ON PRIMITIVITY AND THE UNITAL FULL FREE PRODUCT OF FINITE DIMENSIONAL C*-ALGEBRAS

A Dissertation

by

FRANCISCO JAVIER TORRES AYALA

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2012

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Approved by:

Chair of Committee,	Kenneth Dykema
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ABSTRACT

On Primitivity and the Unital Full Free Product
of Finite Dimensional C*-algebras. (May 2012)

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Chair of Advisory Committee: Dr. Kenneth Dykema

A C^*-algebra is called primitive if it admits a $*$–representation that is both faithful and irreducible. Thus the simplest examples are matrix algebras. The main objective of this work is to classify unital full free products of finite dimensional C*-algebras that are primitive. We prove that given two nontrivial finite dimensional C^*-algebras, $A_1 \neq \mathbb{C}$, $A_2 \neq \mathbb{C}$, the unital C^*-algebra full free product $A = A_1 * A_2$ is primitive except when $A_1 = \mathbb{C}^2 = A_2$.

Roughly speaking, we first show that, except for trivial cases and the case $A_1 = \mathbb{C}^2 = A_2$, there is an abundance of irreducible finite dimensional $*$–representations of A. The latter is accomplished by taking advantage of the structure of Lie group of the unitary operators in a finite dimensional Hilbert space. Later, by means of a sequence of approximations and Kaplansky's density theorem we construct an irreducible and faithful $*$–representation of A. We want to emphasize the fact that unital full free products of C*-algebras are highly abstract objects hence finding an irreducible $*$–representation that is faithfully is an amazing fact.

The dissertation is divided as follows. Chapter I gives an introduction, basic definitions and examples. Chapter II recalls some facts about $*$–automorphisms of finite dimensional C^*-algebras. Chapter III is fully devoted to prove Theorem III.6 which is about perturbing a pair of proper unital C^*-subalgebras of a matrix algebra

in such a way that they have trivial intersection. Theorem III.6 is the cornerstone for the rest of the results in this work. Lastly, Chapter IV contains the proof of the main theorem about primitivity and some consequences.

Dedicado a Bety y Vale. Ustedes le dan sentido a todo mi trabajo.

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CHAPTER I

INTRODUCTION

At some extent, primitive C*-algebras are the building blocks of the theory of C*-algebras. Thus the study of this type of C*-algebras is reasonable. The main objective of this is work is to prove that, except for trivial cases, the unital full free product of two finite dimensional C*-algebras is primitive except when both algebras have dimension 2.

Before we start, we make explicit the notation that will be used in this work.

Notation I.1. Given a Hilbert space H, we denote the set of bounded linear operators by $\mathbb{B}(H)$ and the set of compact operators by $\mathbb{K}(H)$.

For a concrete C*-algebra A, contained in $\mathbb{B}(H)$, A' denotes the commutator of A in $\mathbb{B}(H)$, in other words

$$A' = \{x \in \mathbb{B}(H) : xa = ax \quad \text{for all } a \text{ in } A\ \}.$$

For a unital C^*-algebra A, $*$-SubAlg(A) denotes the set of all unital C^*-subalgebras of A and $\mathbb{U}(A)$ denotes the set of unitary elements of A. For simplicity, given a Hilbert space H we write $\mathbb{U}(H)$ instead of $\mathbb{U}(\mathbb{B}(H))$.

By $\mathrm{Aut}(A)$ we denote the set of $*$–automorphisms of A. For u in $\mathbb{U}(A)$ we let $\mathrm{Ad}\, u$ denote the $*$–automorphism of A given by $\mathrm{Ad}\, u(x) = uxu^*$. The set of all $*$–automorphisms of the form $\mathrm{Ad}\, u$, for some u, is called the set of inner automorphism and it is denoted by $\mathrm{Inn}(A)$.

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For a unital C^*-algebra A, $C(A)$ denotes its center. In other words

$$C(A) = \{x \in A : xa = ax \quad \text{for all } a \in A\}.$$

For a positive integer n, M_n denotes the set of $n \times n$ matrices over $\mathbb{C}$ and S_n denotes the permutation group of the set $\{1, \ldots, n\}$.

A. Primitive C*-algebras

The purpose of this section is to give examples of primitive C*-algebras and show some elementary facts.

Definition I.2. A $*$–representation of a C*-algebra A in the Hilbert space H is a $*$–homomorphism from A into $\mathbb{B}(H)$. A $*$–representation is called faithful if it is injective or, equivalently, it is an isometry. A $*$–representation $\pi : A \to \mathbb{B}(H)$ is called topological irreducible if the only closed invariant subspaces for $\pi(A)$ are $\{0\}$ and H.

The following well known theorem gives an algebraic characterization of topological irreducibility. Hence from now on instead of saying that a $*$–representation is topological irreducible we just say it is irreducible.

Theorem I.3. *Let $\pi : A \to \mathbb{B}(H)$ be a $*$–representation. Then π is topological irreducible if and only if $\pi(A)' = \mathbb{C}id_H$.*

Definition I.4. A C*-algebras A is called primitive if there is a Hilbert space H and a faithful irreducible $*$–representation $\pi : A \to \mathbb{B}(H)$.

As far as we now, the basic approach to prove that a C*-algebra A is primitive is start with a faithful $*$–representation of A, or in some cases a C*-subalgebra of A, and perform some kind of operation that does not destroy faithfulness but as a result gives an irreducible $*$–representation of A. We illustrate this principle by showing that primitive C*-algebras are closed under hereditary C*-subalgebras.

Definition I.5. Let A be a C*-algebra. A C*-subalgebra B of A is called hereditary if b_1ab_2 belongs to B whenever b_1 and b_2 lie in B and a lies in A.

Proposition I.6. *Any hereditary C*-subaglebra of a primitive C*-algebra is again primitive.*

Proof. We start proving that any closed two sided ideal of a primitive C*-algebra is again primitive.

Let A be a primitive C*-algebra and let I be a nonzero closed two sided ideal in A. The existence of a faithful and irreducible $*$–representation of I is easy. We take $\pi : A \to \mathbb{B}(H)$ a faithful and irreducible $*$–representation and prove that its restriction to I is still irreducible.

Firstly let V denote the vector space generated by the family $\{\pi(x)\xi : \xi \in H, x \in I\}$. Notice that V is nonzero and it is $\pi(A)$-invariant (since I is a left ideal). Thus V is dense in H. Take T in $\mathbb{B}(H)$ with the property that $\pi(x)T = T\pi(x)$ for any x in I. We now show that T is a scalar operator. Since π is irreducible it suffices to show $T\pi(a) = \pi(a)T$ for any a in A. Since V is dense in H, $T\pi(a) = \pi(a)T$ is equivalent to show $T\pi(a)v = \pi(a)Tv$ for v in V. Write v as $\pi(x)\xi$ for some x in I and ξ in H. Thus $T\pi(a)v = T\pi(a)\pi(x)\xi = T\pi(ax)\xi = \pi(ax)T\xi$ and $\pi(a)Tv = \pi(a)T\pi(x)\xi = \pi(ax)T\xi$, where in both cases we used T commutes with all the elements in I.

Now assume A is a primitive C*-algebra and let B denote a hereditary C*-subalgebra of A.

Consider the set $I = \{ab : a \in A, \quad b \in B\}$. From the fact that B is a hereditary C*-subalgebra of A and Proposition II.5.3.2 in [3] we obtain I is a closed left ideal in A. As a consequence the set $J = \{ba : b \in B, \quad a \in A\}$ is a closed right ideal in A. Thus $I \cap J$ is a closed two sided ideal in A. Hence $I \cap J$ is a primitive C*-algebra. Using approximate units we conclude $B \subseteq I \cap J$ and since B is a hereditary C*-subalgebra of A, B is a two sided ideal in $I \cap J$.

□

To finish this section we summarize some of the main known results for primitive C*-algebras.

One of the earliest results is due to Choi and Yoshizawa. Independently, in [4] and [15], they showed that the full group C*-algebra of the free group in n generators, $2 \leq n \leq \infty$, is primitive. In [10], Murphy gave numerous conditions for the primitivity of full group C*-algebras, for instance he proved that for amenable discrete groups its full group C*-algebras is primitive if and only if the group is ICC. More recently Bédos and Omland proved in [2] that the modular group is primitive and then generalized this result in [1] and proved that if G_1 and G_2 are non trivial countable discrete amenable groups where at last one of them has more that two elements, then the full group C*-algebra of the free product of G_1 and G_2 is primitive.

B. Unital full free products of C*-algebras

In this section we recall the definition and give the construction of the unital full free product of C*-algebras.

During this section A_1 and A_2 denote two unital C*-algebras. There are many ways to define the unital full free product of A_1 and A_2. One way is using universal properties and another, more constructive way, is using reduced words. We explain both ways.

Definition I.7. The unital full free product of A_1 and A_2, denoted $A_1 * A_2$, is a unital C*-algebra together with unital $*$–homomorphisms $\iota_i : A_i \to A_1 * A_2$, $i = 1, 2$, satisfying the following universal property: given a unital C*-algebra B and unital $*$–homomorphisms $\varphi_i : A_i \to B$, $i = 1, 2$, there is a unique unital $*$–homomorphism $\varphi : A_1 * A_2 \to B$ with the property that $\varphi \circ \iota_i = \varphi_i$, for $i = 1, 2$.

As you can see from the definition $A_1 * A_2$ is a terminal object in the category of C*-algebras and unital $*$–homomorphisms. Another terminology that it is used to refer to unital full free products is push outs, for this see [12].

Next we prove existence of unital full free products.

For $i = 1, 2$, fix two states $\phi_i : A_i \to \mathbb{C}$ and let $A_i^o := \ker(\phi_i)$. For $n \geq 1$, an index $j = (j(1), \cdots, j(n))$, where $j(i) \in \{1, 2\}$, is called admissible if $j(1) \neq j(2) \neq \cdots \neq j(n)$. For an admissible index j define $W_j := A^o_{j(1)} \otimes \cdots \otimes A^o_{j(n)}$, where tensor product is taken over the complex numbers, and define

$$A_1 *_{alg} A_2 = \mathbb{C}1 \oplus \oplus_j W_j$$

where j is taken over all admissible indices and 1 is a distinguished element.

The next step is to give $A_1 *_{alg} A_2$ an structure of $*$–algebra.

First multiplication. The element 1 acts as the multiplicative identity. For admissible indexes j_1 and j_2 and elementary tensors $x_i = a_{j_i(1)} \otimes \cdots \otimes a_{j_i(n_i)} \in W_{j_i}$, $i = 1, 2$, we define x_1x_2 by induction on n_2. If $n_2 = 1$ and $x_2 \in A^o_{j_2(1)}$ we define

$$x_1x_2 = \begin{cases} a_{j_1(1)} \otimes \cdots \otimes a_{j_1(n_1)} \otimes x_2, & \text{if } j_1(n_1) \neq j_2(1), \\ a_{j_1(1)} \otimes \cdots \otimes (a_{j_1(n_1)}x_2 - \varphi_{j_0}(a_{j_1(n_1)}x_2)1_{A_{j_0}}) & \text{if } j_0 = j_1(n_1) = j_2(1). \\ \quad +\varphi_{j_0}(a_{j_1(n_1)}x_2)a_{j_1(1)} \otimes \cdots \otimes a_{j_1(n_1-1)}, & \end{cases}$$

For $n_2 \geq 2$ define $x_1x_2 = (x_1a_{j_2(1)} \otimes \cdots \otimes a_{j_2(n_2-1)})a_{j_2(n_2)}$. One can check this operation is well defined, extends to $W_{j_1} \times W_{j_2}$ and makes $A_1 *_{alg} A_2$ an algebra over the complex numbers.

Now it is turn of adjoint. For a complex number z define $(z1)^* = \overline{z}1$. For an admissible index j and elementary tensor $a_{j(1)} \otimes \cdots \otimes a_{j(n)}$ in W_j define $(a_{j(1)} \otimes \cdots \otimes a_{j(n)})^* = a^*_{j(n)} \otimes \cdots \otimes a^*_{j(1)}$. Then it is easy to check that along with the multiplication and adjoint, $A_1 *_{alg} A_2$ becomes a $*$-algebra. Even more, at the algebraic level $A_1 *_{alg} A_2$ has the universal property that characterize the unital full free product. In specific define maps $\iota_i : A_i \to A_1 *_{alg} A_2$ by $\iota_i(a) = \phi_i(a)1 \oplus (a - \phi_i(a)1_{A_i})$. Then whenever B is a unital $*$–algebra and $\varphi_i : A_i \to B$ are unital $*$–homomorphism of $*$-algebras, there is a unique unital $*$–homomorphism $\varphi : A_1 *_{alg} A_2 \to B$ such that $\varphi \circ \iota_i = \varphi_i$. We denote such a φ as $\varphi_1 * \varphi_2$. Indeed just take $\varphi(1) = 1_B$ and $\varphi(a_{j(1)} \otimes \cdots \otimes a_{j(n)}) = \varphi_{j(1)}(a_{j(1)}) \cdots \varphi_{j(n)}(a_{j(n)})$.

Now we define a norm on $A_1 *_{alg} A_2$ by $\|x\| = \sup_\pi\{\pi(x)\}$, where the sup is taken over all $*$–algebra homomorphisms π from $A_1 *_{alg} A_2$ into bounded operators of Hilbert spaces. After separation and completion we obtain a C^*-algebra that is $*$–isomorphic to the full free product of A_1 and A_2 as defined in I.7.

C. A crucial example

In this section we discuss some aspects of the C*-algebra $\mathbb{C}^2 * \mathbb{C}^2$. In particular we are interested in finding all its irreducible $*$–representations. All the results presented in this section are well known and are written for the convenience of the reader. For the rest of this section $A = \mathbb{C}^2 * \mathbb{C}^2$, $p = \iota_1((1,0))$ and $q = \iota_2((1,0))$, where $\iota_1, \iota_2 : \mathbb{C}^2 \to A$ are the canonical inclusions of $\mathbb{C}^2$ into A.

Lemma I.8. *Show that if $P, Q \in \mathbb{B}(H)$ are projections then $P + Q - PQ - QP$ lies in the center of the unital C^*-algebra generated by P and Q.*

Proof. Since P, Q, PQ and QP are in the algebra generated by P and Q then $P + Q - PQ - QP$ lies in the unital C^*-algebra generated by P and Q.

To prove $P+Q-PQ-QP$ lies in the center of the unital C^*-algebra generated by P and Q, it suffices to prove that it commutes with P and Q. But

$$P(P+Q-PQ-QP) = P - PQP = (P+Q-PQ-QP)P,$$
$$Q(P+Q-PQ-QP) = Q - QPQ = (P+Q-PQ-QP)Q.$$

□

Proposition I.9. *For any* $\pi : A \to \mathbb{B}(K)$ *irreducible* $*$*–representation,* $\dim(K) \leq 2$.

Proof. Firstly we show that if there is a nonzero vector that is not cyclic for π then $\dim(K) = 1$. Indeed, assume x in K is nonzero and $\overline{\{\pi(a)x : a \in A\}} \neq K$. Since π is irreducible and $\overline{\{\pi(a)x : a \in A\}}$ is a closed $\pi(A)$-invariant subspace we must have $\pi(a)x = 0$ for all a in A. Thus if V denotes the one-dimensional subspace generated by x we have that V is $\pi(A)$-invariant. Hence $K = V$.

Thus we may assume all nonzero vector is cyclic for π.

Since A is generated by p, q and the identity element, $\pi(A)$ is generated by P, Q and id_K, where $P = \pi(p)$ and $Q = \pi(q)$. Furthermore, by Lemma I.8 $P+Q-PQ-QP$ lies in the center of $\pi(A)$. Since π is irreducible its center equals $\mathbb{C}$, hence there is a complex number λ such that $P+Q-PQ-QP = \lambda$. Multiplying by P or Q, the last equality implies

$$PQP = (1-\lambda)P \tag{1.1}$$

$$QPQ = (1-\lambda)Q \tag{1.2}$$

From (1.1) and (1.2) follow that any word on P and Q simplifies to an expression of the form $(1-\lambda)^n P, (1-\lambda)^n Q, (1-\lambda)^n PQ, (1-\lambda)^n QP$ for some natural number n.

Then for all x in K, $V = \text{span}\ \{Px, Qx, PQx, QPx\}$ (which is closed being finite dimensional) is $\pi(A)$-invariant.

We deduce that, for $x \neq 0$, $K = V$. So far $\dim(K) \leq 4$ but we can reduce this upper bound for a suitable x.

Notice that if $P = id_k$ and $Q = id_K$ then $\pi(A) = \mathbb{C}$ and in consequence $\pi(A)' = \mathbb{B}(K)$. Since π is irreducible we conclude $\dim(K) = 1$.

Now assume that $P \neq id_k$ or $Q \neq id_k$. In this case we can pick a nonzero x such that $Px = 0$ or $Qx = 0$. It follows that $\dim(K) \leq 2$.

□

Our next objective is to compute, up to unitary equivalence, all irreducible $*$–representations of A.

Notation I.10. Let $f_p, f_q : [0,1] \to M_2$ be the continuous functions given by

$$f_p(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad f_q(t) = \begin{bmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{bmatrix}.$$

Notice that for each t in $[0,1]$, $f_q(t)$ is a projection. Thus, for each t in $(0,1)$ we have a 2-dimensional irreducible $*$–representation $\pi_t : A \to M_2$ given by $\pi_t(p) = f_p(t)$ and $\pi_t(q) = f_q(t)$.

Notice that for $t = 0$ and $t = 1$ we have 1-dimensional $*$–representations that we denote as follows. Let $\pi_1, \pi_p, \pi_q : A \to \mathbb{C}$ be the $*$–representations induced by $\pi_1(p) = \pi_1(p) = id_{\mathbb{C}}$, $\pi_p(p) = id_{\mathbb{C}}, \pi_p(q) = 0$ and $\pi_q(p) = 0, \pi_q(q) = id_{\mathbb{C}}$.

Lemma I.11. *Let $\pi : A \to \mathbb{B}(K)$ be a nonzero irreducible $*$–representation.*

If $\dim(K) = 1$ then π is unitarily equivalent to one of π_1, π_p or π_q.

If $\dim(K) = 2$ then π is unitarily equivalent to π_t for a unique t in $(0,1)$.

Proof. *C*ase $\dim(K) = 1$.

We notice that the only projections in $\mathbb{B}(K)$ are the identity and the zero map. So we have 3 possibilities: $\pi(p) = \pi(q) = id_K$, $\pi_p(p) = id_K$, $\pi_p(q) = 0$ and $\pi(p) =$

$0, \pi_q(q) = id_K$, that are respectively unitarily equivalent to π_1, π_p and π_q.

Case $\dim(K) = 2$.

Fix $\{e_1, e_2\}$ an orthonormal basis for K.

In this case we have that the projections in $\mathbb{B}(K)$ are 0, id_K and of the from P_v, where v is a unit vector and $P_v(w) = \langle w, v\rangle v$.

Since π is irreducible and $\dim(K) = 2$ neither $\pi(p)$ nor $\pi(q)$ equal 0 or id_K. Thus $\pi(p) = P_{v_p}$ and $\pi(q) = P_{v_q}$ for two unit vectors v_p and v_q. Complete $\{v_p\}$ to an orthonormal base β. Thus, with respect to the base β we have

$$[\pi(p)]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad [\pi(q)]_\beta = \begin{bmatrix} a_{1,1} & a_{1,2} \\ \overline{a_{1,2}} & a_{2,2} \end{bmatrix}$$

where $a_{1,1}$ and $a_{1,2}$ are non negative real numbers and $a_{1,2}$ is complex. Notice that from $\pi(q)^2 = \pi(q)$ we deduce $|a_{1,2}|^2 = a_{1,1}(1 - a_{1,1})$.

Since the trace of $\pi(q)$ is 1 we must have $a_{1,1} + a_{2,2} = 1$. Even more, $a_{1,1}$ and $a_{2,2}$ lie in the open interval $(0, 1)$. Indeed, if for instance $a_{2,2} = 1$ then $a_{1,1} = a_{1,2} = 0$. It follows that

$$\pi(p) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \pi(q) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

But this in this situation the vector space generated by v_p in $\pi(A)$-invariant, a contradiction since π is irreducible and $\dim(K) = 2$. A similar argument shows $a_{2,2} \neq 0$, $a_{1,1} \neq 1$ and $a_{1,1} \neq 0$.

Let $t = a_{1,1}$. Then $a_{2,2} = 1 - t$ and $|a_{1,2}| = \sqrt{t(1-t)}$. Now notice that, for a complex number λ in the unit circle,

$$\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \overline{\lambda} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} t & a_{1,2} \\ \overline{a_{2,1}} & 1-t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \overline{\lambda} \end{bmatrix} = \begin{bmatrix} t & \overline{\lambda} a_{1,2} \\ \lambda \overline{a_{1,2}} & 1-t \end{bmatrix}$$

If we take λ such that $\overline{\lambda} a_{1,2} = |a_{1,2}| = \sqrt{t(1-t)}$ we conclude π is unitarily equivalent to π_t.

