OPERATOR IDEALS IN LIPSCHITZ AND OPERATOR SPACES CATEGORIES

A Dissertation

by

JAVIER ALEJANDRO CHÁVEZ DOMÍNGUEZ

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2012

Major Subject: Mathematics

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Approved by:

Chair of Committee,	William B. Johnson
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ABSTRACT

Operator Ideals in Lipschitz and Operator Spaces Categories. (August 2012)

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We study analogues, in the Lipschitz and Operator Spaces categories, of several classical ideals of operators between Banach spaces. We introduce the concept of a Banach-space-valued molecule, which is used to develop a duality theory for several nonlinear ideals of operators including the ideal of Lipschitz p-summing operators and the ideal of factorization through a subset of a Hilbert space. We prove metric characterizations of p-convex operators, and also of those with Rademacher type and cotype. Lipschitz versions of p-convex and p-concave operators are also considered. We introduce the ideal of Lipschitz (q,p)-mixing operators, of which we prove several characterizations and give applications. Finally the ideal of completely (q,p)-mixing maps between operator spaces is studied, and several characterizations are given. They are used to prove an operator space version of Pietsch's composition theorem for p-summing operators.

DEDICATION

To all the people who make the Mathematical Olympiads possible, for without them I would have never become a mathematician.

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I would like to thank my advisor, Prof. William B. Johnson, for his support, guidance and encouragement throughout my graduate studies. I have learned a lot from him, and not just about Mathematics and being a mathematician.

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I thank my family for their support from the very beginning of my journey into Mathematics, even though it meant me going far away to school. I also thank my good friends from all stages of my life, who have been there for me in spite of the distance.

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CHAPTER I

INTRODUCTION

Banach spaces have long been a central object in functional analysis due to their versatility. Their structure, possessing both a linear and a metric/topological component, is simple enough to provide for a unified treatment of many analytic situations and yet rich enough to have a whole theory devoted solely to them. As with any other branch of mathematics, the theory of Banach spaces is not only concerned with the objects but also with the morphisms (normally called *operators* in this context) between them. Of course such operators are usually not studied on a one-by-one basis, but are grouped into families that have common characteristics. Many of the most studied such families of operators are in fact operator ideals, in the sense that they are closed under left and right multiplication. The interested reader can find out more about the general theory of operator ideals in Banach spaces in [DF93, Pie80]. In this dissertation, we study generalizations of well-known ideals of operators acting between Banach spaces to other categories. The study of such ideals of operators has historically provided tools for proving a myriad of interesting results whose utility goes beyond just Banach spaces, with applications to many other areas of analysis. For various reasons to be detailed below, it makes sense to try to generalize some of the ideas from Banach space theory to these other contexts. We will concentrate on two such situations: first the nonlinear functional analysis that arises when Banach spaces are replaced with general metric spaces, and then the operator space theory that concerns the case where Banach spaces are replaced with their noncommutative or quantized counterparts.

This dissertation follows the style of Proceedings of the American Mathematical Society.

1.1 Non-linear functional analysis

Ribe's program, laid down by J. Bourgain [Bou86] as a consequence of a theorem of M. Ribe [Rib76], is the pursuit of purely metrical formulations of the notions from the local theory of Banach spaces. Such formulations have been achieved for a number of properties, like superreflexivity, p-convexity, type and cotype [Bou86, LNP09, MN08a, MN08b, MN07]. In addition to the obvious theoretical importance of these results, many applications have been found to subjects like the study of bilipschitz, uniform and coarse embeddings of metric spaces, metric Ramsey theorems, and Lipschitz quotients; some of these applications are particularly interesting due to their connections to theoretical computer science. Even though the local theory of Banach spaces is not only concerned with the spaces but also with the morphisms between them, so far the maps have been largely absent from the literature on Ribe's program. There is a rich interplay between the local properties of Banach spaces and those of the linear maps between them, and the corresponding results for metric spaces are still mostly unexplored.

One of the most important classes of linear maps between Banach spaces is that of p-summing maps, These operators are widely recognized as one of the most important developments in modern Banach space theory, as attested to by the astonishing number of results and applications that can be found, for example, in [DJT95]. Thus, it is not surprising that the first published paper on Ribe's program for maps [FJ09] dealt with their nonlinear generalization. In this paper, J. Farmer and W.B. Johnson [FJ09] define Lipschitz p-summing operators between metric spaces and show that they generalize p-summing operators between Banach spaces. The paper ends with several interesting open problems, whose essence is summarized in the last one: "what results about p-summing operators have analogues for Lipschitz p-summing operators?".

In Chapter II (see also [CD11]) we answer one of the problems raised by Farmer and Johnson. Specifically, we identify the dual of the space of Lipschitz p-summing maps from a finite metric space to a Banach space. The key contribution is the concept that lends its name to the chapter, that is, the introduction of spaces of Banach-space-valued molecules on a metric space. This idea is the vector-valued case of an idea introduced by Arens and Eells [AE56], and it plays the role of a sort of "tensor product" between a metric space and a Banach space. Along the way, we develop a general theory of these spaces of molecules. In the Banach space realm, there is a close relationship between tensor norms and operator ideals (see the book [DF93]). Although there was some initial resistance in the Banach space community to embrace the tensor product approach, the success that came with thinking in terms of both tensor norms and operator ideals (for example, G. Pisier's solution to the problems stated at the end of Grothendieck's "Resumé") earned tensor norms the respect they enjoy today. Some of the classical results carry over to this new setting, for example: (1) There is a natural notion of a reasonable norm, and among the reasonable norms there is a smallest one and a largest one. (2) A "Hilbertian" norm on spaces of molecules can be defined, and it is in duality with the ideal of maps that factor through a subset of Hilbert space.

In Chapter III we continue the work on Ribe's program from the point of view of operators rather than spaces. Our work is heavily based on several existing papers [LNP09, MN08a, MN08b, MN07], where the classical Banach-space notions of p-convexity, Rademacher cotype and Rademacher type are given metric characterizations. These three Banach-space properties have counterparts for linear maps between Banach spaces, and the chapter is dedicated to showing the proofs of their corresponding metric characterizations.

In Chapter IV we continue along similar lines, this time turning our attention to Banach lattices. These are special Banach spaces endowed with an order structure well-related to the norm, and this extra structure often allows us to obtain better results than those available for general Banach spaces. The relationship between this order structure and operators from or into a Banach lattice has of course been extensively studied, and the most important classes of operators in this context, at least from the point of view of their applications to more general Banach space problems, are the p-convex and q-concave ones. In this chapter we develop nonlinear counterparts of these two concepts, considering Lipschitz maps between a metric space and a Banach lattice, and show how some of the elementary results from the theory of p-convex and q-concave operators admit generalizations to the Lipschitz setting.

In Chapter V (see also [CD12]) we close a circle by going back to the study of Lipschitz p-summing operators. There are two theorems that can be considered as the cornerstones of the theory of p-summing operators and are central to their applications. The first one is the factorization/domination theorem of Pietsch [Pie67], whose version for Lipschitz p-summing operators was proved by Farmer and Johnson [FJ09]. The second one is the composition theorem also proved by Pietsch in his seminal paper [Pie67] which, in spirit, says that if we take two p-summing operators and combine them via composition we obtain a "better" operator (just as when two Hilbert-Schmidt operators are composed we get a trace class operator). Such a way to phrase the theorem suggests why it is useful, and moreover this is not the only instance where such an "improvement" of p-summing operators happens. Inspired by ideas of Maurey [Mau74], Pietsch [Pie80, Chap. 20] systematically studied the subject and called such operators (q, p)-mixing. Another exposition of the subject, with a more "tensorial" point of view, can be found in [DF93, Sec. 32]. Since the spaces of molecules play a role somewhat similar to the tensor product, this second point of view was particularly well suited for a study in the Lipschitz case. In this chapter we introduce the natural notion of Lipschitz (q, p)-mixing operators, modeled after Pietsch's analogous definition in the linear case. After proving some basic properties, three different characterizations of Lipschitz (q, p)-mixing operators are presented. The first two are modeled after results in the linear case, and one of them is used in two different applications: first, a different proof of a nonlinear

Grothendieck theorem of Farmer and Johnson [FJ09] is given, followed by an "interpolation style" result relating different Lipschitz (q, p)-mixing constants. This last result is in turn applied in two situations: it is used to show nontrivial reversed inequalities between Lipschitz p-summing norms, and also to give a different proof of the nonlinear extrapolation theorem of D. Chen and B. Zheng [CZ11] (while also improving the constant appearing in their proof). The third characterization relies on the duality theory for Lipschitz p-summing operators developed in Chapter 2.

1.2 Operator spaces

The words 'quantization' and 'noncommutative' seem to get thrown around everywhere these days, and there is a good reason for that. Ever since physicists came to the realization that our old notions of measurement and geometry do not quite correspond to the real world, the development of new mathematical tools that take into account this extra complexity of a quantized or noncommutative world has proved very fruitful. Whenever a mathematical theory gets this treatment, we come across a new theory that not only is mathematically attractive, but it is also naturally well-positioned to have applications to quantum physics. In line with this, operator spaces were developed in the late 80's as the quantized or noncommutative version of Banach spaces, and their study continues to this day.

In Chapter VI (see [CD]), we conduct an study very similar to the one carried out in Chapter V, but this time in the context of operator spaces. In this setting, G. Pisier's completely p-summing maps [Pis98] correspond to the p-summing operators between Banach spaces. A natural modification of the definition yields the notion of completely (q, p)-mixing maps, already introduced by K.L. Yew [Yew08], which is the subject of this chapter. Some basic properties of these maps are proved, as well as a couple of characterizations. A generalization of Yew's operator space version of the Extrapolation Theorem is obtained, via an interpolation-style theorem relating different completely (q, p)-mixing norms. Finally some composition theorems for completely p-summing maps are proved, including an operator space version of Pietsch's composition theorem.

CHAPTER II

BANACH-SPACE-VALUED MOLECULES*

2.1 Introduction

The origin of the work presented in this chapter was the following question: given a normed space of Lipschitz maps from a metric space X into a Banach space E (e.g. Lipschitz p-summing operators as in [FJ09]) how can one identify its dual?

A natural starting point would be to try to identify the dual of $X^{\#}$, the space of Lipschitz functions from X to $\mathbb{R}$ that vanish at a specified point with the Lipschitz norm. Unfortunately, duals of spaces of Lipschitz functions are known to be rather large and somewhat pathological — e.g. in [Bou86] it is shown that $(\ell_1)^{\#}$ does not have finite cotype, and it is still unknown whether $([0,1]\times[0,1])^{\#}$ has finite cotype —, so ours would appear to be a futile task.

We may, however, flip the table and get back into a workable situation: the space $X^{\#}$ is known to be a dual Banach space (and is sometimes even called the Lipschitz dual of X), so we embark on the slightly different (but related) quest of finding preduals of some spaces of Lipschitz maps from a metric space X into a dual Banach space E^*.

The key element in our work is the introduction of the concept of a Banach-space-valued molecule, a generalization of the concept used by R. Arens and J. Eells [AE56] to construct a predual of $X^{\#}$. Despite the fact that the Arens-Eells space has been used repeatedly in the literature (e.g. [GK03, Kal04]), and Banach-space valued versions of it have been considered (as in [Joh70]), as far as the author knows the idea of Banach-space-valued molecules had escaped attention so far.

Let us fix the notation that will be used throughout the chapter. X, Y, Z will always denote metric spaces, whereas E, F, G will denote real Banach spaces. We use the convention of having *pointed* metric spaces, i.e. with a designated special point always denoted by 0. As customary, B_E denotes the closed unit ball of E and E^* its linear dual, and $\mathcal{L}(E,F)$ is the space of bounded linear maps from E to F. We use the symbol $\equiv$ to indicate that two Banach spaces are isometrically isomorphic. $\mathrm{Lip}_0(X,E)$ is the Banach space of Lipschitz functions $T : X \to E$ such that $T(0) = 0$ with pointwise addition and the Lipschitz norm. As the reader will recall from the previous paragraphs, we use the shorthand $X^\# := \mathrm{Lip}_0(X,\mathbb{R})$. The letters p, r, s will designate elements of $[1,\infty]$, and p' denotes the exponent conjugate to p (i.e. the one that satisfies $1/p + 1/p' = 1$).

2.2 Banach-space-valued molecules on a metric space

We start by recalling the definition and basic properties of the space of Arens and Eells [AE56]. We follow the presentation in [Wea99].

A *molecule* on a metric space X is a finitely supported function $m : X \to \mathbb{R}$ such that $\sum_{x\in X} m(x) = 0$. For $x, x' \in X$ we denote by $m_{xx'}$ the molecule $\chi_{\{x\}} - \chi_{\{x'\}}$. The simplest molecules, i.e. those of the form $am_{xx'}$ with $x, x' \in X$ and a a real number are called *atoms*. It is easy to show that every molecule can be expressed as a sum of atoms (for instance, by induction on the cardinality of the support of the molecule). The Arens-Eells space of X, denoted Æ(X), is the completion of the space of molecules with the norm

$$\|m\|_{\text{Æ}} := \inf\left\{ \sum_{j=1}^{n} |a_j| d(x_j, x_j') \ : \ m = \sum_{j=1}^{n} a_j m_{x_j x_j'} \right\}. \tag{2.2.1}$$

The fundamental properties of the Arens-Eells space are summarized in the following theorem [AE56], [Wea99, pp. 39-41].

Theorem 2.2.1. *(i)* $\|\cdot\|_{Æ}$ *is a norm on the vector space of molecules on* X*.*

(ii) The dual of $Æ(X)$ *is (canonically) isometrically isomorphic to* $X^{\#}$*. Moreover, on bounded subsets of* $X^{\#}$ *the weak* topology coincides with the topology of pointwise convergence.*

(iii) The map $\iota : x \mapsto m_{x0}$ *is an isometric embedding of* X *into* $Æ(X)$*. Moreover, for any Banach space* E *and any Lipschitz map* $T : X \to E$ *with* $T(0) = 0$ *there is a unique linear map* $\widehat{T} : Æ(X) \to E$ *such that* $\widehat{T} \circ \iota = T$*. Furthermore,* $\|\widehat{T}\| = \mathrm{Lip}(T)$*.*

Because of the universal property (iii), the space $Æ(X)$ is sometimes called the free Lipschitz space of X and denoted $\mathscr{F}(X)$ (see [GK03, Kal04]). From that point of view, it is natural to think of the space $Æ(X)$ as the closure in $(X^{\#})^*$ of the linear span of the point evaluations $\delta_x : f \mapsto f(x)$, for $x \in X$ and $f \in X^{\#}$. Such an approach was used by J. Johnson [Joh70] to show that $\mathrm{Lip}_0(X, E^*)$ is always a dual space, without any reference to molecules. Our Theorem 2.6.4 recovers Johnson's result as a particular case of duality for Lipschitz p-summing operators.

In the spirit of Arens and Eells' original formulation [AE56], define an *E-valued molecule* on X to be a finitely supported function $m : X \to E$ such that $\sum_{x\in X} m(x) = 0$. The vector space of all E-valued molecules on X is denoted by $\mathcal{M}(X, E)$. An *E-valued atom* is a function of the form $vm_{xx'}$ with $v \in E$, $x, x' \in X$. Atoms are the building blocks of the space of molecules in the same sense that elementary tensors are the building blocks of the tensor product: every molecule is a sum of atoms. This is proved by induction on the cardinality of the support of the molecule as follows. It is clear if the support has cardinality 0 or 2 (1 is clearly impossible), so now suppose the result holds for molecules with support of size at most n, and let $m : X \to E$ be a molecule with support $\{x_0, x_1, \ldots, x_n\} \subset X$. Note that $\tilde{m} = m - \sum_{j=1}^{n} \frac{1}{n} m(x_0) m_{x_0 x_j}$ is a molecule with support of size at most n (since $\tilde{m}(x_0) = 0$), so $\tilde{m}$ is a sum of atoms and therefore clearly so is m.

Define a pairing $\langle \cdot, \cdot \rangle$ of $\mathrm{Lip}_0(X, E^*)$ and $\mathcal{M}(X, E)$ by

$$\langle T, m \rangle = \sum_{x \in X} \langle T(x), m(x) \rangle, \qquad \text{for } m \in \mathcal{M}(X, E), \quad T \in \mathrm{Lip}_0(X, E^*).$$

Note that this sum makes sense because m is finitely supported, and clearly $\langle \cdot, \cdot \rangle$ is bilinear. For an atom $m = v m_{x'y'}$ and $T \in \mathrm{Lip}_0(X, E^*)$,

$$\begin{aligned}\langle T, m \rangle &= \sum_{x \in X} \langle T(x), v m_{x'y'}(x) \rangle \\ &= \langle T(x'), v m_{x'y'}(x') \rangle + \langle T(y'), v m_{x'y'}(y') \rangle = \langle T(x') - T(y'), v \rangle.\end{aligned}$$

Therefore, for a general molecule $m = \sum_j v_j m_{x_j x'_j}$,

$$\langle T, m \rangle = \sum_j \langle T x_j - T x'_j, v_j \rangle. \tag{2.2.2}$$

Spaces of Banach-space-valued molecules will play a role similar to that of tensor products of Banach spaces in our investigations about duality. We will define several norms on the spaces of molecules that correspond to norms defined on tensor products of Banach spaces, and obtain similar duality results. We start with the analogues of the most basic tensor norms, the projective and injective ones.

2.3 The projective norm

The projective norm is considered the simplest way to endow the tensor product of Banach spaces with a norm. Just as the algebraic tensor product linearizes bilinear mappings, the projective tensor product linearizes bounded bilinear mappings. The following defines a norm on spaces of molecules that is analogous to the projective norm for the tensor product of Banach spaces, and we will also call it projective.

Definition 2.3.1. For $m \in \mathcal{M}(X,E)$ we define its *projective norm* by

$$\|m\|_\pi = \inf\left\{ \sum_{j=1}^{n} d(a_j,b_j)\,\|v_j\| : m = \sum_{j=1}^{n} v_j m_{a_j b_j} \right\}.$$

Lemma 2.3.2. *$\|\cdot\|_\pi$ is a norm on $\mathcal{M}(X,E)$.*

Proof. It is clear that for any molecule $m \in \mathcal{M}(X,E)$ and any scalar λ, $\|m\|_\pi \geq 0$ and $\|\lambda m\|_\pi = |\lambda|\,\|m\|_\pi$.

Now let $m_1, m_2 \in \mathcal{M}(X,E)$ and $\varepsilon > 0$. We can choose a representation $m_1 = \sum_{j=1}^{n} v_j m_{a_j b_j}$ such that

$$\sum_{j=1}^{n} d(a_j,b_j)\,\|v_j\| \leq \|m_1\|_\pi + \varepsilon.$$

Similarly, choose a representation $m_2 = \sum_{j=n+1}^{n+k} v_j m_{a_j b_j}$ such that

$$\sum_{j=n+1}^{n+k} d(a_j,b_j)\,\|v_j\| \leq \|m_2\|_\pi + \varepsilon.$$

Therefore, $m_1 + m_2 = \sum_{j=1}^{n+k} v_j m_{a_j b_j}$ and

$$\sum_{j=1}^{n+k} d(a_j,b_j)\,\|v_j\| \leq \|m_1\|_\pi + \|m_2\|_\pi + 2\varepsilon,$$

so $\|m_1 + m_2\|_\pi \leq + \|m_1\|_\pi + \|m_2\|_\pi + 2\varepsilon$, and by letting $\varepsilon \downarrow 0$ we have the triangle inequality for $\|\cdot\|_\pi$.

Let $T \in \mathrm{Lip}_0(X,E^*)$ be a map that admits a representation as a finite sum of the form $\sum_k v_k^* f_k$ with $(v_k^*)_k \subset E^*$, $(f_k)_k \subset X^\#$ (i.e. such that the linearization $\hat{T} : Æ(X) \to E^*$ has finite rank). For such a T, set

$$\theta(T) = \inf\left\{ \sum_k \|v_k^*\|\,\mathrm{Lip}(f_k) \right\}$$

where the infimum is taken over all representations as above. Now, given $m = \sum_j v_j m_{x_j x'_j} \in \mathcal{M}(X, E)$, we have from the pairing formula (2.2.2) and the triangle inequality

$$\begin{aligned}\left|\langle T, m\rangle\right| = \left|\sum_{j,k} v_k^*(v_j)\big[f_k(x_j) - f_k(x'_j)\big]\right| &\leq \sum_{j,k}\left|v_k^*(v_j)\big[f_k(x_j) - f_k(x'_j)\big]\right| \\ &\leq \sum_{j,k} \|v_k^*\| \cdot \|v_j\| \cdot \operatorname{Lip}(f_k) \cdot d(x_j, x'_j) \\ &\leq \Big(\sum_k \|v_k^*\| \operatorname{Lip}(f_k)\Big) \cdot \Big(\sum_j \|v_j\|\, d(x_j, x'_j)\Big).\end{aligned}$$

Taking the infimum over all representations of both T and m, we deduce $\left|\langle T, m\rangle\right| \leq \|m\|_\pi\, \theta(T)$. In particular, this applies to maps T of the form $v^* \circ f$ with $v^* \in E^*$ and $f \in X^\#$, so if m is such that $\|m\|_\pi = 0$ then we have, using the pairing formula (2.2.2),

$$0 = \langle v^* \circ f, m\rangle = \sum_j v^*(v_j)\big[f(x_j) - f(x'_j)\big] \text{ for all } v^* \in E^*, f \in X^\#.$$

By the duality between Æ(X) and $X^\#$ (see Theorem 2.2.1), this means that the real-valued molecule $v^* \circ m$ is equal to 0 for all $v^* \in E^*$ and consequently $m = 0$. □

We'll denote by $\mathcal{M}_\pi(X, E)$ the normed space $(\mathcal{M}(X, E), \|\cdot\|_\pi)$, and by $\widehat{\mathcal{M}}_\pi(X, E)$ its completion.

2.3.1 The dual norm

We now identify the norm dual to the projective norm on a space of molecules. This is clearly a generalization of the duality result of Arens and Eells [AE56] already stated in Theorem 2.2.1.

Proposition 2.3.3. *The pairing $\langle \cdot, \cdot \rangle$ induces an isometric isomorphism between $\mathcal{M}_\pi(X, E)^*$ and $\operatorname{Lip}_0(X, E^*)$.*

Proof. Define $T_1 : \mathcal{M}_\pi(X,E)^* \to \mathrm{Lip}_0(X,E^*)$ by $\langle (T_1\phi)(a), v\rangle = \phi(vm_{a0})$ for $\phi \in \mathcal{M}_\pi(X,E)^*$, $a \in X$ and $v \in E$. Since clearly $\|vm_{ab}\|_\pi \le d(a,b)\,\|v\|$, we have

$$|\langle (T_1\phi)(a) - (T_1\phi)(b), v\rangle| = |\phi(vm_{a0} - vm_{b0})|$$
$$= |\phi(vm_{ab})| \le \|\phi\|\, d(a,b)\,\|v\| \le \|\phi\|\, d(a,b),$$

so taking the supremum over $v \in B_E$ we get that $\mathrm{Lip}(T_1\phi) \le \|\phi\|$. Also $(T_1\phi)(0) = 0$, so indeed $T_1\phi \in \mathrm{Lip}_0(X,E^*)$. We conclude that T_1 is a nonexpansive linear map from $\mathcal{M}_\pi(X,E)^*$ to $\mathrm{Lip}_0(X,E^*)$.

Now define $T_2 : \mathrm{Lip}_0(X,E^*) \to \mathcal{M}_\pi(X,E)^*$ by $(T_2 f)(m) = \sum_{a\in X} \langle f(a), m(a)\rangle$ for $f \in \mathrm{Lip}_0(X,E^*)$ and $m \in \mathcal{M}(X,E)$. Observe that if $m = \sum_{j=1}^n v_j m_{a_j b_j}$ then

$$|(T_2 f)(m)| = \Big|(T_2 f)\Big(\sum_{j=1}^n v_j m_{a_j b_j}\Big)\Big| = \Big|\sum_{j=1}^n \langle f(a_j) - f(b_j), v_j\rangle\Big|$$
$$\le \sum_{j=1}^n |f(a_j) - f(b_j)|\,\|v_j\| \le \mathrm{Lip}(f) \sum_{j=1}^n d(a_j,b_j)\,\|v_j\|\,.$$

Taking the infimum over all such representations for m yields $|(T_2 f)(m)| \le \mathrm{Lip}(f)\,\|m\|_\pi$. Thus $f \in \mathcal{M}_\pi(X,E)^*$ and $\|T_2 f\| \le \mathrm{Lip}(f)$, so T_2 is a nonexpansive linear map from $\mathrm{Lip}_0(X,E^*)$ to $\mathcal{M}_\pi(X,E)^*$. Finally, a straightforward calculation shows that T_1 and T_2 are inverses, so that $\mathcal{M}_\pi(X,E)^* \equiv \mathrm{Lip}_0(X,E^*)$. □

2.3.2 Other properties

Many of the properties of the projective norm for molecules will be reminiscent of the ones for the projective tensor product of Banach spaces. We begin with a calculation showing that the projective norm of an atom is what one would expect it to be.

Proposition 2.3.4. *For $v \in E$, $a, b \in X$, $\|vm_{ab}\|_\pi = \|v\|\, d(a,b)$.*

Proof. On one hand, it is clear from the definition that $\|vm_{ab}\|_\pi \leq \|v\|\, d(a,b)$. By the Hahn-Banach theorem (linear and metric versions) there exist $v^* \in B_{E^*}$ and $f \in B_{X^\#}$ such that $\|v\| = \langle v^*, v\rangle$ and $f(a) - f(b) = d(a,b)$. Consider $T \in \mathrm{Lip}_0(X, E^*)$ given by $x \mapsto f(x)v^*$. Clearly, $\mathrm{Lip}(T) \leq 1$ and

$$\|vm_{ab}\|_\pi \geq \langle T, vm_{ab}\rangle = \langle v^*, v\rangle \big[f(a) - f(b)\big] = \|v\|\, d(a,b).$$

□

It turns out that there is a very close relationship between the projective norm for molecules and the projective tensor product of Banach spaces. In fact, the projective norm on E-valued molecules on a metric space X can be identified with the projective tensor norm between the free Banach space of X and E. The author wishes to thank Richard Haydon for suggesting that this might be true.

Theorem 2.3.5. *For a metric space X and a Banach space E, $\mathcal{M}_\pi(X,E) \equiv \mathscr{F}(X) \otimes_\pi E$.*

Proof. Define $\varphi : \mathcal{M}_\pi(X,E) \to \mathscr{F}(X) \otimes_\pi E$ by

$$\varphi\Big(\sum_j v_j m_{x_j x'_j}\Big) = \sum_j m_{x_j x'_j} \otimes v_j.$$

First of all, let us note that φ is well-defined. If $\sum_j v_j m_{x_j x'_j} = \sum_i w_i m_{y_i y'_i}$, then the duality result in Proposition 2.3.3 implies that for all $T \in \mathrm{Lip}_0(X, E^*)$,

$$\sum_j \langle Tx_j - Tx'_j, v_j\rangle = \sum_i \langle Ty_i - Ty'_i, w_i\rangle.$$

Since each Lipschitz map $X \to E^*$ extends to a linear map $\mathscr{F}(X) \to E^*$, we have that for any $\widehat{T} \in L(\mathscr{F}(X), E^*)$

$$\sum_j \langle \widehat{T}(m_{x_j x'_j}), v_j\rangle = \sum_i \langle \widehat{T}(m_{y_i y'_i}), w_i\rangle.$$

Recall from the linear theory that $(\mathscr{F}(X) \otimes_\pi E)^* \equiv L(\mathscr{F}(X), E^*)$, so this means that

$$\sum_j m_{x_j x'_j} \otimes v_j = \sum_i m_{y_i y'_i} \otimes w_i.$$

Hence, φ is well-defined.

Now we show that it is continuous. For $m = \sum_j v_j m_{x_j x'_j}$,

$$\|\varphi(m)\| = \left\| \sum_j m_{x_j x'_j} \otimes v_j \right\| \leq \sum_j \left\| m_{x_j x'_j} \right\|_{\mathscr{F}(X)} \cdot \|v_j\| = \sum_j d(x_j, x'_j) \|v_j\| .$$

Taking the infimum over all representations of m, $\|\varphi(m)\| \leq \|m\|$.

From arguments similar to those above, it is clear that φ has an inverse given by

$$\varphi^{-1}\Big(\sum_j m_{x_j x'_j} \otimes v_j \Big) = \sum_j v_j m_{x_j x'_j}.$$

(It should be remarked that every element of $\mathscr{F}(X) \otimes E$ can be written in the form $\sum_j m_{x_j x'_j} \otimes v_j$). Let $w \in \mathscr{F}(X) \otimes_\pi E$, and let $\varepsilon > 0$. By the definition of the projective tensor product, there exist $m^{(i)} \in \mathscr{F}(X)$ and $v_i \in E$ such that $w = \sum_i m^{(i)} \otimes v_i$ and

$$\|w\| (1 + \varepsilon) \geq \sum_i \left\| m^{(i)} \right\| \cdot \|v_i\| .$$

For each i, find a representation $m^{(i)} = \sum_j a_j^{(i)} m_{x_j^{(i)} y_j^{(i)}}$ such that

$$\left\| m^{(i)} \right\| (1 + \varepsilon) \geq \sum_j |a_j^{(i)}| d(x_j^{(i)}, y_j^{(i)}).$$

Putting everything together,

$$\|w\| (1 + \varepsilon)^2 \geq \sum_{ij} |a_j^{(i)}| d(x_j^{(i)}, y_j^{(i)}) \geq \left\| \varphi^{-1}(w) \right\| .$$

Letting ε go to 0, we see that $\|\varphi^{-1}\| \leq 1$. $\square$

Theorem 2.3.5 implies that, given a metric space X, there is a Banach space A such that $\widehat{\mathcal{M}}_\pi(X,E) \equiv A\widehat{\otimes}_\pi E$ for every Banach space E. The author would like to thank Jesús Castillo for pointing out a result in categorical Banach space theory that shows this was to be expected. Without going into all the details, let us outline the argument. First, a theorem of Fuks [Fuk66, Sec. 6] (a nice presentation can be found in [Cas10, Prop. 5.6]) states the following: if $\mathcal{F}$, $\mathcal{G}$ are two covariant Banach functors such that for any Banach spaces E, F we have

$$\mathcal{L}\big(\mathcal{F}(E), F\big) \equiv \mathcal{L}\big(E, \mathcal{G}(F)\big),$$

then there exists a Banach space A such that for every Banach space E, $\mathcal{F}(E) \equiv A\widehat{\otimes}_\pi E$ and $\mathcal{G}(E) \equiv \mathcal{L}(A,F)$. Now consider a fixed metric space X. Note it induces two covariant Banach functors $\widehat{\mathcal{M}}_\pi(X,\cdot)$ and $\mathrm{Lip}_0(X,\cdot)$. Arguments closely related to those in the proof of Theorem 2.3.3 show that for any Banach spaces E and F we have

$$\mathcal{L}\big(\widehat{\mathcal{M}}_\pi(X,E), F\big) \equiv \mathcal{L}\big(E, \mathrm{Lip}_0(X,F)\big),$$

so Fuks' result applies.

This relationship between the projective norm for molecules and projective tensor products allows us to obtain several results similar to those in the linear case. We follow closely the presentation of [Rya02, Sec. 2.1] Let us start with what could be called "projective tensor products of operators". Compare to [Rya02, Prop. 2.3].

Proposition 2.3.6. *Let $S : X \to Z$ be a Lipschitz map mapping 0 to 0, and $T : E \to F$ a bounded linear map. Then there is a unique operator $S \boxtimes_\pi T : \mathcal{M}_\pi(X,E) \to \mathcal{M}_\pi(Z,F)$ such that*

$$(S \boxtimes_\pi T)(vm_{xy}) = (Tv)m_{(Sx)(Sy)}, \qquad \text{for all } v \in E, x,y \in X.$$

Furthermore, $\|S \boxtimes_\pi T\| = \mathrm{Lip}(S)\,\|T\|$.

Proof. Since every molecule can be expressed as a finite sum of atoms, it is clear that if such an operator exists it must be unique. Therefore, all we need to do is show that such an operator is well-defined. For that, let us consider the linear operator $\widehat{S} \otimes T : \mathscr{F}(X) \otimes_\pi E \to \mathscr{F}(Z) \otimes_\pi F$ (see [Rya02, Prop. 2.3]). Applying it to an atom $vm_{x,y}$ with $v \in E$, $x, y \in X$, we get

$$(\widehat{S} \otimes T)(m_{xy} \otimes v) = m_{(Sx)(Sy)} \otimes (Tv)$$

which corresponds to $(Tv)m_{(Sx)(Sy)}$ under the canonical identification between $\mathscr{F}(Z) \otimes_\pi F$ and $\mathcal{M}_\pi(Z, F)$.

Let us now calculate the norm of $S \boxtimes_\pi T$. For any $m = \sum_j v_j m_{x_j x'_j}$ in $\mathcal{M}(X, E)$,

$$\|(S \boxtimes_\pi T)(m)\|_\pi \leq \sum_j \|Tv_j\| \, d(Sx_j, Sx'_j) \leq \mathrm{Lip}(S) \, \|T\| \sum_j \|v_j\| \, d(x_j, x'_j).$$

Taking the infimum over all representations of m we conclude that $\|S \boxtimes_\pi T\| \leq \mathrm{Lip}(S) \, \|T\|$. For the other inequality, it suffices to consider what happens to atoms and use Proposition 2.3.4. □

The choice of the word *projective*, besides being in accordance with the usage in the linear case, is justified by the following result which explains in what sense the norm is actually projective. Before stating it, recall that a linear operator $T : E \to F$ is a 1-*linear quotient* if it is surjective and $\|w\| = \inf \big\{ \|v\| \ : \ v \in E, \ Tv = w \big\}$ for every $w \in F$. On the other hand, a Lipschitz map $S : X \to Z$ is called C-*co-Lipschitz* if for every $x \in X$ and $r > 0$, $f\big(B(x, r)\big) \supseteq B\big(f(x), r/C\big)$. Moreover, it is called a *Lipschitz quotient* if it is surjective, Lipschitz and co-Lipschitz.

Theorem 2.3.7. *Let $S : X \to Z$ be a Lipschitz quotient with Lipschitz and co-Lipschitz constants equal to 1 and mapping 0 to 0, and let $T : E \to F$ be a linear quotient map. Then $S \boxtimes_\pi T : \widehat{\mathcal{M}}_\pi(X, E) \to \widehat{\mathcal{M}}_\pi(Z, F)$ is also a linear quotient map.*

Proof. From the behavior of the projective tensor norm with respect to quotients [Rya02, Prop. 2.5], it suffices to notice that if $S : X \to Z$ is a Lipschitz quotient with Lipschitz and co-Lipschitz constants equal to 1, then the induced map $\widehat{S} : \mathscr{F}(X) \to \mathscr{F}(Z)$ is a linear quotient. It is clear that $\widehat{S}$ is surjective, and we know that $\|\widehat{S}\| = \mathrm{Lip}(S) = 1$. Now let $m \in \mathscr{F}(Z)$, and let $\varepsilon > 0$. There exists a representation $m = \sum_{j=1}^{n} a_j m_{z_j z_j'}$ such that $\|m\|(1+\varepsilon) \geq \sum_{j=1}^{n} |a_j| d(z_j, z_j')$. For each j, choose $x_j, x_j' \in X$ such that $Sx_j = z_j$, $Sx_j' = z_j'$ and $d(x_j, x_j') \leq (1+\varepsilon) d(z_j, z_j')$. Setting $m' = \sum_{j=1}^{n} a_j m_{x_j x_j'}$, clearly $\widehat{S}(m') = m$ and

$$\|m'\| \leq \sum_{j=1}^{n} |a_j| d(x_j, x_j') \leq (1+\varepsilon) \sum_{j=1}^{n} |a_j| d(z_j, z_j') \leq (1+\varepsilon)^2 \|m\| .$$

Since this holds for all $\varepsilon > 0$, it follows that

$$\|m\| = \inf\{\|m'\| \ : \ \widehat{S}(m') = m\}.$$

□

In a similar manner, the projective norm respects complemented subspaces.

Theorem 2.3.8. *Let Z be a Lipschitz retract of X, and let F be a complemented subspace of E. Then $\mathcal{M}_\pi(Z, F)$ is complemented in $\mathcal{M}_\pi(X, E)$ and the norm on $\mathcal{M}_\pi(Z, F)$ induced by the projective norm of $\mathcal{M}_\pi(X, E)$ is equivalent to the projective norm on $\mathcal{M}_\pi(Z, F)$. If Z is Lipschitz complemented with a projection of Lipschitz constant one and F is complemented by a projection of norm one, then $\mathcal{M}_\pi(Z, F)$ is a subspace of $\mathcal{M}_\pi(X, E)$ and is also complemented by a projection of norm one.*

Proof. This follows from the corresponding result for the projective tensor product [Rya02, Prop. 2.4], after noting that a Lipschitz retraction $r : X \to Z$ extends to a linear projection $\widehat{r} : \mathscr{F}(X) \to \mathscr{F}(Z) \subset \mathscr{F}(X)$ with $\|\widehat{r}\| = \mathrm{Lip}(r)$. □

Calculating the projective norm of a tensor in a tensor product of Banach spaces is generally difficult, but there is a particular case where the calculation is relatively easy: for any Banach space E, $\ell_1 \otimes_\pi E \equiv \ell_1(E)$ [Rya02, Ex. 2.6]. In the nonlinear setting trees play a role analogous to that of ℓ_1 in the linear theory, so the following result is not surprising.

Proposition 2.3.9. *Let $T = (X, \mathcal{E})$ be a graph-theoretic tree, with vertex set X and edge set $\mathcal{E}$, G a Banach space. Then $\mathcal{M}_\pi(T, G)$ is isometric to $\ell_1(\mathcal{E}, G)$.*

Proof. Fix an arbitrary vertex y_0 in X. We say that a vertex is positive (resp. negative) if it is at an even (resp. odd) distance from y_0. Note that, since T is a tree, the endpoints of every edge in $\mathcal{E}$ have different parities. Every edge $\{x, y\}$ in $\mathcal{E}$ will be written as (x, y) with x negative and y positive. For a vertex x, denote by $\delta(x)$ its degree in the directed graph so obtained.

Define a map $j : \mathcal{M}_\pi(X, G) \to \ell_1(\mathcal{E}, G)$ given by

$$m \mapsto \left(\frac{m(y)}{\delta(y)} - \frac{m(x)}{\delta(x)} \right)_{(x,y)\in\mathcal{E}}$$

Note that for a vertex $y_0 \in V$,

$$\Big[\sum_{(x,y)\in\mathcal{E}} \Big(\frac{m(y)}{\delta(y)} - \frac{m(x)}{\delta(x)} \Big) m_{yx} \Big](y_0)$$
$$= \text{indegree}(y_0) \frac{m(y_0)}{\delta(y_0)} - \text{outdegree}(y_0) \frac{m(y_0)}{\delta(y_0)} = m(y_0),$$

since by definition of the orientation, either $\delta(y_0) = \text{indegree}(y_0)$ and $\text{outdegree}(y_0) = 0$, or $\delta(y_0) = -\text{outdegree}(y_0)$ and $\text{indegree}(y_0) = 0$. Therefore,

$$m = \sum_{(x,y)\in\mathcal{E}} \Big(\frac{m(y)}{\delta(y)} - \frac{m(x)}{\delta(x)} \Big) m_{yx} \tag{2.3.1}$$

and thus

$$\|m\|_\pi \leq \sum_{(x,y)\in\mathcal{E}} \left\| \frac{m(y)}{\delta(y)} - \frac{m(x)}{\delta(x)} \right\| d(y,x) = \|j(m)\|_{\ell_1(\mathcal{E},G)} .$$

Now, consider $x, y \in X$. Let $n = d(x,y)$ and $\{x = z_0, z_1, \ldots, z_n = y\}$ be the unique minimal-length path joining x and y. Since

$$\|v\| \, d(x,y) = \sum_{i=1}^{n} \|v\| \, d(z_i, z_{i-1}),$$

in order to calculate $\|m\|_\pi$ it suffices to consider only representations with molecules of the form m_{yx} with $(x,y) \in \mathcal{E}$. By the triangle inequality, in the representation we can consolidate all terms corresponding to the same elementary molecule m_{yx}, so we can consider only representations of the form

$$m = \sum_{(x,y)\in\mathcal{E}} v_{yx} m_{yx}.$$

But there is only one such representation (easily seen by induction on the size of the tree), the one given by (2.3.1), so $\|m\|_\pi \geq \|j(m)\|_{\ell_1(\mathcal{E},G)}$. □

More generally, in the linear case we have that $L_1(\mu) \otimes_\pi E \equiv L_1(\mu; E)$ for any measure μ [Rya02, Ex. 2.19]. In our nonlinear setting, a possible analogue will be given by a generalization of Proposition 2.3.9 to a more general class of metric trees. This will depend heavily on the identification of the free Lipschitz space of such trees carried out in [God10]. Before stating the result, let us recall a definition. An $\mathbb{R}$-tree is a metric space X satisfying the following two conditions: (1) For any points a and b in X, there exists a unique isometry ϕ of the closed interval $[0, d(a,b)]$ into X such that $\phi(0) = a$ and $\phi(d(a,b)) = b$; (2) Any one-to-one continuous mapping $\varphi : [0,1] \to X$ has the same range as the isometry ϕ associated to the points $a = \varphi(0)$ and $b = \varphi(1)$.

Corollary 2.3.10. *Let X be an $\mathbb{R}$-tree and E a Banach space. Then there exists a measure μ such that $\mathcal{M}_\pi(X,E)$ is isometric to $L_1(\mu;E)$.*

Proof. By [God10, Cor. 3.3], there exists a measure μ such that $\mathscr{F}(X)$ is isometric to $L_1(\mu)$. From Theorem 2.3.5, $\mathcal{M}_\pi(X,E)$ is isometric to $\mathscr{F}(X)\otimes_\pi E$. Finally, from [Rya02, Ex. 2.19] $L_1(\mu)\otimes_\pi E$ is isometric to $L_1(\mu;E)$. □

2.4 The injective norm

In a sense, the injective norm is the opposite of the projective one in the context of tensor product of Banach spaces. The following defines a norm analogous to the injective norm for the tensor product of Banach spaces.

Definition 2.4.1. For $m \in \mathcal{M}(X,E)$ we define its *injective norm* by

$$\|m\|_\varepsilon = \sup\left\{ \sum_{j=1}^n \big[f(a_j)-f(b_j)\big] v^*(v_j) : m = \sum_{j=1}^n v_j m_{a_j b_j}, f\in B_{X^\#}, v^*\in B_{E^*}\right\}.$$

Note that the injective norm is given by an obvious embedding into $C(B_{X^\#}\times B_{E^*})$. Moreover, the duality $\mathscr{F}(X)^*\equiv X^\#$ makes it clear that this injective norm for E-valued molecules on X is nothing but the injective tensor product of $\mathscr{F}(X)$ and E. We will denote by $\mathcal{M}_\varepsilon(X,E)$ the normed space $(\mathcal{M}(X,E),\|\cdot\|_\varepsilon)$ and by $\widehat{\mathcal{M}_\varepsilon}(X,E)$ its completion.

2.4.1 The dual norm

The identification of $\mathcal{M}_\varepsilon(X,E)$ with $\mathscr{F}(X)\otimes_\varepsilon E$ makes it easy to figure out its dual space. From [Rya02, Prop. 3.14 and Prop. 3.22] $\mathcal{M}_\varepsilon(X,E)$ can be readily identified with $\mathcal{I}_1(\mathscr{F}(X),E^*)$, the space of (linear) 1-integral operators from $\mathscr{F}(X)$ to E^*.

2.4.2 Other properties

As with the projective norm, the injective norm of an atom is precisely what one would expect.

Proposition 2.4.2. *For any $v \in E$ and $a, b \in X$, $\|vm_{ab}\|_\varepsilon = \|v\| \, d(a,b)$.*

Proof. The quantity $\sum_{j=1}^n \big[f(a_j) - f(b_j)\big] v^*(v_j)$ does not, in fact, depend of the representation $m = \sum_{j=1}^n v_j m_{a_j b_j}$, only on m, v^* and f. Thus,

$$\|vm_{ab}\|_\varepsilon = \sup\left\{ v^*(v)[f(a) - f(b)] : v^* \in B_{E^*}, f \in B_{X^\#} \right\} = \|v\| \, d(a,b).$$

□

As in the projective case, the relationship between the injective norm for molecules and injective tensor products allows us to obtain several results similar to those in the linear case. This time we follow closely the presentation of [Rya02, Sec. 3.1] Let us start with “injective tensor products of operators”.

Proposition 2.4.3. *Let $S : X \to Z$ be a Lipschitz map mapping 0 to 0, and $T : E \to F$ a bounded linear map. Then there is a unique operator $S \boxtimes_\varepsilon T : \mathcal{M}_\varepsilon(X,E) \to \mathcal{M}_\varepsilon(Z,F)$ such that*

$$(S \boxtimes_\varepsilon T)(vm_{xy}) = (Tv) m_{(Sx)(Sy)}, \qquad \text{for all } v \in E, x, y \in X.$$

Furthermore, $\|S \boxtimes_\varepsilon T\| = \mathrm{Lip}(S) \, \|T\|$.

Proof. The result follows from the linear result [Rya02, Prop. 3.2] together with arguments very similar to those of Proposition 2.3.6. □

From the linear results, it follows that the injective norm for molecules respects inclusions and in general does not respect quotients. One of the few cases where it is easy to calculate an injective tensor product is when one of the spaces is a $C(K)$

space. Since no metric space X with $\mathscr{F}(X) \equiv C(K)$ is known, there is no significant result we can deduce here in the context of the injective norm for molecules.

2.5 Reasonable norms

After studying the projective and injective norms for spaces of molecules, we are ready to take a look at other possible norms defined on spaces of molecules. Of course, we will only be interested in norms that take into account the nature of a space of molecules. What properties should such a norm have? Inspired by the theory of tensor norms, we provide a possible answer.

Definition 2.5.1. A norm $\|\cdot\|$ on the space $\mathcal{M}(X, E)$ of E-valued molecules on a metric space X is called *reasonable* if

(i) $\|vm_{xx'}\| \leq \|v\| \, d(x, x')$ for all $x, x' \in X$, $v \in E$.

(ii) $|\langle v^* \circ m, f\rangle| \leq \|v^*\| \operatorname{Lip}(f) \|m\|$ for all $v^* \in E^*$, $m \in \mathcal{M}(X, E)$ and $f \in X^\#$.

