DEFORMATIONS OF QUANTUM SYMMETRIC ALGEBRAS EXTENDED BY GROUPS

A Dissertation

by

JEANETTE MATILDE SHAKALLI TANG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2012

Major Subject: Mathematics

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Approved by:

Chair of Committee,	Sarah Witherspoon
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	Andreas Klappenecker
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ABSTRACT

Deformations of Quantum Symmetric Algebras Extended by Groups. (May 2012)
Jeanette Matilde Shakalli Tang, B.S., University of Notre Dame
Chair of Advisory Committee: Dr. Sarah Witherspoon

The study of deformations of an algebra has been a topic of interest for quite some time, since it allows us to not only produce new algebras but also better understand the original algebra. Given an algebra, finding all its deformations is, if at all possible, quite a challenging problem. For this reason, several specializations of this question have been proposed. For instance, some authors concentrate their efforts in the study of deformations of an algebra arising from an action of a Hopf algebra.

The purpose of this dissertation is to discuss a general construction of a deformation of a smash product algebra coming from an action of a particular Hopf algebra. This Hopf algebra is generated by skew-primitive and group-like elements, and depends on a complex parameter. The smash product algebra is defined on the quantum symmetric algebra of a finite-dimensional vector space and a group. In particular, an application of this result has enabled us to find a deformation of such a smash product algebra which is, to the best of our knowledge, the first known example of a deformation in which the new relations in the deformed algebra involve elements of the original vector space. Finally, using Hochschild cohomology, we show that these deformations are nontrivial.

To my parents, my grandparents, and my sister

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First of all, I would like to thank my parents for their unconditional love and constant support. I am particularly grateful to my dad for sharing his passion for mathematics with me. The hard work ethic that I learned from my mom lies at the heart of all my achievements. I would also like to thank my sister for always believing in me.

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Finally, I would like to give my most sincere apologies to anyone who believes should be mentioned here. The omission is by no means a belittlement of the important role that you have played in my life.

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CHAPTER I

INTRODUCTION

A deformation of an algebra is obtained by slightly modifying its multiplicative structure. Deformations arise in many areas of mathematics, such as combinatorics [6], representation theory [9], and orbifold theory [10]. From this assertion, it is clear that the study of deformations of an algebra is of utmost importance. The main interest in this dissertation is to study deformations arising from an action of a Hopf algebra.

According to Giaquinto's survey paper [17], the aim of studying deformations is to organize objects of some type, in our case algebras, into continuous families and then determine how objects within each family are related. The works of Fröhlicher and Nijenhuis [12], and Kodaira and Spencer [25] on deformations of complex manifolds set the basic foundations of the modern theory of deformations. Gerstenhaber's seminal paper [14] marked the beginning of algebraic deformation theory. Although the origins of deformations lie in analytic theory, the algebraic setting provides a much more general framework.

In general, finding all possible deformations of an algebra is quite a challenging task (see Section IV.A for a detailed discussion on this matter). For a certain kind of Hopf algebra, Witherspoon [37] has found an explicit formula that yields a deformation of its Hopf module algebras. By a Hopf module algebra, we mean an algebra that is also a module, for which the two structures are compatible. Applications of this formula have been studied in [20] and [37] for the case of the smash product algebra of the symmetric algebra of a vector space with a group. The purpose of this dissertation is to extend these results to the case of the smash product algebra of

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the *quantum* symmetric algebra of a vector space with a group, thus increasing the number of known examples of deformations.

To place this work in context, let us briefly describe what is known about deformations. First, let us introduce some notation. In this work, the set of natural numbers $\mathbb{N}$ includes 0. By $\mathbf{k}$, we will denote a field of characteristic 0. Unless stated otherwise, by $\otimes$ we mean $\otimes_{\mathbf{k}}$. Let V be a $\mathbf{k}$-vector space with basis $\{w_1, \ldots, w_k\}$. By $T(V)$ we denote the tensor algebra of V given by

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n}, \quad \text{where} \quad V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n}$$

is the nth tensor power of V. Let $q_{ij} \in \mathbf{k}^{\times}$ for which $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for $i, j = 1, \ldots, k$. Set $\mathbf{q} = (q_{ij})$. Then the quantum symmetric algebra $S_{\mathbf{q}}(V)$ of V is defined as

$$S_{\mathbf{q}}(V) = T(V) \;/\; (w_i w_j - q_{ij} w_j w_i \mid 1 \leq i, j \leq k),$$

where the element $w_i \otimes w_j$ is abbreviated as $w_i w_j$. If $q_{ij} = 1$ for all i, j, then we obtain the symmetric algebra $S(V)$. Let G be a group acting linearly on V. If the action of G on V extends to all of $S_{\mathbf{q}}(V)$, then the smash product algebra $S_{\mathbf{q}}(V)\#G$ is obtained by using $S_{\mathbf{q}}(V) \otimes \mathbf{k}G$ as a vector space but with a new multiplication given by

$$(a\#g)(b\#h) = a\; g(b) \;\#\; gh \quad \text{for all } a, b \in S_{\mathbf{q}}(V),\; g, h \in G.$$

For further details, see Definition II.29 and Example II.31.

Graded Hecke algebras, also known as Drinfeld Hecke algebras [9], can be viewed as deformations of $S(V)\#G$ of type

$$(T(V)\#G) \;/\; (w_i w_j - w_j w_i - \sum_{g \in G} a_g(w_i, w_j)\; g), \quad \text{where } a_g(w_i, w_j) \in \mathbf{k}. \qquad \text{(I.1)}$$

The so-called symplectic reflection algebras [10] and rational Cherednik algebras [6] are special cases of Drinfeld Hecke algebras. Analogously, braided Cherednik algebras [2] are deformations of $S_{\mathbf{q}}(V)\#G$ of type

$$(T(V)\#G) \;/\; (w_i w_j - q_{ij} w_j w_i - \sum_{g\in G} a_g(w_i, w_j)\; g), \qquad \text{where } a_g(w_i, w_j) \in \mathbf{k}. \tag{I.2}$$

For other types of deformations of $S(V)\#G$ and $S_{\mathbf{q}}(V)\#G$, not as much is known, and from what is known, not all the resulting algebras are necessarily defined by generators and relations, but via a universal deformation formula, originally introduced by Giaquinto and Zhang in [18], coming from a Hopf algebra action. Roughly speaking, a Hopf algebra is an algebra with some additional structure. For a precise definition, see Definition II.14.

In his fundamental work [14], Gerstenhaber showed that two commuting derivations on an algebra A lead to a deformation of A. Giaquinto and Zhang generalized this idea in [18] with their theory of universal deformation formulas. The question that naturally arises is what about skew derivations. The answer is that sometimes skew derivations do lead to a deformation via a Hopf algebra action, as shown in [20] and [37] for the case of $S(V)\#G$. The work presented here generalizes these results to the quantum version, that is to the case of $S_{\mathbf{q}}(V)\#G$. This is particularly relevant since explicit examples of deformations of $S_{\mathbf{q}}(V)\#G$ have proven to be difficult to find. Moreover, it turns out that some of the deformations constructed in this dissertation are not graded in the sense of Braverman and Gaitsgory [4] (see Remark IV.18 for a detailed discussion). We are confident that this newly found set of examples will be helpful in the quest to understand deformations more generally.

This dissertation is organized as follows: In Chapter II, Hopf algebras are defined and a detailed exposition of the Hopf algebra H_q is given. This Hopf algebra is fundamental for our discussion, as all the deformations that we will present are based on the action of H_q. Chapter III introduces some basic concepts of homological algebra and their applications to Hochschild cohomology. The core of this dissertation is Chapter IV, where we present the basics of algebraic deformation theory, construct some original examples of deformations, and build a general theory that encompasses our new examples as a special case. In Chapter V, using Hochschild cohomology, which was introduced in Chapter III, we prove that the deformations obtained in Chapter IV are nontrivial. Finally, concluding remarks and possible extensions of this work are given in Chapter VI.

