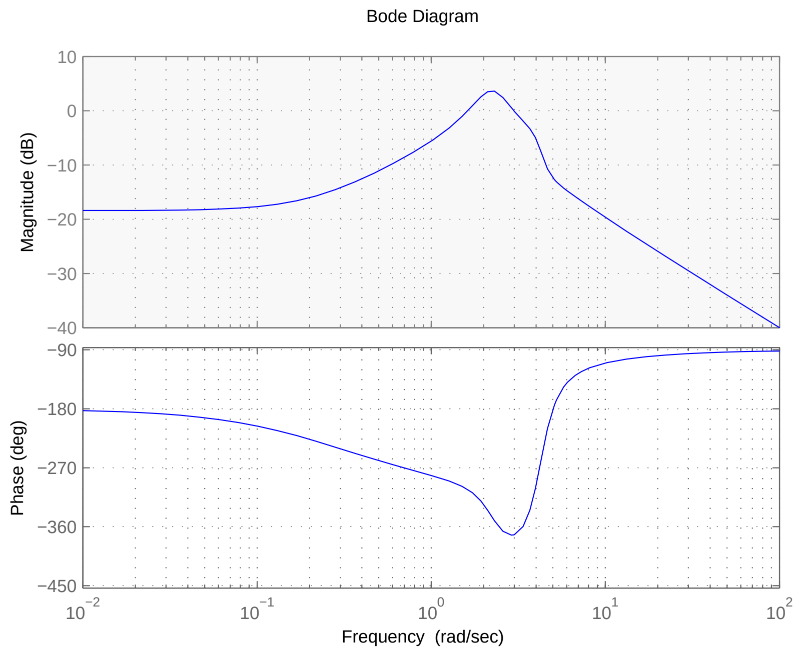



Fig. 22. Frequency response measurements of $P(s)$.

For this first order controller, the functions $\delta_r(w)$ and $\delta_i(w)$ can be expressed as:

$$\delta_r(w,K) = \begin{bmatrix} wH_i(w) & 0 & |H_p(jw)|^2 & H_r(w) \end{bmatrix} \begin{bmatrix} K \\ 1 \end{bmatrix},$$

$$\delta_i(w,K) = \begin{bmatrix} wH_r(w) & w|H_p(jw)|^2 & 0 & -H_i(w) \end{bmatrix} \begin{bmatrix} K \\ 1 \end{bmatrix}.$$

Using the 30 frequency data points, Chebyshev polynomial approximations are constructed for $H_r(w)$, $wH_i(w)$, $|H_p(jw)|^2$ for $\delta_r(w,K)$ and $wH_r(w)$, $w|H(jw)|^2$, $H_i(w)$ for $\delta_i(w,K)$. The maximum measurements errors are bounded by $\mu_1 = \mu_2 = \mu_3 = 0.02$. These approximations are shown in Fig. 23.

The Chebyshev polynomials are orthogonal and thus allows the polynomials formed by the arrays $\Delta_i^*, \Delta_r^*$, and $\Delta_{d,i}^*$ to be put in the form presented in section

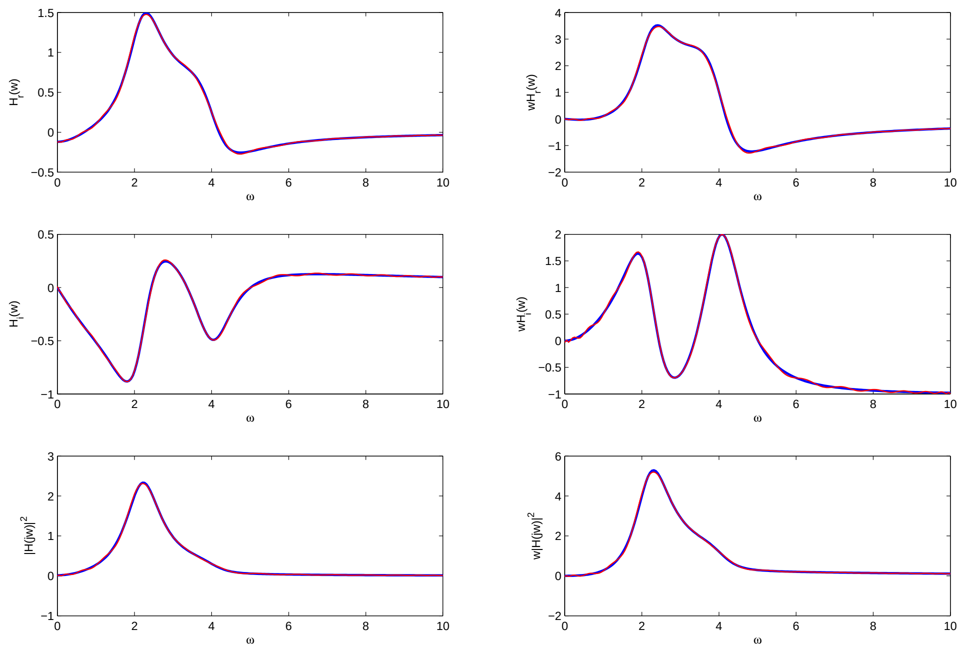


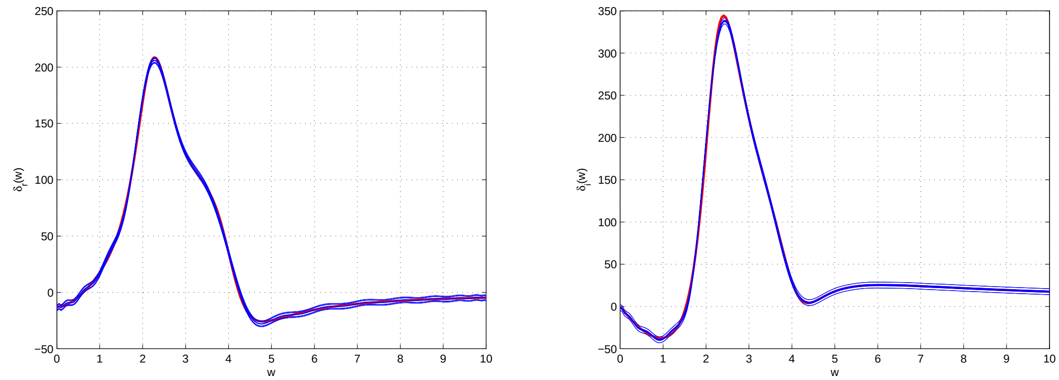

Fig. 23. Chebyshev polynomial approximations.

5.3 of [81]. The non-negativity constraints of theorem 2 are posed as SDP using Markov-Lucaks theorem. Details of how the problem is setup as an SDP is provided in the appendix. The computer packages SeDuMi [83] and YALMIP [84] are used to obtain a solution. The following stabilizing controller was obtained:

$$C_1(s) = \frac{67.2638s + 19.9411}{s + 108.4066}.$$

Using this controller and the Chebyshev approximation found above, the plot of $\delta_r(w, K)$ and $\delta_i(w, K)$ are obtained (Fig. 24).

A projection algorithm is used to obtain an idea about the feasible set of the SDP and hence find a set of stabilizing controllers. This set is shown in Fig. 25 and is generated through the YALMIP's *plot* command.




(a) $\delta_r w$ (b) $\delta_i w$

Fig. 24. The plot of $\delta_r(w)$ and $\delta_i(w)$ for the controller $C_1(s)$.

## D. Summary

In this chapter, a method is presented for constructing a fixed order controller which directly uses the frequency response measurements. The proposed method applies to plants which do not have purely imaginary poles or zeros and are representable with rational, strictly proper transfer functions. It does not require the knowledge of the transfer function $H_p(s)$, of the plant, but only requires a polynomial approximation of the real and imaginary parts of the $H_p(jw)$ in a frequency range $[0, w_b]$, where $w_b$ is a frequency beyond which the amplitude response of the plant is negligible and there is no appreciable change in phase. Using the phase change formula for rational functions, the problem of synthesizing the sets of stabilizing controllers is posed as that of sets of controllers satisfying some robust SDPs. The advantage of this approach is that noise in the frequency response measurements can also be directly handled in the synthesis of controllers. While the technique proposed can be computationally challenging, it indicates the possibility of fixed order controller synthesis using only frequency response measurements.

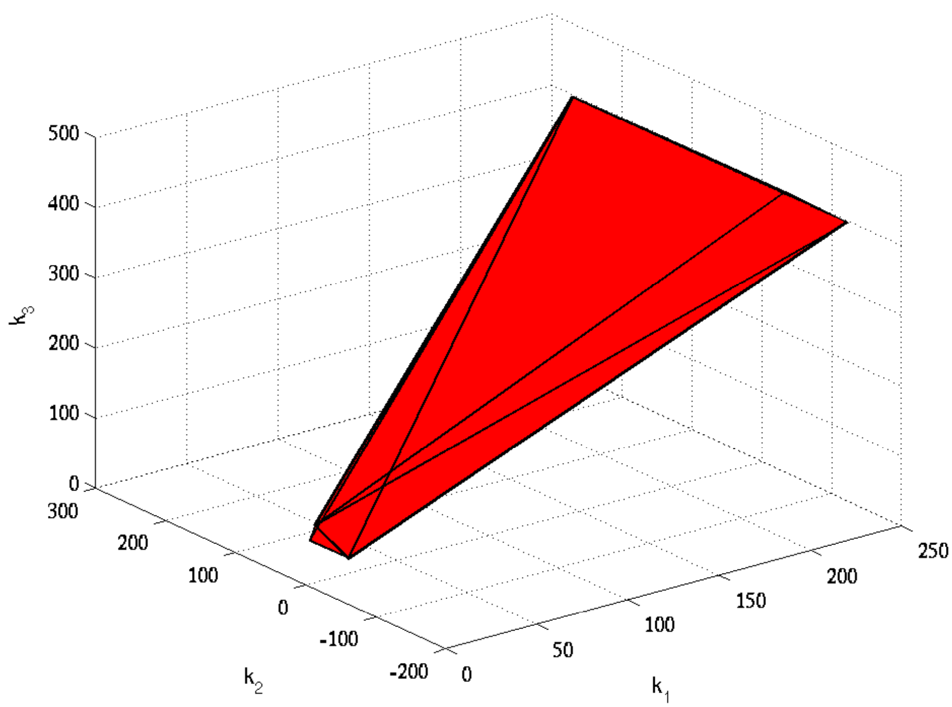



Fig. 25. A set of stabilizing first order controllers.