Of course, the injective and projective norms are reasonable: part (i) in Definition 2.5.1 follows from Corollary 2.3.4 and Proposition 2.4.2. We now show that part (ii) is also satisfied. Let $v^* \in E^*$, $m \in \mathcal{M}(X, E)$ and $f \in X^\#$. Writing $m = \sum_{j=1}^n v_j m_{x_j y_j}$,

$$|\langle v^* \circ m, f\rangle| = \Big| \sum_{j=1}^n v^*(v_j)[f(x_j) - f(y_j)] \Big|$$
$$\leq \sum_{j=1}^n |v^*(v_j)[f(x_j) - f(y_j)]| \leq \|v^*\| \operatorname{Lip}(f) \sum_{j=1}^n \|v_j\| \, d(x_j, y_j)$$

so, taking the infimum over all representations of m,

$$|\langle v^* \circ m, f\rangle| \leq \|v^*\| \operatorname{Lip}(f) \|m\|_\pi \, .$$

Also,

$$
\begin{aligned}
|\langle v^* \circ m, f\rangle| &= \Big| \sum_{j=1}^{n} v^*(v_j)[f(x_j) - f(y_j)] \Big| \\
&= \|v^*\| \operatorname{Lip}(f) \Big| \sum_{j=1}^{n} \frac{v^*}{\|v^*\|}(v_j)[\frac{f}{\operatorname{Lip}(f)}(x_j) - \frac{f}{\operatorname{Lip}(f)}(y_j)] \Big| \leq \|v^*\| \operatorname{Lip}(f) \|m\|_{\varepsilon} .
\end{aligned}
$$

Moreover, just as in the linear case [Rya02, Prop. 6.1.(a)] the projective and injective norms are the extremes among all possible reasonable norms.

Proposition 2.5.2. *A norm $\|\cdot\|$ on $\mathcal{M}(X, E)$ is reasonable if and only if $\|m\|_{\varepsilon} \leq \|m\| \leq \|m\|_{\pi}$ for all $m \in \mathcal{M}(X, E)$.*

Proof. Suppose that $\|\cdot\|$ is reasonable. Write $m = \sum_{j=1}^{n} v_j m_{x_j y_j}$. Then

$$
\|m\| = \Big\| \sum_{j=1}^{n} v_j m_{x_j y_j} \Big\| \leq \sum_{j=1}^{n} \|v_j m_{x_j y_j}\| \leq \sum_{j=1}^{n} \|v_j\| \, d(x_j, y_j),
$$

so by taking the infimum over all representations of m, $\|m\| \leq \|m\|_{\pi}$. Also, by definition of reasonability,

$$
\|m\|_{\varepsilon} = \sup \big\{ |\langle v^* \circ m, f\rangle| : \|v^*\| \leq 1, \operatorname{Lip}(f) \leq 1 \big\} \leq 1 \cdot 1 \cdot \|m\| .
$$

Conversely, suppose that $\|\cdot\|_{\varepsilon} \leq \|\cdot\| \leq \|\cdot\|_{\pi}$. Since both $\|\cdot\|_{\pi}$ and $\|\cdot\|_{\varepsilon}$ are reasonable, for any $x, y \in X$, $v \in E$, $v^* \in E^*$, $m \in \mathcal{M}(X, E)$ and $f \in X^{\#}$ we have

$$
\|v m_{xy}\| \leq \|v m_{xy}\|_{\pi} \leq \|v\| \, d(x, y)
$$

and

$$
|\langle v^* \circ m, f\rangle| \leq \|v^*\| \operatorname{Lip}(f) \|m\|_{\varepsilon} \leq \|v^*\| \operatorname{Lip}(f) \|m\| ,
$$

so $\|\cdot\|$ is reasonable. $\square$

It turns out that the inequalities in the definition of a reasonable norm for molecules can be strengthened to equalities. This corresponds to [Rya02, Prop. 6.1.(b)]

Proposition 2.5.3. *Suppose that* $\|\cdot\|$ *is a reasonable norm on* $\mathcal{M}(X,E)$. *Then:*

(i) $\|vm_{xx'}\| = \|v\|\, d(x,x')$ *for all* $x, x' \in X$, $v \in E$.

(ii) $\|v^*\| \operatorname{Lip}(f) = \sup\left\{\langle v^* \circ m, f\rangle : \|m\| \leq 1\right\}$ *for all* $v^* \in E^*$ *and* $f \in X^\#$.

Proof. For all $x, y \in X$ and $v \in E$, by Corollary 2.3.4, Proposition 2.4.2 and Proposition 2.5.2

$$\|v\|\, d(x,y) = \|vm_{xy}\|_\varepsilon \leq \|vm_{xy}\| \leq \|vm_{xy}\|_\pi = \|v\|\, d(x,y),$$

so $\|vm_{xy}\| = \|v\|\, d(x,y)$. Now fix $v^* \in E^*$ and $f \in X^\#$. Clearly, $\|v^*\| \operatorname{Lip}(f) \geq \sup\left\{\langle v^* \circ m, f\rangle : \|m\| \leq 1\right\}$ from the definition of reasonable norm. Given $\delta > 0$, there exist $v \in E$ with $\|v\| = 1$ and $v^*(v) \geq \|v^*\| - \delta$, and $x \neq y$ in X with $|f(x) - f(y)|/d(x,y) \geq \operatorname{Lip}(f) - \delta$. Let $m = \frac{v}{d(x,y)} m_{xy}$. Then $\|m\| \leq \frac{1}{d(x,y)} d(x,y) = 1$ because $\|\cdot\|$ is reasonable, and

$$|\langle v^* \circ m, f\rangle| = \left| v^*(v) \frac{[f(x) - f(y)]}{d(x,y)} \right| \geq (\|v^*\| - \delta)(\operatorname{Lip}(f) - \delta).$$

Letting $\delta \downarrow 0$, the result follows. □

In the remaining sections of this chapter we study several other reasonable norms and their properties, most importantly identifying their dual spaces as nonlinear ideals of operators between a metric space and a Banach space.

2.6 The Chevet-Saphar norms and duality for Lipschitz p-summing operators

Absolutely summing operators are by now widely recognized as one of the most important developments in modern Banach space theory, as attested to by the astonishing number of results and applications that can be found, for example, in [DJT95]. Let us recall that for $1 \leq p < \infty$, a linear map $T : E \to F$ is p-summing if there exists a constant $C \geq 0$ such that regardless of the choice of vectors $v_1 \dots, v_n$ in E we have

$$\left[\sum_{j=1}^{n} \|Tv_j\|^p \right]^{1/p} \leq C \sup_{v^* \in B_{X^*}} \left[\sum_{j=1}^{n} |v^*(v_j)|^p \right]^{1/p}.$$

The infimum of such constants C is denoted by $\pi_p(T)$ and called the p-summing norm of T. Inspired by this useful concept, J. Farmer and W. B. Johnson introduced in [FJ09] the following definition: a Lipschitz map $T : X \to Y$ is called *Lipschitz p-summing* if there exists a constant $C \geq 0$ such that regardless of the choice of points $x_1, \dots, x_n, x'_1, \dots, x'_n$ in X and the choice of positive reals $\lambda_1, \dots, \lambda_n$ we have the inequality

$$\left[\sum_{j=1}^{n} \lambda_j d(Tx_j, Tx'_j)^p \right]^{1/p} \leq C \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^{n} \lambda_j \big| f(x_j) - f(x'_j) \big|^p \right]^{1/p}.$$

The infimum of such constants is denoted by $\pi_p^L(T)$. This is a true generalization of the concept of linear p-summing operator, since it is shown in [FJ09, Thm. 2] that the Lipschitz p-summing norm of a linear operator is the same as its p-summing norm. For the sequel, it will be useful to note that the above definition is the same if we restrict to $\lambda_j = 1$ (see [FJ09] for the proof).

In order to shorten the notation and avoid having to treat the case $p = \infty$ separately, we introduce some more symbols and terminology. $\|\cdot\|_p$ denotes the norm on ℓ_p of a sequence of real numbers. All sequences (of numbers and vectors) under consideration in this chapter will be finite, so there will be no issues of convergence.

For a sequence of vectors $(v_j)_j$ in a Banach space E, its *strong p-norm* is the ℓ_p-norm of the sequence $(\|v_j\|)_j$ and we denote its *weak p-norm* (cf. [DF93, p. 91]) by

$$w_p\big((v_j)_j\big) := \sup_{v^* \in B_{E^*}} \left\| \big(v^*(v_j)\big)_j \right\|_p .$$

Analogously, for sequences of the same length $(\lambda_j)_j$ of real numbers and $(x_j)_j$, $(x'_j)_j$ of points in X, we denote their *weak Lipschitz p-norm* by

$$w_p^{\mathrm{Lip}}\big((\lambda_j, x_j, x'_j)_j\big) := \sup_{f \in B_{X^\#}} \left\| \big(\lambda_j [f(x_j) - f(x'_j)]\big)_j \right\|_p .$$

When the tensor product $E \otimes F$ of two Banach spaces is endowed with a tensor norm, its dual space can be interpreted as linear operators from E to F^*. Under (some of) the Chevet-Saphar tensor norms, introduced independently by S. Chevet [Che69] and P. Saphar [Sap70] as generalizations of earlier work of Saphar [Sap65], the operators from E to F^* obtained in this way are precisely the p-summing operators. The main result of this section (Theorem 2.6.4) is the analogous result in the setting of Lipschitz p-summing operators between a metric space and a Banach space, with the space of molecules playing the role of the tensor product in the linear theory.

2.6.1 Definition and elementary properties

For a molecule $m \in \mathcal{M}(X, E)$ we define its *p-Chevet-Saphar norm* by

$$cs_p(m) = \inf \left\{ \left\| \big(\lambda_j \|v_j\| \big)_j \right\|_p w_{p'}^{\mathrm{Lip}}\big((\lambda_j^{-1}, x_j, x'_j)_j\big) \ : \ m = \sum_j v_j m_{x_j x'_j}, \lambda_j > 0 \right\}. \tag{2.6.1}$$

The reader familiar with the theory of Chevet-Saphar norms on tensor products of Banach spaces will recall that there are two versions of those norms for a given index p; a *left* one and a *right* one. Such variants are also possible in the present context, but we stick with only one for now and postpone the study of the other one until section 2.7, when we tackle the more general Lapresté norms. Let us start by showing that our use of the word "norm" is justified.

Theorem 2.6.1. *cs_p is a norm on $\mathcal{M}(X, E)$.*

Proof. It is clear that for any molecule $m \in \mathcal{M}(X, E)$ and any scalar λ, $cs_p(m) \geq 0$ and $cs_p(\lambda m) = |\lambda| cs_p(m)$. Let $m_1, m_2 \in \mathcal{M}(X, E)$ and $\varepsilon > 0$. By definition of the cs_p norm we can find a representation $m_1 = \sum_j v_j m_{x_j x'_j}$ and a sequence of positive reals $(\lambda_j)_j$ such that

$$\left\| \big(\lambda_j \left\| v_j \right\| \big)_j \right\|_p w_{p'}^{\mathrm{Lip}}\big((\lambda_j^{-1}, x_j, x'_j)_j\big) \leq cs_p(m_1) + \varepsilon.$$

By replacing $(\lambda_j)_j$ by an appropriate multiple of it, we may in fact assume that

$$\left\| \big(\lambda_j \left\| v_j \right\| \big)_j \right\|_p \leq \big(cs_p(m_1) + \varepsilon\big)^{1/p}, \qquad w_{p'}^{\mathrm{Lip}}\big((\lambda_j^{-1}, x_j, x'_j)_j\big) \leq \big(cs_p(m_1) + \varepsilon\big)^{1/p'}. \tag{2.6.2}$$

Similarly, there exist a representation $m_2 = \sum_i w_i m_{y_i y'_i}$, and positive reals $(\kappa_i)_i$ such that

$$\left\| \big(\kappa_i \left\| w_i \right\| \big)_i \right\|_p \leq \big(cs_p(m_2) + \varepsilon\big)^{1/p}, \qquad w_{p'}^{\mathrm{Lip}}\big((\kappa_i^{-1}, y_i, y'_i)_i\big) \leq \big(cs_p(m_2) + \varepsilon\big)^{1/p'}. \tag{2.6.3}$$

We now "glue" together these representations of m_1 and m_2 to get a representation of $m_1 + m_2$: let $(u_k)_k$ be the sequence obtained from concatenating $(v_j)_j$ and $(w_i)_i$; similarly obtain $(z_k, z'_k)_k$ from concatenating $(x_j, x'_j)_j$ and (y_i, y'_i); and construct $(\eta_k)_k$ from $(\lambda_j)_j$ and $(\kappa_i)_i$. Then the strong p-norm of $(\eta_k u_k)_k$ is just the p-sum of the strong p-norms of $(\lambda_j v_j)_j$ and $(\kappa_i w_i)_i$, so from (2.6.2) and (2.6.3) we have

$$\left\| \big(\eta_k \left\| u_k \right\| \big)_k \right\|_p \leq \big(cs_p(m_1) + cs_p(m_2) + 2\varepsilon\big)^{1/p}. \tag{2.6.4}$$

Similarly, the weak Lipschitz p'-norm of $(\eta_k^{-1}, z_k, z_k')_k$ is bounded above by the p'-sum of the weak Lipschitz p'-norms of $(\lambda_j^{-1}, x_j, x_j')_j$ and $(\kappa_i^{-1}, y_i, y_i')$, so once more from (2.6.2) and (2.6.3) we obtain

$$w_{p'}\big((\eta_k^{-1}, z_k, z_k')_k\big) \leq \big(cs_p(m_1) + cs_p(m_2) + 2\varepsilon\big)^{1/p'}. \tag{2.6.5}$$

But clearly $m_1 + m_2 = \sum_k u_k m_{z_k z_k'}$, so the product of (2.6.4) and (2.6.5) together with the definition of cs_p give $cs_p(m_1 + m_2) \leq cs_p(m_1) + cs_p(m_2) + 2\varepsilon$. By letting ε tend to zero we have the triangle inequality for cs_p.

Let $T \in \mathrm{Lip}_0(X, E^*)$ be a map that admits a representation as a finite sum of the form $\sum_k v_k^* f_k$ with $(v_k^*)_k \subset E^*$, $(f_k)_k \subset X^\#$ (i.e. such that the linearization $\widehat{T} : Æ(X) \to E^*$ has finite rank). For such a T, set

$$\theta_p(T) = \inf\left\{ \big\|\big(\,\|v_k^*\|\,\big)_k\big\|_p \, \big\|\big(\,\mathrm{Lip}(f_k)\big)_k\big\|_{p'} \right\}$$

where the infimum is taken over all representations as above. Now, given $m = \sum_j v_j m_{x_j x_j'} \in \mathcal{M}(X, E)$, and $(\lambda_j)_j$ a sequence of positive real numbers, we have from the pairing formula (2.2.2) and Hölder's inequality

$$\begin{aligned} \big|\langle T, m\rangle\big| = \left| \sum_{j,k} v_k^*(v_j)\big[f_k(x_j) - f_k(x_j')\big]\right| &\leq \sum_{j,k} \left|\lambda_j v_k^*(v_j)\lambda_j^{-1}\big[f_k(x_j) - f_k(x_j')\big]\right| \\ &\leq \left\|\big(\lambda_j v_k^*(v_j)\big)_{j,k}\right\|_p \left\|\Big(\lambda_j^{-1}\big[f_k(x_j) - f_k(x_j')\big]\Big)_{j,k}\right\|_{p'}. \end{aligned} \tag{2.6.6}$$

For finite p, the definition of the ℓ_p-norm gives

$$\left\|\big(\lambda_j v_k^*(v_j)\big)_{j,k}\right\|_p^p = \sum_k \sum_j |\lambda_j|^p |v_k^*(v_j)|^p \leq \sum_k \|v_k^*\|^p \sum_j |\lambda_j|^p \, \|v_j\|^p$$

so after taking the p-th root we get

$$\Big\| \big(\lambda_j v_k^*(v_j)\big)_{j,k} \Big\|_p \leq \Big\| \big(\lambda_j \, \|v_j\| \,\big)_j \Big\|_p \, \big\| \big(\, \|v_k^*\| \,\big)_k \big\|_p \tag{2.6.7}$$

and the same inequality is also trivially valid for $p = \infty$. On the other hand, by an analogous argument,

$$\Bigg\| \Big(\lambda_j^{-1} \big[f_k(x_j) - f_k(x_j') \big] \Big)_{j,k} \Bigg\|_{p'} \leq \big\| \big(\operatorname{Lip}(f_k) \big)_k \big\|_{p'} \, w_{p'}^{\operatorname{Lip}} \Big(\big(\lambda_j^{-1}, x_j, x_j' \big)_j \Big). \tag{2.6.8}$$

Together, equations (2.6.6), (2.6.7) and (2.6.8) imply

$$\big| \langle T, m \rangle \big| \leq \Big\| \big(\lambda_j \, \|v_j\| \,\big)_j \Big\|_p \, w_{p'}^{\operatorname{Lip}} \Big(\big(\lambda_j^{-1}, x_j, x_j' \big)_j \Big) \, \big\| \big(\, \|v_k^*\| \,\big)_k \big\|_p \, \big\| \big(\operatorname{Lip}(f_k) \big)_k \big\|_{p'} ,$$

so after taking the infimum over all representations, $\big| \langle T, m \rangle \big| \leq cs_p(m) \theta_p(T)$. In particular, this applies to maps T of the form $v^* f$ with $v^* \in E^*$ and $f \in X^\#$, so if m is such that $cs_p(m) = 0$ then we have, using the pairing formula (2.2.2),

$$0 = \langle v^* f, m \rangle = \sum_j v^*(v_j) \big[f(x_j) - f(x_j') \big] \text{ for all } v^* \in E^*, f \in X^\#.$$

By the duality between Æ(X) and $X^\#$ (see Theorem 2.2.1), this means that the real-valued molecule $v^* \circ m$ is equal to 0 for all $v^* \in E^*$ and consequently $m = 0$. $\square$

We will denote by $\mathcal{CS}_p(X, E)$ the normed space $\big(\mathcal{M}(X, E), cs_p \big)$. Notice that when X is a finite set the space $\mathcal{CS}_p(X, E)$ is complete, since it is isomorphic to $E^{|X|-1}$. On the other hand, when the set X is infinite the elements of the completion of $\mathcal{CS}_p(X, E)$ correspond to infinite representations as sums of atoms that are analogous to the ones considered in (2.6.1), but we need not concern ourselves with such technicalities for our present purposes.

Let us now show that the Chevet-Saphar norms for molecules are reasonable.

Proposition 2.6.2. *For $1 \leq p \leq \infty$, the p-Chevet-Saphar norm on spaces of molecules is reasonable.*

Proof. Let $v \in E$, $x, x' \in X$. Then clearly, just from the trivial representation of $vm_{xx'}$, $cs_p(vm_{xx'}) \leq \|v\| \, d(x,x')$. Now, let $v^* \in E^*$, $m \in \mathcal{M}(X,E)$ and $f \in X^\#$. Writing $m = \sum_{j=1}^n v_j m_{x_j x'_j}$ we have

$$|\langle v^* \circ m, f \rangle| \leq \sum_{j=1}^n |v^*(v_j)| \cdot \left| f(x_j) - f(x'_j) \right|,$$

so

$$|\langle v^* \circ m, f \rangle| \leq \|v^*\| \left(\sum_{j=1}^n \lambda_j^p \, \|v_j\|^p \right)^{1/p} \mathrm{Lip}(f) \sup_{g \in B_{X^\#}} \left(\sum_{j=1}^n \lambda_j^{-p'} |g(x_j) - g(x'_j)|^{p'} \right)^{1/p'}$$

and hence $|\langle v^* \circ m, f \rangle| \leq \|v^*\| \, \mathrm{Lip}(f) cs_p(m)$. □

The next proposition shows that in the extreme cases $p = 1$ and $p = \infty$, cs_p can be calculated using a simpler formula. In particular, we obtain that the cs_1 norm is just the straightforward generalization of the Arens-Eells norm to the Banach-valued case (cf. (2.2.1)), that is, the projective tensor norm for molecules.

Proposition 2.6.3. *For a molecule $m \in \mathcal{M}(X,E)$,*

$$cs_1(m) = \inf \left\{ \sum_j \|v_j\| \, d(x_j, x'_j) : m = \sum_j v_j m_{x_j x'_j} \right\} \tag{2.6.9}$$

and

$$cs_\infty(m) = \inf \left\{ \sup_{f \in B_{X^\#}} \sum_j \|v_j\| \left| f(x_j) - f(x'_j) \right| : m = \sum_j v_j m_{x_j x'_j} \right\}. \tag{2.6.10}$$

Proof. Start by noting that given positive numbers λ_j and points x_j, x'_j in X,

$$w_\infty^{\mathrm{Lip}}\big((\lambda_j, x_j, x'_j)_j\big) = \sup_{f \in B_{X^\#}} \left\| \big(\lambda_j [f(x_j) - f(x'_j)]\big)_j \right\|_\infty = \max_j \lambda_j d(x_j, x'_j), \tag{2.6.11}$$

because for each j, $\left|f(x_j) - f(x'_j)\right|$ is at most $d(x_j, x'_j)$ whenever $f \in B_{X^\#}$ and this upper bound is in fact achieved: given any two points $x, x' \in X$, the function $f : X \to \mathbb{R}$ given by $f(\cdot) = d(\cdot, x') - d(x', 0)$ is in $\mathrm{Lip}_0(X, \mathbb{R})$, has Lipschitz constant 1 and satisfies $\left|f(x) - f(x')\right| = d(x, x')$. Now, given a molecule $m = \sum_j v_j m_{x_j x'_j}$ and positive reals $(\lambda_j)_j$, (2.6.11) gives

$$\begin{aligned}\left\|\left(\lambda_j \left\|v_j\right\|\right)_j\right\|_1 w_\infty^{\mathrm{Lip}}\left((\lambda_j^{-1}, x_j, x'_j)\right) &= \left(\sum_j \lambda_j \left\|v_j\right\|\right)\left(\max_j \lambda_j^{-1} d(x_j, x'_j)\right) \\ &\geq \sum_j \lambda_j^{-1} d(x_j, x'_j)\lambda_j \left\|v_j\right\| = \sum_{j=1}^n \left\|v_j\right\| d(x_j, x'_j).\end{aligned}$$

Taking the infimum over all representations of m we get the inequality $\geq$ in (2.6.9). On the other hand, note that we may assume without loss of generality that $x_j \neq x'_j$ for all j and thus (2.6.11) with the particular choice $\lambda_j = d(x_j, x'_j) > 0$ gives

$$\begin{aligned}cs_1(m) &\leq \left\|\left(d(x_j, x'_j) \left\|v_j\right\|\right)_j\right\|_1 w_\infty^{\mathrm{Lip}}\left(\left(d(x_j, x'_j)^{-1}, x_j, x'_j\right)_j\right. \\ &= \left(\sum_j d(x_j, x'_j) \left\|v_j\right\|\right) \max_j \frac{d(x_j, x'_j)}{d(x_j, x'_j)} = \sum_j d(x_j, x'_j) \left\|v_j\right\|\end{aligned}$$

and after taking the infimum over all representations of m we obtain $\leq$ in (2.6.9).

Now, given a molecule $m = \sum_j v_j m_{x_j x'_j}$ and positive numbers λ_j,

$$\begin{aligned}\left\|\left(\lambda_j \left\|v_j\right\|\right)_j\right\|_\infty w_1^{\mathrm{Lip}}\left((\lambda_j^{-1}, x_j, x'_j)\right) &= \left(\max_j \lambda_j \left\|v_j\right\|\right) \sup_{f \in B_{X^\#}} \sum_j \lambda_j^{-1}\left|f(x_j) - f(x'_j)\right| \\ &\geq \sup_{f \in B_{X^\#}} \sum_j \lambda_j \left\|v_j\right\| \lambda_j^{-1}\left|f(x_j) - f(x'_j)\right| = \sup_{f \in B_{X^\#}} \sum_j \left\|v_j\right\| \left|f(x_j) - f(x'_j)\right|\end{aligned}$$

so taking the infimum over all representations gives $\geq$ in (2.6.10). On the other hand, note that we can also assume without loss of generality that $v_j \neq 0$ for all j, so

$$cs_\infty(m) \leq \left\| \left(\|v_j\|^{-1} \|v_j\| \right)_j \right\|_\infty w_1^{\mathrm{Lip}}\big((\|v_j\|, x_j, x_j')\big) = 1 \cdot \sup_{f \in B_{X^\#}} \sum_j \|v_j\| \left| f(x_j) - f(x_j') \right|$$

and taking the infimum yet again rewards us with $\leq$ in (2.6.10). □

2.6.2 Examples

We can use Proposition 2.6.3 to calculate explicitly the space $\mathcal{CS}_1$ in the case when X is a graph-theoretic tree. First note that (2.6.9) can be interpreted as saying that in general the space $\mathcal{CS}_1(X,E)$ is a quotient of a weighted (with weight given by the distance d) ℓ_1-sum of copies of E; in fact the quotient map $Q : \big(\bigoplus_{x,x' \in X} E\big)_{\ell_1,d} \to \mathcal{M}(X,E)$ is given by

$$Q\big((v_{xx'})_{x,x' \in X}\big) = \sum_{x,x' \in X} v_{xx'} m_{xx'}.$$

When X is a graph-theoretic tree, we will show that $\mathcal{CS}_1(X,E)$ is again a weighted ℓ_1-sum of copies of E. In fact, by the fact that the cs_1 norm and the projective one are the same, this has already been shown in Proposition 2.3.9. Moreover, in Corollary 2.3.10 we have already calculated $\mathcal{CS}_1(X,E)$ in the case when X is an $\mathbb{R}$-tree.

2.6.3 Duality

We show now that the duals of the Chevet-Saphar spaces of molecules can be canonically identified as spaces of Lipschitz p-summing operators.

Theorem 2.6.4. *The spaces $\mathcal{CS}_p(X,E)^*$ and $\Pi^L_{p'}(X,E^*)$ are isometrically isomorphic via the canonical pairing. Moreover, on the unit ball of $\Pi^L_{p'}(X,E^*)$ the weak* topology coincides with the topology of pointwise $\sigma(E^*,E)$-convergence.*

Proof. First, let $T \in \Pi^L_{p'}(X,E^*)$. Consider a molecule $m = \sum_j v_j m_{x_j x'_j} \in \mathcal{M}(X,E)$ and positive numbers λ_j. The pairing formula 2.2.2, Hölder's inequality and the definition of Lipschitz p'-summing naturally come together to give us

$$\begin{aligned}\big|\langle T,m\rangle\big| &= \Big|\sum_j \langle Tx_j - Tx'_j, v_j\rangle\Big| \le \sum_j |\langle Tx_j - Tx'_j, v_j\rangle| \\ &\le \sum_j \left\|Tx_j - Tx'_j\right\| \|v_j\| \le \|(\lambda_j \|v_j\|)_j\|_p \left\|\big(\lambda_j^{-1} \left\|Tx_j - Tx'_j\right\|\big)_j\right\|_{p'} \\ &\le \|(\lambda_j \|v_j\|)_j\|_p \, \pi^L_{p'}(T) w^{\mathrm{Lip}}_{p'}\big((\lambda_j^{-1}, x_j, x'_j)_j\big).\end{aligned}$$

Taking the infimum over all representations of m and positive λ_j we conclude that

$$\big|\langle T,m\rangle\big| \le \pi^L_{p'}(T) cs_p(m).$$

Conversely, let $\varphi \in \mathcal{CS}_p(X,E)^*$ with $\|\varphi\| = C$. Then we have $|\varphi(m)| \le C cs_p(m)$ for any $m \in \mathcal{M}(X,E)$. Note that φ can be identified with a mapping $T : X \mapsto E^*$ via the formula $\langle Tx, v\rangle = \varphi(vm_{x0})$. Indeed, for $x \in X$ and $v \in E$,

$$|\langle Tx,v\rangle| = |\langle \varphi, vm_{x0}\rangle| \le C cs_p(vm_{x0}) \le C \|v\| \sup_{f\in B_{X^\#}} |f(x) - f(0)| = C \|v\| d(x,0)$$

so $Tx \in E^*$. Now, fix points x_j, x'_j in X and positive numbers λ_j, $j = 1, \dots, n$. Let $\varepsilon > 0$. For each j pick $v_j \in E$ such that $\langle Tx_j - Tx'_j, v_j\rangle = \|Tx_j - Tx'_j\|$ and $\|v_j\| \le 1 + \varepsilon$. Then, for any sequence $(\alpha_j)_j$ of real numbers,

$$\Big|\sum_j \alpha_j\lambda_j\langle Tx_j - Tx'_j, v_j\rangle\Big| = \Big|\langle T, \sum_j \alpha_j\lambda_j v_j m_{x_jx'_j}\rangle\Big| \le C \cdot cs_p\Big(\sum_j \alpha_j\lambda_j v_j m_{x_jx'_j}\Big)$$
$$\le C \left\|\big(|\alpha_j|\,\|v_j\|\,\big)_j\right\|_p w_{p'}^{\mathrm{Lip}}\big(\lambda_j, x_j, x'_j\big) \le C(1+\varepsilon)\left\|\big(|\alpha_j|\big)_j\right\|_p w_{p'}^{\mathrm{Lip}}\big(\lambda_j, x_j, x'_j\big).$$

Taking the supremum over all sequences with $\left\|\big(|\alpha_j|\big)_j\right\|_p \le 1$,

$$\left\|\big(\lambda_j\langle Tx_j - Tx'_j, v_j\rangle\big)_j\right\|_{p'} \le C(1+\varepsilon)w_{p'}^{\mathrm{Lip}}\big(\lambda_j, x_j, x'_j\big).$$

Letting ε go to 0,

$$\left\|\big(\lambda_j\,\|Tx_j - Tx'_j\|\,\big)_j\right\|_{p'} \le C w_{p'}^{\mathrm{Lip}}\big(\lambda_j, x_j, x'_j\big),$$

i.e. T is Lipschitz p'-summing with $\pi_{p'}^L(T) \le C$.

For the second part, suppose $(T_\alpha)_\alpha \subset \Pi_{p'}^L(X, E^*)$ converges in the weak* topology to $T \in \Pi_{p'}^L(X, E^*)$. Then, for any $x \in X$ and any $v \in E$, $\langle T_\alpha, vm_{x0}\rangle \to \langle T, vm_{x0}\rangle$, i.e. $\langle T_\alpha(x), v\rangle \to \langle T(x), v\rangle$. This means that (T_α) converges to T in the topology of pointwise $\sigma(E^*, E)$-convergence. Therefore, the identity on $\Pi_{p'}^L(X, E^*)$ is a continuous bijection from the weak* topology to the topology of pointwise $\sigma(E^*, E)$-convergence. On the unit ball, the former is compact and the latter is Hausdorff, so they must coincide. □

In order to answer Question 3 from [FJ09], i.e. identify the dual of the space of Lipschitz p-summing operators from a finite metric space to a Banach space, we will need to “reverse” the duality given by Theorem 2.6.4. Unsurprisingly, the principle of local reflexivity will play a crucial role.

Lemma 2.6.5. *When X is a finite metric space, $\Pi_p^L(X,E)^{**}$ and $\Pi_p^L(X,E^{**})$ are (canonically) isometrically isomorphic.*

Proof. As vector spaces, both spaces can be identified with the space of functions from X to E^{**} that vanish at 0, so it will suffice to show equality of their unit balls. By Goldstein's theorem, $B_{\Pi_p^L(X,E)^{**}}$ is the weak*-closure of $B_{\Pi_p^L(X,E)}$. Since X is finite, the weak* topology on $\Pi_p^L(X,E)^{**}$ is the topology of pointwise $\sigma(E^{**},E^*)$-convergence. Since the Lipschitz p-summing norm does not change if the codomain is enlarged, $B_{\Pi_p^L(X,E)}$ embeds isometrically into $B_{\Pi_p^L(X,E^{**})}$. By Theorem 2.6.4 the weak*-topology in $\Pi_p^L(X,E^{**})$ (as the dual of $\mathcal{CS}_{p'}(X,E^*)$) is also the topology of pointwise $\sigma(E^{**},E^*)$-convergence. Therefore, $B_{\Pi_p^L(X,E)^{**}} \subseteq B_{\Pi_p^L(X,E^{**})}$.

Now fix $T \in \mathrm{Lip}_0(X,E^{**})$. Let F be a finite dimensional subspace of E^{**} containing the span of the image of T such that $F \cap E \neq \{0\}$, and let $\mathcal{A}$ be the directed set of all finite-dimensional subspaces of E^*. Given $\varepsilon \in (0,1)$, by the principle of local reflexivity (say, in the form given in [DJT95, p. 178]) for every $A \in \mathcal{A}$ there exists an injective linear map $u_A : F \to E$ such that: (a) $u_A v = v$ for all $v \in F \cap E$; (b) $\|u_A\| \cdot \left\|u_A^{-1}\right\| \leq 1+\varepsilon$; and (c) $\langle u_A v^{**}, v^*\rangle = \langle v^{**}, v^*\rangle$ for all $v^{**} \in F$ and $v^* \in A$. Note that since $F \cap E$ is not trivial, condition (a) guarantees that $\left\|u_A^{-1}\right\| \geq 1$ and thus $\|u_A\| \leq 1+\varepsilon$ from condition (b). If we set $T_A := u_A \circ T : X \to E$, Then $\pi_p^L(T_A) \leq \|u_A\|\,\pi_p^L(T) \leq (1+\varepsilon)\pi_p^L(T)$ and for every $v^* \in E^*$, since v^* is eventually in $A \in \mathcal{A}$ condition (c) implies that

$$\lim_{A\in\mathcal{A}} \langle T_A x, v^*\rangle = \lim_{A\in\mathcal{A}} \langle u_A T x, v^*\rangle = \langle Tx, v^*\rangle,$$

i.e. the net $(T_A)_{A\in\mathcal{A}}$ converges to T with respect to the topology of pointwise $\sigma(E^{**},E^*)$-convergence. Since $T \in \mathrm{Lip}_0(X,E^{**})$ was arbitrary, this implies that $B_{\Pi_p^L(X,E^{**})}$ is contained in the closure of $(1+\varepsilon)B_{\Pi_p^L(X,E)}$ with respect to the topology of pointwise $\sigma(E^{**},E^*)$-convergence, that is, $B_{\Pi_p^L(X,E^{**})} \subseteq (1+\varepsilon)B_{\Pi_p^L(X,E)^{**}}$. Letting ε go to 0 we conclude that $B_{\Pi_p^L(X,E^{**})} \subseteq B_{\Pi_p^L(X,E)^{**}}$. $\square$

Corollary 2.6.6. *When X is a finite metric space,* $\Pi_p^L(X,E)^* \equiv \mathcal{CS}_{p'}(X,E^*)$.

Proof. From Theorem 2.6.4 we have $\mathcal{CS}_{p'}(X,E^*)^* \equiv \Pi_p^L(X,E^{**})$ and Lemma 2.6.5 gives us $\Pi_p^L(X,E)^{**} \equiv \Pi_p^L(X,E^{**})$, so $\mathcal{CS}_{p'}(X,E^*)^* \equiv \Pi_p^L(X,E)^{**}$. Moreover, the isometry implied in this last inequality is weak*-to-weak* continuous (reasoning as in the proof of Theorem 2.6.4, weak*-convergence in $\mathcal{CS}_{p'}(X,E^*)^*$ implies pointwise $\sigma(E^{**},E^*)$ convergence, that is, weak*-convergence in $\Pi_p^L(X,E)^{**}$), so it is the adjoint of an isometry between $\mathcal{CS}_{p'}(X,E^*)$ and $\Pi_p^L(X,E)^*$. □

2.6.4 An application: a characterization of Lipschitz p-summing operators between metric spaces

Even though we have been considering only Lipschitz p-summing operators from a metric space into a Banach space, the Chevet-Saphar spaces of molecules can be used to get a new characterization of Lipschitz p-summing operators between metric spaces. Moreover, this characterization has the (potential) advantage of being expressed only in terms of linear operators.

A Lipschitz map $T : X \to Y$ naturally induces a linear map $T_E : \mathcal{M}(X,E) \to \mathcal{M}(Y,E)$ given by

$$T_E\left(\sum_{j=1}^n v_j m_{x_j x_j'}\right) = \sum_{j=1}^n v_j m_{Tx_j Tx_j'}.$$

First, let us note that $T_E : \mathcal{M}(X,E) \to \mathcal{M}(Y,E)$ is well-defined, i.e. it does not depend on the given representation of a molecule. For that, suppose that a molecule $m : X \to E$ has two representations $\sum_j v_j m_{x_j x_j'}$ and $\sum_i w_i m_{y_i y_i'}$. Then for all $v^* \in E^*$ the real-valued molecule $v^* \circ m$ has representations $\sum_j v^*(v_j) m_{x_j x_j'}$ and $\sum_i v^*(w_i) m_{y_i y_i'}$. Hence, by duality between Æ(X) and $X^\#$ (see Theorem 2.2.1), for all $f \in X^\#$ we have that

$$\sum_j v^*(v_j)\big[f(x_j) - f(x_j')\big] = \sum_i v^*(w_i)\big[f(y_i) - f(y_i')\big].$$

In particular, for any $g \in Y^\#$ we have $g \circ T \in X^\#$ and thus

$$\sum_j v^*(v_j)\big[g(Tx_j) - g(Tx'_j)\big] = \sum_i v^*(w_i)\big[g(Ty_i) - g(Ty'_i)\big],$$

which means that $\sum_j v_j m_{Tx_jTx'_j} = \sum_i w_i m_{Ty_iTy'_i}$ (applying the same arguments in reverse order).

Theorem 2.6.7. *Let $T : X \to Y$ be a Lipschitz map. The following are equivalent:*

(a) T is Lipschitz p-summing.

(b) For every Banach space E (or only $E = Y^\#$), the operator

$$T_E : \mathcal{CS}_{p'}(X, E) \to \mathcal{CS}_1(Y, E)$$

is continuous.

In this case,

$$\pi_p^L(T) = \big\|T_{Y^\#} : \mathcal{CS}_{p'}(X, Y^\#) \to \mathcal{CS}_1(Y, Y^\#)\big\| \geq \|T_E : \mathcal{CS}_{p'}(X, E) \to \mathcal{CS}_1(Y, E)\|\,.$$

Proof. Suppose that $T : X \to Y$ is Lipschitz p-summing. Let $\varphi \in \big(\mathcal{CS}_1(Y, E)\big)^*$ with $\|\varphi\| \leq 1$. Since $\big(\mathcal{CS}_1(Y, E)\big)^* \equiv \mathrm{Lip}_0(Y, E^*)$, we can identify φ with a function $L_\varphi \in \mathrm{Lip}_0(Y, E^*)$ with $\mathrm{Lip}(L_\varphi) = \|\varphi\| \leq 1$. Let $m = \sum v_j m_{x_jx'_j} \in \mathcal{M}(X, E)$. Then $T_E(m) = \sum v_j m_{Tx_jTx'_j}$, so

$$\langle \varphi, T_E(m)\rangle = \sum_j \langle L_\varphi(Tx_j) - L_\varphi(Tx'_j), v_j\rangle = \langle L_\varphi \circ T, m\rangle,$$

and thus

$$\begin{aligned}\big|\langle \varphi, T_E(m)\rangle\big| = \big|\langle L_\varphi \circ T, m\rangle\big| &\leq \pi_p^L(L_\varphi \circ T) cs_{p'}(m)\\ &\leq \mathrm{Lip}(L_\varphi)\pi_p^L(T) cs_{p'}(m) \leq \pi_p^L(T) cs_{p'}(m).\end{aligned}$$

Taking the supremum over all such φ,

$$cs_1(T_E(m)) \leq \pi_p^L(T) cs_{p'}(m),$$

so $T_E : \mathcal{CS}_{p'}(X,E) \to \mathcal{CS}_1(Y,E)$ is continuous and $\|T_E\| \leq \pi_p^L(T)$.

Now, suppose that $T_{Y^\#} : \mathcal{CS}_{p'}(X,Y^\#) \to \mathcal{CS}_1(Y,Y^\#)$ is continuous and has norm C. Let $j_Y : Y \to (Y^\#)^*$ be the canonical isometric embedding. From the definition of Lipschitz p-summing, it suffices to show that $j_Y \circ T$ is Lipschitz p-summing. Let $m \in \mathcal{M}(X,Y^\#)$. Write $m = \sum_j g_j m_{x_j x'_j}$ with $g_j \in Y^\#$. Then

$$\begin{aligned}\langle j_Y \circ T, m\rangle = \sum_j \langle j_Y \circ T(x_j) - j_Y \circ (Tx'_j), g_j\rangle &= \sum_j \left[g_j(Tx_j) - g_j(Tx'_j)\right] \\ &= \sum_{y \in Y} \langle j_Y(y), \sum_j g_j m_{Tx_j Tx'_j}(y)\rangle = \langle j_Y, T_{Y^\#}(m)\rangle,\end{aligned}$$

so

$$\big|\langle j_Y \circ T, m\rangle\big| = \big|\langle j_Y, T_{Y^\#}(m)\rangle\big| \leq \mathrm{Lip}(j_Y) cs_1\big(T_{Y^\#}(m)\big) \leq 1 \cdot C cs_{p'}(m).$$

Therefore, from the duality between the p'-Chevet-Saphar norm and the Lipschitz p-summing norm, after taking the supremum over all m with $cs_{p'}(m) \leq 1$ we get

$$\pi_p^L(T) \leq C$$

and the proof is over because, now that we know that T is Lipschitz p-summing, from the first part we get $\pi_p^L(T) \geq C$. □

2.7 Lapresté norms and duality for Lipschitz (p,r,s)-summing operators

In [Lap76], J.T. Lapresté defined a generalization of the Chevet-Saphar tensor norms. In this section we study the corresponding definition for spaces of molecules.

2.7.1 Definition and elementary properties

For a molecule $m \in \mathcal{M}(X, E)$, let

$$\mu_{p,r,s}(m) =$$
$$\inf \left\{ \|(\lambda_j)_j\|_p \, w_r^{\mathrm{Lip}}\big((\kappa_j^{-1}\lambda_j^{-1}, x_j, x_j')_j\big) w_s\big((\kappa_j v_j)_j\big) \ : \ m = \sum_j v_j m_{x_j x_j'}, \ \lambda_j, \kappa_j > 0 \right\}.$$

Recall that for $0 < \beta \leq 1$, a non-negative positively homogeneous functional μ defined on a vector space U is called a *β-seminorm* if $\mu(u_1 + u_2)^\beta \leq \mu(u_1)^\beta + \mu(u_2)^\beta$ for all $u_1, u_2 \in U$. If in addition μ vanishes only at 0, it is called a *β-norm.*

Theorem 2.7.1. *Suppose $1/\beta := 1/p + 1/r + 1/s \geq 1$. Then $\mu_{p,r,s}$ is a β-norm on $\mathcal{M}(X, E)$.*

Proof. It is clear that for any molecule $m \in \mathcal{M}(X, E)$ and any scalar λ, $\mu_{p,r,s}(m) \geq 0$ and $\mu_{p,r,s}(\lambda m) = |\lambda| \mu_{p,r,s}(m)$.

Let $m_1, m_2 \in \mathcal{M}(X, E)$ and $\varepsilon > 0$. Choose a representation $m_1 = \sum_j v_j m_{x_j x_j'}$ and positive reals λ_j, κ_j such that

$$\|(\lambda_j)_j\|_p \, w_r^{\mathrm{Lip}}\big((\kappa_j^{-1}\lambda_j^{-1}, x_j, x_j')_j\big) \, w_s\big((\kappa_j v_j)_j\big) \leq \mu_{p,r,s}(m_1) + \varepsilon.$$

Multiplying $(\lambda_j)_j$ and $(\kappa_j)_j$ by appropriate positive constants we may in fact assume that

$$\begin{aligned} \|(\lambda_j)_j\|_p &\leq \big(\mu_{p,r,s}(m_1)^\beta + \varepsilon\big)^{1/p}, \\ w_s\big((\kappa_j v_j)_j\big) &\leq \big(\mu_{p,r,s}(m_1)^\beta + \varepsilon\big)^{1/s}, \\ w_r^{\mathrm{Lip}}\big((\kappa_j^{-1}\lambda_j^{-1}, x_j, x_j')_j\big) &\leq \big(\mu_{p,r,s}(m_1)^\beta + \varepsilon\big)^{1/r}. \end{aligned}$$

Similarly, choose a representation $m_2 = \sum_i w_i m_{y_i y_i'}$ and positive reals η_i, γ_i such that

$$\begin{aligned} \|(\eta_i)_i\|_p &\leq \big(\mu_{p,r,s}(m_2)^\beta + \varepsilon\big)^{1/p}, \\ w_s\big((\gamma_i w_i)_i\big) &\leq \big(\mu_{p,r,s}(m_2)^\beta + \varepsilon\big)^{1/s}, \\ w_r^{\mathrm{Lip}}\big((\gamma_i^{-1}\eta_i^{-1}, y_i, y_i')_i\big) &\leq \big(\mu_{p,r,s}(m_2)^\beta + \varepsilon\big)^{1/r}. \end{aligned}$$

As in the proof of Theorem 2.6.1, concatenate these representations and accompanying positive reals to get a representation of $m_1 + m_2$ and sequences of positive reals that witness the fact that

$$\mu_{p,r,s}(m_1 + m_2) \leq \big(\mu_{p,r,s}(m_1)^\beta + \mu_{p,r,s}(m_2)^\beta + 2\varepsilon\big)^{1/\beta}$$

and hence, letting $\varepsilon \downarrow 0$

$$\mu_{p,r,s}(m_1 + m_2)^\beta \leq \mu_{p,r,s}(m_1)^\beta + \mu_{p,r,s}(m_2)^\beta.$$

For a function $T \in \mathrm{Lip}_0(X, E^*)$ that admits a representation as a finite sum of the form $T = \sum_k \tilde{\lambda}_k v_k^* f_k$ with $\tilde{\lambda}_k \in \mathbb{R}$, $v_k^* \in E^*$ and $f_k \in X^\#$ (i.e. such that the linearization $\widehat{T} : \text{Æ}(X) \to E^*$ has finite rank) set

$$\theta_{p,r,s}(T) = \inf \Big\{ \Big\| \big(\tilde{\lambda}_k\big)_k \Big\|_p \big\| \big(\|v_k^*\| \big)_k \big\|_r \big\| \big(\mathrm{Lip}(f_k) \big)_k \big\|_s \Big\}$$

where the infimum is taken over all representations of T as above. For any such T and $m = \sum_j v_j m_{x_j x'_j}$, $\lambda_j, \kappa_j > 0$ using the fact that $0 < \beta \le 1$ and Hölder's inequality

$$\begin{aligned}
\big|\langle T, m\rangle\big| &= \Big|\sum_{j,k} \tilde{\lambda}_k v_k^*(v_j)\big[f_k(x_j) - f_k(x'_j)\big]\Big| \\
&\le \sum_{j,k} \big|\tilde{\lambda}_k \lambda_j \kappa_j v_k^*(v_j)\lambda_j^{-1}\kappa_j^{-1}\big[f_k(x_j) - f_k(x'_j)\big]\big| \\
&\le \Big(\sum_{j,k} \big|\tilde{\lambda}_k \lambda_j \kappa_j v_k^*(v_j)\lambda_j^{-1}\kappa_j^{-1}\big[f_k(x_j) - f_k(x'_j)\big]\big|^\beta\Big)^{1/\beta} \\
&\le \Big\|\big(\tilde{\lambda}_k\lambda_j\big)_{j,k}\Big\|_p \Big\|\big(\lambda_j^{-1}\kappa_j^{-1}\big[f_k(x_j) - f_k(x'_j)\big]\big)_{j,k}\Big\|_r \Big\|\big(\kappa_j v_k^*(v_j)\big)_{j,k}\Big\|_s .
\end{aligned}$$

Note that

$$\begin{aligned}
\Big\|\big(\tilde{\lambda}_k\lambda_j\big)_{j,k}\Big\|_p &\le \Big\|\big(\tilde{\lambda}_k\big)_k\Big\|_p \big\|(\lambda_j)_j\big\|_p \\
\Big\|\big(\lambda_j^{-1}\kappa_j^{-1}\big[f_k(x_j) - f_k(x'_j)\big]\big)_{j,k}\Big\|_r &\le \big\|\big(\operatorname{Lip}(f_k)\big)_k\big\|_r \, w_r^{\operatorname{Lip}}\big((\lambda_j^{-1}, \kappa_j^{-1}, x_j, x'_j)_j\big) \\
\Big\|\big(\kappa_j v_k^*(v_j)\big)_{j,k}\Big\|_s &\le \big\|\big(\, \|v_k^*\|\, \big)_k\big\|_s \, w_s\big((\kappa_j v_j)_j\big)
\end{aligned}$$

so by taking the infimum over all representations of both m and T, and all positive numbers λ_j, κ_j we obtain

$$\big|\langle T, m\rangle\big| \le \mu_{p,r,s}(m)\theta_{p,r,s}(T).$$

Therefore, if $\mu_{p,r,s}(m) = 0$ we have $\langle v^* f, m\rangle = 0$ for all $v^* \in E^*$, $f \in X^\#$. By duality between Æ(X) and $X^\#$, that means the real-valued molecule $v^* \circ m$ is equal to 0 for all $v^* \in E^*$, so we conclude that $m = 0$ and thus $\mu_{p,r,s}$ is a β-norm rather than just a β-seminorm. □

The β-normed space $\big(\mathcal{M}(X, E), \mu_{p,r,s}\big)$ will be denoted by $M_{p,r,s}$. When $\beta = 1$, what we get is not only a norm but even a reasonable norm on spaces of molecules.