CHAPTER II

HOPF ALGEBRAS

In this chapter, we will present some definitions and results that are necessary to define a Hopf algebra. According to [1], the origins of the theory of Hopf algebras can be traced back to two main sources: algebraic topology and algebraic group theory. Armand Borel was the first to use the expression *Hopf algebra* in [3], in honor of the work of Heinz Hopf [23] on algebraic topology. In [5], Pierre Cartier gave the first formal definition of a Hopf algebra, under the name of *hyperalgebra*, inspired by the work of Jean Dieudonné [7, 8] on algebraic group theory. The theory of Hopf algebras became an independent part of abstract algebra with the publication of Sweedler's book [35] in 1969.

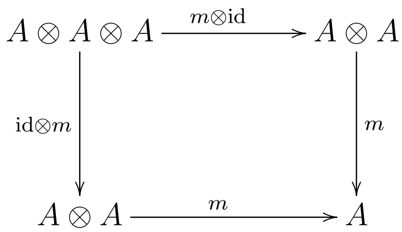
Unless stated otherwise, the ideas discussed here can be found in [30] and [35]. In Section A, we recall the basic concepts of an algebra and a coalgebra. Section B introduces the notion of a Hopf algebra, which is fundamental for this work, and gives a detailed description of the Hopf algebra H_q. Section C discusses modules, Hopf module algebras and smash product algebras. In subsequent chapters, we will be interested in the deformations of the smash product algebras that are H_q-module algebras. Finally, graded and filtered algebras are introduced in Section D.

A. Algebras and Coalgebras

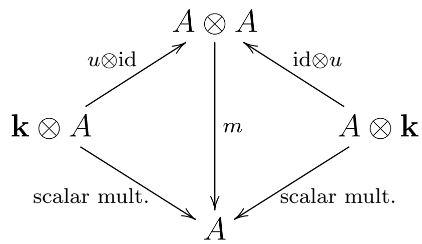
The concept of an algebra is fundamental in mathematics. Its definition and basic properties can be found in several references, for instance see [24]. The definition that we present here can be easily dualized.

Definition II.1. A $\mathbf{k}$-*algebra* is a $\mathbf{k}$-vector space A together with two $\mathbf{k}$-linear maps, multiplication $m : A \otimes A \to A$ and unit $u : \mathbf{k} \to A$, such that the following diagrams commute:

a) associativity



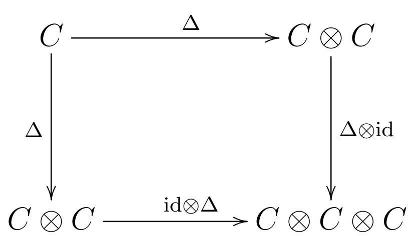
b) unit



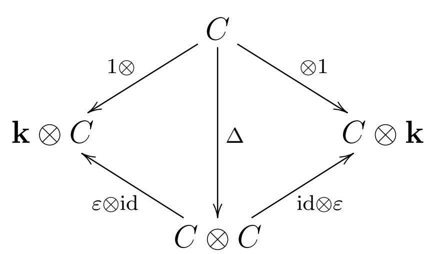
If the underlying field is understood from the context, a **k**-algebra will simply be called an algebra. By dualizing the notion of an algebra, we obtain the following:

Definition II.2. A **k**-*coalgebra* is a **k**-vector space C together with two **k**-linear maps, comultiplication $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to \mathbf{k}$, such that the following diagrams commute:

a) coassociativity



b) counit



We will use the terms comultiplication and coproduct interchangeably.

For any **k**-spaces V and W, the *twist map* $\tau : V \otimes W \to W \otimes V$ is given by $\tau(v \otimes w) = w \otimes v$. We say that an algebra A is commutative if and only if $m \circ \tau = m$. Similarly, a coalgebra C is cocommutative if and only if $\tau \circ \Delta = \Delta$.

Definition II.3. Let C and D be coalgebras, with comultiplications Δ_C and Δ_D, and counits ε_C and ε_D, respectively. Then

- A map $f : C \to D$ is a *coalgebra morphism* if the following two conditions hold:

$$\Delta_D \circ f = (f \otimes f) \circ \Delta_C \quad \text{and} \quad \varepsilon_C = \varepsilon_D \circ f.$$

- A subspace $I \subseteq C$ is a *coideal* if the following two conditions hold:

$$\Delta I \subseteq I \otimes C + C \otimes I \quad \text{and} \quad \varepsilon(I) = 0.$$

Next, let us look at some examples.

Example II.4. Let A be an algebra and C a coalgebra. Then the *opposite algebra* A^{op} is obtained by using A as a vector space but with new multiplication m' given by $m' = m \circ \tau$, where τ is the twist map on $A \otimes A$. Similarly, the *coopposite coalgebra* C^{cop} is obtained by using C as a vector space but with new comultiplication Δ' given by $\Delta' = \tau \circ \Delta$, where τ is the twist map on $C \otimes C$.

Notation. Let C be a coalgebra. Then the *sigma notation* for Δ is given by

$$\Delta(c) = \sum c_1 \otimes c_2 \qquad \text{for all } c \in C. \tag{II.5}$$

The subscripts on the right hand side of (II.5) are just symbolic and are not meant to designate particular elements of C. Notice that, under the sigma notation,

- the coassociativity diagram in Definition II.2 gives

$$\Delta_{n-1}(c) = \sum c_1 \otimes \cdots \otimes c_n,$$

  where $\Delta_{n-1}(c)$ is the (necessarily unique) element obtained by applying the comultiplication $(n-1)$ times.

- the counit diagram in Definition II.2 says that for all $c \in C$,

$$c = \sum \varepsilon(c_1)\, c_2 = \sum \varepsilon(c_2)\, c_1.$$

Finally, we will introduce the notion of group-like and skew-primitive elements.

Definition II.6. Let C be a coalgebra and let $c \in C$.

- We say that the element c is *group-like* if

$$\Delta(c) = c \otimes c \quad \text{and} \quad \varepsilon(c) = 1.$$

  The set of group-like elements in C is denoted by $G(C)$.

- The element c is *g, h-primitive* if there exist $g, h \in G(C)$ such that

$$\Delta(c) = c \otimes g + h \otimes c.$$

  The set of g, h-primitive elements is denoted by $P_{g,h}(C)$. An element is called *skew primitive* if it is g, h-primitive for some g, h.

B. Bialgebras, Convolution, and Hopf Algebras

The purpose of this section is to define a Hopf algebra, which is essential for this dissertation. Roughly speaking, a Hopf algebra is an object that has the structure of an algebra and a coalgebra, these two structures are compatible, and in addition, there exists a map known as the antipode with very specific properties. The following definitions make this notion more precise.

Definition II.7. Let B be a $\mathbf{k}$-vector space. We say that $(B, m, u, \Delta, \varepsilon)$ is a *bialgebra* if

- (B, m, u) is an algebra.
- (B, Δ, ε) is a coalgebra.
- Either of the following (equivalent) conditions holds:

- $\star$ Δ and ε are algebra morphisms.
- $\star$ m and u are coalgebra morphisms.

When no confusion arises, we will simply say that B is a bialgebra.

Definition II.8. Let B and B' be bialgebras. Then

- A map $f : B \to B'$ is a *bialgebra morphism* if it is both an algebra morphism and a coalgebra morphism.
- A subspace $I \subseteq B$ is a *biideal* if it is both an ideal and a coideal.

Remark II.9. It can be shown that the quotient of a bialgebra by a biideal is again a bialgebra.