Lastly we prove that if s and t lie in $(0,1)$ and π_t is unitarily equivalent to π_s then $s = t$. Assume U is a unitary matrix such that $U\pi_t(p)U^* = \pi_s(p)$ and $U\pi_t(q)U^* = \pi_s(q)$. Notice that $\pi_t(p) = P_{e_1}$ and $\pi_t(q) = P_{v_t}$ where $v_t = \sqrt{t}e_1 + \sqrt{1-t}e_2$. It follows that $UP_{e_1}U^* = P_{e_1}$ and $UP_{v_t}U^* = P_{v_s}$ and in consequence $Ue_1 = e_1$ and $Uv_t = v_s$. Thus $\langle Ue_1, Uv_t \rangle = \langle e_1, v_s \rangle$ and since U is unitary we also have $\langle Ue_1, Uv_t \rangle = \langle e_1, v_t \rangle$. We conclude $t = s$.

□

As we mentioned before computing full free products is, in general, a difficult task. Nevertheless using the fact that we know all the irreducible $*$–representations of $\mathbb{C}^2 * \mathbb{C}^2$ we have a nice description.

Proposition I.12. *A is $*$–isomorphic to the C^*-algebra of M_2-valued continuous functions over the unit interval with the property that its values at 0 and 1 are diagonal matrices.*

Proof. Let

$$B = \{f : [0,1] \to M_2 : f \text{ is continous and } f(0),\, f(1) \text{ are diagonal } \}$$

Then f_p and f_q belong to B and they are projections. By the universal property of A, there is a unital $*$-algebra homomorphism $\phi : A \to B$ such that $\phi(p) = f_p$ and $\phi(q) = f_q$. We claim ϕ is an isometric $*$–isomorphism.

Using Bernestein's polynomials, we notice that B is the unital C^*-algebra generated by $\{I_1, I_2, f_1, f_2, f_{j,k} : j, k \geq 1\}$, where

$$I_1(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad I_2(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad f_1(t) = \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix}, \quad f_2(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1-t \end{bmatrix}$$

and

$$f_{j,k} = \begin{bmatrix} 0 & t^j(1-t)^k \\ 0 & 0 \end{bmatrix}.$$

But taking sums, products and adjoints of the elements $1_B, f_p, f_q$ we obtain that $\{I_1, I_2, f_1, f_2, f_{j,k} : j, k \geq 1\} \subseteq \phi(A)$. We conclude $\phi(A) = B$.

Next we prove ϕ is injective. In order to prove ϕ is injective first we show that every irreducible $*$–representation $\pi : A \to \mathbb{B}(K)$ factors through B i.e. there is a $*$–representation $\sigma : B \to \mathbb{B}(K)$ such that $\sigma \circ \phi = \pi$.

Take $\pi : A \to \mathbb{B}(K)$ a nonzero irreducible $*$–representation. Then $\dim(K) = 1$ or $\dim(K) = 2$.

If $\dim(K) = 1$ from Lemma I.11 there are tree irreducible $*$–representations, π_1, π_p and π_q, where each $*$–representation is determined by

$$\pi_1(p) = \pi_1(q) = id_K, \quad \pi_p(p) = id_K, \pi_p(q) = 0, \quad \pi_q(p) = 0, \pi_q(q) = id_K.$$

In the case π_1, let $\sigma : B \to \mathbb{B}(K)$ be given by $\sigma(f) = f(1)[1,1]$, where $f(1)[1,1]$ denotes the (1,1)-entry of the matrix $f(1)$.

In the case π_p, let $\sigma : B \to \mathbb{B}(K)$ be given by $\sigma(f) = f(0)[1,1]$.

In the case π_q, let $\sigma : B \to \mathbb{B}(K)$ be given by $\sigma(f) = f(0)[2,2]$.

In the case $\dim(K) = 2$, from Lemma I.11, any irreducible $*$–representations is unitarily equivalent to π_t, for a unique $t \in (0,1)$, where $\pi_t(p) = f_p(t)$ and $\pi_t(q) = f_q(t)$.

Thus in this case we may take σ to be the evaluation at t.

Lastly, take a in $\ker(\phi)$ and let $\pi : A \to \mathbb{B}(K)$ be an irreducible $*$–representation such that $\|\pi(a)\| = \|a\|$. If σ is defined as above we have $\|\sigma(\phi(a))\| = \|\pi(a)\| = \|a\|$ but $\phi(a) = 0$ hence $a = 0$ and we conclude ϕ is injective.

□

CHAPTER II

AUTOMORPHISMS

By a $*$-automorphism of a C^*-algebra we mean a bijective map, from the algebra onto itself, that preserves sums, products and adjoints.

In this chapter we recall some basic results concerning $*$-automorphisms of finite dimensional C^*-algebras, in particular we are interested is determine a precise algebraic relation between the group of $*$-automorphism and the subgroup of inner $*$-automorphisms. In concrete see Propositions II.3 and II.4.

Remark II.1. Any $*$-homomorphism from a simple C^*-algebra is either zero or injective (since its kernel is an ideal). Even more, any non-zero $*$-endomorphism of a finite dimensional simple C^*-algebra is a $*$-automorphism. Indeed, any such $*$-endomorphism is injective and thus it is bijective (by finite dimensionality) and a straightforward computation shows its inverse is a $*$-endomorphism.

As a consequence any $*$-automorphism of a finite dimensional C^*-algebra move, without breaking, each one of its simple C^*-subalgebras with the same dimension (we may think these as blocks). Thus modulo an inner $*$-automorphism, a $*$-automorphism is just a permutation. The rest of this chapter is formalizing this ideas.

Proposition II.2. *Let B be a finite dimensional C^*-algebra and assume B decomposes as $\oplus_{j=1}^{J} B_j$, where all B_j are $*$-isomorphic to the same matrix algebra i.e. there is a positive integer n such that, for all j, B_j is $*$-isomorphic to M_n.*

Then for any α in $\mathrm{Aut}(B)$, there is a permutation σ in S_J and a family of $*$-isomorphisms, $\{\alpha_j : B_j \to B_{\sigma(j)}\}_{1\leq j\leq J}$, such that

$$\alpha(b_1,\ldots,b_J) = \left(\alpha_{\sigma^{-1}(1)}(b_{\sigma^{-1}(1)}),\ldots,\alpha_{\sigma^{-1}(J)}(b_{\sigma^{-1}(J)})\right).$$

Proof. For $1 \leq j_1, j_2 \leq J$ write

$$\alpha[j_1, j_2] = \pi_{j_2} \circ \alpha \circ \iota_{j_1} : B_{j_1} \to B_{j_2},$$

where $\iota_{j_1} : B_{j_1} \to B$ is the canonical inclusion and $\pi_{j_2} : B \to B_{j_2}$ is the canonical projection. Thus $\alpha[j_1, j_2]$ is a $*$-homomorphism.

Since all B_j have the same dimension, Remark II.1 implies that either $\alpha[j_1, j_2]$ is zero or a $*$-isomorphism. For fixed j let

$$F_j = \{k \in \{1, \ldots, J\} : \alpha[j, k] \neq 0\}.$$

Next we show the sets $\{F_j\}_{1 \leq j \leq J}$ are pair wise disjoint.

Assume $j_1 < j_2$. Take $b_1, c_1 \in B_{j_1}$ and $b_2, c_2 \in B_{j_2}$. From

$$\alpha(\iota_{j_1}(b_1)) = (\alpha[j_1, 1](b_1), \ldots, \alpha[j_1, J](b_1)),$$
$$\alpha(\iota_{j_2}(b_2)) = (\alpha[j_2, 1](b_2), \ldots, \alpha[j_2, J](b_2)),$$

we get

$$\alpha(\iota_{j_1}(b_1) + \iota_{j_2}(b_2)) = (\alpha[j_1, 1](b_1) + \alpha[j_2, 1](b_2), \ldots, \alpha[j_1, J](b_1) + \alpha[j_2, J](b_2)).$$

Since

$$\alpha(\iota_{j_1}(b_1) + \iota_{j_2}(b_2))\alpha(\iota_{j_1}(c_1) + \iota_{j_2}(c_2)) = \alpha(\iota_{j_1}(b_1 c_1) + \iota_{j_2}(b_2 c_2))$$

we conclude that for all $1 \leq j \leq J$,

$$(\alpha[j_1, j](b_1) + \alpha[j_2, j](b_2))(\alpha[j_1, j](c_1) + \alpha[j_2, j](c_2)) = \alpha[j_1, j](b_1 c_1) + \alpha[j_2, j](b_2 c_2)$$

which implies

$$\alpha[j_2, j](b_2)\alpha[j_1, j](c_1) + \alpha[j_1, j](b_1)\alpha[j_2, j](c_2) = 0. \tag{2.1}$$

Take $j \in F_{j_1}$ so that $\alpha[j_1, j]$ is a $*$-isomorphism. Since $\alpha[j_1, j](1_{B_{j_1}}) = 1_{B_j}$, making $b_1 = c_1 = 1_{B_{j_1}}$ and $b_2 = c_2$ in (2.1) we get $\alpha[j_2, j](b_2) = 0$. We conclude $j \notin F_{j_2}$. This proves the sets F_j are pair wise disjoint.

We also notice each F_j is not empty. Otherwise $\alpha \circ \iota_j$ is zero, a contradiction since both are injective maps.

In conclusion we have each F_j contains exactly one element, call it $\sigma(j)$.

Now we show the map $j \mapsto \sigma(j)$ is a bijection. Since we are dealing with finite sets it is enough to show it is injective. Assume $j_1 < j_2$ and $\sigma(j_1) = \sigma(j_2) = k$. Using that $\alpha[j_1, k]$ and $\alpha[j_2, k]$ are onto we can pick $b \in B_k$ non-zero and $b_1 \in Bj_1$, $b_2 \in B_{j_2}$ both non-zero such that

$$\begin{aligned}
\alpha(0, \ldots, \underbrace{b_1}_{j_1\text{-th entry}}, \ldots, 0) &= (0, \ldots, \underbrace{b}_{k\text{-th entry}}, \ldots, 0) \\
\alpha(0, \ldots, \underbrace{b_2}_{j_2\text{-th entry}}, \ldots, 0) &= (0, \ldots, \underbrace{b}_{k\text{-th entry}}, \ldots, 0)
\end{aligned}$$

But this implies

$$\alpha(0, \ldots, \underbrace{b_1}_{j_1\text{-th entry}}, \ldots, \underbrace{-b_2}_{j_2\text{-th entry}}, 0) = 0$$

a contradiction.

The maps we are looking for are $\alpha_j = \alpha[j, \sigma(j)]$.

□

Proposition II.3. *Let B be a finite dimensional C^*-algebra, assume B decomposes as $\oplus_{j=1}^{J} B_j$ and there is a positive integer n such that all B_j are $*$-isomorphic to M_n.*

Fix $\{\beta_j : B_j \to M_n\}_{1 \le j \le J}$ a set of $*$-isomorphisms.

1. *For a permutation σ in S_J define $\psi_\sigma : B \to B$ by*

$$\psi_\sigma(b_1, \ldots, b_J) = (\beta_1^{-1} \circ \beta_{\sigma^{-1}(1)}(b_{\sigma^{-1}(1)}), \ldots, \beta_J^{-1} \circ \beta_{\sigma^{-1}(J)}(b_{\sigma^{-1}(J)}))$$

Then ψ_σ *lies in* $\mathrm{Aut}(B)$ *and the map* $\sigma \mapsto \psi_\sigma$ *defines a group embedding of* S_J *into* $\mathrm{Aut}(B)$*.*

2. *Every element* α *in* $\mathrm{Aut}(B)$ *factors as*

$$\left(\oplus_{j=1}^{J} \mathrm{Ad}\, u_j\right) \circ \psi_\sigma$$

for some permutation σ *in* S_J *and unitaries* u_j *in* $\mathbb{U}(B_j)$*.*

3. *There is a exact sequence*

$$0 \to \mathrm{Inn}(B) \to \mathrm{Aut}(B) \to S_J \to 0.$$

Proof. Part 1:

A straight forward computation shows that ψ_σ is a $*$-homomorphism.

The next step is to show

$$\psi_\sigma \circ \psi_\varsigma = \psi_{\sigma \circ \varsigma} \tag{2.2}$$

Pick b an element of B and let $c = \psi_\varsigma(b)$.

Take $k = \sigma^{-1}(j)$. From the equations

$$\psi_\sigma(c)_j = \beta_j^{-1} \circ \beta_{\sigma^{-1}(j)}(c_{\sigma^{-1}(j)})$$

$$\psi_\varsigma(b)_k = \beta_k^{-1} \circ \beta_{\varsigma^{-1}(k)}(b_{\varsigma^{-1}(k)})$$

we get

$$\left(\psi_\sigma \circ \psi_\varsigma(b)\right)_j = \beta_j^{-1} \circ \beta_{\varsigma^{-1}(\sigma^{-1}(j))}(b_{\varsigma^{-1}(\sigma^{-1}(j))}) = \psi_{\sigma\circ\varsigma}(b)_j.$$

Equation (2.2) implies ψ_σ belongs to $\mathrm{Aut}(B)$ and it also shows the map $\sigma \mapsto \psi_\sigma$ is a group homomorphism.

Now assume $\psi_\sigma = id_B$ but $\sigma \neq id_{S_J}$. Then we can find j_0 with $\sigma^{-1}(j_0) \neq j_0$. Define an element b in B via $b_{j_0} = 1_{B_{j_0}}$ and $b_j = 0$ for $j \neq j_0$.

Since $\psi_\sigma = ib_B$ we have

$$1_{B_{j_0}} = b_{j_0} = \psi_\sigma(b)_{j_0} = \beta_{j_0}^{-1} \circ \beta_{\sigma_{j_0}^{-1}}(b_{\sigma^{-1}(j_0)}) = 0.$$

Thus $\sigma = id_{S_J}$.

□

Proof. Part 2:

By Proposition II.2, there is a permutation σ in S_j and a set of $*$-isomorphisms $\{\alpha_j : B_j \to B_{\sigma(j)}\}_{1\leq j\leq J}$ with

$$\alpha(b) = \left(\alpha_{\sigma^{-1}(1)}(b_{\sigma^{-1}(1)}), \ldots, \alpha_{\sigma^{-1}(J)}(b_{\sigma^{-1}(J)})\right).$$

Since $\beta_{\sigma(j)} \circ \alpha_j \circ \beta_j^{-1}$ lies in $\mathrm{Aut}(M_n)$, it equals $\mathrm{Ad}\, v_j$ for some unitary v_j in $\mathbb{U}(M_n)$. Thus for all $b_{\sigma^{-1}(k)}$ we have

$$\alpha_{\sigma^{-1}(k)}(b_{\sigma^{-1}(k)}) = \beta_k^{-1}(v_{\sigma^{-1}(k)})\beta_k^{-1}(\beta_{\sigma^{-1}(k)}(b_{\sigma^{-1}(k)}))\beta_k^{-1}(v_{\sigma^{-1}(k)})^*$$

Hence if we take take $u_j = \beta_j^{-1}(v_{\sigma^{-1}(j)})$ we have the result.

□

Proof. Part 3:

We show $\mathrm{Inn}(B)$ is normal in $\mathrm{Aut}(B)$ and $\mathrm{Aut}(B)/\mathrm{Inn}(B)$ is isomorphic to S_J.

Thanks to part 2, to show normality, it suffices to show that given any ψ_σ and unitary v_j in $\mathbb{U}(B_j)$, there are unitaries w_j in $\mathbb{U}(B_j)$ such that

$$\psi_{\sigma^{-1}} \circ \left(\oplus_{j=1}^J \mathrm{Ad}\, v_j\right) \circ \psi_\sigma = \oplus_{j=1}^J \mathrm{Ad}\, w_j.$$

A direct computation shows

$$\psi_{\sigma^{-1}} \circ \left(\oplus_{j=1}^J \mathrm{Ad}\, v_j\right) \circ \psi_\sigma = \mathrm{Ad}\, \psi_{\sigma^{-1}} \left(\oplus_{j=1}^J \mathrm{Ad}\, v_j\right),$$

and by definition

$$\psi_{\sigma^{-1}}\left(\oplus_{j=1}^{J}\operatorname{Ad}v_j\right) = (\beta_1^{-1}\circ\beta_{\sigma(1)}(v_{\sigma(1)}),\ldots,\beta_J^{-1}\circ\beta_{\sigma(J)}(v_{\sigma(J)})).$$

Hence take $w_j = \beta_j^{-1}\circ\beta_{\sigma(j)}(v_{\sigma(j)})$. This completes the proof that $\operatorname{Inn}(B)$ is normal in $\operatorname{Aut}(B)$.

By part 2, to show $\operatorname{Aut}(B)/\operatorname{Inn}(B)$ is isomorphic to S_J, it is enough to prove

$$\{\psi_\sigma : \sigma\in S_J\}\cap\operatorname{Inn}(B) = \{id_B\}.$$

Thus assume there is a unitary u in $\mathbb{U}(B)$ such that $\psi_\sigma(b) = ubu^*$ for all elements b in B. It follows that for all $1\le j\le J$,

$$\beta_j^{-1}\circ\beta_{\sigma^{-1}(j)}(b_{\sigma^{-1}(j)}) = u_j b_j u_j^*.$$

Since we can choose elements b_j independently from each other we must have $\sigma^{-1}(j) = j$ for all j, and we are done.

□

So far we have consider C^*-algebras with only one type of block subalgebra, so to speak. Next proposition shows that a $*$-automorphism can not mix blocks of different dimensions. As a consequence, and along with Proposition II.3, we get a general decomposition of $*$-automorphisms of finite dimensional C^*-algebras.

Proposition II.4. *Let B be a finite dimensional C^*-algebra and decompose B as $\oplus_{i=1}^{I}\oplus_{j=1}^{J_i}B(i,j)$, where for each i, there is a positive integer n_i such that $B(i,j)$ is isomorphic to M_{n_i} for all $1\le j\le J_i$, i.e. we group subalgebras that are isomorphic to the same matrix algebra.*

Then any α in $\operatorname{Aut}(B)$ factors as $\alpha = \oplus_{i=1}^{I}\alpha_i$ where

$$\alpha_i : \oplus_{j=1}^{J_i}B(i,j)\to\oplus_{j=1}^{J_i}B(i,j)$$

is a $*$-isomorphism.

Proof. Let's start with a rough decomposition of α. For $1 \leq i_1, i_2 \leq I$, $1 \leq j_1 \leq J_{i_1}$ and $1 \leq j_2 \leq J_{i_2}$ let

$$\alpha[(i_1, j_1), (i_2, j_2)] = \pi_{(i_2,j_2)} \circ \alpha \circ \iota_{(i_1,j_1)}$$

where $\iota_{(i_1,j_1)}$ denote the canonical inclusion of $B(i_1, j_1)$ into B and $\pi_{(i_2,j_2)}$ denote the canonical projection of B onto $B(i_2, j_2)$. Then $\alpha[(i_1, j_1), (i_2, j_2)]$ is a $*$-homomorphism from $B(i_1, j_1)$ into $B(i_2, j_2)$.

Now we proceed by induction on I.

The case $I = 1$ is trivial.

Now assume the result is true for k and let $I = k + 1$.

With no loss of generality we may assume $n_1 < \cdots < n_k < n_{k+1}$.

Take $1 \leq l \leq J_{k+1}$. By remark II.1 $\alpha[(k+1, l), (i_2, j_2)]$ either is zero or injective. But for $1 \leq i_2 \leq k$, it must be zero, because in this case $\dim B(i_2, j_2)) < \dim B(k + 1, l)$.

As in proposition II.2, one can show that there is $1 \leq \sigma_{k+1}(l) \leq J_{k+1}$ unique such that $\alpha[(k + 1, l), (k + 1, \sigma_{k+1}(l))]$ is not zero and the map $l \mapsto \sigma_{k+1}(l)$ is a bijection. Thus it follows that α restricted to $\oplus_{j=1}^{J_{k+1}} B(k + 1, j)$ gives a $*$-isomorphism onto $\oplus_{j=1}^{J_{k+1}} B(k + 1, j)$.

Next we show that $\alpha[(i_1, j_1), (k + 1, l)] = 0$ for $1 \leq i_1 \leq k$ and $1 \leq l \leq J_{k+1}$. Take $b_1 \in B(i_1, j_1)$. The $(k + 1, \sigma_{k+1}(l))$-entry of the following identity (which holds because $i_1 < k + 1$)

$$\begin{aligned} \alpha\left(\iota_{(i_1,j_1)}(b_1) + \iota_{(k+1,l)}(1_{B(k+1,l)})\right) \alpha\left(\iota_{(i_1,j_1)}(b_1) + \iota_{(k+1,l)}(1_{B(k+1,l)})\right) \\ = \alpha\left(\iota_{(i_1,j_1)}(b_1 b_1) + \iota_{(k+1,l)}(1_{B(k+1,l)})\right) \end{aligned}$$

along with the fact that $\alpha[(k+1,l),(k+1,\sigma_{k+1}(l))]$ is a $*$-isomorphism imply

$$\alpha[(i_1,j_1),(k+1,\sigma_{k+1}(l))](b_1)=0.$$

Since σ is a bijection we conclude $\alpha[(i_1,j_1),(k+1,l)]=0$ for all $1\leq l\leq J_{k+1}$.

Hence we conclude that the image of $\oplus_{i=1}^{k}\oplus_{j=1}^{J_i}B(i,j)$ under α is contained in $\oplus_{i=1}^{k}\oplus_{j=1}^{J_i}B(i,j)$. But α injective and thus finite dimensionality gives that this restriction is a $*$-isomorphism. Lastly we apply induction hypothesis to this restriction get the desired result. $\square$

CHAPTER III

PERTURBATIONS

A. Useful results from Lie groups

In this section we summarize some result that, later on, will be repeatedly used. Definitions and proofs of results mentioned in this section can be found in [9] and [8].