# CHAPTER VIII

# SYNTHESIS OF FIXED ORDER CONTROLLERS FOR NONLINEAR SYSTEMS WITH SECTOR BOUNDED NONLINEARITIES

The previous chapters have considered various procedures for synthesizing stabilizing and performance attaining controllers of a fixed order/structure, for Linear-Time Invariant (LTI) plants. In this chapter, the synthesis of fixed order controllers for nonlinear systems with sector bounded nonlinearities is considered. An inner and outer approximation of the set of absolutely stabilizing linear controllers is constructed by casting the closed loop system as a Lure-Postnikov system. The inner approximation is based on the well-known sufficient conditions that require Strict Positive Realness (SPR) of open loop transfer function (possibly with some multipliers) and a characterization of SPR transfer functions that require a family of complex polynomials to be Hurwitz. The outer approximation is based on the condition that the open loop transfer function must have infinite gain margin, which translates to a family of real polynomials being Hurwitz.

## A. Introduction

Absolute stability of Lure-Postnikov systems have been studied quite extensively, see the books of Aizerman [85], Popov [86], Hahn [87], Lefschetz [88], Siljak [31], Narendra and Taylor [89] and Safanov [90]. The problem of absolute stability is that of ensuring the asymptotic stability in the large of a nonlinear system of the form given in Fig. 26 for every nonlinearity in the first and third quadrants.

The seminal result of Popov subsumes earlier results of Lure and others concerning the problem and all subsequent results on this problem have the same flavor of requiring a transfer function, which is usually the product of the transfer function

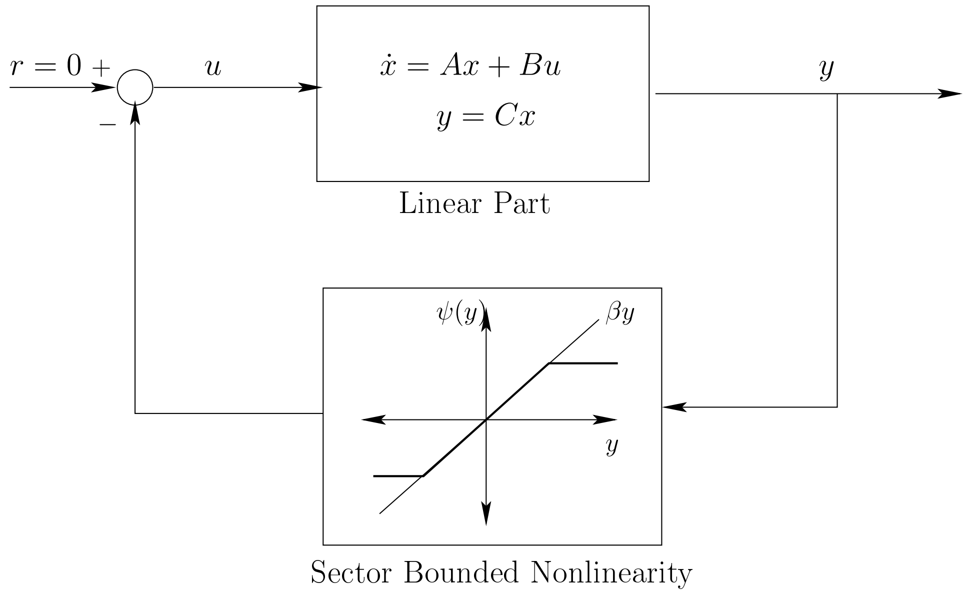



Fig. 26. Lure-Postnikov system.

of the linear part of the Lure-Postnikov system and an appropriate multiplier to be strictly positive real.

The problem of the synthesis of absolutely stabilizing controllers is important for two reasons - absolute stability naturally comes with a robustness guarantee that the zero solution of the closed loop is asymptotically stable for every nonlinearity satisfying the sector condition. In some nonlinear systems, the nonlinearity in the system is provided in terms of empirical data and only crude information about the nonlinearity is available, i.e., that it lies in the first and third quadrants. In such a case, the problem of synthesis of absolutely stabilizing controllers is relevant while being conservative. The reason for conservatism is that one is designing a controller that stabilizes the closed loop for every nonlinearity in the first and third quadrants as opposed to the specific nonlinearity provided in terms of empirical data. In some applications, the assumptions involved in developing a lumped model of a system, render the coefficient of a nonlinearity parametrically uncertain. The classic example is that of a pendulum - whether one *assumes the mass of the pendulum lumped or uniformly distributed*, the structure of the resulting equations is similar; while the nonlinearity is sector bounded, its coefficient may not be known.

In the case when the nonlinearity is known, but the coefficient is not exactly known, the situation may be remedied using nonlinear design techniques developed in [91, 92] to design a nonlinear controller which are tailor-made for the specific nonlinearity. However, the constraint on the order of the controller cannot be handled by the existing design techniques. This chapter explores the synthesis of linear absolutely stabilizing controllers of a given order. Although the procedure adopted here is conservative and applies only to systems with sector bounded nonlinearities, the proposed method allows for imposing structure (such as the order) on the controller.

The problem of synthesizing absolutely stabilizing controllers has been considered in the literature, for example, see [93, 94, 95, 96]. In [93, 94], the focus is on the synthesis of fixed order controllers. In [93], *a controller* is synthesized in terms of the solution to coupled Riccatti and Lyapunov equations, while in [94, 96], the focus was on the use of LMIs to synthesize *a controller*. In [95], the problem of synthesizing an inner approximation of the *a set* of absolutely stabilizing PID (fixed structure) controllers was considered. In this paper, we consider the problem of constructing an inner and an outer approximation of the set of stabilizing controllers of fixed order/structure for Single-Input, Single-Output (SISO) Lure-Postnikov systems. The construction of an approximation of *set* of stabilizing controllers is accomplished through the use of Hermite-Biehler theorem and a characterization of strict positive real transfer functions through the requirement of a one-parameter family of complex polynomials being Hurwitz [8]. The novelty and usefulness of the procedure in this chapter lies in the construction of an approximation to the set of absolute stabilizing controllers as a control engineer can restrict the search for a controller satisfying multiple objectives from the given set.

## B. Synthesis of the of Set Absolutely Stabilizing Controller

Consider the problem of synthesizing an absolutely stabilizing controller for the following system:

$$\dot{x} = Ax - B_1 u - B_2 \phi(y), \tag{8.1}$$

$$y = Cx, \tag{8.2}$$

where the nonlinear function $\phi(y)$ satisfies $0 \leq y\phi(y)$ for all $y \in \Re$.

Let $G_1(s) := C(sI - A)^{-1}B_1$ and $G_2 := C(sI - A)^{-1}B_2$. Consider a controller $G_c(s) = \frac{N_c(s)}{D_c(s)}$, where the polynomial, $D_c(s)$ is monic and of degree $r$, while the degree of the polynomial $N_c(s)$ is assumed to be at most $r$. Let $(A_c, B_c, K_1, K_2)$ be a minimal realization of $G_c(s)$. Hence, $G_c(s) = K_1(sI - A_c)^{-1}B_c + K_2$. We will assume that they are in the controllable canonical form. The coefficients of the polynomials $N_c(s)$ and $D_c(s)$ are free parameters that must be chosen so as to make the zero solution of the closed loop absolutely stable:

$$\dot{x} = Ax - B_1 u - B_2 \phi(y), \tag{8.3}$$

$$\dot{x}_c = A_c x_c + B_c y, \tag{8.4}$$

$$y = Cx, \tag{8.5}$$

$$u = (K_1 x_c + K_2 y). \tag{8.6}$$

In the above equation, $x_c(t) \in \Re^r$ represents the state of the controller.

The closed loop system may be expressed as a Lure-Postnikov system as follows:

$$\dot{z} = A_{cl} z - B_{cl} \phi(y), \tag{8.7}$$

$$y \;=\; C_{cl} z, \tag{8.8}$$

for some $A_{cl}$, $B_{cl}$ and $C_{cl}$ which constitute a realization of the transfer function $H(s) = (1 + G_1 G_c(s))^{-1} G_2$. If one were to write $G_1(s) = \frac{N_1(s)}{D_p(s)}$, $G_2(s) = \frac{N_2(s)}{D_p(s)}$ and $G_c(s) = \frac{N_c(s)}{D_c(s)}$, then the transfer function $H(s)$ may be expressed as $\frac{N_2(s)D_c(s)}{D_p(s)D_c(s)+N_1(s)N_c(s)}$. It is clear that the coefficients of the numerator and denominator of $H(s)$ are affine in the parameters of the controller.

The following lemma provides the necessary conditions for absolute stability of the Lure-Postnikov system:

Let the transfer function $H(s) = \frac{N_{cl}(s)}{D_{cl}(s)}$ for some co-prime polynomials, where $N_{cl}(s) = N_2(s)D_c(s)$ and $D_{cl}(s) = D_p(s)D_c(s) + N_1(s)N_c(s)$.

**Lemma 30.** *The requirement that $D_{cl}(s) + \lambda N_{cl}(s)$ be Hurwitz for every $\lambda \geq 0$ is a necessary condition for the absolute stability of the zero solution of the Lure-Postnikov System considered above.*

*Proof.* Since $\phi(y) = \lambda y$ and $\lambda \geq 0$ is an admissible function for $\phi$, the resulting Lure-Postnikov system represents a linear output feedback system. In this case, the characteristic polynomial of the closed loop system is $D(s) + \lambda N(s) = 0$. One can uniquely associate $\lambda \geq 0$ with a $\mu \in [0, 1)$ through the transformation $\lambda = \frac{\mu}{1-\mu}$. Then, one can state the necessary condition for absolute stability in terms of requiring that every convex combination of $D_{cl}(s)$ and $N_{cl}(s)$ be Hurwitz. □

*Remark* 13. Brockett and Willems [97] showed that $D_{cl}(s) + \lambda N_{cl}(s) = 0$ is Hurwitz for every $\lambda \geq 0$ if and only if there exists a strictly positive real transfer function $Q(s)$ such that $Q(s)\frac{N_{cl}(s)}{D_{cl}(s)}$ is also strictly positive real. For the purpose of outer approximation, only the result of Lemma 30 is used and not the characterization of Brockett and Willems.

A sufficient condition for absolute stability is given in terms of Popov's criterion [85].

**Theorem 12.** *If there exists a $q \geq 0$ such that $(1+qs)H(s)$ is Strictly Positive Real (SPR), then the zero solution of the Lure-Postnikov system is absolutely stable. This is also a necessary and sufficient condition for the existence of a Lyapunov function of the form $x^T Px + q\int_0^y \phi(\eta)d\eta$.*

A proof of the theorem can be found in [86, 88, 85, 87, 31, 89, 90].