Example II.10. Let G be a group. The *group algebra* $\mathbf{k}G$ is defined to be

$$\mathbf{k}G = \left\{ \sum_{g \in G} a_g \, g \;\middle|\; a_g \in \mathbf{k} \text{ with } a_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

The group algebra can be given the structure of a bialgebra by defining

$$\begin{aligned} \left(\sum_{g \in G} a_g \, g\right)\left(\sum_{g \in G} b_g \, g\right) &= \sum_{g,h \in G} (a_g \, b_h) \, gh \\ a\left(\sum_{g \in G} a_g \, g\right) &= \sum_{g \in G} (a \, a_g) \, g \\ \Delta(g) &= g \otimes g \\ \varepsilon(g) &= 1 \end{aligned}$$

for all $g \in G, \; a_g, b_g, a \in \mathbf{k}$.

Remark II.11. In Definition II.6, if B is a bialgebra and $g = h = 1$, the multiplicative identity of B, then the elements of $P(B) = P_{1,1}(B)$ are called the *primitive elements* of B.

Let us now define the convolution product. For two **k**-vector spaces V and W, we denote by $\mathrm{Hom}_{\mathbf{k}}(V, W)$ the set of all **k**-linear maps from V to W.

Definition II.12. Let A be an algebra and C a coalgebra. Then the *convolution product* $* : \mathrm{Hom}_{\mathbf{k}}(C, A) \otimes \mathrm{Hom}_{\mathbf{k}}(C, A) \to \mathrm{Hom}_{\mathbf{k}}(C, A)$ is given by

$$(f * g)(c) = (m \circ (f \otimes g) \circ \Delta)(c),$$

for all $f, g \in \mathrm{Hom}_{\mathbf{k}}(C, A),\ c \in C$.

Remark II.13. It is possible to show that $\mathrm{Hom}_{\mathbf{k}}(C, A)$ can be given the structure of an algebra by setting the convolution product $*$ as the multiplication, and $u \circ \varepsilon$ as the unit element. Moreover, under the sigma notation II.5, the convolution product can be written as

$$(f * g)(c) = \sum f(c_1)\ g(c_2) \quad \text{for all } f, g \in \mathrm{Hom}_{\mathbf{k}}(C, A),\ c \in C.$$

Now we are ready to define a Hopf algebra and its antipode.

Definition II.14. Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra. Then H is a *Hopf algebra* if there exists an element $S \in \mathrm{Hom}_{\mathbf{k}}(H, H)$ which is an inverse to the identity map id_H under the convolution product $*$, i.e. S satisfies

$$\sum S(h_1)\ h_2 = \varepsilon(h)\ 1_H = \sum h_1\ S(h_2) \quad \text{for all } h \in H.$$

S is called an *antipode* for H.

Definition II.15. Let H and K be Hopf algebras. Then

- A map $f : H \to K$ is a *Hopf morphism* if it is a bialgebra morphism and $(f \circ S_H)(h) = (S_K \circ f)(h)$ for all $h \in H$.
- A subspace $I \subseteq H$ is a *Hopf ideal* if it is a biideal and $S(I) \subseteq I$.

Remark II.16. It is possible to show that the quotient of a Hopf algebra by a Hopf ideal is again a Hopf algebra.

Let us illustrate the concept of a Hopf algebra by means of two simple examples.

Example II.17. The group algebra $\mathbf{k}G$ presented in Example II.10 can be given the structure of a Hopf algebra by defining

$$S(g) = g^{-1} \quad \text{for all } g \in G.$$

Example II.18. Let n be a positive integer ($n \geq 2$) and let $q \in \mathbb{C}$ be a primitive nth root of unity. The *Taft algebra* is defined to be

$$T_n = \mathbb{C}\langle g, x \mid g^n = 1,\ x^n = 0,\ xg = q\,gx\rangle.$$

It is possible to show that T_n is a Hopf algebra with

$$\begin{aligned} \Delta(g) &= g \otimes g, & \Delta(x) &= x \otimes 1 + g \otimes x, \\ \varepsilon(g) &= 1, & \varepsilon(x) &= 0, \\ S(g) &= g^{-1}, & S(x) &= -g^{-1}x. \end{aligned}$$

Some elementary properties of the antipode are stated in the following:

Proposition II.19. *Let H be a Hopf algebra with antipode S. Then*

- *S is an* anti-algebra morphism, *i.e.*

$$S(hk) = S(k)S(h) \quad \text{and} \quad S(1_H) = 1_H \qquad \text{for all } h, k \in H.$$

- *S is an* anti-coalgebra morphism, *i.e.*

$$\sum (S(h))_1 \otimes (S(h))_2 = \sum S(h_2) \otimes S(h_1) \quad \text{and} \quad \varepsilon(S(h)) = \varepsilon(h) \qquad \text{for all } h \in H.$$

Proof. See Proposition 4.0.1 in [35]. □

We end this section with an example of a Hopf algebra that will be needed throughout this work. This Hopf algebra appeared in [37], where it was used to derive a deformation formula.

1. The Hopf Algebra H_q

Let $q \in \mathbb{C}^{\times}$. The quantum integer $(i)_q$ is given by

$$(i)_q = 1 + q + q^2 + \cdots + q^{i-1} \quad \text{with } (0)_q = 0. \tag{II.20}$$

Let $n \in \mathbb{N}$, $n \geq 2$. Notice that if q is a primitive nth root of unity, then $(n)_q = 0$. The quantum factorial $(i)_q!$ is defined as

$$(i)_q! = (i)_q\,(i-1)_q \cdots (1)_q \quad \text{with } (0)_q! = 1.$$

The quantum binomial coefficient $\binom{i}{k}_q$ is

$$\binom{i}{k}_q = \frac{(i)_q!}{(k)_q!\,(i-k)_q!}. \tag{II.21}$$

Notice that if q is a primitive nth root of unity, then

$$\binom{n}{k}_q = 0 \quad \text{for } k = 1, \ldots, n-1. \tag{II.22}$$

Let H be the algebra generated by D_1, D_2, σ and σ^{-1} subject to the following relations:

$$D_1 D_2 = D_2 D_1,$$

$$\sigma\sigma^{-1} = \sigma^{-1}\sigma = 1,$$

$$q\sigma D_i = D_i \sigma \quad \text{for } i = 1, 2.$$

Proposition II.23. *H is a Hopf algebra with*

$$\Delta(D_1) = D_1 \otimes \sigma + 1_H \otimes D_1, \qquad \varepsilon(D_1) = 0, \qquad S(D_1) = -D_1\, \sigma^{-1},$$
$$\Delta(D_2) = D_2 \otimes 1_H + \sigma \otimes D_2, \qquad \varepsilon(D_2) = 0, \qquad S(D_2) = -\sigma^{-1}\, D_2,$$
$$\Delta(\sigma) = \sigma \otimes \sigma, \qquad \varepsilon(\sigma) = 1, \qquad S(\sigma) = \sigma^{-1}.$$

Sketch of Proof. Let us check that

$$\sum S(h_1)\, h_2 = \varepsilon(h)\, 1_H \quad \text{for all } h \in H.$$

It suffices to show that this condition holds for the generators of H.

Since $\Delta(D_1) = D_1 \otimes \sigma + 1_H \otimes D_1$,

$$S(D_1)\, \sigma + S(1_H)\, D_1 = -D_1\, \sigma^{-1}\, \sigma + D_1 = 0 = \varepsilon(D_1)\, 1_H.$$

Since $\Delta(D_2) = D_2 \otimes 1_H + \sigma \otimes D_2$,

$$S(D_2)\, 1_H + S(\sigma)\, D_2 = -\sigma^{-1}\, D_2 + \sigma^{-1}\, D_2 = 0 = \varepsilon(D_2)\, 1_H.$$

Since $\Delta(\sigma) = \sigma \otimes \sigma$,

$$S(\sigma)\, \sigma = \sigma^{-1}\, \sigma = 1_H = \varepsilon(\sigma)\, 1_H.$$

The rest of the conditions can be checked similarly.