The next two theorems are quite important and will be used in the next section.

Theorem III.1. *Any closed subgroup of a Lie group is a Lie subgroup.*

Theorem III.2. *Let G be a Lie group of dimension n and $H \subseteq G$ be a Lie subgroup of dimension k.*

1. *Then the left coset space G/H has a natural structure of a manifold of dimension $n-k$ such that the canonical quotient map $\pi : G \to G/H$, is a fiber bundle, with fiber diffeomorphic to H.*
2. *If H is a normal Lie subgroup then G/H has a canonical structure of a Lie group.*

The next proposition is from Corollary 2.21 in [9].

Proposition III.3. *Let G denote a Lie group and assume it acts smoothly on a manifold M. For $m \in M$ let $\mathcal{O}(m)$ denote its orbit and $\mathrm{Stab}(m)$ denote its stabilizer i.e.*

$$\begin{aligned} \mathcal{O}(m) &= \{g.m : g \in G\}, \\ \mathrm{Stab}(m) &= \{g \in G : g.m = m\}. \end{aligned}$$

The orbit $\mathcal{O}(m)$ is an immersed submanifold of M. If $\mathcal{O}(m)$ is compact, then the map $g \mapsto g.m$, is a diffeomorphism from $G/\mathrm{Stab}(m)$ onto $\mathcal{O}(m)$. (In this case we say $\mathcal{O}(m)$ is an embedded submanifold of M.)

Corollary III.4. *Let G be a compact Lie group and let K and L be closed subgroups of G. The subspace $KL = \{kl : k \in K, l \in L\}$ is an embedded submanifold of G of dimension*

$$\dim K + \dim L - \dim(L \cap K).$$

Proof. First of all KL is compact. This follows from the fact that multiplication is continuous and both K and L are compact. Consider the action of $K \times L$ on G given by $(k, l).g = kgl^{-1}$. Notice that the orbit of e is precisely KL. By Proposition III.3, KL is an immersed submanifold diffeomorphic to $K \times L/\mathrm{Stab}(e)$. Since it is compact, it is an embedded submanifold. But $\mathrm{Stab}(e) = \{(x, x) : x \in K \cap L\}$ and we conclude

$$\dim KL = \dim(K \times L) - \dim \mathrm{Stab}(e) = \dim K + \dim L - \dim(K \cap L).$$

□

Proposition III.5. *Let G be a compact Lie group and let H be a closed subgroup. Let π denote the quotient map onto G/H.*

There are:

1. *$\mathcal{N}_G$, a compact neighborhood of e in G,*
2. *$\mathcal{N}_H$, a compact neighborhood of e in H,*
3. *$\mathcal{N}_{G/H}$, a compact neighborhood of $\pi(e)$ in G/H,*
4. *a continuous function $s : \mathcal{N}_{G/H}(\pi(e)) \to G$ satisfying*

   (a) $s(\pi(e)) = e$ and $\pi(s(y)) = y$ for all y in $\mathcal{N}_{G/H}(\pi(e))$,

(b) The map

$$\begin{aligned} \mathcal{N}_H \times \mathcal{N}_{G/H} &\to \mathcal{N}_G, \\ (h, y) &\mapsto hs_g(y) \end{aligned}$$

is a homeomorphism.

Proof. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote, respectively, the Lie algebras of G and H. Take $\mathfrak{m}$ a vector subspace such that $\mathfrak{g}$ is the direct sum of $\mathfrak{h}$ and $\mathfrak{m}$. By Lemmas 2.4 and 4.1 in [8], chapter 2, there are compact neighborhoods $U_{\mathfrak{g}}$, $U_{\mathfrak{h}}$ and $U_{\mathfrak{m}}$ of 0 in $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{m}$, respectively, such that the map

$$\begin{aligned} U_{\mathfrak{m}} \times U_{\mathfrak{h}} &\to U_{\mathfrak{g}}, \\ (a, b) &\mapsto \exp(a)\exp(b) \end{aligned}$$

is an homeomorphism and π maps homeomorphically $\exp(U_{\mathfrak{m}})$ onto a compact neighborhood of $\pi(e)$. Call the latter neighborhood $\mathcal{N}_{G/H}$. Take $\mathcal{N}_G = exp(U_{\mathfrak{g}})$, $\mathcal{N}_H = \exp(U_{\mathfrak{h}})$ and s the inverse of π restricted to $\exp(U_{\mathfrak{m}})$.

□

B. Intersections and perturbations

In this section we fix a positive integer N and, unless stated otherwise, $B_1 \subsetneq M_N$ and $B_2 \subsetneq M_N$ denote proper unital C^*-subalgebras of M_N.

The main purpose if this section is give a proof of the following theorem (recall that for a C^*-algebra A, $C(A)$ denotes its center).

Theorem III.6. *Assume one of the following conditions holds:*

1. $\dim C(B_1) = 1 = \dim C(B_2)$,

2. $\dim C(B_1) \geq 2$*,* $\dim C(B_2) = 1$ *and* B_1 *is* $*$*–isomorphic to*

$$M_{N/\dim C(B_1)} \oplus \cdots \oplus M_{N/\dim C(B_1)},$$

3. $\dim C(B_1) = 2 = \dim C(B_2)$*,* B_1 *is* $*$*–isomorphic to*

$$M_{N/2} \oplus M_{N/2},$$

and B_2 *is* $*$*–isomorphic to*

$$M_{N/2} \oplus M_{N/(2k)}$$

where $k \geq 2$*,*

4. $\dim C(B_1) \geq 2$*,* $\dim C(B_2) \geq 3$ *and, for* $i = 1, 2$*,* B_i *is* $*$*–isomorphic to*

$$M_{N/\dim C(B_i)} \oplus \cdots \oplus M_{N/\dim C(B_i)}.$$

Then

$$\Delta(B_1, B_2) := \{u \in \mathbb{U}(M_N) : B_1 \cap uB_2u^* = \mathbb{C}\}$$

is dense in $\mathbb{U}(M_N)$*.*

The C^*-algebra uB_2u^* is what we call a perturbation of B_2 by u. With this nomenclature we are trying to prove that, in the cases mentioned above, almost always we can perturb one C^*-subalgebra a little bit in such a way that the intersection with the other one is the smallest possible.

Roughly speaking, the idea behind is to show that the complement of $\Delta(B_2, B_2)$ can be locally parametrized with strictly fewer variables than $\dim \mathbb{U}(M_N) = N^2$. Thus, the complement of $\Delta(B_1, B_2)$ is, topologically speaking, small.

We start with some definitions. The group $\mathbb{U}(B_1)$ acts on $*$-SubAlg(B_1) via $(u, B) \mapsto uBu^*$ and the equivalence relation on $*$-SubAlg(B_1) induced by this action will be denoted by $\sim_{B_1}$. Specifically, we have

$$B \sim_{B_1} C \Leftrightarrow \exists u \in \mathbb{U}(B_1) : uBu^* = C.$$

We denote by $[B]_{B_1}$ the $\sim_{B_1}$-equivalence class of a subalgebra B in $*$-SubAlg(B_1).

Notation III.7. For B in $*$-SubAlg(B_1) let

$$\begin{aligned} X(B_1, B_2; B) &= \{u \in \mathbb{U}(M_N) : uB_2u^* \cap B_1 = B\}, \\ Y(B_2; B) &= \{u \in \mathbb{U}(M_N) : u^*Bu \subseteq B_2\}, \\ Z(B_1, B_2; [B]_{B_1}) &= \{u \in \mathbb{U}(M_N) : uB_2u^* \cap B_1 \sim_{B_1} B\}. \end{aligned}$$

It is straightforward that the complement of $\Delta(B_1, B_2)$ is precisely the union of the sets $Z(B_1, B_2; [B]_{B_1})$, where B runs over all unital C^*-subalgebras of B_1 and $B \neq \mathbb{C}$. Just for a moment, with out being formal, we may think $Z(B_1, B_2; [B]_{B_1})$ as being parametrized by two coordinates. The first one is an algebra $\sim_{B_1}$-equivalent to B. Hence the first coordinate lives in $[B]_{B_1}$. The second, is a unitary u that realizes the first coordinate as $uB_2u^* \cap B_1$. $X(B_1, B_2; B)$ comes into play in order to parametrize this second coordinate. The problem is that $X(B_1, B_2; B_{B_1})$ is complicated to handle (for instance it may not be closed). This is way we introduce the friendlier set $Y(B_2; B)$. Good properties about $Y(B_2; B)$ is that it is a closed subset of $\mathbb{U}(M_N)$, in fact we will show it is a finite union of enbedded compact submanifolds of $\mathbb{U}(M_N)$, and it contains $X(B_1, B_2; B)$.

The rest of this section is the formalization of the previous idea. In concrete our first goal is to show $[B]_{B_1}$ has a structure of manifold and we are particularly interested in finding its dimension.

Let Stab(B_1, B) denote the $\sim_{B_1}$-stabilizer of B i.e.

$$\text{Stab}(B_1, B) = \{u \in \mathbb{U}(B_1) : uBu^* = B\}.$$

Remark III.8. Given B in $*$-SubAlg(B_1) we can endow $[B]_{B_1}$ with a structure of manifold. Indeed, let $\mathbb{U}(B_1)/\mathrm{Stab}(B_1, B)$ denote the set of left-cosets and consider the map

$$\begin{aligned} \beta_B : [B]_{B_1} &\to \mathbb{U}(B_1)/\mathrm{Stab}(B_1, B), \\ \beta_B(uBu^*) &= u\mathrm{Stab}(B_1, B). \end{aligned}$$

One can check β_B is well defined and bijective. Since $\mathbb{U}(B_1)/\mathrm{Stab}(B_1, B)$ is a manifold, β_B induces a structure of manifold on $[B]_{B_1}$. To avoid ambiguity we have to check the topology does not depend on the representative B. In fact, we will show the topology induced by β_B is the same as the topology induced by the Hausdorff distance.

For C_1 and C_2 in $[B]_{B_1}$ define

$$\mathrm{d_H}(C_1, C_2) = \max\left\{\sup_{x_2}\inf_{x_1}\{\|x_1 - x_2\|\}, \sup_{x_1}\inf_{x_2}\{\|x_1 - x_2\|\}\right\},$$

where x_i is taken in the unit ball of C_i, $i = 1, 2$. Since unit balls of unital C^*-subalgebras of B_1 are compact subsets (in the norm topology), $\mathrm{d_H}$ defines a metric on $[B]_{B_1}$. Let τ and τ_H denote, respectively, the topologies on $[B]_{B_1}$ induced by β_B and $\mathrm{d_H}$. We are going to show $\tau = \tau_H$. Consider the identity map $\mathrm{id} : ([B]_{B_1}, \tau) \to ([B]_{B_1}, \tau_H)$. First we show id is continuous. Since $\mathbb{U}(B_1)/\mathrm{Stab}(B_1, B)$ is endowed with the pull back topology from the quotient map $\pi : \mathbb{U}(B_1) \to \mathbb{U}(B_1)/\mathrm{Stab}(B_1, B)$ where $\mathbb{U}(B_1)$ is taken with the norm topology, id is continuous if and only if the map

$$\beta_B^{-1} \circ \pi : \mathbb{U}(B_1) \to ([B]_{B_1}, \tau_H)$$

is continuous. Take $(u_n)_{n\geq 1}$ a sequence in $\mathbb{U}(B_1)$ and a unitary u in $\mathbb{U}(B_1)$ such that $\lim_n \|u_n - u\| = 0$. We need to show

$$\lim_n \mathrm{d_H}(\beta_B^{-1} \circ \pi(u_n), \beta_B^{-1} \circ \pi(u)) = \lim_n \mathrm{d_H}(u_n B u_n^*, uBu^*) = 0.$$

Take n_0 such that $\|u_n - u\| < \varepsilon/2$ for all $n \geq n_0$. For any b in the unit ball of B and any $n \geq n_0$, we have

$$\|u_n b u_n^* - u b u^*\| < \varepsilon.$$

Thus, for $n \geq n_0$

$$\sup_{x_2} \inf_{x_1} \|x_1 - x_2\| < \varepsilon$$

and

$$\sup_{x_1} \inf_{x_2} \|x_1 - x_2\| < \varepsilon,$$

where x_2 is taken in the unit ball of $u_n B u_n^*$ and x_1 is taken in the unit ball of uBu^*. Hence $\mathrm{id} : ([B]_{B_1}, \tau) \to ([B]_{B_1}, \tau_H)$ is continuous. Lastly, since id is bijective, $([B]_{B_1}, \tau)$ is compact and $([B]_{B_1}, \tau_H)$ is Hausdorff, we conclude that id is a homeomorphism. Thus $\tau = \tau_H$.

Now that we know $[B]_{B_1}$ is a manifold, we want to find its dimension. Since by construction $[B]_{B_1}$ is diffeomorphic to $\mathbb{U}(B_1)/\mathrm{Stab}(B_1, B)$, $\dim[B]_{B_1} = \dim \mathbb{U}(B_1) - \dim \mathrm{Stab}(B_1, B)$. Thus we only need to find $\dim \mathrm{Stab}(B_1, B)$.

Notation III.9. Whenever we take commutators they will be with respect to the ambient algebra M_N, in other words for a subalgebra A in $*$-SubAlg(M_N)

$$A' = \{x \in M_N : xa = ax, \quad \text{for all } a \text{ in } A\}.$$

Recall that $C(A)$ denotes the center of A i.e.

$$C(A) = A \cap A' = \{a \in A : xa = ax \quad \text{for all } x \text{ in A}\}.$$

Proposition III.10. *For any B_1 in $*$-SubAlg(M_N) and for any B in $*$-SubAlg(B_1), we have*

$$\dim \mathrm{Stab}(B_1, B) = \dim \mathbb{U}(B) + \dim \mathbb{U}(B_1 \cap B') - \dim \mathbb{U}(C(B)).$$

Proof. We'll find a normal subgroup of $\mathrm{Stab}(B_1, B)$, for which we can compute its dimension and that partitions $\mathrm{Stab}(B_1, B)$ into a finite number of cosets. Let G denote the subgroup of $\mathrm{Stab}(B_1, B)$ generated by $\mathbb{U}(B_1 \cap B')$ and $\mathbb{U}(B)$. Since the elements of $\mathbb{U}(B)$ commute with the elements of $\mathbb{U}(B_1 \cap B')$, a typical element of G looks like vw, where v lies in $\mathbb{U}(B)$ and w lies in $\mathbb{U}(B_1 \cap B')$. Taking into account compactness of $\mathbb{U}(B)$ and $\mathbb{U}(B_1 \cap B')$, we deduced G is compact.

Now we show G is normal in $\mathrm{Stab}(B_1, B)$. Take u an element in $\mathrm{Stab}(B_1, B)$. For a unitary v in $\mathbb{U}(B)$ it is immediate that uvu^* lies in $\mathbb{U}(B)$. For a unitary w in $\mathbb{U}(B_1 \cap B')$, the following computation shows uwu^* belongs to $\mathbb{U}(B_1 \cap B')$. For any element b in B we have:

$$(uwu^*)b = uw(u^*bu)u^* = u(u^*bu)wu^* = b(uwu^*),$$

where in the second equality we used u^*bu lies in B. In conclusion uGu^* is contained in G for all u in $St(B_1, B)$ i.e. G is normal in $\mathrm{Stab}(B_1, B)$.

As a result $\mathrm{Stab}(B_1, B)/G$ is a Lie group. The next step is to show $\mathrm{Stab}(B_1, B)/G$ is finite. Decompose B as

$$B = \oplus_{i=1}^{I} \oplus_{j=1}^{J_i} B(i, j),$$

where for all i there is k_i such that for $1 \leq j \leq J_i$, $B(i, j)$ is $*$–isomorphic to M_{k_i}. For the rest of our proof we fix a family, $\beta(i, j) : B(i, j) \to M_{k_i}$, of $*$–isomorphisms.

An element u in $\mathrm{Stab}(B_1, B)$ defines a $*$–automorphism of B by conjugation. As a consequence, Propositions II.3 and II.4 imply there are permutations σ_i in S_{J_i} and unitaries v_i in $\mathbb{U}(\oplus_{j=1}^{J_i} B(i, j))$ such that

$$\forall b \in B : ubu^* = v\psi(b)v^* \tag{3.1}$$

where $v = \oplus_{i=1}^{I} v_i$ is a uitary in $\mathbb{U}(B)$ and $\psi = \oplus_{i=1}^{I} \psi_{\sigma_i}$ is a $*$–automorphism in

$Aut(B)$ (the maps ψ depends on the family of $*$–isomorphisms $\beta(i,j)$ we fixed earlier). Equation (3.1) is telling us important information. Firstly, that ψ extends to an $*$–isomorphism of B_1 and most importantly, this extension is an inner $*$–automorphism. Fix a unitary U_ψ in $\mathbb{U}(B_1)$ such that $\psi(b) = AdU_\psi(b)$ for all b in B (note that U_ψ may not be unique but we just pick one and fix it for rest of the proof). From equation (3.1) we deduce there is a unitary w in $\mathbb{U}(B_1 \cap B')$ satisfying $u = vU_\psi w$. Since the number of functions ψ, that may arise from (3.1), is at most $J_1! \cdots J_I!$, we conclude

$$|\mathrm{Stab}(B_1, B)/G| \leq J_1! \cdots J_I!$$

Now that we know $\mathrm{Stab}(B_1, B)/G$ is finite we have $\dim \mathrm{Stab}(B_1, B) = \dim G$, and Corollary III.4 gives the result. □

From Proposition III.10 and Remark III.8, we get the following corollary.

Corollary III.11. *For any B_1 in $*$-SubAlg(M_N) and any B in $*$-SubAlg(B_1), we have*

$$\dim[B]_{B_1} = \dim \mathbb{U}(B_1) - \dim \mathbb{U}(B' \cap B_1) + \dim \mathbb{U}(C(B)) - \dim \mathbb{U}(B)$$

Now we focus our efforts on $Y(B_2; B)$.

Proposition III.12. *Assume $Y(B_2; B) \neq \emptyset$. Then $Y(B_2; B)$ is a finite disjoint union of embedded submanifolds of $\mathbb{U}(M_N)$. For each one of these submanifolds there is $u \in Y(B_2; B)$ such that the submanifold's dimension is*

$$\dim \mathrm{Stab}(M_N, B) + \dim \mathbb{U}(B_2) - \dim \mathrm{Stab}(B_2, u^* B u).$$

Using Proposition III.10 the later equals

$$\dim \mathbb{U}(B') + \dim \mathbb{U}(B_2) - \dim \mathbb{U}(B_2 \cap u^* B' u). \tag{3.2}$$

Proof. We'll define an action on $Y(B_2; B)$ which will partition $Y(B_2; B)$ into a finite number of orbits, each orbit an embedded submanifold of dimension (3.2) for a corresponding unitary. Define an action of $\mathrm{Stab}(M_N, B) \times \mathbb{U}(B_2)$ on $Y(B_2; B)$ via

$$(w, v).u = wuv^*.$$

For $u \in Y(B_2; B)$ let $\mathcal{O}(u)$ denote the orbit of u and let $\mathcal{O}$ denote the set of all orbits. To prove $\mathcal{O}$ is finite consider the function

$$\begin{aligned} \varphi : \mathcal{O} &\to *\text{-SubAlg}(B_2)/\sim_{B_2}, \\ \varphi(\mathcal{O}(u)) &= [u^* B u]_{B_2}. \end{aligned}$$

Firstly, we need to show φ is well defined. Assume $u_2 \in \mathcal{O}(u_1)$ and take $(w, v) \in \mathrm{Stab}(M_n, B) \times \mathbb{U}(B_2)$ such that $u_2 = wu_1v^*$. From the identities

$$u_2^* B u_2 = vu_1 w^* B w u_1 v^* = vu_1 B u_1 v^*$$

we obtain $[u_2 B u_2^*]_{B_2} = [u_1 B u_1^*]_{B_2}$. Hence φ is well defined.

The next step is to show φ is injective. Assume $\varphi(\mathcal{O}(u_1)) = \varphi(\mathcal{O}(u_2))$, for $u_1, u_2 \in Y(B_2; B)$. Since $[u_1^* B u_1]_{B_2} = [u_2^* B u_2]_{B_2}$, we have $u_2^* B u_2 = vu_1^* B u_1 v^*$ for some $v \in \mathbb{U}(B_2)$. But this implies $u_1 v^* u_2^* \in \mathrm{Stab}(M_N, B)$ so if $w = u_1 v^* u_2^*$ we conclude $(w, v).u_2 = u_1$ which yields $\mathcal{O}(u_1) = \mathcal{O}(u_2)$. We conclude $|\mathcal{O}| \leq |*\text{-SubAlg}(B_2)/\sim_{B_2}| < \infty$.