The following characterization of SPR functions [8] for reducing the problem of synthesizing SPR functions to that of controllers rendering a family of polynomials Hurwitz:

**Lemma 31.** *A rational transfer function $\frac{N(s)}{D(s)}$ is SPR if and only if*

1. $\frac{N(0)}{D(0)} > 0$.
2. *The polynomials $N(s)$ and $D(s)$ are Hurwitz.*
3. *The family of complex polynomials, $D(s) + j\alpha N(s), \quad \alpha \in \Re$ is Hurwitz.*

This necessary and sufficient conditions respectively involve a family of real and complex polynomials being Hurwitz. Chapters III and IV provide a characterization of Hurwitz polynomials suitable for the construction of outer and inner approximation using Linear Programming (LP) techniques.

The closed loop may be expressed as a linear system with transfer function, $H(s)$, perturbed by a sector-bounded non-linearity, $\phi(y)$, in the feedback path.

Let $N_c(s) = n_0 + n_1 s + \cdots + n_r s^r$ and $D_c(s) = d_0 + d_1 s + \cdots + d_{r-1}s^{r-1} + s^r$. Let $K$ be the $(2r+1)-$tuplet, $(n_0,\ n_1,\ \cdots, n_r,\ d_0,\ d_1, \cdots, d_{r-1})$. Let $\Delta_1(s,K) = N_1(s)N_c(s) + D_p(s)D_c(s)$, where the coefficients of $\Delta_1(s,K)$ are affine functions of $K$. For a given $\mu \in [0,1]$, let $\Delta_2(s,K,\mu) = \mu\Delta_1(s,K) + (1-\mu)D_c(s)N_2(s)$ and let

$Q(s, K, \mu)$ denote a one-parameter family of polynomials as $\mu$ varies from 0 to 1. Let $\mathcal{A}$ be the set of all $K$ that render the closed loop absolutely stable. If $\mathcal{A}_{outer}$ is any set containing $\mathcal{A}$, we refer to $\mathcal{A}$ as an outer approximation and similarly if $\mathcal{A}_{inner}$ is a set contained in $\mathcal{A}$, it will be referred to as an inner approximation.

A way to construct an outer approximation of $\mathcal{A}$ is provided below.

Let $\Delta_2(jw, K, \mu) = \delta_r(w^2, K) + jw\delta_i(w^2, K)$ for some real polynomials $\delta_r$ and $\delta_i$. Let the degrees of the polynomials $\delta_r(\lambda, K)$ and $\delta_i(\lambda, K)$ be $n_r$ and $n_i$ respectively.

**Lemma 32.** *Let $\mathcal{S}(p, \mu)$ be the set of $K$ satisfying the following conditions for a given non-negative integer $p$ and a $\mu \in [0, 1]$: The number of sign changes in the coefficients of the polynomials $(1 + \lambda)^p \delta_r(\lambda, K)$ and $(1 + \lambda)^p \delta_i(\lambda, K)$ are respectively $n_r$ and $n_i$. Then, $\mathcal{S}(p, \mu)$ is an outer approximation of $\mathcal{A}$ for every $p$ and for every $\mu \in [0, 1]$. Moreover, $\mathcal{S}(p + 1, \mu) \subset \mathcal{S}(p, \mu)$.*

*Proof.* If $K$ is any absolutely stabilizing controller, then for every $\mu$, the polynomial $\Delta_2(s, K, \mu)$ is Hurwitz. By the Hermite-Biehler theorem, the polynomials $\delta_r, \delta_i$ must have $n_r$, $n_i$ real, positive respectively. By the generalization of the Descartes' rule of signs, for any $p$, the polynomials $(1 + \lambda)^p \delta_r$, $(1 + \lambda)^p \delta_i$ must have exactly $n_r$ and $n_i$ sign changes respectively in their coefficients. Hence, $K \in \mathcal{S}(p, \mu)$. Therefore, $\mathcal{A} \subset \mathcal{S}(p, \mu)$.

Observe that the maximum number of sign changes in the coefficients of $(1+\lambda)^p \delta_r$ is $n_r$ as the number of real positive roots can at most be $n_r$. By the generalization of Descartes' rule of signs, the number of sign changes in the coefficients of $(1 + \lambda)^p \delta_r$ is a non-increasing function of $p$. Therefore, if the number of sign changes in the coefficients of $(1 + \lambda)^{p+1} \delta_r$ is $n_r$, it must follow that the number of sign changes in the coefficients of $(1 + \lambda)^p \delta_r$ must also be $n_r$. A similar case holds for $(1 + \lambda)^p \delta_i$. Hence, if $K \in \mathcal{S}(p + 1, \mu)$, it must be that $K \in \mathcal{S}(p, \mu)$ for every non-negative integer $p$. □

*Remark* 14. For any given $p$, checking the feasibility of a specified number of sign changes in the coefficients of $(1+\lambda)^p\delta_r(\lambda, K)$ is equivalent to checking the feasibility of Linear Programs (LPs) that can be constructed as follows:

- Arrange the coefficients of the polynomial $(1+\lambda)^p\delta_r(\lambda, K)$ according to increasing powers of $\lambda$.
- Choose $(n_r + 1)$ coefficients of the $(n_r + p + 1)$ possible coefficients of $(1 + \lambda)^p\delta_r(\lambda, K)$ in the order in which they appear in the polynomial, and
- assign the sign of the coefficients to be alternating so that there are in all $n_r$ sign changes.
- One will have a choice of the sign for the first coefficient in the $n_r+1$ coefficients chosen if the degree of $D_p$ equals that of $N_1$ or $N_2$; otherwise, the sign of the first coefficient is fixed and is the same as that of the leading coefficient of $\Delta_1(s, K)$. In the former case, one gets two LPs for every choice of $n_r+1$ coefficients whose feasibility must be checked.

Let $\mathcal{S}_r$ be union of the feasible sets of all the LPs thus constructed is an outer approximation of $\mathcal{A}$ as it satisfies a necessary condition of the Hermite-Biehler Theorem. A similar remark may be made concerning the polynomial $(1 + \lambda)^p\delta_i(s, K)$ and let $\mathcal{S}_i$ be the corresponding union of feasible sets of LPs. Then, it is clear that $\mathcal{S}(p, \mu) = \mathcal{S}_r \cap \mathcal{S}_i$ is also an outer approximation of $\mathcal{A}$.

*Remark* 15. If, for a given $\mu \in [0, 1]$, the set $\mathcal{S}(p, \mu)$ is determined for some $p$, it is automatically contained in $\mathcal{S}(q, \mu)$ for every $q < p$. In other words, one does not get a refinement of the outer approximation by considering any $q \leq p$.

The set $\mathcal{S}(p, \mu)$ can be refined further taking into account the requirement of interlacing in the following way: Let $K \in \mathcal{S})(p, \mu)$ be such that $\Delta_2(s, K, \mu)$ is not Hurwitz. Then $\mathcal{S}(p, \mu)$ can be refined in the following steps:

1. Find a $\eta > 0$ such that $Q(\lambda) = \delta_r(\lambda, K) - \eta\delta_i(\lambda, K)$ does not have all real roots.

2. Find a $q$ such that $(1 + \lambda)^q Q(\lambda)$ has fewer than $n_r$ changes in the sign of its coefficients.

3. Consider the LPs associated with requiring the coefficients of $(1 + \lambda)^p Q(\lambda, K)$ to have $n_r$ sign changes subject to $K \in \mathcal{S}(p, \mu)$. Let the corresponding set be $\mathcal{S}_{ref}$.

**Lemma 33.** *The set $\mathcal{S}_{ref}$ is a refinement of $\mathcal{S}(p, \mu)$ and is an outer approximation.*

*Proof.* Clearly, every $K \in \mathcal{A}$ is such that the polynomials $(1 + \lambda)^p \delta_r$ and $(1 + \lambda)^p \delta_i$ have $n_r$ and $n_i$ sign changes respectively in their coefficients. Since the polynomial $Q(\lambda)$, corresponding to the controller $K$, must have $n_r$ real positive roots, it must also have $n_r$ sign changes in its coefficients. By the generalization of Descartes' rule of signs, it follows that $(1+\lambda)^q Q(\lambda)$ must have exactly $n_r$ sign changes. Hence, $K \in \mathcal{S}_{ref}$ and $\mathcal{A} \subset \mathcal{S}_{ref}$. Therefore, $\mathcal{S}_{ref}$ is an outer approximation. From the construction, it is clear that $\mathcal{S}_{ref} \subset \mathcal{S}(p, \mu)$. Hence, it is a refinement of $\mathcal{S}(p, \mu)$. □

*Remark* 16.

1. One can construct outer approximations corresponding to various values of $\mu \in [0, 1]$. Since each of them is an outer approximation, $\cap_\mu \mathcal{S}_{ref}(\mu)$, is also an outer approximation. Such an outer approximation is a refinement of the outer approximation obtained for each $\mu$ and can again be determined using a linear programming approach.

2. In fact, the above lemma can be used to provide an arbitrarily tight approximation of the set $\mathcal{S} = \{K : \Delta_2(s, K, \mu) \text{ } is \text{ } Hurwitz \text{ } \forall \mu \in [0, 1]\}$. However, the computational burden can be quite significant.

3. If it is determined that an outer approximation is an empty set, clearly there cannot be any stabilizing controller.

Using the characterization of SPR transfer functions given by Lemma 31 and the characterization of Hurwitz polynomials given by the Hermite-Biehler theorem, one obtains an inner approximation for $\mathcal{A}$. The transfer function $G_T(s,K) = \frac{N_T(s,K)}{D_T(s,K)} = (1+qs)H(s,K) = (1+qs)\frac{N_{cl}(s,K)}{D_{cl}(s,K)}$ is required to be SPR. Consider the complex polynomial $\Delta_c(s,K,\alpha) := D_T(s) + j\alpha N_T(s)$ and further let $\Delta_c(jw,K,\alpha) := \Delta_r(w,K) + jD_i(w,K)$ for some real polynomials $\Delta_r(w,K)$ and $\Delta_i(w,K)$. Let the degree of $\Delta_c(s,K,\alpha)$ be $N = n + r$. Further, let $\Delta_r(w,K) = \delta_{r,N}w^N + \delta_{r,N-1}w^{N-1} + \cdots + \delta_{r,0}$ and similarly, let $\Delta_i(w,K) = \delta_{i,N}w^N + \delta_{i,N-1}w^{N-1} + \cdots + \delta_{i,0}$.