Let us now show that

$$\Delta(D_1 D_2) = \Delta(D_2 D_1).$$

Since we require Δ to be an algebra homomorphism, we have that

$$\begin{aligned}\Delta(D_1D_2) &= \Delta(D_1)\, \Delta(D_2) \\ &= (D_1 \otimes \sigma + 1_H \otimes D_1)(D_2 \otimes 1_H + \sigma \otimes D_2) \\ &= D_1D_2 \otimes \sigma + D_2 \otimes D_1 + D_1\sigma \otimes \sigma D_2 + \sigma \otimes D_1D_2\end{aligned}$$

and

$$\begin{aligned}\Delta(D_2D_1) &= \Delta(D_2)\,\Delta(D_1)\\ &= (D_2 \otimes 1_H + \sigma \otimes D_2)(D_1 \otimes \sigma + 1_H \otimes D_1)\\ &= D_2D_1 \otimes \sigma + D_2 \otimes D_1 + \sigma \otimes D_2D_1 + \sigma D_1 \otimes D_2\sigma\\ &= D_1D_2 \otimes \sigma + D_2 \otimes D_1 + \sigma \otimes D_1D_2 + D_1\sigma \otimes \sigma D_2\end{aligned}$$

and therefore, $\Delta(D_1D_2) = \Delta(D_2D_1)$. The rest of the relations can be verified in a similar fashion. □

Remark II.24. By Definition II.6, σ is group-like, D_1 is $\sigma, 1_H$-primitive, and D_2 is $1_H, \sigma$-primitive.

Recall that $n \in \mathbb{N}$, $n \geq 2$. Let I be the ideal of H generated by D_1^n and D_2^n.

Proposition II.25. *If q is a primitive nth root of unity, then I is a Hopf ideal.*

Proof. By Definition II.15, we need to show that I is a coideal, that is $\Delta(I) \subseteq I \otimes H + H \otimes I$ and $\varepsilon(I) = 0$, and that I satisfies $S(I) \subseteq I$. Notice that it suffices to show that these conditions hold for the generators of I. Recall that the quantum binomial coefficient was defined in (II.21).

Since $\Delta(D_1) = D_1 \otimes \sigma + 1_H \otimes D_1$, it can be shown by induction that

$$\Delta(D_1^n) = D_1^n \otimes \sigma^n + \sum_{k=1}^{n-1} \binom{n}{k}_{q^{-1}} D_1^k \otimes D_1^{n-k}\,\sigma^k + 1_H \otimes D_1^n,$$

which can be simplified by (II.22) to

$$\Delta(D_1^n) = D_1^n \otimes \sigma^n + 1_H \otimes D_1^n.$$

Notice that $\Delta(D_1^n) \in I \otimes H + H \otimes I$. Similarly, since $\Delta(D_2) = D_2 \otimes 1_H + \sigma \otimes D_2$, it can be shown by induction that

$$\Delta(D_2^n) = D_2^n \otimes 1_H + \sum_{k=1}^{n-1} \binom{n}{k}_{q^{-1}} D_2^k \, \sigma^{n-k} \otimes D_2^{n-k} + \sigma^n \otimes D_2^n$$
$$= D_2^n \otimes 1_H + \sigma^n \otimes D_2^n.$$

Again notice that $\Delta(D_2^n) \in I \otimes H + H \otimes I$.

Since $\varepsilon(D_1) = \varepsilon(D_2) = 0$, clearly

$$\varepsilon(D_1^n) = \varepsilon(D_2^n) = 0.$$

Since $S(D_1) = -D_1\, \sigma^{-1}$, it can be shown by induction that

$$S(D_1^n) = (-1)^n \, q^{(n^2-n)/2} \, D_1^n \, \sigma^{-n}.$$

Notice that $S(D_1^n) \in I$. Similarly, since $S(D_2) = -\sigma^{-1}\, D_2$, it can be shown by induction that

$$S(D_2^n) = (-1)^n \, q^{(n^2+n)/2} \, D_2^n \, \sigma^{-n}.$$

Again notice that $S(D_2^n) \in I$. Therefore, I is a Hopf ideal. □

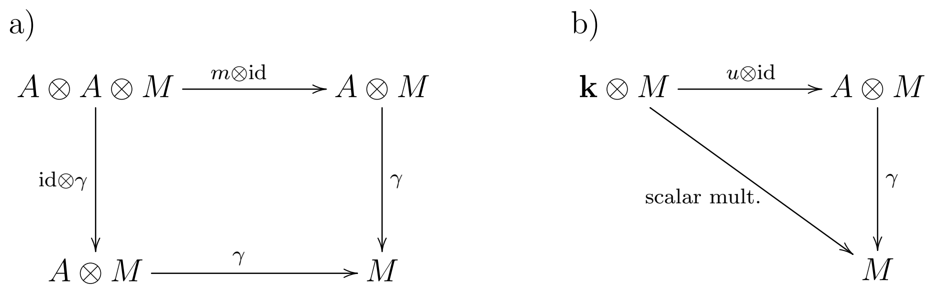
If q is a primitive nth root of unity, then by Proposition II.25 and Remark II.16, the quotient H/I is a Hopf algebra. Define

$$H_q = \begin{cases} H/I, & \text{if } q \text{ is a primitive } n\text{th root of unity } (n \geq 2), \\ H, & \text{if } q = 1 \text{ or } q \text{ is not a root of unity.} \end{cases}$$

C. Modules, Hopf Module Algebras, and Smash Product Algebras

In Chapter IV, we will study the smash product algebras that have the structure of a Hopf module algebra for a particular kind of Hopf algebra. In this section, we present the definition of a Hopf module algebra and a smash product algebra. In order to define a Hopf module algebra, we first need to introduce the concept of a module.

Definition II.26. Let A be a $\mathbf{k}$-algebra. A *left A-module* is a $\mathbf{k}$-vector space M together with a $\mathbf{k}$-linear map $\gamma : A \otimes M \to M$ such that the following diagrams commute:



Similarly, we can define a right A-module. For the sake of brevity, we skip the details.

Having defined a module, we can now introduce the tensor product of modules. Let H be a Hopf algebra, and V and W left H-modules with structure maps γ_V and γ_W, respectively. Then $V \otimes W$ is a left H-module via

$$\gamma_{V\otimes W} = (\gamma_V \otimes \gamma_W) \circ (\mathrm{id}_H \otimes \tau \otimes \mathrm{id}_W) \circ (\Delta \otimes \mathrm{id}_V \otimes \mathrm{id}_W) : H \otimes V \otimes W \to V \otimes W,$$

given by

$$h(v \otimes w) = \sum h_1(v) \otimes h_2(w) \quad \text{for all } h \in H,\ v \in V,\ w \in W,$$

where τ is the twist map on $H \otimes V$. By a similar procedure, if V and W are right H-modules, then $V \otimes W$ can be given the structure of a right H-module.

Roughly speaking, a Hopf module algebra is an algebra that is also a module, for which the two structures are compatible. The following makes this statement more precise.

Definition II.27. Let H be a Hopf algebra. An algebra A is a *left H-module algebra* if the following conditions hold:

- A is a left H-module, via $h \otimes a \mapsto h(a)$.
- m_A and u_A are H-module maps, i.e.

$$h(ab) = \sum h_1(a)\, h_2(b) \text{ and } h(1_A) = \varepsilon(h)\, 1_A \text{ for all } h \in H,\ a, b \in A. \tag{II.28}$$

Next, we introduce the concept of a smash product algebra.

Definition II.29. Let H be a Hopf algebra and A a left H-module algebra. The *smash product algebra* $A\#H$ is obtained by using $A \otimes H$ as a vector space but with new multiplication given by

$$(a\#h)\,(b\#k) = \sum a\, h_1(b) \;\#\; h_2 k \quad \text{for all } a, b \in A,\ h, k \in H,$$

where we write $a\#h$ for the element $a \otimes h$ to emphasize that the multiplication is different from the usual tensor product of algebras.