Now we prove each orbit is an embedded submanifold of $\mathbb{U}(M_N)$ of dimension (3.2). Since $\mathrm{Stab}(M_n, B) \times \mathbb{U}(B_2)$ is compact, every orbit $\mathcal{O}(u)$ is compact. Thus, Proposition III.3 implies $\mathcal{O}(u)$ is an embedded submanifold of $\mathbb{U}(M_N)$, diffeomorphic to

$$(\mathrm{Stab}(M_N, B) \times \mathbb{U}(B_2))/\mathrm{Stab}(u)$$

where

$$\mathrm{Stab}(u) = \{(w,v) \in \mathrm{Stab}(M_N, B) \times \mathbb{U}(B_2) : (w,v).u = u\}.$$

Since

$$(w,v).u = u \quad \Leftrightarrow \quad wuv^* = u \quad \Leftrightarrow \quad u^*wu = v,$$

we deduce the group $\mathrm{Stab}(u)$ is isomorphic to

$$\mathbb{U}(B_2) \cap [u^*\mathrm{Stab}(M_N, B)u],$$

via the map $(w,v) \mapsto v$. A straightforward computation shows

$$u^*\mathrm{Stab}(M_N, B)u = \mathrm{Stab}(M_N, u^*Bu),$$

for any $u \in \mathbb{U}(M_N)$. Hence, for any $u \in Y(B_2; B)$,

$$\dim \mathcal{O}(u) = \dim \mathrm{Stab}(M_N, B) + \mathbb{U}(B_2) - \dim \mathbb{U}(B_2) \cap \mathrm{Stab}(M_N, u^*Bu).$$

Lastly, one can check $\mathbb{U}(B_2) \cap \mathrm{Stab}(M_N, u^*Bu) = \mathrm{Stab}(B_2, u^*Bu)$. □

Notation III.13. For a unital C^*-subalgebra B of B_1, with the property that B is unitarily equivalent to a C^*-subalgebra of B_2, or in other words $Y(B_2; B)$ is nonempty, define

$$d(B) := \dim[B]_{B_1} + \max_i\{\dim Y_i(B_2; B)\},$$

where $Y_1(B_2, B), \ldots, Y_r(B_2; B)$ are disjoint submanifolds of $\mathbb{U}(M_N)$ whose union is $Y(B_2; B)$.

As we mention at the beginning of this section, in order to prove Theorem III.6, we need to parametrize each $Z(B_1, B_2; [B]_{B_1})$ with a number of coordinates less than N^2. The number of coordinates will be given by $d(B)$. Thus the next step is to show that, under the hypothesis of Theorem III.6, we have $d(B) < N^2$ for $B \neq \mathbb{C}$. We will later see that it suffices to show $d(B) < N^2$ for $B \neq \mathbb{C}$ and B abelian.

Before we proceed, we recall definition of multiplicity of of a representation. The following lemma combines Lemma III.2.1 in [5] and Theorem 11.9 in [14].

Lemma III.14. *Suppose $\varphi : A_1 \to A_2$ is a unital $*$-homomorphism and A_i is isomorphic to $\bigoplus_{j=1}^{l_i} M_{k_i(j)}$, $(i = 1, 2)$. Then φ is determined, up to unitary equivalence in A_2, by an $l_2 \times l_1$ matrix, written $\mu = \mu(\phi) = \mu(A_2, A_1)$, having nonnegative integer entries such that*

$$\mu \begin{bmatrix} k_1(1) \\ \vdots \\ k_1(l_1) \end{bmatrix} = \begin{bmatrix} k_2(1) \\ \vdots \\ k_2(l_2) \end{bmatrix}.$$

We call this the matrix of partial multiplicities. In the special case when φ is a unital $*$–representation of A_1 into M_N, μ is a row vector and this vector is called the multiplicity of the representation. One constructs μ as follows: decompose A_p as

$$A_p = \oplus_{j=1}^{l_p} A_p(j)$$

where each $A_p(j)$ is simple, $p = 1, 2$, $1 \leq j \leq l_p$. Taking projections, π induces unital $*$–representations $\pi_i : A_1 \to A_2(i)$, $1 \leq i \leq l_2$. But up to unitary equivalence, π_i equals

$$\underbrace{\mathrm{id}_{A_1(1)} \oplus \cdots \oplus \mathrm{id}_{A_1(1)}}_{m_{i,1}-times} \oplus \cdots \oplus \underbrace{\mathrm{id}_{A_1(l_1)} \oplus \cdots \oplus \mathrm{id}_{A_1(l_1)}}_{m_{i,l_1}-times}$$

for some nonnegative integer $m_{i,j}$, $1 \leq j \leq l_1$. Set $\mu[i,j] := m_{i,j}$. In particular, $\mu[i,j]$ equals the rank of $\pi_i(p) \in A_2(i)$, where p is a minimal projection in $A_1(j)$. Clearly, π is injective if and only if for all j there is i such that $\mu[i,j] \neq 0$.

Furthermore, the C^*-subalgebra

$$A_2 \cap \varphi(A_1)' = \{x \in A_2 : x\varphi(a) = \varphi(a)x \quad \text{for all } a \in A_1\}$$

is $*$–isomorphic to $\bigoplus_{i=1}^{l_2}\bigoplus_{j=1}^{l_1} M_{\mu[i,j]}$ *and if we have morphisms* $A_1 \to A_2 \to A_3$*, then* $\mu(A_3, A_2)\mu(A_2, A_1) = \mu(A_3, A_1)$ *for the corresponding matrices.*

Our next task is to show $d(B) < N^2$, for abelian $B \neq C$. We prove it by cases, so let us start.

Lemma III.15. *Assume* B_i *is $*$–isomorphic to* M_{k_i}*,* ($i = 1, 2$) *and let* $k = \gcd(k_1, k_2)$*. Take* B *a unital* C^**-subalgebra of* B_1 *such that it is unitarily equivalent to a* C^**-subalgebra of* B_2*. Then there is an injective unital $*$–representation of* B *into* M_k*.*

Proof. Take u in $Y(B_2; B)$ so that $u^*Bu \subseteq B_2$. Let $m_i := \mu(M_N, B_i)$, so that $m_ik_i = N$, $(i = 1, 2)$. Find positive integers p_1 and p_2 such that $k_1 = kp_1$ and $k_2 = kp_2$ Assume B is $*$–isomorphic to $\bigoplus_{j=1}^{l} M_{n_j}$. To prove the result it is enough to show there are positive integers $(m(1), \ldots m(l))$ such that

$$n_1m(1) + \cdots + n_lm(l) = k.$$

Let

$$\mu(B_1, B) = [m_1(1), \ldots, m_1(l)],$$
$$\mu(B_2, u^*Bu) = [m_2(1), \ldots, m_2(l)].$$

Since $\mu(M_N, B_1)\mu(B_1, B) = \mu(M_N, B_2)\mu(B_2, u^*Bu)$ we deduce that $m_1m_1(j) = m_2m_2(j)$ for all $1 \leq j \leq l$. Multiplying by k and using $N = m_1k_1 = m_2k_2$ we conclude

$$\frac{N}{p_1}m_1(j) = km_1m_1(j) = km_2m_2(j) = \frac{N}{p_2}m_2(j),$$

so $p_2m_1(j) = p_1m_2(j)$. Since $\gcd(p_1, p_2) = 1$, the number $\frac{m_1(j)}{p_1} = \frac{m_2(j)}{p_2}$ is a positive integer whose value we name $m(j)$. From

$$kp_1 = k_1 = \sum_{j=1}^{l} n_jm_1(j) = \sum_{j=1}^{l} n_jm(j)p_1,$$

we conclude $k = \sum_{j=1}^{l} n_j m(j)$. □

Proposition III.16. *Assume B_1 and B_2 are simple. Take $B \neq \mathbb{C}$ an abelian unital C^*-subalgebra of B_1, that is unitarily equivalent to a C^*-subalgebra of B_2. Then $d(B) < N^2$.*

Proof. Assume B_i is $*$–isomorphic to M_{k_i}, $(i = 1, 2)$ and B is $*$–isomorphic to $\mathbb{C}^l$, $l \geq 2$. Using Corollary III.11 and Proposition III.12, we may take u in $Y(B_2, B)$ such that $d(B)$ equals the sum of the following terms,

$$\begin{aligned} S_1(B) &:= \dim \mathbb{U}(B_1) - \dim \mathbb{U}(B_1 \cap B'), \\ S_2(B) &:= \dim \mathbb{U}(B_2) - \dim \mathbb{U}(B_2 \cap u^* B' u), \\ S_3(B) &:= \dim \mathbb{U}(B'), \end{aligned}$$

Let $k = \gcd(k_1, k_2)$ and write $k_1 = kp_1$, $k_2 = kp_2$. From proof of Lemma III.15, there are positive integers $m(j)$, $1 \leq j \leq l$, such that

$$\mu(B_1, B) = [m(1)p_1, \ldots, m(l)p_1]$$
$$\mu(B_2, B) = [m(1)p_2, \ldots, m(l)p_2].$$

Hence

$$S_1(B) = k_1^2 - \sum_{i=1}^{l} m(i)^2 p_1^2 = k^2 p_1^2 - \sum_{i=1}^{l} m(i)^2 p_1^2$$
$$S_2(B) = k_2^2 - \sum_{i=1}^{l} m(i)^2 p_2^2 = k^2 p_2^2 - \sum_{i=1}^{l} m(i)^2 p_2^2.$$

Let $m_i = \mu(M_N, B_i)$, $(i = 1, 2)$. Since

$$\mu(M_N, B_1)\mu(B_1, B) = \mu(M_N, B_2)\mu(B_2, u^* B u),$$

we get

$$\mu(M_N, B) = [m_1p_1m(1), \ldots, m_1p_1m(l)] \qquad (3.3)$$
$$= [m_2p_2m(1), \ldots, m_2p_2m(l)].$$

Hence

$$S_3(B) = \sum_{i=1}^{l}(m(i)p_1m_1)(m(i)p_2m_2) = \left(\sum_{i=1}^{l} m(i)^2\right) p_1p_2m_1m_2.$$

Factoring the term $\sum_{i=1}^{l} m(i)^2$ we get $d(B)$ equals

$$\left(\sum_{i=1}^{l} m(i)^2\right)\left(p_1p_2m_1m_2 - p_1^2 - p_2^2\right) + k^2(p_1^2 + p_2^2).$$

On the other hand, using $N = m_1k_1 = m_1kp_1 = m_2k_2 = m_2kp_2$, we get $N^2 = k^2p_1p_2m_1m_2$. Hence $d(B) < N^2$ if and only if

$$\left(\sum_{i=1}^{l} m(i)^2\right)\left(p_1p_2m_1m_2 - p_1^2 - p_2^2\right) < k^2(p_1p_2m_1m_2 - p_1^2 - p_2^2). \qquad (3.4)$$

We want to cancel $(p_1p_2m_1m_2 - p_1^2 - p_2^2)$, in equation (3.4), so we prove it is positive. First we divide it by p_1p_2 to get $m_1m_2 - \frac{p_1}{p_2} - \frac{p_2}{p_1}$. But from equation (3.3) we have $\frac{p_1}{p_2} = \frac{m_2}{m_1}$. Thus we need to show $m_1m_2 - \frac{m_1}{m_2} - \frac{m_2}{m_1}$ is positive. If we divide it by m_1m_2 we get $1 - \frac{1}{m_1^2} - \frac{1}{m_2^2}$, which is clearly positive (recall that $m_1 \geq 2$ and $m_2 \geq 2$ since $B_1 \neq M_N$ and $B_2 \neq M_N$). Therefore, equation (3.4) is equivalent to

$$\sum_{i=1}^{l} m(i)^2 < k^2.$$

But $\sum_{i=1}^{l} m(i) = k$, $l \geq 2$ and each $m(i)$ is positive. □

In the nonsimple case in Theorem III.6, we will need some minimization lemmas to show $d(B) < N^2$, for abelian $B \neq \mathbb{C}$. A straightfroward use of Lagrange multipliers proves the following lemma, and the one after that is even more elementary.

Lemma III.17. *Fix a positive integer n and let $r_1, \ldots, r_n$ be positive real numbers.*

Then

$$\min\left\{\sum_{j=1}^{n}\frac{x_j^2}{r_j}\;\middle|\;\sum_{j=1}^{n}x_j=1\right\}=\frac{1}{\sum_{j=1}^{n}r_j},$$

where the minimum is taken over all n-tuples of real numbers that sum up to 1.

Lemma III.18. *For an integer $k \geq 2$ define*

$$h(x,y)=2xy-\left(1+\frac{1}{k^2}\right)y^2-\frac{1}{2}x^2.$$

Then

$$\max\{h(x,y)\mid 0\leq x\leq 1,\, 0\leq y\leq 1/2\}=\frac{1}{4}-\frac{1}{4k^2}.$$

Proposition III.19. *Suppose $\dim C(B_1) \geq 2$ and B_1 is $*$–isomorphic to*

$$M_{N/\dim C(B_1)}\oplus\cdots\oplus M_{N/\dim C(B_1)}. \tag{3.5}$$

Assume one of the following cases holds:

1. $\dim C(B_2) = 1$,
2. B_1 *is $*$–isomorphic to*

   $$M_{N/2}\oplus M_{N/2},$$

   B_2 *is $*$–isomorphic to*

   $$M_{N/2}\oplus M_{N/(2k)}$$

   where $k \geq 2$.
3. $\dim C(B_2) \geq 3$ *and B_2 is $*$–isomorphic to*

   $$M_{N/\dim C(B_2)}\oplus\cdots\oplus M_{N/\dim C(B_2)}.$$

Then for any $B \neq \mathbb{C}$ an abelian unital C^*-subalgebra of B_1 that is unitarily equivalent to a C^*-subalgebra of B_2, we have that $d(B) < N^2$.

Proof. Let $l_i = \dim C(B_i)$, $(i = 1, 2)$, $l = \dim(B)$. Take u in $Y(B_2; B)$ such that $d(B)$ is the sum of the following terms:

$$S_1(B) := \dim \mathbb{U}(B_1) - \dim \mathbb{U}(B_1 \cap B'), \tag{3.6}$$

$$S_2(B) := \dim \mathbb{U}(B_2) - \dim \mathbb{U}(B_2 \cap u^* B' u), \tag{3.7}$$

$$S_3(B) := \dim \mathbb{U}(B'). \tag{3.8}$$

Write

$$\begin{aligned}
\mu(B_1, B) &= [a_{i,j}]_{1 \le i \le l_1, 1 \le j \le l}, \\
\mu(B_2, u^* B u) &= [b_{i,j}]_{1 \le i \le l_2, 1 \le j \le l}, \\
\mu(M_N, B_1) &= [m_1(1), \ldots, m_1(l_1)], \\
\mu(M_N, B_2) &= [m_2(1), \ldots, m_2(l_2)], \\
\mu(M_N, B) &= [m(1), \ldots, m(l)].
\end{aligned}$$

Then

$$\begin{aligned}
S_1(B) &= \frac{N^2}{l_1} - \sum_{i=1}^{l} \sum_{j=1}^{l_1} a_{i,j}^2, \\
S_2(B) &= \dim \mathbb{U}(B_2) - \sum_{i=1}^{l} \sum_{j=1}^{l_2} b_{i,j}^2, \\
S_3(B) &= \sum_{j=1}^{l} m(j)^2.
\end{aligned}$$

Since the sum of the ranks appearing in (3.5) is N, we have $m_1(i) = 1$ for all $1 \le i \le l_1$. Since

$$\mu(M_N, B) = \mu(M_N, B_1)\mu(B_1, B) = \mu(M_N, B_2)\mu(B_2, u^* B u),$$

we must have

$$m(j) = \sum_{i=1}^{l_1} a_{i,j} = \sum_{i=1}^{l_2} m_2(i) b_{i,j}$$

for all $1 \leq j \leq l$. Hence there are nonnegative numbers $\alpha_{i,j}$ and $\beta_{i,j}$ such that $\sum_{i=1}^{l_1} \alpha_{i,j} = \sum_{i=1}^{l_2} \beta_{i,j} = 1$ and $a_{i,j} = \alpha_{i,j} m(j)$, $m_2(i) b_{i,j} = \beta_{i,j} m(j)$. On the other hand, since B is a unital C^*-subalgebra of M_N we must have

$$\sum_{j=1}^{l} m(j) = N.$$

Thus, there are positive numbers γ_j, $(1 \leq j \leq l)$, such that $\sum_{j=1}^{l} \gamma_j = 1$ and $m(j) = \gamma_j N$. It will be important to notice that $\gamma_j > 0$ for all $1 \leq j \leq l$ (otherwise B is not a unital C^*-algebra of M_N). In consequence,

$$\begin{aligned}
S_1(B) &= \frac{N^2}{l_1} - N^2 \Bigg(\sum_{j=1}^{l} \gamma_j^2 \Bigg(\sum_{i=1}^{l_1} \alpha_{i,j}^2 \Bigg) \Bigg), \\
S_2(B) &= \dim \mathbb{U}(B_2) - N^2 \Bigg(\sum_{j=1}^{l} \gamma_j^2 \Bigg(\sum_{i=1}^{l_2} \frac{\beta_{i,j}^2}{m_2(i)^2} \Bigg) \Bigg), \\
S_3(B) &= N^2 \Bigg(\sum_{j=1}^{l} \gamma_j^2 \Bigg).
\end{aligned}$$

Case (1). B_2 is simple, let us say it is $*$–isomorphic to M_{k_2}. In this case $\mu(M_N, B_2) = [m_2]$ is just one number and we must have $m_2 k_2 = N$. Notice that $m_2 \geq 2$, since by our standing assumption, $B_2 \neq M_N$. Also notice that from $\mu(M_N, B_2)\mu(B_2, u^* B u) = \mu(M_N, B)$ we obtain $m_2 b_{i,1} = m(i)$ and $\beta_{i,1} = 1$ for all $1 \leq i \leq l$. In consequence

$$\begin{aligned}
S_1(B) &= \frac{N^2}{l_1} - N^2 \Bigg(\sum_{j=1}^{l} \gamma_j^2 \Bigg(\sum_{i=1}^{l_1} \alpha_{i,j}^2 \Bigg) \Bigg), \\
S_2(B) &= \frac{N^2}{m_2^2} - \frac{N^2}{m_2^2} \Bigg(\sum_{j=1}^{l} \gamma_j^2 \Bigg), \\
S_3(B) &= N^2 \Bigg(\sum_{j=1}^{l} \gamma_j^2 \Bigg).
\end{aligned}$$

From Lemma III.17, we deduce

$$S_1(B) \leq \frac{N^2}{l_1} - \frac{N^2}{l_1}\Big(\sum_{j=1}^{l} \gamma_j^2\Big).$$

Thus, it suffices to show

$$N^2\Big(\frac{1}{l_1} + \frac{1}{m_2^2} + \sum_{j=1}^{l}\gamma_j^2 - \frac{1}{l_1}\Big(\sum_{j=1}^{l}\gamma_j^2\Big) - \frac{1}{m_2^2}\Big(\sum_{j=1}^{l}\gamma_j^2\Big)\Big) < N^2$$

or equivalently

$$\Big(\sum_{j=1}^{l}\gamma_j^2\Big)\Big(1 - \frac{1}{l_1} - \frac{1}{m_2^2}\Big) < 1 - \frac{1}{l_1} - \frac{1}{m_2^2}.$$

Since $l_1 \geq 2$ and $m_2 \geq 2$ we can cancel the term $1 - \frac{1}{l_1} - \frac{1}{m_2^2}$. Thus we need to show $\sum_{j=1}^{l} \gamma_j^2 < 1$. But the latter follows from the fact that $l \geq 2$, each γ_j is positive and $\sum_{j=1}^{l} \gamma_j = 1$.

Case (2). We have

$$\mu(M_N, B_1) = [1, 1],$$

$$\mu(M_N, B_2) = [1, k].$$

Thus

$$S_1(B) = \frac{N^2}{2} - N^2\Big(\sum_{j=1}^{l}\gamma_j^2\Big(\alpha_{1,j}^2 + \alpha_{2,j}^2\Big)\Big),$$

$$S_2(B) = \frac{N^2}{4} + \frac{N^2}{4k^2} - N^2\Big(\sum_{j=1}^{l}\gamma_j^2\Big(\beta_{1,j}^2 + \frac{\beta_{2,j}^2}{k^2}\Big)\Big),$$

$$S_3(B) = N^2\Big(\sum_{j=1}^{l}\gamma_j^2\Big).$$

From Lemma III.17 we obtain

$$S_1(B) \quad \leq \quad \frac{N^2}{2} - \frac{N^2}{2}\left(\sum_{j=1}^{l} \gamma_j^2\right).$$

Thus, it suffices to show

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{4k^2} + \sum_{j=1}^{l} \gamma_j^2 \left(\frac{1}{2} - \beta_{1,j}^2 - \frac{1}{k^2}\beta_{2,j}^2\right) < 1$$

or, equivalently,

$$\sum_{j=1}^{l} \gamma_j^2 \left(\frac{1}{2} - \beta_{1,j}^2 - \frac{1}{k^2}\beta_{2,j}^2\right) < \frac{1}{4} - \frac{1}{4k^2}.$$

Define

$$r = \sum_{j=1}^{l} \gamma_j^2 \left(\frac{1}{2} - \beta_{1,j}^2 - \frac{1}{k^2}\beta_{2,j}^2\right). \tag{3.9}$$

Now we use the constraints on the variables γ_j and $\beta_{i,j}$. First of all we have $\beta_{1,j}+\beta_{2,j} = 1$ for all $1 \leq i \leq l$. Thus, r simplifies to

$$r = \sum_{j=1}^{l} \gamma_j^2 \left(2\beta_{2,j} - \left(1 + \frac{1}{k^2}\right)\beta_{2,j}^2 - \frac{1}{2}\right).$$

We also have

$$\sum_{j=1}^{l} \beta_{2,j}\gamma_j \quad = \quad \frac{1}{2}. \tag{3.10}$$

Indeed, since all blocks of B are one dimensional, we must have

$$\sum_{j=1}^{l} b_{2,j} = \frac{N}{2k}.$$

But $kb_{2,j} = \beta_{2,j}m(j) = \beta_{2,j}\gamma_j N$, which implies (3.10). The final constraint is $\sum_{j=1} \gamma_j = 1$.