**Theorem 13.** *There exists a controller, $C(s)$, of order $r$, that renders the transfer function $G_T(s,K)$ SPR if and only if there exists a $K$ such that*

1. $H(0,K) = \frac{N_{cl}(0,K)}{D_{cl}(0,K)} > 0$,
2. *the polynomials $N_{cl}(s,K)$ and $D_{cl}(s,K)$ are Hurwitz,*
3. *for every $\alpha \in \Re$, there exists a set of $2N-1$ frequencies, $-\infty < w_1(\alpha) < w_2(\alpha) < \cdots < w_{2N-1}(\alpha) < \infty$ such that $K$ is a feasible solution of one of the following four linear programs:*

$$C_k \begin{bmatrix} \delta_{r,N}(\alpha) \\ \Delta_r(w_1,\alpha) \\ \Delta_r(w_2,\alpha) \\ \vdots \\ \Delta_r(w_{2N-1},\alpha) \end{bmatrix} > 0, \qquad S_k \begin{bmatrix} \delta_{i,N}(\alpha) \\ \Delta_i(w_1,\alpha) \\ \Delta_i(w_2,\alpha) \\ \vdots \\ \Delta_i(w_{2N-1},\alpha) \end{bmatrix} > 0.$$

These conditions are obtained by applying Lemma 31 to the transfer function $G_T(s,K)$ and application of the Hermite-Biehler theorem for complex polynomials to render the family of complex polynomials to be Hurwitz. By Theorem 12, the set $K$ obtained as the solution to the above theorem, is an inner approximation for the set of absolutely stabilizing controllers.

C. Illustrative Example

Consider a one-link robot shown in Fig. 27 with a flexible joint as an example for absolute stabilization. The governing equations of motion may be written as:

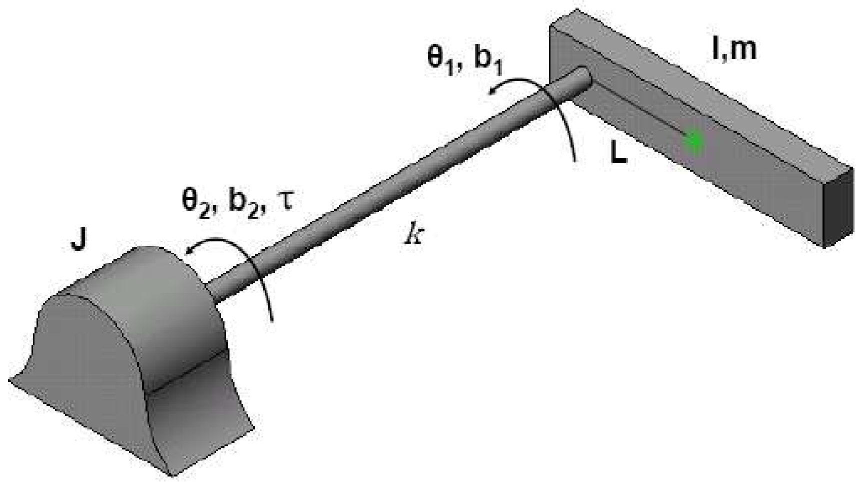



Fig. 27. One-link robot with a flexible joint.

$$
\begin{aligned}
I_1\ddot{\theta}_1 + b_1\dot{\theta}_1 + mgL\sin\theta_1 + k(\theta_1 - \theta_2) &= 0, \\
J\ddot{\theta}_2 + b_2\dot{\theta}_2 + k(\theta_2 - \theta_1) &= \tau
\end{aligned}
\tag{8.9}
$$

One can obtain a state space representation of the system (8.9) by choosing state variables :

$$
\begin{aligned}
x_1 = \theta_1, \quad x_2 = \dot{\theta}_1, \\
x_3 = \theta_2, \quad x_4 = \dot{\theta}_2.
\end{aligned}
\tag{8.10}
$$

The state space representation is:

$$\dot{x} = Ax + B_1 u - B\psi(y),$$

$$y \;\; = \;\; Cx, \tag{8.11}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{I} & -\frac{b_1}{I} & \frac{k}{I} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J} & 0 & -\frac{k}{J} & -\frac{b_2}{J} \end{bmatrix}, \;\; B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \;\; B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \tag{8.12}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \tag{8.13}$$

$$\psi(y) = \frac{mgL}{I} \sin y, \;\; u = \frac{\tau}{J}. \tag{8.14}$$

The system parameters are given as follows :

$$J = 0.5kg \cdot m^2, \; b1 = 0.0Nm \cdot s/rad, \; k = 50.0Nm/rad,$$

$$I = 25.0kg \cdot m^2, \; b2 = 1.0Nm \cdot s/rad, \; m = 1.0kg, \; L = 5.0m.$$

1. PID Controller

Consider a PID controller :

$$C(s) = k_p + \frac{k_i}{s} + k_d s \tag{8.15}$$

$$u = k_p(r - y) + k_d(\dot{r} - \dot{y}) + k_i w \tag{8.16}$$

$$\dot{w} = r - y, \tag{8.17}$$

where $C(s)$ is the PID controller, $w$ is the integral of the error and $r$ is reference which is set to be 0. Fig. 28 shows a control structure for the one-link robot with a flexible joint which has a sector-bounded nonlinearity.

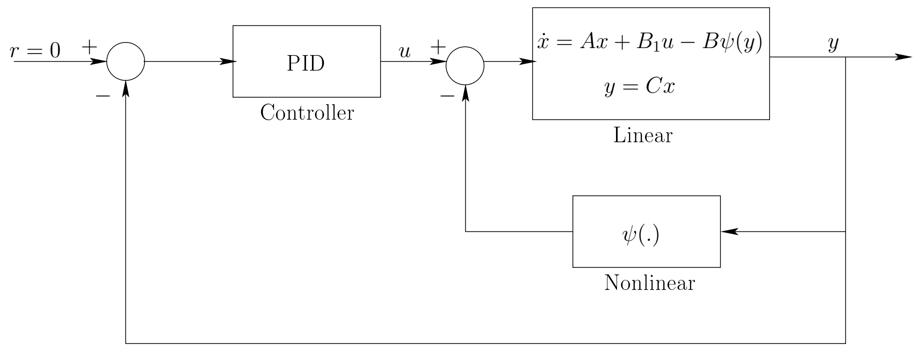



Fig. 28. Control structure of one-link robot with a flexible Joint.

The closed loop system can be represented as an augmented system as follows :

$$\begin{aligned} \dot{z} &= \mathbf{A}z - \mathbf{B}\psi(y) \\ y &= \mathbf{C}z, \end{aligned}$$

where $z = \begin{bmatrix} x & w \end{bmatrix}'$.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{k}{I} & -\frac{b_1}{I} & \frac{k}{I} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{k}{J} - k_p & -k_d & -\frac{k}{J} & -\frac{b_2}{J} & k_i \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{8.18}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\ , \end{bmatrix} \tag{8.19}$$

$$\psi(y) = \frac{mgL}{I} \sin y, \tag{8.20}$$

which constitutes a realization of the transfer function,

$$G(s) \quad = \quad \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B}$$

$$
\begin{aligned}
&= \frac{N_{cl}(s)}{D_{cl}(s)} \\
&= \frac{s^3 + 2s^2 + 100s}{s^5 + 2s^4 + 102s^3 + (4 + 2k_d)s^2 + 2k_p s + 2k_i} \qquad (8.21)
\end{aligned}
$$

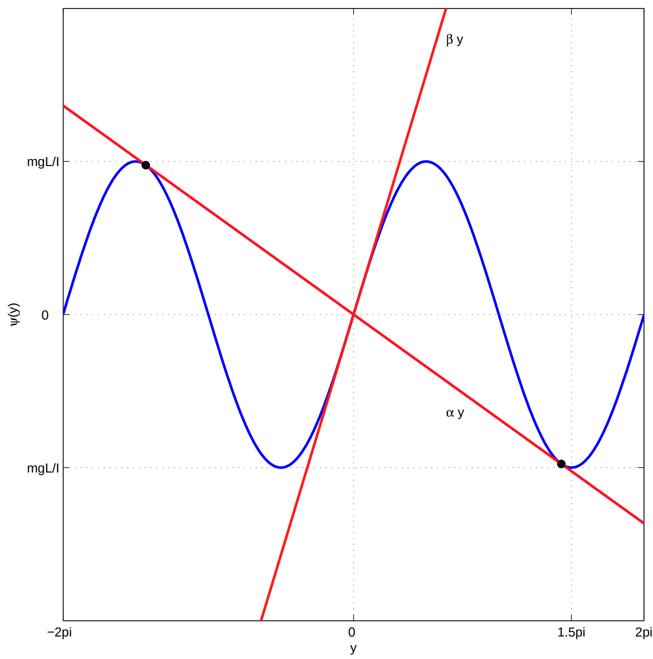



Fig. 29. Plot of the nonlinearity, $\psi(y) = \frac{mgL}{I} sin(y)$.

Note that the nonlinearity, $\psi(.) = \frac{mgL}{I} sin(.)$ though sector bounded, is not restricted to the first and third quadrants. A nonlinearity, $\psi(.)$ is said to belong to a sector $[\alpha, \beta]$, if the graph of this function belongs to a sector whose boundaries are the lines $\alpha y$ and $\beta y$. Fig. 29 shows the nonlinearity, $\psi(y) = \frac{mgL}{I} \sin y$ and the associated sector. From the figure it is clear that $\psi(.) \in [\alpha, \beta]$, where $\beta = \frac{mgL}{I}$ and $\alpha = \beta \cos(y^*)$, $y^* \approx 1.5\pi$ is the solution of the equation $y \cos(y) - \sin(y) = 0$.

Since the theory developed in the previous section requires the nonlinearity to lie in the first and third quadrants, one needs to transform the above system to the appropriate form. The following loop transformation (see Ex.6.1 in [98]) basically transforms the nonlinearity $\psi(.) \in [\alpha, \beta]$ to a case where the nonlinearity belongs to the sector $[0, \infty]$ (Fig. 30).

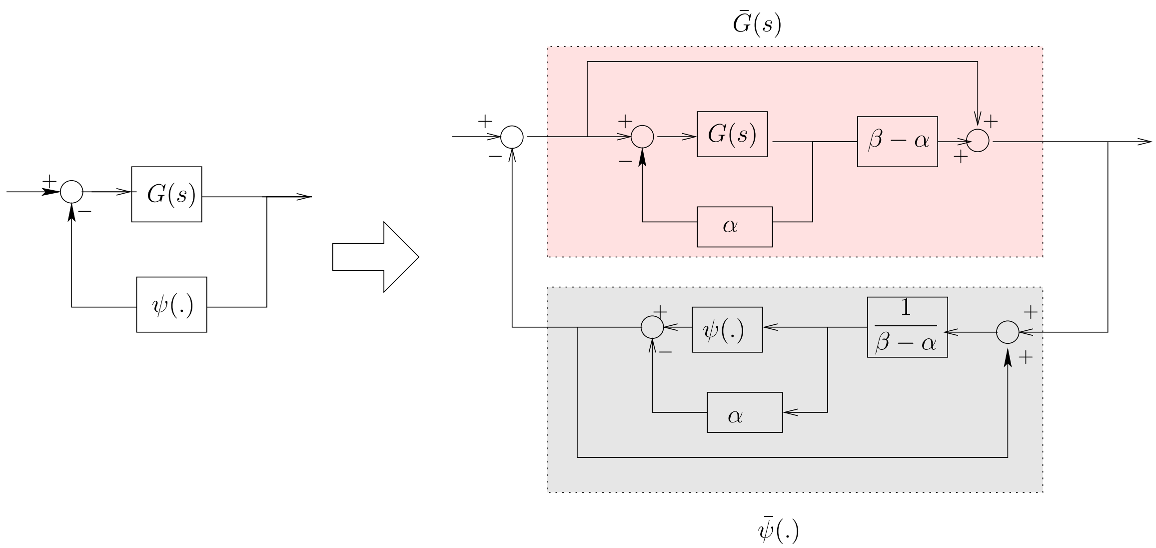



Fig. 30. Loop transformation.