Proposition 2.7.2. *When* $1/p+1/r+1/s=1$, $\mu_{p,r,s}$ *is a reasonable norm on spaces of molecules.*

Proof. For an atom $vm_{xx'}$ with $v \in E$ and $x, x' \in X$, the trivial representation shows that $\mu_{p,r,s}(vm_{xx'}) \leq d(x,x')\,\|v\|$. Now let $m \in \mathcal{M}(X,E)$, and consider a representation $m = \sum_j v_j m_{x_j x_j'}$ and positive reals λ_j, κ_j. Fix $v^* \in E^*$. Then Hölder's inequality implies

$$\begin{aligned}|\langle v^* \circ m, f\rangle| &\leq \sum_j \lambda_j \kappa_j |v^*(v_j)| \kappa_j^{-1} \lambda_j^{-1} |f(x_j) - f(x_j')| \\ &\leq \|(\lambda_j)_j\|_p \,\|v^*\|\, w_s\big((\kappa_j v_j)_j\big)\, \mathrm{Lip}(f) w_r^{\mathrm{Lip}}\big((\kappa_j^{-1}\lambda_j^{-1}, x_j, x_j')_j\big).\end{aligned}$$

Taking the infimum over all representations of m and all positive reals λ_j, κ_j, we conclude that $|\langle v^* \circ m, f\rangle| \leq \|v^*\|\, \mathrm{Lip}(f) \mu_{p,r,s}(m)$. □

2.7.2 Duality

Just as in the linear case, the dual of the (p,r,s)-Lapresté norm is the (p',r,s)-summing norm. An operator $T : X \to E$ is called *Lipschitz* (p,r,s)*-summing* if there is a constant C such that for all $x_j, x_j' \in X$, $v_j \in E^*$, and $\lambda_j, \kappa_j > 0$ we have

$$\Big\|\big(\lambda_j \langle v_j, Tx_j - Tx_j'\rangle\big)_j\Big\|_p \leq C w_r^{\mathrm{Lip}}\big((\lambda_j \kappa_j^{-1}, x_j, x_j')_j\big) w_s\big((\kappa_j v_j)_j\big) \tag{2.7.1}$$

The smallest such constant C will be denoted by $\pi^L_{p,r,s}(T)$, and $\Pi^L_{p,r,s}(X,E)$ will denote the set of all such operators. A few remarks about this definition are in order. First, when $E = F^*$ it suffices to consider only $v_j \in F$. Also, the case (p,p,∞) corresponds to Lipschitz p-summing operators from X to E as in [FJ09], whereas the case (q,p,∞) corresponds to the Lipschitz (q,p)-summing operators from X to E as in [JS09]. Moreover, by the same arguments as in [FJ09], we may take $\lambda_j = 1$ for all j in (2.7.1). Finally, it is easy to check that $\left(\Pi^L_{p,r,s}(X,E), \pi^L_{p,r,s}\right)$ is a normed space.

Theorem 2.7.3. *The spaces $M_{p,r,s}(X,E)^*$ and $\Pi^L_{p',r,s}(X,E^*)$ are isometrically isomorphic via the canonical pairing. Moreover, on the unit ball of $\Pi^L_{p',r,s}(X,E^*)$ the weak* topology coincides with the topology of pointwise $\sigma(E^*,E)$-convergence.*

Proof. First, let $T \in \Pi^L_{p',r,s}(X,E^*)$. Then, for any $m = \sum_j v_j m_{x_j x'_j} \in \mathcal{M}(X,E)$ and $\lambda_j, \kappa_j > 0$, by the pairing formula (2.2.2) and Hölder's inequality

$$\begin{aligned}
\left|\langle T, m\rangle\right| &= \left|\sum_j \langle Tx_j - Tx'_j, v_j\rangle\right| \le \sum_j \left|\langle Tx_j - Tx'_j, v_j\rangle\right| \\
&\le \|(\lambda_j)_j\|_p \left\|\left(\lambda_j^{-1}\langle Tx_j - Tx'_j, v_j\rangle\right)_j\right\|_{p'} \\
&\le \|(\lambda_j)_j\|_p \, \pi^L_{p',r,s}(T) w_r^{\mathrm{Lip}}\big((\lambda_j^{-1}\kappa_j^{-1}, x_j, x'_j)_j\big) w_s((\kappa_j v_j)_j).
\end{aligned}$$

Taking the infimum over all representations of m and $\lambda_j, \kappa_j > 0$ we conclude that $\left|\langle T, m\rangle\right| \le \pi^L_{p',r,s}(T)\mu_{p,r,s}(m)$. Conversely, let $\varphi \in M_{p,r,s}(X,E)^*$ with $\|\varphi\| = C$, so we have $|\varphi(m)| \le C\mu_{p,r,s}(m)$ for any $m \in \mathcal{M}(X,E)$. Note that φ can be identified with a mapping $T : X \mapsto E^*$ via the formula $\langle Tx, v\rangle = \varphi(v m_{x0})$. Indeed, for $x \in X$ and $v \in E$,

$$\begin{aligned}
|\langle Tx, v\rangle| = |\langle \varphi, v m_{x0}\rangle| &\le C\mu_{p,r,s}(v m_{x0}) \\
&\le C \sup_{v^* \in B_{E^*}} |v^*(v)| \sup_{f \in B_{X^\#}} |f(x) - f(0)| = C\,\|v\|\, d(x,0)
\end{aligned}$$

so $Tx \in E^*$. Now, suppose $x_j, x'_j \in X$, $\lambda_j, \kappa_j > 0$. For any sequence $(\alpha_j)_j$ of real numbers with $\|(\alpha_j)_j\|_p \le 1$,

$$\begin{aligned}\Big|\sum_j \alpha_j \lambda_j \langle Tx_j - Tx'_j, v_j\rangle\Big| &= \Big|\langle T, \sum_j \alpha_j \lambda_j v_j m_{x_j x'_j}\rangle\Big| \\ &\le C\mu_{p,r,s}\Big(\sum_j \alpha_j \lambda_j v_j m_{x_j x'_j}\Big) \\ &\le C\,\|(\alpha_j)_j\|_p\, w_r^{\mathrm{Lip}}\big((\kappa_j^{-1}\lambda_j, x_j, x'_j)_j\big) w_s((\kappa_j v_j)_j) \\ &\le C w_r^{\mathrm{Lip}}\big((\kappa_j^{-1}\lambda_j, x_j, x'_j)_j\big) w_s((\kappa_j v_j)_j).\end{aligned}$$

Taking the supremum over all such α,

$$\Big\|\big(\lambda_j \langle Tx_j - Tx'_j, v_j\rangle\big)_j\Big\|_{p'} \le C w_r^{\mathrm{Lip}}\big((\kappa_j^{-1}\lambda_j, x_j, x'_j)_j\big) w_s((\kappa_j v_j)_j),$$

i.e. T is Lipschitz (p', r, s)-summing with $\pi^L_{p',r,s}(T) \le C$. For the statement about the weak*-topology, we use the exact same argument as in the proof of Theorem 2.6.4. □

2.7.3 A special case

Just as in the linear case, when $1/p + 1/r + 1/s = 1$ the dual of $M_{p,r,s}(X, E)$ has another interesting characterization in terms of summing operators. We will make use of the following elementary identity.

Lemma 2.7.4. *Suppose $1 \le p, r, s < \infty$ and $1 = 1/p + 1/r + 1/s$. Then For $a, b, c \ge 0$,*

$$abc = \inf_{\lambda,\kappa>0}\left\{\frac{\lambda^p}{p}a^p + \frac{\kappa^s}{s}b^s + \frac{\lambda^{-r}\kappa^{-r}}{r}c^r\right\}.$$

Proof. It is an easy calculus exercise to show that

$$a(bc) = \inf_{\lambda>0}\left\{\frac{\lambda^p}{p}a^p + \frac{\lambda^{-p'}}{p'}(bc)^{p'}\right\}.$$

Applying the same idea again to the product bc we get the result. □

The following theorem identifies the dual of $M_{p,r,s}(X,E)$ in this special case.

Theorem 2.7.5 (Domination/Factorization). *Suppose* $1/p+1/r+1/s=1$ *and let* $T \in \mathrm{Lip}_0(X,E^*)$, $C>0$. *The following are equivalent:*

(a) $|\langle T,m\rangle| \le C\mu_{p,r,s}(m)$ *for all* $m \in \mathcal{M}(X,E)$.

(b) There exist regular Borel probability measures μ *and* ν *on the weak***-compact unit balls* $B_{X^\#}$, B_{E^*} *(considering* $X^\# = Æ(X)^*$*) such that for all* $x,x' \in X$ *and* $v \in E$,

$$|\langle Tx-Tx',v\rangle| \le C\left[\int_{B_{X^\#}} \left|f(x)-f(x')\right|^r d\mu(f)\right]^{1/r}\left[\int_{B_{E^*}} |v^*(v)|^s\, d\nu(v^*)\right]^{1/s}.$$

(c) There exist a Banach space Z, *a Lipschitz* r*-summing operator* $R: X \to Z^*$ *and a linear* s*-summing operator* $S: E \to Z$ *such that* $\pi_r^L(R)\cdot\pi_s(S) \le C$ *and*

$$\langle Tx,v\rangle = \langle Rx,Sv\rangle \qquad \text{for all } x \in X, v \in E;$$

that is, $T = S^* \circ R$.

Note that condition (c) can be considered as a Lipschitz version of (linear) (r,s)-dominated operators, i.e. those that can be factored as a composition of an r-summing operator and the adjoint of an s-summing operator (see, e.g. [DF93, p. 241]).

Proof. We will assume $p,r,s<\infty$ for the sake of simplicity; the other cases have similar proofs (for instance, the case $s=\infty$ follows from the domination theorem for Lipschitz p-summing operators [FJ09, Thm. 1] and Theorem 2.7.3).

(a) $\Rightarrow$ (b) Consider a molecule $m = \sum_j v_j m_{x_j x_j'}$, $x_j, x_j' \in X$, $v_j \in E$. By the definition of $\mu_{p,r,s}$, for any $\lambda_j, \kappa_j > 0$

$$\Big|\sum_j \langle Tx_j - Tx_j', v_j\rangle\Big| \leq$$
$$C\Big(\sum_j \lambda_j^p\Big)^{1/p} \sup_{f\in B_X^\#}\Big(\sum_j \lambda_j^{-r}\kappa_j^{-r}\big|f(x_j)-f(x_j')\big|^r\Big)^{1/r} \sup_{v^*\in B_{E^*}}\Big(\sum_j \kappa_j^s|v^*(v_j)|^s\Big)^{1/s}.$$

Lemma 2.7.4 gives for any $\gamma, \delta > 0$,

$$\Big|\sum_j \langle Tx_j - Ty_j, v_j\rangle\Big| \leq$$
$$C \sup_{f\in B_{X^\#}, v^*\in B_{E^*}} \sum_j \Big[\frac{\gamma^p}{p}\lambda_j^p + \frac{\delta^s}{s}\kappa_j^s|v^*(v_j)|^s + \frac{\gamma^{-r}\delta^{-r}}{r}\lambda_j^{-r}\kappa_j^{-r}\big|f(x_j)-f(x_j')\big|^r\Big].$$

This means, after renaming variables, that for all $\lambda_j, \kappa_j > 0$

$$\Big|\sum_j \langle Tx_j - Tx_j', v_j\rangle\Big| \leq$$
$$C \sup_{f\in B_{X^\#}, v^*\in B_{E^*}} \sum_j \Big[\frac{\lambda_j^p}{p} + \frac{\kappa_j^s}{s}|v^*(v_j)|^s + \frac{\lambda_j^{-r}\kappa_j^{-r}}{r}\big|f(x_j)-f(x_j')\big|^r\Big]. \quad (2.7.2)$$

We now use the same idea as in the proof of the Pietsch Domination Theorem [Pie67, Thm. 2] to find the measures μ and ν. Working on the space $C(B_{X^\#}\times B_{E^*})$, consider the set L consisting of functions of the form

$$g_A(f,v^*) = \Big|\sum_{(x,x',v,\lambda,\kappa)\in A} \langle Tx - Tx', v\rangle\Big| -$$
$$C \sum_{(x,x',v,\lambda,\kappa)\in A}\Big[\frac{\lambda^p}{p} + \frac{\kappa^s}{s}|v^*(v)|^s + \frac{\lambda^{-r}\kappa^{-r}}{r}\big|f(x)-f(x')\big|^r\Big],$$

where A is a finite subset of $X\times X\times E\times\mathbb{R}^+\times\mathbb{R}^+$. Then L is a convex set and every function in L takes at least one non-positive value by (2.7.2). In particular, L

is disjoint from the open positive cone P of $C(B_{X^\#} \times B_{E^*})$, and hence there exists a regular (finite) Borel measure μ_0 on $B_{X^\#} \times B_{E^*}$ that separates L and P. Arguing as usual (see, for example, the proof of Theorem 2.8.3), we may assume that μ_0 is a probability measure and $\langle f_A, \mu_0 \rangle \leq 0$ for every $f_A \in L$. Taking a singleton $A = \big\{(x, x', v, \lambda, \kappa)\big\}$ we get

$$\begin{aligned}
&|\langle Tx - Tx', v\rangle| \leq \\
&\qquad C \int_{B_{X^\#} \times B_{E^*}} \left[\frac{\lambda^p}{p} + \frac{\kappa^s}{s} |v^*(v)|^s + \frac{\lambda^{-r}\kappa^{-r}}{r} \big|f(x) - f(x')\big|^r \right] d\mu_0(f, v^*) \\
&\qquad\qquad = C \Bigg[\frac{\lambda^p}{p} + \frac{\kappa^s}{s} \int_{B_{X^\#} \times B_{E^*}} |v^*(v)|^s \, d\mu_0(f, v^*) \\
&\qquad\qquad\qquad + \frac{\lambda^{-r}\kappa^{-r}}{r} \int_{B_{X^\#} \times B_{E^*}} \big|f(x) - f(x')\big|^r \, d\mu_0(f, v^*) \Bigg].
\end{aligned}$$

Another application of Lemma 2.7.4 gives

$$|\langle Tx - Tx', v\rangle| \leq C \left[\int_{B_{E^*}} |v^*(v)|^s \, d\nu(v^*) \right]^{1/s} \left[\int_{B_{X^\#}} \big|f(x) - f(x')\big|^r \, d\mu(f) \right]^{1/r}$$

where μ and ν are the marginals of μ_0.

(b) $\Rightarrow$ *(c)* Let $j_X : X \to L_r(\mu)$ and $j_E : E \to L_s(\nu)$ be given by

$$(j_X x)(f) = f(x), \quad (j_E v)(v^*) = v^*(v) \qquad \text{for all } x \in X, v \in E, f \in B_{X^\#}, v^* \in B_{E^*}.$$

Note that j_X is Lipschitz r-summing (resp. j_E is linear s-summing) since it factors through the canonical injection $C(B_{X^\#}) \to L_r(\mu)$ (resp. through $C(B_{E^*}) \to L_s(\nu)$) and moreover $\pi_r^L(j_X) \leq 1$ (resp. $\pi_s(j_E) \leq 1$).

Let $\tilde{X} := j_X(X) \subset L_r(\mu)$ and $Z := \overline{j_E(E)} \subset L_s(\nu)$. Define $U : \tilde{X} \to Z^*$ by

$$\langle U j_X(x), j_E(v) \rangle = \langle Tx, v \rangle \qquad \text{for all } x \in X, v \in E.$$

First note that this indeed defines an element of Z^*, since by condition (b) we have for all $x \in X$ and $v \in E$

$$\left|\langle Uj_X(x), j_E(v)\rangle\right| = |\langle Tx, v\rangle| \leq C \left\|j_X(x)\right\|_{L_r(\mu)} \left\|j_E(v)\right\|_{L_s(\nu)} = C \left\|j_X(x)\right\|_{L_r(\mu)} \left\|j_E(v)\right\|_Z$$

and then we extend to all of Z by continuity. Moreover, U is Lipschitz with $\mathrm{Lip}(U) \leq C$: for any $x, x' \in X$, by definition of U and condition (b)

$$\begin{aligned}
\left\|Uj_X(x) - Uj_X(x')\right\|_{Z^*} &= \sup_{\|j_E(v)\|_{L_s(\nu)}\leq 1} \left|\langle Uj_X(x) - Uj_X(x'), j_E(v)\rangle\right| \\
&= \sup_{\|j_E(v)\|_{L_s(\nu)}\leq 1} \left|\langle Tx - Tx', v\rangle\right| \\
&\leq \sup_{\|j_E(v)\|_{L_s(\mu)}\leq 1} C \left\|j_X(x) - j_X(x')\right\|_{L_r(\mu)} \left\|j_E(v)\right\|_{L_s(\mu)} \\
&= C \left\|j_X(x) - j_X(y)\right\|_{L_r(\mu)}.
\end{aligned}$$

Therefore, we have (c) with $S = j_E : E \to Z$ and $R = Uj_X : X \to Z^*$, since clearly $\langle Tx, v\rangle = \langle Rx, Sv\rangle$, and

$$\pi_r^L(R)\pi_s(S) = \pi_r^L(Uj_X)\pi_s(S) \leq \mathrm{Lip}(U)\pi_r^L(j_X)\pi_s(j_E) \leq C \cdot 1 \cdot 1 = C.$$

(c) $\Rightarrow$ (a) Suppose there exist operators R and S as in (c). Then for any molecule $m = \sum_j v_j m_{x_j x_j'}$ and any $\lambda_j, \kappa_j > 0$ the pairing formula 2.2.2 and Hölder's inequality give

$$\begin{aligned}
\left|\langle T, m\rangle\right| &= \left|\sum_j \langle Sv_j, Rx_j - Rx_j'\rangle\right| \leq \sum_j \left|\langle Sv_j, Rx_j - Rx_j'\rangle\right| \\
&\leq \sum_j \|Sv_j\| \cdot \left\|Rx_j - Rx_j'\right\| = \sum_j \lambda_j\kappa_j \|Sv_j\| \cdot \lambda_j^{-1}\kappa_j^{-1} \left\|Rx_j - Rx_j'\right\| \\
&\leq \left(\sum_j \lambda_j^p\right)^{1/p} \left(\sum_j \kappa_j^s \|Sv_j\|^s\right)^{1/s} \left(\sum_j \lambda_j^{-r}\kappa_j^{-r} \left\|Rx_j - Rx_j'\right\|^r\right)^{1/r}.
\end{aligned}$$

Since R is Lipschitz r-summing and S is s-summing, the last expression is at most

$$\pi_r^L(R)\pi_s(S)\Big(\sum_j \lambda_j^p\Big)^{1/p} \cdot \sup_{v^*\in B_{E^*}}\Big(\sum_j \kappa_j^s|v^*(v_j)|^s\Big)^{1/s} \sup_{f\in B_X^\#}\Big(\sum_j \lambda_j^{-r}\kappa_j^{-r}\big|f(x_j)-f(x'_j)\big|^r\Big)^{1/r}.$$

Taking the infimum over all representations of m and all $\lambda_j, \kappa_j > 0$, we conclude that $\big|\langle T, m\rangle\big| \le C\mu_{p,r,s}(m)$. □

2.8 The norm of Lipschitz factorization through subsets of Hilbert space

One very common way of defining operator ideals is through factorization schemes. A particularly interesting one, known as Γ_2, is the ideal of operators that factor through a Hilbert space. In this section we introduce and study a nonlinear counterpart, and use molecules to identify its dual space. We start with a definition of when a sequence of pairs of points in a metric space "dominates" another one. Compare to [Pis86, Page 22].

Definition 2.8.1. Let $x_j, x'_j, y_i, y'_i \in X$ and $\mu_j, \lambda_i \in \mathbb{R}$. We write

$$(\lambda_i, y_i, y'_i)_{i=1}^n \prec (\mu_j, x_j, x'_j)_{j=1}^m$$

if for all $f \in X^\#$,

$$\sum_{i=1}^n \lambda_i^2|f(y_i)-f(y'_i)|^2 \le \sum_{j=1}^m \mu_j^2|f(x_j)-f(x'_j)|^2.$$

In similarity with the linear case, there is an alternate characterization of dominance that involves contractions between finite-dimensional Hilbert spaces. This corresponds to [Pis86, Prop. 2.2]

Lemma 2.8.2. *Let $x_j, x'_j, y_i, y'_i \in X$ and $\mu_j, \lambda_i \in \mathbb{R}$. Then $(\lambda_i, y_i, y'_i)_{i=1}^n \prec (\mu_j, x_j, x'_j)_{j=1}^m$ if and only if there exists a matrix (a_{ij}) such that*

$$\sum_{i=1}^{n} \Big| \sum_{j=1}^{m} a_{ij} t_j \Big|^2 \leq \sum_{j=1}^{m} |t_j|^2 \text{ for all } (t_j)_{j=1}^m \in \mathbb{R}^m \tag{2.8.1}$$

and for $1 \leq i \leq n$,

$$\lambda_i m_{y_i y'_i} = \sum_{j=1}^{m} a_{ij} \mu_j m_{x_j x'_j}.$$

Proof. Suppose that $(\lambda_i, y_i, y'_i)_{i=1}^n \prec (\mu_j, x_j, x'_j)_{j=1}^m$. Let

$$S = \big\{ \big(\mu_j \big[f(x_j) - f(x'_j) \big] \big)_{j=1}^m : f \in X^{\#} \big\} \subseteq \ell_2^m.$$

Define a linear operator $A : S \to \ell_2^n$ by

$$A\big(\big(\mu_j \big[f(x_j) - f(x'_j) \big] \big)_{j=1}^m \big) = \big(\lambda_i \big[f(y_i) - f(y'_i) \big] \big)_{i=1}^n.$$

Note that the condition $(\lambda_i, y_i, y'_i)_{i=1}^n \prec (\mu_j, x_j, x'_j)_{j=1}^m$ implies that A is well-defined, whereas (2.8.1) implies that $\|A\| \leq 1$. By Hahn-Banach, we can extend A to an operator $\tilde{A} : \ell_2^m \to \ell_2^n$ with $\|\tilde{A}\| \leq 1$. The matrix representation (a_{ij}) of $\tilde{A}$ with respect to the canonical bases of ℓ_2^m and ℓ_2^n clearly satisfies (2.8.1), and by definition we have for all $f \in X^{\#}$ and $1 \leq i \leq n$

$$\lambda_i \big[f(y_i) - f(y'_i) \big] = \sum_{j=1}^{m} a_{ij} \mu_j \big[f(x_j) - f(x'_j) \big].$$

By the duality between $\mathscr{F}(X)$ and $X^{\#}$, this means precisely that

$$\lambda_i m_{y_i y'_i} = \sum_{j=1}^{m} a_{ij} \mu_j m_{x_j x'_j}.$$

For the converse, we just reverse the preceding argument. $\square$

We now proceed to prove a characterization of Lipschitz maps that factor through a subset of a Hilbert space. In the linear case one actually gets factorizations through the whole Hilbert space, and that is because every subspace of a Hilbert space is complemented. Since not every subset of a Hilbert space is a Lipschitz retract of it, factoring through a subset or through the whole Hilbert space are different concepts in the Lipschitz category. The proof follows very closely that of [Pis86, Thm. 2.4]

Theorem 2.8.3. *Let X and Z be metric spaces, and $C > 0$. For a map $T : X \to Z$ the following are equivalent:*

(i) There exist a Hilbert space H and Lipschitz maps $R : X \to H$, $S : R(X) \to Z$ such that $T = SR$ and $\mathrm{Lip}(R) \cdot \mathrm{Lip}(S) \leq C$.

(ii) Whenever $x_j, x'_j, y_i, y'_i \in X$ and $\mu_j, \lambda_i \in \mathbb{R}$ satisfy $(\lambda_i, y_i, y'_i)_{i=1}^n \prec (\mu_j, x_j, x'_j)_{j=1}^m$, we have that

$$\sum_{i=1}^n \lambda_i^2 d_Z(Ty_i, Ty'_i)^2 \leq C^2 \sum_{j=1}^m \mu_j^2 d_X(x_j, x'_j)^2.$$

We denote by $\gamma_2^{\mathrm{Lip}}(T)$ be the infimum of such constants C.

Proof. *(i)* $\Rightarrow$ *(ii)* Suppose we have such a factorization. By rescaling the Hilbert space, we may assume that $\mathrm{Lip}(S) = 1$ and $\mathrm{Lip}(R) \leq C$. Let $x_j, x'_j, y_i, y'_i \in X$ and $\lambda_i, \mu_j \in \mathbb{R}$ be such that $(\lambda_i, y_i, y'_i)_{i=1}^n \prec (\mu_j, x_j, x'_j)_{j=1}^m$. Note that for any $v \in H$ the function $x \mapsto \langle Rx, v\rangle$ is in $X^\#$, so

$$\sum_{i=1}^n \lambda_i^2 |\langle Ry_i - Ry'_i, v\rangle|^2 \leq \sum_{j=1}^m \mu_j^2 |\langle Rx_j - Rx'_j, v\rangle|^2 \qquad \text{for all } v \in H.$$

Let $(v_\alpha)_{\alpha\in A}$ be an orthonormal basis for H. Since $\|v\|^2 = \sum_{\alpha\in A} |\langle v, v_\alpha\rangle|^2$ for any $v \in H$, we conclude that

$$\sum_{i=1}^n \lambda_i^2 \left\| Ry_i - Ry'_i \right\|^2 \leq \sum_{j=1}^m \mu_j^2 \left\| Rx_j - Rx'_j \right\|^2.$$

Hence,

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^2 d_Z(Ty_i, Ty_i')^2 &= \sum_{i=1}^{n} \lambda_i^2 d_Z(SRy_i, SRy_i')^2 \le \sum_{i=1}^{n} \lambda_i^2 \operatorname{Lip}(S)^2 \left\|Ry_i - Ry_i'\right\|^2 \\ &= \sum_{i=1}^{n} \lambda_i^2 \left\|Ry_i - Ry_i'\right\|^2 \le \sum_{j=1}^{m} \mu_j^2 \left\|Rx_j - Rx_j'\right\|^2 \\ &\le \sum_{j=1}^{m} \mu_j^2 \operatorname{Lip}(R)^2 d_X(x_j, x_j')^2 \le C^2 \sum_{j=1}^{m} \mu_j^2 d_X(x_j, x_j')^2.\end{aligned}$$

$(ii) \Rightarrow (i)$ For each $x \in X$, denote by δ_x its corresponding evaluation function in $C(B_{X^\#})$. Consider the following subsets of $C(B_{X^\#})$:

$$\begin{aligned}K_1 &:= \Big\{ \sum_{i=1}^{n} \lambda_i^2 |\delta_{y_i} - \delta_{y_i'}|^2 \ : \ \lambda_i \in \mathbb{R}, \ \sum_{i=1}^{n} \lambda_i^2 d_Z(Ty_i, Ty_i')^2 \ge 1 \Big\} \\ K_2 &:= \Big\{ \sum_{j=1}^{m} \mu_j^2 |\delta_{x_j} - \delta_{x_j'}|^2 \ : \ \mu_j \in \mathbb{R}, \ \sum_{j=1}^{m} \mu_j^2 d_X(x_j, x_j')^2 \le 1 \Big\}.\end{aligned}$$

Clearly, both K_1 and K_2 are convex. Set

$$K = \bigcup_{\rho > C} (\rho K_1 - K_2).$$

Note that K is also convex: Let $h \in \rho_1 K_1 - K_2$, $h' \in \rho' K_1 - K_2$ with $\rho' > \rho$. Then $h = \rho h_1 - h_2$, $h' = \rho' h_1' - h_2$ with $h_r, h_r' \in K_r$, $r = 1, 2$. Note that $\rho' h_1' = \rho(\rho'/\rho) h_1'$ and $\rho'/\rho > 1$, so in fact $\rho' h_1' \in \rho K_1$ (since $\tilde{h} \in K_1, \eta \ge 1$ imply $\eta \tilde{h} \in K_1$). Therefore, we in fact have $h, h' \in \rho K_1 - K_2$ from where, using the convexity of K_1 and K_2, it is obvious that $\omega h + (1 - \omega) h \in \rho K_1 - K_2 \subset K$ for any $\omega \in [0, 1]$. Moreover, the condition (ii) implies that every function $h \in K$ has a maximum ≥ 0 on $B_{X^\#}$. Otherwise, we would have $\rho > C$, $x_j, x_j', y_i, y_i' \in X$, $\lambda_i, \mu_j \in \mathbb{R}$ such that

$$\sum_{i=1}^{n} \lambda_i^2 d_Z(Ty_i, Ty_i')^2 \ge 1 \ge \sum_{j=1}^{m} \mu_j^2 d_X(x_j, x_j')^2$$

and for all $f \in B_{X^\#}$,

$$\rho^2 \sum_{i=1}^{n} \lambda_i^2 |f(y_i) - f(y_i')|^2 - \sum_{j=1}^{m} \mu_j^2 |f(x_j) - f(x_j)|^2 \leq 0.$$

But then

$$\sum_{i=1}^{n} \rho^2 \lambda_i^2 |f(y_i) - f(y_i')|^2 \leq \sum_{j=1}^{m} \mu_j^2 |f(x_j) - f(x_j)|^2 \text{ for all } f \in X^\#$$

despite the fact that

$$\sum_{i=1}^{n} \rho^2 \lambda_i^2 d_Z(Ty_i, Ty_i')^2 > C^2 \sum_{j=1}^{m} \mu_j^2 d_X(x_j, x_j')^2,$$

in plain contradiction with (ii).

Therefore, K is disjoint from the open cone N of negative functions in $C(B_{X^\#})$. By the Hahn-Banach and Riesz representation theorems, there exists a signed Borel measure ν that separates N and K, i.e. there exists a real number α such that for all $h \in K$, $g \in N$ $\int h d\nu \geq \alpha \geq g d\nu$. Since N is closed under multiplication by positive constants, $\alpha \geq 0$. Then $\int g d\nu \leq 0$ for all $g \in N$, so ν is a positive measure such that $\int h d\nu \geq 0$ for all $h \in K$. Define $R : X \to L_2(\nu)$ by $R(x) = \delta_x$ and $S : R(X) \subset L_2(\nu) \to E$ by $S(\delta_x) = Tx$. Note that $T = SR$ and multiplying ν by an appropiate positive constant we may assume that $\text{Lip}(R) = C$.

Let $x, x', y, y' \in X$ be such that $x \neq x'$ and $Ty \neq Ty'$. From the definition of ν we have

$$C^2 \frac{1}{d_Z(Ty, Ty')^2} \int_{B_X^\#} |f(y) - f(y')|^2 d\nu(f) \geq \frac{1}{d_X(x, x')^2} \int_{B_X^\#} |f(x) - f(x')|^2 d\nu(f)$$

or equivalently

$$\frac{C}{d_Z(Ty - Ty')} \|Ry - Ry'\|_{L_2(\nu)} \geq \frac{1}{d_X(x, x')} \|Rx - Rx'\|_{L_2(\nu)} .$$

Choosing $x, x' \in X$ so that $\|Rx - Rx'\|_{L_2(\nu)} / d_X(x, x')$ is arbitrarily close to $\mathrm{Lip}(R) = C$, we conclude

$$\|\delta_y - \delta_{y'}\|_{L_2(\nu)} = \|Ry - Ry'\|_{L_2(\nu)} \geq d_Z(Ty, Ty') = d_Z(S(\delta_y), S(\delta_{y'})).$$

Therefore $\mathrm{Lip}(S) \leq 1$, so we have condition (i). $\square$

A few remarks are in order. First, by the Farmer/Johnson/Mendel/Schechtman argument already referred to in the discussion of Lipschitz p-summing operators (see Section 2.6), it suffices to consider the case where all λ_i and μ_j are equal to 1. Second, as in the linear case, the measure in the previous proof is not necessarily a probability measure. Finally, it is clear that γ_2^{Lip} has the ideal property.

2.8.1 Duality

Let us now give the definition of the norm that is in duality with γ_2^{Lip}.

Definition 2.8.4. Let m be an E-valued molecule on X. Define

$$\|m\|_* = \inf \Bigg\{ \Big(\sum_{i=1}^n \|v_i\|^2 \Big)^{1/2} \Big(\sum_{j=1}^m \mu_j^2 d(x_j, x_j')^2 \Big)^{1/2} \ : \ x_j, x_j', y_i, y_i' \in X, \lambda_i, \mu_j \in \mathbb{R}, \\ v_i \in E, m = \sum_{i=1}^n \lambda_i v_i m_{y_i y_i'} \text{ and } (\lambda_i, y_i, y_i')_{i=1}^n \prec (\mu_j, x_j, x_j')_{j=1}^m \Bigg\}$$

One could try to take a slightly different condition more closely related to the one in the linear case, namely

$$m = \sum_{i=1}^n \sum_{j=1}^m a_{ij} v_i \mu_j m_{x_j x_j'} \text{ with } (a_{ij}) \text{ satisfying (2.8.1)},$$

which is equivalent to

$$|\langle v^* \circ m, f, \rangle| \le \Big(\sum_{i=1}^n |v^*(v_i)|^2\Big)^{1/2}\Big(\sum_{j=1}^m \mu_j^2 |f(x_j) - f(x_j')|^2\Big)^{1/2} \text{ for all } f \in X^{\#}, v^* \in E^*.$$

Unfortunately that is not the right choice, it turns out that we also need each of the sums $\sum_{j=1}^m a_{ij}\mu_j m_{x_j x_j'}$ to be an elementary molecule.

Lemma 2.8.5. $\|\cdot\|_*$ *is a norm on* $\mathcal{M}(X, E)$.

Proof. It is clear that for any molecule $m \in \mathcal{M}(X, E)$ and any scalar λ, $\|m\|_* \ge 0$ and $\|\lambda m\|_* = |\lambda| \, \|m\|_*$.

Let $m_1, m_2 \in \mathcal{M}(X, E)$ and $\varepsilon > 0$. Choose a representation $m_1 = \sum_{i=1}^n \lambda_i v_i m_{y_i y_i'}$ and $(\mu_j, x_j, x_j')_{j=1}^m \succ (\lambda_i, y_i, y_i')_{i=1}^n$ such that

$$\Big(\sum_{i=1}^n \|v_i\|^2\Big)^{1/2}\Big(\sum_{j=1}^m \mu_j^2 d(x_j, x_j')^2\Big)^{1/2} \le \|m_1\|_* + \varepsilon.$$

By absorbing a constant into the v_i's, we may assume that

$$\Big(\sum_{i=1}^n \|v_i\|^2\Big)^{1/2} \le \big(\|m_1\|_* + \varepsilon\big)^{1/2} \quad \text{and} \quad \Big(\sum_{j=1}^m \mu_j^2 d(x_j, x_j')^2\Big)^{1/2} \le \big(\|m_1\|_* + \varepsilon\big)^{1/2}.$$

Similarly, choose a representation $m_2 = \sum_{i=n+1}^{n+k} \lambda_i v_i m_{y_i y_i'}$ and $(\mu_j, x_j, x_j')_{j=m+1}^{m+l} \succ (\lambda_i, y_i, y_i')_{i=n+1}^{n+k}$ such that

$$\Big(\sum_{i=n+1}^{n+k} \|v_i\|^2\Big)^{1/2} \le \big(\|m_2\|_* + \varepsilon\big)^{1/2} \quad \text{and} \quad \Big(\sum_{j=m+1}^{m+l} \mu_j^2 d(x_j, x_j')^2\Big)^{1/2} \le \big(\|m_2\|_* + \varepsilon\big)^{1/2}.$$

Then $m_1 + m_2 = \sum_{i=1}^{n+k} \lambda_i v_i m_{y_i y_i'}$, $(\mu_j, x_j, x_j')_{j=1}^{m+l} \succ (\lambda_i, y_i, y_i')_{i=1}^{n+k}$ and

$$\Big(\sum_{i=1}^{n+k} \|v_i\|^2\Big)^{1/2}\Big(\sum_{j=1}^{m+l} \mu_j^2 d(x_j, x_j')^2\Big)^{1/2} \le \|m_1\|_* + \|m_2\|_* + 2\varepsilon$$

so $\|m_1 + m_2\|_* \leq \|m_1\|_* + \|m_2\|_* + 2\varepsilon$, and by letting $\varepsilon \downarrow 0$ we have the triangle inequality for $\|\cdot\|_*$.

Let $T \in \mathrm{Lip}_0(X, E^*)$ be a map that admits a representation as a finite sum of the form $\sum_k v_k^* f_k$ with $(v_k^*)_k \subset E^*$, $(f_k)_k \subset X^\#$ (i.e. such that the linearization $\hat{T} : \text{Æ}(X) \to E^*$ has finite rank). For such a T, set

$$\theta(T) = \inf \Big\{ \sum_k \|v_k^*\| \, \mathrm{Lip}(f_k) \Big\}$$

where the infimum is taken over all representations as above. Now, given $m = \sum_{i=1}^n \lambda_i v_i m_{y_i y_i'} \in \mathcal{M}(X, E)$, and assume $(\lambda_i, y_i, y_i')_{i=1}^n \prec (\mu_j, x_j, x_j')_{j=1}^m$. From Lemma 2.8.2, there exists a matrix (a_{ij}) satisfying (2.8.1) and such that for $1 \leq i \leq n$, $\lambda_i m_{y_i y_i'} = \sum_{j=1}^m a_{ij} \mu_j m_{x_j x_j'}$. We then have from the pairing formula (2.2.2), the Cauchy-Schwartz inequality and the property (2.8.1) of the matrix (a_{ij}),

$$\begin{aligned}
\big|\langle T, m\rangle\big| &= \Big| \sum_{i,j,k} v_k^*(v_i) a_{ij} \mu_j \big[f_k(x_j) - f_k(x_j')\big] \Big| \\
&\leq \sum_k \sum_i \Big| v_k^*(v_i) \sum_j a_{ij} \mu_j \big[f_k(x_j) - f_k(x_j')\big] \Big| \\
&\leq \sum_k \Big(\sum_i |v_k^*(v_i)|^2 \Big)^{1/2} \Big(\sum_i \Big| \sum_j a_{ij} \mu_j \big[f_k(x_j) - f_k(x_j')\big] \Big|^2 \Big)^{1/2} \\
&\leq \sum_k \|v_k^*\| \Big(\sum_i \|v_i\|^2 \Big)^{1/2} \Big(\sum_j \mu_j^2 \big| f_k(x_j) - f_k(x_j') \big|^2 \Big)^{1/2} \\
&\leq \sum_k \|v_k^*\| \, \mathrm{Lip}(f_k) \Big(\sum_i \|v_i\|^2 \Big)^{1/2} \Big(\sum_j \mu_j^2 d(x_j, x_j')^2 \Big)^{1/2}.
\end{aligned}$$

Taking the infimum over all representations of both T and m, we deduce $\big|\langle T, m\rangle\big| \leq \|m\|_\pi \, \theta(T)$. In particular, this applies to maps T of the form $v^* \circ f$ with $v^* \in E^*$

and $f \in X^{\#}$, so if m is such that $\|m\|_* = 0$ then we have, using the pairing formula (2.2.2),

$$0 = \langle v^* \circ f, m \rangle = \sum_j v^*(v_j)\big[f(x_j) - f(x'_j)\big] \text{ for all } v^* \in E^*, f \in X^{\#}.$$

By the duality between Æ(X) and $X^{\#}$ (see Theorem 2.2.1), this means that the real-valued molecule $v^* \circ m$ is equal to 0 for all $v^* \in E^*$ and consequently $m = 0$. □

Moreover, let us now show that this norm is a reasonable one.

Proposition 2.8.6. *The norm* $\|\cdot\|_*$ *is a reasonable norm.*

Proof. As usual, the obvious representation of an atom shows that $\|vm_{xx'}\|_* \leq \|v\|\, d(x,x')$. Now, suppose $m = \sum_{i=1}^n \lambda_i v_i m_{y_i y'_i}$ and $(\lambda_i, y_i, y'_i)_{i=1}^n \prec (\mu_j, x_j, x'_j)_{j=1}^m$. Then

$$\begin{aligned}
|\langle v^* \circ m, f\rangle| &\leq \|v^*\| \sum_{i=1}^n \|v_i\| \cdot |\lambda_i| \cdot |f(y_i) - f(y'_i)| \\
&\leq \|v^*\| \Big(\sum_{i=1}^n \|v_i\|^2\Big)^{1/2} \Big(\sum_{i=1}^n \lambda_i^2 |f(y_i) - f(y'_i)|^2\Big)^{1/2} \\
&\leq \|v^*\| \Big(\sum_{i=1}^n \|v_i\|^2\Big)^{1/2} \Big(\sum_{j=1}^m \mu_j^2 |f(x_j) - f(x'_j)|^2\Big)^{1/2} \\
&\leq \|v^*\| \Big(\sum_{i=1}^n \|v_i\|^2\Big)^{1/2} \operatorname{Lip}(f) \Big(\sum_{j=1}^m \mu_j^2 d(x_j, x'_j)^2\Big)^{1/2}
\end{aligned}$$

so taking the infimum over all representations of m we obtain the desired inequality: $|\langle v^* \circ m, f\rangle| \leq \|v^*\| \operatorname{Lip}(f)\, \|m\|_*$. □

The following theorem is the main result of this section, and gives the duality for the norm of Lipschitz factorization through subsets of a Hilbert space.

Theorem 2.8.7. *Let* $T : X \to E^*$ *and* $C > 0$. *The following are equivalent:*

(i) $\gamma_2^{\mathrm{Lip}}(T) \leq C$.

(ii) $|\langle T, m\rangle| \leq C \|m\|_*$ *for all* $m \in \mathcal{M}(X, E)$.

Proof. (*i*) $\Rightarrow$ (*ii*) Suppose that $\gamma_2^{\mathrm{Lip}}(T) \leq C$. Let $m \in \mathcal{M}(X, E)$. Let $x_j, x_j', y_i, y_i' \in X$, $\lambda_i, \mu_j \in \mathbb{R}$ and $v_i \in E$ such that $m = \sum_{i=1}^n \lambda_i v_i m_{y_i y_i'}$ and $(\lambda_i, y_i, y_i')_{i=1}^n \prec (\mu_j, x_j, x_j')_{j=1}^m$. Then, using Theorem 2.8.3,

$$\begin{aligned}
|\langle T, m\rangle| &\leq \sum_{i=1}^n \lambda_i \big|\langle Ty_i - Ty_i', v_i\rangle\big| \\
&\leq \Big(\sum_{i=1}^n \|v_i\|^2\Big)^{1/2} \Big(\sum_{i=1}^n \lambda_i^2 \|Ty_i - Ty_i'\|^2\Big)^{1/2} \\
&\leq \Big(\sum_{i=1}^n \|v_i\|^2\Big)^{1/2} \Big(\sum_{j=1}^m \mu_j^2 d(x_j, x_j')^2\Big)^{1/2}
\end{aligned}$$

and therefore $|\langle T, m\rangle| \leq C \|m\|_*$ for all $m \in \mathcal{M}(X, E)$.

(*ii*) $\Rightarrow$ (*i*) Assume condition (*ii*). Suppose $x_j, x_j', y_i, y_i \in X$ and $\mu_j, \lambda_i \in \mathbb{R}$ satisfy $(\lambda_i, y_i, y_i')_{i=1}^n \prec (\mu_j, x_j, x_j')_{j=1}^m$. Then, by Lemma 2.8.2, there exists a matrix (a_{ij}) satisfying (2.8.1) such that for $1 \leq i \leq n$,

$$\lambda_i m_{y_i y_i'} = \sum_{j=1}^m a_{ij} \mu_j m_{x_j x_j'}.$$

Fix $\varepsilon > 0$. For each $1 \leq i \leq n$, choose $v_i \in E^*$ with $\|v_i\| \leq 1+\varepsilon$ and $\langle Ty_i - Ty_i', v_i\rangle = \|Ty_i - Ty_i'\|$. Let $\alpha_i \in \mathbb{R}$ be such that $\sum_{i=1}^n \alpha_i^2 = 1$ and

$$\sum_{i=1}^n \alpha_i \lambda_i \|Ty_i - Ty_i'\| = \Big(\sum_{i=1}^n \lambda_i^2 \|Ty_i - Ty_i'\|^2\Big)^{1/2}.$$

Define a molecule by

$$m = \sum_{i=1}^n \sum_{j=1}^m a_{ij} \alpha_i v_i \mu_j m_{x_j x_j'} = \sum_{i=1}^n \alpha_i \lambda_i v_i m_{y_i y_i'}.$$

Then, by condition (ii),

$$\begin{aligned}\Big(\sum_{i=1}^{n}\lambda_i^2\left\|Ty_i-Ty_i'\right\|^2\Big)^{1/2} &= \sum_{i=1}^{n}\alpha_i\lambda_i\left\|Ty_i-Ty_i'\right\| \\ &= \sum_{i=1}^{n}\alpha_i\lambda_i\langle Ty_i-Ty_i',v_i\rangle = \langle T,m\rangle \\ &\le C\Big(\sum_{i=1}^{n}\alpha_i^2\left\|v_i\right\|^2\Big)^{1/2}\Big(\sum_{j=1}^{m}\mu_j^2 d(x_jx_j')^2\Big)^{1/2} \\ &\le C(1+\varepsilon)\Big(\sum_{i=1}^{n}\alpha_i^2\Big)^{1/2}\Big(\sum_{j=1}^{m}\mu_j^2 d(x_jx_j')^2\Big)^{1/2} \\ &= C(1+\varepsilon)\Big(\sum_{j=1}^{m}\mu_j^2 d(x_jx_j')^2\Big)^{1/2}.\end{aligned}$$

Letting $\varepsilon \downarrow 0$

$$\sum_{i=1}^{n}\lambda_i^2\left\|Ty_i-Ty_i'\right\|^2 \le C^2\sum_{j=1}^{m}\mu_j^2 d(x_j,x_j')^2,$$

so by Theorem 2.8.3, $\gamma_2^{\mathrm{Lip}}(T) \le C$. □

Let us finish the section by noting that Theorem 2.8.7 means that $\Gamma_2^{\mathrm{Lip}}(X,E^*) \equiv (\mathcal{M}(X,E),\left\|\cdot\right\|_*)^*$.