Now we make the change of variables $q_j := \gamma_j\beta_{2,j}$ and r becomes

$$r = 2\left(\sum_{j=1}^{l} q_j\gamma_j\right) - \left(1+\frac{1}{k^2}\right)\left(\sum_{j=1}^{l} q_j^2\right) - \frac{1}{2}\left(\sum_{j=1}^{l} \gamma_j^2\right).$$

Letting $\gamma = (\gamma_1, \dots, \gamma_l)$ and $q = (q_1, \dots, q_l)$ and using the Cauchy-Schwartz inequality, we get

$$r \leq 2\|q\|_2\|\gamma\|_2 - \left(1+\frac{1}{k^2}\right)\|q\|_2^2 - \frac{1}{2}\|\gamma\|_2^2$$

Set $x = \|\gamma\|$, $y = \|q\|$. Notice that $0 \leq x \leq 1$ and $0 \leq y \leq 1/2$. Take

$$h(x,y) = 2xy - \left(1+\frac{1}{k^2}\right)y^2 - \frac{1}{2}y^2$$

apply Lemma III.18 to get

$$r \leq h(\|\gamma\|, \|q\|) \leq \frac{1}{4} - \frac{1}{4k^2}.$$

Now we will rule out equality. Assuming, for contradiction, $r = \frac{1}{4} - \frac{1}{4k^2}$, we must have equality in the instince of the Cauchy-Schwartz inequality. Hence $q = z\gamma$ for some real number z. Summing over the coordinates we deduce $z = 1/2$ and then, for all $1 \leq j \leq l$,

$$\frac{1}{2}\gamma_j = q_j = \gamma_j\beta_{2,j}.$$

Since $\gamma_j > 0$ we can cancel and get $\beta_{2,j} = 1/2$. Thus, using the original formulation (3.9) of r, we get

$$r = \left(\frac{1}{4} - \frac{1}{4k^2}\right)\left(\sum_{j=1}^{l} \gamma_j^2\right)$$

which is strictly less that $1/4 - 1/(4k^2)$, because $k \geq 2$, $l \geq 2$, all γ_j are strictly positive and $\sum_{j=1}^{l} \gamma_j = 1$.

Case (3). Then B_2 is $*$–isomorphic to

$$\underbrace{M_{N/l_2} \oplus \cdots \oplus M_{N/l_2}}_{l_2-\text{times}}.$$

Arguing as we did before for $m_1(i)$, we have $m_2(i) = 1$, for all $1 \leq i \leq l_2$. Hence

$$\begin{aligned} S_1(B) &= \frac{N^2}{l_1} - N^2\Big(\sum_{j=1}^{l} \gamma_j^2 \Big(\sum_{i=1}^{l_1} \alpha_{i,j}^2\Big)\Big), \\ S_2(B) &= \frac{N^2}{l_2} - N^2\Big(\sum_{j=1}^{l} \gamma_j^2 \Big(\sum_{i=1}^{l_2} \beta_{i,j}^2\Big)\Big), \\ S_3(B) &= N^2\Big(\sum_{j=1}^{l} \gamma_j^2\Big). \end{aligned}$$

From Lemma III.17 we deduce

$$\begin{aligned} S_1(B) &\leq \frac{N^2}{l_1} - \frac{N^2}{l_1}\Big(\sum_{j=1}^{l} \gamma_j^2\Big), \\ S_2(B) &\leq \frac{N^2}{l_2} - \frac{N^2}{l_2}\Big(\sum_{j=1}^{l} \gamma_j^2\Big). \end{aligned}$$

Thus, it suffices to show

$$N^2\Big(\frac{1}{l_1} + \frac{1}{l_2} + \sum_{j=1}^{l} \gamma_j^2 - \frac{1}{l_1}\Big(\sum_{j=1}^{l} \gamma_j^2\Big) - \frac{1}{l_2}\Big(\sum_{j=1}^{l} \gamma_j^2\Big)\Big) < N^2$$

or equivalently

$$\Big(\sum_{j=1}^{l} \gamma_j^2\Big)\Big(1 - \frac{1}{l_1} - \frac{1}{l_2}\Big) < 1 - \frac{1}{l_1} - \frac{1}{l_2}.$$

Since $l_1 \geq 2$ and $l_2 \geq 3$ we can cancel the term $1 - \frac{1}{l_1} - \frac{1}{l_2}$ in the above equation and finish the proof as in the previous case. □

The next step is to find parameterizations of $Z(B_1, B_2; [B]_{B_1})$.

Lemma III.20. *Take $B \neq \mathbb{C}$ a unital C^*-subalgebra of B_1 that is unitarily equivalent to a C^*-subalgebra of B_2. If $\dim \mathbb{U}(B_1) + \dim \mathbb{U}(B_2) \leq N^2$, B is simple and C in*

$*$-SubAlg(B) *is* $*$*–isomorphic to* $\mathbb{C}^2$*, then* $d(B) \leq d(C)$*.*

Proof. Assume B is $*$–isomorphic to M_k and let m denote the multiplicity of B in M_N. Thus we must have $km = N$. Take a unitary u in the submanifold of maximum dimension in $Y(B_2; B)$, so that $d(B)$ is the sum of the terms

$$\begin{aligned} S_1(B) &:= \dim \mathbb{U}(B_1) - \dim \mathbb{U}(B_1 \cap B'), \\ S_2(B) &:= \dim \mathbb{U}(B_2) - \dim \mathbb{U}(B_2 \cap u^* B' u), \\ S_3(B) &:= \dim \mathbb{U}(B'), \\ S_4(B) &:= \dim \mathbb{U}(B \cap B') - \dim \mathbb{U}(B). \end{aligned}$$

and let v lie in the submanifold of maximum dimension in $Y(B_2, C)$ so that $d(C)$ is the sum of the terms

$$\begin{aligned} S_1(C) &:= \dim \mathbb{U}(B_1) - \dim \mathbb{U}(B_1 \cap C'), \\ S_2(C) &:= \dim \mathbb{U}(B_2) - \dim \mathbb{U}(B_2 \cap v^* C' v), \\ S_3(C) &:= \dim \mathbb{U}(C'). \end{aligned}$$

Clearly, $S_4(B) = 1 - k^2$. We write

$$\begin{aligned} B_1 &\simeq \bigoplus_{i=1}^{l_1} M_{k_1(i)}, \\ B_2 &\simeq \bigoplus_{i=1}^{l_2} M_{k_2(i)}. \end{aligned}$$

and

$$\begin{aligned} \delta(B_1) &= [k_1(1), \ldots, k_1(l_1)]^t, \\ \delta(B_2) &= [k_2(1), \ldots, k_2(l_2)]^t. \end{aligned}$$

From definition of multiplicity and the fact that it is invariant under unitary equivalence we get

$$\begin{aligned}\mu(B_1, B)k &= \delta(B_1), \\ \mu(B_2, u^*Bu)k &= \delta(B_2), \\ \mu(M_N, B_1)\delta(B_1) &= \mu(M_N, B_2)\delta(B_2) = N, \\ \mu(M_N, B_1)\mu(B_1, B) &= \mu(M_N, B_2)\mu(B_2, u^*Bu) = m.\end{aligned} \tag{3.11}$$

From Lemma III.14 and equation (3.11) we get

$$\dim \mathbb{U}(B_1 \cap B') = \frac{1}{k^2} \dim \mathbb{U}(B_1). \tag{3.12}$$

Hence

$$S_1(B) = \left(1 - \frac{1}{k^2}\right) \dim \mathbb{U}(B_1).$$

Similarly

$$S_2(B) = \left(1 - \frac{1}{k^2}\right) \dim \mathbb{U}(B_2).$$

Now it is the turn of C. To ease notation let

$$\mu(B, C) = [x_1, x_2]$$

Notice that $x_1 + x_2 = k$. We claim

$$S_1(C) = \left(1 - \frac{x_1^2 + x_2^2}{k^2}\right) \dim \mathbb{U}(B_1).$$

Using $\mu(B_1, C) = \mu(B_1, B)\mu(B, C)$ we get

$$\dim \mathbb{U}(B_1 \cap C') = (x_1^2 + x_2^2) \dim \mathbb{U}(B_1 \cap B').$$

Furthermore using (3.12) we obtain

$$\dim \mathbb{U}(B_1 \cap C') = \frac{x_1^2 + x_2^2}{k^2} \dim \mathbb{U}(B_1).$$

Hence our claim follows from definition of $S_1(C)$. Similarly

$$S_2(C) = \left(1 - \frac{x_1^2 + x_2^2}{k^2}\right) \dim \mathbb{U}(B_2).$$

Lastly from $\mu(M_N, C) = [mx_1, mx_2]$ and $mk = N$ we get

$$\begin{aligned} S_3(C) &= (x_1^2 + x_2^2)\frac{N^2}{k^2}, \\ S_3(B) &= \frac{N^2}{k^2}. \end{aligned}$$

To prove $d(B) \leq d(C)$ we'll show

$$S_1(B) - S_1(C) + S_2(B) - S_2(C) + S_4(B) \leq S_3(C) - S_3(B). \tag{3.13}$$

Using the description of each summand we have that left hand side of (3.13) equals

$$\frac{x_1^2 + x_2^2 - 1}{k^2}\left(\dim \mathbb{U}(B_1) + \dim \mathbb{U}(B_2)\right) + 1 - k^2.$$

The right hand side of (3.13) equals

$$\frac{x_1^2 + x_2^2 - 1}{k^2} N^2.$$

But x_1 and x_2 are strictly positive, because C is a unital subalgebra of B. Hence we can cancel $x_1^2 + x_2^2 - 1$ and finish the proof by using that $1 - \delta(B)^2 < 0$ and the assumption $\dim \mathbb{U}(B_1) + \dim \mathbb{U}(B_2) \leq N^2$. □

We recall an important perturbation result that can be found in Lemma III.3.2 from [5].

Lemma III.21. *Let A be a finite dimensional C^*-algebra. Given any positive num-*

ber ε there is a positive number $\delta = \delta(\varepsilon)$ so that whenever B and C are unital C^*-subalgebras of A and such that C has a system of matrix units $\{e_C(s,i,j)\}_{s,i,j}$, satisfying $\operatorname{dist}(e_C(s,i,j), B) < \delta$ *for all s, i and j, then there is a unitary u in $\mathbb{U}(C^*(B,C))$ with $\|u-1\| < \varepsilon$ so that $uCu^* \subseteq B$.*

Notation III.22. For an element x in M_N and a positive number ε, $\mathcal{N}_\varepsilon(x)$ denotes the open ε-neighborhood around x (i.e. open ball of radius ε centered at x), where the distance is from the operator norm in M_N.

The next proposition is quite technical and is mainly a consequence of Lemma III.21. The set $[B]_{B_1}$ is endowed with the equivalent topologies described in Remark III.8.

Lemma III.23. *Take B in* $*$-SubAlg(B_1) *and assume $Z(B_1, B_2; [B]_{B_1})$ is nonempty. Then the function*

$$\begin{aligned} Z(B_1, B_2; [B]_{B_1}) &\rightarrow [B]_{B_1} \\ u &\mapsto uB_2u^* \cap B_1 \end{aligned} \tag{3.14}$$

is continuous.

Proof. Assume B is $*$–isomorphic to

$$\bigoplus_{s=1}^{l} M_{k_s}.$$

First we recall that the topology of $[B]_{B_1}$ is induced by the bijection

$$\begin{aligned} \beta : [B]_{B_1} &\rightarrow \mathbb{U}(B_1)/\mathrm{Stab}(B_1, B), \\ \beta(uBu^*) &= u\mathrm{Stab}(B_1, B). \end{aligned}$$

For convenience let $\pi : \mathbb{U}(B_1) \rightarrow \mathbb{U}(B_1)/\mathrm{Stab}(B_1, B)$ denote the canonical quotient map. Pick u_0 in $Z(B_1, B_2; [B]_{B_1})$. With no loss of generality we may assume $B = u_0 B_2 u_0^* \cap B_1$.

We prove the result by contradiction. Suppose the function in (3.14) is not continuous at u_0. Then there is a sequence $(u_k)_{k\geq 1} \subset Z(B_1, B_2, [B]_{B_1})$ and an open neighborhood $\mathcal{N}$ of B in $[B]_{B_1}$ such that

1. $\lim_k u_k = u_0$,
2. for all k, $u_k B_2 u_k^* \cap B_1 \notin \mathcal{N}$.

On the other hand, let $\varepsilon > 0$ be such that $\pi(\mathcal{N}_\varepsilon(1_{B_1})) \subseteq \beta(\mathcal{N})$. Let $\{e_k(s,i,j)\}_{1\leq s\leq l, 1\leq i,j\leq k_s}$ denote a system of matrix units for $u_k B_2 u_k^* \cap B_1$. Fix elements $f_k(s,i,j)$ in B_2 such that $e_k(s,i,j) = u_k f_k(s,i,j) u_k^*$. Since B_2 is finite dimensional, passing to a subsequence if necessary, we may assume that $\lim_k f_k(s,i,j) = f(s,i,j)$, for all s, i and j. Using property (1) of the sequence $(u_k)_{k\geq 1}$, we deduce

$$\lim_k e_k(s,i,j) = \lim_k u_k f_k(s,i,j) u_k^* = u_0 f(s,i,j) u_0^*.$$

Hence the element $e(s,i,j) = u_0 f(s,i,j) u^*$ belongs to $u_0 B_1 u_0^* \cap B_1 = B$. Use Lemma III.21 and take δ_1 positive such that whenever C is a subalgebra in $*$-SubAlg(B_1) having a system of matrix units $\{e_C(s,i,j)\}_{s,i,j}$ satisfying $\mathrm{dist}(e_C(s,i,j), B) < \delta_1$, for all s, i and j, then there is a unitary Q in $\mathbb{U}(B_1)$ such that $\|Q - 1_{B_1}\| < \varepsilon$ and $QCQ^* \subseteq B$. Take k such that $\|e_k(s,i,j) - e(s,i,j)\| < \delta_1$ for all s, i and j. This implies $\mathrm{dist}(e_k(s,i,j), B) < \delta_1$ for all s, i and j. We conclude there is a unitary Q in $\mathbb{U}(B_1)$ such that $\|Q - 1_{B_1}\| < \varepsilon$ and $Q^*(u_k B_2 u_k^* \cap B_1)Q \subseteq B$. But

$$\dim B = \dim u_k B_2 u_k^* \cap B_1 = \dim Q^*(u_k B_2 u_k^* \cap B_1)Q,$$

where in the first equality we used that u_k lies in $Z(B_1, B_2; [B]_{B_1})$. Hence $Q^*(u_k B_2 u_k^* \cap$

$B_1)Q = B$. As a consequence,

$$\beta(u_k B_2 u_k^* \cap B_1) = \beta(QBQ^*) = \pi(Q) \in \beta(\mathcal{N}).$$

But the latter contradicts property (2) of $(u_k)_{k\geq 1}$. □

Lemma III.24. *For B in $*$-SubAlg(B), the function $c : [B]_{B_1} \to [C(B)]_{B_1}$ given by $c(uBu^*) = uC(B)u^*$ is continuous.*

Proof. First, we must show the function c is well defined. In other words we have to show $\mathrm{Stab}(B_1, B) \subseteq \mathrm{Stab}(B_1, C(B))$. But this follows directly from the fact that any u in $\mathrm{Stab}(B_1, B)$ defines a $*$–automorphism of B and any $*$–automorphism leaves the center fixed. Since $[B]_{B_1}$ and $[C(B)]_{B_1}$ are homeomorphic to $\mathbb{U}(B_1)/\mathrm{Stab}(B_1, B)$ and $\mathbb{U}(B_1)/\mathrm{Stab}(B_1, C(B))$ respectively, it follows that c is continuous if and only if the function $\tilde{c} : \mathbb{U}(B_1)/\mathrm{Stab}(B_1, B) \to \mathbb{U}(B_1)/\mathrm{Stab}(B_1, C(B))$ given by $\tilde{c}(u\mathrm{Stab}(B_1, B)) = u\mathrm{Stab}(B_1, C(B))$ is continuous. But the spaces $\mathbb{U}(B_1)/\mathrm{Stab}(B_1, B)$ and $\mathbb{U}(B_1)/\mathrm{Stab}(B_1, C(B))$ have the quotient topology induced by the canonical projections

$$\pi_B : \mathbb{U}(B_1) \to \mathrm{Stab}(B_1, B), \quad \pi_{C(B)} : \mathbb{U}(B_1) \to \mathbb{U}(B_1)/\mathrm{Stab}(B_1, C(B)).$$

Thus $\tilde{c}$ is continuous if and only if $\pi_B \circ \tilde{c}$ is continuous. But $\pi_B \circ \tilde{c} = \pi_{C(B)}$, which is indeed continuous. □

We are ready to find local parameterizations of $Z(B_1, B_2; [B]_{B_1})$.

Proposition III.25. *Take B a unital C^*-subalgebra in B_1 that is unitarily equivalent to a C^*-subalgebra of B_2. Fix an element u_0 in $Z(B_1, B_2; [B]_{B_1})$. Then there is a positive number r and a continuous injective function*

$$\Psi : \mathcal{N}_r(u_0) \cap Z(B_1, B_2; [B]_{B_1}) \to \mathbb{R}^{d(C(B))}.$$

Proof. Using that $Z(B_1, B_2; [B]_{B_1}) = Z(B_1, B_2, [u_0B_2u_0^* \cap B_1]_{B_1})$, with no loss of generality we may assume $u_0B_2u_0^* \cap B_1 = B$. Now, we use the manifold structure of $[C(B)]_{B_1}$ and $Y(B_2; C(B))$ to construct Ψ. Note that if $Y(B_2, B)$ is nonempty then $Y(B_2, C(B))$ is nonempty as well. Let d_1 denote the dimension of $[C(B)]_{B_1}$ and let d_2 denote the dimension of the submanifold of $Y(B_2; C(B))$ that contains u_0. Of course, we have $d_1 + d_2 \leq d(C(B))$.

We use the local cross section result from previous section to parametrize $[C(B)]_{B_1}$. To ease notation take $G = \mathbb{U}(B_1)$, $H = \mathrm{Stab}(B_1, C(B))$ and let π denote the canonical quotient map from G onto the left-cosets of H. By Proposition III.5 there are

1. $\mathcal{N}_G$, a compact neighborhood of 1 in G,
2. $\mathcal{N}_H$, a compact neighborhood of 1 in H,
3. $\mathcal{N}_{G/H}$, a compact neighborhood of $\pi(1)$ in G/H,
4. a continuous function $s : \mathcal{N}_{G/H} \to \mathcal{N}_G$ satisfying
   (a) $s(\pi(1)) = 1$ and $\pi(s(\pi(g))) = \pi(g)$ whenever $\pi(g)$ lies in $\mathcal{N}_{G/H}$,
   (b) the function

$$\begin{aligned} \mathcal{N}_H \times \mathcal{N}_{G/H} &\to \mathcal{N}_G, \\ (h, \pi(g)) &\mapsto hs(\pi(g)), \end{aligned}$$

   is an homeomorphism.

Since G/H is a manifold of dimension d_1, we may assume there is a continuous injective map $\Psi_1 : \mathcal{N}_{G/H} \to \mathbb{R}^{d_1}$.

Parametrizing $Y(B_2; C(B))$ is easier. Since $u_0B_2u_0^* \cap B_1 = B$, u_0 belongs to $Y(B_2; B)$. Take r_1 positive and a diffeomorphism Ψ_2 from $Y(B_2; C(B)) \cap \mathcal{N}_{r_1}(u_0)$ onto an open subset of $\mathbb{R}^{d_2}$.

Now that we have fixed parametrizations Ψ_1 and Ψ_2, we can parametrize $Z(B_1, B_2; [B]_{B_1})$ around u_0. Recall $[C(B)]_{B_1}$ has the topology induced by the bijection $\beta : [C(B)]_{B_1} \to G/H$, given by $\beta(uC(B)u^*) = \pi(u)$. The function

$$\begin{aligned} Z(B_1, B_2; [B]_{B_1}) &\to [C(B)]_{B_1}, \\ u &\mapsto c(uB_2u^* \cap B_1) \end{aligned}$$

is continuous by Lemma III.23 and Lemma III.24. Hence there is δ_2 positive such that $\beta(c(uB_2u^* \cap B_1))$ belongs to $\mathcal{N}_{G/H}$, whenever u lies in the intersection $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta_2}(u_0)$. For a unitary u in $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta_2}(u_0)$ define

$$q(u) := s(\beta(c(uB_2u^* \cap B_1))).$$

We note that $q(u_0) = 1$, $q(u)$ lies in G and that the map $u \mapsto q(u)$ is continuous. The main property of $q(u)$ is that

$$c(uB_2u^* \cap B_1) = q(u)c(B)q(u)^*. \tag{3.15}$$

Indeed, for u in $Z(B_1, B_2; [B]_1) \cap \mathcal{N}_{\delta_2}(u_0)$ there is a unitary v in G with the property $uB_2u^* \cap B_1 = vBv^*$. Hence $c(uB_2 \cap B_1) = vC(B)v^*$. Since $\|u - u_0\| < \delta_2$, $\beta(c(uB_2u^* \cap B_1))$ lies in $\mathcal{N}_{G/H}$. Hence $\beta(c(uB_2u^* \cap B_1)) = \pi(v)$ lies in $\mathcal{N}_{G/H}$. Using the fact that s is a local section on $\mathcal{N}_{G/H}$ (property (4a) above) we deduce $\pi(s(\pi(v))) = \pi(v)$.