The nonlinearity $\bar{\psi}(.)$ now lies in the sector $[0, \infty]$. The modified plant is given by

$$
\begin{aligned}
\bar{G}(s) &= \frac{\bar{N}(s,K)}{\bar{D}(s,K)} = 1 + (\beta - \alpha)\frac{G(s)}{1 + \alpha G(s)} = \frac{1 + \beta G(s)}{1 + \alpha G(s)} \\
&= \frac{s^5 + 2s^4 + 102.2s^3 + (2k_d + 4.3924)s^2 + (2k_p + 19.62)s + 2k_i}{s^5 + 2s^4 + 101.96s^3 + (2k_d + 3.9148)s^2 + (2k_p - 4.2621)s + 2k_i}
\end{aligned}
$$

The results developed in the previous section are now applied to this modified system.

### a. Outer Approximation

For a given $\mu \in [0, 1]$, let $\Delta(s, K, \mu) = \mu \bar{N}(s, K) + (1 - \mu)\bar{D}(s, K)$ and let $Q(s, K, \mu)$ denote a one-parameter family of polynomials as $\mu$ varies from 0 to 1.

Let

$$\Delta(s,K,\mu) = \underbrace{(2w^4 - (0.4776\mu + 2k_d + 3.9148)w^2 + 2k_i)}_{\delta_r(w^2,K)} +$$

$$jw\,\underbrace{(w^4 - (101.96 + 0.2388\mu)w^2 + 23.882\mu - 4.2621 + 2k_p)}_{\delta_i(w^2,K)}$$

To construct the outer approximation, we consider different values of $\mu \in [0,1]$, and require the polynomials $\delta_r(w^2,K)$ and $\delta_i(w^2,K)$ to have exactly two sign changes. Application of Lemma 32, generates an outer approximation to the set of absolutely stabilizing controllers (Fig. 31).

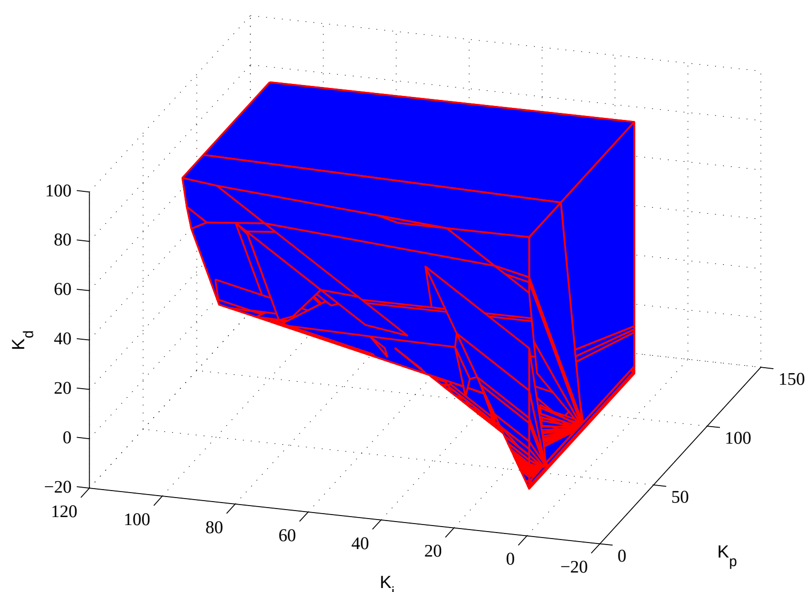



Fig. 31. Outer approximation of the set of absolutely stabilizing PID controllers.

b. Inner Approximation

Using Theorem 12, the system is absolutely stable if there is a $q \geq 0$ such that $G_T(s,K) = \dfrac{N_{GT}(s,K)}{D_{GT}(s,K)} = (1+qs)\bar{G}(s,K)$ is strictly positive real.

For strictly positive realness of the $G_T(s)$, the following conditions should be satisfied: (Theorem 13)

1. $G_T(0) = \frac{N_{GT}(s)}{D_{GT}(s)} > 0$,
2. $N_{GT}(s, K)$ and $D_{GT}(s, K)$ are Hurwitz for some $q \geq 0$, and
3. $P(s, K) = D_{GT}(s, K) + j\alpha N_{GT}(s, K)$ is Hurwitz for some $q \geq 0$, $\forall \alpha \in \Re$.

Let $q = 1$.

1. For condition 1:

$$G_T(s) = \frac{N_{GT}(s)}{D_{GT}(s)}$$

where

$$N_{GT}(s) = s^6 + 3s^5 + 104.2s^4 + (2k_d + 106.6)s^3 + (2k_p + 24.01 + 2k_d)s^2 + (2k_p + 19.62 + 2k_i)s + 2k_i$$

$$D_{GT}(s) = s^5 + 2s^4 + 102.0s^3 + (2k_d + 3.915)s^2 + (2k_p - 4.262)s + 2k_i$$

and we clearly see that $G_T(0) = 1 > 0$

2. For condition 2: $N_{GT}(s) = (1 + qs)\bar{N}(s)$ is Hurwitz if $\bar{N}(s)$ is Hurwitz.

$$\bar{N}(s) = s^5 + 2s^4 + 102.2s^3 + (2k_d + 4.3924)s^2 + (2k_p + 19.62)s + 2k_i$$

The real and imaginary parts of the $\bar{N}(s)$ at $s = jw$ are given by

$$\bar{N}(jw, K) = \underbrace{(2w^4 - (2k_d + 4.392)w^2 + 2k_i)}_{\bar{N}_e(w,K)} + jw\underbrace{(w^4 - 102.2w^2 + 2k_p + 19.62)}_{\bar{N}_o(w,K)}$$

For the polynomial $\bar{N}$ to be Hurwitz, there must exist a set of frequencies $0 = w_0 < w_1 < w_2 < w_3 < w_4$ for which at least one of the following two LPs is feasible $k = (1, 3)$:

$$C_k \begin{bmatrix} 1 & 0 & 0 \\ 1 & w_1^2 & w_1^4 \\ 1 & w_2^2 & w_2^4 \\ 1 & w_3^2 & w_3^4 \\ 1 & w_4^2 & w_4^4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 0 \\ -4.3924 & 0 & 0 & -2 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_p \\ k_i \\ k_d \end{bmatrix} > 0,$$

$$S_k \begin{bmatrix} 1 & 0 & 0 \\ 1 & w_1^2 & w_1^4 \\ 1 & w_2^2 & w_2^4 \\ 1 & w_3^2 & w_3^4 \\ 1 & w_4^2 & w_4^4 \end{bmatrix} \begin{bmatrix} 19.62 & 2 & 0 & 0 \\ -102.1962 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_p \\ k_i \\ k_d \end{bmatrix} > 0$$

A similar procedure is applied to find the set of controllers for which $D_T(s, K) = \bar{D}(s, K)$ is Hurwitz.

Fig. 32 shows the set of controller for which the second condition is satisfied.

3. For condition 3: The family of polynomials

$$\begin{aligned} P(s, \alpha) &= D_{GT}(s) + j\alpha N_{GT}(s) \\ &= -j\alpha w^6 + (j - 3\alpha)w^5 + (2 + 104.2\alpha j)w^4 + (2\alpha k_d + 106.6\alpha - 102j)w^3 \\ &\qquad + (-2k_d - 3.915 - 2j\alpha(k_p + k_d) - 24.01j\alpha)w^2 \\ &\qquad\quad + (-4.262j + 2jk_p - 2\alpha(k_i + k_p) - 19.62\alpha)w + 2k_i + 2j\alpha k_i \end{aligned}$$

should be Hurwitz $\forall \alpha \in \Re$.

This polynomial can be decomposed as:

$$P(jw, K) = P_r(w, K) + jP_i(w, K)$$

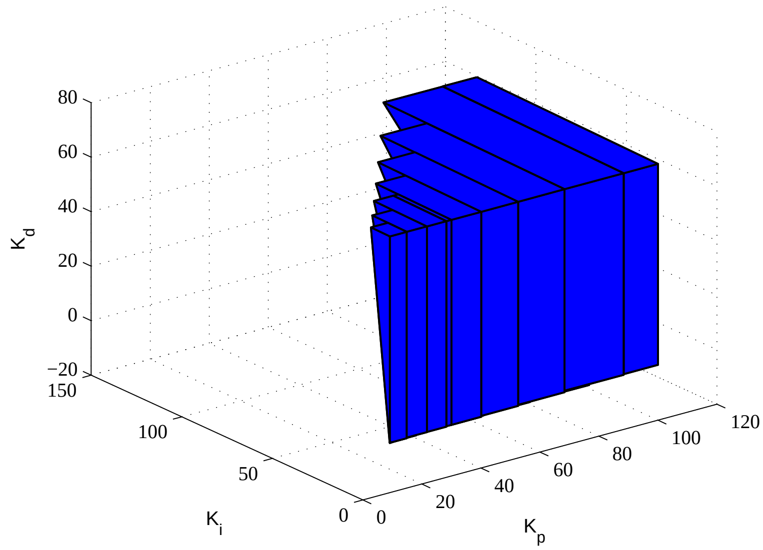



Fig. 32. Set of PID controllers satisfying SPR condition 2.

$$P_r(w, K) = -3\alpha w^5 + 2w^4 + (106.6\alpha + 2\alpha k_d)w^3 + (-2k_d - 3.915)w^2 + (-19.62\alpha - 2\alpha k_i - 2\alpha k_p)w + 2k_i$$

$$P_i(w, K) = -\alpha w^6 + w^5 + 104.2\alpha w^4 - 102.0w^3 + (-2\alpha k_d - 2\alpha k_p - 24.01\alpha)w^2 + (2k_p - 4.262)w + 2\alpha k_i$$

For the polynomial $P(s, K)$ to be Hurwitz, there must exist a set of frequencies $-1 = w_0 < w_1 < w_2 < \cdots < w_{10} < w_{11}$ for which at least one of the following four

LPs is feasible:

$$C_k \begin{bmatrix} 0 & 0 & \dots & -1 \\ 1 & w_1 & \dots & w_1^5 \\ 1 & w_2 & \dots & w_2^5 \\ \vdots & & & \\ \vdots & & & \\ 1 & w_{11} & \dots & w_{11}^5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 0 \\ -19.62\alpha & -2\alpha & -2\alpha & 0 \\ -3.915 & 0 & 0 & -2 \\ 106.6\alpha & 0 & 0 & 2\alpha \\ 2 & 0 & 0 & 0 \\ -3\alpha & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_p \\ k_i \\ k_d \end{bmatrix} > 0,$$

and

$$S_k \begin{bmatrix} 0 & 0 & \dots & -1 \\ 1 & w_1 & \dots & w_1^5 \\ 1 & w_2 & \dots & w_2^5 \\ \vdots & & & \\ \vdots & & & \\ 1 & w_9 & \dots & w_9^5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2\alpha & 0 \\ -4.262 & 2 & 0 & 0 \\ -24.01\alpha & -2\alpha & 0 & -2\alpha \\ -102.0 & 0 & 0 & 0 \\ 104.2\alpha & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1\alpha & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_p \\ k_i \\ k_d \end{bmatrix} > 0$$

Fig. 33 shows the set of controller for which the transfer function is SPR and this set of controller absolutely stabilizes the one-link robot with a flexible joint.