Remark II.30. Skew group algebras are a special case of smash product algebras in which the Hopf algebra is a group algebra (see Example II.31). Crossed product algebras are a generalization of smash product algebras.

Example II.31. Recall from Example II.17 the definition of the group algebra. Let G be a group and $\mathbf{k}G$ its group algebra. Let A be a left $\mathbf{k}G$-module algebra, that is

$$g(ab) = g(a)\, g(b) \text{ and } g(1_A) = 1_A \quad \text{for all } a, b \in A,\ g \in G.$$

In this case, the multiplication in $A\#\mathbf{k}G$ is given by

$$(a\#g)(b\#h) = a\, g(b) \;\#\; gh \quad \text{for all } a, b \in A,\ g, h \in G.$$

For simplicity, we denote the smash product algebra $A\#\mathbf{k}G$ by $A\#G$.

Finally, we define the notion of a skew derivation. For an algebra A, we denote by $\mathrm{Aut}_{\mathbf{k}}(A)$ the group of all algebra automorphisms of A that preserve the multiplicative identity.

Definition II.32. Let $g, h \in \mathrm{Aut}_{\mathbf{k}}(A)$. A *$g, h$-skew derivation* of A is a $\mathbf{k}$-linear function $F : A \to A$ such that

$$F(ab) = F(a)\, g(b) + h(a)\, F(b) \quad \text{ for all } a, b \in A.$$

If $g = h = \mathrm{id}_A$, then we say that F is a *derivation* of A.

Recall that skew primitive and primitive elements were introduced in Definition II.6 and Remark II.11.

Example II.33. Let H be a Hopf algebra and A an H-module algebra. Then every primitive element of H acts as a derivation of A. More generally, if $g, h \in G(H)$, then every g, h-primitive element of H acts as a g, h-skew derivation of A.

D. Graded and Filtered Algebras

In the literature, many of the deformations that are described are graded in the sense of Braverman and Gaitsgory [4]. The deformations that are constructed in this dissertation are not necessarily graded in this sense (see Remark IV.18). Here we define when an algebra is called graded and present some examples. Then we define a filtered algebra and give several basic properties.

Definition II.34. An algebra A is *graded* if there exist subspaces $A_0, A_1, \ldots$ of A such that

$$A = \bigoplus_{i \in \mathbb{N}} A_i \quad \text{and} \quad A_i\, A_j \subseteq A_{i+j} \quad \text{for all } i, j.$$

Let us illustrate this definition by means of two examples.

Example II.35. The polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ is graded. To see this, it suffices to note that

$$\mathbb{C}[x_1, \ldots, x_n] = \bigoplus_{k \in \mathbb{N}} P_k,$$

where P_k denotes the subspace of homogeneous polynomials of degree k.

Example II.36. From the definition of $T(V)$, introduced in Chapter I, it directly follows that the tensor algebra is graded.

Next, we introduce the notion of a filtered algebra.

Definition II.37. An algebra A is *filtered* if there exists an increasing sequence of subspaces

$$\{0\} \subset F_0 \subset F_1 \subset \cdots \subset F_i \subset \cdots \subset A$$

such that

$$A = \bigcup_{i \in \mathbb{N}} F_i \quad \text{and} \quad F_i\, F_j \subseteq F_{i+j} \quad \text{for all } i, j.$$

Example II.38. The polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ is filtered. To see this, it suffices to notice that

$$\mathbb{C}[x_1, \ldots, x_n] = \bigcup_{\ell \in \mathbb{N}} \mathbb{C}_\ell[x_1, \ldots, x_n],$$

where $\mathbb{C}_\ell[x_1, \ldots, x_n]$ denotes the subspace of polynomials of degree no greater than ℓ.

In Chapter IV, we will make use of the following example:

Example II.39. Let V be a $\mathbf{k}$-vector space and let G be a group acting linearly on V. Extend the action of G on V to all of $T(V)$ by algebra automorphisms. Then the resulting smash product algebra $T(V)\#G$ is filtered. To see this, set the degree of the vector space elements to be 1 and the degree of the group elements to be 0. Then define

$$
\begin{aligned}
S_0 &= \mathbf{k}G \\
S_1 &= \mathbf{k}G \oplus (V \otimes \mathbf{k}G) \\
S_2 &= \mathbf{k}G \oplus (V \otimes \mathbf{k}G) \oplus (V \otimes V \otimes \mathbf{k}G) \\
&\vdots \\
S_i &= \mathbf{k}G \oplus (V \otimes \mathbf{k}G) \oplus \cdots \oplus (V^{\otimes i} \otimes \mathbf{k}G) \\
&\vdots
\end{aligned}
$$

where the notation $V^{\otimes i}$ was introduced in Chapter I. It is straightforward to show that

$$T(V)\#G = \bigcup_{i \in \mathbb{N}} S_i \quad \text{and} \quad S_i\, S_j \subseteq S_{i+j} \quad \text{for all } i, j.$$

Remark II.40. Notice that every graded algebra is filtered. To see this, define

$$F_i = \bigoplus_{k=0}^{i} A_k,$$

where F_i are the subspaces coming from Definition II.37 and A_k are the subspaces in Definition II.34.

Remark II.41. The deformations (I.1) and (I.2) are filtered algebras.

We are now ready to define an associated graded algebra of a filtered algebra.

Definition II.42. The *associated graded algebra* of a filtered algebra A is given by

$$\mathrm{gr}(A) = \bigoplus_{i \in \mathbb{N}} F_i(A) \,/\, F_{i-1}(A),$$

with multiplication induced by the multiplication on A.

Remark II.43. If A is a graded algebra, then $\mathrm{gr}(A) \cong A$.

CHAPTER III

HOMOLOGICAL ALGEBRA

The goal of this chapter is to define Hochschild cohomology, which we will use in Chapter V to show that the deformations obtained in Chapter IV are nontrivial. For this purpose, in Section A, we give a, by far nonexhaustive, review of some basic concepts from homological algebra. For further details, we refer the reader to [21]. Section B introduces the notion of Hochschild cohomology of an algebra and shows how to determine it using the bar resolution. Hochschild cocycles are described in Section C and Hochschild coboundaries are discussed in Section D. The main ideas of Hochschild cohomology can be found in [22] and [13].

A. Free Resolutions and the Functor Ext

In this short section, we recall the definition of a free resolution of a module over a ring. Then we construct the functor Ext, which will be used to define Hochschild cohomology in the next section.

Let R be a ring.

Definition III.1. Let M be an R-module. A *free R-resolution* of M is an exact sequence of R-module homomorphisms

$$\cdots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\varepsilon} M \to 0,$$

i.e. $\mathrm{Im}(\delta_{i+1}) = \mathrm{Ker}(\delta_i)$ for all $i \geq 0$ with $\delta_0 = \varepsilon$, where P_i is a free R-module for all $i \geq 0$.