On the other hand, by definition of $q(u)$ we have

$$\pi(s(\pi(v))) = \pi(s(\beta(uB_2u^* \cap B_1))) = \pi(q(u)).$$

As a consequence, $\pi(v) = \pi(q(u))$ i.e. $v^*q(u)$ belongs to $\mathrm{Stab}(B_1, B)$ which is just another way to say (3.15) holds. At last we are ready to find r. Continuity of the map $u \mapsto q(u)$ gives a positive δ_3, less that δ_2, such that $\|q(u) - 1\| < \frac{\delta_1}{2}$ whenever u lies in

$Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta_3}(u_0)$. Define $r = \min\{\frac{\delta_1}{2}, \delta_3\}$. The first thing we notice is that $q(u)^*u$ belongs to $Y(B_2; C(B)) \cap \mathcal{N}_{\delta_1}(u_0)$ whenever u lies in $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta}(u_0)$. Indeed, from

$$q(u)c(B)q(u)^* = c(uB_2u^* \cap B_1) \subseteq uB_2u^*$$

we obtain $q(u)^*u \in Y(B_2; c(B))$ and a standard computation, using $\|q(u) - 1\| < \frac{\delta_1}{2}$, shows $\|q(u)^*u - u_0\| < \delta_1$. Hence we are allowed to take $\Psi_2(q(u)^*u)$. Lastly, for u in $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta}(u_0)$ define

$$\Psi(u) := (\Psi_1(\beta(c(uB_2u^* \cap B_1))), \Psi_2(q(u)u^*)).$$

It is clear that Ψ is continuous.

Now we show Ψ is injective. If $\Psi(u_1) = \Psi(u_2)$, for two element u_1 and u_2 in $Z(B_1, B_2; [B]_{B_1})$, then

$$\Psi_1(\beta(c(u_1B_2u_1^* \cap B_1))) = \Psi_1(\beta(c(u_2B_2u_2^* \cap B_1))), \tag{3.16}$$

$$\Psi_2(q(u_1)u_1^*) = \Psi_2(q(u_2)u_2^*). \tag{3.17}$$

From (3.16) and definition of $q(u)$ it follows that $q(u_1) = q(u_2)$ and from equation (3.17) we conclude $u_1 = u_2$. □

Proposition III.26. *Take B a unital C^*-subalgebra of B_1 such that it is unitarily equivalent to a C^*-subalgebra of B_2. Fix an element u_0 in $Z(B_1, B_2; [B]_{B_1})$.*

There is a positive number r and a continuous injective function

$$\Psi : \mathcal{N}_r(u_0) \cap Z(B_1, B_2; [B]_{B_1}) \to \mathbb{R}^{d(B)}$$

.

The proof of Proposition III.26 is similar to that of Proposition III.25, so we omit it.

We now begin showing density in $\mathbb{U}(M_N)$ of certain sets of unitaries.

Lemma III.27. *Assume B_1 and B_2 are simple. If $B \neq \mathbb{C}$ is a unital C^*-subalgebra of B_1 and it is unitarily equivalent to a C^*-subalgebra of B_2 then $Z(B_1, B_2; [B]_{B_1})^c$ is dense.*

Proof. Firstly we notice that $\dim \mathbb{U}(B_1) + \dim \mathbb{U}(B_2) < N^2$. Indeed, if B_i is $*-$isomorphic to M_{k_i}, $i = 1, 2$ and $m_i = \mu(M_N, B_i)$ then $\dim \mathbb{U}(B_1) + \dim \mathbb{U}(B_2) = N^2(1/m_2^2 + 1/m_2^2) < N^2$. Secondly we will prove that for any u in $Z(B_1, B_2; [B]_{B_1})$ there is a natural number d_u, with $d_u < N^2$, a positive number r_u and a continuous injective function $\Psi_u : \mathcal{N}_{r_u}(u) \cap Z(B_1, B_2; [B]_{B_1}) \to \mathbb{R}^{d_u}$. We will consider two cases.

Case (1): B is not simple. Take $d_u = d(C(B))$. Since $C(B) \neq \mathbb{C}$, Proposition III.16 implies $d(C(B)) < N^2$. Take r_u and Ψ_u as required to exist by Proposition III.25.

Case (2): B is simple. Take $d_u = d(B)$. Since $B \neq \mathbb{C}$, B contains a unital C^*-subalgebra isomorphic to $\mathbb{C}^2$, call it C. Lemma III.20 implies $d(B) \leq d(C)$ and Lemma III.16 implies $d(C) < N^2$. Take r_u and Ψ_u the positive number and continuous injective function from Proposition III.26.

We will show that $U \cap Z(B_1, B_2; [B]_{B_1})^c \neq \emptyset$, for any nonempty open subset $U \subseteq \mathbb{U}(M_N)$. First notice that if the intersection $U \cap (\bigcup_{u \in Z(B_1, B_2; [B]_{B_1})} \mathcal{N}_{r_u}(u))^c$ is nonempty then we are done. Thus we may assume $U \subseteq \bigcup_{u \in Z(B_1, B_2; [B]_{B_1})} \mathcal{N}_{r_u}(u)$. Furthermore, by making U smaller, if necessary, we may assume there is u in $Z(B_1, B_2; [B]_{B_1})$ such that $U \subseteq \mathcal{N}_{r_u}(u)$.

For sake of contradiction assume $U \subseteq Z(B_1, B_2; [B]_{B_1})$. We may take an open subset V, contained in U, small enough so that V is diffeomorphic to an open connected set $\mathcal{O}$ of $\mathbb{R}^{N^2}$. Let $\varphi : \mathcal{O} \to V$ be a diffeomorphism. It follows we have a continuous injective function

$$\mathbb{R}^{N^2} \supseteq \mathcal{O} \xrightarrow{\varphi} V \xrightarrow{\Psi_u} \mathbb{R}^{d_u} \hookrightarrow \mathbb{R}^{N^2} .$$

By the Invariance of Domain Theorem, the image of this map must be open in $\mathbb{R}^{N^2}$. But this is a contradiction since the image is contained in $\mathbb{R}^{d_u}$ and $d_u < N^2$. We conclude $U \cap Z(B_1, B_2; [B]_{B_1})^c \neq \emptyset$. □

Lemma III.28. *Suppose* $\dim C(B_1) \geq 2$ *and* B_1 *is* $*$*–isomorphic to*

$$M_{N/\dim C(B_1)} \oplus \cdots \oplus M_{N/\dim C(B_1)}.$$

Assume one of the following cases holds:

1. $\dim C(B_2) = 1$,
2. B_1 *is* $*$*–isomorphic to*

   $$M_{N/2} \oplus M_{N/2}$$

   and B_2 *is* $*$*–isomorphic to*

   $$M_{N/2} \oplus M_{N/(2k)},$$

   where $k \geq 2$.
3. $\dim C(B_2) \geq 3$ *and* B_2 *is* $*$*–isomorphic to*

   $$M_{N/\dim C(B_2)} \oplus \cdots \oplus M_{N/\dim C(B_2)}.$$

Then for any $B \neq \mathbb{C}$ *unital* C^**-subalgebra of* B_1 *such that it is unitarily equivalent to a* C^**-subalgebra of* B_2*,* $Z(B_1, B_2; [B]_{B_1})^c$ *is dense.*

Proof. The proof of Lemma III.28 is exactly as the proof of III.27 but using Lemma III.19 instead of Lemma III.16 . □

At this point if the sets $Z(B_1, B_2; [B]_{B_1})$ were closed one could conclude immediately that $\Delta(B_1, B_2)$ is dense. Unfortunately they may not be closed. What saves the day is the fact that we can control the closure of $Z(B_1, B_2; [B]_{B_1})$ with sets of the same form i.e. sets like $Z(B_1, B_2; [C]_{B_1})$ for a suitable finite family of subalgebras C. We make this statement clearer with the definition of an order on $*$-SubAlg(B_1).

Definition III.29. On $*\text{-SubAlg}(B_1)/\sim_{B_1}$ we define a partial order as follows:

$$[B]_{B_1} \leq [C]_{B_1} \Leftrightarrow \exists D \in *\text{-SubAlg}(C) : D \sim_{B_1} B.$$

Proposition III.30. *For any B in* $*$-SubAlg(B_1),

$$\overline{Z(B_1, B_2; [B]_{B_1})} \subseteq \bigcup_{[C]_{B_1} \geq [B]_{B_1}} Z(B_1, B_2; [C]_{B_1}).$$

Proof. Let $(u_k)_{k\geq 1}$ be a sequence in $Z(B_1, B_2; [B]_{B_1})$ and u in $\mathbb{U}(M_N)$ such that $\lim_k \|u_k - u\| = 0$. Pick q_k in $\mathbb{U}(M_N)$ such that $q_k B q_k^* = u_k B_2 u_k^* \cap B_1$. Let $\{f_k(s,i,j)\}_{s,i,j}$ be a matrix unit for $u_k B_2 u_k^* \cap B_1$ and take elements $e_k(s,i,j)$ in B_2 such that $f_k(s,i,j) = u_k e_k(s,i,j) u_k^*$. Since B_2 is finite dimensional, passing to a subsequence if necessary, we may assume $\lim_k f_k(s,i,j) = f(s,i,j) \in B_2$ and $\lim_k u_k e_k(s,i,j) u_k^* = u e(s,i,j) u^*$ for some $e(s,i,j) \in B_1$, for all s, i and j. It follows that $\lim_k \operatorname{dist}(f_k(s,i,j), uB_2u^* \cap B_1) = 0$. Hence, from Lemma III.21, for large k, there is q in $\mathbb{U}(M_N)$ so that $q(u_k B_2 u_k^* \cap B_1) q^* = q q_k B q_k^* q^*$ is contained in $uB_2u^* \cap B_1$. We conclude $[uB_2u^* \cap B_1]_{B_1} \geq [B]_{B_1}$ and since u lies in $Z(B_1, B_2; [uB_2u^* \cap B_1])$ the proof is complete. □

Lemma III.31. *Assume one of the following conditions holds:*

1. $\dim C(B_1) = 1 = \dim C(B_2)$,

2. $\dim C(B_1) \geq 2$, $\dim C(B_2) = 1$ *and* B_1 *is* $*$*–isomorphic to*

$$M_{N/\dim C(B_1)} \oplus \cdots \oplus M_{N/\dim C(B_1)},$$

3. $\dim C(B_1) = 2 = \dim C(B_2)$, B_1 *is* $*$*–isomorphic to*

$$M_{N/2} \oplus M_{N/2},$$

and B_2 *is* $*$*–isomorphic to*

$$M_{N/2} \oplus M_{N/(2k)}$$

where $k \geq 2$*,*

4. $\dim C(B_1) \geq 2$, $\dim C(B_2) \geq 3$ *and, for* $i = 1, 2$, B_i *is* $*$*–isomorphic to*

$$M_{N/\dim C(B_i)} \oplus \cdots \oplus M_{N/\dim C(B_i)}.$$

Take B *a unital* C^**-subalgebra of* B_1 *such that it is unitarily equivalent to a* C^**-subalgebra of* B_2 *. If* $\overline{Z(B_1, B_2; [B]_{B_1})}^c$ *is not dense and* $B \neq \mathbb{C}$ *then there is a subalgebra* C *in* $*\text{-SubAlg}(B_1)$ *such that* $[C]_{B_1} > [B]_{B_1}$ *and* $\overline{Z(B_1, B_2; [C]_{B_1})}^c$ *is not dense.*

Proof. We proceed by contrapositive. Thus, assume $\overline{Z(B_1, B_2; [C]_{B_1})}^c$ is dense for all $[C]_{B_1} > [B]_{B_1}$. Since the set $\{[C]_{B_1} : [C]_{B_1} > [B]_{B_1}\}$ is finite,

$$\bigcap_{[C]_{B_1} > [B]_{B_1}} \overline{Z(B_1, B_2; [C]_{B_1})}^c$$

is open and dense. Furthermore, Lemma III.27 or Lemma III.28 implies $Z(B_1, B_2; [B]_{B_1})^c$ is dense. Hence the intersection

$$Z(B_1, B_2; [B]_{B_1})^c \cap \bigcap_{[C]_{B_1} > [B]_{B_1}} \overline{Z(B_1, B_2; [C]_{B_1})}^c$$

is dense. But this along with Proposition III.30 implies $\overline{Z(B_1, B_2; [B]_{B_1})}^c$ is dense. □

Lemma III.32. *Assume one of the conditions (1)–(4) of Lemma III.31 holds. Then for any $B \neq \mathbb{C}$, unital C^*-subalgebra of B_1 that is unitarily equivalent to a C^*-subalgebra of B_2, the set $\overline{Z(B_1, B_2; [B]_{B_1})}^c$ is dense.*

Proof. Assume $\overline{Z(B_1, B_2; [B]_{B_1})}^c$ is not dense. By Lemma III.31 there is $[C]_{B_1} > [B]_{B_1}$ such that $\overline{Z(B_1, B_2; [C]_{B_1})}^c$ is not dense. We notice that again we are in the same condition to apply Lemma III.31, since $[C]_{B_1} > [B]_{B_1} > [\mathbb{C}]_{B_1}$. In this way we can construct chains, in $*$-SubAlg$(B_1)/\sim_{B_1}$, of length arbitrarily large, but this can not be since it is finite. □

At last we can give a proof of Theorem III.6.

Proof of Theorem III.6. A direct computation shows that

$$\Delta(B_1, B_2) = \bigcap_{[B]_{B_1} > [\mathbb{C}]_{B_1}} Z(B_1, B_2, [B]_{B_1})^c.$$

Thus

$$\Delta(B_1, B_2) \supseteq \bigcap_{[B]_{B_1} > [\mathbb{C}]_{B_1}} \overline{Z(B_1, B_2, [B]_{B_1})}^c.$$

Now, by Lemma III.32, whenever $[B]_{B_1} > [\mathbb{C}]_{B_1}$, the set $\overline{Z(B_1, B_2, [B]_{B_1})}^c$ is dense. Hence $\Delta(B_1, B_2)$ is dense. □

CHAPTER IV

PRIMITIVITY

During this section, unless stated otherwise, $A_1 \neq \mathbb{C}$ and $A_2 \neq \mathbb{C}$ denote two nontrivial finite dimensional C^*-algebras. Our goal is to prove $A_1 * A_2$ is primitive, except for the case $A_1 = \mathbb{C}^2 = A_2$. Two main ingredients are used. Firstly, the perturbation results from previous chapter. Secondly, the fact that $A_1 * A_2$ has a separating family of finite dimensional $*$–representations, a result due to Excel and Loring, [7].

Before we start proving results about primitivity, we want to consider the case $\mathbb{C}^2 * \mathbb{C}^2$. This is a well studied C^*-algebra; see for instance [3], [11] and [13]. From Proposition I.12 $\mathbb{C}^2 * \mathbb{C}^2$ is $*$–isomorphic to the C^*-algebra of continuous M_2-valued functions on the closed interval $[0, 1]$, whose values at 0 and 1 are diagonal matrices. As a consequences its center is not trivial. Since the center of any primitive C^*-algebra is trivial, we conclude $\mathbb{C}^2 * \mathbb{C}^2$ is not primitive.

Definition IV.1. We denote by ι_j the inclusion homomorphism from A_j into $A_1 * A_2$. Given a unital $*$–representation $\pi : A_1 * A_2 \to \mathbb{B}(H)$, we define $\pi^{(1)} = \pi \circ \iota_1$ and $\pi^{(2)} = \pi \circ \iota_2$. Thus, with this notation, we have $\pi = \pi^{(1)} * \pi^{(2)}$. For a unitary u in $\mathbb{U}(H)$ we call the $*$–representation $\pi^{(1)} * (\mathrm{Ad}\, u \circ \pi^{(2)})$, a perturbation of π by u.

Remark IV.2. The $*$–representation $\pi^{(1)} * (\mathrm{Ad}\, u \circ \pi^{(2)})$ is irreducible if and only if

$$u\pi^{(2)}(A_2)'u^* \cap \pi^{(1)}(A_1)' = \mathbb{C}.$$

where $(\pi^{(1)}(A_1))'$ denotes de commutant of $\pi^{(1)}(A_1)$ in $\mathbb{B}(H)$.

Proposition IV.3. *Assume A_1 and A_2 are simple. Given any unital finite dimensional $*$–representation $\pi : A_1 * A_2 \to \mathbb{B}(H)$ and a positive number ε, there is u in $\mathbb{U}(H)$ such that $\|u - \mathrm{id}_H\| < \varepsilon$ and $\pi^{(1)} * (\mathrm{Ad}\, u \circ \pi^{(2)})$ is irreducible.*

Proof. Since $\pi^{(i)}(A_i)'$ is again simple, $(i = 1, 2)$, the result is a direct consequence of Remark IV.2 and part (1) of Theorem III.6. □

If A_1 or A_2 fail to be simple, then is it not always possible to perturb any given finite dimensional $*$–representation of $A_1 * A_2$ into an irreducible one, even if $A_1 \neq \mathbb{C}^2$ and $A_2 \neq \mathbb{C}^2$. The key method for the nonsimple case is to repeat blocks of A_1 and A_2.

Lemma IV.4. *Assume A is a finite dimensional C^*-algebra $*$–isomorphic to $\oplus_{j=1}^{l} M_{n(j)}$ and take $\pi : A \to \mathbb{B}(H)$ a unital finite dimensional $*$–representation. Let $\mu(\pi) = [m(1), \ldots, m(l)]$ and let $\tilde{\pi}$ be the restriction of π to the center of A. Then*

$$\mu(\tilde{\pi}) = [m(1)n(1), \ldots, m(l)n(l)].$$

Proof. Write

$$A = \oplus_{j=1}^{l} A(j)$$

where $A(j)$ is $*$-isomorphic to $M_{n(j)}$. Up to unitary equivalence in $\mathbb{U}(H)$, π equals

$$\underbrace{id_{A(1)} \oplus \cdots \oplus id_{A(1)}}_{m(1)-\text{times}} \oplus \cdots \oplus \underbrace{id_{A(l)} \oplus \cdots \oplus id_{A(l)}}_{m(l)-\text{times}}.$$

It follows that, up to unitary equivalence in $\mathbb{U}(H)$, $\tilde{\pi}$ equals

$$\underbrace{\underbrace{id_{\mathbb{C}} \oplus \cdots \oplus id_{\mathbb{C}}}_{n(1)-\text{times}} \oplus \cdots \oplus \underbrace{id_{\mathbb{C}} \oplus \cdots \oplus id_{\mathbb{C}}}_{n(1)-\text{times}}}_{m(1)-\text{times}} \oplus \cdots$$
$$\cdots \oplus \underbrace{\underbrace{id_{\mathbb{C}} \oplus \cdots \oplus id_{\mathbb{C}}}_{n(l)-\text{times}} \oplus \cdots \oplus \underbrace{id_{\mathbb{C}} \oplus \cdots \oplus id_{\mathbb{C}}}_{n(l)-\text{times}}}_{m(l)-\text{times}}.$$

and the result follows. □

Lemma IV.5. *Assume A is a finite dimensional C^*-algebra and $\pi : A \to \mathbb{B}(H)$ is a unital finite dimensional $*$–representation. Let*

$$\mu(\pi) = [m(1), \dots, m(l)].$$

For any nonnegative integers $q(1), \dots, q(l)$ there is a finite dimensional unital $*$–representation $\rho : A \to \mathbb{B}(K)$ such that

$$\mu(\pi \oplus \rho) = [m(1) + q(1), \dots, m(l) + q(l)].$$

Proof. Write A as

$$A = \bigoplus_{i=1}^{l} A(i)$$

where $A(i) = \mathbb{B}(V_i)$ for V_i finite dimensional. For $1 \leq i \leq l$, let $p_i : A \to A(i)$ denote the canonical projection onto $A(i)$. Notice that p_i is a unital $*$–representation of A. Define

$$\rho := \bigoplus_{i=1}^{l} \underbrace{(p_i \oplus \cdots \oplus p_i)}_{q(i)-\text{times}} : A \to \bigoplus_{i=1}^{l} A(i)^{q(i)} \subseteq \mathbb{B}(K),$$

where $K = \bigoplus_{i=1}^{l} (V_i^{\oplus q_i})$. Then ρ is a unital $*$–representation of A on K and

$$\mu(\pi \oplus \rho) = [m(1) + q(1), \dots, m(l) + q(l)].$$

□

Definition IV.6. Let $\rho : A_1 * A_2 \to \mathbb{B}(H)$ be a unital, finite dimensional representation. We say that ρ satisfies the *Rank of Central Projections condition* (or *RCP condition*) if for both $i = 1, 2$, the rank of $\rho(p)$ is the same for all minimal projections p of the center $C(A_i)$ of A_i, (but they need not agree for different values of i).