Fig. 34 shows the outer and inner approximation for the set of absolutely stabilizing controller on the same plot.

## D. Summary

This chapter outlines a procedure for the synthesis of fixed order controllers for nonlinear systems with sector bounded nonlinearities. The procedure constructs an inner and outer approximation of the set of absolutely stabilizing linear controllers by cast-

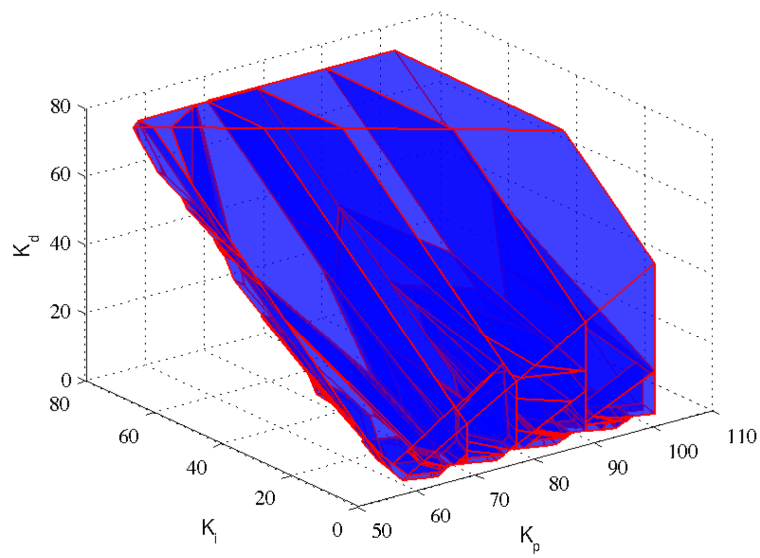



Fig. 33. Inner approximation of the set of absolutely stabilizing PID controllers.

ing the closed loop system as a Lure-Postnikov system. The inner approximation is based on the well-known sufficient conditions that require Strict Positive Realness (SPR) of open loop transfer function (possibly with some multipliers) and a characterization of SPR transfer functions that require a family of complex polynomials to be Hurwitz. The outer approximation is based on the condition that the open loop transfer function must have infinite gain margin, which translates to a family of real polynomials being Hurwitz.

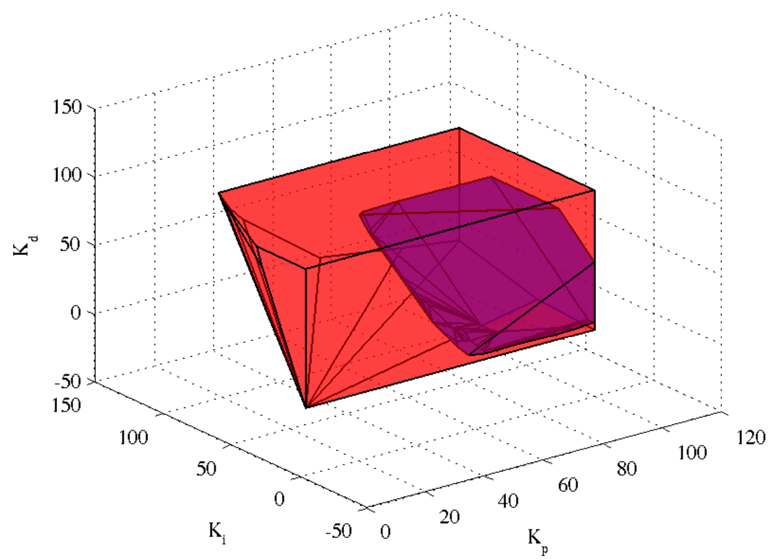



Fig. 34. Outer and inner approximation for the set of absolutely stabilizing PID controllers.

# CHAPTER IX

## CONCLUSION AND RECOMMENDATION FOR FUTURE WORK

This dissertation considers the open problem of the synthesis of the sets fixed order and structure controllers, where the coefficients of the closed loop polynomial are linear in the parameters of the controller. A novel feature of the algorithm is the *systematic exploitation* of the interlacing property of Hurwitz polynomials and the use of Descartes' rule of signs to construct LPs in the parameters of a fixed order controller. The feasible set of any LP constructed for an inner approximation of the set of all stabilizing controllers, can be indexed by a set of $n-1$ increasing frequencies, $0 = w_0 < w_1 < w_2 < \cdots < w_n$; in particular, any controller in the feasible set of LPs places the roots of the even and odd polynomials of $P(s, K)$ alternately in the intervals $(w_i, w_{i+1}),\ i = 0, \ldots, n-1$. The problem of inner approximation of the set of stabilizing controllers is then posed as the search for all sets of ordered $n-1$-tuples of frequencies for which the associated LP is feasible; the union of all feasible LPs is an inner approximation for the set of all stabilizing controllers. For constructing the outer approximation, we use the fact that a necessary condition for a polynomial to be Hurwitz is that the roots of even and odd parts of the polynomial have all real, positive and interlacing roots. The Descartes' rule of signs and its generalization due to Poincaré were used to construct the LPs for the outer approximation. Robust stability and performance specifications such as gain and phase margins are accommodated by imposing further linear inequality constraints. This involves finding the set of controllers which renders a family of polynomials to be Hurwitz.

An algorithm is next developed to construct an outer approximation of the set of $K$'s that corresponding to the transfer function $\frac{N(s,K)}{D(s,K)}$ having a non-negative and decaying impulse response. A broad class of transient response control problems can be formulated in this way.

In this dissertation, the structure of the set of minimal order stabilizing and performance attaining controllers for continuous time LTI plants in the controller parameter space is also studied. The minimal order of a controller that guarantee specified performance is $l$ if and only if (1) there is a controller of order $l$ guaranteeing the specified performance and (2) the set of strictly proper robustly stabilizing controllers guaranteeing the performance is bounded. Moreover, if the order of the controller is increased, the set of higher order controllers which satisfies the specified performance, will necessarily be unbounded. These characterization are provided for performance specifications which can be posed as finding the set of controllers which renders a one parameter family of polynomials Hurwitz. Also, if the set of proper stabilizing controllers of order $r$ is not empty and the set of strictly proper robustly stabilizing controllers of order $r$ is bounded *iff* $r$ is the minimal order of stabilization for the plant.

A procedure is presented for controller order reduction through the construction of an under-determined system of linear equations. The system of linear equations is obtained by canceling the poles of the closed loop system obtained by a controller of higher order and replacing it with one less pole. The free parameter in the solution of the under-determined system is then used to search for stability and performance.

It is widely recognized that an accurate analytical model of the plant may not be available to a control designer. However, it is reasonable in many applications that one will have an empirical model of the plant in terms of its frequency response data and from physical considerations or from the empirical time response data, one may have some coarse information about the plant such as the number of non-minimum phase zeros of the plant etc.. A method is developed for constructing a fixed order controller which directly uses the frequency response measurements. This method applies to plants which do not have purely imaginary poles or zeros and are representable with rational, strictly proper transfer functions. It does not require the knowledge of the transfer function $H_p(s)$, of the plant, but only requires a polynomial approximation of the real and imaginary parts of the $H_p(jw)$ in a frequency range $[0, w_b]$, where $w_b$ is a frequency beyond which the amplitude response of the plant is negligible and there is no appreciable change in phase. Using the phase change formula for rational functions, the problem of synthesizing the sets of stabilizing controllers is posed as that of sets of controllers satisfying some robust SDPs. The advantage of this approach is that noise in the frequency response measurements can also be directly handled in the synthesis of controllers. While the technique proposed can be computationally challenging, it indicates the possibility of fixed order controller synthesis using only frequency response measurements.

A procedure for the synthesis of fixed order controllers for nonlinear systems with sector bounded nonlinearities is also developed. The procedure constructs an inner and outer approximation of the set of absolutely stabilizing linear controllers by casting the closed loop system as a Lure-Postnikov system. The inner approximation is based on the well-known sufficient conditions that require Strict Positive Realness (SPR) of open loop transfer function (possibly with some multipliers) and a characterization of SPR transfer functions that require a family of complex polynomials to be Hurwitz. The outer approximation is based on the condition that the open loop transfer function must have infinite gain margin, which translates to a family of real polynomials being Hurwitz.

The problem considered in this dissertation is fundamental and is a longstanding and difficult open problem. The systematic procedure developed in this dissertation might facilitate the development of emergent applications such as decentralized control algorithms for formations of unmanned vehicles.

In this dissertation, mostly continuous-time systems are considered. Most of the results of this dissertation can be extended for the discrete-time systems.

This algorithms developed in this dissertation applies only to SISO, SIMO and MISO systems and cannot be used to synthesize controllers for MIMO systems. The synthesis of fixed structure or fixed order controllers for MIMO systems require additional research, and most probably, a different approach needs to be taken to synthesize the set of fixed structure controllers for such systems.

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# APPENDIX

# SEMIDEFINITE REPRESENTATIONS FOR A NONNEGATIVE POLYNOMIAL

## A. Polynomial Approximation

Chapter VII dealt with synthesizing sets of stabilizing controllers of strictly proper, delay-free, Single Input, Single Output Linear Time Invariant (LTI) plants directly from their empirical frequency response data. The method of synthesizing stabilizing controllers involved the use of generalized Hermite-Biehler theorem for rational functions for counting the roots and the use of recently developed Sum-of-Squares techniques for checking the non-negativity of a polynomial in an interval through the Markov-Lucaks theorem. In this appendix, I'll explain how the non-negativity of a polynomial can be represented as a semi-definite program

For simplicity, we consider a first order controller which stabilizes the closed loop characteristic polynomial. Higher order controller design also can be applied with the same procedure.