Using a free resolution, we can define the functor Ext as follows:

Definition III.2. Let M, N be two R-modules and let

$$\cdots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

be a free R-resolution of M. Applying $\mathrm{Hom}_R(-, N)$ and dropping the term $\mathrm{Hom}_R(M, N)$, we obtain the sequence

$$0 \to \mathrm{Hom}_R(P_0, N) \xrightarrow{\delta_1^*} \mathrm{Hom}_R(P_1, N) \xrightarrow{\delta_2^*} \mathrm{Hom}_R(P_2, N) \xrightarrow{\delta_3^*} \cdots$$

which may no longer be exact. The maps δ_i^* are given by $\delta_i^*(f) = f \circ \delta_i$ for all $i \geq 1$, and δ_0^* is the zero map. It can be shown that $\delta_{i+1}^* \circ \delta_i^* = 0$. Then $\mathrm{Ext}_R^i(M, N)$ is defined as the following quotient of vector spaces:

$$\mathrm{Ext}_R^i(M, N) = \mathrm{Ker}(\delta_{i+1}^*) \;/\; \mathrm{Im}(\delta_i^*)$$

and $\mathrm{Ext}_R^\bullet(M, N)$ is defined as

$$\mathrm{Ext}_R^\bullet(M, N) = \bigoplus_{i \geq 0} \mathrm{Ext}_R^i(M, N).$$

Remark III.3. It may be shown that this definition does not depend on the choice of resolution. Moreover, it is enough to have a resolution by projective modules. For a detailed discussion, we refer the reader to Section 3.2 in [21].

B. Hochschild Cohomology

In this section, we define the Hochschild cohomology of an algebra over a field. Let A be a **k**-algebra. Recall from Example II.4 the definition of the opposite algebra A^{op} of A. Let $A^e = A \otimes A^{op}$. Then A is a left A^e-module (or equivalently, an A-bimodule under multiplication) via

$$(a \otimes b)(c) = acb \quad \text{for all } a \otimes b \in A^e,\ c \in A. \tag{III.4}$$

Notice that for $n \geq 2$, $A^{\otimes n}$ can be given the structure of a left A^e-module by extending (III.4), namely if $a \otimes b \in A^e$ and $a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n \in A^{\otimes n}$, then

$$(a \otimes b)(a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n) = a\, a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n\, b. \tag{III.5}$$

Definition III.6. Let M be a left A^e-module. The *Hochschild cohomology* of A is defined as

$$\mathrm{HH}^\bullet(A, M) = \mathrm{Ext}^\bullet_{A^e}(A, M).$$

If $M = A$, then we denote $\mathrm{HH}^\bullet(A, A)$ simply by $\mathrm{HH}^\bullet(A)$.

It is possible to find the Hochschild cohomology of A by considering the free A^e-resolution of A, which is given by

$$\cdots \xrightarrow{\delta_3} A^{\otimes 4} \xrightarrow{\delta_2} A^{\otimes 3} \xrightarrow{\delta_1} A^e \xrightarrow{m} A \to 0, \tag{III.7}$$

where m is the multiplication in A and the maps δ_i are defined as

$$\delta_i(a_0 \otimes a_1 \otimes \cdots \otimes a_{i+1}) = \sum_{j=0}^{i} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}. \tag{III.8}$$

This resolution is known as the *bar resolution* of A. Applying $\mathrm{Hom}_{A^e}(-, A)$ and dropping the term $\mathrm{Hom}_{A^e}(A, A)$, we obtain

$$0 \to \mathrm{Hom}_{A^e}(A^e, A) \xrightarrow{\delta_1^*} \mathrm{Hom}_{A^e}(A^{\otimes 3}, A) \xrightarrow{\delta_2^*} \mathrm{Hom}_{A^e}(A^{\otimes 4}, A) \xrightarrow{\delta_3^*} \cdots$$

where $\delta_i^*(f) = f \circ \delta_i$ for $i \geq 1$ and $f \in \mathrm{Hom}_{A^e}(A^{\otimes(i+1)}, A)$. Then

$$\mathrm{HH}^i(A) = \mathrm{Ker}(\delta_{i+1}^*) \,/\, \mathrm{Im}(\delta_i^*) \quad \text{and} \quad \mathrm{HH}^\bullet(A) = \bigoplus_{i \geq 0} \mathrm{HH}^i(A). \tag{III.9}$$

Remark III.10. When G is finite, the Hochschild cohomology of $S(V)\#G$ was computed in [11] and [19]. If G is finite and acts diagonally on V, then the Hochschild cohomology ring of $S_{\mathbf{q}}(V)\#G$ is described in [31]. This last result will be used in

Chapter V.

C. Hochschild Cocycles

Using the bar resolution (III.7) of A, we will define Hochschild cocycles. Let us begin by introducing Hochschild 1-cocycles. Let $f \in \mathrm{Hom}_{A^e}(A^{\otimes 3}, A)$. It is straightforward to check that the map

$$\mathrm{Hom}_{A^e}(A^{\otimes 3}, A) \to \mathrm{Hom}_{\mathbf{k}}(A, A)$$
$$h \mapsto (a \mapsto h(1 \otimes a \otimes 1))$$

is an isomorphism of vector spaces. Thus, we may identify f with a $\mathbf{k}$-linear map from A to A. By definition, if $f \in \mathrm{Ker}(\delta_2^*)$, then

$$0 = \delta_2^*(f)(a \otimes b \otimes c \otimes d) = (f \circ \delta_2)(a \otimes b \otimes c \otimes d) \quad \text{for all } a, b, c, d \in A.$$

Since $\delta_2^*(f) \in \mathrm{Hom}_{A^e}(A^{\otimes 4}, A)$ and it is possible to check that the map

$$\mathrm{Hom}_{A^e}(A^{\otimes 4}, A) \to \mathrm{Hom}_{\mathbf{k}}(A^{\otimes 2}, A)$$
$$h \mapsto ((a \otimes b) \mapsto h(1 \otimes a \otimes b \otimes 1))$$

induces an isomorphism of vector spaces, $\delta_2^*(f)$ may be identified with an element of $\mathrm{Hom}_{\mathbf{k}}(A^{\otimes 2}, A)$. Since $f \in \mathrm{Ker}(\delta_2^*)$, f must then satisfy

$$(f \circ \delta_2)(1 \otimes b \otimes c \otimes 1) = 0 \quad \text{for all } b, c \in A.$$

By (III.8) and since f is a module morphism, we get that

$$\begin{aligned} 0 = (f \circ \delta_2)(1 \otimes b \otimes c \otimes 1) &= f(b \otimes c \otimes 1 - 1 \otimes bc \otimes 1 + 1 \otimes b \otimes c) \\ &= f(b \otimes c \otimes 1) - f(1 \otimes bc \otimes 1) + f(1 \otimes b \otimes c) \\ &= b\ f(1 \otimes c \otimes 1) - f(1 \otimes bc \otimes 1) + f(1 \otimes b \otimes 1)\ c \end{aligned}$$

Therefore, an element $f \in \operatorname{Ker}(\delta_2^*)$ can be identified with a linear map $A \to A$ that satisfies

$$a\ f(b) - f(ab) + f(a)\ b = 0 \quad \text{for all } a, b \in A. \tag{III.11}$$

Such a map is called a *Hochschild* 1*-cocycle*.

Before moving on to Hochschild 2-cocycles, let us state the relation between Hochschild 1-cocycles and derivations. Recall the definition of derivations and skew derivations introduced in Definition II.32. Let f be a Hochschild 1-cocycle. Then notice that (III.11) is precisely what defines a derivation. Therefore, f can be identified with a derivation.