The RCP condition for ρ, of course, is really about the pair of representations $(\rho^{(1)}, \rho^{(2)})$. However, it will be convenient to express it in terms of $A_1 * A_2$. In any case, the following two lemmas are clear.

Lemma IV.7. *Suppose $\rho : A_1 * A_2 \to \mathbb{B}(H)$ is a finite dimensional representation that satisfies the RCP condition and $u \in \mathbb{U}(H)$. Then the representation $\rho^{(1)} * (\operatorname{Ad} u \circ \rho^{(2)})$ of $A_1 * A_2$ also satisfies the RCP condition.*

Lemma IV.8. *Suppose $\rho : A_1 * A_2 \to \mathbb{B}(H)$ and $\sigma : A_1 * A_2 \to \mathbb{B}(K)$ are finite dimensional representations that satisfy the RCP condition. Then $\rho \oplus \sigma : A_1 * A_2 \to \mathbb{B}(H \oplus K)$ also satisfies the RCP condition.*

The next lemma takes slightly more work and is essential to our construction.

Lemma IV.9. *Given a unital finite dimensional $*$–representation $\pi : A_1 * A_2 \to \mathbb{B}(H)$, there is a finite dimensional Hilbert space $\hat{H}$ and a unital $*$–representation*

$$\hat{\pi} : A_1 * A_2 \to \mathbb{B}(\hat{H})$$

such that $\pi \oplus \hat{\pi}$ satisfies the RCP condition.

Proof. For $i = 1, 2$, let $l_i = \dim C(A_i)$, let A_i be $*$–isomorphic to $\bigoplus_{j=1}^{l_i} M_{n_i(j)}$ and write

$$\mu(\pi^{(i)}) = [m_i(1), \ldots, m_i(l_i)].$$

Take $n_i = \operatorname{lcm}(n_i(1), \ldots, n_i(l_i))$ and integers $r_i(j)$, such that $r_i(j)n_i(j) = n_i$, for $1 \leq j \leq l_i$. Take a positive integer s such that $sr_i(j) \geq m_i(j)$ for all $i = 1, 2$ and $1 \leq j \leq l_i$. Use Lemma IV.5 to find a unital finite dimensional $*$–representation $\rho_i : A_i \to \mathbb{B}(K_i)$, $i = 1, 2$ such that

$$\mu(\pi^{(i)} \oplus \rho_i) = [sr_i(1), \ldots, sr_i(l_i)].$$

Letting κ_i denote the restriction of $\pi^{(i)} \oplus \rho_i$ to $C(A_i)$, from Lemma IV.4 we have

$$\mu(\kappa_i) = [sr_i(1)n_i(1), \ldots, sr_i(l_i)n_i(l_i)] = [sn_i, sn_i, \ldots, sn_i].$$

The $*$–representations $(\pi^{(1)} \oplus \rho_1)$ and $(\pi^{(2)} \oplus \rho_2)$ are almost what we want, but they may take values in Hilbert spaces with different dimensions. To take care of this, we take multiples of them. Let $N = \mathrm{lcm}(\dim(H \oplus K_1), \dim(H \oplus K_2))$, find positive integers k_1 and k_2 such that

$$N = k_1 \dim(H \oplus K_1) = k_2 \dim(H \oplus K_2)$$

and consider the Hilbert spaces $(H \oplus K_i)^{\oplus k_i}$, whose dimensions agree for $i = 1, 2$. Then

$$\dim(K_1 \oplus (H \oplus K_1)^{\oplus(k_1-1)}) = \dim(K_2 \oplus (H \oplus K_2)^{\oplus(k_2-1)})$$

and there is a unitary operator

$$U : K_2 \oplus (H \oplus K_2)^{\oplus(k_2-1)} \to K_1 \oplus (H \oplus K_1)^{\oplus(k_1-1)}.$$

Take

$$\begin{aligned}
\hat{H} &:= K_1 \oplus (H + K_1)^{\oplus(k_1-1)}, \\
\hat{\pi}_1 &:= \rho_1 \oplus (\pi^{(1)} \oplus \rho)^{\oplus(k_1-1)}, \\
\sigma_1 &:= \pi^{(1)} \oplus \hat{\pi}_1, \\
\hat{\pi}_2 &:= \mathrm{Ad}\, U \circ (\rho_2 \oplus (\pi^{(2)} \oplus \rho)^{\oplus(k_2-1)}), \\
\sigma_2 &:= \pi^{(2)} \oplus \hat{\pi}_2, \\
\hat{\pi} &:= \hat{\pi}_1 * \hat{\pi}_2.
\end{aligned}$$

Then $\sigma_1 * \sigma_2 = (\pi^{(1)} \oplus \hat{\pi}_1) * (\pi^{(2)} \oplus \hat{\pi}_2) = \pi \oplus \hat{\pi}$. We have $\mu(\sigma_i) = [k_i s r_i(1), \ldots, k_i s r_i(l_i)]$. Let $\tilde{\sigma}_i$ denote the restriction of σ_i to $C(A_i)$. From Lemma IV.4 we have

$$\mu(\tilde{\sigma}_i) = [k_i s r_i(1) n_i(1), \ldots, k_i s r_i(l_i) n_i(l_i)] = [k_i s n_i, \ldots, k_i s n_i].$$

□

Proposition IV.10. *Suppose* $A_1 \neq \mathbb{C}^2$ *or* $A_2 \neq \mathbb{C}^2$ *and* $\rho : A_1 * A_2 \to \mathbb{B}(H)$ *is a finite dimensional* $*$*–representation that satisfies the RCP condition. Then for any* $\varepsilon > 0$ *there is a unitary* u *in* $\mathbb{U}(H)$ *such that* $\|u - \mathrm{id}_H\| < \varepsilon$ *and* $\rho^{(1)} * (\mathrm{Ad}\, u \circ \rho^{(2)})$ *is irreducible.*

Proof. After interchanging A_1 and A_2, if necessary, one of the following must hold:

(1) A_1 and A_2 are simple,

(2) $\dim C(A_1) \geq 2$ and A_2 is simple,

(3) for $i = 1, 2$, $A_i = M_{n_i(1)} \oplus M_{n_i(2)}$, with $n_2(2) \geq 2$,

(4) $\dim C(A_1) \geq 2$, $\dim C(A_2) \geq 3$.

In all cases, we will show using Theorem III.6 that $\Delta(\rho^{(1)}(A_1)', \rho^{(2)}(A_2)')$ is dense in $\mathbb{U}(H)$, from which the result follows by Remark IV.2.

In case (1), this is just as in Proposition IV.3.

In case (2), let $B_1 = \rho^{(1)}(C(A_1))'$ and $B_2 = \rho^{(2)}(A_2)'$. Notice that $\dim C(B_2) = 1$, $\dim C(B_1) = \dim C(A_1) \geq 2$ and, by the RCP assumption, B_1 is $*$–isomorphic to $M_{\dim H/\dim C(B_1)} \oplus \cdots \oplus M_{\dim H/\dim C(B_1)}$. By Theorem III.6m, part (2), the set $\Delta(B_1, B_2)$ is dense. But since $\rho^{(i)}(A_i)' \subseteq B_i$, we have $\Delta(B_1, B_2) \subseteq \Delta(\rho^{(1)}(A_1)', \rho^{(2)}(A_2)')$.

In case (3), let $B_1 = \rho^{(1)}(C(A_1))'$ and $B_2 = \rho^{(2)}(\mathbb{C} \oplus M_{n_2(2)})'$. By the RCP assumption, B_1 is $*$–isomorphic to

$$M_{\dim H/2} \oplus M_{\dim H/2}$$

and B_2 is $*$–isomorphic to

$$M_{\dim H/2} \oplus M_{\dim H/(2n_2(2))}.$$

By Theorem III.6, part (3), the set $\Delta(B_1, B_2)$ is dense. But $\Delta(B_1, B_2) \subseteq \Delta(\rho^{(1)}(A_1)', \rho^{(2)}(A_2)')$.

In case (4), let $B_i = \rho^{(i)}(C(A_i))'$ for $i = 1, 2$. Then $\dim C(B_1) = \dim C(A_1) \geq 2$, $\dim C(B_2) = \dim C(A_2) \geq 3$ and, for $i = 1, 2$, B_i is $*$–isomorphic to

$$M_{\dim H/\dim C(B_i)} \oplus \cdots \oplus M_{\dim H/\dim C(B_i)}.$$

By Theorem III.6, part (4), the set $\Delta(B_1, B_2)$ is dense. But again we have $\Delta(B_1, B_2) \subseteq \Delta(\rho^{(1)}(A_1)', \rho^{(2)}(A_2)')$. $\square$

Combining Lemma IV.9 and Proposition IV.10, together with Proposition IV.3, and so long as A_1 and A_2 are not both $\mathbb{C}^2$, we construct irreducible finite dimensional $*$–representations of the form

$$(\pi^{(1)} \oplus \hat{\pi}^{(1)}) * (\mathrm{Ad}\, u \circ (\pi^{(2)} \oplus \hat{\pi}^{(2)})),$$

starting with any finite dimensional represenation π of $A_1 * A_2$ and where u is a unitary that can be chosen arbitrarily close to the identity. The next proposition shows that with sufficient control on u, the values of σ on any given finite subset can be as close as desired to the corresponding values of $\pi \oplus \hat{\pi}$.

Proposition IV.11. *Let A_1 and A_2 be two unital C^*-algebras. Given a nonzero element x in $A_1 * A_2$ and a positive number ε, there is a positive number $\delta = \delta(x, \varepsilon)$ such that for any u and v in $\mathbb{U}(H)$ satisfying $\|u - v\| < \delta$ and any unital $*$–representations $\pi : A_1 * A_2 \to \mathbb{B}(H)$, we have*

$$\|(\pi^{(1)} * (\mathrm{Ad}\, v \circ \pi^{(2)}))(x) - (\pi^{(1)} * (\mathrm{Ad}\, u \circ \pi^{(2)}))(x)\| < \varepsilon.$$

Proof. Fix π, a unital $*$-representation of $A_1 * A_2$ into $\mathbb{B}(H)$ and two unitaries u and v in $\mathbb{U}(H)$.

To ease notation let $\rho_u = \pi^{(1)} * (Adu \circ \pi^{(2)})$ and $\rho_v = \pi^{(1)} * (Adv \circ \pi^{(2)})$.

Case 1: x is a word with letters from A_1 and A_2.

Here we use induction on the length of x.

Assume the length of x is 1. We have two cases. Either x is in A_1 or it is in A_2.

If x lies in A_1 we can take δ any positive number.

If x lies in A_2 take $\delta(\varepsilon, x) = \frac{\varepsilon}{2\|x\|}$. A standard computation shows that, if u and v satisfy $\|u - v\| < \delta$ then

$$\|\rho_v(x) - \rho_u(x)\| < \varepsilon.$$

Now, assume the result true for words of length l and take $x = x_1 \cdots x_{l+1}$ where x_j is a non zero element in A_{i_j}, $1 \leq j \leq l+1$ and $i_1 \neq \cdots \neq i_{l+1}$.

As before we have two cases, x_{l+1} lies in A_1 or it lies in A_2.

For convenience let $y = x_1 \ldots x_l$.

If x_{l+1} happens to be in A_1, then using the identities

$$\begin{aligned} \rho_u(x) &= \rho_u(y)\pi^{(1)}(x_{l+1}), \\ \rho_v(x) &= \rho_v(y)\pi^{(1)}(x_{l+1}) \end{aligned}$$

we obtain

$$\|\rho_v(x) - \rho_u(x)\| \leq \|x_{l+1}\| \|\rho_v(y) - \rho_u(y)\|.$$

Therefore the δ that works in this case is $\delta(\varepsilon, x) = \delta(\frac{\varepsilon}{\|x_{l+1}\|}, y)$.

The last possibility is that x_{l+1} lies in A_2. If so, we use the identities

$$\begin{aligned} \rho_u(x) &= \rho_u(y)u\pi^{(2)}(x_{l+1})u^*, \\ \rho_v(x) &= \rho_v(y)v\pi^{(2)}(x_{l+1})v^*, \end{aligned}$$

to obtain

$$
\begin{aligned}
\|\rho_v(x)-\rho_u(x)\| &\le \|\rho_v(y)v\pi^{(2)}(x_{l+1})v^* - \rho_v(y)v\pi^{(2)}(x_{l+1})u^*\| \\
&+ \|\rho_v(y)v\pi^{(2)}(x_{l+1})u^* - \rho_u(y)v\pi^{(2)}(x_{l+1})u^*\| \\
&+ \|\rho_u(y)v\pi^{(2)}(x_{l+1})u^* - \rho_u(y)u\pi^{(2)}(x_{l+1})u^*\| \\
&\le 2\|x_1\|\cdots\|x_{l+1}\|\|v-u\| + \|x_{l+1}\|\|\rho_v(y)-\rho_u(y)\|.
\end{aligned}
$$

Thus we take $\delta(\varepsilon, x) = \min\{\frac{\varepsilon}{3\|x_1\|\cdots\|x_{l+1}\|}, \delta(\frac{\varepsilon}{3\|x_{l+1}\|}, y)\}$.

Case 2: General case.

Since the algebraic unital full free product of A_1 and A_2 is norm-dense in $A_1 * A_2$, we can find words $w_1, \ldots, w_n$ with letters from A_1 and A_2 such that

$$\left\| x - \sum_{j=1}^{n} w_j \right\| < \frac{\varepsilon}{3}.$$

By case 1 there are positive numbers $\delta(w_1, \frac{\varepsilon}{3n}), \ldots, \delta(w_n, \frac{\varepsilon}{3n})$ such that

$$\left\| \rho_v(w_j) - \rho_u(w_j) \right\| < \frac{\varepsilon}{3n},$$

whenever $\|u-v\| < \delta(w_j, \frac{\varepsilon}{3n})$.

Take $\delta = \min\{\delta(w_1, \frac{\varepsilon}{3n}), \ldots, \delta(w_n, \frac{\varepsilon}{3n})\}$.

If u and v satisfy $\|u-v\| < \delta$, then the identity

$$
\begin{aligned}
\rho_v(x)-\rho_u(x) &= \rho_v\left(x - \sum_{j=1}^{n} w_j\right) \\
&+ \sum_{j=1}^{n} (\rho_v - \rho_u)(w_j) \\
&- \rho_u\left(x - \sum_{j=1}^{n} w_j\right)
\end{aligned}
$$

along with triangle inequality completes the proof.

□

Now our objective is to perturb the direct sum of a sequence of unital finite dimensional $*$–representations of $A_1 * A_2$ into an irreducible one. The construction is long and uses several intermediate results.

Recall that if π and σ are two irreducible representations of a C*–algebra A on the same Hilbert space H such that π and σ are not unitarily equivalent, then there are no nonzero operators $T \in \mathbb{B}(H)$ that intertwine the representations, i.e. such that $\pi(A)T = T\sigma(a)$ for all $a \in A$. From this fact, one quickly gets the following standard result:

Proposition IV.12. *Let A be a C^*-algebra and suppose $(\pi_j)_{j\geq 1}$ is a sequence of irreducible $*$–representations $\pi_j : A \to \mathbb{B}(H_j)$ that are pairwise not unitarily equivalent. Then, for $\pi = \oplus_{j\geq 1}\pi_j$, we have*

$$\pi(A)' = \{\oplus_{j\geq 1} z_j \,\mathrm{id}_{H_j} : z_j \in \mathbb{C}, \sup\{|z_j|\} < \infty\}.$$

Lemma IV.13. *Let A be a C^*-algebra and assume we have $\pi : A \to \mathbb{B}(H)$, a finite dimensional $*$–representation. Given a positive number ε there is a finite set F, contained in the closed unit ball of A, fulfilling the condition for all y in the closed unit ball of A there is x in F with $\|\pi(x) - \pi(y)\| < \varepsilon$.*

Proof. Let E denote the norm closure, in $\mathbb{B}(H)$, of the set $\{\pi(a) : \|a\| \leq 1\}$. Since H is finite dimensional, E is compact. Thus there exists $\{T_1, \ldots, T_k\}$, a finite $\frac{\varepsilon}{2}$-net for E. For each T_i, take x_i in the closed unit ball of A such that $\|x_i - T_i\| < \frac{\varepsilon}{2}$. Then the set F we are looking for is $\{x_1, \ldots, x_k\}$.

□

Lemma IV.14. *Let $(H_j)_{j\geq 1}$ be a sequence of finite dimensional Hilbert spaces and let H denote its direct sum. Assume we have bounded operators T_j in $\mathbb{B}(H_j)$ and let T denote its direct sum. T is a compact operator in $\mathbb{B}(H)$ if and only if $\lim_j \|T_j\| = 0$.*

Proof. Assume T is compact and in order to get a contradiction assume assume there is a positive number ε and a subsequence $(j_k)_{k\geq 1}$ such that $\|T_{j_k}\| > \varepsilon$ for all k. Take h_{j_k} a unit vector in H_{j_k} with $\|T_{j_k} h_{j_k}\| \geq \varepsilon$. Consider the sequence $(\xi_k)_{k\geq 1}$, of unit vectors in H given by

$$\xi_k(i) = \begin{cases} h_{j_k} & \text{if } i = j_k \\ 0 & \text{otherwise} \end{cases}$$

Since T is compact there is a subsequence $(k_l)_{l\geq 1}$ such that $(T\xi_{k_l})_{l\geq 1}$ converges in norm. In particular it is Cauchy and then there is l_0 such that $\|T\xi_{k_{l_1}} - T\xi_{k_{l_2}}\| < \frac{\varepsilon}{2}$ for all $l_1, l_2 \geq l_0$. But this implies $\|T_{j_{k_l}} h_{j_{k_l}}\| < \frac{\varepsilon}{2}$ whenever $l \geq l_0$, a contradiction.

Now assume $\lim_j \|T_j\| = 0$. To show T is compact just notice T is the norm limit of the sequence of finite rank operators $(S_k)_{k\geq 1}$ where S_k equals $T_1 \oplus \cdots \oplus T_k$ on $\oplus_{j=1}^k H_j$ and it is zero on $\oplus_{j\geq k+1} H_j$. □

The following result follows from the very nice fact that a $*$–representation is faithful if and only if it is an isometry.

Lemma IV.15. *Let A denote a C^*-algebra and let $(\pi_k : A \to \mathbb{B}(H))_{k\geq 1}$ be a sequence of faithful $*$–representations. If π is a $*$–representation such that for all a in A, $\lim_k \|\pi_k(a) - \pi(a)\| = 0$ then π is faithful.*

At last, we can prove $A_1 * A_2$ is primitive when not both of A_1 and A_2 are $\mathbb{C}^2$.

Theorem IV.16. *Assume A_1 and A_2 are nontrivial finite dimensional C^*-algebras. If $A_1 \neq \mathbb{C}^2$ or $A_2 \neq \mathbb{C}^2$, then $A_1 * A_2$ is primitive.*

Proof. Write $A_i = \bigoplus_{j=1}^{l_i} M_{n_i(j)}$. By a result of Exel and Loring [7], there is a separating sequence $(\vartheta_j : A_1 * A_2 \to \mathbb{B}(K_j))_{j\geq 1}$, of finite dimensional unital $*$–representations. By Lemma IV.9, there are finite dimensional Hilbert spaces $\hat{K}_j$

and unital $*$–representation $\hat{\vartheta}_j : A_1 * A_2 \to \mathbb{B}(\hat{K}_j)$ such each that $\vartheta_j \oplus \hat{\vartheta}_j$ satisfies the RCP condition. Let $\pi_j = \vartheta_j \oplus \tilde{\vartheta}_j$ and $H_j = K_j \oplus \hat{K}_j$.

We may modify the original sequence $(\vartheta_j)_{j\geq 1}$, if necessary, so that each representation that appears is repeated infinitely many times and, thus, we may also assume

$$\pi(A_1 * A_2) \cap \mathbb{K}(H) = \{0\}, \tag{4.1}$$

where $\pi = \oplus_{j\geq 1}\pi_j$ and $H = \bigoplus_{j\geq 1} H_j$.

We will show that given $\varepsilon > 0$, there is a unitary u on $\mathbb{U}(H)$ such that $\|u - \mathrm{id}_H\| < \varepsilon$ and the representation $\pi^{(1)} * (\mathrm{Ad}\, u \circ \pi^{(2)})$ of $A_1 * A_2$ is irreducible and faithful. Find a strictly increasing sequence of natural numbers $(l(j))_{j\geq 0}$ with the property that $l(0) = 0, l(1) = 1$ and for all $k \geq 1$,

$$\sum_{j=l(k-1)+1}^{l(k)} \dim H_j < \sum_{j=l(k)+1}^{l(k+1)} \dim H_j. \tag{4.2}$$

Let $G_1 = H_1$ and for $k \geq 2$ define $G_k = \bigoplus_{j=l(k-1)+1}^{l(k)} H_j$ and fix a sequence of positive numbers $(\delta_j)_{j\geq 1}$ such that $\sum_{j\geq 1} \delta_j < \frac{\varepsilon}{2}$. By Lemma IV.8, for each $k \geq 1$ the direct sum

$$\lambda_k := \oplus_{j=l(k-1)+1}^{l(k)} \pi_j$$

satisfies the RCP condition. So by Proposition IV.10, there is a unitary v_k in $\mathbb{U}(G_k)$ with the property that $\|v_k - \mathrm{id}_{G_k}\| < \delta_k$ and the $*$–representation

$$\rho_k := \lambda_k^{(1)} * (\mathrm{Ad}\, v_k \circ \lambda_k^{(2)})$$

is irreducible and, by Lemma IV.7, satisfies the RCP condition. To ease notation let $\rho = \oplus_{j\geq 1}\rho_j$ and for $k \geq 1$ let $\rho_{[k]} = \oplus_{j=1}^{k}\rho_j$. If $v = \oplus_{k\geq 1} v_k$ then $\|v - \mathrm{id}_H\| < \frac{\varepsilon}{2}$ and, as a direct computation shows, we have $\rho = \pi^{(1)} * (\mathrm{Ad}\, v \circ \pi^{(2)})$. By dimension considerations, the irreducible representations ρ_k are pairwise not unitarily equivalent, and

Proposition IV.12 implies that the commutant of ρ consists of all diagonal operators of the form $\oplus_{k\geq 1} z_k \,\mathrm{id}_{G_k}$. We will perturb ρ a little more to finally get an irreducible representation.