Recall, that for the first order controller, the functions $\delta_r(w)$ and $\delta_i(w)$ can be expressed as:

$$\delta_r(w, K) = \begin{bmatrix} wH_i(w) & 0 & |H_p(jw)|^2 & H_r(w) \end{bmatrix} \begin{bmatrix} K \\ 1 \end{bmatrix},$$

$$\delta_i(w, K) = \begin{bmatrix} wH_r(w) & w|H_p(jw)|^2 & 0 & -H_i(w) \end{bmatrix} \begin{bmatrix} K \\ 1 \end{bmatrix}.$$

Chebyshev polynomial approximations were constructed for $H_r(w)$, $wH_i(w)$, $|H_p(jw)|^2$

for $\delta_r(w, K)$ and $wH_r(w)$, $w|H(jw)|^2$, $H_i(w)$ for $\delta_i(w, K)$. The next subsection provides an overview of these approximations.

### 1. Chebyshev Polynomials of the First Kind

**Theorem 14.** *(Weierstrass Approximation: ) If $f$ is a continuous real-valued function on $[a, b]$ and if any $\epsilon > 0$ is given, then there exists a polynomial $P(x)$ on $[a, b]$ such that*

$$| \, f(x) - P(x) \, | < \; \epsilon \;\; \forall \, x \; \in \; [a, b]$$

In words, any continuous function on a closed and bounded interval can be uniformly approximated on that interval by polynomials to any degree of accuracy. Several proofs of the Weierstrass Approximation theorem can be found in [99].

The algebraic polynomials, $T_n(x)$, satisfying

$$T_n(\cos x) = \cos(nx), \quad for \; n = 0, 1, 2, \ldots$$

are called the Chebyshev polynomials of the first kind. The algebraic polynomials, $U_n(x)$, satisfying

$$U_n(\cos x) = \frac{\sin(n+1)x}{\sin x}, \quad for \; n = 0, 1, 2, \ldots$$

are called the Chebyshev polynomials of the second kind. These formulas uniquely defines $T_n$ and $U_n$ as a polynomials of degree exactly $n$. So, the Chebyshev polynomial of the first kind, $T_n$, is the unique real polynomial of degree $n$ whose leading coefficient is 1, if $n = 0$, and $2^{n-1}$, if $n \geq 1$, such that $T_n(\cos\theta) = \cos(n\theta)$. Some properties of Chebyshev polynomials of the first kind are listed below. These properties help with the semi-definite representation.

1. Since $\cos(nx) = 2\cos x \;\; \cos(n-1)x - \cos(n-2)x$, $T_n(x)$ has the following

recurrence relation.

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}, \quad n \geq 2,$$

where $T_0(x) = 1$, $T_1(x) = x$. This recurrence relation may be taken as a definition for the Chebyshev polynomial of the first kind.

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
&\vdots
\end{aligned}$$

2. Chebyshev Polynomials of the first kind are orthogonal with respect to the weight function $(1-x^2)^{-1/2}$ on the interval $(-1, 1)$.

$$\begin{aligned}
\int_{-1}^{1} \frac{T_n(x)\, T_m(x)}{\sqrt{1-x^2}} dx &= \int_0^{\pi} \cos n\theta \ \cos m\theta \ d\theta \\
&= \left\{ \begin{array}{c} 0, \quad n \neq m \\ \pi, \quad n = m = 0 \\ \pi/2, \quad n = m \neq 0 \end{array} \right\}
\end{aligned}$$

3. The polynomial $T_n(x)$ has $n$ zeros in the interval $[-1,\ 1]$, and they are located at the points

$$x = \cos\left(\frac{\pi(k - 1/2)}{n}\right), \quad k = 1, 2 \ldots, n \tag{A.1}$$

4. The Chebyshev Polynomials satisfy a discrete orthogonality as well as the con-

tinuous one. If $x_k$ $(k = 1, 2, \ldots, m)$ are the $m$ zeros of $T_m(x)$ given by (A.1) and if $i, j < m$, then

$$\sum_{k=1}^{m} T_i(x_k)T_j(x_k) = \left\{ \begin{array}{c} 0, \quad i \neq j \\ m, \quad i = j = 0 \\ m/2, \quad n = m \neq 0 \end{array} \right\} \tag{A.2}$$

2. Chebyshev Polynomial Approximation

**Theorem 15.** *(Chebyshev Approximation)*
*If $f(x)$ is an arbitrary function in the interval $[-1, 1]$ then $f(x)$ can be approximated as follows.*

$$f(x) \approx \left[ \sum_{k=0}^{N-1} C_k T_k(x) \right] - \frac{1}{2} C_0,$$

*where*

$$\begin{aligned} C_j &\equiv \frac{2}{N} \sum_{k=1}^{N} f(x_k) T_j(x_k) \\ &= \frac{2}{N} \sum_{k=1}^{N} f \left[ \cos \left( \frac{\pi(k - \frac{1}{2})}{N} \right) \right] \cos \left( \frac{\pi j(k - \frac{1}{2})}{N} \right) \end{aligned}$$

For our application it is necessary to normalize the frequency range $w \in [a, b]$ to $x \in [-1, 1]$ as follows:

$$x = -1 + 2\frac{w - a}{b - a}, \quad w \in [a, b]$$

Now, we are ready to approximate $\dfrac{\delta_r(jw, K)}{| D_p(jw) |^2}$ and $\dfrac{\delta_i(jw, K)}{| D_p(jw) |^2}$ with finite frequency data with Chebyshev polynomials upto degree $N$.

- The real part was given by :

$$\frac{\delta_r(jw, K)}{| D_p(jw) |^2} = \left[ \begin{array}{cccc} wH_i(w) & 0 & |H(jw)|^2 & H_r(w) \end{array} \right] [K'],$$

The approximation of the real part $f_r(x,K) \approx \frac{\delta_r(jw,K)}{|D_p(jw)|^2}$ is achieved by approximating each of the individual quantities

$$
\begin{aligned}
wH_i(w) \approx& C_0^1 T_0(x) + C_1^1 T_1(x) + \ldots + C_N^1 T_N(x) - \frac{1}{2}C_0^1,\\
|H(jw)|^2 \approx& C_0^2 T_0(x) + C_1^2 T_1(x) + \ldots + C_N^2 T_N(x) - \frac{1}{2}C_0^2,\\
H_r(w) \approx& C_0^3 T_0(x) + C_1^3 T_1(x) + \ldots + C_N^3 T_N(x) - \frac{1}{2}C_0^3.
\end{aligned}
$$

The final approximation $f_r(x,K)$ can be expressed compactly as:

$$
\begin{aligned}
f_r(x,K) =& \frac{1}{2}(C_0^1 + k_2 C_0^2 + k_3 C_0^3)T_0(x) + \ldots + (C_N^1 + k_2 C_N^2 + k_3 C_N^3)T_N(x),\\
=& C_0^r(K)T_0(x) + C_1^r(K)T_1(x) + \ldots + C_N^r(K)T_N(x),
\end{aligned}
$$

where,

$$
\begin{aligned}
C_0^r(K) &= \frac{1}{2}(C_0^1 + k_2 C_0^2 + k_3 C_0^3),\\
C_1^r(K) &= C_1^1 + k_2 C_1^2 + k_3 C_1^3,\\
&\vdots\\
C_N^r(K) &= C_N^1 + k_2 C_N^2 + k_3 C_N^3.
\end{aligned}
$$

- The approximation of the imaginary part $f_i(x,K) \approx \frac{\delta_i(jw,K)}{|D_p(jw)|^2}$

$$
\begin{aligned}
\frac{\delta_i(jw,K)}{|\ D_p(jw)\ |^2} &= \begin{bmatrix} wH_r(w) & w|H(jw)|^2 & 0 & -H_i(w) \end{bmatrix}[K'],\\
wH_r(w) &\approx C_0^4 T_0(x) + C_1^4 T_1(x) + \ldots + C_N^4 T_N(x) - \frac{1}{2}C_0^4,\\
w|H(jw)|^2 &\approx C_0^5 T_0(x) + C_1^5 T_1(x) + \ldots + C_N^5 T_N(x) - \frac{1}{2}C_0^5,\\
-H_i(w) &\approx C_0^6 T_0(x) + C_1^6 T_1(x) + \ldots + C_N^6 T_N(x) - \frac{1}{2}C_0^6,
\end{aligned}
$$

$$
f_i(x,K) = \frac{1}{2}(C_0^4 + k_1 C_0^5 + k_3 C_0^6)T_0(x) + \ldots + (C_N^4 + k_1 C_N^5 + k_3 C_N^6)T_N(x),
$$

$$=C_0^i(K)T_0(x) + C_1^i(K)T_1(x) + \ldots + C_N^i(K)T_N(x),$$

where,

$$\begin{aligned}
C_0^i(K) =& \frac{1}{2}(C_0^4 + k_1 C_0^5 + k_3 C_0^6),\\
C_1^i(K) =& C_1^4 + k_1 C_1^5 + k_3 C_1^6,\\
&\vdots\\
C_N^i(K) =& C_N^4 + k_1 C_N^5 + k_3 C_N^6.
\end{aligned}$$

## B. Semidefinite Representation

It is well known that nonnegative polynomials can be represented as sums of squares (SOS) [80, 100]. A polynomial, that can be expressed as sum of squares, can be formulated as a linear inequality over the cone of positive semidefinite matrices [81, 55, 101, 33, 82, 35].

### 1. Sum of Squares

A basic problem that appears in many areas of control is that of checking global, or local nonnegativity of a function of several variables [82, 35].

**Theorem 16.** *If a polynomial $f(x)$ is real and nonnegative for all $x \in \Re$ of degree $n$, then $f(x)$ it can be written as sum of squares as shown in [20]*

$$f(x) = f_1^2(x) + f_2^2(x), \tag{A.3}$$

where $deg(f_1) \leq n/2$ and $deg(f_2) \leq n/2$.

*Proof.* A polynomial $f(x)$ of degree $n = 2m$ which is real and nonnegative can be decomposed into factors of the form.

$$f(x) = \prod_{i=1}^{m}((x - x_i)^2 + y_i^2),$$

where, $x_i$ and $y_i$ are some constants.

If we apply the following identity repeatedly, $f(x)$ can be represented as sum of squares (A.3).

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2) = (a_1 a_2 + b_1 b_2)^2 - (a_1 b_2 + a_2 b_1)^2.$$

□

Since we have to find nonnegative conditions of a real polynomial in the specific frequency intervals, local nonnegativity of a polynomial will be considered next..