Similarly, we can realize skew derivations by changing the action (III.4) of A^e on A using an automorphism ϕ. Let us explain this in more detail. To be more precise, let us denote by ${}_{\phi}A$ the A^e-module that has A as the underlying vector space but with the action

$$(a \otimes b)(c) = \phi(a)\ cb \quad \text{for all } a \otimes b \in A^e,\ c \in {}_{\phi}A. \tag{III.12}$$

Let us now show that under this new action, a Hochschild 1-cocycle with image in ${}_{\phi}A$ can be identified with a $1, \phi$-skew derivation. First, recall the bar resolution (III.7) of A. Apply $\operatorname{Hom}_{A^e}(-, {}_{\phi}A)$ and drop the term $\operatorname{Hom}_{A^e}(A, {}_{\phi}A)$. Denote the induced maps by $\delta_{i,\phi}^*$. Then $f \in \operatorname{Ker}(\delta_{2,\phi}^*)$ implies that

$$f(b \otimes c \otimes 1) - f(1 \otimes bc \otimes 1) + f(1 \otimes b \otimes c) = 0 \quad \text{for all } b, c \in {}_{\phi}A.$$

Using the actions (III.12) and (III.5), we obtain

$$\begin{aligned} f(b \otimes c \otimes 1) &= f((b \otimes 1)\ (1 \otimes c \otimes 1)) \\ &= (b \otimes 1)\ f(1 \otimes c \otimes 1) \\ &= \phi(b)\ f(1 \otimes c \otimes 1)\ 1 \\ &= \phi(b)\ f(1 \otimes c \otimes 1) \end{aligned}$$

and

$$\begin{aligned} f(1 \otimes b \otimes c) &= f((1 \otimes c)\ (1 \otimes b \otimes 1)) \\ &= (1 \otimes c)\ f(1 \otimes b \otimes 1) \\ &= \phi(1)\ f(1 \otimes b \otimes 1)\ c \\ &= 1\ f(1 \otimes b \otimes 1)\ c \\ &= f(1 \otimes b \otimes 1)\ c. \end{aligned}$$

Thus,

$$\phi(b)\ f(1 \otimes c \otimes 1) - f(1 \otimes bc \otimes 1) + f(1 \otimes b \otimes 1)\ c = 0.$$

Since we have seen that $\mathrm{Hom}_{A^e}(A^{\otimes 3}, A)$ is isomorphic to $\mathrm{Hom}_{\mathbf{k}}(A, A)$, f may be identified with a $1, \phi$-skew derivation.

In a similar fashion, it is possible to show that if we set the action of A^e on A to be

$$(a \otimes b)(c) = ac\ \phi(b) \quad \text{ for all } a \otimes b \in A^e,\ c \in A_\phi,$$

where A_ϕ denotes the A^e-module that has A as the underlying vector space but with this new action, then a Hochschild 1-cocycle with image in A_ϕ is a $\phi, 1$-skew derivation. Therefore, both derivations and skew derivations are Hochschild 1-cocycles.

To define Hochschild 2-cocycles, let us again consider the bar resolution (III.7) of A. Let $f \in \mathrm{Ker}(\delta_3^*)$. By a similar argument as before, we can think of f as being a bilinear map from $A \times A$ to A that satisfies

$$a\ f(b,c) - f(ab,c) + f(a,bc) - f(a,b)\ c = 0 \quad \text{for all } a,b,c \in A. \tag{III.13}$$

Such a map is called a *Hochschild* 2*-cocycle.*

Similarly, a *Hochschild* 3*-cocycle* is a trilinear map $f : A \times A \times A \to A$ such that

$$a\ f(b,c,d) - f(ab,c,d) + f(a,bc,d) - f(a,b,cd) + f(a,b,c)\ d = 0 \tag{III.14}$$

for all $a,b,c,d \in A$. An analogous description for Hochschild i-cocycles ($i \geq 4$) can be given. For the sake of brevity, we skip the details.

D. Hochschild Coboundaries

Using the bar resolution (III.7) of A, we will now define Hochschild coboundaries. Let us begin by introducing Hochschild 1-coboundaries. Let $g \in \mathrm{Hom}_{A^e}(A^e, A)$. It is straightforward to check that the map

$$\mathrm{Hom}_{\mathbf{k}}(\mathbf{k}, A) \to \mathrm{Hom}_{A^e}(A^e, A)$$
$$h \mapsto (a \otimes b \mapsto a\ h(1)\ b)$$

is an isomorphism of vector spaces. Thus, g may be identified with a $\mathbf{k}$-linear map from $\mathbf{k}$ to A. By definition, if $f \in \mathrm{Im}(\delta_1^*)$, then there exists $g \in \mathrm{Hom}_{A^e}(A^e, A)$ such that $\delta_1^*(g) = f$. Let $a \in A$. Then

$$\begin{aligned} f(1 \otimes a \otimes 1) &= \delta_1^*(g)(1 \otimes a \otimes 1) \\ &= (g \circ \delta_1)(1 \otimes a \otimes 1) \\ &= g(a \otimes 1 - 1 \otimes a) \\ &= g(a \otimes 1) - g(1 \otimes a) \\ &= a\ g(1 \otimes 1) - g(1 \otimes 1)\ a \\ &= a\ g(1) - g(1)\ a, \end{aligned}$$

where, in the last equality, we have used the isomorphism between $\mathrm{Hom}_{A^e}(A^e, A)$ and $\mathrm{Hom}_{\mathbf{k}}(\mathbf{k}, A)$. Denote $g(1)$ by b. Then we may conclude that f may be identified with a linear map $A \to A$ that satisfies

$$f(a) = ab - ba \quad \text{for all } a \in A. \tag{III.15}$$

A Hochschild 1-cocycle that satisfies such a condition is called a *Hochschild* 1-*coboundary*.

To define Hochschild 2-coboundaries, let us again consider the bar resolution (III.7) of A. By definition, if $f \in \mathrm{Im}(\delta_2^*)$, then there exists $g \in \mathrm{Hom}_{A^e}(A^{\otimes 3}, A)$ such that $\delta_2^*(g) = f$. By a similar argument as before, we can think of f as being a bilinear map from $A \times A$ to A and g as a linear map from A to A, which satisfy

$$f(a, b) = a\ g(b) - g(ab) + g(a)\ b \quad \text{for all } a, b \in A. \tag{III.16}$$

A Hochschild 2-cocycle that satisfies such a condition is called a *Hochschild 2-coboundary*.

Similarly, a Hochschild 3-cocycle f is a *Hochschild 3-coboundary* if there exists a map $g : A \times A \to A$ such that

$$f(a, b, c) = a\ g(b, c) - g(ab, c) + g(a, bc) - g(a, b)\ c \quad \text{for all } a, b, c \in A. \tag{III.17}$$

In an analogous fashion, Hochschild i-coboundaries ($i \geq 4$) can be defined.

Remark III.18. Recall the Hochschild cohomology (III.9) of A obtained by using the bar resolution of A. Notice that $\mathrm{HH}^i(A) = \mathrm{Ker}(\delta_{i+1}^*) \,/\, \mathrm{Im}(\delta_i^*)$ may be rephrased as follows:

$$\mathrm{HH}^i(A) = \{\text{Hochschild } i\text{-cocycles}\} \,/\, \{\text{Hochschild } i\text{-coboundaries}\}.$$

CHAPTER IV

DEFORMATIONS

This chapter is the core of this dissertation. In Section A, we define a deformation of an algebra and briefly discuss the difficulties that arise when trying to find all possible deformations of a given algebra. We end this section by introducing universal deformation formulas. Section B describes a motivational example of a deformation of the smash product algebra of the quantum symmetric algebra of a vector space with a group. To the best of our knowledge, this is the first known example of such a deformation in which the new relations in the deformed algebra involve elements of the original vector space. We also present a generalization of the motivational example, which in turn generates numerous new examples of deformations of $S_{\mathbf{q}}(V)\#G$. In Section C, we generalize the results of Section B by finding the necessary and sufficient conditions that give $S_{\mathbf{q}}(V)\#G$ the structure of an H_q-module algebra, under some hypotheses.

A. Preliminaries

Here we present the definition of a deformation of an algebra, some examples, and a step-by-step construction to find a deformation. In this section, we also introduce universal deformation formulas. The basics of algebraic deformation theory can be found in [14], [15] and [17].

Definition IV.1. Let t be an indeterminate. A *formal deformation* of a $\mathbf{k}$-algebra A is an associative algebra $A[[t]]$ over the formal power series ring $\mathbf{k}[[t]]$ with multiplication

$$a * b = ab + \mu_1(a \otimes b)\, t + \mu_2(a \otimes b)\, t^2 + \dots$$

for all $a, b \in A$, where ab denotes the multiplication in the original algebra A and the maps $\mu_i : A \otimes A \to A$ are $\mathbf{k}$-linear extended to be $\mathbf{k}[[t]]$-linear.