CHAPTER III

RIBE'S PROGRAM FOR MAPS

As in the rest of mathematics, classification problems for Banach spaces are fundamental within the theory. One kind of classification problem can be paraphrased as follows: if two Banach spaces are the same in some category, are they the same as Banach spaces (that is, linearly isomorphic/isometric)? As a first example, an old result of S. Mazur and S. Ulam [MU32] shows that the metric structure of a Banach space determines its linear structure. On the other extreme, the topological structure does not determine the linear one: M. Kadec [Kad67] proved that any two separable Banach spaces are homeomorphic. Somewhere in between these two extremes, we may ask what happens if two Banach spaces are uniformly homeomorphic. A very important result of M. Ribe [Rib76] states that then the two Banach spaces have the same finite dimensional subspaces in the following sense: there is a number $C > 0$ with the property that every finite-dimensional subspace of one of the spaces is embeddable in the other by means of a linear mapping T such that $\|T\| \cdot \|T^{-1}\| \leq C$.

J. Bourgain [Bou86] observed that, in particular, Ribe's result implies that the notions from local theory of normed spaces are determined by the metric structure of the space and thus have a purely metrical formulation. Furthermore, he proposed a "next step": studying these metrical concepts in general metric spaces in an attempt to develop an analogue of the linear theory. This is nowadays known as *Ribe's program*, and it has seen several remarkable successes. A number of *linear* properties of Banach spaces — like superreflexivity, p-convexity, type and cotype — have been characterized in *nonlinear* terms, often giving rise to a new and useful metric concept inspired by the linear theory [Bou86,LNP09,MN08a,MN,MN08b,MN07]. In addition to the obvious theoretical importance of these results, many applications have been found to subjects like the study of bilipschitz, uniform and coarse embeddings of metric spaces, metric Ramsey theorems, and Lipschitz quotients. This applications to the study of metric spaces are particularly interesting due to their connections to theoretical computer science.

As with any other mathematical theory, the local theory of Banach spaces is not only concerned with the objects but also with the morphisms between them. Although Bourgain did mention the local theory in general when laying down the program, so far the emphasis in the literature has been on properties of spaces. The first such result was obtained by Bourgain himself [Bou86], who proved that a Banach space E is not superreflexive if and only if hyperbolic trees of arbitrary height admit uniformly Lipschitz embeddings in E. More recently, works of J. Lee, A. Naor and Y. Peres [LNP09] and M. Mendel and A. Naor [MN08a, MN] give a nonlinear characterization of p-convexity, where the martingales of the linear characterization are replaced by Markov chains. Further work of Mendel and Naor [MN08b, MN07] gives nonlinear characterizations of the classical notions of Rademacher type and cotype.

Not much has been done for operators, but a look back at the history of Banach spaces shows that there is potential in pursuing such an avenue of research. For example, a well-known result of S. Kwapień [Kwa72] states that a Banach space that has both type 2 and cotype 2 is isomorphic to a Hilbert space. The operator version of this states that the composition of a type 2 operator followed by a cotype 2 operator factors through a Hilbert space (see, for example, [TJ89, Cor. 25.11]). Moreover, the operator versions of this and other results are slightly stronger than the corresponding ones for spaces [TJ89, §25]. More generally, paraphrasing A. Pietsch and J. Wenzel [PW98], spaces are needed to understand operators and operators are needed to understand spaces. There is a rich interplay between the local properties of Banach spaces and those of the linear operators between them, and the corresponding results for metric spaces are still mostly unexplored.

All the Banach-space properties for which Ribe's program has been successful (superreflexivity, p-convexity, Rademacher type and cotype) can be generalized to properties of linear operators between Banach spaces, so the rest of this chapter is devoted to showing how the characterizations from [LNP09, MN08a, MN, MN08b, MN07] admit generalizations to operators.

3.1 Markov p-convexity for operators

3.1.1 Introduction

A linear operator $T : E \to F$ is said to be uniformly p-convex with constant C if for all $x, y \in E$ we have

$$\left\| \frac{Tx - Ty}{2} \right\| \leq C \left(\frac{\|x\|^p + \|y\|^p}{2} - \left\| \frac{x+y}{2} \right\|^p \right)^{1/p}. \tag{3.1.1}$$

Uniformly p-convex linear operators are characterized by beautiful martingale inequalities [Wen05] in the spirit of Pisier's classical work [Pis75], but that point of view will not play a direct role for us. Nevertheless, it should be mentioned that Pisier's martingale techniques were the inspiration for the non-linear arguments of Mendel and Naor [MN08a, MN] that we are generalizing here.

More generally, the linear operator $T : E \to F$ is said to be p-convex if there exists an equivalent norm on E such that T considered as an operator from E equipped with the new norm into F is uniformly p-convex. Although it may seem somewhat mysterious at first sight to consider only renormings of the domain, the choice is justified by the fact that this definition of p-convexity for operators is equivalent to having Haar cotype p. We refer the interested reader to [DJP01, Sec. 10] and [PW98, Sec. 7.9] for the details.

3.1.2 Definition and elementary properties

By a *Markov chain* with state space Ω we will mean a sequence of Ω-valued random variables $\{X_t\}_{t\in\mathbb{Z}}$ with the Markov property, i.e.

$$\mathbb{P}(X_{t+1} = x | X_s = x_s \text{ for all } s \le t) = \mathbb{P}(X_{t+1} = x | X_t = x_t)$$

for all $t \in \mathbb{Z}$ and $x, x_j \in \Omega$. In what follows, the state space will always be assumed to be finite. Given a non-negative integer k, $\{\widetilde{X}_t(k)\}_{t\in\mathbb{Z}}$ denotes a process such that $\{\widetilde{X}_t\}_{t\le k}$ equals $\{X_t\}_{t\le k}$, and $\{\widetilde{X}_t\}_{t>k}$ and $\{X_t\}_{t>k}$ are independent and identically distributed.

The following definition is the natural adaptation to mappings of the concept of Markov p-convexity for metric spaces [LNP09].

Definition 3.1.1. A mapping $T : X \to Y$ is called *Markov p-convex with constant C* if for every Markov chain $\{Z_t\}_{t\in\mathbb{Z}}$ on a state space Ω, and every $f : \Omega \to X$, we have

$$\sum_{k=0}^{\infty}\sum_{t\in\mathbb{Z}} \frac{\mathbb{E}\Big[d_Y\big(Tf(Z_t), Tf(\widetilde{Z}_t(t-2^k))\big)^p\Big]}{2^{kp}} \le C^p \cdot \sum_{t\in\mathbb{Z}} \mathbb{E}\big[d_X(f(Z_t), f(Z_{t-1}))^p\big]. \quad (3.1.2)$$

The least constant C above is called the *Markov p-convexity constant of T*, and is denoted $C_p(T)$. We will say that T is Markov p-convex if $C_p(T) < \infty$.

Although it is technically necessary to consider Ω-valued Markov chains and functions $f : \Omega \to X$, whenever possible we will strive for simpler notation to get better readability and consider X-valued Markov chains.

The set of all Markov p-convex maps from X to Y will be denoted by $\mathfrak{C}_p(X, Y)$. From the definition, it is easy to observe that Markov p-convex operators possess the ideal property: that is, $C_p(A \circ T \circ B) \le \mathrm{Lip}(A) \cdot C_p(T) \cdot \mathrm{Lip}(B)$ whenever the composition makes sense. Indeed, if $B : X_0 \to X$, $T : X \to Y$, $A : Y \to Y_0$ are Lipschitz maps, and $\{X_t\}_{t\in\mathbb{Z}}$ is an X_0-valued Markov chain,

$$\sum_{k=0}^{\infty}\sum_{t\in\mathbb{Z}}\frac{\mathbb{E}\Big[d_{Y_0}\big(ATB(X_t),ATB(\widetilde{X}_t(t-2^k))\big)^p\Big]}{2^{kp}}$$
$$\le \operatorname{Lip}(A)^p\sum_{k=0}^{\infty}\sum_{t\in\mathbb{Z}}\frac{\mathbb{E}\Big[d_{Y}\big(TB(X_t),TB(\widetilde{X}_t(t-2^k))\big)^p\Big]}{2^{kp}}$$
$$\le \operatorname{Lip}(A)^pC_p(T)^p\sum_{t\in\mathbb{Z}}\mathbb{E}\big[d_X(B(X_t),B(X_{t-1}))^p\big]$$
$$\le \operatorname{Lip}(A)^pC_p(T)^p\operatorname{Lip}(B)^p\sum_{t\in\mathbb{Z}}\mathbb{E}\big[d_{X_0}(X_t,X_{t-1})^p\big].$$

Also, a simple argument shows that when the codomain is a normed space, the set of Markov p-convex maps with a fixed domain is also a normed space. Indeed, let X be a metric space and E a Banach space, and consider Markov p-convex operators $T, S : X \to E$. Then for any X-valued Markov chain $\{X_t\}_{t\in\mathbb{Z}}$, using the triangle inequality in $\ell_p(E)$,

$$\left[\sum_{k=0}^{\infty}\sum_{t\in\mathbb{Z}}\frac{\mathbb{E}\|(S+T)(X_t)-(S+T)(\widetilde{X}_t(t-2^k)\|^p}{2^{kp}}\right]^{1/p}$$
$$=\left\|\Big(2^{-k}\big((S+T)(X_t)-(S+T)(\widetilde{X}_t(t-2^k)\big)\Big)_{t\in\mathbb{Z},k\ge 0}\right\|_{\ell_p(E)}$$
$$\le\left\|\Big(2^{-k}\big(S(X_t)-S(\widetilde{X}_t(t-2^k)\big)\Big)_{t\in\mathbb{Z},k\ge 0}\right\|_{\ell_p(E)}$$
$$+\left\|\Big(2^{-k}\big(T(X_t)-T(\widetilde{X}_t(t-2^k)\big)\Big)_{t\in\mathbb{Z},k\ge 0}\right\|_{\ell_p(E)}$$
$$\le C_p(S)\left[\sum_{t\in\mathbb{Z}}\mathbb{E}d_X(X_t,X_{t-1})^p\right]^{1/p}+C_p(T)\left[\sum_{t\in\mathbb{Z}}\mathbb{E}d_X(X_t,X_{t-1})^p\right]^{1/p}$$
$$=\big(C_p(S)+C_p(T)\big)\left[\sum_{t\in\mathbb{Z}}\mathbb{E}d_X(X_t,X_{t-1})^p\right]^{1/p},$$

which implies that $S+T$ is Markov p-convex and $C_p(S+T) \leq C_p(S) + C_p(T)$, so $C_p(\cdot)$ satisfies the triangle inequality. Quite obviously $C_p(\cdot)$ is non-negative and positively homogeneous, so it is a seminorm. In order for it to be a bona fide norm we resort to the usual trick: consider X as a pointed metric space with a designated special point denoted by 0, and restrict our attention to the maps that send $0 \in X$ to $0 \in E$.

Figuring out the dual of $\mathfrak{C}_p(X,E)$ (when X is a finite metric space, say) could give a clue as to what a conceivable non-linear notion of smoothness is. Unfortunately, it does not seem that the point of view of molecules introduced in 2 is going to be useful for this.

3.1.3 p-convexity implies Markov p-convexity

The arguments in this section follow closely those of [MN]. It should be mentioned that the author's efforts to adapt the arguments given previously in [LNP09] to the operator case were unsuccessful, but it is still unknown whether or not that is possible.

We start with a technical lemma, where the inequality that defines a p-convex linear operator is adapted to a slightly different one much better suited to the metric setting in that it only involves norms of differences of vectors rather than sums.

Lemma 3.1.2. (Compare to [MN, Lemma 2.3].) *Let $T : E \to F$ be a uniformly p-convex linear operator with constant C. Then for any x, y, z, w in E we have*

$$2\left\|y-x\right\|^p + \left\|y-w\right\|^p + \left\|z-y\right\|^p \geq \frac{\left\|x-w\right\|^p + \left\|z-x\right\|^p}{2^{p-1}} + \frac{\left\|Tz-Tw\right\|^p}{4^{p-1}C^p}$$

Proof. For every x, y, z, w in E, the definition of uniformly p-convex operator with constant C implies

$$\left\|y-x\right\|^p + \left\|y-w\right\|^p \geq \frac{\left\|x-w\right\|^p}{2^{p-1}} + \frac{2}{C^p}\left\|Ty - \frac{Tx+Tw}{2}\right\|^p$$

and

$$\|z-y\|^p + \|y-x\|^p \geq \frac{\|z-x\|^p}{2^{p-1}} + \frac{2}{C^p}\left\|Ty - \frac{Tz+Tx}{2}\right\|^p.$$

Adding together both inequalities and using the convexity of the map $u \mapsto \|u\|^p$ we obtain

$$\begin{aligned}
&2\|y-x\|^p + \|y-w\|^p + \|z-y\|^p \\
&\quad\geq \frac{\|x-w\|^p + \|z-x\|^p}{2^{p-1}} + \frac{4}{C^p} \cdot \frac{\left\|Ty - \frac{Tx+Tw}{2}\right\|^p + \left\|Ty - \frac{Tz+Tx}{2}\right\|^p}{2} \\
&\qquad\geq \frac{\|x-w\|^p + \|z-x\|^p}{2^{p-1}} + \frac{4}{C^p} \cdot \left\|\frac{Tz-Tw}{4}\right\|^p
\end{aligned}$$

from which the conclusion follows. □

Next, the technical Lemma 3.1.2 is used to give a rather straightforward proof of the fact that a p-convex linear operator is Markov p-convex.

Theorem 3.1.3. (Compare to [MN, Prop. 2.1].) *Let $T : E \to F$ be a uniformly p-convex linear operator with constant C. Then T is Markov p-convex with constant $4C$, and thus every p-convex linear operator is Markov p-convex.*

Proof. Consider an E-valued Markov chain $\{X_t\}_{t\in\mathbb{Z}}$. Using Lemma 3.1.2 we see that for every t and k,

$$\begin{aligned}
&\frac{\|X_{t-2^k} - \widetilde{X}_t(t-2^{k-1})\|^p + \|X_t - X_{t-2^k}\|^p}{2^{p-1}} + \frac{\|TX_t - T\widetilde{X}_t(t-2^{k-1})\|^p}{4^{p-1}C^p} \\
&\qquad\leq 2\|X_{t-2^{k-1}} - X_{t-2^k}\|^p + \|X_{t-2^{k-1}} - \widetilde{X}_t(t-2^{k-1})\|^p + \|X_t - X_{t-2^{k-1}}\|^p.
\end{aligned}$$

Taking expectation, and remembering the definition of $\widetilde{X}_s(k)$,

$$\begin{aligned}
&\frac{\mathbb{E}\|X_t - X_{t-2^k}\|^p}{2^{p-2}} + \frac{\mathbb{E}\|TX_t - T\widetilde{X}_t(t-2^{k-1})\|^p}{4^{p-1}C^p} \\
&\qquad\leq 2\mathbb{E}\|X_{t-2^{k-1}} - X_{t-2^k}\|^p + 2\mathbb{E}\|X_t - X_{t-2^{k-1}}\|^p.
\end{aligned}$$

Dividing by $2^{(k-1)p+2}$ we obtain

$$\frac{\mathbb{E}\|X_t - X_{t-2^k}\|^p}{2^{kp}} + \frac{\mathbb{E}\|TX_t - T\widetilde{X}_t(t-2^{k-1})\|^p}{2^{p(k+1)}C^p} \leq \frac{\mathbb{E}\|X_{t-2^{k-1}} - X_{t-2^k}\|^p}{2^{(k-1)p+1}} + \frac{\mathbb{E}\|X_t - X_{t-2^{k-1}}\|^p}{2^{(k-1)p+1}}.$$

Adding the inequalities corresponding to $k = 1, 2, \ldots, m$ and $t \in \mathbb{Z}$ we have

$$\begin{aligned}\sum_{k=1}^{m}\sum_{t\in\mathbb{Z}} \frac{\mathbb{E}\|X_t - X_{t-2^k}\|^p}{2^{kp}} + \sum_{k=1}^{m}\sum_{t\in\mathbb{Z}} \frac{\mathbb{E}\|TX_t - T\widetilde{X}_t(t-2^{k-1})\|^p}{2^{p(k+1)}C^p} \\ \leq \sum_{k=1}^{m}\sum_{t\in\mathbb{Z}} \frac{\mathbb{E}\|X_{t-2^{k-1}} - X_{t-2^k}\|^p}{2^{(k-1)p+1}} + \sum_{k=1}^{m}\sum_{t\in\mathbb{Z}} \frac{\mathbb{E}\|X_t - X_{t-2^{k-1}}\|^p}{2^{(k-1)p+1}} \\ = \sum_{j=0}^{m-1}\sum_{s\in\mathbb{Z}} \frac{\mathbb{E}\|X_s - X_{s-2^j}\|^p}{2^{jp}}. \quad (3.1.3)\end{aligned}$$

In order to prove the inequality (3.1.2) (that is, the inequality defining Markov p-convexity), we may assume without loss of generality that $\sum_{t\in\mathbb{Z}} \mathbb{E}\|X_t - X_{t-1}\|^p < \infty$. By the triangle inequality, for every $k \in \mathbb{N}$ we then have $\sum_{t\in\mathbb{Z}} \mathbb{E}\|X_t - X_{t-2^k}\|^p < \infty$. Therefore, it is possible to cancel terms in (3.1.3) to arrive at

$$\begin{aligned}\sum_{k=1}^{m}\sum_{t\in\mathbb{Z}} \frac{\mathbb{E}\|TX_t - T\widetilde{X}_t(t-2^{k-1})\|^p}{2^{p(k+1)}C^p} \\ \leq \sum_{t\in\mathbb{Z}} \mathbb{E}\|X_t - X_{t-1}\|^p - \sum_{t\in\mathbb{Z}} \frac{\mathbb{E}\|X_t - X_{t-2^m}\|^p}{2^{mp}} \\ \leq \sum_{t\in\mathbb{Z}} \mathbb{E}\|X_t - X_{t-1}\|^p.\end{aligned}$$

Shifting the index k and multiplying by $(4C)^p$,

$$\sum_{k=0}^{m-1}\sum_{t\in\mathbb{Z}} \frac{\mathbb{E}\|TX_t - T\widetilde{X}_t(t-2^k)\|^p}{2^{kp}} \leq (4C)^p \sum_{t\in\mathbb{Z}} \mathbb{E}\|X_t - X_{t-1}\|^p.$$

By letting m go to infinity, we conclude that every uniformly p-convex linear operator with constant C is Markov p-convex with constant $4C$. The second part of the conclusion follows immediately. □

3.1.4 Markov p-convexity implies p-convexity

The arguments in this section follow closely those of [MN08a].

Theorem 3.1.4. (Compare to [MN08a, Thm. 4].) *If $T : E \to F$ be a linear operator which is Markov p-convex with constant C, then T is p-convexifiable. More precisely, for every $\varepsilon \in (0,1)$ there exists a norm $|||\cdot|||$ on E such that for all $x, y \in E$,*

$$(1-\varepsilon)\|x\| \le |||x||| \le \|x\|,$$

and

$$\left\|\frac{Tx+Ty}{2}\right\|^p \le \frac{|||x|||^p + |||y|||^p}{2} - \frac{1-(1-\varepsilon)^p}{4C^p(p+1)} \cdot \left|\left|\left|\frac{x-y}{2}\right|\right|\right|^p.$$

Thus, the operator $T : (E, |||\cdot|||) \to F$ satisfies (3.1.1) *with constant $K = O(C/\varepsilon^{1/p})$.*

Proof. Recall that the fact that $T : E \to F$ is Markov p-convex with constant C implies that for every Markov chain $\{X_t\}_{t\in\mathbb{Z}}$ with values in E we have

$$\sum_{k=0}^{m}\sum_{t=1}^{2^m} \frac{\mathbb{E}\left\|TX_t - T\widetilde{X}_t(t-2^k)\right\|^p}{2^{kp}} \le C^p \sum_{t=1}^{2^m} \mathbb{E}\|X_t - X_{t-1}\|^p. \tag{3.1.4}$$

For $x \in E$ we shall say that a Markov chain $\{X_t\}_{t=-\infty}^{2^m}$ is an m-admissible representation of x if $X_t = 0$ for $t \le 0$ and $\mathbb{E}X_t = tx$ for $t \in \{1, 2, \dots, 2^m\}$. Fix $\varepsilon \in (0,1)$, and denote $\eta = 1 - (1-\varepsilon)^p$. For every $m \in \mathbb{N}$ define

$$|||x|||_m = \inf\left\{\left(\frac{1}{2^m}\sum_{t=1}^{2^m}\mathbb{E}\left\|X_t - X_{t-1}\right\|^p - \frac{\eta}{C^p}\cdot\frac{1}{2^m}\sum_{k=0}^{m}\sum_{t=1}^{2^m}\frac{\mathbb{E}\|TX_t - T\widetilde{X}_t(t-2^k)\|^p}{2^{kp}}\right)^{1/p}\right\} \tag{3.1.5}$$

where the infimum in (3.1.5) is taken over all m-admissible representations of x. Note that such a representation of x always exists, since we can define $X_t = 0$ for $t \le 0$ and $X_t = tx$ for $t \in \{1, 2, \dots, 2^m\}$. Moreover, this example shows that

$$|||x|||_m \le \left(\frac{1}{2^m}\sum_{t=1}^{2^m}\|tx - (t-1)x\|^p\right)^{1/p} = \|x\|.$$

On the other hand if $\{X_t\}_{t=-\infty}^{2^m}$ is an m-admissible representation of x then

$$\begin{aligned}
\sum_{t=1}^{2^m}\mathbb{E}\left\|X_t - X_{t-1}\right\|^p - \frac{\eta}{C^p}\sum_{k=0}^{m}\sum_{t=1}^{2^m}\frac{\mathbb{E}\|TX_t - T\widetilde{X}_t(t-2^k)\|^p}{2^{kp}}& \\
\ge (1-\eta)\sum_{t=1}^{2^m}\mathbb{E}\left\|X_t - X_{t-1}\right\|^p & \quad (3.1.6)\\
\ge (1-\varepsilon)^p\sum_{t=1}^{2^m}\left\|\mathbb{E}X_t - \mathbb{E}X_{t-1}\right\|^p & \quad (3.1.7)\\
= (1-\varepsilon)^p\sum_{t=1}^{2^m}\|tx - (t-1)x\|^p = 2^m(1-\varepsilon)^p\left\|x\right\|^p &
\end{aligned}$$

where in (3.1.6) we used (3.1.4), and in (3.1.7) we used the convexity of the function $z \mapsto \|z\|^p$ (and Jensen's inequality). In conclusion we see that for all $x \in E$,

$$(1-\varepsilon)\left\|x\right\| \le |||x|||_m \le \left\|x\right\|. \tag{3.1.8}$$

Now take $x, y \in E$ and fix $\delta \in (0,1)$. Let $\{X_t\}_{t=-\infty}^{2m}$ be an m-admissible representation of x and $\{Y_t\}_{t=-\infty}^{2m}$ be an m-admissible representation of y which is stochastically independent of $\{X_t\}_{t=-\infty}^{2m}$, such that

$$\sum_{t=1}^{2^m} \mathbb{E}\,\|X_t - X_{t-1}\|^p - \frac{\eta}{C^p}\sum_{k=0}^{m}\sum_{t=1}^{2^m}\frac{\mathbb{E}\|TX_t - T\widetilde{X}_t(t-2^k)\|^p}{2^{kp}} \le 2^m(|||x|||_m + \delta) \quad (3.1.9)$$

and

$$\sum_{t=1}^{2^m} \mathbb{E}\,\|Y_t - Y_{t-1}\|^p - \frac{\eta}{C^p}\sum_{k=0}^{m}\sum_{t=1}^{2^m}\frac{\mathbb{E}\|TY_t - T\widetilde{Y}_t(t-2^k)\|^p}{2^{kp}} \le 2^m(|||y|||_m + \delta). \quad (3.1.10)$$

Define a Markov chain $\{Z_t\}_{t=-\infty}^{2^{m+1}}$ in E as follows. For $t \le -2^m$ set $Z_t = 0$ while with probability 1/2 we let $(Z_{-2^m+1}, Z_{-2^m+2}, \ldots, Z_{2^{m+1}})$ equal

$$(\underbrace{0,\ldots,0}_{2^m\text{ times}}, X_1, X_2, \ldots, X_{2^m}, X_{2^m}+Y_1, X_{2^m}+Y_2, \ldots, X_{2^m}+Y_{2^m})$$

and with probability 1/2 we let $(Z_{-2^m+1}, Z_{-2^m+2}, \ldots, Z_{2^{m+1}})$ equal

$$(\underbrace{0,\ldots,0}_{2^m\text{ times}}, Y_1, Y_2, \ldots, Y_{2^m}, Y_{2^m}+X_1, Y_{2^m}+X_2, \ldots, Y_{2^m}+X_{2^m}).$$

Hence, $Z_t = 0$ for $t \le 0$; for $t \in \{1, 2, \ldots, 2^m\}$ we have $\mathbb{E}Z_t = \frac{1}{2}\big(\mathbb{E}X_t + \mathbb{E}Y_t\big) = t \cdot \frac{1}{2}(x+y)$ and for $t \in \{2^m+1, 2^m+2\ldots, 2^{m+1}\}$ we have

$$\begin{aligned}\mathbb{E}Z_t &= \frac{1}{2}\mathbb{E}\big(X_{2^m} + Y_{t-2^m}\big) + \frac{1}{2}\mathbb{E}\big(Y_{2^m} + X_{t-2^m}\big)\\ &= \frac{1}{2}\big(2^m x + (t-2^m)y\big) + \frac{1}{2}\big(2^m y + (t-2^m)x\big) = t\cdot\frac{x+y}{2}.\end{aligned}$$

Thus $\{Z_t\}_{t=-\infty}^{2^{m+1}}$ is an $(m+1)$-admissible representation of $\frac{x+y}{2}$. The definition of $|||\cdot|||_m$ (that is, equation (3.1.5)) implies that

$$2^{m+1}\left|\left|\left|\frac{x+y}{2}\right|\right|\right|_{m+1}^{p} \leq \sum_{t=1}^{2^{m+1}} \mathbb{E}\left\|Z_t - Z_{t-1}\right\|^p - \frac{\eta}{C^p}\sum_{k=0}^{m+1}\sum_{t=1}^{2^{m+1}} \frac{\mathbb{E}\|TZ_t - T\widetilde{Z}_t(t-2^k)\|^p}{2^{kp}}. \tag{3.1.11}$$

Note that, from the definition of $\{Z_t\}$,

$$\sum_{t=1}^{2^{m+1}} \mathbb{E}\left\|Z_t - Z_{t-1}\right\|^p = \sum_{t=1}^{2^{m+1}} \mathbb{E}\left\|X_t - X_{t-1}\right\|^p + \sum_{t=1}^{2^{m+1}} \mathbb{E}\left\|Y_t - Y_{t-1}\right\|^p. \tag{3.1.12}$$

Moreover,

$$\sum_{k=0}^{m+1}\sum_{t=1}^{2^{m+1}} \frac{\mathbb{E}\|TZ_t - T\widetilde{Z}_t(t-2^k)\|^p}{2^{kp}} = \frac{1}{2^{(m+1)p}}\sum_{t=1}^{2^{m+1}} \mathbb{E}\|TZ_t - T\widetilde{Z}_t(t-2^{m+1})\|^p + \sum_{k=0}^{m}\sum_{t=1}^{2^{m+1}} \frac{\mathbb{E}\|TZ_t - T\widetilde{Z}_t(t-2^k)\|^p}{2^{kp}} \tag{3.1.13}$$

We bound each of the terms in (3.1.13) separately. Note that by construction we have for every $t \in \{1, 2, \ldots, 2^m\}$

$$Z_t - \widetilde{Z}_t(t-2^{m+1}) = Z_t - \widetilde{Z}_t(1-2^{m+1}) = \begin{cases} X_t - Y_t & \text{with probability } 1/4, \\ Y_t - X_t & \text{with probability } 1/4, \\ X_t - \widetilde{X}_t(0) & \text{with probability } 1/4, \\ Y_t - \widetilde{Y}_t(0) & \text{with probability } 1/4. \end{cases}$$

Thus the first term in (3.1.13) can be bounded from below as follows: using the previous remark, Jensen's inequality and some calculus (in particular, we can exchange the expectation and T because X_t is a finite-state Markov chain.)

$$\begin{aligned}\sum_{t=1}^{2^{m+1}} \|TZ_t - T\widetilde{Z}_t(t-2^{m+1})\|^p &\geq \frac{1}{2}\sum_{t=1}^{2^m} \mathbb{E}\,\|TX_t - TY_t\|^p \\ &\geq \frac{1}{2}\sum_{t=1}^{2^m} \|\mathbb{E}TX_t - \mathbb{E}TY_t\|^p = \frac{1}{2}\sum_{t=1}^{2^m} \|T\mathbb{E}X_t - T\mathbb{E}Y_t\|^p \\ &= \frac{\|Tx-Ty\|^p}{2}\sum_{t=1}^{2^m} t^p \geq \frac{\|Tx-Ty\|^p}{2}\int_0^{2^m} t^p\,dt \\ &= \frac{\|Tx-Ty\|^p}{2}\frac{1}{p+1}(2^m)^{p+1} \geq \frac{\|Tx-Ty\|^p}{2}\frac{1}{p+1}(2^m-1)^{p+1} \\ &= \frac{\|Tx-Ty\|^p}{2}\frac{2^{(m+1)(p+1)}}{p+1}\left(\frac{1}{2}-\frac{1}{2^{m+1}}\right)^{p+1} = \frac{2^{(m+1)(p+1)}}{2p+2}\left(\frac{1}{2}-\frac{1}{2^{m+1}}\right)^{p+1}\|Tx-Ty\|^p .\end{aligned} \tag{3.1.14}$$

We now bound the second term in (3.1.13). Note first that for every $k \in \{0, 1, \dots, m\}$ and every $t \in \{2^m + 1, \dots, 2^{m+1}\}$ we have

$$Z_t - \widetilde{Z}_t(t-2^k) = \begin{cases} (X_{2^m} + Y_{t-2^m}) - \big(\widetilde{X}_{2^m}(t-2^k) + \widetilde{Y}_{t-2^m}(t-2^m-2^k)\big) & \text{with probability } 1/2, \\ (Y_{2^m} + X_{t-2^m}) - \big(\widetilde{Y}_{2^m}(t-2^k) + \widetilde{X}_{t-2^m}(t-2^m-2^k)\big) & \text{with probability } 1/2. \end{cases}$$

By Jensen's inequality, if U, V are X-valued independent random variables with $\mathbb{E}V = 0$, then

$$\mathbb{E}\,\|U+V\|^p \geq \mathbb{E}\,\|U+\mathbb{E}V\|^p \geq \mathbb{E}\,\|U\|^p .$$

Thus, since $\{X_t\}_{t=-\infty}^{2^m}$ and $\{Y_t\}_{t=-\infty}^{2^m}$ are independent,

$$\mathbb{E}\left\|TY_{t-2^m} - T\widetilde{Y}_{t-2^m}(t-2^m-2^k) + TX_{2^m} - T\widetilde{X}_{2^m}(t-2^k)\right\|^p$$
$$\geq \mathbb{E}\left\|TY_{t-2^m} - T\widetilde{Y}_{t-2^m}(t-2^m-2^k)\right\|^p$$

and

$$\mathbb{E}\left\|TX_{t-2^m} - T\widetilde{X}_{t-2^m}(t-2^m-2^k) + TY_{2^m} - T\widetilde{Y}_{2^m}(t-2^k)\right\|^p$$
$$\geq \mathbb{E}\left\|TX_{t-2^m} - T\widetilde{X}_{t-2^m}(t-2^m-2^k)\right\|^p.$$

It follows that for every $k \in \{0, 1, \dots, m\}$ and every $t \in \{2^m + 1, \dots, 2^{m+1}\}$ we have

$$\mathbb{E}\left\|TZ_t - TZ_t(t-2^k)\right\|^p$$
$$\geq \frac{1}{2}\mathbb{E}\left\|TX_{t-2^m} - T\widetilde{X}_{t-2^m}(t-2^m-2^k)\right\|^p$$
$$+ \frac{1}{2}\mathbb{E}\left\|TY_{t-2^m} - T\widetilde{Y}_{t-2^m}(t-2^m-2^k)\right\|^p.$$

Hence,

$$\sum_{k=0}^{m}\sum_{t=1}^{2^{m+1}} \frac{\mathbb{E}\|TZ_t - T\widetilde{Z}_t(t-2^k)\|^p}{2^{kp}} \tag{3.1.15}$$
$$\geq \sum_{k=0}^{m}\sum_{t=1}^{2^m} \frac{\frac{1}{2}\mathbb{E}\|TX_t - T\widetilde{X}_t(t-2^k)\|^p + \frac{1}{2}\mathbb{E}\|TY_t - T\widetilde{Y}_t(t-2^k)\|^p}{2^{kp}}$$
$$+ \sum_{k=0}^{m}\sum_{t=2^m+1}^{2^{m+1}} \frac{\frac{1}{2}\mathbb{E}\|TX_{t-2^m} - T\widetilde{X}_{t-2^m}(t-2^m-2^k)\|^p}{2^{kp}} \tag{3.1.16}$$
$$+ \sum_{k=0}^{m}\sum_{t=2^m+1}^{2^{m+1}} \frac{\frac{1}{2}\mathbb{E}\|TY_{t-2^m} - T\widetilde{Y}_{t-2^m}(t-2^m-2^k)\|^p}{2^{kp}} \tag{3.1.17}$$
$$= \sum_{k=0}^{m}\sum_{t=1}^{2^m} \frac{\mathbb{E}\|TX_t - T\widetilde{X}_t(t-2^k)\|^p}{2^{kp}} + \sum_{k=0}^{m}\sum_{t=1}^{2^m} \frac{\mathbb{E}\|TY_t - T\widetilde{Y}_t(t-2^k)\|^p}{2^{kp}} \tag{3.1.18}$$

Combining (3.1.9), (3.1.10), (3.1.11), (3.1.12), (3.1.13), (3.1.14) and (3.1.18), and letting δ tend to 0, we see that

$$\left|\left|\left|\frac{x+y}{2}\right|\right|\right|_{m+1}^{p} \leq \frac{|||x|||_m^p + |||y|||_m^p}{2} - \frac{\eta}{C^p}\cdot\frac{1}{2p+2}\left(\frac{1}{2}-\frac{1}{2^{m+1}}\right)^{p+1}\|Tx-Ty\|^p. \quad (3.1.19)$$

Define for $w \in E$,

$$|||w||| = \limsup_{m\to\infty} |||w|||_m .$$

Then a combination of (3.1.8) and (3.1.19) yields that

$$(1-\varepsilon)\|x\| \leq |||x||| \leq \|x\|,$$

and

$$\begin{aligned}\left|\left|\left|\frac{x+y}{2}\right|\right|\right|^{p} &\leq \frac{|||x|||^p + |||y|||^p}{2} - \frac{\eta}{C^p(p+1)2^{p+2}}\|Tx-Ty\|^p \\ &\leq \frac{|||x|||^p + |||y|||^p}{2} - \frac{\eta}{4C^p(p+1)}\left\|\frac{Tx-Ty}{2}\right\|^p \qquad (3.1.20)\end{aligned}$$

Note that (3.1.20) implies that the set $\{x \in E : |||x||| \leq 1\}$ is mid-point convex, so that $|||\cdot|||$ is a norm on E. This concludes the proof of the theorem. □

3.2 Metric cotype

3.2.1 Introduction

The classical concepts of (Rademacher) type and cotype of Banach spaces emerged in the early 70's and quickly earned a prominent place within the theory because of its various applications. Its seemingly innocent quantification of the interplay between geometry and probability in Banach spaces turned out to be a powerful tool in the study of Banach spaces and the operators between them. Thus, it was a natural step in Ribe's program to find a metric characterization of both type and cotype.

The breakthrough was finally achieved by Mendel and Naor [MN08b], who came up with a definition of metric cotype for metric spaces and gave various remarkable applications of it. The following definition is the natural adaptation to mappings of their definition for spaces. Let $q > 0$. A Lipschitz map $T : X \to Y$ is said to have *metric cotype q with constant* Γ if for every integer $n \in \mathbb{N}$ there exists an even integer $m \in \mathbb{N}$ such that for every $f : \mathbb{Z}_m^n \to X$

$$\sum_{j=1}^{n} \mathbb{E}_x d_Y\big(Tf(x + \tfrac{m}{2}e_j), Tf(x)\big)^q \leq \Gamma^q m^q \mathbb{E}_{\varepsilon,x} d_X\big(f(x+\varepsilon), f(x)\big)^q, \tag{3.2.1}$$

where the expectations are taken with respect to uniformly chosen $x \in \mathbb{Z}_m^n$ and $\varepsilon \in \{-1, 0, 1\}^n$, and $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbb{R}^n$. The smallest constant for which inequality (3.2.1) holds is denoted by $\Gamma_q(T)$ and called the metric cotype q constant of T.

The main result of this section is the following theorem.

Theorem 3.2.1. (Compare to [MN08b, Thm. 1.2].) *Let E, F be Banach spaces, $T : E \to F$ a linear map and $q \in [2, \infty)$. Then T has metric cotype q if and only if it has Rademacher cotype q. Moreover,*

$$\frac{1}{2\pi} C_q(T) \leq \Gamma_q(T) \leq 20 C_q(T).$$

Furthermore, following the footsteps of Mendel and Naor we also define a weak variant of metric cotype in the spirit of Bourgain, Milman and Wolfson [BMW86]. Let $1 \leq p \leq q$ and $T : X \to Y$ a Lipschitz map. We say that T has *weak metric cotype q with exponent p and constant* Γ if for every integer $n \in \mathbb{N}$ there exists an even integer $m \in \mathbb{N}$ such that for every $f : \mathbb{Z}_m^n \to X$

$$\sum_{j=1}^{n} \mathbb{E}_x d_Y\big(Tf(x + \tfrac{m}{2}e_j), Tf(x)\big)^p \leq \Gamma^p m^p n^{1-p/q} \mathbb{E}_{\varepsilon,x} d_X\big(f(x+\varepsilon), f(x)\big)^p, \tag{3.2.2}$$

where the expectations are taken with respect to uniformly chosen $x \in \mathbb{Z}_m^n$ and $\varepsilon \in \{-1,0,1\}^n$. The smallest constant Γ in 3.2.2 is denoted by $\Gamma_q^{(p)}(T)$. Clearly, $\Gamma_q^{(q)}(T) = \Gamma_q(T)$.

The following theorem is analogous to Theorem 3.2.1.

Theorem 3.2.2. (Compare to [MN08b, Thm. 1.4].) *Let E, F be Banach spaces and $T : E \to F$ a linear map. Suppose that T has weak metric cotype q with exponent p for some $1 \leq p < q$. Then T has Beauzamy-Rademacher cotype; that is, for some $p \geq 1$ (equivalently, any $p \geq 1$) the sequence $a_n^{(p)}(T)$ defined by*

$$a_n^{(p)}(T) = \inf\left\{ \left[\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j v_j \right\|^p \right]^{1/p} \; : \; \|Tv_1\|, \ldots, \|Tv_n\| \geq 1 \right\}$$

converges to 0. If $p \geq 2$, then T has weak Rademacher cotype q and hence has Rademacher cotype r for every $r > q$. On the other hand,

$$\Gamma_q^{(p)}(T) \leq c_{pq} C_q(T)$$

where c_{pq} is a constant depending only on p and q.

It should be mentioned that Theorem 3.2.2 is clearly less satisfying than Theorem 3.2.1. In the corresponding theorem for spaces [MN08b, Thm. 1.4], Mendel and Naor achieve Rademacher cotype r for every $r > q$ (and even cotype 2 when $q = 2$) where we only obtain the much weaker notion of Beauzamy-Rademacher cotype, but that is not a complete surprise. Even in the linear setting, the results involving cotype for operators also suffer from analogous shortcomings when compared with what one can get for spaces (compare, for example, 4.6.7 and 4.6.18 in [PW98]).

3.2.2 Notation and preliminaries

X, Y will always denote metric spaces, whereas E, F denote Banach spaces. The letters ε and δ will always denote elements of $\{-1,1\}^n$ and $\{-1,0,1\}^n$ respectively.

We denote by μ the uniform probability measure on $\mathbb{Z}_m^n$, and by σ the uniform probability measure on $\{-1,0,1\}^n$. The notation $\mathbb{E}_\varepsilon$ will denote expectation with respect to ε uniformly distributed in $\{-1,1\}^n$.

Given a linear operator $T : E \to F$ and $1 \le p, q < \infty$, we denote by $C_q^{(p)}$ the infimum over all constants $C > 0$ such that for every $n \in \mathbb{N}$ and every $v_1, \ldots, v_n$ in E,

$$\left(\sum_{j=1}^n \|Tv_j\|_F^q \right)^{1/q} \le C \left(\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j v_j \right\|_E^p \right)^{1/p}. \tag{3.2.3}$$

We also denote $C_q(T) := C_q^{(q)}(T)$. Note that, from the Kahane-Khintchine inequality, $C_q^{(p)}(T) \le c_{pq} C_q(T)$ where the constant c_{pq} depends on p and q only.

Following [PW98, 4.5.7], for $2 < q$ a linear operator $T : E \to F$ is said to have *weak Rademacher cotype* q if there is a constant $C > 0$ such that for any $v_1, \ldots, v_n$ in E

$$\left(\sum_{j=1}^n \|Tv_j\|_F^2 \right)^{1/2} \le C n^{1/2 - 1/q} \left(\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j v_j \right\|_E^2 \right)^{1/2}.$$

The infimum of such constants C will be denoted by $WC_q(T)$. Equivalently [PW98, Prop. 4.5.9], for any $2 \le p < q$ there exists $C' > 0$ such that for any $v_1, \ldots, v_n$ in E

$$\left(\sum_{j=1}^n \|Tv_j\|_F^p \right)^{1/p} \le C' n^{1/p - 1/q} \left(\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j v_j \right\|_E^p \right)^{1/p}. \tag{3.2.4}$$

Recall from the introduction that a linear operator $T : E \to F$ is said to have Beauzamy-Rademacher cotype [Bea76] [1] if for some $p \ge 1$ (equivalently, any $p \ge 1$) we have that the sequence $a_n^{(p)}(T)$ defined by

$$a_n^{(p)}(T) = \inf \left\{ \left[\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j v_j \right\|^p \right]^{1/p} \;:\; \|Tv_1\|, \ldots, \|Tv_n\| \ge 1 \right\}$$

[1] This is not standard terminology. Beauzamy used the term *opérateurs de cotype Rademacher*.

converges to 0. These operators have a nice characterization due, of course, to Beauzamy [Bea76, Thm. 2]: T has Beauzamy-Rademacher cotype if and only if T does not uniformly factor the identity operators of ℓ_∞^n.

3.2.3 Metric cotype implies Rademacher cotype

The structure of this section follows very closely that of [MN08b]. Thus, in this section we prove the "easy" directions of Theorems 3.2.1 and 3.2.2: the metric notions of cotype imply the linear ones.

Mendel and Naor emphasize that the intuition behind the proof is relatively simple: given a linear operator $T : E \to F$ of metric cotype q, if we apply inequality 3.2.1 to functions $f : \mathbb{Z}_m^n \to E$ of the form $f(x) = \sum_{j=1}^n x_j v_j$ (where $v_1, \ldots, v_n$ in E are fixed), then by homogeneity the m would cancel and we would obtain the inequality that defines Rademacher cotype q. This argument does not work as is because the addition in $\mathbb{Z}_m^n$ is modulo m, so we will resort to using functions of the form $f(x) = \sum_{j=1}^n \exp\left(\frac{2\pi i x_j}{m}\right) v_j$, where the structure of $\mathbb{Z}_m$ is clearly present. Of course, for this to make sense we need to use *complex* Banach spaces. That is not a problem because one can complexify an operator between real spaces without changing its metric cotype or Rademacher cotype constants.