We will construct a sequence $(u_k, F_k)_{k\geq 1}$ where

(a) u_k is a unitary in $\mathbb{U}(\oplus_{j=1}^{k} G_j)$ satisfying

$$\|u_k - \mathrm{id}_{\oplus_{j=1}^{k} G_j}\| < \frac{\varepsilon}{2^{k+1}}. \tag{4.3}$$

(b) letting

$$u_{(j,k)} = u_j \oplus \mathrm{id}_{G_{j+1}} \oplus \cdots \oplus \mathrm{id}_{G_k} \in \mathbb{U}(\oplus_{i=1}^{k} G_i)$$

for $1 \leq j \leq k-1$, letting

$$U_k = u_k u_{(k-1,k)} u_{(k-2,k)} \cdots u_{(1,k)} \tag{4.4}$$

and taking the unital irreducible $*$–representation

$$\theta_k = \rho_{[k]}^{(1)} * (\mathrm{Ad}\, U_k \circ \rho_{[k]}^{(2)}) \tag{4.5}$$

of $A_1 * A_2$ on $\bigoplus_{i=1}^{k} G_i$, we have that θ_k is irreducible

(c) F_k is a finite subset of the closed unit ball of $A_1 * A_2$ and for all y in the closed unit ball of $A_1 * A_2$ there is an element x in F_k such that

$$\|\theta_k(x) - \theta_k(y)\| < \frac{1}{2^{k+1}}$$

(d) if $k \geq 2$, then for any element x in the union $\bigcup_{j=1}^{k-1} F_j$, we have

$$\big\|\theta_k(x) - (\theta_{k-1} \oplus \rho_k)(x)\big\| < \frac{1}{2^k}.$$

Note that (4.5) together with Lemmas IV.7 and IV.8 will ensure that θ_k satisfies the RCP condition.

We construct such a sequence $(u_k, F_k)_{k\geq 1}$ by recursion. To start, we construct (u_1, F_1) by letting $\theta_1 = \rho_1$ and $u_1 = \mathrm{id}_{G_1}$. Then conditions (a) and (b) hold trivially. Since $\rho_1 : A_1 * A_2 \to \mathbb{B}(H_1)$ is finite dimensional, Lemma IV.13 implies there is a finite set F_1 contained in the closed unit ball of $A_1 * A_2$ so that condition (c) is satisfied. At this stage condition (d) does not apply.

Let $k \geq 2$ and let us construct (u_k, F_k) from (u_j, F_j), $1 \leq j \leq k-1$. A consequence of (4.4) and (4.5) is the formula

$$\theta_k = (\theta_{k-1} \oplus \rho_k)^{(1)} * (\mathrm{Ad}\, u_k \circ (\theta_{k-1} \oplus \rho_k)^{(2)}).$$

Since θ_{k-1} and ρ_k satisfy the RCP condition, Proposition IV.10 yields a unitary u_k as close as we like to the identity, so that θ_k is irreducible and (4.3) holds. Applying Proposition IV.11 and choosing u_k even closer to the identity, if necessary, we also get that condition (d) holds. Finally, Lemma IV.13 guarantees the existence of a finite set F_k contained in the closed unit ball of $A_1 * A_2$ so that condition (c) is satisfied. This completes the recursive construction of $(u_k, F_k)_{k\geq 1}$ so that (a)–(d) hold.

Now, letting

$$\sigma_k = \theta_k \oplus \oplus_{j\geq k+1} \rho_j. \tag{4.6}$$

we will show that σ_k converges pointwise to an irreducible $*$–representation σ of $A_1 * A_2$. We extend the unitaries u_k to all of H by defining

$$\tilde{u}_k = u_k \oplus_{j\geq k+1} \mathrm{id}_{G_j},$$

and then from (4.5) we obtain

$$\sigma_k = \rho^{(1)} * (\mathrm{Ad}\, \tilde{U}_k \circ \rho^{(2)}),$$

where $\tilde{U}_k = \tilde{u}_k \cdots \tilde{u}_1$. Thanks to condition (4.3) we have

$$\|\tilde{U}_k - \mathrm{id}_H\| \leq \sum_{j=1}^{k} \|\tilde{u}_k - \mathrm{id}_H\| < \sum_{j=1}^{k} \frac{\varepsilon}{2^{k+1}},$$

and for $l \geq 1$

$$\|\tilde{U}_{k+l} - \tilde{U}_k\| = \|\tilde{u}_{k+l} \cdots \tilde{u}_{k+1} - \mathrm{id}_H\| \leq \sum_{j=k+1}^{k+l} \frac{\varepsilon}{2^{j+1}}.$$

Hence Cauchy's criteria implies there is an unitary $U \in \mathbb{U}(H)$ such that the sequence $(\tilde{U}_k)_{k\geq 1}$ converges in norm to U and $\|U - \mathrm{id}_H\| < \frac{\varepsilon}{2}$. Now, by Proposition IV.11, the sequence σ_k converges pointwise to the $*$–representation

$$\sigma = \rho^{(1)} * (\mathrm{Ad}\, U \circ \rho^{(2)}). \tag{4.7}$$

Thus, we have $\lim_k \|\sigma_k - \sigma\| = 0$, where

$$\|\sigma_k - \sigma\| = \sup_{a \in A,\, \|a\|=1} \|\sigma_k(a) - \sigma(a)\|.$$

Our next goal is to show that σ is irreducible. To ease notation let $A = A_1 * A_2$. From (4.2) and Proposition IV.12 we get

$$\sigma_k(A)'' = \mathbb{B}(\oplus_{j=1}^{k} G_j) \oplus \bigoplus_{j\geq k+1} \mathbb{B}(G_j). \tag{4.8}$$

Hence, for all $k \geq 1$, $\sigma_k(A)'' \subseteq \sigma_{k+1}(A)''$. Let B be the norm closure, in $\mathbb{B}(H)$, of $\bigcup_{k\geq 1} \sigma_k(A)''$. Next we will show B'' is contained in $\sigma(A)''$. Take $T \in B''$. Since $\sigma(A)$ is a unital C^*-algebra, showing T lies in $\sigma(A)''$ is equivalent to showing T is in the strong operator topology closure of $\sigma(A)$. Recall that a neighborhood basis for the strong operator topology around T is given by the sets

$$\mathcal{N}_T(\xi_1, \ldots, \xi_i; r) = \{S \in \mathbb{B}(H) : \|T\xi_j - S\xi_j\| < r \quad \text{for all } 1 \leq j \leq i\},$$

where $\xi_1, \ldots, \xi_i$ are unit vectors in H and $0 < r < 1$. We will show that given $\xi_1, \ldots, \xi_i$ and r as above, there is an element z in $A_1 * A_2$ such that $\sigma(z)$ lies in $\mathcal{N}_T(\xi_1, \ldots, \xi_i; r)$. This involves several approximations, so let's start.

By Kaplansky's Density Theorem there is an operator S in B such that $\|S\| \leq \|T\|$ and for all $1 \leq j \leq i$,

$$\|S\xi_j - T\xi_j\| < \frac{r}{100}. \tag{4.9}$$

Since S lies in B, there is k_0 and an operator R in $\sigma_{k_0}(A)''$ such that

$$\|S - R\| < \frac{r}{100}. \tag{4.10}$$

Thus, we have $\|R\| \leq 1 + \|S\| \leq \|T\| + 1$. We can pick $k_1 \geq k_0$ such that

$$\frac{1}{2^{k_1 - 1}} < \frac{r}{100(\|T\| + 2)} \tag{4.11}$$

and for all $1 \leq j \leq i$ we have

$$\|\xi_j - P_{[k_1]}(\xi_j)\| < \frac{r}{100(\|T\| + 2)}, \tag{4.12}$$

where $P_{[m]}$ denotes the orthogonal projection from H onto $\oplus_{j=1}^m G_j$. Since R commutes with $P_{[k_1]}$, this implies

$$\|P_{[k_1]} R P_{[k_1]} \xi_j - R\xi_j\| < \frac{r\,\|R\|}{100(\|T\| + 2)} < \frac{r}{100}. \tag{4.13}$$

Since R lies in $\sigma_{k_0}(A)''$ and $\sigma_{k_0}(A)'' \subseteq \sigma_{k_1}(A)''$ and $\sigma_{k_1}(\cdot) P_{[k_1]} = \theta_{k_1}$, Kadison's transitivity theorem implies there is y in A such that $\|y\| \leq \|R\| + 1$ and for all $1 \leq j \leq i$

$$P_{[k_1]} R P_{[k_1]} \xi_j = \theta_{k_1}(y)(P_{[k_1]}(\xi_j)). \tag{4.14}$$

By construction, there is $x \in F_{k_1}$ such that

$$\|\theta_{k_1}(y) - \theta_{k_1}(\|y\| x)\| < \frac{\|y\|}{2^{k_1}}. \tag{4.15}$$

Take $z = \|y\|x$ and note that we have $\|z\| \leq \|y\| \leq \|T\| + 2$. We will show $\sigma(z)$ lies in $\mathcal{N}_T(\xi_1, \ldots, \xi_i; r)$.

Fix $1 \leq j \leq i$ and for simplicity set $\xi = \xi_j$ and $\eta = P_{[k_1]}(\xi_j)$. We also write $\xi = (\xi(k))_{k\geq 1}$, $\eta = (\eta(k))_{k\geq 1}$ where $\xi(k)$ and $\eta(k)$ are in G_k. Thus $\xi(k) = \eta(k)$ for $1 \leq k \leq k_1$ and $\eta(k) = 0$ for $k > k_1$. Cleary, $\|T\xi - \sigma(z)\xi\|$ is bounded above by the sum of the following terms:

$$\|T\xi - S\xi\| \tag{4.16}$$

$$\|S\xi - R\xi\| \tag{4.17}$$

$$\|R\xi - \sigma_{k_1}(y)\xi\| \tag{4.18}$$

$$\|\sigma_{k_1}(y)\xi - \sigma_{k_1}(z)\xi\| \tag{4.19}$$

$$\|\sigma_{k_1}(z)\xi - \sigma(z)\xi\| \tag{4.20}$$

From (4.9) and (4.10), the terms (4.16) and (4.17) are both less than $\frac{r}{100}$. For the third term (4.18), we have

$$\begin{aligned} \|R\xi - \sigma_{k_1}(y)\xi\| &\leq \|R\xi - P_{[k_1]}RP_{[k_1]}\xi\| \\ &+ \|P_{[k_1]}RP_{[k_1]}\xi - \theta_{k_1}(y)(P_{[k_1]}\xi)\| \\ &+ \|(\oplus_{j>k_1}\rho_j(y))(\xi - P_{[k_1]}(\xi))\| \end{aligned}$$

and from (4.13), (4.14) and (4.12) we deduce that (4.18) is less than $\frac{2r}{100}$. For the fourth term (4.19) we have

$$\begin{aligned} \|\sigma_{k_1}(y)\xi - \sigma_{k_1}(z)\xi\| &\leq \|\sigma_{k_1}(y)(\xi - \eta)\| \\ &+ \|\sigma_{k_1}(y)\eta - \sigma_{k_1}(z)\eta\| \\ &+ \|\sigma_{k_1}(z)(\eta - \xi)\| \end{aligned}$$

which, along with (4.12) and the upper bounds for $\|z\|$ and $\|y\|$, yield

$$\|\sigma_{k_1}(y)\xi - \sigma_{k_1}(z)\xi\| \leq \frac{2r}{100} + \|\sigma_{k_1}(y)\eta - \sigma_{k_1}(z)\eta\|.$$

Now we will show

$$\|\sigma_{k_1}(y)\eta - \sigma_{k_1}(z)\eta\| < \frac{r}{100}. \tag{4.21}$$

From definition of σ_{k_1} (see (4.6)) we get

$$\sigma_{k_1}(y)\eta = (\theta_{k_1}(y)(\eta(1), \dots, \eta(k_1)), 0, \dots)$$

and

$$\sigma_{k_1}(z)\eta = (\theta_{k_1}(z)(\eta(1), \dots, \eta(k_1)), 0, \dots)$$

Hence from condition (4.15) and (4.11) we deduce (4.21). Thus, term (4.19) is less that $\frac{3r}{100}$.

For the fifth term (4.20), since $\lim_k \|\sigma_k(z) - \sigma(z)\| = 0$, there is $k_2 > k_1$ such that $\|\sigma_{k_2}(z) - \sigma(z)\| < \frac{r}{100}$. Hence

$$\|\sigma_{k_1}(z)\xi - \sigma(z)\xi\| \quad < \quad \sum_{k=k_1}^{k_2-1} \|\sigma_i(z)\xi - \sigma_{i+1}(z)\xi\| + \frac{r}{100} \tag{4.22}$$

For $k_I \leq k \leq k_2 - 1$ we have

$$\begin{aligned} \sigma_k(z)\xi &= \|y\|\Big(\theta_k(x)(P_{[k]}\xi), \rho_{k+1}(x)\xi(k+1), \rho_{k+2}(x)\xi(k+2), \dots\Big) \\ \sigma_{k+1}(z)\xi &= \|y\|\Big(\theta_{k+1}(x)(P_{[k+1]}\xi), \rho_{k+2}(x)\xi(k+1), \dots\Big) \end{aligned}$$

Hence condition (d) from the construction of the sequence (u_k, F_k) and (4.11) imply

$$\sum_{k=k_1}^{k_2-1} \|\sigma_k(z)\xi - \sigma_{k+1}(z)\xi\| \leq$$
$$\leq \|y\| \left(\sum_{k=k_1}^{k_2-1} \|(\theta_k \oplus \rho_{k+1})(x)(P_{[k+1]}\xi) - \theta_{k+1}(x)(P_{[k+1]}\xi)\| \right) < \frac{r}{100}$$

Thus, from (4.22) we conclude that the fifth term (4.20) is less that $\frac{2r}{100}$. Putting together all these estimates, we obtain $\|\sigma(z)\xi - T\xi\| < r$.

Thus we have proved $B'' \subseteq \sigma(A)''$. But $B'' = \mathbb{B}(H)$ follows from the fact that $\sigma_k(A)''$ is contained in B'' along with (4.8). In conclusion $\sigma(A)'' = \mathbb{B}(H)$ which implies $\sigma(A)' = \mathbb{C}\,\mathrm{id}_H$ i.e., σ is irreducible.

Now we will show σ is faithful. Recall that, by construction, π is faithful. Using the property (4.1) of π, we will show, inspired by Choi's technique (see Theorem 6 in [4]), that ρ is faithful and

$$\rho(A) \cap \mathbb{K}(H) = \{0\}. \tag{4.23}$$

Recall that we contructed

$$\rho = \pi^{(1)} * (\mathrm{Ad}\, v \circ \pi^{(2)})$$

and $v = \oplus_{k\geq 1} v_k$ where $v_k \in \mathbb{B}(G_k)$. Moreover, $\|v_k - \mathrm{id}_{G_k}\| < \delta_k$ and $\lim_k \delta_k = 0$. So by Lemma IV.14, V differs from the identity operator by a compact operator. It follows that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \pi\ } & \mathbb{B}(H) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi_C} \\ \mathbb{B}(H) & \xrightarrow{\ \pi_C\ } & \mathbb{B}(H)/\mathbb{K}(H) \end{array} \tag{4.24}$$

commutes, where π_C denotes the canonical quotient map onto the Calkin Algebra. Indeed, we see directly that $\pi_C \circ \pi$ and $\pi_C \circ \rho$ agree on elements of $A = A_1 * A_2$ that are words of finite length in elements of A_1 and A_2. However, such words span a dense subalgebra of A. Since $\pi_C \circ \pi$ is faithful and the diagram (4.24) commutes, it follows that ρ is faithful and (4.23) holds.

A second application of Choi's technique will give us faithfulness of σ. Indeed, from construction, for all x in A, $\sigma(x) = \lim_k \sigma_k(x)$. Thus if each σ_k is faithful Lemma IV.15 would imply σ is faithful. But faithfulness of σ_k follows from the commutativity of the diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho} & \mathbb{B}(H) \\ \downarrow{\scriptstyle \sigma_k} & & \downarrow{\scriptstyle \pi_C} \\ \mathbb{B}(H) & \xrightarrow{\pi_C} & \mathbb{B}(H)/\mathbb{K}(H), \end{array}$$

which is implied by (4.6), and the fact that $\pi_C \circ \rho$ is faithful. □

We finish with some straightforward consequences of our main theorem.

Definition IV.17. A C^*-algebra A is called *liminal* if for all irreducible $*$–representations $\pi : A \to \mathbb{B}(H)$ and for all elements a in A, $\pi(a)$ is compact.

Example IV.18. From Proposition I.9, all irreducible $*$–representation of $\mathbb{C}^2 * \mathbb{C}^2$ are of dimension 1 or 2. Hence $\mathbb{C}^2 * \mathbb{C}^2$ is liminal.

Definition IV.19. A C^*-algebra A is called *antiliminal* if $\{0\}$ is the only closed two sided liminal ideal.

Part of Lemma 3.2 of [1] is the following:

Proposition IV.20. *Any infinite dimensional primitive C^*-algebra that admits a faithful tracial state is antiliminal.*

Proof. Assume A is a infinite dimensional primitive C^*-algebra and let I be a closed two sided liminal ideal.

Let $\pi : A \to \mathbb{B}(H)$ be a faithful infinite dimensional irreducible $*$-representation. One can check that π restricted to I is a faithful irreducible $*$-representation of I.

Liminality implies $\pi(I)$ is contained in the compact operators. In addition, if $I \neq \{0\}$, irreducibility implies $\pi(I)$ contains all compact operators.

Thus, the restriction to I of a faithful tracial state of A gives a faithful tracial state on the compacts operators of H, a contradiction since H is infinite dimensional.

□

Corollary IV.21. *Assume A_1 and A_2 are nontrivial finite dimensional C^*-algebras. $A_1 * A_2$ is antiliminal except when $A_1 = \mathbb{C}^2 = A_2$.*

Proof. By a theorem of Exel and Loring [7], a unital C^*-algebra full free product of residually finite dimensional C*–algebras is again residually finite dimensional. Thus, by taking a convergent weighted infinite sum of matrix traces composed with finite dimensional representations (from a separating family of them), the free product C*–algebra $\mathcal{A}_1 * \mathcal{A}_2$ admits a faithful tracial state. □

We finish with a corollary derived from a proposition of Dixmier. The following proposition is Lemma 11.2.4 in [6].

Proposition IV.22. *If A is a unital primitive antiliminal C^*-algebra then pure states are w^*-dense in state space.*

Corollary IV.23. *Assume A_1 and A_2 are nontrivial finite dimensional C^*-algebras. If $A_1 \neq \mathbb{C}^2$ or $A_2 \neq \mathbb{C}^2$, then pure states of $A_1 * A_2$ are w^*-dense in the state space.*

CHAPTER V

CONCLUSION

The main contribution of this dissertation was determining all unital full free product of finite dimensional C*-algebras that are primitive. At a philosophical level it seems there is a basic theme underlying primitive C*-algebras. Namely some type of perturbation or deformation of $*$-representations. This feature is manifested in the works of Choi, Murphy, Bédos and Omland where completely different notions of perturbation are used. We summarize the basic principle behind our approach.

The cornerstone is a theorem that we may call density of C*-subalgebras in general position, Theorem III.6. This theorem is particularly hard to grasp and this is due o the fact that we had to break it into several parts. We mention that at some point we thought we had generalized this theorem as follows: with the same notation as Theorem III.6, if $\dim(B_1)+\dim(B_2) < N^2$ then $\Delta(B_1, B_2)$ is dense. Unfortunately computations turned out to be much harder. We leave this as a conjecture for future research.

Let A_1 and A_2 denote two non-trivial finite dimensional C*-algebras. With Theorem III.6 at our disposition it is easy to prove that, except for the case $A_1 = \mathbb{C}^2 = A_2$, one can find finite dimensional irreducible $*$-representations of dimensions arbitrary large. At this point is worth to mention that $\mathbb{C}^2 * \mathbb{C}^2$ is an illuminating C*-algebra. Not only because it has been studied by many people but because in our investigation it was always a good test case for our claims.

Lastly we took a faithful $*$-representation, constructed as a direct sum of a separating family of finite dimensional $*$-representations and we perturb it, using as a main tool Kaplansky's density theorem, to finally obtain a faithful and irreducible $*$-representation.

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VITA

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