**Theorem 17.** *Markov-Lukacs*

*Let $f$ be a polynomial of degree $n$ with real coefficients. Suppose $f(x) \geq 0$ for all $x \in [a, b]$, then one of the following holds.*

(1) *If $deg(f) = n = 2m$ is even, then*

$$f(x) = f_1^2(x) + (x - a)(b - x)f_2^2(x)$$

*where $deg(f_1) \leq m$ and $deg(f_2) \leq m - 1$*

(2) *If $deg(f) = n = 2m + 1$ is odd, then*

$$f(x) = (x - a)f_1^2(x) + (b - x)f_2^2(x)$$

*where $deg(f_1) \leq m$ and $deg(f_2) \leq m$*

Proofs can be found in [80, 100].

## 2. Discrete Polynomial Transforms

Let $p_n(x)$, $n = 0, 1, \ldots$, be a system of orthogonal and normalized polynomials on a bounded or unbounded interval $I \subseteq \mathbf{R}$, with respect to a nonnegative weight function $w(x)$.

$$\int_I p_n(x)p_m(x)w(x)dx = \begin{cases} 0, & \text{n} \neq \text{m} \\ 1, & \text{n=m.} \end{cases}$$

The Chebyshev Polynomials of the first kind use the weight function $(1-x^2)^{-1/2}$ and are orthogonal on the interval $(-1, 1)$, but they are not normalized polynomials on the interval $(-1, 1)$.

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \left\{ \begin{array}{c} 0, \quad n \neq m \\ \pi, \quad n = m = 0 \\ \pi/2, \quad n = m \neq 0 \end{array} \right\}$$

So it is necessary to normalize the polynomials as follows:

$$\begin{aligned} p_0(x) &= \sqrt{\frac{1}{\pi}}\,T_0(x) \\ p_1(x) &= \sqrt{\frac{2}{\pi}}\,T_1(x) \\ p_2(x) &= \sqrt{\frac{2}{\pi}}\,T_2(x) \\ &\vdots \end{aligned}$$

Now, the approximation polynomials $f_r(x, K)$ and $f_i(x, K)$ can be rewritten in terms of $p_i(x)$.

$$\begin{aligned} f_r(x, K) &= \frac{\delta_r(jw, K)}{\mid D_p(jw) \mid^2} \\ &= \begin{bmatrix} wH_i(w) & 0 & |H(jw)|^2 & H_r(w) \end{bmatrix} [K'] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(C_0^1 + k_2 C_0^2 + k_3 C_0^3)T_0(x) + \ldots + (C_N^1 + k_2 C_N^2 + k_3 C_N^3)T_N(x) \\
&= C_0^r(K)T_0(x) + C_1^r(K)T_1(x) + \ldots + C_N^r(K)T_N(x) \\
&= \sqrt{\pi}C_0^r(K)p_0(x) + \sqrt{\frac{\pi}{2}}C_1^r(K)p_1(x) + \ldots + \sqrt{\frac{\pi}{2}}C_N^r(K)p_N(x)
\end{aligned}
$$

$$
\begin{aligned}
f_i(x, K) &= \frac{\delta_r(jw, K)}{\mid D_p(jw) \mid^2} \\
&= \begin{bmatrix} wH_i(w) & 0 & |H(jw)|^2 & H_r(w) \end{bmatrix} [K'] \\
&= \frac{1}{2}(C_0^4 + k_1 C_0^5 + k_3 C_0^6)T_0(x) + \ldots + (C_N^4 + k_1 C_N^5 + k_3 C_N^6)T_N(x) \\
&= C_0^i(K)T_0(x) + C_1^i(K)T_1(x) + \ldots + C_N^i(K)T_N(x) \\
&= \sqrt{\pi}C_0^i(K)p_0(x) + \sqrt{\frac{\pi}{2}}C_1^i(K)p_1(x) + \ldots + \sqrt{\frac{\pi}{2}}C_N^i(K)p_N(x)
\end{aligned}
$$

We define the discrete polynomial transforms $V_{DPT}$ for $f(x) = C_0p_0(x) + C_1p_1(x) + \ldots + C_Np_N(x)$ which offers a way to map the coefficients of a polynomial to its polynomial values.

**Definition 1.** Let $\lambda_0, \lambda_1, \ldots, \lambda_M$ are the roots of $p_{M+1}$. Then we define $V_{DPT}$ as:

$$
V_{DPT} = \begin{bmatrix} p_0(\lambda_0) & p_1(\lambda_0) & \cdots & p_M(\lambda_0) \\ p_0(\lambda_1) & p_1(\lambda_1) & \cdots & p_M(\lambda_1) \\ \vdots & \vdots & & \vdots \\ p_0(\lambda_M) & p_1(\lambda_M) & \cdots & p_M(\lambda_M) \end{bmatrix} \tag{A.4}
$$

Now, suppose $M \geq N$ and let $V$ be the matrix formed by first $N+1$ columns of $V_{DPT}$. and define $W$ which holds $W^TV = I$. Then the linear transformations $VC,\ C = [C_0,\ C_1,\ \ldots, C_N]$ and $W^Ty,\ y = [f(\lambda_0),\ f(\lambda_1)\ , \ldots, \lambda_M]$ map the coefficients of the polynomial

$$f(x) = C_0 p_0(x) + C_1 p_1(x) + \ldots + C_N p_N(x)$$

to $M+1$ values at $\lambda_0, \lambda_1, \cdots, \lambda_M$ and vice-versa.

$$y = VC = \begin{bmatrix} p_0(\lambda_0) & p_1(\lambda_0) & \cdots & p_N(\lambda_0) \\ p_0(\lambda_1) & p_1(\lambda_1) & \cdots & p_N(\lambda_1) \\ \vdots & \vdots & & \vdots \\ p_0(\lambda_M) & p_1(\lambda_M) & \cdots & p_N(\lambda_M) \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_N \end{bmatrix}$$

## 3. Semidefinite Programming

Let $q_1, q_2, \ldots, g_s$ be all monomials of degree $r$ or less.

**Theorem 18.** *A polynomial $f(x)$ of degree $n$ is a sum of squares if and only if there exist a positive semidefinite matrix $X$ and a vector of monomials $g(x)$ containing monomials in $x$ with degree no more than $n/2$ such that*

$$f(x) = g^T(x) X g(x), \quad \textit{for some } X \succeq 0$$

Let $q(x) = [q_1(2)\ q_1(2)\ \ldots] = Lg(x)$. $L$ is a compatible coefficient matrix and $g(x)$ is a vector of monomials. Then

$$f(x) = q^T(x) q(x) = g^T(x) L^T L g(x)$$

and $X = L^T L\ X \succeq 0$. Now suppose there exists $f(x) = g^T(x) X g(x)$. A positive semidefinite matrix $X$ can be represented by eigenvalue decomposition $X = M^T \Lambda M$. Then

$$f(x) = g^T(x) M^T \Lambda M g(x) = \sum_{i=1} \lambda_i (M g(x))_i^2$$

The existence of $X, X \succeq 0$ satisfying $f(x) = g^T(x) X g(x)$ can be expressed as a set of linear equality constraints relating the coefficients of $f(x)$ and the matrix $X$. This implies that the condition that $f(x)$ be nonnegative for $x \in [x_1, x_2]$ becomes the existence problem of $X, X \succeq 0$ satisfying equivalence between a set of linear equalities and the coefficients of the $f(x)$ [81, 55].

**Definition 2.** $A \ o \ B$ denotes the Hadamard product of two matrices A and B of the same dimension, i.e., the matrix with elements $(A \ o \ B)_{ik} = A_{ik}B_{ik}$. The same notation is used for vectors : $(x \ o \ y)_i = x_i y_i$. For real matrices $sqr(A) = A \ o \ A$, for complex matrices $sqr(A) = A \ o \ \bar{A}$, ($\bar{A}$ is complex conjugate of $A$).

**Theorem 19.** *$f(x) \geq 0$ on $[x_1, x_2]$ iff there exist $X_1 \in S^{m_1+1}, X_2 \in S^{m_2+1}$ such that*

$$C(K) = W^T \left[d_1 \ o \ diag(V_1 X_1 {V_1}^T) + d_2 \ o \ diag(V_2 X_2 {V_2}^T)\right], X_1 \succeq 0, X_2 \succeq 0.$$

where, $m_1 = \lfloor N/2 \rfloor$, $m_2 = \lfloor \frac{N-1}{2} \rfloor$, and $V_1$ and $V_2$ are the matrices formed by the first $m_1 + 1$, and $m_2 + 1$, columns of $V_{DPT}$ respectively.
The vectors $d_1, d_2 \in \Re^{N+1}$ are defined as:

$$d_1 = \left\{ \begin{array}{cc} \bar{1}, & N \ is \ even \\ \lambda - x_1\bar{1}, & N \ is \ odd \end{array} \right\}$$

$$d_2 = \left\{ \begin{array}{c} (\lambda - x_1\bar{1}) \ o \ (x_2\bar{1} - \lambda), \quad N \ is \ even \\ x_2\bar{1} - \lambda \quad N \ is \ odd \end{array} \right\}.$$

The notation, $\lfloor z \rfloor$, is the largest integer which does not exceed $z$.

The non-negativeness of $f(x)$ for $x \in [x_1, x_2]$ can be posed as a feasibility problem.

$$\text{Minimize} \quad u$$

$$\text{subject to} \quad \left\{ \begin{array}{c} -u \ \leq \ W^T \left[d_1 \ o \ diag(V_1 X_1 {V_1}^T) + d_2 \ o \ diag(V_2 X_2 {V_2}^T)\right] - C(K) \ \leq \ u \\ X_1 \ \geq \ 0, \quad X_2 \ \geq \ 0 \end{array} \right\}$$

where, $C(K) = [C_0(K),\ C_1(K), \ldots,\ C_N(K)]$

We have to consider a additional condition to satisfy the positiveness or negativeness of the $f(x)$ at specific value ($x = x_3$) of frequency. Finally, this leads to a new feasibility problem combined with the positiveness or negativeness of $f(x)$ in an interval, $x \in\ [x_1, x_2]$

$$\text{Minimize} \quad u$$

$$\text{subject to} \quad \left\{ \begin{array}{c} C_0(K)p_0(x) + C_1(K)p_1(x) + \ldots + C_N(K)p_N(x) - u\ >\ 0, \quad for\ x = x_3 \\ C(K)\ =\ W^T \left[d_1\ o\ diag(V_1X_1{V_1}^T) + d_2\ o\ diag(V_2X_2{V_2}^T)\right] \\ u\ \geq\ 0,\ \ X_1\ \geq\ 0,\ \ X_2\ \geq\ 0 \end{array} \right\}$$

The computer packages SeDuMi [83] and YALMIP [84] were used to obtain a solution.

# VITA

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This document was typeset in LaTeX by Waqar A. Malik.