Before proceeding, let us recall the contraction principle from [LT91, p. 95]: for any $p \geq 1$, $a_1, \ldots, a_n \in \mathbb{R}$ and $v_1, \ldots, v_n \in E$,

$$\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j a_j v_j \right\|_E^p \leq \left(\max_{1\leq j\leq n} |a_j|^p \right) \cdot \mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j v_j \right\|_E^p,$$

where the expectations are taken with respect to $\varepsilon \in \{-1,1\}^n$ uniformly distributed. If the coefficients $a_1,\dots,a_n$ are complex, separating them into real and imaginary parts plus the convexity of $\|\cdot\|^p$ give

$$\mathbb{E}_\varepsilon\Big\|\sum_{j=1}^n \varepsilon_j a_j v_j\Big\|_E^p \le 2^p\Big(\max_{1\le j\le n}|a_j|^p\Big)\cdot \mathbb{E}_\varepsilon\Big\|\sum_{j=1}^n \varepsilon_j v_j\Big\|_E^p.$$

Lemma 3.2.3. *Let E, F be Banach spaces and $T: E\to F$ a linear operator of weak metric cotype q with exponent p for some $1\le p\le q$. Then for any $v_1,\dots,v_n$ in E we have*

$$\sum_{j=1}^n \|Tv_j\|_F^p \le \big(2\pi\Gamma_q^{(p)}(T)\big)^p n^{1-p/q}\mathbb{E}_\varepsilon\Big\|\sum_{j=1}^n \varepsilon_j v_j\Big\|_E^p.$$

Proof. Fix $\Gamma > \Gamma_q^{(p)}$, vectors $v_1,\dots,v_n$ in X and let m be any even positive integer. Define $f:\mathbb{Z}_m^n \to E$ by

$$f(x_1,\dots,x_n) = \sum_{j=1}^n \exp\big(\tfrac{2\pi i x_j}{m}\big)v_j.$$

Then

$$\sum_{j=1}^n \int_{\mathbb{Z}_m^n} \big\|Tf(x+\tfrac{m}{2}e_j) - Tf(x)\big\|_F^p\, d\mu(x)$$
$$= \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \big|\exp\big(\tfrac{2\pi i x_j}{m}+\pi i\big) - \exp\big(\tfrac{2\pi i x_j}{m}\big)\big|^p\,\|Tv_j\|_F^p\, d\mu(x) = 2^p\sum_{j=1}^n \|Tv_j\|_F^p. \quad (3.2.5)$$

On the other hand,

$$\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n} \|f(x+\delta)-f(x)\|_E^p\, d\mu(x)d\sigma(\delta)$$
$$= \int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n} \Big\|\sum_{j=1}^n \exp\big(\tfrac{2\pi i x_j}{m}\big)\Big(\exp\big(\tfrac{2\pi i \delta_j}{m}\big)-1\Big)v_j\Big\|_E^p d\mu(x)d\sigma(\delta). \quad (3.2.6)$$

Observe that for every $\varepsilon \in \{-1,1\}^n$, when $(x_j)_{j=1}^n$ runs over $\mathbb{Z}_m^n$ so does $\big(x_j + \frac{m}{2}\frac{(1-\varepsilon_j)}{2}\big)_{j=1}^n$. Therefore,

$$\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n}\Big\|\sum_{j=1}^n \exp\big(\tfrac{2\pi i x_j}{m}\big)\Big(\exp\big(\tfrac{2\pi i\delta_j}{m}\big)-1\Big)v_j\Big\|_E^p d\mu(x)d\sigma(\delta)$$
$$=\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n}\Big\|\sum_{j=1}^n \exp\big(\tfrac{2\pi i}{m}(x_j+m(1-\varepsilon_j)/4)\big)\Big(\exp\big(\tfrac{2\pi i\delta_j}{m}\big)-1\Big)v_j\Big\|_E^p d\mu(x)d\sigma(\delta).$$

Noting that $\exp\big(\frac{2\pi i m(1-\varepsilon_j)}{4m}\big)=\varepsilon_j$, we obtain

$$\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n}\Big\|\sum_{j=1}^n \exp\big(\tfrac{2\pi i x_j}{m}\big)\Big(\exp\big(\tfrac{2\pi i\delta_j}{m}\big)-1\Big)v_j\Big\|_E^p d\mu(x)d\sigma(\delta)$$
$$=\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n}\Big\|\sum_{j=1}^n \varepsilon_j\exp\big(\tfrac{2\pi i x_j}{m}\big)\Big(\exp\big(\tfrac{2\pi i\delta_j}{m}\big)-1\Big)v_j\Big\|_E^p d\mu(x)d\sigma(\delta).$$

Taking expectation with respect to $\varepsilon \in \{-1,1\}^n$ and using the contraction principle

$$\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n}\Big\|\sum_{j=1}^n \exp\big(\tfrac{2\pi i x_j}{m}\big)\Big(\exp\big(\tfrac{2\pi i\delta_j}{m}\big)-1\Big)v_j\Big\|_E^p d\mu(x)d\sigma(\delta)=$$
$$\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n}\mathbb{E}_\varepsilon\Big\|\sum_{j=1}^n \varepsilon_j\exp\big(\tfrac{2\pi i x_j}{m}\big)\Big(\exp\big(\tfrac{2\pi i\delta_j}{m}\big)-1\Big)v_j\Big\|_E^p d\mu(x)d\sigma(\delta)$$
$$\le\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n}2^p\max_{1\le j\le n}\Big|\exp\big(\tfrac{2\pi i\delta_j}{m}\big)-1\Big|^p\mathbb{E}_\varepsilon\Big\|\sum_{j=1}^n\varepsilon_j v_j\Big\|_E^p d\mu(x)d\sigma(\delta). \quad (3.2.7)$$

Observe that for $\theta\in[0,\pi]$, $|e^{i\theta}-1|\le\theta$. Therefore, we have

$$\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n}2^p\max_{1\le j\le n}\Big|\exp\big(\tfrac{2\pi i\delta_j}{m}\big)-1\Big|^p\mathbb{E}_\varepsilon\Big\|\sum_{j=1}^n\varepsilon_j v_j\Big\|_E^p d\mu(x)d\sigma(\delta)$$
$$\le\Big(\frac{4\pi}{m}\Big)^p\mathbb{E}_\varepsilon\Big\|\sum_{j=1}^n\varepsilon_j v_j\Big\|_E^p. \quad (3.2.8)$$

Combining (3.2.2), (3.2.5), (3.2.6), (3.2.7) and (3.2.8) we get

$$2^p \sum_{j=1}^{n} \|Tv_j\|_F^p \leq \Gamma^p m^p n^{1-p/q} \left(\frac{4\pi}{m}\right)^p \mathbb{E}_\varepsilon \left\| \sum_{j=1}^{n} \varepsilon_j v_j \right\|_E^p$$

or equivalently,

$$\sum_{j=1}^{n} \|Tv_j\|_F^p \leq (2\pi\Gamma)^p n^{1-p/q} \mathbb{E}_\varepsilon \left\| \sum_{j=1}^{n} \varepsilon_j v_j \right\|_E^p.$$

□

Armed with the previous lemma, we will prove the easy implications of Theorems 3.2.1 and 3.2.2. If a linear operator $T : E \to F$ has metric cotype q, Lemma 3.2.3 implies immediately that T has Rademacher cotype q and moreover $C_q(T) \leq 2\pi\Gamma_q(T)$. If $p < q$, note that when $\|Tv_1\|_F, \dots, \|Tv_n\|_F \geq 1$ we obtain

$$\left[\mathbb{E}_\varepsilon \left\| \sum_{j=1}^{n} \varepsilon_j v_j \right\|^p\right]^{1/p} \geq \frac{n^{1/q}}{2\pi\Gamma_q^{(p)}(T)} \xrightarrow{n\to\infty} \infty,$$

showing that T has Beauzamy-Rademacher cotype.

Although Beauzamy-Rademacher cotype of an operator is equivalent to uniformly preserving ℓ_∞^n's, in the operator case this does not imply having non-trivial cotype (unlike in the case of spaces). This is one of the unavoidable shortcomings that are encountered when one goes from spaces to operators, a fact mentioned in the introduction.

In the case $p \geq 2$, Lemma 3.2.3 implies that T has weak Rademacher cotype q (compare to equation (3.2.4)). This implies, in turn, that T has Rademacher cotype q' for every $q' > q$ by [PW98, Thm. 4.5.10]. Unfortunately, quantitative estimates do not appear to be easy to come by. The only exception is the case $p = 2$, where we get immediately $WC_q(T) \leq 2\pi\Gamma_q^{(2)}(T)$.

3.2.4 Rademacher cotype implies metric cotype

Mendel and Naor start by treating the case of K-convex spaces first, before proceeding to the general case. Not only is the proof easier, also the dependance of m with respect to n is sharper. This improved dependance is crucial for several applications, so it is not a matter of getting better estimates just for the sake of it. One of the most important problems left open in [MN08b] is whether this sharper estimate holds in the general case. We also start with the K-convex case, and come across something interesting. At least with the current proof, it is not sufficient to have K-convexity of the operator to obtain the improved estimates. We need the much stronger assumption of K-convexity of the domain of the operator.

The K-convex case

Theorem 3.2.4 (Compare to Thm 4.1 in [MN08b])**.** *Let E be a K-convex Banach space and $T : E \to F$ a linear operator with Rademacher cotype q. Then for every integer n and every m an integer multiple of 4 such that $m \geq \frac{2n^{1/q}}{C_q^{(p)}(T)K_p(E)}$, we have $\Gamma_q^{(p)}(T; n, m) \leq 15C_q^{(p)}(T)K_p(E)$.*

Proof. For $f : \mathbb{Z}_m^n \to E$ we define the following operators:

$$\widetilde{\partial}_j f(x) = f(x + e_j) - f(x - e_j)$$
$$\mathcal{E}_j f(x) = \mathbb{E}_\varepsilon f\Big(x + \sum_{\ell \neq j} \varepsilon_\ell e_\ell\Big),$$

and for $\varepsilon \in \{-1, 0, 1\}^n$,

$$\partial_\varepsilon f(x) = f(x + \varepsilon) - f(x).$$

From [MN08b, eqn. (17)], it follows that

$$\int_{\mathbb{Z}_m^n} \mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j \big[\mathcal{E}_j f(x+e_j) - \mathcal{E}_j f(x-e_j)\big] \right\|_E^p d\mu(x) \\ \le K_p(E)^p \int_{\mathbb{Z}_m^n} \mathbb{E}_\varepsilon \left\| \partial_\varepsilon f(x) \right\|_E^p d\mu(x). \quad (3.2.9)$$

By the previous equation and the definition of $C_q^{(p)}(T)$, for every $C > C_q^{(p)}(T)$ we have that

$$K_p(E)^p C^p \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon) - f(x)\|_E^p d\mu(x) \\ \ge C^p \cdot \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^n \varepsilon_j \big[\mathcal{E}_j f(x+e_j) - \mathcal{E}_j f(x-e_j)\big] \right\|_E^p d\mu(x) \\ \ge \int_{\mathbb{Z}_m^n} \left(\sum_{j=1}^n \|\mathcal{E}_j T f(x+e_j) - \mathcal{E}_j T f(x-e_j)\|_F^q \right)^{p/q} d\mu(x) \\ \ge \frac{1}{n^{1-p/q}} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|\mathcal{E}_j T f(x+e_j) - \mathcal{E}_j T f(x-e_j)\|_F^p d\mu(x). \quad (3.2.10)$$

Note that we have made crucial use of the fact that T is linear. Now, for $j \in \{1, \ldots, n\}$,

$$\int_{\mathbb{Z}_m^n} \left\|\mathcal{E}_j T\big(x + \tfrac{m}{2} e_j\big) - \mathcal{E}_j T f(x)\right\|_F^p d\mu(x) \\ \le \left(\frac{m}{4}\right)^{p-1} \sum_{s=1}^{m/4} \int_{\mathbb{Z}_m^n} \left\|\mathcal{E}_j T f\big(x + 2s e_j\big) - \mathcal{E}_j T f\big(x + 2(s-1)e_j\big)\right\|_F^p d\mu(x) \\ = \left(\frac{m}{4}\right)^p \int_{\mathbb{Z}_m^n} \|\mathcal{E}_j T f(x+e_j) - \mathcal{E}_j T f(x-e_j)\|_F^p d\mu(x). \quad (3.2.11)$$

Putting together (3.2.10) and (3.2.11),

$$
\begin{aligned}
\left(\frac{m}{4}\right)^p n^{1-p/q} K_p(E)^p C^p \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon) - f(x)\|_E^p \, d\mu(x) \\
&\geq \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j T\big(x + \tfrac{m}{2} e_j\big) - \mathcal{E}_j T f(x) \right\|_F^p d\mu(x) \\
&\geq \frac{1}{3^{p-1}} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| T\big(x + \tfrac{m}{2} e_j\big) - T f(x) \right\|_F^p d\mu(x) \\
&\quad - 2 \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|\mathcal{E}_j T f(x) - T f(x)\|_F^p \, d\mu(x) \\
&= \frac{1}{3^{p-1}} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| T\big(x + \tfrac{m}{2} e_j\big) - T f(x) \right\|_F^p d\mu(x) \\
&\quad - 2 \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| \mathbb{E}_\varepsilon \left(T f\Big(x + \sum_{\ell \neq j} \varepsilon_\ell e_\ell\Big) - T f(x) \right) \right\|_F^p d\mu(x) \\
&\geq \frac{1}{3^{p-1}} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| T\big(x + \tfrac{m}{2} e_j\big) - T f(x) \right\|_F^p d\mu(x) \\
&\quad - 2 \sum_{j=1}^n \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| \left(T f\Big(x + \sum_{\ell \neq j} \varepsilon_\ell e_\ell\Big) - T f(x) \right) \right\|_F^p d\mu(x) \\
&\geq \frac{1}{3^{p-1}} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| T\big(x + \tfrac{m}{2} e_j\big) - T f(x) \right\|_F^p d\mu(x) \\
&\quad - 2^p n \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|T f(x+\varepsilon) - T f(x)\|_F^p \, d\mu(x) \\
&\quad - 2^p \sum_{j=1}^n \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|T f(x + \varepsilon_j e_j) - T f(x)\|_F^p \, d\mu(x). \qquad (3.2.12)
\end{aligned}
$$

Thus the desired result follows from Lemma 3.2.6 below, since the assumption on m implies that

$$n = n^{1-p/q} (n^{1/q}) p \leq n^{1-p/q} \tfrac{1}{2^p} m^p C_q^{(p)}(T)^p K_p(E)^p.$$

□

The general case

The reverse implications of Theorems 3.2.1 and 3.2.2, are much harder to prove, just as in the case of spaces. First we will need several lemmas, some taken verbatim from [MN08b] and some others adapted.

Lemma 3.2.5 (Lemma 2.6 in [MN08b]). *For every $n, m \in \mathbb{N}$, any metric space X, and any $f : \mathbb{Z}_m^n \to X$ and any $p \geq 1$,*

$$\sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} d_X\big(f(x+e_j), f(x)\big)^p d\mu(x) \leq 3 \cdot 2^{p-1} n \int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} d_X\big(f(x+\delta), f(x)\big)^p d\mu(x) d\sigma(\delta).$$

Lemma 3.2.6 (Compare to Lemma 2.7 in [MN08b]). *Let $T : X \to Y$ be a Lipschitz map. Assume that for an integer n and an even integer m we have for every integer $\ell \leq n$ and every $f : \mathbb{Z}_m^\ell \to X$,*

$$\sum_{j=1}^{\ell} \int_{\mathbb{Z}_m^\ell} d_Y\big(Tf(x + \tfrac{m}{2} e_j), Tf(x)\big)^p d\mu(x) \leq C^p m^p n^{1-p/q} \Bigg(\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^\ell} d_X\big(f(x+\varepsilon), f(x)\big)^p d\mu(x) + \frac{1}{\ell} \sum_{j=1}^{\ell} \int_{\mathbb{Z}_m^\ell} d_X\big(f(x+e_j), f(x)\big)^p d\mu(x) \Bigg).$$

Then $\Gamma_q^{(p)}(T; m, n) \leq 4C$.

Proof. Fix $f : \mathbb{Z}_m^n \to X$ and A a nonempty subset of $\{1, \ldots, n\}$. The assumption implies that

$$\sum_{j\in A} \int_{\mathbb{Z}_m^n} d_Y\big(Tf(x + \tfrac{m}{2}e_j), Tf(x)\big)^p d\mu(x)$$
$$\le C^p m^p n^{1/p/q}\Bigg(\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} d_X\big(f(x + \sum_{j\in A}\varepsilon_j e_j), f(x)\big)^p d\mu(x)$$
$$+ \frac{1}{|A|}\sum_{j\in A}\int_{\mathbb{Z}_m^\ell} d_X\big(f(x+e_j), f(x)\big)^p d\mu(x)\Bigg). \quad (3.2.13)$$

For a fixed $j \in \{1, \ldots, n\}$ note that

$$\sum_{A\ni j} \frac{2^{|A|}}{3^n} = \frac{1}{3^n}\sum_{k=0}^{n-1}\binom{n-1}{k}2^{k+1} = \frac{2}{3^n}3^{n-1} = \frac{2}{3}, \quad (3.2.14)$$

so multiplying (3.2.13) by $2^{|A|}/3^n$ and summing over all nonempty subsets of $\{1, \ldots, n\}$, we obtain

$$\frac{2}{3}\sum_{j=1}^n \int_{\mathbb{Z}_m^n} d_Y\big(Tf(x + \tfrac{m}{2}e_j), Tf(x)\big)^p d\mu(x)$$
$$= \sum_{\emptyset\ne A\subseteq\{1,\ldots,n\}} \frac{2^{|A|}}{3^n}\sum_{j\in A}\int_{\mathbb{Z}_m^n} d_Y\big(Tf(x + \tfrac{m}{2}e_j), Tf(x)\big)^p d\mu(x)$$
$$\le C^p m^p n^{1-p/q}\Bigg(\sum_{\emptyset\ne A\subseteq\{1,\ldots,n\}} \frac{2^{|A|}}{3^n}\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} d_X\big(f(x + \sum_{j\in A}\varepsilon_j e_j), f(x)\big)^p d\mu(x)$$
$$+ \sum_{\emptyset\ne A\subseteq\{1,\ldots,n\}} \frac{2^{|A|}}{|A|3^n}\sum_{j\in A}\int_{\mathbb{Z}_m^\ell} d_X\big(f(x+e_j), f(x)\big)^p d\mu(x)\Bigg). \quad (3.2.15)$$

Observe that by a simple counting argument,

$$\sum_{\emptyset\ne A\subseteq\{1,\ldots,n\}} \frac{2^{|A|}}{3^n}\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} d_X\big(f(x + \sum_{j\in A}\varepsilon_j e_j), f(x)\big)^p d\mu(x)$$
$$= \int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n} d_X\big(f(x+\delta), f(x)\big)^p d\mu(x) d\sigma(\delta). \quad (3.2.16)$$

Now, for a fixed $j \in \{1, \ldots, n\}$ we have

$$\sum_{A \ni j} \frac{2^{|A|}}{|A| 3^n} \leq \frac{1}{n}, \tag{3.2.17}$$

since

$$\sum_{k=1}^{n} \frac{n 2^k}{k \cdot 3^n} \binom{n-1}{k-1} = \sum_{k=1}^{n} \frac{2^k}{3^n} \binom{n}{k} = \sum_{k=1}^{n} \left(\frac{1}{3}\right)^{n-k} \left(\frac{2}{3}\right)^{k} \binom{n}{k} \leq \left(\frac{1}{3} + \frac{2}{3}\right)^{n} = 1.$$

Thus, (3.2.15), (3.2.16) and (3.2.17) yield

$$\begin{aligned} &\frac{2}{3} \sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} d_Y\big(Tf(x + \tfrac{m}{2} e_j), Tf(x)\big)^p d\mu(x) \\ &\qquad \leq C^p m^p n^{1-p/q} \bigg(\int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} d_X\big(f(x+\delta), f(x)\big)^p d\mu(x) d\sigma(\delta) \\ &\qquad\qquad + \frac{1}{n} \sum_{j=1}^{n} \int_{\mathbb{Z}_m^\ell} d_X\big(f(x+e_j), f(x)\big)^p d\mu(x) \bigg). \end{aligned} \tag{3.2.18}$$

An application of Lemma 3.2.5 now gives

$$\begin{aligned} &\sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} d_Y\big(Tf(x + \tfrac{m}{2} e_j), Tf(x)\big)^p d\mu(x) \\ &\qquad \leq \frac{3}{2} C^p m^p n^{1-p/q} (3 \cdot 2^{p-1} + 1) \int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} d_X\big(f(x+\delta), f(x)\big)^p d\mu(x) d\sigma(\delta), \end{aligned} \tag{3.2.19}$$

which together with

$$\frac{3}{2}(3 \cdot 2^{p-1} + 1) \leq \frac{3}{2} \cdot 2 \cdot 3 \cdot 2^{p-1} = 3 \cdot 3 \cdot 2^p \leq 3 \cdot 3^p \leq 4^p$$

show that $\Gamma_q^{(p)}(T; n, m) \leq 4C$.

□

Lemma 3.2.7 (Lemma 5.1 in [MN08b]). *For every integer $n \in \mathbb{N}$, any even integer $m \in \mathbb{N}$, every Banach space E, every function $f : \mathbb{Z}_m^n \to E$, every $j \in \{1, \ldots, n\}$, every odd integer $k < m/2$ and every $p \geq 1$,*

$$\int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j^{(k)} f(x) - f(x) \right\|_E^p d\mu(x) \leq 2^p k^p \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon) - f(x)\|_E^p \, d\mu(x) + 2^{p-1} \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_E^p \, d\mu(x)$$

Before proceeding, a little bit of notation is in order. Having fixed an integer $n \in \mathbb{N}$ and an even integer $m \in \mathbb{N}$, for an odd integer $k < m/2$ and a $j \in \{1, \ldots, n\}$, we define $S(j,k)$ as the set of all $y \in \mathbb{Z}_m^n$ all of whose coordinates are between $-k$ and k, with the j-th coordinate being even and all the others being odd. For $f : \mathbb{Z}_m^n \to E$ define

$$\mathcal{E}_j^{(k)}(x) := \frac{1}{\mu\big(S(j,k)\big)} \int_{S(j,k)} f(x+y) d\mu(y)$$

These definitions will not play a direct role in our arguments here, and are only needed to state and use certain lemmas from [MN08b].

Lemma 3.2.8 (Lemma 5.4 in [MN08b]). *For every integer $n \in \mathbb{N}$, any even integer $m \in \mathbb{N}$, every Banach space E, every function $f : \mathbb{Z}_m^n \to E$, every $\varepsilon \in \{-1,1\}^n$, every odd integer $k < m/2$, and every $p \geq 1$,*

$$\int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^n \varepsilon_j \left[\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j) \right] \right\|_E^p d\mu(x)$$
$$\leq 3^{p-1} \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon) - f(x-\varepsilon)\|_E^p \, d\mu(x) + \frac{24^p n^{2p-1}}{k^p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_E^p \, d\mu(x)$$

Proof of Theorems 3.2.1 and 3.2.2. The easy implications have already been proven in Section 3.2.3. Taking expectations with respect to $\varepsilon \in \{-1,1\}^n$ in Lemma 3.2.8 we get that

$$\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^n \varepsilon_j \left[\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j) \right] \right\|_E^p d\mu(x)$$
$$\leq 3^{p-1} \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon) - f(x-\varepsilon)\|_E^p \, d\mu(x)$$
$$+ \frac{24^p n^{2p-1}}{k^p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_E^p \, d\mu(x). \quad (3.2.20)$$

Note that by convexity of $\|\cdot\|^p$ we have for any $x \in \mathbb{Z}_m^n$ and $\varepsilon \in \{-1,1\}^n$

$$\left\| \frac{f(x+\varepsilon) - f(x-\varepsilon)}{2} \right\|_E^p \leq \frac{\|f(x+\varepsilon) - f(x)\|_E^p + \|f(x) - f(x-\varepsilon)\|_E^p}{2},$$

So

$$\|f(x+\varepsilon) - f(x-\varepsilon)\|_E^p \leq 2^{p-1} \big(\|f(x+\varepsilon) - f(x)\|_E^p + \|f(x) - f(x-\varepsilon)\|_E^p \big)$$

and thus (3.2.20) gives

$$\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^n \varepsilon_j \left[\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j) \right] \right\|_E^p d\mu(x)$$
$$\leq \frac{6^p}{3} \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon) - f(x)\|_E^p \, d\mu(x)$$
$$+ \frac{24^p n^{2p-1}}{k^p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_E^p \, d\mu(x). \quad (3.2.21)$$

Fix $x \in \mathbb{Z}_m^n$ and let m be an integer multiple of 4 such that $m \geq 6n^{2+1/q}$. Fixing $C > C_q^{(p)}(T)$ and applying the definition of $C_q^{(p)}(T)$ (see equation (3.2.3)) to the vectors $\left\{\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right\}_{j=1}^n$, and also noting that $T\mathcal{E}_j^{(k)} f = \mathcal{E}_j^{(k)} Tf$ because T is linear, we obtain

$$\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j \left[\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right] \right\|_E^p \geq \frac{1}{C^p n^{1-p/q}} \sum_{j=1}^n \left\| \mathcal{E}_j^{(k)} Tf(x+e_j) - \mathcal{E}_j^{(k)} Tf(x-e_j) \right\|_F^p. \quad (3.2.22)$$

Now, for every $j \in \{1, \ldots, n\}$ the convexity of $\|\cdot\|^p$ gives

$$\sum_{s=1}^{m/4} \left\| \mathcal{E}_j^{(k)} Tf(x+2se_j) - \mathcal{E}_j^{(k)} Tf(x+2(s-1)e_j) \right\|_F^p \geq \left(\frac{4}{m}\right)^{p-1} \left\| \mathcal{E}_j^{(k)} Tf\big(x+\tfrac{m}{2}e_j\big) - \mathcal{E}_j^{(k)} Tf(x) \right\|_F^p. \quad (3.2.23)$$

Integrating the previous inequality over $x \in \mathbb{Z}_m^n$ we have

$$\frac{m}{4} \int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j^{(k)} Tf(x+e_j) - \mathcal{E}_j^{(k)} Tf(x-e_j) \right\|_F^p d\mu(x) \geq \left(\frac{4}{m}\right)^{p-1} \int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j^{(k)} Tf\big(x+\tfrac{m}{2}e_j\big) - \mathcal{E}_j^{(k)} Tf(x) \right\|_F^p d\mu(x). \quad (3.2.24)$$

Integrating (3.2.22) over $x \in \mathbb{Z}_m^n$ in combination with (3.2.24) imply

$$\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^n \varepsilon_j \left[\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right] \right\|_E^p d\mu(x) \geq \frac{1}{C^p n^{1-p/q}} \left(\frac{4}{m}\right)^p \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j^{(k)} Tf\big(x+\tfrac{m}{2}e_j\big) - \mathcal{E}_j^{(k)} Tf(x) \right\|_F^p d\mu(x). \quad (3.2.25)$$

Now, for every $j \in \{1, \dots, n\}$, the convexity of $\|\cdot\|^p$ implies

$$\begin{aligned}
\int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j^{(k)} Tf\big(x + \tfrac{m}{2} e_j\big) - \mathcal{E}_j^{(k)} Tf(x) \right\|_F^p d\mu(x) \\
&\geq \frac{1}{3^{p-1}} \int_{\mathbb{Z}_m^n} \left\| Tf\big(x + \tfrac{m}{2} e_j\big) - Tf(x) \right\|_F^p d\mu(x) \\
&- \int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j^{(k)} Tf\big(x + \tfrac{m}{2} e_j\big) - f\big(x + \tfrac{m}{2} e_j\big) \right\|_F^p d\mu(x) \\
&- \int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j^{(k)} Tf\big(x\big) - Tf(x) \right\|_F^p d\mu(x) \\
&= \frac{1}{3^{p-1}} \int_{\mathbb{Z}_m^n} \left\| Tf\big(x + \tfrac{m}{2} e_j\big) - Tf(x) \right\|_F^p d\mu(x) \\
&\qquad - 2 \int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j^{(k)} Tf\big(x + \tfrac{m}{2} e_j\big) - Tf(x) \right\|_F^p d\mu(x).
\end{aligned}$$

By Lemma 3.2.7, the previous inequality implies

$$\begin{aligned}
\int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j^{(k)} Tf\big(x + \tfrac{m}{2} e_j\big) - \mathcal{E}_j^{(k)} Tf(x) \right\|_F^p d\mu(x) \\
&\geq \frac{1}{3^{p-1}} \int_{\mathbb{Z}_m^n} \left\| Tf\big(x + \tfrac{m}{2} e_j\big) - Tf(x) \right\|_F^p d\mu(x) \\
&- 2^{p+1} k^p \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| Tf(x + \varepsilon) - Tf(x) \right\|_F^p d\mu(x) \\
&\qquad - 2^p \int_{\mathbb{Z}_m^n} \left\| Tf(x + e_j) - Tf(x) \right\|_F^p d\mu(x). \qquad (3.2.26)
\end{aligned}$$

Equations (3.2.26) and (3.2.25) together yield

$$\begin{aligned}
&\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^n \varepsilon_j \Big[\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j) \Big] \right\|_E^p d\mu(x) \\
&\quad \geq \frac{1}{C^p n^{1-p/q}} \left(\frac{4}{m} \right)^p \sum_{j=1}^n \Bigg[\frac{1}{3^{p-1}} \int_{\mathbb{Z}_m^n} \big\| Tf\big(x + \tfrac{m}{2} e_j\big) - Tf(x) \big\|_F^p \, d\mu(x) \\
&\qquad\qquad - 2^{p+1} k^p \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \| Tf(x+\varepsilon) - Tf(x) \|_F^p \, d\mu(x) \\
&\qquad\qquad\qquad - 2^p \int_{\mathbb{Z}_m^n} \| Tf(x+e_j) - Tf(x) \|_F^p \, d\mu(x) \Bigg],
\end{aligned}$$

which, after some rearranging and simplification becomes

$$\begin{aligned}
&\sum_{j=1}^n \int_{\mathbb{Z}_m^n} \big\| Tf\big(x + \tfrac{m}{2} e_j\big) - Tf(x) \big\|_F^p \, d\mu(x) \\
&\quad \leq \frac{(3Cm)^p n^{1-p/q}}{3 \cdot 4^p} \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^n \varepsilon_j \Big[\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j) \Big] \right\|_E^p d\mu(x) \\
&\qquad + \frac{2}{3} \cdot 6^p k^p n \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \| Tf(x+\varepsilon) - Tf(x) \|_F^p \, d\mu(x) \\
&\qquad\qquad + \frac{6^p}{3} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \| Tf(x+e_j) - Tf(x) \|_F^p \, d\mu(x). \quad (3.2.27)
\end{aligned}$$

Using (3.2.21) we obtain

$$\begin{aligned}
\sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} \left\|Tf\big(x+\tfrac{m}{2}e_j\big) - Tf(x)\right\|_F^p d\mu(x) \\
&\le \frac{(3Cm)^p n^{1-p/q}}{3\cdot 4^p}\Bigg[\frac{6^p}{3}\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon)-f(x)\|_F^p\, d\mu(x) \\
&\quad + \frac{24^p n^{2p-1}}{k^p}\sum_{j=1}^{n}\int_{\mathbb{Z}_m^n} \|f(x+e_j)-f(x)\|_F^p\, d\mu(x)\Bigg] \\
&\quad + \frac{2}{3}\cdot 6^p k^p n \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|Tf(x+\varepsilon)-Tf(x)\|_F^p\, d\mu(x) \\
&\quad + \frac{6^p}{3}\sum_{j=1}^{n}\int_{\mathbb{Z}_m^n} \|Tf(x+e_j)-Tf(x)\|_F^p\, d\mu(x) \\
= \left[\frac{(18Cm)^p n^{1-p/q}}{9\cdot 4^p} + \frac{2}{3}\cdot 6^p k^p n \|T\|^p\right]\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \|f(x+\varepsilon)-f(x)\|_F^p\, d\mu(x) \\
+ \left[\frac{(3Cm)^p n^{1-p/q}}{3\cdot 4^p}\cdot\frac{24^p n^{2p-1}}{k^p} + \frac{6^p}{3}\|T\|^p\right]\sum_{j=1}^{n}\int_{\mathbb{Z}_m^n} \|f(x+e_j)-f(x)\|_F^p\, d\mu(x)
\end{aligned} \tag{3.2.28}$$

From here, the choice of m large enough will give good results to feed into Lemma 3.2.6. Explicitly, $m \ge 6n^{2+1/q}$ allows us to arrange $4n^2 \le k \le \frac{3m}{4n^{1/q}}$ and therefore

$$\begin{aligned}
\frac{(18Cm)^p n^{1-p/q}}{9\cdot 4^p} + \frac{2}{3}\cdot 6^p k^p n \|T\|^p &\le \frac{(18Cm)^p n^{1-p/q}}{9\cdot 4^p} + \frac{2}{3}\cdot 6^p k^p n C^p \\
&\le \frac{(18Cm)^p n^{1-p/q}}{9\cdot 4^p} + \frac{2}{3}\cdot\frac{(18Cm)^p}{4^p} n^{1-p/q} \le (5Cm)^p n^{1-p/q}
\end{aligned} \tag{3.2.29}$$

and

$$\begin{aligned}
\frac{(3Cm)^p n^{1-p/q}}{3\cdot 4^p}\cdot\frac{24^p n^{2p-1}}{k^p} + \frac{6^p}{3}\|T\|^p &\le \frac{(18Cm)^p n^{1-p/q+2p-1}}{3k^p} + \frac{6^p}{3}C^p \\
&\le \frac{(18Cm)^p n^{1-p/q+2p-1}}{3\cdot 4^p n^{2p}} + \frac{2}{3}\cdot(5Cm)^p \le \frac{1}{n}(5Cm)^p n^{1-p/q}.
\end{aligned} \tag{3.2.30}$$

Equations (3.2.28), (3.2.29) and (3.2.30) together with Lemma 3.2.6 lead us to conclude that $\Gamma_q^{(p)}(T) \leq 20C_q^{(p)}(T)$, including the case $\Gamma_q(T) \leq 20C_q(T)$. $\square$

3.3 Metric type

After obtaining their results on metric cotype, Mendel and Naor went further and gave a metric characterization of type [MN07]. The following definition is the natural adaptation to mappings of their definition of scaled Enflo type. Let $p > 0$. A Lipschitz map $T : X \to Y$ is said to have *scaled Enflo type* p *with constant* τ if for every integer $n \in \mathbb{N}$ there exists an even integer $m \in \mathbb{N}$ such that for every $f : \mathbb{Z}_m^n \to X$

$$\mathbb{E}_{\varepsilon,x} d_Y\big(Tf\big(x + \tfrac{m}{2}\varepsilon\big), Tf(x)\big)^q \leq \tau^p m^p \sum_{j=1}^{n} \mathbb{E}_x d_X\big(f(x+e_j), f(x)\big)^p, \tag{3.3.1}$$

where, as before, the expectations are taken with respect to uniformly chosen $x \in \mathbb{Z}_m^n$ and $\varepsilon \in \{-1,1\}^n$, and $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbb{R}^n$. The smallest constant for which inequality (3.3.1) holds is denoted by $\tau_p(T)$ and called the scaled Enflo type p constant of T.

The main result of this section is the following theorem.

Theorem 3.3.1 (Compare to Thm. 1.1 in [MN07]). *Let E, F be Banach spaces, $T : E \to F$ a linear map and $p \in [1,2]$. Then T has scaled Enflo type p if and only if it has Rademacher type p. Moreover,*

$$\frac{1}{2\pi} T_p(T) \leq \tau_p(T) \leq 15 T_p(T).$$

As with the case of metric cotype, we define a corresponding weak notion. It should be noted that Mendel and Naor did not consider the corresponding weak notion for spaces in [MN07]. Let $1 \leq p < q$. A Lipschitz map $T : X \to Y$ is said to have *scaled Enflo type p with exponent q and constant τ* if for every integer $n \in \mathbb{N}$ there exists an even integer $m \in \mathbb{N}$ such that for every $f : \mathbb{Z}_m^n \to X$

$$\mathbb{E}_{\varepsilon,x} d_Y\big(Tf\big(x + \tfrac{m}{2}\varepsilon\big), Tf(x)\big)^q \le \tau^q m^q n^{q/p-1} \sum_{j=1}^{n} \mathbb{E}_x d_X\big(f(x+e_j), f(x)\big)^q. \quad (3.3.2)$$

The smallest constant for which inequality (3.3.2) holds is denoted by $\tau_p^{(q)}(T)$ and is called the scaled Enflo type p with exponent q constant of T.

The next theorem relates this weak metric type to Rademacher type. It should be mentioned that there is no analogous result in [MN07]. Before proceeding, let us recall a couple of definitions. First, a linear operator $T : E \to F$ is said to have *Beauzamy-Rademacher type* [2] if for some $p \ge 1$ (equivalently, any $p \ge 1$) we have that the sequence $b_n^{(p)}(T)$ defined by

$$b_n^{(p)}(T) = \sup \left\{ \left[\mathbb{E}_\varepsilon \left\| \sum_{j=1}^{n} \varepsilon_j T v_j \right\|^p \right]^{1/p} \; : \; \|v_1\|, \dots, \|v_n\| \le 1 \right\}$$

converges to 0. These operators have a nice characterization due, of course, to Beauzamy [Bea76, Thm. 1]: T has Beauzamy-Rademacher type if and only if T does not uniformly factor the identity operators of ℓ_1^n. Following [PW98, 4.3.7], for $1 < p < 2$ we say that a linear operator $T : E \to F$ has *weak Rademacher type p* if there exists a constant C such that for every $v_1, \dots, v_n$ in E we have

$$\left(\mathbb{E}_\varepsilon \left\| \sum_{j=1}^{n} \varepsilon_j T v_j \right\|_E^2 \right)^{1/2} \le C n^{1/2-1/p} \left(\sum_{j=1}^{n} \|v_j\|_F^p \right)^{1/p}.$$

Equivalently [PW98, Prop. 4.3.9], for any $p < q$ there is a constant C such that for every $v_1, \dots, v_n$ in E we have

$$\left(\mathbb{E}_\varepsilon \left\| \sum_{j=1}^{n} \varepsilon_j T v_j \right\|_E^q \right)^{1/q} \le C n^{1/p-1/q} \left(\sum_{j=1}^{n} \|v_j\|_F^q \right)^{1/q}. \quad (3.3.3)$$

[2] Again, this is not standard terminology. Beauzamy used the term *opérateurs de type Rademacher*.

Theorem 3.3.2. *Let E, F be Banach spaces and $T : E \to F$ a linear map. Suppose that T has weak scaled Enflo type p with exponent q for some $1 \leq p < q$. Then T has Beauzamy-Rademacher type. If $q \leq 2$, then T has weak Rademacher type p and hence has Rademacher type r for every $1 < r < p$. On the other hand, Rademacher type p implies weak scaled Enflo type p with exponent q. To be precise,*

$$\tau_p^{(q)}(T) \leq c_{pq} T_p(T)$$

where c_{pq} is a constant depending only on p and q.

3.3.1 Scaled Enflo type implies Rademacher type

As in the case of metric cotype, the proof of the easy implication will proceed by applying the metric inequality to functions of a specific form. Again, we will assume that the Banach spaces are complex in order for the arguments to work.

Lemma 3.3.3 (Compare to Lemma 2.1 in [MN07])**.** *Let E, F be Banach spaces and $T : E \to F$ a linear operator of weak scaled Enflo type p with exponent q for some $1 \leq p \leq q$. Then for any $v_1, \ldots, v_n$ in E we have*

$$\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j T v_j \right\|_F^q \leq \left(2\pi \tau_p^{(q)}(T)\right)^q n^{q/p-1} \sum_{j=1}^n \|v_j\|_E^q .$$

Proof. Fix $\tau > \tau_p^{(q)}(T)$ and vectors $v_1, \ldots, v_n$ in E. For an even integer m, define $f : \mathbb{Z}_m^n \to E$ by

$$f(x_1, \ldots, x_n) := \sum_{j=1}^n \exp\left(\tfrac{2\pi i x_j}{m}\right) v_j.$$

Then, using the fact that $|e^{\theta}-1| \le \theta$ for $\theta \in [0,\pi]$

$$\sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} \|f(x+e_j)-f(x)\|_E^q \, d\mu(x) = \big|\exp(\tfrac{2\pi i}{m})-1\big|^q \sum_{j=1}^{n} \|v_j\|_E^q \le \left(\frac{2\pi}{m}\right)^q \sum_{j=1}^{n} \|v_j\|_E^q. \quad (3.3.4)$$

and on the other hand

$$\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \big\|Tf\big(x+\tfrac{m}{2}\varepsilon\big)-Tf(x)\big\|_F^q \, d\mu(x) = 2^q \int_{\mathbb{Z}_m^n} \Big\| \sum_{j=1}^{n} \exp\big(\tfrac{2\pi i x_j}{m}\big) T v_j \Big\|_F^q d\mu(x). \quad (3.3.5)$$

Whereas using the contraction principle and the same arguments as in the proof of Lemma 3.2.3,

$$\begin{aligned} &\int_{\mathbb{Z}_m^n} \Big\| \sum_{j=1}^{n} \exp\big(\tfrac{2\pi i x_j}{m}\big) T v_j \Big\|_F^q d\mu(x) \\ &\qquad = \int_{\mathbb{Z}_m^n} \mathbb{E}_\varepsilon \Big\| \sum_{j=1}^{n} \exp\big(\tfrac{2\pi i}{m}(x_j + m(1-\varepsilon_j)/4)\big) T v_j \Big\|_F^q d\mu(x) \\ &\qquad = \int_{\mathbb{Z}_m^n} \mathbb{E}_\varepsilon \Big\| \sum_{j=1}^{n} \exp\big(\tfrac{2\pi i x_j}{m}\big) \varepsilon_j T v_j \Big\|_F^q d\mu(x) \\ &\qquad \ge \frac{1}{2^q} \mathbb{E}_\varepsilon \Big\| \sum_{j=1}^{n} \varepsilon_j T v_j \Big\|_F^q. \end{aligned} \quad (3.3.6)$$

Combining (3.3.2), (3.3.4), (3.3.5) and (3.3.6) gives the desired result. □

Now, let us prove the easy implications of Theorems 3.3.1 and 3.3.2. If a linear $T: E \to F$ has scaled Enflo type p, Lemma 3.3.3 implies immediately that T has

Rademacher type p and moreover $T_p(T) \leq 2\pi\tau_p(T)$. If $1 < p < q$, note that when $\|v_1\|_E, \ldots, \|v_n\|_E \leq 1$ we obtain

$$\frac{1}{n}\left[\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j T v_j \right\|^q\right]^{1/q} \leq 2\pi\tau_p^{(q)}(T) n^{1/p-1} \xrightarrow{n\to\infty} 0,$$

showing that T has Beauzamy-Rademacher type. In the case $q \leq 2$, Lemma 3.3.3 implies that T has weak Rademacher type p (compare to equation (3.3.3)). This implies, in turn, that T has Rademacher type r for every $1 < r < p$ by [PW98, Thm. 4.3.10].

3.3.2 Rademacher type implies scaled Enflo type

In order to prove the harder implications of Theorems 3.3.1 and 3.3.2 we will use a Lemma from [MN07], and in order to state it let us introduce some notation. Given m an integer multiple of 4, k an odd integer, $\varepsilon \in \{-1, 1\}^n$ and $f : \mathbb{Z}_m^n \to E$, define $\mathcal{A}^{(k)} f : \mathbb{Z}_m^n \to E$ by

$$\mathcal{A}^{(k)} f(x) := \frac{1}{k^n} \sum_{z \in (-k,k)^n \cap (2\mathbb{Z})^n}$$

Lemma 3.3.4 (Lemma 2.2 from [MN07]). *For every $p \geq 1$ and every $f : \mathbb{Z}_m^n \to E$ we have*

$$\int_{\mathbb{Z}_m^n} \left\| \mathcal{A}^{(k)} f(x) - f(x) \right\|_E^p d\mu(x) \leq (k-1)^p n^{p-1} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_E^p \, d\mu(x).$$

Proof of the hard implications of Theorems 3.3.1 and 3.3.2. For the sake of simplicity, we will only do the calculations for scaled Enflo type p. The argument works equally well for the weak version just by adding an appropriate factor of $n^{1/q-1/p}$ in the step where Rademacher type is used, much as in the proof of Theorem 3.2.2.

Suppose that $T : E \to F$ is a linear operator with Rademacher type p. In [MN08b] (see equation (39) there) it is shown that for every $x \in \mathbb{Z}_m^n$ and every $\varepsilon \in \{-1,1\}^n$,

$$\left(\frac{k}{k+1}\right)^{n-1}\left(\mathcal{A}^{(k)} f(x+\varepsilon) - \mathcal{A}^{(k)} f(x-\varepsilon)\right) \\ = \sum_{j=1}^n \varepsilon_j \left[\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right] + U(x,\varepsilon) + V(x,\varepsilon),$$

where, by inequalities (41) and (42) in [MN08b],

$$\max\left\{\int_{\mathbb{Z}_m^n} \|U(x)\|_E^p \, d\mu(x), \int_{\mathbb{Z}_m^n} \|U(x)\|_E^p \, d\mu(x)\right\} \\ \le \frac{8^q n^{2p-1}}{k^p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_E^p \, d\mu(x).$$

Thus, we have by the convexity of $\|\cdot\|_F^p$

$$\left(\frac{k}{k+1}\right)^{p(n-1)} \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\|\mathcal{A}^{(k)} Tf(x+\varepsilon) - \mathcal{A}^{(k)} Tf(x-\varepsilon)\right\|_F^p d\mu(x) \\ \le 3^{p-1} \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| \sum_{j=1}^n \varepsilon_j \left[\mathcal{E}_j^{(k)} Tf(x+e_j) - \mathcal{E}_j^{(k)} Tf(x-e_j)\right] \right\|_F^p d\mu(x) \\ + \frac{24^p n^{2p-1}}{k^p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|Tf(x+e_j) - Tf(x-e_j)\|_F^p \, d\mu(x). \quad (3.3.7)$$

Since T is linear, $T\mathcal{E}_j^{(k)} = \mathcal{E}_j^{(k)} T$ and thus for every $C > T_q(T)$

$$\left(\frac{k}{k+1}\right)^{p(n-1)} \mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\|\mathcal{A}^{(k)} Tf(x+\varepsilon) - \mathcal{A}^{(k)} Tf(x-\varepsilon)\right\|_F^p d\mu(x) \\ \le 3^{p-1} C^p \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \left\|\mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j)\right\|_E^p d\mu(x) \\ + \frac{24^p n^{2p-1} \|T\|^p}{k^p} \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x-e_j)\|_E^p \, d\mu(x). \quad (3.3.8)$$

Note that for each fixed $j \in \{1, \ldots, n\}$,

$$\begin{aligned}
&\int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x-e_j) \right\|_E^p d\mu(x) \\
&\quad \le 2^{p-1} \int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x) \right\|_E^p + \left\| \mathcal{E}_j^{(k)} f(x) - \mathcal{E}_j^{(k)} f(x-e_j) \right\|_E^p d\mu(x) \\
&\quad = 2^p \int_{\mathbb{Z}_m^n} \left\| \mathcal{E}_j^{(k)} f(x+e_j) - \mathcal{E}_j^{(k)} f(x) \right\|_E^p d\mu(x) \\
&\quad \le 2^p \int_{\mathbb{Z}_m^n} \| f(x+e_j) - f(x) \|_E^p \, d\mu(x), \qquad (3.3.9)
\end{aligned}$$

where the last inequality follows from the fact that $\mathcal{E}_j^{(k)}$ is an averaging operator and hence has norm 1. Combining (3.3.7), (3.3.8) and (3.3.9)

$$\begin{aligned}
&\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \left\| \mathcal{A}^{(k)} T f(x+\varepsilon) - \mathcal{A}^{(k)} T f(x-\varepsilon) \right\|_F^p d\mu(x) \\
&\quad \le \left(1 + \frac{1}{k}\right)^{p(n-1)} \left[\frac{6^p C^p}{3} + \frac{24^p n^{2p-1} \|T\|^p}{k^p} \right] \sum_{j=1}^n \int_{\mathbb{Z}_m^n} \| f(x+e_j) - f(x) \|_E^p \, d\mu(x).
\end{aligned}$$

(3.3.10)

On the other hand, the convexity of $\|\cdot\|_F^p$ once again gives

$$\begin{aligned}
&\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \big\|Tf\big(x+\tfrac{m}{2}\varepsilon\big) - Tf(x)\big\|_F^p \, d\mu(x) \\
&\qquad \le 3^{p-1}\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \big\|\mathcal{A}^{(k)}Tf\big(x+\tfrac{m}{2}\varepsilon\big) - \mathcal{A}^{(k)}Tf(x)\big\|_F^p \, d\mu(x) \\
&\qquad + 3^{p-1}\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \big\|Tf\big(x+\tfrac{m}{2}\varepsilon\big) - \mathcal{A}^{(k)}Tf\big(x+\tfrac{m}{2}\varepsilon\big)\big\|_F^p \, d\mu(x) \\
&\qquad + 3^{p-1}\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \big\|\mathcal{A}^{(k)}Tf(x) - Tf(x)\big\|_F^p \, d\mu(x) \\
&= 3^{p-1}\Bigg[\left(\frac{m}{4}\right)^{p-1}\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \sum_{t=1}^{m/4} \big\|\mathcal{A}^{(k)}Tf\big(x+2t\varepsilon\big) - \mathcal{A}^{(k)}Tf\big(x+2(t-1)\varepsilon\big)\big\|_F^p \, d\mu(x) \\
&\qquad + 2\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \big\|\mathcal{A}^{(k)}Tf(x) - Tf(x)\big\|_F^p \, d\mu(x)\Bigg] \\
&= 3^{p-1}\Bigg[\left(\frac{m}{4}\right)^{p}\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \big\|\mathcal{A}^{(k)}Tf\big(x+\varepsilon\big) - \mathcal{A}^{(k)}Tf\big(x-\varepsilon\big)\big\|_F^p \, d\mu(x) \\
&\qquad + 2\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \big\|\mathcal{A}^{(k)}Tf(x) - Tf(x)\big\|_F^p \, d\mu(x)\Bigg]
\end{aligned} \tag{3.3.11}$$

Since T is linear, $T\mathcal{A}^{(k)} = \mathcal{A}^{(k)}T$ and thus Lemma 3.3.4, (3.3.10) and (3.3.11) imply

$$\begin{aligned}
&\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} \big\|Tf\big(x+\tfrac{m}{2}\varepsilon\big) - Tf(x)\big\|_F^p \, d\mu(x) \\
&\le 3^{p-1}\Bigg[\left(\frac{m}{4}\right)^{p}\left(1+\frac{1}{k}\right)^{p(n-1)}\left[\frac{6^pC^p}{3} + \frac{24^p n^{2p-1}\|T\|^p}{k^p}\right] + 2(k-1)^p n^{p-1}\|T\|^p\Bigg] \\
&\qquad \cdot \sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} \|f(x+e_j) - f(x)\|_E^p \, d\mu(x)
\end{aligned}$$

Recall that $C > \|T\|$. Now, if $m \ge 3n^{3-2/p}$ we may choose k such that $4n^{2-1/p} \le k \le 3m/(2n^{1-1/p})$ and thus

$$(k-1)^p n^{p-1} \le (3/2)^p m^p; \qquad \frac{n^{2p-1}}{k^p} \le \frac{1}{4^p}; \qquad (1+1/k)^p(n-1) \le (1+1/k)^{kp} \le 4^p.$$

A bit of elementary algebra then reveals that T has scaled Enflo type p with constant $15T_q(T)$ □

CHAPTER IV

LIPSCHITZ p-CONVEX AND p-CONCAVE OPERATORS

4.1 Introduction

The classical examples of Banach spaces of functions or sequences (say, L_p or c_0) come naturally endowed with an order structure that is compatible with the norm, and this is often a useful tool. Banach spaces with such "extra" order structure are called Banach lattices, and this additional ingredient makes the theory of Banach lattices in some regards simpler, cleaner and more complete than the theory for general Banach spaces [LT79]. There is of course a theory of linear operators involving Banach lattices, and two of the most important classes of such operators are the p-convex and p-concave ones. These two notions play an important role in the study of isomorphic properties of lattices, for example uniform convexity in Banach lattices [LT79, Sec. 1.f] and the study of rearrangement invariant function spaces [LT79, Sec. 2.e]. Let us recall their definitions. Consider $1 \leq p \leq \infty$. A linear map $T : E \to L$ from a Banach space E to a Banach lattice L is called *p-convex* if there exists a constant $M < \infty$ such that for all $v_1, \dots, v_n \in E$

$$\left\| \Big(\sum_{j=1}^{n} |Tv_j|^p \Big)^{1/p} \right\|_L \leq M \Big(\sum_{j=1}^{n} \|v_j\|_E^p \Big)^{1/p}, \quad \text{if } 1 \leq p < \infty$$

or

$$\left\| \bigvee_{j=1}^{n} |Tv_j| \right\|_L \leq M \max_{1 \leq j \leq n} \|v_j\|_E \,, \quad \text{if } p = \infty.$$

The smallest such constant M is denoted $M^{(p)}(T)$. On the other hand, a linear operator $S : L \to E$ from a Banach lattice L to a Banach space E is called p-concave if there exists a constant $M < \infty$ such that for all $v_1, \ldots, v_n \in L$

$$\Big(\sum_{j=1}^{n} \|Sv_j\|_E^p\Big)^{1/p} \leq M \left\| \Big(\sum_{j=1}^{n} |v_j|^p\Big)^{1/p} \right\|_L, \quad \text{if } 1 \leq p < \infty$$

or

$$\max_{1 \leq j \leq n} \|Sv_j\|_E \leq \left\| \bigvee_{j=1}^{n} |v_j| \right\|_L, \quad \text{if } p = \infty.$$

The smallest such constant M is denoted $M_{(p)}(T)$. The constants $M^{(p)}(T)$ and $M_{(p)}(T)$ are called the *p-convexity*, respectively *p-concavity* constant of T.

In this chapter we develop nonlinear counterparts of these two concepts, considering Lipschitz maps between a metric space and a Banach lattice, and show how some of the elementary results from the theory of p-convex and q-concave operators admit generalizations to the Lipschitz setting. The basic background and notation not covered in this introduction can be found in [LT79].

4.2 Lipschitz p-convex operators

The concept of Lipschitz p-convex operator was inspired by our discovery of the following non-linear version of the Maurey-Nikishin factorization theorem. The proof presented here follows very closely that of [AK06, Thm. 7.1.2]

Theorem 4.2.1. *Let X be a metric space and μ be a σ-finite measure on some measurable space (Ω, Σ) and $1 \leq p < q < \infty$. Suppose that $T : X \to L_p(\mu)$ is a Lipschitz operator and $C > 0$. The following are equivalent:*

(a) There exists a density function h on Ω such that

$$\left[\int_{\{h>0\}} \left| \frac{Tx - Tx'}{h^{1/p}} \right|^q h \, d\mu \right]^{1/q} \leq C d(x, x'), \qquad x, x' \in X \tag{4.2.1}$$

and

$$\mu\{|Tx - Tx'| > 0, h = 0\} = 0 \qquad x, x' \in X. \tag{4.2.2}$$

(b) For every $x_1, \ldots, x_n, x_1', \ldots, x_n' \in X$ *and* $\lambda_1, \ldots, \lambda_n \geq 0$,

$$\left\| \left(\sum_{j=1}^{n} \lambda_j |Tx_j - Tx_j'|^q \right)^{1/q} \right\|_{L_p(\mu)} \leq C \left(\sum_{j=1}^{n} \lambda_j d(x_j, x_j')^q \right)^{1/q} \tag{4.2.3}$$

As in the linear case, condition (a) is equivalent to the existence of a factorization diagram

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & L_p(\mu) \\ \Big\downarrow{\scriptstyle S} & & \Big\uparrow{\scriptstyle j} \\ L_q(hd\mu) & \xrightarrow{\ i_{q,p}\ } & L_p(hd\mu) \end{array}$$

where S is a Lipschitz function with $\mathrm{Lip}(S) \leq C$ and the isometry j has, in fact, range $L_p(A, \mu)$ where $A = \{h > 0\}$. Also, if we consider X as a pointed metric space with a designated point $0 \in X$ and impose the condition $T(0) = 0$, condition (4.2.2) can be replaced by the somewhat simpler one

$$\mu\{|Tx| > 0, h = 0\} = 0 \qquad x \in X \setminus \{0\}.$$

Proof of Theorem 4.2.1. Without loss of generality, via a first change of density, we may assume that μ is in fact a probability measure.

$(a) \Rightarrow (b)$ Let $x_1, \ldots, x_n, x'_1, \ldots, x'_n \in X$ and $\lambda_1, \ldots, \lambda_n \geq 0$. Since μ is a probability space and $p < q$, the $L_p(hd\mu)$ norm is smaller than the $L_q(hd\mu)$ norm and thus

$$\begin{aligned}
\left\| \left(\sum_{j=1}^n \lambda_j |Tx_j - Tx'_j|^q \right)^{1/q} \right\|_{L_p(\mu)} &= \left\| \left(\sum_{j=1}^n \lambda_j |Sx_j - Sx'_j|^q \right)^{1/q} \right\|_{L_p(hd\mu)} \\
&\leq \left\| \left(\sum_{j=1}^n \lambda_j |Sx_j - Sx'_j|^q \right)^{1/q} \right\|_{L_q(hd\mu)} \\
&= \left(\sum_{j=1}^n \lambda_j \left\| Sx_j - Sx'_j \right\|^q_{L_q(hd\mu)} \right)^{1/q} \\
&\leq C \left(\sum_{j=1}^n \lambda_j d(x_j, x'_j)^q \right)^{1/q}.
\end{aligned}$$

$(b) \Rightarrow (a)$ Assume C is the best constant in (4.2.3). Without loss of generality, we can assume $C = 1$ (by considering T/C instead of T).

Let

$$W_0 = \left\{ f : \Omega \to \mathbb{R} \,:\, 0 \leq f \leq \left(\sum_{j=1}^n \lambda_j |Tx_j - Tx'_j|^q \right)^{p/q}, \;\; \sum_{j=1}^n \lambda_j d(x_j, x'_j)^q \leq 1 \right\},$$

and let W be the closure of W_0 in $L_1(\mu)$. Since 1 is the best constant in (4.2.3),

$$\sup \left\{ \int_\Omega f \, d\mu : f \in W_0 \right\} = \sup \left\{ \int_\Omega f \, d\mu : f \in W \right\} = 1. \tag{4.2.4}$$

CLAIM 1: $W^{q/p}$ is a convex set.

It suffices to show that $W_0^{q/p}$ is a convex set. Let $f, g \in W_0$ and $a, b \geq 0$ with $a + b = 1$. From the definition of W_0, there exist $x_1, \ldots, x_n, x'_1, \ldots, x'_n \in X$ and $\lambda_1, \ldots, \lambda_n \geq 0$ such that

$$0 \leq f \leq \left(\sum_{j=1}^n \lambda_j |Tx_j - Tx'_j|^q \right)^{p/q} \quad \text{and} \quad \sum_{j=1}^n \lambda_j d(x_j, x'_j)^q \leq 1,$$

and there also exist $y_1, \dots, y_m, y'_1, \dots, y'_m \in X$ and $\sigma_1, \dots, \sigma_m \geq 0$ such that

$$0 \leq g \leq \left(\sum_{k=1}^{m} \sigma_k |Ty_k - Ty'_k|^q \right)^{p/q} \quad \text{and} \quad \sum_{k=1}^{m} \sigma_k d(y_k, y'_k)^q \leq 1.$$

Now

$$\begin{aligned} 0 \leq \left(af^{q/p} + b^{q/p} \right)^{p/q} &\leq \left(a \sum_{j=1}^{n} \lambda_j |Tx_j - Tx'_j|^q + b \sum_{k=1}^{m} \sigma_k |Ty_k - Ty'_k|^q \right)^{p/q} \\ &\leq \left(\sum_{j=1}^{n} a\lambda_j |Tx_j - Tx'_j|^q + \sum_{k=1}^{m} b\sigma_k |Ty_k - Ty'_k|^q \right)^{p/q}, \end{aligned}$$

and since

$$\sum_{j=1}^{n} a\lambda_j d(x_j, x'_j)^q + \sum_{k=1}^{m} b\sigma_k d(y_k, y'_k)^q \leq a + b = 1,$$

we conclude that $\left(af^{q/p} + b^{q/p} \right)^{p/q} \in W_0$ and therefore $W_0^{q/p}$ is a convex set.

CLAIM 2: There exists $h \in W$ such that $\int h d\mu = 1$.

Since μ is a probability measure, the map $f \mapsto \int f d\mu$ is a continuous linear functional and therefore it will suffice to show that W is a weakly compact set in $L_1(\mu)$. By definition, W is norm closed. Moreover, it is convex so W is weakly closed.

In order to show that W is weakly compact, all that is left to check is equi-integrability. Suppose that W is not equi-integrable. Then there exist $\delta > 0$, a sequence $(E_n)_{n=1}^{\infty}$ of disjoint subsets of Ω and a sequence $(f_n)_{n=1}^{\infty}$ in W such that for all $n \in \mathbb{N}$,

$$\int_{E_n} f_n \, d\mu > \delta.$$

Thus given any $N \in \mathbb{N}$, since the sets (E_n) are disjoint,

$$N\delta \leq \sum_{n=1}^{N} \int_{E_n} f_n \, d\mu \leq \int \max\{f_1, \ldots, f_n\} \, d\mu$$
$$\leq N^{p/q} \int \left(\sum_{n=1}^{N} \frac{1}{N} f_n^{q/p} \right)^{p/q} d\mu.$$

By Claim 1, this last integral is at most 1, so $\delta \leq N^{p/q-1}$. Since $p/q < 1$, this is a contradiction for large enough N.

Now, let $f \in W$ and $\tau > 0$. By Claim 1,

$$\frac{1}{1+\tau}\left(h^{q/p} + \tau f^{q/p}\right) \in W^{q/p},$$

so from (4.2.4)

$$(1+\tau)^{p/q} \geq \int \left(h^{q/p} + \tau f^{q/p}\right)^{p/q} d\mu. \tag{4.2.5}$$

But

$$\int \left(h^{q/p} + \tau f^{q/p}\right)^{p/q} d\mu. \geq \int_{\{h>0\}} h \, d\mu + \tau^{p/q} \int_{\{h=0\}} f \, d\mu \; = \; 1 + \tau^{p/q} \int_{\{h=0\}} f \, d\mu.$$

so, since $0 < p/q < 1$,

$$0 \leq \int_{\{h=0\}} f \, d\mu \leq \frac{(1+\tau)^{p/q} - 1}{\tau^{p/q}} \underset{\tau \to 0^+}{\longrightarrow} 0,$$

from where we get (4.2.2). by considering f of the form $|Tx - Tx'|/d(x, x')$.

From (4.2.5),

$$\frac{(1+\tau)^{p/q} - 1}{\tau} \geq \int_{\{h>0\}} \left[\frac{\left(1 + \tau (f/h)^{q/p}\right)^{p/q} - 1}{\tau} \right] h d\mu. \tag{4.2.6}$$

Letting $\tau \to 0^+$, the left-hand side of (4.2.6) converges to p/q. By Fatou's lemma, the right-hand side is at least

$$\frac{p}{q}\int_{\{h>0\}}\left(\frac{f}{h}\right)^{q/p} hd\mu.$$

By considering once more f of the form $|Tx - Tx'|/d(x,x')$ we get (4.2.1). □

Since condition (a) in Theorem 4.2.1 is nothing but the fact that the linear extension $\hat{T} : \mathscr{F}(X) \to L_p(\mu)$ of $T : X \to L_p(\mu)$ is q-convex, the following definition is a natural one:

Definition 4.2.2. Let X be a metric space and L a Banach lattice. A Lipschitz map $T : X \to L$ is called *Lipschitz p-convex* if there exists a constant $C \geq 0$ such that for any $x_j, x'_j \in X$,

$$\left\| \left(\sum_{j=1}^n |Tx_j - Tx'_j|^p \right)^{1/p} \right\|_L \leq C \left(\sum_{j=1}^n d(x_j, x'_j)^p \right)^{1/p}.$$

The smallest such constant C is the *Lipschitz p-convexity constant* of T and is denoted by $M^{(p)}_{\text{Lip}}(T)$.

One could be tempted to follow the footsteps of [FJ09] and "add constants" to the Lipschitz p-convexity condition; that is, checking that the condition is equivalent to having inequality (4.2.3):

$$\left\| \left(\sum_{j=1}^n \lambda_j |Tx_j - Tx'_j|^p \right)^{1/p} \right\|_L \leq C \left(\sum_{j=1}^n \lambda_j d(x_j, x'_j)^p \right)^{1/p}.$$

for any $x_j, x'_j \in X$ and $\lambda_j \geq 0$. Unfortunately, the convergence issues in the context of general Banach lattices are more delicate and we will not explore that route.

The situation of Theorem 4.2.1, where a Lipschitz map turned out to be Lipschitz p-convex if and only if its linearization is p-convex, is in fact the general case as demonstrated below.

Theorem 4.2.3. *Let X be a metric space and L a Banach lattice. A Lipschitz map $T : X \to L$ is Lipschitz p-convex if and only if $\hat{T} : \mathscr{F}(X) \to L$ is p-convex. Moreover, in this case the p-convexity constants are the same.*

Proof. The "if" part is trivial: p-convexity of $\hat{T}$ clearly implies Lipschitz p-convexity of T with no increment in the constant, since $\|m_{xx'}\|_{\mathscr{F}(X)} = d(x, x')$ and $\hat{T}m_{xx'} = Tx - Tx'$.

Now suppose that T is Lipschitz p-convex. Let $\varphi_j^* \in L^*$ be arbitrary. For any $x_j, x_j' \in X$ with $x_j \neq x_j'$ we obviously have

$$\Big(\sum_j \Big|\frac{\langle \varphi_j^*, Tx_j - Tx_j'\rangle}{d(x_j, x_j')}\Big|^{p'}\Big)^{1/p'} = \sup_{\sum_j |\alpha_j|^p \leq 1} \sum_j \alpha_j \frac{\langle \varphi_j^*, Tx_j - Tx_j'\rangle}{d(x_j, x_j')}.$$

Using [LT79, Prop. 1.d.2.(iii)], the latter is bounded by

$$\begin{aligned}
&\sup_{\sum_j |\alpha_j|^p \leq 1} \Bigg(\Big(\sum_j |\varphi_j^*|^{p'}\Big)^{1/p'}\Bigg)\Bigg(\Big(\sum_j |\alpha_j|^p \frac{\big|Tx_j - Tx_j'\big|^p}{d(x_j, x_j')^p}\Big)^{1/p}\Bigg) \\
&\qquad \leq \Bigg\|\Big(\sum_j |\varphi_j^*|^{p'}\Big)^{1/p'}\Bigg\|_{L^*} \sup_{\sum_j |\alpha_j|^p \leq 1} \Bigg\|\Big(\sum_j |\alpha_j|^p \frac{\big|Tx_j - Tx_j'\big|^p}{d(x_j, x_j')^p}\Big)^{1/p}\Bigg\|_L
\end{aligned}$$

The Lipschitz p-convexity of T allows us to bound this by

$$\begin{aligned}
&\Bigg\|\Big(\sum_j |\varphi_j^*|^{p'}\Big)^{1/p'}\Bigg\|_{L^*} M_{\mathrm{Lip}}^{(p)}(T) \sup_{\sum_j |\alpha_j|^p \leq 1} \Big(\sum_j |\alpha_j|^p \frac{d(x_j, x_j')^p}{d(x_j, x_j')^p}\Big)^{1/p} \\
&\qquad = M_{\mathrm{Lip}}^{(p)}(T) \Bigg\|\Big(\sum_j |\varphi_j^*|^{p'}\Big)^{1/p'}\Bigg\|_{L^*}.
\end{aligned}$$

Therefore,

$$\Big(\sum_j \Big|\frac{(\hat{T}^*\varphi_j^*)(x_j) - (\hat{T}^*\varphi_j^*)(x_j')}{d(x_j, x_j')}\Big|^{p'}\Big)^{1/p'} \leq M_{\mathrm{Lip}}^{(p)}(T) \Bigg\|\Big(\sum_j |\varphi_j^*|^{p'}\Big)^{1/p'}\Bigg\|_{L^*},$$

so taking the supremum over all pairs $x_j, x_j' \in X$ with $x_j \neq x_j'$ we conclude

$$\left(\sum_j \left\| \hat{T}^* \varphi_j^* \right\|_{X^\#}^{p'} \right)^{1/p'} \leq M_{\mathrm{Lip}}^{(p)}(T) \left\| \left(\sum_j |\varphi_j^*|^{p'} \right)^{1/p'} \right\|_{L^*}.$$

Since the $\varphi_j^* \in L^*$ were arbitrary, this means that $\hat{T}^* : L^* \to X^\#$ is p'-concave with $M_{(p')}(\hat{T}^*) \leq M_{\mathrm{Lip}}^{(p)}(T)$, and by duality [LT79, Prop. 1.d.4] $\hat{T} : \mathscr{F}(X) \to L$ is p-convex with $M^{(p)}(\hat{T}) \leq M_{\mathrm{Lip}}^{(p)}(T)$. □

Let us note that the argument in the previous result is based on the duality between p-convexity and p'-concavity, so it seems unlikely that it could be used to prove a more similar result for other classes of operators obtained by replacing the expression $\left(\sum_j |x_j|^p \right)^{1/p}$ by other homogeneous functions given by the Krivine functional calculus for Banach lattices.

4.3 Lipschitz p-concave operators

Following up on the previous work on p-convexity we now point our attention to the natural companion concept, that of Lipschitz p-concavity.

Definition 4.3.1. Let X be a metric space and L a Banach lattice. A Lipschitz map $T : L \to X$ is called *Lipschitz p-concave* if there exists a constant $C \geq 0$ such that for any $v_j, v_j' \in L$,

$$\left(\sum_{j=1}^n d(Tv_j, Tv_j')^p \right)^{1/p} \leq C \left\| \left(\sum_{j=1}^n |v_j - v_j'|^p \right)^{1/p} \right\|_L.$$

The smallest such constant C is the *Lipschitz p-concavity constant* of T and is denoted by $M_{(p)}^{\mathrm{Lip}}(T)$.

We will primarily be interested in the case when X is a Banach space. Note that when X is a Banach space and T is linear, clearly T is p-concave if and only if it is Lipschitz p-concave.

The following factorization theorem and its proof are inspired by [LT79, Thm. 1.d.11].

Theorem 4.3.2. *Let X, Y be metric spaces with Y complete and L a Banach lattice. Suppose that $T : X \to L$ is Lipschitz p-convex and $S : L \to Y$ is Lipschitz p-concave. Then the operator ST can be factorized through an $L_p(\mu)$ space. Moreover, we may arrange to have $ST = S_1 T_1$ with $T_1 : X \to L_p(\mu)$, $S_1 : L_p(\mu) \to Y$,* $\mathrm{Lip}(T_1) \leq M^{(p)}_{\mathrm{Lip}}(T)$ *and* $\mathrm{Lip}(S_1) \leq M^{\mathrm{Lip}}_{(p)}(S)$.

Proof. Let I_T be the (in general non-closed) ideal of L generated by the range of T. We define new operations on I_T as in the usual p-concavification procedure, that is for $x, y \in I_T$ and real α put

$$x \oplus y := (x^p + y^p)^{1/p}, \qquad \alpha \odot x := \alpha^{1/p} x,$$

and let $\check{I}_T$ denote the vector lattice obtained when I_T is endowed with the original order and the operations $\oplus, \odot$. Set

$$F_1 := \operatorname{conv} \Big\{ x \in \check{I}_T : |x| \leq \lambda |Tv - Tv'| \\ \text{for some } v, v' \in X, \lambda > 0 \text{ with } \lambda d(v, v') < 1/M^{(p)}_{\mathrm{Lip}}(T) \Big\}$$

and

$$F_2 := \operatorname{conv} \Big\{ x \in \check{I}_T : x > 0 \text{ and } \eta d(Sy, Sy') \geq M^{\mathrm{Lip}}_{(p)}(S) \\ \text{for some } y, y' \in L, \eta > 0 \text{ with } \eta |y - y'| \leq x \Big\}.$$

where both convex hulls are taken in the sense of $\check{I}_T$, i.e. using the operations $\oplus, \odot$.

If x belongs to F_1, then it can be written the form $\bigoplus_j \alpha_j \odot x_j$ where $\alpha_j \geq 0$, $\sum_j \alpha_j = 1$ and $|x_j| \leq \lambda_j |Tv_j - Tv_j'|$ with $\lambda_j d(v_j, v_j') < 1/M^{(p)}_{\mathrm{Lip}}(T)$. Therefore,

$$\|x\| = \left\| \Big(\sum_j |\alpha_j^{1/p} x_j|^p \Big)^{1/p} \right\| \leq \left\| \Big(\sum_j \alpha_j \lambda_j^p |Tv_j - Tv_j'|^p \Big)^{1/p} \right\|$$
$$\leq M^{(p)}_{\mathrm{Lip}}(T) \Big(\sum_j \alpha_j \lambda_j^p d(v_j, v_j')^p \Big)^{1/p} < 1.$$

On the other hand, if x belongs to F_2 then it can be written as $\bigoplus_j \beta_j \odot x_j$ where $\beta_j \geq 0$, $\sum_j \beta_j = 1$ and $x_j \geq \eta_j |y_j - y_j'|$ with $\eta_j \geq 0$ and $\eta_j d(Sy_j, Sy_j') \geq M^{\mathrm{Lip}}_{(p)}(S)$. Therefore

$$\|x\| = \left\| \Big(\sum_j |\beta_j^{1/p} x_j|^p \Big)^{1/p} \right\| \geq \left\| \Big(\sum_j \beta_j \eta_j^p |y_j - y_j'|^p \Big)^{1/p} \right\|$$
$$\geq \frac{1}{M^{\mathrm{Lip}}_{(p)}(S)} \Big(\sum_j \beta_j \eta_j^p d(Sy_j, Sy_j')^p \Big) \geq 1.$$

Hence, $F_1 \cap F_2 = \emptyset$ and since 0 is an internal point of F_1 it follows from the separation theorem that there exists a linear functional φ on $\check{I}_T$ such that $\varphi(x) \leq 1$ for all $x \in F_1$ and $\varphi(x) \geq 1$ for all $x \in F_2$. Note that from the definition of F_2, for any positive real α, any positive x in $\check{I}_T$ and any $x_0 \in F_2$ we have that $\alpha \odot x \oplus x_0$ belongs to F_2. It follows that $\varphi(x) \geq 0$ whenever $0 < x \in \check{I}_T$ and, thus, we can define a seminorm on I_T by putting

$$\|x\|_0 := \varphi(|x|)^{1/p}, \qquad x \in I_T.$$

Let α be a real number and $x \in I_T$. Then

$$\|\alpha x\|_0 = \varphi(|\alpha||x|)^{1/p} = \varphi(|\alpha|^p \odot |x|)^{1/p} = \big[|\alpha|^p \varphi(|x|)\big]^{1/p} = |\alpha| \, \|x\|_0 \, .$$

Let $x, y \in I_T$. Note that $|x| + |y| = \big(|x|^{1/p} \oplus |y|^{1/p}\big)^p$. On the other hand, from the lattice functional calculus and Hölder's inequality, whenever α and β are positive reals with $\alpha^{p'} + \beta^{p'} = 1$ we have

$$\big(|x|^{1/p} \oplus |y|^{1/p}\big)^p \leq \alpha^{-p} \odot |x| + \beta^{-p} \odot |y|.$$

Hence

$$\begin{aligned} \|x+y\|_0^p &= \varphi\big(|x+y|\big) \leq \varphi\big(|x|+|y|\big) = \varphi\big(\big(|x|^{1/p} \oplus |y|^{1/p}\big)^p\big) \\ &\leq \varphi\big(\alpha^{-p} \odot |x| + \beta^{-p} \odot |y|\big) = \alpha^{-p}\varphi(|x|) + \beta^{-p}\varphi(|y|) \\ &= \alpha^{-p}\, \|x\|_0^p + \beta^{-p}\, \|y\|_0^p\,. \end{aligned}$$

Therefore, setting

$$\alpha := \frac{\|x\|_0^{1/p'}}{(\|x\|_0 + \|y\|_0)^{1/p'}} \qquad \text{and} \qquad \beta := \frac{\|y\|_0^{1/p'}}{(\|x\|_0 + \|y\|_0)^{1/p'}}$$

we satisfy the condition $\alpha^{p'} + \beta^{p'} = 1$, while

$$\alpha^{-p} = \frac{(\|x\|_0 + \|y\|_0)^{p-1}}{\|x\|_0^{p-1}} \qquad \text{and} \qquad \beta^{-p} = \frac{(\|x\|_0 + \|y\|_0)^{p-1}}{\|y\|_0^{p-1}}$$

so we conclude

$$\|x+y\|_0^p \leq \frac{(\|x\|_0 + \|y\|_0)^{p-1}}{\|x\|_0^{p-1}}\, \|x\|_0^p + \frac{(\|x\|_0 + \|y\|_0)^{p-1}}{\|y\|_0^{p-1}}\, \|y\|_0^p = \big(\, \|x\|_0 + \|y\|_0\,\big)^p,$$

and thus $\|x+y\|_0 \leq \|x\|_0 + \|y\|_0$.

Observe now that for any $x, y \in I_T$ we have

$$|x| + |y| \geq \big(|x|^p + |y|^p\big)^{1/p} \geq |x| \vee |y|$$

since these inequalities are valid for reals. By the fact that φ is non-negative, we get that

$$\begin{aligned}\big\||x|+|y|\big\|_0^p &= \varphi\big(|x|+|y|\big) \geq \varphi\big((|x|^p+|y|^p)^{1/p}\big) = \varphi(|x|\oplus|y|) = \varphi(|x|)+\varphi(|y|) \\ &= \|x\|_0^p + \|y\|_0^p \geq \varphi\big(|x|\vee|y|\big) = \big\||x|\vee|y|\big\|_0^p.\end{aligned}$$

This inequality concerning $\|\cdot\|_0$ clearly remains valid in the completion Z of I_T modulo the ideal of all $x \in I_T$ for which $\|x\|_0 = 0$. Therefore, if $|x| \wedge |y| = 0$ for some x and y in the lattice Z then (recalling that $|x|\wedge|y| = |x|+|y|-|x|\vee|y|$) we obtain

$$\big\||x|+|y|\big\|_0^p = \|x\|_0^p + \|y\|_0^p\,,$$

i.e. Z is an abstract L_p space. It follows from the L_p version of Kakutani's representation theorem that Z is order isometric to an $L_p(\mu)$ space for a suitable measure μ.

Let $T_1 : X \to Z$ be defined by $T_1 v = Tv$, $v \in X$, i.e. the same as T but considered as an operator into Z. For $v, v' \in X$ and $\lambda > 0$, if $\lambda d(v,v') < 1/M_{\mathrm{Lip}}^{(p)}(T)$ then $\lambda(T_1 v - T_1 v') \in F_1$, which implies that $\varphi(\lambda(T_1 v - T_1 v')) \leq 1$ and thus $\|\lambda(T_1 v - T_1 v')\|_0 \leq 1$, from where it follows that $\|T_1 v - T_1 v'\|_0 \leq M_{\mathrm{Lip}}^{(p)}(T) d(v,v')$, i.e. $\mathrm{Lip}(T_1) \leq M_{\mathrm{Lip}}^{(p)}(T)$.

Let $S_1 : I_T/\ker(\|\cdot\|_0) \to Y$ be defined by $S_1 x = Sx$, $x \in I_T$. Note that this is well defined: if $Sx \neq Sx'$, then $\frac{M_{(p)}^{\mathrm{Lip}}(S)}{d(Sx,Sx')}|x-x'|$ belongs to F_2, so $\varphi\Big(\frac{M_{(p)}^{\mathrm{Lip}}(S)}{d(Sx,Sx')}|x-x'|\Big) \geq 1$ and in particular $\|x-x'\|_0 \neq 0$. By an argument similar to the one for T_1, this defines a Lipschitz map from $I_T/\ker(\|\cdot\|_0)$ to Y with Lipschitz constant at most $M_{(p)}^{\mathrm{Lip}}(S)$. Since $I_T/\ker(\|\cdot\|_0)$ is dense in Z and Y is complete, this can be extended to a Lipschitz map $S_1 : Z \to Y$ with the same Lipschitz constant, giving the desired factorization. □

CHAPTER V

LIPSCHITZ (q,p)-MIXING OPERATORS*

5.1 Introduction

The theory of p-summing operators plays a very important role in modern Banach space theory, not only for its intrinsic beauty but also for its far-reaching applications among a wide spectrum of subjects like Banach space geometry, harmonic analysis, approximation theory, operator theory and others. When working with p-summing operators, it is not unusual to come across an operator T with the property that $S \circ T$ is p-summing whenever S is q-summing. One example of such situation appears in A. Pietsch's composition theorem, a very useful tool already present in his seminal paper [Pie67]: whenever $p, q, r \in [1, \infty]$ satisfy $1/p = 1/q + 1/r$, the composition of a q-summing operator followed by an r-summing operator is p-summing. Another example with T being the identity on an L_1 space is provided by a celebrated theorem of A. Grothendieck, stating that every continuous linear operator from L_1 into Hilbert space is 1-summing; therefore, any 2-summing operator with an L_1 space as domain is 1-summing. More generally, by a theorem of B. Maurey any 2-summing operator defined on a cotype 2 space is 1-summing. Similarly, any continuous linear operator from a $C(K)$ space into a cotype 2 space is 2-summing.

Inspired by ideas of Maurey [Mau74], Pietsch [Pie80, Chap. 20] systematically studied the situation described in the previous paragraph and called such operators (q,p)-mixing. Another exposition of the subject, with a more "tensorial" point of view, can be found in [DF93, Sec. 32]. On the other hand, J. Farmer and W. B. Johnson [FJ09] recently introduced the concept of a Lipschitz p-summing operator between metric spaces. They proved that this is a true extension of the linear

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concept, and obtained a nonlinear counterpart of one of the cornerstones of the theory of (linear) p-summing operators: Pietsch's celebrated domination/factorization theorem.

In the present chapter, the corresponding concept of Lipschitz (q,p)-mixing operators is defined and studied. We start by recalling the necessary theory of Lipschitz p-summing operators, and then introduce the main definition. Afterwards three different characterizations of Lipschitz (q,p)-mixing operators are presented. The first one is an integral inequality along the lines of Pietsch's domination theorem, while the second one corresponds to his (q,p)-mixed sequences. The third one relies on the recently developed [CD11] duality theory for Lipschitz p-summing operators. Finally these characterizations are used to prove relationships between (q,p)-mixing constants and s-summing norms in various situations, in particular obtaining reversed inequalities for Lipschitz p-summing norms.

5.2 Notation and preliminaries

The letters X, Y, Z will denote metric spaces, whereas E, F, G will denote Banach spaces. All metric spaces under consideration will be *pointed*, i.e. each one has a special point designated by 0. For a mapping T between metric spaces, $\mathrm{Lip}(T)$ denotes its Lipschitz constant. Given a metric space X, the Banach space of real valued Lipschitz functions defined on X that send 0 to 0 with the Lipschitz norm $\mathrm{Lip}(\cdot)$ will be denoted by $X^{\#}$. As customary, B_E denotes the closed unit ball of a Banach space E. The letters p, q, r, s will designate elements of $[1,\infty]$, and p' denotes the exponent conjugate to p (i.e. the one that satisfies $1/p + 1/p' = 1$).

The remainder of this section is all from [FJ09]. Recall that for $1 \leq p < \infty$ a linear operator $T : E \to F$ is called *p-summing* if there is a non negative constant C such that for any vectors v_j in E, the inequality

$$\sum_j \|Tv_j\|^p \leq C^p \sup_{v^* \in B_{E^*}} \sum_j |v^*(v_j)|^p$$

holds. In this case, the *p-summing norm* $\pi_p(T)$ of T is the infimum of such constants C. Inspired by this useful concept, Farmer and Johnson defined the *Lipschitz p-summing norm* π_p^L of a (non necessarily linear) mapping $T : X \to Y$ as the smallest non negative constant C such that for any x_j, x_j' in X and any positive reals a_j,

$$\sum_j a_j d(Tx_j, Tx_j')^p \leq C^p \sup_{f \in B_{X^\#}} \sum_j a_j \big|f(x_j) - f(x_j')\big|^p.$$

This definition remains unchanged if we consider only the case $a_j = 1$, a very useful observation in [FJ09] also credited to M. Mendel and G. Schechtman. The set of all Lipschitz p-summing maps from X to Y is denoted by $\Pi_p^L(X, Y)$. Note that the condition that would naturally correspond to being Lipschitz ∞-summing is just the Lipschitz condition, and we adopt this convention for notational convenience.

It is clear from the definition that the Lipschitz p-summing norm of a mapping is equal to the supremum of the Lipschitz p-summing norms of all the restrictions of said mapping to finite subsets of its domain. Also directly from the definition, it is clear that the Lipschitz p-summing norm has the ideal property: $\pi_p^L(A \circ T \circ B) \leq \mathrm{Lip}(A) \cdot \pi_p^L(T) \cdot \mathrm{Lip}(B)$ whenever the composition makes sense. We next state the domination/factorization theorem for Lipschitz p-summing operators [FJ09, Thm. 1], a particular case of the general Pietsch-type domination theorems considered in [BPR10].

Theorem 5.2.1. *For a mapping $T : X \to Y$ and a constant $C \geq 0$, the following are equivalent:*

(a) $\pi_p^L(T) \leq C$.

(b) There is a probability μ *on* $B_{X^\#}$ *such that for any* $x, x' \in X$

$$d(Tx, Tx') \leq C \left[\int_{B_{X^\#}} \left| f(x) - f(x') \right|^p d\mu(f) \right]^{1/p}.$$

(c) For some (or any) isometric embedding J *of* Y *into a* 1*-injective space* Z*, there is a factorization*

$$\begin{array}{ccccc} L_\infty(\mu) & & \xrightarrow{I_{\infty,p}} & & L_p(\mu) \\ {\scriptstyle A}\uparrow & & & & \downarrow{\scriptstyle B} \\ X & \xrightarrow{T} & Y & \xrightarrow{J} & Z. \end{array}$$

with μ *a probability and* $\mathrm{Lip}(A) \cdot \mathrm{Lip}(B) \leq C$.

The domination theorem immediately implies the monotonicity of the Lipschitz p-summing norms, that is, $\pi_p^L(T) \geq \pi_q^L(T)$ whenever $p \leq q$.

It is important to stress that the concept of a Lipschitz p-summing operator is a true generalization of that of a (linear) p-summing operator: for a bounded linear operator T between Banach spaces, T is Lipschitz p-summing if and only if it is (linearly) p-summing, and moreover $\pi_p(T) = \pi_p^L(T)$ [FJ09, Thm. 2].

5.3 Definition and elementary properties

Let $1 \leq p, q \leq \infty$. An operator $T : X \to Y$ is said to be *Lipschitz* (q,p)*-mixing with constant* K if for any metric space Z and any Lipschitz q-summing operator $S : Y \to Z$, the composition $S \circ T$ is a Lipschitz p-summing operator and $\pi_p^L(S \circ T) \leq K \pi_q^L(S)$. The smallest such K will be denoted by $\mathfrak{m}_{q,p}^L(T)$.

A first example of such an operator already appears in [FJ09], where a nonlinear Grothendieck inequality is proved. Namely, any Lipschitz map T from a metric tree X into a Hilbert space is Lipschitz 1-summing and in fact $\pi_1^L(T) \leq K_G \,\mathrm{Lip}(T)$ where K_G is Grothendieck's constant. This result together with the factorization theorem

5.2.1, imply that the identity on X is Lipschitz $(2,1)$-mixing with constant at most K_G. D. Chen and B. Zheng [CZ11] gave another proof of this nonlinear Grothendieck inequality, showing that $\mathfrak{m}^L_{2,1}(id_X) \leq A_1^{-1}$ where A_1 is the constant in Khintchine's inequality.

Note that in order to determine if a mapping $T : X \to Y$ is Lipschitz (q,p)-mixing, it suffices to consider its compositions with mappings from Y to ℓ_q (or any other infinite-dimensional L_q space, in fact). First, we may assume without loss of generality that X and Y are finite metric spaces. Now suppose that

$$\pi_p^L(R \circ T) \leq C\pi_q^L(R) \quad \text{for any} \quad R : Y \to \ell_q, \tag{$\star$}$$

and let $S : Y \to Z$ be a Lipschitz q-summing map. Let $J : Z \to W$ be an isometric embedding of Z into a 1-injective space W. By the factorization theorem for Lipschitz q-summing operators, we can find a factorization

$$\begin{array}{ccccc} L_\infty(\mu) & & \xrightarrow{I_{\infty,q}} & & L_q(\mu) \\ {\scriptstyle A}\uparrow & & & & \downarrow{\scriptstyle B} \\ Y & \xrightarrow{S} & Z & \xrightarrow{J} & W. \end{array}$$

with $\mathrm{Lip}(A) \cdot \mathrm{Lip}(B) = \pi_q^L(S)$. Since Y is a finite set, the range of $I_{\infty,q} \circ A$ is a finite subset of $L_q(\mu)$ and therefore is almost isometric to a subset of ℓ_q. Thus, for the purposes of computing Lipschitz summing norms we may assume that $I_{\infty,q} \circ A$ is a map from Y into ℓ_q, so condition $(\star)$ applies and therefore $\pi_p^L(I_{\infty,q} \circ A \circ T) \leq C\pi_q^L(I_{\infty,q} \circ A)$. The ideal property for Lipschitz q-summing operators implies $\pi_q^L(I_{\infty,q} \circ A) \leq \mathrm{Lip}(A) \cdot \pi_q^L(I_{\infty,q}) \leq \mathrm{Lip}(A) \cdot 1$, whereas the ideal property for Lipschitz p-summing operators gives us

$$\begin{aligned} \pi_p^L(J \circ S \circ T) &= \pi_p^L(B \circ I_{\infty,q} \circ A \circ T) \\ &\leq \mathrm{Lip}(B) \cdot \pi_q^L(I_{\infty,q} \circ A \circ T) \leq \mathrm{Lip}(B) \cdot C \cdot \mathrm{Lip}(A) = C\pi_q^L(S). \end{aligned}$$

But since J is an isometric embedding $J \circ S \circ T$ and $S \circ T$ have the same Lipschitz p-summing norm, so we conclude that $\pi_p^L(S \circ T) \leq C\pi_q^L(S)$, i.e. T is Lipschitz (q, p)-mixing with constant C.

The ideal property for Lipschitz p-summing operators implies that for any operator T, $\mathfrak{m}_{q,p}^L(T) = \text{Lip}(T)$ whenever $q \leq p$ and $\mathfrak{m}_{\infty,p}^L(T) = \pi_p^L(T)$, so only the case $1 \leq p < q < \infty$ gives something new. Moreover, Lipschitz (q, p)-mixing operators also satisfy the ideal property and $\mathfrak{m}_{q,p}^L(A \circ T \circ B) \leq \text{Lip}(A) \cdot \mathfrak{m}_{q,p}^L(T) \cdot \text{Lip}(B)$ whenever the composition makes sense.

Just from the definition, we obtain a trivial composition formula for Lipschitz (q, p)-mixing operators: regardless of the values of p, q and r in $[1, \infty]$, the composition of a Lipschitz (p, r)-mixing operator T followed by a Lipschitz (q, p)-mixing operator S is Lipschitz (q, r)-mixing and moreover $\mathfrak{m}_{q,r}^L(ST) \leq \mathfrak{m}_{q,p}^L(S) \cdot \mathfrak{m}_{p,r}^L(T)$.

Additionally, the monotonicity of the Lipschitz p-summing norms implies a monotonicity condition for the Lipschitz (q, p)-mixing constants: whenever $p_1 \leq p_2$ and $q_2 \leq q_1$, $\mathfrak{m}_{q_2,p_2}^L(T) \leq \mathfrak{m}_{q_1,p_1}^L(T)$ for any T.

5.4 Characterizations

In this section three different characterizations of Lipschitz (q, p)-mixing operators are presented, all of them somewhat inspired by analogous results in the linear theory.

5.4.1 Domination

The first characterization is close in spirit to the characterization of Lipschitz p-summing operators via a dominating measure [FJ09]. Compare with [DF93, Prop. 32.4].

Theorem 5.4.1. *Let $1 \leq p \leq q \leq \infty$, $T : X \to Y$ Lipschitz and $C \geq 0$. The following are equivalent:*

(a) T is Lipschitz (q, p)-mixing with $\mathfrak{m}_{q,p}^L(T) \leq C$.

(b) For any probability measure μ on $B_{Y^\#}$ there exists a probability measure ν on $B_{X^\#}$ such that for all $x, x' \in X$,

$$\left[\int_{B_{Y^\#}} |g(Tx) - g(Tx')|^q \, d\mu(g)\right]^{1/q} \leq C \left[\int_{B_{X^\#}} |f(x) - f(x')|^p \, d\nu(f)\right]^{1/p}.$$

(c) For any $x_1, \ldots, x_m, x'_1, \ldots, x'_m \in X$ and $g_1, \ldots, g_n \in Y^\#$,

$$\left[\sum_{j=1}^m \left[\sum_{k=1}^n \left|g_k(Tx_j) - g_k(Tx'_j)\right|^q\right]^{p/q}\right]^{1/p} \leq C \left[\sum_{k=1}^n \mathrm{Lip}(g_k)^q\right]^{1/q} \cdot \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m \left|f(x_j) - f(x'_j)\right|^p\right]^{1/p}.$$

(d) For any $x_1, \ldots, x_m, x'_1, \ldots, x'_m \in X$ and any probability measure μ on $B_{Y^\#}$,

$$\left[\sum_{j=1}^m \left(\int_{B_{Y^\#}} \left|g(Tx_j) - g(Tx'_j)\right|^q d\mu(g)\right)^{p/q}\right]^{1/p} \leq C \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m \left|f(x_j) - f(x'_j)\right|^p\right]^{1/p}. \quad (5.4.1)$$

In this case, $\mathfrak{m}^L_{q,p}(T)$ is equal to the infimum of such constants C in either (b), (c) or (d).

Proof. The case $q = \infty$ reduces to the Domination Theorem for Lipschitz p-summing operators (Thm. 5.2.1), so we will assume $1 \leq p \leq q < \infty$.

$(a) \Rightarrow (b)$: Suppose that $T : X \to Y$ is Lipschitz (q,p)-mixing, and let μ be a probability measure on $B_{Y^\#}$. By restricting to Y the canonical inclusion $C(B_{Y^\#}) \hookrightarrow L_q(\mu)$, we get a Lipschitz q-summing operator $j_\mu : Y \to L_q(\mu)$ with Lipschitz q-summing norm at most 1. Hence, since T is Lipschitz (q,p)-mixing, the composition $j_\mu \circ T : X \to L_q(\mu)$ is Lipschitz p-summing. By the Pietsch domination theorem for Lipschitz p-summing operators (Thm. 5.2.1), there is a probability measure ν on $B_{X^\#}$ such that for all $x, x' \in X$,

$$\|j_\mu(Tx) - j_\mu(Tx')\|_{L_q(\mu)} \le \pi_p^L(j_\mu \circ T) \left[\int_{B_{X^\#}} \left| f(x) - f(x') \right|^p d\nu(f) \right]^{1/p},$$

i.e.

$$\left[\int_{B_{Y^\#}} \left| g(Tx) - g(Tx') \right|^q d\mu(g) \right]^{1/q} \le \pi_p^L(j_\mu \circ T) \left[\int_{B_{X^\#}} \left| f(x) - f(x') \right|^p d\nu(f) \right]^{1/p},$$

so we have condition (b) with $C = \pi_p^L(j_\mu \circ T) \le \mathfrak{m}_{q,p}^L(T)\pi_q^L(j_\mu) \le \mathfrak{m}_{q,p}^L(T)$.

$(b) \Rightarrow (c)$: By homogeneity, we may assume without loss of generality that $\sum_{k=1}^n \mathrm{Lip}(g_k)^q = 1$. Then $\mu := \sum_{k=1}^n \mathrm{Lip}(g_k)^q \delta_{g_k/\,\mathrm{Lip}(g_k)}$ (where δ_g is the Dirac measure at $g \in Y^\#$) is a probability measure on $B_{Y^\#}$, so there exists a corresponding ν as in (b). Therefore,

$$\begin{aligned} \sum_{j=1}^m \left[\sum_{k=1}^n \left| g_k(Tx_j) - g_k(Tx'_j) \right|^q \right]^{p/q} &= \sum_{j=1}^m \left[\int_{B_{Y^\#}} \left| g(Tx_j) - g(Tx'_j) \right|^q d\mu(g) \right]^{p/q} \\ &\le C^p \sum_{j=1}^m \int_{B_{X^\#}} |f(x) - f(x')|^p \, d\nu(f) \le C^p \sup_{f \in B_{X^\#}} \sum_{j=1}^m |f(x) - f(x')|^p, \end{aligned}$$

so we have (c) with the same constant C.

$(c) \Rightarrow (d)$: Condition (c) means that all finitely supported probability measures μ on $B_{Y^\#}$ already satisfy (5.4.1). Since the set of all finitely supported probability measures on $B_{Y^\#}$ is $\sigma\big(C(B_{Y^\#})^*, C(B_{Y^\#})\big)$-dense in the set of all probability measures on $B_{Y^\#}$, it follows that inequality (5.4.1) holds for all probability measures μ on $B_{Y^\#}$.

$(d) \Rightarrow (a)$: Now let $S : Y \to Z$ be Lipschitz q-summing. Appealing to the domination theorem again, there is a measure μ on $B_{Y^\#}$ such that for all $y, y' \in Y$,

$$d_Z\big(Sy, Sy\big)^p \le \pi_q^L(S)^p \left[\int_{B_{Y^\#}} \left| g(y) - g(y') \right|^q d\mu(g) \right]^{p/q}.$$

Fix $x_1, \dots, x_m, x'_1, \dots, x'_m \in X$. Then, from the previous inequality

$$\left[\sum_{j=1}^m d_Z\big(S(Tx_j), S(Tx'_j)\big)^p\right]^{1/p} \leq \pi_q^L(S) \left[\sum_{j=1}^m \left[\int_{B_{Y^\#}} \left|g(Tx_j) - g(Tx'_j)\right|^q d\mu(g)\right]^{p/q}\right]^{1/p},$$

which together with (5.4.1) implies

$$\left[\sum_{j=1}^m d_Z\big(STx_j, STx'_j\big)^p\right]^{1/p} \leq C\pi_q^L(S) \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m \left|f(x) - f(x')\right|^p\right]$$

so $S \circ T$ is Lipschitz p-summing and $\pi_p^L(S \circ T) \leq C\pi_q^L(S)$. Therefore, T is Lipschitz (q,p)-mixing and $\mathfrak{m}_{q,p}^L(T) \leq C$. □

5.4.2 Lipschitz (q,p)-mixed sequences

Linear (q,p)-mixing operators were given such a name by Pietsch [Pie80] because a linear operator is linearly (q,p)-mixing if and only if it maps every weakly p-summable sequence into a (q,p)-mixed sequence, i.e. one that can be expressed as the pointwise product of a weakly q-summable sequence and an r-summable scalar sequence where $1/p = 1/q + 1/r$. The analogous result in the nonlinear case will follow from Theorem 5.4.1 as soon as we find an appropiate nonlinear counterpart of (q,p)-mixing sequences. We will use Ky Fan's minimax lemma as stated in [Pie80, Lemma E.4.2]. A collection of real-valued functions $\mathscr{A}$ defined on a set K is called *concave* if given $\Phi_1, \dots, \Phi_n \in \mathscr{A}$ and $\alpha_1, \dots, \alpha_n \geq 0$ such that $\sum_{j=1}^n \alpha_j = 1$, there is $\Phi \in \mathscr{A}$ satisfying $\Phi(x) \geq \sum_{j=1}^n \alpha_j \Phi_j(x)$ for all $x \in K$. Now we prove a result analogous to [Pie80, Thm. 16.4.3] (credited mostly to [Mau74]).

Proposition 5.4.2. *Let $1 \le p < q < \infty$ and $1/p = 1/q + 1/r$. Then, for any points $x_1, \ldots, x_n, x'_1, \ldots, x'_n$ in X,*

$$\sup\left\{\left[\sum_{j=1}^{n}\left[\int_{B_{X^\#}} \left|f(x_j) - f(x'_j)\right|^q d\mu(f)\right]^{p/q}\right]^{1/p} : \mu \text{ is a probability on } B_{X^\#}\right\}$$
$$= \inf\left\{\left[\sum_{j=1}^{n}\lambda_j^r\right]^{1/r} \sup_{f\in B_{X^\#}}\left[\sum_{j=1}^{n}\lambda_j^{-q}\left|f(x_j) - f(x'_j)\right|^q\right]^{1/q} : \lambda_j > 0\right\}. \quad (5.4.2)$$

Proof. Define σ to be the supremum on the left hand side of (5.4.2) (noting that it is finite). Let $u = r/p$ and $v = q/p$, so that $1/u + 1/v = 1$. We now consider the compact, convex subset

$$K = \left\{\xi = (\xi_j)_{j=1}^n : \sum_{j=1}^{n}\xi_j^u \le \sigma^p \text{ and } \xi_j \ge 0\right\}$$

of ℓ_u^n. For $\varepsilon > 0$ and μ a probability on $B_{X^\#}$, observe that the equation

$$\Phi(\xi) = \sum_{j=1}^{n}(\xi_j + \varepsilon)^{-v}\int_{B_{X^\#}} \left|f(x_j) - f(x'_j)\right|^q d\mu(f)$$

defines a continuous convex function Φ on K. Take the special vector $\xi \in \mathbb{R}^n$ with

$$\xi_j = \left(\int_{B_{X^\#}} \left|f(x_j) - f(x'_j)\right|^q d\mu(f)\right)^{1/uv}.$$

Then $\xi \in K$ and $\Phi(\xi) \le \sigma^p$. Since the collection $\mathscr{A}$ of all functions Φ obtained in this way is concave, by Ky Fan's lemma we can find $\xi^0 \in K$ such that $\Phi(\xi^0) \le \sigma^p$ for all $\Phi \in \mathscr{A}$ simultaneously. In particular, considering the Dirac measure δ_f at a function $f \in B_{X^\#}$ we obtain

$$\sum_{j=1}^{n}(\xi_j^0 + \varepsilon)^{-v}\left|f(x_j) - f(x'_j)\right|^q \le \sigma^p.$$

Set $\lambda_j(\varepsilon) := (\xi_j^0 + \varepsilon)^{1/p}$. Then

$$\lim_{\varepsilon\downarrow 0}\left[\sum_{j=1}^n \lambda_j(\varepsilon)^r\right]^{1/r} = \left[\sum_{j=1}^n \xi_j^{r/p}\right]^{1/r} = \left[\sum_{j=1}^n \xi_j^u\right]^{1/r} \leq \sigma^{p/r} = \sigma^{1/u}$$

and, for $f \in B_{X\#}$

$$\left[\sum_{j=1}^n \lambda_j(\varepsilon)^{-q}\left|f(x_j) - f(x_j')\right|^q\right]^{1/q} = \left[\sum_{j=1}^n (\xi_j^0 + \varepsilon)^{-v}\left|f(x_j) - f(x_j')\right|^q\right]^{1/q} \leq \sigma^{p/q} = \sigma^{1/v}.$$

Therefore, the right-hand side of (5.4.2) is less than or equal to the left-hand side.

Conversely, let $\lambda_j > 0$ be arbitrary. Then, by Hölder's inequality for any probability measure μ on $B_{X\#}$ we have

$$\begin{aligned}
&\left[\sum_{j=1}^n \left[\int_{B_{X\#}} \left|f(x_j) - f(x_j')\right|^q d\mu(f)\right]^{p/q}\right]^{1/p} \\
&\quad = \left[\sum_{j=1}^n \left[\lambda_j \left(\int_{B_{X\#}} \lambda_j^{-q}\left|f(x_j) - f(x_j')\right|^q d\mu(f)\right)^{1/q}\right]^p\right]^{1/p} \\
&\quad \leq \left[\sum_{j=1}^n \lambda_j^r\right]^{1/r} \left(\sum_{j=1}^n \int_{B_{X\#}} \lambda_j^{-q}\left|f(x_j) - f(x_j')\right|^q d\mu(f)\right)^{1/q} \\
&\quad = \left[\sum_{j=1}^n \lambda_j^r\right]^{1/r} \left(\int_{B_{X\#}} \sum_{j=1}^n \lambda_j^{-q}\left|f(x_j) - f(x_j')\right|^q d\mu(f)\right)^{1/q} \\
&\quad \leq \left[\sum_{j=1}^n \lambda_j^r\right]^{1/r} \sup_{f\in B_{X\#}} \left(\sum_{j=1}^n \lambda_j^{-q}\left|f(x_j) - f(x_j')\right|^q\right)^{1/q}.
\end{aligned}$$

□

Together, Theorem 5.4.1 and Proposition 5.4.2 immediately give us another characterization of Lipschitz (q, p)-mixing operators, stated below.

Corollary 5.4.3. *Let $1 \leq p < q < \infty$ and $1/p = 1/q + 1/r$. A Lipschitz map $T : X \to Y$ is (q,p)-mixing if and only if there exists a constant C such that for all $x_1, \dots, x_n, x'_1, \dots, x'_n \in X$,*

$$\inf \left\{ \left[\sum_{j=1}^{n} \lambda_j^r \right]^{1/r} \sup_{g \in B_{Y^\#}} \left[\sum_{j=1}^{n} \lambda_j^{-q} \big| g(Tx_j) - g(Tx'_j) \big|^q \right]^{1/q} : \lambda_j > 0 \right\} \\ \leq C \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^{n} \big| f(x_j) - f(x'_j) \big|^p \right]^{1/p}.$$

In this case, $\mathfrak{m}^L_{q,p}(T)$ is equal to the infimum of such constants C.

5.4.3 Chevet-Saphar spaces

The expression on the right-hand side of (5.4.2) looks reminiscent of the Chevet-Saphar norms introduced in [CD11]. This section is devoted to a characterization of Lipschitz (q,p)-mixing operators in terms of such norms. Let us recall the pertinent definitions first.

An *E-valued molecule on X* is a finitely supported function $m : X \to E$ such that $\sum_{x \in X} m(x) = 0$. The space of E-valued molecules on X, denoted $\mathcal{M}(X, E)$ is clearly a vector space under pointwise addition. Given $x, x' \in X$, define $m_{xx'} := \chi_{\{x\}} - \chi_{\{x'\}}$. The simplest non-zero molecules, i.e. those of the form $v m_{xx'}$ for some $x, x' \in X$ and $v \in E$, are called *atoms*. Note that any molecule may be expressed (in a non-unique way) as a finite sum of atoms. The p-th Chevet-Saphar norm of a molecule m is given by

$$cs_p(m) := \inf \left\{ \left(\sum_j \lambda_j^p \, \|v_j\|^p \right)^{1/p} \sup_{f \in B_{X^\#}} \left(\sum_j \lambda_j^{-p'} \big| f(x_j) - f(x'_j) \big| \right)^{1/p'} \\ : \ m = \sum_j v_j m_{x_j x'_j}, \lambda_j > 0 \right\}.$$

The space of E-valued molecules on E, endowed with the norm $cs_p(\cdot)$, is denoted by $\mathcal{CS}_p(X,E)$. There is a canonical way of inducing a pairing between E-valued molecules on X and functions from X to E^*: given $m \in \mathcal{M}(X,E)$ and a function $T : X \to E^*$, define $\langle T, m\rangle := \sum_{x\in X} \langle T(x), m(x)\rangle$. If we know an expression of the molecule as a sum of atoms, say $m = \sum_j v_j m_{x_j x'_j}$, then $\langle T, m\rangle = \sum_j \langle Tx_j - Tx'_j, v_j\rangle$. The main theorem in [CD11] states that with this pairing, the dual space of $\mathcal{CS}_p(X,E)$ is canonically identified with the space of Lipschitz p'-summing operators from X into E^*. Also from [CD11], recall that for any Banach space E a Lipschitz map $T : X \to Y$ naturally induces a well-defined linear map $T_E : \mathcal{M}(X,E) \to \mathcal{M}(Y,E)$ given by

$$T_E\bigg(\sum_{j=1}^n v_j m_{x_j x'_j}\bigg) = \sum_{j=1}^n v_j m_{Tx_j Tx'_j}.$$

Now we come to the third characterization of Lipschitz (q,p)-mixing operators.

Theorem 5.4.4. *Let $T : X \to Y$ be a Lipschitz map. The following are equivalent:*

(a) T is Lipschitz (q,p)-mixing.

(b) For every Banach space G (or only $G = \ell_{q'}$), the operator

$$T_G : \mathcal{CS}_{p'}(X,G) \to \mathcal{CS}_{q'}(Y,G)$$

is continuous.

In this case,

$$\mathfrak{m}^L_{q,p}(T) = \Big\| T_{\ell_{q'}} : \mathcal{CS}_{p'}(X,\ell_{q'}) \to \mathcal{CS}_{q'}(Y,\ell_{q'}) \Big\| \geq \| T_G : \mathcal{CS}_{p'}(X,G) \to \mathcal{CS}_{q'}(Y,G) \| .$$

Proof. First, suppose that T is Lipschitz (q,p)-mixing. Let $\varphi \in \big(\mathcal{CS}_{q'}(Y,G)\big)^*$ with $\|\varphi\| \leq 1$. Since $\big(\mathcal{CS}_{q'}(Y,G)\big)^* \equiv \Pi^L_q(Y,G^*)$, we can identify φ with a map $L_\varphi \in$

$\Pi_q^L(Y, G^*)$ with $\pi_q^L(L_\varphi) = \|\varphi\| \le 1$. Let $m = \sum v_j m_{x_j x'_j} \in \mathcal{M}(X, G)$. Then $T_G(m) = \sum v_j m_{Tx_j Tx'_j}$, so

$$\langle \varphi, T_G(m) \rangle = \sum_j \langle L_\varphi(Tx_j) - L_\varphi(Tx'_j), v_j \rangle = \langle L_\varphi \circ T, m \rangle,$$

and thus

$$\begin{aligned} \big|\langle \varphi, T_G(m) \rangle\big| = \big|\langle L_\varphi \circ T, m \rangle\big| &\le \pi_p^L(L_\varphi \circ T) cs_{p'}(m) \\ &\le \pi_q^L(L_\varphi) \mathfrak{m}_{q,p}^L(T) cs_{p'}(m) \le \mathfrak{m}_{q,p}^L(T) cs_{p'}(m). \end{aligned}$$

Taking the supremum over all such φ we obtain, $cs_{q'}\big(T_G(m)\big) \le \mathfrak{m}_{q,p}^L(T) cs_{p'}(m)$, i.e. $T_G : \mathcal{CS}_{p'}(X, G) \to \mathcal{CS}_{q'}(Y, G)$ is continuous and $\|T_G\| \le \mathfrak{m}_{q,p}^L(T)$.

Now, suppose that $T_{\ell_{q'}} : \mathcal{CS}_{p'}(X, \ell_{q'}) \to \mathcal{CS}_{q'}(Y, \ell_{q'})$ is continuous and has norm C. and let $S : Y \to \ell_q$ be a q-summing operator. Let m be an $\ell_{q'}$-valued molecule on X, say $m = \sum_j v_j m_{x_j x'_j}$ with $v_j \in \ell_{q'}$ and $x_j, x'_j \in X$. Then

$$\langle S \circ T, m \rangle = \sum_j \langle v_j, STx_j - STx'_j \rangle = \Big\langle S, \sum_j v_j m_{Tx_j Tx'_j} \Big\rangle = \langle S, T_{\ell_{q'}}(m) \rangle.$$

By the duality between the Lipschitz q-summing norm and the q'-Chevet-Saphar norm, together with the boundedness of $T_{\ell_{q'}}$,

$$\big|\langle S \circ T, m \rangle\big| = \big|\langle S, T_{\ell_{q'}}(m) \rangle\big| \le \pi_q^L(S) cs_{q'}\big(T_{\ell_{q'}}(m)\big) \le \pi_q^L(S) \cdot C \cdot cs_{p'}(m).$$

Taking the supremum over all m with $cs_{p'}(m) \le 1$ and invoking the duality between the Lipschitz p-summing norm and the p'-Chevet-Saphar norm, we conclude that $\pi_p^L(S \circ T) \le C \pi_q^L(S)$. By the remarks in Section 5.3, we conclude that T is Lipschitz (q, p)-mixing with $\mathfrak{m}_{q,p}^L(T) \le C$. □

Of course, the space $\ell_{q'}$ in the preceding theorem may be replaced by any other infinite dimensional $L_{q'}$ space.

5.5 Applications

5.5.1 The Lipschitz $(2,1)$-mixing constant of the identity on a tree

As already mentioned in Section 5.3, Farmer and Johnson [FJ09] proved a non-linear Grothendieck inequality which, in our language, means that the identity on a metric tree is Lipschitz $(2,1)$-mixing with constant at most Grothendieck's constant. While both their proof and the one given in [CZ11] make explicit use of the lifting property for trees, using Theorem 5.4.1 we can reobtain the same bound without explicitly appealing to the lifting property.

Lemma 5.5.1. *When T is an unweighted graph-theoretic tree on $n+1$ points and H is a Hilbert space,* $\mathrm{Lip}(T,H)$ *is isometric to* $\ell_\infty^n(H)$*.*

Proof. From [CD11, Sec. 4.2], $\mathcal{CS}_1(T,H)$ is isometric to $\ell_1^n(H)$ in a natural way. By the duality result [CD11, Thm. 4.3], $\mathrm{Lip}(T,H)$ is then isometric to $\ell_\infty^n(H)$. □

Proposition 5.5.2. *Let T be a finite unweighted graph-theoretic tree. Then the identity on T is Lipschitz* $(2,1)$*-mixing with constant at most K_G.*

Proof. Let $x_1,\dots,x_m,x_1',\dots,x_m' \in T$ and let μ be a probability measure on $B_{T^\#}$. Note that

$$\sup_{f\in B_{T^\#}} \sum_{j=1}^m \left|f(x_j)-f(x_j')\right|$$

is the norm of the linear operator A from $T^\#$ to ℓ_1^m given by $f \mapsto \big(f(x_j)-f(x_j')\big)_{j=1}^m$. By Lemma 5.5.1, $T^\#$ can be identified with ℓ_∞^N for some N, so the operator A under consideration goes from ℓ_∞^N to ℓ_1^m. The classical Grothendieck inequality gives us

$$\left\|A:\ell_\infty^N(L_2(\mu)) \to \ell_1^m(L_2(\mu))\right\| \le K_G \left\|A:\ell_\infty^N \to \ell_1^m\right\|.$$

But another application of Lemma 5.5.1 reveals that $\ell_\infty^N(L_2(\mu))$ can be identified with the space of Lipschitz functions from T to $L_2(\mu)$, so in fact one has

$$\sup_{\mathrm{Lip}(F:T\to L_2(\mu))\le 1} \sum_{j=1}^m \left\|F(x_j)-F(x'_j)\right\|_{L_2(\mu)} \le K_G \sup_{f\in B_{T^\#}} \sum_{j=1}^m \left|f(x_j)-f(x'_j)\right|.$$

In particular, consider the pointwise evaluation $\delta : T \to L_2(\mu)$. For any $x, x' \in T$ we have

$$\|\delta(x)-\delta(x')\|_{L_2(\mu)} = \left[\int_{g\in B_{T^\#}} \left|g(x)-g(x')\right|^2 d\mu(g)\right]^{1/2} \le d(x,x'),$$

hence $\mathrm{Lip}(\delta : T \to L_2(\mu)) \le 1$ and thus

$$\sum_{j=1}^m \left[\int_{B_{T^\#}} \left|g(x_j)-g(x'_j)\right|^2 d\mu(g)\right]^{1/2} \le K_G \sup_{f\in B_{T^\#}} \sum_{j=1}^m \left|f(x_j)-f(x'_j)\right|.$$

By Theorem 5.4.1, we conclude that the identity on T is Lipschitz $(2,1)$-summing with constant at most K_G. □

5.5.2 An "interpolation style" theorem

As it so often happens with many constants associated to mappings, it is not easy to calculate the Lipschitz (q,p)-mixing constant of a specific map. The following "interpolation style" theorem is based on [Puh77, Lemma 5] and gives useful bounds that are sufficient in some cases.

Theorem 5.5.3. *Let $1 \le p, q, r \le \infty$ with $1/r + 1/q = 1/p$. Then every Lipschitz p-summing map $T : X \to Y$ is Lipschitz (q,p)-mixing and satisfies*

$$\mathfrak{m}^L_{q,p}(T) \le \pi^L_p(T)^{p/r}\,\mathrm{Lip}(T)^{p/q}.$$

Proof. The fact that T is (q,p)-mixing is obvious from the ideal property of Lipschitz p-summing operators. Now, let $x_1, \ldots, x_n, x'_1, \ldots, x'_n \in X$. For any probability measure μ on $B_{Y^\#}$, from the pointwise inequality $|g(y)-g(y')| \le \mathrm{Lip}(g)\cdot d(y,y')$ for any $y,y' \in Y$ and $g \in Y^\#$ we have that

$$\left[\sum_{j=1}^n \left(\int_{B_{Y^\#}} \left|g(Tx_j)-g(Tx_j')\right|^q \, d\mu(g)\right)^{p/q}\right]^{1/p}$$
$$\le \left[\sum_{j=1}^n \left(\int_{B_{Y^\#}} \left|g(Tx_j)-g(Tx_j')\right|^p \, d\mu(g)\right)^{p/q} d(Tx_j,Tx_j')^{(q-p)p/q}\right]^{1/p}. \quad (5.5.1)$$

Noting that $(q-p)r/q = p$, Hölder's inequality lets us bound the latter expression by

$$\left[\sum_{j=1}^n \int_{B_{Y^\#}} \left|g(Tx_j)-g(Tx_j')\right|^p \, d\mu(g)\right]^{1/q} \left[\sum_{j=1}^n d(Tx_j,Tx_j')^p\right]^{1/r}. \quad (5.5.2)$$

On one hand, the fact that T is Lipschitz p-summing means that

$$\left[\sum_{j=1}^n d(Tx_j,Tx_j')^p\right]^{1/r} \le \pi_p^L(T)^{p/r} \sup_{f\in B_{X^\#}} \left[\sum_{j=1}^n \left|f(x_j)-f(x_j')\right|^p\right]^{1/r}, \quad (5.5.3)$$

whereas on the other a simple pointwise estimate gives

$$\left[\sum_{j=1}^n \int_{B_{Y^\#}} \left|g(Tx_j)-g(Tx_j')\right|^p \, d\mu(g)\right]^{1/q}$$
$$\le \mathrm{Lip}(T)^{p/q} \sup_{f\in B_{X^\#}} \left[\sum_{j=1}^n \left|f(x_j)-f(x_j')\right|^p\right]^{1/q}. \quad (5.5.4)$$

Bringing (5.5.1), (5.5.2), (5.5.3) and (5.5.4) together we have

$$\left[\sum_{j=1}^n \left(\int_{B_{Y^\#}} \left|g(Tx_j)-g(Tx_j')\right|^q \, d\mu(g)\right)^{p/q}\right]^{1/p}$$
$$\le \pi_p^L(T)^{p/r}\,\mathrm{Lip}(T)^{p/q} \sup_{f\in B_{X^\#}} \left[\sum_{j=1}^n \left|f(x_j)-f(x_j')\right|^p\right]^{1/p}$$

and thus the desired conclusion follows from Theorem 5.4.1. □

The identity on a finite discrete metric space

Denote by D_n the discrete metric space on n points. Theorem 5.5.3 allows us to explicitly evaluate the (q,p)-mixing norm of the identity on D_n. In fact, if $1 \leq p \leq q \leq \infty$ then the Lipschitz (q,p)-mixing norm of the identity on D_n is equal to $(2-2/n)^{1/p-1/q}$. To see it, let $1 \leq r \leq \infty$ satisfy $1/r + 1/q = 1/p$. From [FJ09] we have that $\pi_s^L(id_{D_n}) = (2-2/n)^{1/s}$ for any $s \in [1,\infty]$, and therefore

$$\mathfrak{m}_{q,p}^L(id_{D_n}) \geq \frac{\pi_p^L(id_{D_n} \circ id_{D_n})}{\pi_q^L(id_{D_n})} = \frac{(2-2/n)^{1/p}}{(2-2/n)^{1/q}} = (2-2/n)^{1/p-1/q}.$$

On the other hand, from Theorem 5.5.3,

$$\mathfrak{m}_{q,p}^L(id_{D_n}) \leq \pi_p^L(id_{D_n})^{p/r} \operatorname{Lip}(id_{D_n})^{p/q} = (2-2/n)^{1/r} \cdot 1 = (2-2/n)^{1/p-1/q}$$

and thus $\mathfrak{m}_{q,p}^L(id_{D_n}) = (2-2/n)^{1/p-1/q}$. Let us remark what this means: for every metric space X and any $T : D_n \to X$, $\pi_p^L(T) \leq (2-2/n)^{1/p-1/q}\pi_q^L(T)$ and this inequality is sharp.

Reversed inequalities between Lipschitz p-summing norms

The next result goes along the same theme: using Theorem 5.5.3 together with known estimates for Lipschitz p-summing norms.

Theorem 5.5.4. *(a) For any $n \in \mathbb{N}$ and $1 \leq p \leq q$,*

$$\mathfrak{m}_{q,p}^L(id_{\ell_2^n}) \leq c_{p,n}^{p/q-1} \qquad \textit{where} \qquad c_{p,n} = \left[\int_{S_{n-1}} |x_1|^p d\lambda(x)\right]^{1/p},$$

λ being the normalized rotation invariant measure on S_{n-1}. Hence, $\pi_p^L(T) \leq c_{p,n}^{p/q-1}\pi_q^L(T)$ for any Lipschitz map $T : \ell_2^n \to Y$.

(b) For any finite-dimensional normed space E and $2 \leq q$,

$$\mathfrak{m}_{q,2}^L(id_E) \leq \big[\dim(E)\big]^{1/2-1/q}$$

Hence, $\pi_2^L(T) \leq \big[\dim(E)\big]^{1/2-1/q}\pi_q^L(T)$ for any Lipschitz map $T : E \to Y$.

(c) There exists an universal constant C so that for any finite metric space X on n points and $1 \leq q$,

$$\mathfrak{m}_{q,1}^L(id_X) \leq C^{1/q'}\big[\log n\big]^{1/q'}$$

Hence, $\pi_1^L(T) \leq C^{1/q'}\big[\log n\big]^{1/q'}\pi_q^L(T)$ for any Lipschitz map $T : X \to Y$.

Proof. Everything follows from Theorem 5.5.3, together with the fact that the Lipschitz p-summing norm and the linear p-summing norm of a linear operator between Banach spaces coincide (see [FJ09, Theorem 2]), and the following estimates on p-summing norms:

(a) $\pi_p(id_{\ell_2^n}) = c_{p,n}^{-1}$ (see, for instance, [TJ89, Theorem 10.3]).

(b) $\pi_2(id_E) = \big[\dim(E)\big]^{1/2}$ for any finite-dimensional space E (see, for instance [TJ89, Proposition 9.11]).

(c) $\pi_1(id_X) \leq C\log n$, essentially proved in [Bou85] as remarked in [FJ09].

□

5.5.3 The general "interpolation style" theorem

Theorem 5.5.3 is in fact a particular case of the following more general one.

Theorem 5.5.5. *Let $0 < \theta < 1$ and $1 \le p \le q_0, q_1 \le \infty$. Define $1/q := (1-\theta)/q_0 + \theta/q_1$. For a Lipschitz map $T : X \to Y$,*

$$\mathfrak{m}^L_{q,p}(T) \le \mathfrak{m}^L_{q_0,p}(T)^{1-\theta}\mathfrak{m}^L_{q_1,p}(T)^{\theta}.$$

Proof. Set $1/r := 1/p - 1/q$, $1/r_0 := 1/p - 1/q_0$ and $1/r_1 := 1/p - 1/q_1$. Note that $1/r := (1-\theta)/r_0 + \theta/r_1$. Let $x_1, \ldots, x_n, x'_1, \ldots, x'_n \in X$. Given $\varepsilon > 0$, from Corollary 5.4.3 for each $k = 0, 1$ there exist $\lambda_{j,k} > 0$, $1 \le j \le n$ such that

$$\left[\sum_{j=1}^{n} \lambda_{j,k}^{r_k}\right]^{1/r_k} \sup_{g \in B_{Y^\#}} \left[\sum_{j=1}^{n} \lambda_{j,k}^{-q_k} \big|g(Tx_j) - g(Tx'_j)\big|^{q_k}\right]^{1/q_k}$$
$$\le (1+\varepsilon)\mathfrak{m}^L_{q_k,p}(T) \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^{m} \big|f(x_j) - f(x'_j)\big|^{p}\right]^{1/p}.$$

Moreover, dividing by the appropriate constant we may assume that in fact

$$\left[\sum_{j=1}^{n} \lambda_{j,k}^{r_k}\right]^{1/r_k} \le (1+\varepsilon)\mathfrak{m}^L_{q_k,p}(T) \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^{m} \big|f(x_j) - f(x'_j)\big|^{p}\right]^{1/p}$$
$$\text{and} \quad \sup_{g \in B_{Y^\#}} \left[\sum_{j=1}^{n} \lambda_{j,k}^{-q_k} \big|g(Tx_j) - g(Tx'_j)\big|^{q_k}\right]^{1/q_k} \le 1.$$

For $1 \le j \le n$, set $\lambda_j = \lambda_{j,0}^{1-\theta}\lambda_{j,1}^{\theta}$. Then, by Hölder's inequality,

$$\left[\sum_{j=1}^{n} \lambda_j^{r}\right]^{1/r} \le \left[\sum_{j=1}^{n} \lambda_{j,0}^{r_0}\right]^{(1-\theta)/r_0} \cdot \left[\sum_{j=1}^{n} \lambda_{j,1}^{r_1}\right]^{\theta/r_1}$$
$$\le (1+\varepsilon)\mathfrak{m}^L_{q_0,p}(T)^{1-\theta}\mathfrak{m}^L_{q_1,p}(T)^{\theta} \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^{m} \big|f(x_j) - f(x'_j)\big|^{p}\right]^{1/p}.$$

On the other hand, it follows from

$$\lambda_j^{-1}\big|f(x_j) - f(x'_j)\big| = \lambda_{j,0}^{-(1-\theta)}\big|f(x_j) - f(x'_j)\big|^{1-\theta}\lambda_{j,1}^{-\theta}\big|f(x_j) - f(x'_j)\big|^{\theta}$$

that

$$\sup_{g\in B_{Y^\#}}\left[\sum_{j=1}^{n}\lambda_j^{-q}\big|g(Tx_j)-g(Tx'_j)\big|^q\right]^{1/q}$$

$$\leq \prod_{k=0,1}\sup_{g\in B_{Y^\#}}\left[\sum_{j=1}^{n}\lambda_{j,k}^{-q_k}\big|g(Tx_j)-g(Tx'_j)\big|^{q_k}\right]^{1/q_k}\leq 1.$$

Therefore, using the other direction of Corollary 5.4.3,

$$\mathfrak{m}_{q,p}^L(T)\leq(1+\varepsilon)\mathfrak{m}_{q_0,p}^L(T)^{1-\theta}\mathfrak{m}_{q_1,p}^L(T)^{\theta}$$

and by letting $\varepsilon\downarrow 0$, the proof is finished. □

For $q>p\geq 1$, we say that a metric space X is (q,p)-mixing if the identity on X is (q,p)-mixing. The following lemma shows that the class of (q,p)-mixing spaces does not depend on p. This result is basically the nonlinear extrapolation theorem of Chen and Zheng [CZ11, Thm. 2.2], presented in a different language.

Corollary 5.5.6. *Let X be a metric space and $1\leq p_0<p_1<q$. Then X is (q,p_0)-mixing if and only if it is (q,p_1)-mixing. Moreover,*

$$\mathfrak{m}_{q,p_1}^L(id_X)\leq\mathfrak{m}_{q,p_0}^L(id_X)\leq\mathfrak{m}_{q,p_1}^L(id_X)^{1/\theta},$$

where θ is defined by $1/p_1=(1-\theta)/q+\theta/p_0$.

Proof. The monotonicity property for (q,p)-mixing constants from Section 5.3 gives $\mathfrak{m}_{q,p_1}^L(id_X)\leq\mathfrak{m}_{q,p_0}^L(id_X)$, whereas the composition property from the same section provides us with the inequality $\mathfrak{m}_{q,p_0}^L(id_X)\leq\mathfrak{m}_{q,p_1}^L(id_X)\cdot\mathfrak{m}_{p_1,p_0}^L(id_X)$. Now, from Theorem 5.5.5

$$\mathfrak{m}_{p_1,p_0}^L(id_X)\leq\mathfrak{m}_{q,p_0}^L(id_X)^{1-\theta}\cdot\mathfrak{m}_{p_0,p_0}^L(id_X)^{\theta}=\mathfrak{m}_{q,p_0}^L(id_X)^{1-\theta}\cdot 1.$$

So we obtain

$$\mathfrak{m}^L_{q,p_0}(id_X) \leq \mathfrak{m}^L_{q,p_1}(id_X) \cdot \mathfrak{m}^L_{q,p_0}(id_X)^{1-\theta}$$

from which the result follows. □

CHAPTER VI

COMPLETELY (q,p)-MIXING MAPS

6.1 Introduction

Operator spaces are a quantized or noncommutative version of Banach spaces, and can be thought of as the result of combining Banach space theory with the noncommuting nature of operator algebra theory. Many of the concepts and results of Banach space theory have counterparts for operator spaces, and in particular p-summing operators are replaced by the completely p-summing maps of Pisier [Pis98]. Just as in Chapter V, there is of course a natural notion of completely (q,p)-mixing maps that has already been introduced in [Yew08]. Unfortunately, no systematic study of these maps was done there. The present chapter aims to fill that void, and it is structured as follows. We start by recalling some basic notation and results from operator space theory, before formally introducing the definition of completely p-summing maps and proving some of their elementary properties. Afterwards, two different characterizations of completely (q,p)-mixing maps are presented. The first one is a "domination" result along the lines of the Pietsch domination theorem for completely p-summing maps due to Pisier [Pis98]. The second one does not clearly correspond to any of the characterizations in the classical case that can be found in [DF93, Sec. 32], but nevertheless it is used to prove an "interpolation" theorem relating different completely (q,p)-mixing norms which actually is inspired by the classical case. As a byproduct, a strengthening of Yew's quantized extrapolation theorem [Yew08, Thm. 8] is obtained. In the final section several composition theorems are proved, culminating with a composition theorem for completely p-summing maps: if $1/r = 1/p + 1/q$, then the composition of a completely p-summing map and a completely q-summing one is completely r-summing.

6.2 Notation and preliminaries

We only assume familiarity with the basic theory of operator spaces; Pisier's book [Pis03] is an excellent reference for that. We will follow very closely Pisier's notation from [Pis98, Pis03]. The letters E, F and G will always denote operator spaces. For an operator space E, a Hilbert space K and $1 \le p \le \infty$, let us define the spaces S_p, $S_p[E]$ and $S_p(K)$. For $1 < p < \infty$, S_p (resp. $S_p(K)$) denotes the space of Schatten class operators in ℓ_2 (resp. on K). In the case $p = \infty$, we denote by S_∞ (resp. $S_p(K)$) the space of all compact operators on ℓ_2 (resp. on K) with the operator space structure inherited from $B(\ell_2)$ (resp. $B(K)$). We define $S_\infty[E]$ as the minimal operator space tensor product of S_∞ and E, and $S_1[E]$ as the operator space projective tensor product of S_1 and E. In the case $1 < p < \infty$, $S_p[E]$ is defined via complex interpolation between $S_\infty[E]$ and $S_1[E]$.

Let E, F be operator spaces and $u : E \to F$ a linear map. For $1 \le p \le \infty$, we will say that u is *completely p-summing* if the mapping

$$I_{S_p} \otimes u : S_p \otimes_{\min} E \to S_p[F]$$

is bounded, and we denote its norm by $\pi_p^o(u)$. By a result of Pisier [Pis98, Corollary 5.5], in the case $1 \le p < \infty$ we in fact have that the cb-norm and the norm of the map $I_{S_p} \otimes u$ are equal. For notational convenience, we will use the convention $\pi_\infty^o(\cdot) = \|\cdot\|_{\mathrm{cb}}$. Completely p-summing maps satisfy the ideal property (that is, $\pi_p^o(uvw) \le \|u\|_{\mathrm{cb}}\, \pi_p^o(v)\, \|w\|_{\mathrm{cb}}$ whenever the composition makes sense), and being completely p-summing is a local property: the completely p-summing norm of $u : E \to F$ is equal to the supremum of the completely p-summing norms of the restrictions of u to finite-dimensional operator subspaces of E. In fact,

$$\pi_p^o(u : E \to F) = \sup\left\{\pi_p^o(uT) \ : \ T : S_{p'}^n \to E, n \ge 1, \|T\|_{\mathrm{cb}} \le 1\right\}.$$

The following theorem, due to Pisier [Pis98, Thm. 5.1] is an important characterization of completely p-summing maps.

Theorem 6.2.1 (Pietsch domination). *Assume $E \subseteq B(H)$. Let $u : E \to F$ be a completely p-summing map ($1 \leq p < \infty$) and let $C = \pi_p^o(u)$. Then there is an ultrafilter $\mathcal{U}$ over an index set I and families $(a_\alpha)_{\alpha \in I}$, $(b_\alpha)_{\alpha \in I}$ in the unit ball of $S_{2p}(H)$ such that for all $n \in \mathbb{N}$ and all (x_{ij}) in $M_n(E)$ we have*

$$\left\| \left[(u x_{ij}) \right] \right\|_{S_p^n[F]} \leq C \lim_{\mathcal{U}} \left\| \left[(a_\alpha x_{ij} b_\alpha) \right] \right\|_{S_p(\ell_2^n \otimes H)} \tag{6.2.1}$$

and

$$\left\| \left[(u x_{ij}) \right] \right\|_{M_n[F]} \leq C \lim_{\mathcal{U}} \left\| \left[(a_\alpha x_{ij} b_\alpha) \right] \right\|_{M_n(S_p(H))}. \tag{6.2.2}$$

Conversely, if an operator u satisfies either (6.2.1) *or* (6.2.2) *then it is completely p-summing with $\pi_p^o(u) \leq C$.*

One consequence of the domination theorem is the monotonicity of p-summing norms: if $1 \leq p \leq q$ and u is completely p-summing, then u is completely q-summing and moreover $\pi_q^o(u) \leq \pi_p^o(u)$. The standard (although not canonical) example of a completely p-summing map is a multiplication map. To be precise, we have [Pis98, Prop. 5.6]

Theorem 6.2.2. *Let K be any Hilbert space. Consider a, b in $S_{2p}(K)$ and let $M(a,b) : B(K) \to S_p(K)$ be the operator defined by $M(a,b)x = axb$ for all x in $B(K)$. Then $\pi_p^o\big(M(a,b)\big) \leq \|a\|_{S_{2p}(K)} \|b\|_{S_{2p}(K)}$.*

Following [Jun96], we say that a linear map $u : E \to F$ is *completely p-nuclear* (denoted $u \in \mathcal{N}_p^o(E,F)$) if there exists a factorization of u as

$$E \xrightarrow{\alpha} S_\infty \xrightarrow{M(a,b)} S_p \xrightarrow{\beta} F$$

with $a, b \in S_{2p}$ and α, β completely bounded maps. The completely p-nuclear norm of u is defined as

$$\nu_p^o(u) = \inf \left\{ \|\alpha\|_{\mathrm{cb}} \|a\|_{S_{2p}} \|b\|_{S_{2p}} \|\beta\|_{\mathrm{cb}} \right\}$$

where the infimum is taken over all factorizations of u as above.

6.3 Definition and elementary properties

Let $1 \leq p, q \leq \infty$. A map $u : E \to F$ is said to be *completely* (q,p)*-mixing with constant* K if for any operator space G and any completely q-summing map $v : F \to G$, the composition $v \circ u$ is a completely p-summing map and $\pi_p^o(v \circ u) \leq K\pi_q^o(v)$. The *completely* (q,p)*-mixing norm* of u is the smallest such K and will be denoted by $\mathfrak{m}_{q,p}^o(u)$. Note that it is indeed a norm.

This definition (albeit worded in a different way) appears in [Yew08], where several upper and lower bounds for the completely $(2,p)$-mixing norms of the identity on OH_n are computed (for $1 < p < 2$). For an infinite-dimensional example of a completely mixing map, Junge and Parcet prove in [JP10, Corollary A2] that the identity map on the operator Hilbert space OH is completely $(q,2)$-mixing for any $1 < q < 2$ (in sharp contrast with the commutative case, Yew [Yew08] proved that this same map is not completely $(2,1)$-mixing). In fact Junge and Parcet proved a more general result, and in order to state it we will need some definitions. A map $u : E \to F$ is called *completely* $(q,1)$*-summing* if

$$\pi_{q,1}^{\mathrm{cb}}(u) := \|id \otimes u : \ell_1 \otimes_{\min} E \to \ell_q(F)\|_{\mathrm{cb}} < \infty,$$

and it is said to have cb-*cotype* q if

$$c_q^{\mathrm{cb}}(u) = \|\iota \otimes \mathrm{Rad}_q(E) \to \ell_q(F)\|_{\mathrm{cb}} < \infty,$$

with

$$\mathrm{Rad}_q(E) = \Big\{ \sum_j \varepsilon_j x_j \ : \ x_j \in E \Big\} \subset L_q(E)$$

where the ε_j's are independent ± 1 Bernoulli random variables, and $\iota(\varepsilon_j) = \delta_j$ where the δ_j's form the canonical basis of ℓ_q. If a map u has cb-cotype q then it is completely $(q,1)$-summing, and moreover $\pi^{\mathrm{cb}}_{q,1}(u) \leq c^{\mathrm{cb}}_q(u)$ [JP10, Lemma 3.1]. The following result is a straightforward generalization of [JP10, Cor. 3.7].

Theorem 6.3.1. *Let $p \geq 2$. If $u : E \to F$ is completely $(p,1)$-summing (in particular, if u has* cb*-cotype p), then it is $(q',2)$-mixing for any $q > p$. Moreover $\mathfrak{m}^o_{q',2}(u) \leq c(p,q)\pi^{\mathrm{cb}}_{p,1}(u)$, where $c(p,q)$ is a constant depending on p and q only.*

Just from the definition, we obtain a trivial composition formula for completely (q,p)-mixing maps: regardless of the values of p, q and r in $[1,\infty]$, the composition of a completely (p,r)-mixing operator u followed by a completely (q,p)-mixing operator v is completely (q,r)-mixing and moreover $\mathfrak{m}^o_{q,r}(vu) \leq \mathfrak{m}^o_{q,p}(v) \cdot \mathfrak{m}^o_{p,r}(u)$. Many of the properties of completely p-summing maps immediately give rise to corresponding properties of completely (q,p)-mixing maps. For starters, the domination characterization (in its factorization version, as in [Pis98, Rem. 5.7]) for completely p-summing maps implies that for any map u, $\mathfrak{m}^o_{q,p}(u) = \|u\|_{\mathrm{cb}}$ whenever $q \leq p$ and $\mathfrak{m}^o_{\infty,p}(u) = \pi^o_p(u)$, so only the case $1 \leq p < q < \infty$ gives something new. Moreover, completely (q,p)-mixing maps also satisfy the ideal property and $\mathfrak{m}^o_{q,p}(v \circ u \circ w) \leq \|v\|_{\mathrm{cb}} \cdot \mathfrak{m}^o_{q,p}(u) \cdot \|w\|_{\mathrm{cb}}$ whenever the composition makes sense. Additionally, the monotonicity of the completely p-summing norms implies a monotonicity condition for the completely (q,p)-mixing norms: whenever $p_1 \leq p_2$ and $q_2 \leq q_1$, $\mathfrak{m}^o_{q_2,p_2}(u) \leq \mathfrak{m}^o_{q_1,p_1}(u)$ for any u. Finally, being completely (q,p)-mixing is a *local* concept. As in the proof of [Yew08, Prop. 5.(2)], for any map $u : E \to F$,

$$\mathfrak{m}^o_{q,p}(u) = \sup\{\mathfrak{m}^o_{q,p}(u|_{E_0}) \ : \ E_0 \subseteq E, \dim(E_0) < \infty\}.$$

6.4 Characterizations

6.4.1 Domination

The following theorem is the completely (q,p) mixing counterpart of the domination theorem for completely p-summing maps of Pisier.

Theorem 6.4.1. *Let $E \subseteq B(H)$ and $F \subseteq B(K)$ be concrete operator spaces. Let $1 \leq p \leq q < \infty$, $u : E \to F$ a linear map and $C \geq 0$. The following are equivalent:*

(a) u is completely (q,p)-mixing with $\mathfrak{m}^o_{q,p}(T) \leq C$.

(b) For any ultrafilter $\mathcal{U}$ over an index set I and families $(a_\alpha)_{\alpha\in I}$, $(b_\alpha)_{\alpha\in I}$ in the unit ball of $S_{2q}(K)$ there exist an index set J, an ultrafilter $\mathcal{V}$ over J and families $(c_\beta)_{\beta\in J}$, $(d_\beta)_{\beta\in J}$ in the unit ball of $S_{2p}(H)$ such that for all n and all (x_{ij}) in $M_n(E)$ we have

$$\lim_{\mathcal{U}} \left\| \left[(a_\alpha(ux_{ij})b_\alpha) \right] \right\|_{M_n(S_q(K))} \leq C \lim_{\mathcal{V}} \left\| \left[(c_\beta x_{ij} d_\beta) \right] \right\|_{M_n(S_p(H))}$$

(c) For any ultrafilter $\mathcal{U}$ over an index set I and families $(a_\alpha)_{\alpha\in I}$, $(b_\alpha)_{\alpha\in I}$ in the unit ball of $S_{2q}(K)$ there exist an index set J, an ultrafilter $\mathcal{V}$ over J and families $(c_\beta)_{\beta\in J}$, $(d_\beta)_{\beta\in J}$ in the unit ball of $S_{2p}(H)$ such that for all n and all (x_{ij}) in $M_n(E)$ we have

$$\lim_{\mathcal{U}} \left\| \left[(a_\alpha(ux_{ij})b_\alpha) \right] \right\|_{S_p^n[S_q(K)]} \leq C \lim_{\mathcal{V}} \left\| \left[(c_\beta x_{ij} d_\beta) \right] \right\|_{S_p(\ell_2^n \otimes H)}.$$

Proof. We only show that (a) and (b) are equivalent, the equivalence with (c) follows similarly (as in Pisier's [Pis98] proof of Theorem 6.2.1).

$(a) \Rightarrow (b)$ Suppose that u is completely (q,p)-mixing, and let I be an index set, $\mathcal{U}$ an ultrafilter over I and $(a_\alpha)_{\alpha\in I}$, $(b_\alpha)_{\alpha\in I}$ families in the unit ball of $S_{2q}(K)$. The ultraproduct m of the multiplication maps $M(a_\alpha, b_\alpha) : B(K) \to S_q(K)$ is completely

q-summing with completely q-summing norm at most one and therefore, if j is the completely isometric injection of $B(K)$ into the ultrapower $B(K)^{\mathcal{U}}$, $m \circ j \circ u$ is completely p-summing with $\pi_p^o(m \circ j \circ u) \leq C$. By the domination theorem for completely p-summing maps (Theorem 6.2.1), there exists an ultrafilter $\mathcal{V}$ over an index set J and families $(c_\beta)_{\beta \in J}$, $(d_\beta)_{\beta \in J}$ in the unit ball of $S_{2p}(H)$ such that for any $n \in \mathbb{N}$ and any (x_{ij}) in $M_n(E)$,

$$\left\| \left[((mju)x_{ij}) \right] \right\|_{M_n[S_q(K)^{\mathcal{U}}]} \leq C \lim_{\mathcal{V}} \left\| \left[(c_\beta x_{ij} d_\beta) \right] \right\|_{M_n(S_p(H))},$$

that is,

$$\lim_{\mathcal{U}} \left\| \left[(a_\alpha (u x_{ij}) b_\alpha) \right] \right\|_{M_n(S_q(K))} \leq C \lim_{\mathcal{V}} \left\| \left[(c_\beta x_{ij} d_\beta) \right] \right\|_{M_n(S_p(H))}.$$

$(b) \Rightarrow (a)$ Let $v : F \to G$ be a completely q-summing map. By the domination theorem for completely q-summing maps, there exists an ultrafilter $\mathcal{U}$ over an index set I and families $(a_\alpha)_{\alpha \in I}$, $(b_\alpha)_{\alpha \in I}$ in the unit ball of $S_{2q}(K)$ such that for any $n \in \mathbb{N}$ and any (y_{ij}) in $M_n(F)$,

$$\left\| \left[(v y_{ij}) \right] \right\|_{M_n[G]} \leq \pi_q^o(v) \lim_{\mathcal{U}} \left\| \left[(a_\alpha y_{ij} b_\alpha) \right] \right\|_{M_n(S_q(K))}.$$

By hypothesis, there exist an index set J, an ultrafilter $\mathcal{V}$ over J and families $(c_\beta)_{\beta \in J}$, $(d_\beta)_{\beta \in J}$ in the unit ball of $S_{2p}(H)$ such that for all n and all (x_{ij}) in $M_n(E)$ we have

$$\lim_{\mathcal{U}} \left\| \left[(a_\alpha (u x_{ij}) b_\alpha) \right] \right\|_{M_n(S_q(K))} \leq C \lim_{\mathcal{V}} \left\| \left[(c_\beta x_{ij} d_\beta) \right] \right\|_{M_n(S_p(H))}.$$

The two previous inequalities put together give us

$$\left\| \left[(v u x_{ij}) \right] \right\|_{M_n[G]} \leq C \pi_q^o(v) \lim_{\mathcal{V}} \left\| \left[(c_\beta x_{ij} d_\beta) \right] \right\|_{M_n(S_p(H))},$$

which means, by the domination theorem for completely p-summing maps, that $v \circ u$ is completely p-summing and $\pi_p^o(v \circ u) \leq C\pi_q^o(v)$, meaning that u is completely (q,p)-mixing with $\mathfrak{m}_{q,p}^o(u) \leq C$.

□

6.4.2 Mixed norms

We will now prove another characterization of completely (q,p)-summing maps, based on mixed S_p-norm inequalities (Theorem 6.4.3). First we need the following lemma, which is a generalization of [Pis98, Theorem 1.5].

Lemma 6.4.2. *Suppose* $1/p = 1/q + 1/r$. *Let* $X \in S_p[E]$ *(resp.* $X \in S_p^n[E]$*) and let* $(x_{ij}) \in M_\infty(E)$ *(resp.* $(x_{ij}) \in M_n(E)$*) be the corresponding matrix with* $x_{ij} \in E$. *Then* $\|X\|_{S_p[E]}$ *(resp.* $\|X\|_{S_p^n[E]}$*) is equal to*

$$\inf \left\{ \|A\|_{S_{2r}} \|V\|_{S_q[E]} \|B\|_{S_{2r}} \right\}$$

where the infimum runs over all representations of the form

$$(x_{ij}) = A \cdot V \cdot B$$

with $A, B \in S_{2r}$ *and* $V \in S_q[E]$ *(resp.* $A, B \in S_{2r}^n$ *and* $V \in M_n(E)$*).*

Proof. If $(x_{ij}) = A \cdot V \cdot B$, then by [Pis98, Lemma 1.6.(ii)], we have that

$$\|(x_{ij})\|_{S_p[E]} \leq \|A\|_{S_{2r}} \|V\|_{S_q[E]} \|B\|_{S_{2r}},$$

and hence

$$\|(x_{ij})\|_{S_p[E]} \leq \inf \left\{ \|A\|_{S_{2r}} \|V\|_{S_q[E]} \|B\|_{S_{2r}} \right\}.$$

For the opposite inequality, recall from [Pis98, Theorem 1.5] that

$$\|(x_{ij})\|_{S_p[E]} = \inf \left\{ \|\mathbf{A}\|_{S_{2p}} \|Y\|_{M_\infty[E]} \|\mathbf{B}\|_{S_{2p}} : (x_{ij}) = \mathbf{A} \cdot Y \cdot \mathbf{B} \right\}.$$

Therefore, given $\varepsilon > 0$ there exists a factorization $(x_{ij}) = \mathbf{A} \cdot Y \cdot \mathbf{B}$ such that

$$\|(x_{ij})\|_{S_p[E]} + \varepsilon \geq \|\mathbf{A}\|_{S_{2p}} \|Y\|_{M_\infty[E]} \|\mathbf{B}\|_{S_{2p}}.$$

By [DJT95, Thm. 6.3], we can choose $A', B' \in S_{2q}$ and $A'', B'' \in S_{2r}$ such that $\mathbf{A} = A'' \cdot A'$ and $\|\mathbf{A}\|_{S_{2p}}$ is equal to $\|A'\|_{S_{2q}} \|A''\|_{S_{2r}}$, and $\mathbf{B} = B' \cdot B''$ and $\|\mathbf{B}\|_{S_{2p}}$ is equal to $\|B'\|_{S_{2q}} \|B''\|_{S_{2r}}$. Then using [Pis98, Theorem 1.5] again,

$$\begin{aligned}\|\mathbf{A}\|_{S_{2p}} \|Y\|_{M_\infty[E]} \|\mathbf{B}\|_{S_{2p}} &= \|A''\|_{S_{2r}} \|A'\|_{S_{2q}} \|Y\|_{M_\infty[E]} \|B'\|_{S_{2q}} \|B''\|_{S_{2r}} \\ &\geq \|A''\|_{S_{2r}} \|A' \cdot Y \cdot B'\|_{S_q[E]} \|B''\|_{S_{2r}} \\ &\geq \inf \left\{ \|A\|_{S_{2r}} \|V\|_{S_q[E]} \|B\|_{S_{2r}} \ : \ (x_{ij}) = A \cdot V \cdot B \right\},\end{aligned}$$

where the last inequality follows from the fact that $A'' \cdot A' \cdot Y \cdot B' \cdot B'' = \mathbf{A} \cdot Y \cdot \mathbf{B} = (x_{ij})$. Letting ε go to zero, we get the desired inequality. □

With this lemma we can prove the announced characterization of completely (q,p)-mixing maps, one that has the advantage of not having any ultrafilters involved. As far as we can tell, it does not directly correspond to a known characterization of (q,p)-mixing operators (in the Banach space case).

Theorem 6.4.3. *Let $E \subseteq B(H)$ and $F \subseteq B(K)$ be concrete operator spaces. Let $1 \leq p \leq q < \infty$, $u : E \to F$ a linear map and $C \geq 0$. The following are equivalent:*

(a) u is completely (q,p)-mixing with $\mathfrak{m}^o_{q,p}(T) \leq C$.

(b) For all n and all (x_{ij}) in $M_n(E)$ we have

$$\sup \left\{ \left\| \left(a(ux_{ij})b \right) \right\|_{S_p[S_q(K)]} \ : \ a, b \in B_{S_{2q}(K)}, \ a, b \geq 0 \right\} \leq C \|(x_{ij})\|_{S_p \otimes_{\min} E}.$$

Proof. $(a) \Rightarrow (b)$: Suppose that u is completely (q,p)-mixing with $\mathfrak{m}^o_{q,p}(T) \le C$. Let a, b be positive elements in the unit ball of $S_{2q}(H)$. By [Pis98, Proposition 5.6], the multiplication map $M(a,b) : B(K) \to S_q(K)$ is completely q-summing with constant at most one, and thus so is its restriction to F. Therefore, the composition $M(a,b) \circ u : E \to S_q(K)$ is completely p-summing with $\pi^o_p(M(a,b) \circ u) \le C$, that is, the norm of the map

$$I_{S_p} \otimes (M(a,b) \circ u) : S_p \otimes_{\min} E \to S_p[S_q(K)]$$

is at most C. This means that for any (x_{ij}) in $M_n(E)$ we have

$$\left\| \big(a(ux_{ij})b\big) \right\|_{S_p[S_q(K)]} \le C \left\| (x_{ij}) \right\|_{S_p \otimes_{\min} E}.$$

Taking the supremum over all a and b we obtain the desired conclusion.

$(b) \Rightarrow (a)$: Suppose that for all n and all (x_{ij}) in $M_n(E)$ we have

$$\sup\left\{ \left\| \big(a(ux_{ij})b\big) \right\|_{S_p[S_q(K)]} \ : \ a, b \in B_{S_{2p}(K)},\ a, b \ge 0 \right\} \le C \left\| (x_{ij}) \right\|_{S_p \otimes_{\min} E}. \quad (6.4.1)$$

Let $v : F \to G$ be a completely q-summing map. By the domination theorem for completely q-summing maps (Theorem 6.2.1) and [Pis98, Theorem 1.9], there exist an ultrafilter $\mathcal{U}$ over an index set I and families $(a_\alpha)_{\alpha \in I}$, $(b_\alpha)_{\alpha \in I}$ in the unit ball of $S_{2q}(K)$ such that for all $n \in \mathbb{N}$ and all (y_{ij}) in $M_n(F)$ we have

$$\left\| (vy_{ij}) \right\|_{S^n_q[G]} \le \pi^o_q(v) \lim_{\mathcal{U}} \left\| (a_\alpha y_{ij} b_\alpha) \right\|_{S^n_q[S_q(K)]}. \quad (6.4.2)$$

In particular, for every (x_{ij}) in $M_n(E)$ we have

$$\left\| (vux_{ij}) \right\|_{S^n_q[G]} \le \pi^o_q(v) \lim_{\mathcal{U}} \left\| \big(a_\alpha (ux_{ij}) b_\alpha\big) \right\|_{S^n_q[S_q(K)]}. \quad (6.4.3)$$

Let r be such that $1/p = 1/q + 1/r$, and let $\varepsilon > 0$. For each $\alpha \in I$, Lemma 6.4.2 implies the existence of A_α and B_α positive matrices in the unit sphere of S_{2r}^n such that

$$\left\|A_\alpha \cdot \big(a_\alpha(ux_{ij})b_\alpha\big) \cdot B_\alpha\right\|_{S_q^n[S_q(K)]} \leq (1+\varepsilon)\left\|\big(a_\alpha(ux_{ij})b_\alpha\big)\right\|_{S_p^n[S_q(K)]}.$$

By compactness, the limits $A = \lim_{\mathcal{U}} A_\alpha$ and $B = \lim_{\mathcal{U}} B_\alpha$ exist in the positive part of the unit sphere of S_{2r}^n. It follows then from the previous inequality that

$$\lim_{\mathcal{U}} \left\|A \cdot \big(a_\alpha(ux_{ij})b_\alpha\big) \cdot B\right\|_{S_q^n[S_q(K)]} \leq (1+\varepsilon) \lim_{\mathcal{U}} \left\|\big(a_\alpha(ux_{ij})b_\alpha\big)\right\|_{S_p^n[S_q(K)]}. \tag{6.4.4}$$

Now, using Lemma 6.4.2 again together with (6.4.3), (6.4.4), (6.4.2) and (6.4.1) we have

$$\begin{aligned}
\|(vux_{ij})\|_{S_p^n[G]} &\leq \|A \cdot (vux_{ij}) \cdot B\|_{S_q^n[G]} \\
&\leq \pi_q^o(v) \lim_{\mathcal{U}} \left\|A \cdot \big(a_\alpha(ux_{ij})b_\alpha\big) \cdot B\right\|_{S_q^n[S_q(K)]} \\
&\leq \pi_q^o(v)(1+\varepsilon) \lim_{\mathcal{U}} \left\|\big(a_\alpha(ux_{ij})b_\alpha\big)\right\|_{S_p^n[S_q(K)]} \\
&\leq \pi_q^o(v)(1+\varepsilon) C \left\|(x_{ij})\right\|_{S_p \otimes_{\min} E}.
\end{aligned}$$

Letting ε go to zero, this shows that vu is completely p-summing with $\pi_p^o(vu) \leq C\pi_q^o(v)$. Therefore, $\mathfrak{m}_{q,p}^o(u) \leq C$. □

6.5 The "interpolation" result

The main result of this section is the following operator space version of [Pie80, Prop. 20.1.13], which will imply a strengthtening of Yew's quantized extrapolation theorem.

Theorem 6.5.1. *Let* $0 < \theta < 1$ *and* $1 \leq p \leq q_0, q_1 < \infty$*. Define* $1/q := (1-\theta)/q_0 + \theta/q_1$*. For a map* $u : E \to F \subseteq B(K)$*,*

$$\mathfrak{m}^o_{q,p}(u) \leq \mathfrak{m}^o_{q_0,p}(u)^{1-\theta} \mathfrak{m}^o_{q_1,p}(u)^{\theta}.$$

Proof. Let $C_0 = \mathfrak{m}^o_{q_0,p}(u)$ and $C_1 = \mathfrak{m}^o_{q_1,p}(u)$. By Theorem 6.4.3,

$$\sup\left\{ \left\| \left(a(ux_{ij})b\right) \right\|_{S_p[S_{q_0}(K)]} \ : \ a, b \in B_{S_{2q_0}(K)}, \ a, b \geq 0 \right\} \quad \leq \quad C_0 \left\| (x_{ij}) \right\|_{S_p \otimes_{\min} E}.$$

and

$$\sup\left\{ \left\| \left(a(ux_{ij})b\right) \right\|_{S_p[S_{q_1}(K)]} \ : \ a, b \in B_{S_{2q_1}(K)}, \ a, b \geq 0 \right\} \quad \leq \quad C_1 \left\| (x_{ij}) \right\|_{S_p \otimes_{\min} E}.$$

Now, by [Yew08, Lemma 7] (or alternatively, as Yew himself says, by a typical application of the Generalized Hadamard three line theorem and the fact that the spaces $S_p(K)$ form an interpolation chain; see [Yew05, Lemma 3.5] for the detailed proof), for any positive a, b in $S_1(K)$ we have

$$\left\| \left(a^{1/2q}(ux_{ij})b^{1/2q}\right) \right\|_{S_p[S_q(K)]} \leq \left\| \left(a^{1/2q_0}(ux_{ij})b^{1/2q_0}\right) \right\|^{1-\theta}_{S_p[S_{q_0}(K)]} \left\| \left(a^{1/2q_1}(ux_{ij})b^{1/2q_1}\right) \right\|^{\theta}_{S_p[S_{q_1}(K)]}.$$

Therefore,

$$\begin{aligned} &\sup\left\{ \left\| \left(a(ux_{ij})b\right) \right\|_{S_p[S_q(K)]} \ : \ a, b \in B_{S_{2q}(K)}, \ a, b \geq 0 \right\} \\ &\quad \leq \sup\left\{ \left\| \left(a(ux_{ij})b\right) \right\|_{S_p[S_{q_0}(K)]} \ : \ a, b \in B_{S_{2q_0}(K)}, \ a, b \geq 0 \right\}^{1-\theta} \\ &\qquad \cdot \sup\left\{ \left\| \left(a(ux_{ij})b\right) \right\|_{S_p[S_{q_1}(K)]} \ : \ a, b \in B_{S_{2q_1}(K)}, \ a, b \geq 0 \right\}^{\theta} \end{aligned}$$

and thus

$$\sup\left\{\left\|\left(a(ux_{ij})b\right)\right\|_{S_p[S_q(K)]} \,:\, a,b\in B_{S_{2q}(K)},\ a,b\geq 0\right\} \leq C_0^{1-\theta}C_1^{\theta}\left\|(x_{ij})\right\|_{S_p\otimes_{\min}E}.$$

Another appeal to Theorem 6.4.3 gives the desired conclusion. □

Let E be an operator space and $1\leq p_0<p_1<q$. In [Yew08, Thm. 8] it is shown that

$$\mathfrak{m}^o_{q,p_0}(id_E)\leq\left[2^{1/p_0}\mathfrak{m}^o_{q,p_1}(id_E)\right]^{1/\theta}.$$

Our next corollary improves on this result by removing the power of 2, while also emphasizing the fact that for identity maps being completely (q,p)-mixing ($q>p$) is independent of p.

Corollary 6.5.2. *Let E be an operator space and $1\leq p_0<p_1<q$. Then id_E is (q,p_0)-mixing if and only if it is (q,p_1)-mixing. Moreover,*

$$\mathfrak{m}^o_{q,p_1}(id_E)\leq\mathfrak{m}^o_{q,p_0}(id_E)\leq\mathfrak{m}^o_{q,p_1}(id_E)^{1/\theta},$$

where θ is defined by $1/p_1=(1-\theta)/q+\theta/p_0$.

Proof. The monotonicity property for (q,p)-mixing constants from Section 6.3 gives $\mathfrak{m}^o_{q,p_1}(id_E)\leq\mathfrak{m}^o_{q,p_0}(id_E)$, whereas the composition property from the same section provides us with the inequality $\mathfrak{m}^o_{q,p_0}(id_E)\leq\mathfrak{m}^o_{q,p_1}(id_E)\cdot\mathfrak{m}^o_{p_1,p_0}(id_E)$. Now, from Theorem 6.5.1

$$\mathfrak{m}^o_{p_1,p_0}(id_E)\leq\mathfrak{m}^o_{q,p_0}(id_E)^{1-\theta}\cdot\mathfrak{m}^o_{p_0,p_0}(id_E)^{\theta}=\mathfrak{m}^o_{q,p_0}(id_E)^{1-\theta}\cdot 1.$$

So we obtain

$$\mathfrak{m}^o_{q,p_0}(id_E)\leq\mathfrak{m}^o_{q,p_1}(id_E)\cdot\mathfrak{m}^o_{q,p_0}(id_E)^{1-\theta}$$

from which the result follows. □

6.6 Composition theorems

We now proceed to prove various composition theorems for completely p-summing and completely p-nuclear operators. Our starting point is the following duality due to M. Junge:

Theorem 6.6.1. *[Jun96, Cor. 3.1.3.9] When E and F are operator spaces and $1 \leq p < \infty$, trace duality yields an isometric isomorphism between $\mathcal{N}_p^o(E,F)^*$ and $\Pi_{p'}^o(F, E^{**})$. In the finite-dimensional case, the duality is also true for $p = \infty$.*

From here we can deduce our first composition result, stating that in the finite dimensional setting, the composition of a completely p-summing map and a completely p'-nuclear one is completely 1-nuclear.

Theorem 6.6.2. *Let $u : E \to F$ and $v : F \to G$ be linear maps between finite-dimensional operator spaces. Then $\nu_1^o(vu) \leq \nu_{p'}^o(v)\pi_p^o(u)$ and $\nu_1^o(vu) \leq \pi_p^o(v)\nu_{p'}^o(u)$.*

Proof. We only prove the first inequality, the second one may be obtained using an analogous argument. Consider a linear map $w : G \to E$. Then by Theorem 6.6.1,

$$|\operatorname{tr}(wvu)| \leq \pi_p^o(u)\nu_{p'}^o(wv) \leq \pi_p^o(u)\nu_{p'}^o(v)\,\|w\|_{\mathrm{cb}}\,.$$

Taking the supremum over all w of cb-norm at most 1, another appeal to Theorem 6.6.1 (recalling that completely ∞-summing is the same as completely bounded) shows that $\nu_1^o(vu) \leq \pi_p^o(u)\nu_{p'}^o(v)$. □

A proof very similar to that of Theorem 6.6.2, together with the fact that π_2^o is in trace duality with itself [Lee08, Lemma 2.5], allow us to prove the following.

Theorem 6.6.3. *Let $u : E \to F$ and $v : F \to G$ be completely 2-summing maps. When the operator spaces are finite-dimensional, $\nu_1^o(vu) \leq \pi_2^o(v)\pi_2^o(u)$. In the infinite-dimensional case, localization gives $\pi_1^o(vu) \leq \pi_2^o(v)\pi_2^o(u)$.*

The following lemma is at the heart of the proof of the composition theorem for completely p-summing operators with conjugate indices.

Lemma 6.6.4. *Let $u : E \to F \subseteq B(\ell_2)$ be a completely p-summing map and a, b in $S_{2p'}$. Let $M := M(a,b) : B(\ell_2) \to S_{2p'}$ be the multiplication map induced by a and b. Then $\pi_1^o(M \circ u) \leq \|a\|_{2p'} \|b\|_{2p'} \pi_p^o(u)$.*

Proof. Let $\varepsilon > 0$. There exist orthonormal sequences $(e_j), (f_j)$ in ℓ_2 and a sequence of nonnegative numbers (τ_j) such that $a = \sum_j \tau_j e_j \otimes f_j$, and $(\sum_j \tau_j^{2p'})^{1/(2p')} = \|a\|_{2p'}$. Let (λ_j) be a sequence of real numbers greater than one and increasing to infinity such that $(\sum_j \lambda_j^{2p'} \tau_j^{2p'})^{1/(2p')} \leq (1+\varepsilon) \|a\|_{2p'}$. Define $a' = \sum_j \lambda_j \tau_j e_j \otimes f_j$ and let k_1 be the composition of the orthogonal projection onto the span of (e_j) followed by the operator that sends e_j to $\lambda_j^{-1} e_j$. Then we have a decomposition $a = a'k_1$ where k_1 is compact with $\|k_1\| \leq 1$ and $\|a'\|_{2p'} \leq (1+\varepsilon) \|a\|_{2p'}$. Similarly, we can find a decomposition $b = k_2 b'$ where k_2 is compact with norm at most 1 and $\|b'\|_{2p'} \leq (1+\varepsilon) \|b\|_{2p'}$. Therefore, we may factor $M \circ u = M' \circ M'' \circ u$, where $M'' := M(k_1, k_2) : B(\ell_2) \to S_\infty$ and $M' := M(a', b') : S_\infty \to S_{2p'}$. Note that M' is completely p'-nuclear, and $\nu_{p'}^o(M') \leq \|a'\|_{2p'} \|b'\|_{2p'}$. From [Oik10], $\|M''\|_{\text{cb}} \leq \|k_1\| \|k_2\|$. By localization we may assume that E is finite-dimensional, and thus by the proof of Theorem 6.6.2, $\nu_1^o(M' \circ M'' \circ u) \leq \nu_{p'}^o(M') \pi_p^o(M'' \circ u)$. Since $\pi_1^o(M \circ u) \leq \nu_1^o(M' \circ M'' \circ u)$, we have

$$\begin{aligned} \pi_1^o(M \circ u) &\leq \|a'\|_{2p'} \|b'\|_{2p'} \|M''\|_{\text{cb}} \pi_p^o(u) \\ &\leq \|a'\|_{2p'} \|b'\|_{2p'} \|k_1\| \|k_2\| \pi_p^o(u) \leq (1+\varepsilon)^2 \|a\|_{2p'} \|b\|_{2p'} \pi_p^o(u). \end{aligned}$$

Letting ε go to 0, we get the desired result. □

Now we can prove the composition theorem for completely p-summing operators in the case of conjugate indices.

Theorem 6.6.5. *Let $u : E \to F$ be completely p-summing and $v : F \to G$ be completely p'-summing. Then vu is completely 1-summing, and moreover $\pi_1^o(vu) \leq \pi_{p'}^o(v)\pi_p^o(u)$.*

Proof. By localization, we can assume that the operator spaces are finite-dimensional and thus $F \subset B(\ell_2)$. Hence, the result follows immediately from Theorem 6.4.3 and Lemma 6.6.4. □

We will obtain the full composition theorem from the particular case of conjugate indices using interpolation. Before proceeding to the argument, let us recall [Pis03, Corollary 2.7.7], which states that

$$(X \otimes_{\min} E_0, X \otimes_{\min} E_1)_\theta = X \otimes_{\min} (E_0, E_1)_\theta$$

whenever X is a completely complemented subspace of S_∞.

Lemma 6.6.6. *Let $1 \leq p, q, r \leq \infty$ with $1/r = 1/p + 1/q$. For a completely p-summing map $u : S_\infty \to F$ and any completely q-summing map $v : F \to G$ we have $\pi_r^o(vu) \leq \pi_q^o(v)\pi_p^o(u)$.*

Proof. If $r = 1$ the result follows from Theorem 6.6.5, so we may assume $r > 1$. Note that then $p' < q$, so $\theta = p'/q$ is in $(0, 1)$. Consider a completely isometric embedding $J : F \to B(K)$. Define a multilinear map $\Phi : (S_t^n \otimes_{\min} S_\infty) \times S_{2s}(K) \times S_{2s}(K) \to S_t^n[S_s(K)]$ by

$$\Phi\big((x_{ij}), a, b\big) = \big(a(Jux_{ij})b\big).$$

By Theorem 6.6.5 we have

$$\big\|\Phi\big((x_{ij}), a, b\big)\big\|_{S_1^n[S_{p'}(K)]} \leq \|(x_{ij})\|_{S_1^n \otimes_{\min} S_\infty} \|a\|_{S_{2p'}(K)} \|b\|_{S_{2p'}(K)}$$

for any $(x_{ij}) \in S_1^n \otimes S_\infty$ and $a, b \in S_{2p'}(K)$, that is, Φ has norm at most 1 when $t = 1$ and $s = p'$. Similarly, by the ideal property for completely p-summing operators Φ

has norm at most 1 when $t = p$, $s = \infty$. Observe that $1/q = (1-\theta)/\infty + \theta/p'$ and $1/r = (1-\theta)/p + \theta/1$. Therefore, multilinear complex interpolation gives that Φ has norm ≤ 1 when $t = r$ and $s = q$. From Theorem 6.4.3, we obtain that u has completely (q,r)-mixing norm at most $\pi_p^o(u)$, the desired result. □

Let us now apply the previous lemma to estimate the completely (q,r)-mixing norm of completely p-nuclear operators.

Lemma 6.6.7. *Let $1 \leq p,q,r \leq \infty$ with $1/r = 1/p + 1/q$, and $u : E \to F$ be a completely p-nuclear map. Then $\mathfrak{m}_{q,r}^o(u) \leq \nu_p^o(u)$.*

Proof. Consider a completely p-nuclear factorization of $u : E \to F$ as

$$E \xrightarrow{\alpha} S_\infty \xrightarrow{M(a,b)} S_p \xrightarrow{\beta} F$$

with $a, b \in S_{2p}$, and let $v : F \to G$ be a completely q-summing map. By Lemma 6.6.6,

$$\pi_r^o\big(v\beta M(a,b)\big) \leq \pi_q^o(v)\pi_p^o\big(\beta M(a,b)\big) \leq \pi_q^o(v)\, \|\beta\|_{\mathrm{cb}}\, \|a\|_{S_{2p}}\, \|b\|_{S_{2p}}\,.$$

Thus, by the ideal property for completely p-summing operators,

$$\pi_r^o(vu) \leq \|\alpha\|_{\mathrm{cb}}\, \pi_r^o\big(v\beta M(a,b)\big) \leq \pi_q^o(v)\, \|\alpha\|_{\mathrm{cb}}\, \|\beta\|_{\mathrm{cb}}\, \|a\|_{S_{2p}}\, \|b\|_{S_{2p}}\,.$$

Taking the infimum over all such representations of u we obtain $\pi_r^o(vu) \leq \pi_q^o(v)\nu_p^o(u)$ giving the desired result. □

Together with the duality theorem, the previous lemmas will yield the full composition theorem.

Theorem 6.6.8. *Let $1 \leq p,q,r, \leq \infty$ with $1/r = 1/p + 1/q$. Let $u : E \to F$ be completely p-summing and $v : F \to G$ be completely q-summing. Then vu is completely r-summing, and moreover $\pi_r^o(vu) \leq \pi_q^o(v)\pi_p^o(u)$.*

Proof. By localization, we may assume that all the operator spaces involved are finite-dimensional, so in particular we can assume $F \subseteq B(\ell_2)$. By Theorem 6.4.3 we may assume that v is of the form $M(a,b) : B(\ell_2) \to S_q$ where a and b are in the unit ball of S_{2q}, and thus $\nu_q^o\big(M(a,b)\big) \leq 1$. Let $w : S_q \to E$ be completely r'-nuclear with $\nu_{r'}^o(w) \leq 1$. By Theorem 6.6.1,

$$|\operatorname{tr}(vuw)| \leq \nu_q^o(v)\pi_{q'}^o(uw).$$

Since $1/q' = 1/p + 1/r'$, Lemma 6.6.7 implies that

$$|\operatorname{tr}(vuw)| \leq \nu_q^o(v)\pi_p^o(u)\nu_{r'}^o(w) \leq \pi_p^o(u).$$

Taking the supremum over all w with $\nu_{r'}^o(w) \leq 1$, the duality theorem 6.6.1 gives $\pi_r^o(vu) \leq \pi_p^o(u)$, and the result follows. □

As an application, we now prove an operator space version of [DF93, 32.2.(3)], which in turn is part of a result of Saphar [Sap72].

Corollary 6.6.9. *For an operator space E and $1 \leq q \leq \infty$, id_E is completely $(q,1)$-mixing if and only if $CB(S_\infty, E) = \Pi_{q'}^o(S_\infty, E)$.*

Proof. First, suppose that id_E is completely $(q,1)$-mixing. By localization, it suffices to prove that there is a constant C such that for all n and all $w : M_n \to E$ we have $\pi_{q'}^o(w) \leq C\,\|w\|_{\mathrm{cb}}$.

We need to show that w is completely q'-summing, so we might as well assume that E is finite-dimensional. Let $v : E \to M_n$ be a completely q-nuclear map (hence completely q-summing). Since E is completely $(q,1)$-mixing, v is completely

1-summing and moreover $\pi_1^o(v) \le \nu_q^o(v)\mathfrak{m}_{q,1}^o$. Applying the duality theorem 6.6.1 for two different pairs of conjugate indices (q and q', 1 and ∞) we have

$$\begin{aligned}\pi_{q'}^o(w) \le \nu_{q'}^o(w) &= \sup\left\{|\operatorname{tr}(vw)| \;:\; \pi_q^o(v : E \to M_n) \le 1\right\}\\ &\le \mathfrak{m}_{q,1}^o(E)\sup\left\{|\operatorname{tr}(vw)| \;:\; \pi_1^o(v : E \to M_n) \le 1\right\}\\ &= \mathfrak{m}_{q,1}^o(E)\nu_\infty^o(w) \le \mathfrak{m}_{q,1}^o(E)\left\|w\right\|_{\mathrm{cb}},\end{aligned}$$

where in the last step we have used that $\nu_\infty^o(w) = \|w\|_{\mathrm{cb}}$, obvious since w has domain M_n.

Now suppose that $CB(S_\infty, E) = \Pi_{q'}^o(S_\infty, E)$. By the closed graph theorem, there exists a constant C such that for all $w : S_\infty \to E$ we have $\pi_{q'}^o(w) \le C\left\|w\right\|_{\mathrm{cb}}$. Let $v : E \to F$ be a completely q-summing map. Let $n \in \mathbb{N}$ and $w : M_n \to E$ be a completely bounded map. By the assumption, $\pi_{q'}^o(w) \le C\left\|w\right\|_{\mathrm{cb}}$. By the composition theorem 6.6.8, $\pi_1^o(vw) \le \pi_q^o(v)\pi_{q'}^o(w) \le \pi_q^o(v)C\left\|w\right\|_{\mathrm{cb}}$. Taking the supremum over all n and all such maps w with cb-norm at most one, we find that $\pi_1^o(v) \le C\pi_q^o(v)$. Therefore, id_E is completely $(q,1)$-mixing with constant at most C. □

We finish the section with a natural open question. In the Banach space setting, there are other composition formulas for p-summing, p-nuclear and p-integral maps (see [PP69]). Do their operator space analogues hold? Specifically, do we have $\nu_r^o(vu) \le \pi_q^o(v)\nu_p^o(u)$, $\nu_r^o(vu) \le \nu_q^o(v)\pi_p^o(u)$, $\iota_r^o(vu) \le \pi_q^o(v)\iota_p^o(u)$, $\iota_r^o(vu) \le \iota_q^o(v)\pi_p^o(u)$ whenever the compositions make sense?

CHAPTER VII

SUMMARY

In the final chapter we give a brief summary of the results proved in this dissertation, together with some natural open questions that are in need of further inquiry. For the notation and terminology please refer to the corresponding chapter.

7.1 Banach-space-valued molecules

The main point in this chapter was the introduction of the concept of Banach-space-valued molecules, which allowed us to obtain the following three duality theorems for several nonlinear ideals of Lipschitz maps.

Theorem (cf. Thm. 2.6.4)**.** The spaces $\mathcal{CS}_p(X,E)^*$ and $\Pi^L_{p'}(X,E^*)$ are isometrically isomorphic via the canonical pairing. Moreover, on the unit ball of $\Pi^L_{p'}(X,E^*)$ the weak* topology coincides with the topology of pointwise $\sigma(E^*,E)$-convergence.

Theorem (cf. Thm. 2.7.3)**.** The spaces $M_{p,r,s}(X,E)^*$ and $\Pi^L_{p',r,s}(X,E^*)$ are isometrically isomorphic via the canonical pairing. Moreover, on the unit ball of $\Pi^L_{p',r,s}(X,E^*)$ the weak* topology coincides with the topology of pointwise $\sigma(E^*,E)$-convergence.

Theorem (cf. Thm. 2.8.7)**.** The spaces $(\mathcal{M}(X,E),\|\cdot\|_*)^*$ and $\Gamma_2^{\mathrm{Lip}}(X,E^*)$ are isometrically isomorphic via the canonical pairing. Let $T : X \to E^*$ and $C > 0$. The following are equivalent:

Naturally, this chapter only lays down the basics of the study of spaces of Banach-space-valued molecules on a metric space. Further research is still needed to exploit the duality that they provide, and any other extra properties they may have. Among the natural open questions in this regard, we have:

(1) Of all the normed spaces of molecules introduced in this chapter, the only specific examples that have been identified so far are the $\mathcal{CS}_1$ spaces for certain kinds of

metric trees (see Proposition 2.3.9 and Corollary 2.3.10). The identification of other specific examples would be useful for practical calculations.

(2) As Banach spaces, what properties do these spaces of Banach-space-valued molecules have? For example, Arens-Eells spaces are special cases of Chevet-Saphar spaces and they have the Schur property under certain conditions [Kal04].

(3) What can we say when the metric space involved is in fact a Banach space? What properties (if any) does, for example, a Chevet-Saphar space inherit from it? It is known that a Banach space has the bounded approximation property if and only if its Arens-Eells space has the bounded approximation property [GK03], so similar results might hold for other norms on spaces of molecules.

(4) Does any of these classes of spaces of molecules behave well under some Banach-space operation (for example, interpolation)?

7.2 Ribe's program for maps

The focus of this chapter was to prove metrical characterizations of several properties of linear operators between Banach spaces, namely p-convexity, Rademacher cotype and Rademacher type.

The equivalence between p-convexity and Markov p-convexity is shown in the following two theorems.

Theorem (cf. Thm. 3.1.3)**.** Let $T : E \to F$ be a uniformly p-convex linear operator with constant C. Then T is Markov p-convex with constant $4C$, and thus every p-convex linear operator is Markov p-convex.

Theorem (cf. Thm. 3.1.4)**.** If $T : E \to F$ be a linear operator which is Markov p-convex with constant C, then T is p-convexifiable. More precisely, for every $\varepsilon \in (0,1)$ there exists a norm $|||\cdot|||$ on E such that for all $x, y \in E$,

$$(1-\varepsilon)\|x\| \le |||x||| \le \|x\|,$$

and

$$\left\|\frac{Tx+Ty}{2}\right\|^p \le \frac{|||x|||^p + |||y|||^p}{2} - \frac{1-(1-\varepsilon)^p}{4C^p(p+1)} \cdot \left|\left|\left|\frac{x-y}{2}\right|\right|\right|^p.$$

Thus, the operator $T : (E, |||\cdot|||) \to F$ satisfies (3.1.1) with constant $K = O(C/\varepsilon^{1/p})$.

In the case of Rademacher cotype, the next two results show the relationship between Rademacher type both the regular and weak versions of metric cotype.

Theorem (cf. Thm. 3.2.1)**.** Let E ,F be Banach spaces, $T : E \to F$ a linear map and $q \in [2, \infty)$. Then T has metric cotype q if and only if it has Rademacher cotype q. Moreover,

$$\frac{1}{2\pi} C_q(T) \le \Gamma_q(T) \le 20 C_q(T).$$

Theorem (cf. Thm. 3.2.2)**.** Let E ,F be Banach spaces and $T : E \to F$ a linear map. Suppose that T has weak metric cotype q with exponent p for some $1 \le p < q$. Then T has Beauzamy-Rademacher cotype, that is, for some $p \ge 1$ (equivalently, any $p \ge 1$) the sequence $a_n^{(p)}(T)$ defined by

$$a_n^{(p)}(T) = \inf\left\{ \left[\mathbb{E}_\varepsilon \left\| \sum_{j=1}^n \varepsilon_j v_j \right\|^p \right]^{1/p} \ : \ \|Tv_1\|, \dots, \|Tv_n\| \ge 1 \right\}$$

converges to 0. If $p \ge 2$, then T has weak Rademacher cotype q and hence has Rademacher cotype r for every $r > q$. On the other hand,

$$\Gamma_q^{(p)}(T) \le c_{pq} C_q(T)$$

where c_{pq} is a constant depending only on p and q.

Finally, for Rademacher type the situation is reminiscent of what we just saw for cotype. The following two results express the relations between Rademacher type and both scaled Enflo type and its weak version.

Theorem (cf. Thm. 3.3.1)**.** Let E, F be Banach spaces, $T : E \to F$ a linear map and $p \in [1, 2]$. Then T has scaled Enflo type p if and only if it has Rademacher type p. Moreover,

$$\frac{1}{2\pi} T_p(T) \leq \tau_p(T) \leq 15 T_p(T).$$

Theorem (cf. Thm. 3.3.2)**.** Let E, F be Banach spaces and $T : E \to F$ a linear map. Suppose that T has weak scaled Enflo type p with exponent q for some $1 \leq p < q$. Then T has Beauzamy-Rademacher type. If $q \leq 2$, then T has weak Rademacher type p and hence has Rademacher type r for every $1 < r < p$. On the other hand, Rademacher type p implies weak scaled Enflo type p with exponent q. To be precise,

$$\tau_p^{(q)}(T) \leq c_{pq} T_p(T)$$

where c_{pq} is a constant depending only on p and q.

One piece of the Ribe program that has resisted all attempts to solve it so far is giving a metrical characterization of q-smoothness in Banach spaces. In the linear case, both q-smoothness and p-convexity can be characterized in terms of the beautiful martingale inequalities due to Pisier [Pis75]. In the case of p-convexity the arguments of Pisier were the inspiration behind the argument by Mendel and Naor [MN08a] in the nonlinear setting, replacing the martingales by Markov chains. I believe it should be possible to give a metrical characterization of q-smoothness in terms of inequalities involving Markov chains in a somewhat similar fashion. Some further evidence in favor of this idea is given by the results of [NPSS06], where it is proved that a q-smooth Banach space satisfies an inequality involving Markov chains known as Markov type q, introduced by K. Ball [Bal92]. In fact, Ball himself conjectured that Markov type 2 implies 2-smoothness.

7.3 Lipschitz p-concave and p-convex operators

In this chapter we introduced the nonlinear notions of Lipschitz p-convex and p-concave operators. For the p-convex case, the relation between the classical and Lipschitz concepts is expressed in the following theorem.

Theorem (cf. Thm. 4.2.3)**.** Let X be a metric space and L a Banach lattice. A Lipschitz map $T : X \to L$ is Lipschitz p-convex if and only if $\hat{T} : \mathscr{F}(X) \to L$ is p-convex. Moreover, in this case the p-convexity constants are the same.

For Lipschitz p-concavity, the equivalence with the classical concept in the case of linear operators is a trivial matter. Thus, here the most important result is not one showing an equivalence between the concepts, but rather a nonlinear version of a factorization theorem through L_p.

Theorem (cf. Thm. 4.3.2)**.** Let X, Y be metric spaces with Y complete and L a Banach lattice. Suppose that $T : X \to L$ is Lipschitz p-convex and $S : L \to Y$ is Lipschitz p-concave. Then the operator ST can be factorized through an $L_p(\mu)$ space. Moreover, we may arrange to have $ST = S_1 T_1$ with $T_1 : X \to L_p(\mu)$, $S_1 : L_p(\mu) \to Y$, $\mathrm{Lip}(T_1) \le M^{(p)}_{\mathrm{Lip}}(T)$ and $\mathrm{Lip}(S_1) \le M^{\mathrm{Lip}}_{(p)}(S)$.

It would be interesting to investigate whether a result similar to Theorem 4.2.3 is true if one replaces $\left(\sum_j |x_j|^p\right)^{1/p}$ by other expressions in the Krivine functional calculus for lattices.

7.4 Lipschitz (q,p)-mixing operators

In this chapter the concept of Lipschitz (q,p)-mixing operators was introduced, and several characterizations of it were proved. They are summarized in the following theorem.

Theorem (cf. Thm. 5.4.1, Cor. 5.4.3 and Thm. 5.4.4)**.** Let $1 \le p \le q \le \infty$, $T : X \to Y$ Lipschitz and $C \ge 0$. The following are equivalent:

(a) T is Lipschitz (q,p)-mixing with $\mathfrak{m}^L_{q,p}(T) \le C$.

(b) For any probability measure μ on $B_{Y^\#}$ there exists a probability measure ν on $B_{X^\#}$ such that for all $x, x' \in X$,

$$\left[\int_{B_{Y^\#}} |g(Tx) - g(Tx')|^q \, d\mu(g)\right]^{1/q} \le C \left[\int_{B_{X^\#}} |f(x) - f(x')|^p \, d\nu(f)\right]^{1/p}.$$

(c) For any $x_1, \ldots, x_m, x'_1, \ldots, x'_m \in X$ and $g_1, \ldots, g_n \in Y^\#$,

$$\left[\sum_{j=1}^m \left[\sum_{k=1}^n \left|g_k(Tx_j) - g_k(Tx'_j)\right|^q\right]^{p/q}\right]^{1/p} \le C \left[\sum_{k=1}^n \mathrm{Lip}(g_k)^q\right]^{1/q} \cdot \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m \left|f(x_j) - f(x'_j)\right|^p\right]^{1/p}.$$

(d) For any $x_1, \ldots, x_m, x'_1, \ldots, x'_m \in X$ and any probability measure μ on $B_{Y^\#}$,

$$\left[\sum_{j=1}^m \left(\int_{B_{Y^\#}} \left|g(Tx_j) - g(Tx'_j)\right|^q d\mu(g)\right)^{p/q}\right]^{1/p} \le C \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^m \left|f(x_j) - f(x'_j)\right|^p\right]^{1/p}.$$

(e) For all $x_1, \ldots, x_n, x'_1, \ldots, x'_n \in X$,

$$\inf \left\{ \left[\sum_{j=1}^n \lambda_j^r\right]^{1/r} \sup_{g \in B_{Y^\#}} \left[\sum_{j=1}^n \lambda_j^{-q} \left|g(Tx_j) - g(Tx'_j)\right|^q\right]^{1/q} : \lambda_j > 0 \right\} \le C \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^n \left|f(x_j) - f(x'_j)\right|^p\right]^{1/p}.$$

(f) For every Banach space G (or only $G = \ell_{q'}$), the operator

$$T_G : \mathcal{CS}_{p'}(X, G) \to \mathcal{CS}_{q'}(Y, G)$$

is continuous.

In this case, $\mathfrak{m}^L_{q,p}(T)$ is equal to the infimum of such constants C in either (b), (c), (d) or (e).

As a consequence, we proved the following "interpolation" theorem relating different Lipschitz (q, p)-mixing constants for the same operator.

Theorem (cf. Thm. 5.5.5)**.** Let $0 < \theta < 1$ and $1 \leq p \leq q_0, q_1 \leq \infty$. Define $1/q := (1-\theta)/q_0 + \theta/q_1$. For a Lipschitz map $T : X \to Y$,

$$\mathfrak{m}^L_{q,p}(T) \leq \mathfrak{m}^L_{q_0,p}(T)^{1-\theta}\mathfrak{m}^L_{q_1,p}(T)^{\theta}.$$

The most important open problem in this context is the one that inspired the work of this chapter [FJ09, Question 3]: if $1 \leq p, q, r, \leq \infty$ satisfy $1/r = 1/p + 1/q$, is the composition of a Lipschitz p-summing operator and a Lipschitz q-summing one in fact Lipschitz r-summing?

7.5 Completely (q, p)-mixing maps

In this chapter, similar to the previous one but in the context of operator spaces rather than metric spaces, we studied the concept of completely (q, p)-mixing maps. The several characterizations we obtained are summarized in the following theorem.

Theorem (cf. Thm. 6.4.1 and Thm. 6.4.3)**.** Let $E \subseteq B(H)$ and $F \subseteq B(K)$ be concrete operator spaces. Let $1 \leq p \leq q < \infty$, $u : E \to F$ a linear map and $C \geq 0$. The following are equivalent:

(a) u is completely (q, p)-mixing with $\mathfrak{m}^o_{q,p}(T) \leq C$.

(b) For any ultrafilter $\mathcal{U}$ over an index set I and families $(a_\alpha)_{\alpha\in I}$, $(b_\alpha)_{\alpha\in I}$ in the unit ball of $S_{2q}(K)$ there exist an index set J, an ultrafilter $\mathcal{V}$ over J and families $(c_\beta)_{\beta\in J}$, $(d_\beta)_{\beta\in J}$ in the unit ball of $S_{2p}(H)$ such that for all n and all (x_{ij}) in $M_n(E)$ we have

$$\lim_{\mathcal{U}} \big\| \big[(a_\alpha(ux_{ij})b_\alpha)\big] \big\|_{M_n(S_q(K))} \leq C \lim_{\mathcal{V}} \big\| \big[(c_\beta x_{ij} d_\beta)\big] \big\|_{M_n(S_p(H))}$$

(c) For any ultrafilter $\mathcal{U}$ over an index set I and families $(a_\alpha)_{\alpha\in I}$, $(b_\alpha)_{\alpha\in I}$ in the unit ball of $S_{2q}(K)$ there exist an index set J, an ultrafilter $\mathcal{V}$ over J and families $(c_\beta)_{\beta\in J}$, $(d_\beta)_{\beta\in J}$ in the unit ball of $S_{2p}(H)$ such that for all n and all (x_{ij}) in $M_n(E)$ we have

$$\lim_{\mathcal{U}} \big\| \big[(a_\alpha(ux_{ij})b_\alpha)\big] \big\|_{S_p^n[S_q(K)]} \leq C \lim_{\mathcal{V}} \big\| \big[(c_\beta x_{ij} d_\beta)\big] \big\|_{S_p(\ell_2^n\otimes H)} .$$

(d) For all n and all (x_{ij}) in $M_n(E)$ we have

$$\sup \Big\{ \big\| \big(a(ux_{ij})b\big) \big\|_{S_p[S_q(K)]} \ : \ a,b \in B_{S_{2q}(K)},\ a,b \geq 0 \Big\} \leq C \left\| (x_{ij}) \right\|_{S_p\otimes_{\min} E} .$$

As a consequence, we were able to prove a version of Pietsch's composition theorem for completely p-summing maps.

Theorem (cf. Thm. 6.6.8)**.** Let $1 \leq p,q,r, \leq \infty$ with $1/r = 1/p + 1/q$. Let $u : E \to F$ be completely p-summing and $v : F \to G$ be completely q-summing. Then vu is completely r-summing, and moreover $\pi_r^o(vu) \leq \pi_q^o(v)\pi_p^o(u)$.

Some logical open questions stem from this theorem, since in the Banach space setting there are other composition formulas for p-summing, p-nuclear and p-integral maps (see [PP69]). Do their operator space analogues hold? Specifically, do we have $\nu_r^o(vu) \leq \pi_q^o(v)\nu_p^o(u)$, $\nu_r^o(vu) \leq \nu_q^o(v)\pi_p^o(u)$, $\iota_r^o(vu) \leq \pi_q^o(v)\iota_p^o(u)$, $\iota_r^o(vu) \leq \iota_q^o(v)\pi_p^o(u)$ whenever the compositions make sense?

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VITA

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