Remark IV.2. In the literature, the map μ_1 is sometimes referred to as the *infinitesimal* of the deformation. See, for example, [15] and [17].

From now on, we will use the terms formal deformation and deformation interchangeably. Notice that a linear map from $A \otimes A$ to A can be identified with a bilinear map from $A \times A$ to A. Thus, the element $\mu_i(a \otimes b)$ will also be denoted by $\mu_i(a, b)$ for all $a, b \in A$.

To illustrate Definition IV.1, consider the following two examples of deformations:

Example IV.3. The quantum plane

$$\mathbb{C}_q[x, y] = \mathbb{C}\langle x, y \mid yx = qxy \rangle$$

is a specialization of a deformation of the algebra

$$A = \mathbb{C}[x, y] = \mathbb{C}\langle x, y \mid yx = xy \rangle.$$

To see this, it is sufficient to define

$$\mu_i(x, x) = \mu_i(y, y) = \mu_i(x, y) = 0 \quad \text{and} \quad \mu_i(y, x) = \frac{1}{i!}\, xy.$$

Then $x * y = xy$ and

$$y * x = xy + t\, xy + \frac{1}{2!}t^2\, xy + \frac{1}{3!}t^3\, xy + \cdots = \exp(t)\, xy.$$

Specializing to $t = t_0 \in \mathbb{C}$ and setting $q = \exp(t_0)$ yields the new algebra $\mathbb{C}_q[x, y]$.

Example IV.4. The Weyl algebra

$$W = \mathbb{C}\langle x, y \mid xy - yx = 1 \rangle$$

is a specialization of a deformation of the algebra

$$A = \mathbb{C}[x, y] = \mathbb{C}\langle x, y \mid yx = xy \rangle.$$

To see this, it again suffices to define

$$\mu_i(x, x) = \mu_i(y, y) = \mu_i(x, y) = 0 \quad \text{and} \quad \mu_i(y, x) = \begin{cases} -1, & \text{for } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$x * y = xy \quad \text{and} \quad y * x = xy - t.$$

Specializing to $t = 1$ yields the new algebra W.

Given an algebra A, a natural question that arises is to determine all possible deformations of A. By Definition IV.1, we may restate this problem as follows: Find all sequences $\{\mu_i\}$ such that the corresponding map $*$ is associative on $A[[t]]$, i.e.

$$a * (b * c) = (a * b) * c \qquad \text{for all } a, b, c \in A[[t]].$$

This associativity condition allows us to match coefficients of t^n for $n \in \mathbb{N}$, which in turn gives conditions that the maps μ_i must satisfy. For instance, matching the coefficients of the constant term tells us that A must be associative, which is already given. Next, comparing the coefficients of t, we obtain that μ_1 must satisfy

$$a\ \mu_1(b, c) - \mu_1(ab, c) + \mu_1(a, bc) - \mu_1(a, b)\ c = 0,$$

that is μ_1 must be a Hochschild 2-cocycle (compare to (III.13)). In general, matching the coefficients of t^n yields the following condition:

$$\sum_{i=0}^{n} \mu_i(\mu_{n-i}(a, b), c) \ = \ \sum_{i=0}^{n} \mu_i(a, \mu_{n-i}(b, c)),$$

where μ_0 denotes the multiplication in A. This expression can be rewritten as

$$\mu_n(a,b)\ c + \sum_{i=1}^{n-1} \mu_i(\mu_{n-i}(a,b),c) + \mu_n(ab,c) =$$

$$a\ \mu_n(b,c) + \sum_{i=1}^{n-1} \mu_i(a,\mu_{n-i}(b,c)) + \mu_n(a,bc).$$

Rearranging the terms, we obtain

$$\sum_{i=1}^{n-1} \Big(\mu_i(\mu_{n-i}(a,b),c) - \mu_i(a,\mu_{n-i}(b,c)) \Big) = a\ \mu_n(b,c) - \mu_n(ab,c) + \mu_n(a,bc) - \mu_n(a,b)\ c. \tag{IV.5}$$

To simplify notation, define γ_n to be the left hand side of (IV.5), that is

$$\gamma_n(a,b,c) = \sum_{i=1}^{n-1} \Big(\mu_i(\mu_{n-i}(a,b),c) - \mu_i(a,\mu_{n-i}(b,c)) \Big).$$

Notice that by (IV.5), γ_n is a Hochschild 3-coboundary via μ_n and hence a Hochschild 3-cocycle. Recall that Hochschild 3-cocycles and Hochschild 3-coboundaries were defined in (III.14) and (III.17), respectively.

The fact that the maps γ_n are Hochschild 3-coboundaries motivates a recursive procedure to find the maps μ_i. The first step of this process is as follows: Suppose μ_1 is a Hochschild 2-cocycle. Then we have

$$\gamma_2(a,b,c) = \mu_1(\mu_1(a,b),c) - \mu_1(a,\mu_1(b,c)) = a\ \mu_2(b,c) - \mu_2(ab,c) + \mu_2(a,bc) - \mu_2(a,b)\ c,$$

that is γ_2 must be a Hochschild 3-coboundary via μ_2. So γ_2 is the obstruction to finding μ_2.

Therefore, a step-by-step construction to find a deformation is as follows:

- Pick μ_1 such that μ_1 is a Hochschild 2-cocycle and γ_2 is a Hochschild 3-coboundary.
- For $n \geq 2$, pick μ_n such that γ_n is a Hochschild 3-coboundary via μ_n and γ_{n+1} is a Hochschild 3-coboundary.

Thus, trying to find all deformations of a given algebra is a very difficult process, if at all possible. In fact, this is a potentially infinite procedure. For this reason, many authors, such as Guccione et al. [20] and Witherspoon [37], concentrate their efforts in the study of deformations of an algebra coming from an action of a Hopf algebra.

Giaquinto and Zhang [18] developed the theory of universal deformation formulas. Here we define a universal deformation formula based on a bialgebra.

Definition IV.6. A *universal deformation formula* based on a bialgebra B is an element $F \in (B \otimes B)[[t]]$ of the form

$$F = 1_B \otimes 1_B + t\, F_1 + t^2\, F_2 + \dots$$

where $F_i \in B \otimes B$, satisfying the following three conditions:

$$(\varepsilon \otimes \mathrm{id}_B)\,(F) = 1 \otimes 1_B,$$

$$(\mathrm{id}_B \otimes \varepsilon)\,(F) = 1_B \otimes 1,$$

$$[\,(\Delta \otimes \mathrm{id}_B)\,(F)\,]\,(F \otimes 1_B) = [\,(\mathrm{id}_B \otimes \Delta)\,(F)\,]\,(1_B \otimes F).$$

Such a formula is *universal* in the sense that it applies to any B-module algebra A to yield a deformation of A (see Theorem 1.3 in [18] for a detailed proof). In particular, $m \circ F$ is the multiplication in the deformed algebra of A, where m denotes the multiplication in A.

Let us recall the Hopf algebra H_q introduced in Subsection II.B.1. Witherspoon proved in [37] that

$$\exp_q(t\, D_1 \otimes D_2) = \begin{cases} \sum_{i=0}^{n-1} \frac{1}{(i)_q!}\,(tD_1 \otimes D_2)^i, & \text{if } q \text{ is a primitive } n\text{th root of unity,} \\ \sum_{i=0}^{\infty} \frac{1}{(i)_q!}\,(tD_1 \otimes D_2)^i, & \text{if } q = 1 \text{ or } q \text{ is not a root of unity,} \end{cases}$$

is a universal deformation formula based on H_q. Therefore, for every H_q-module algebra A,

$$m \circ \exp_q(t\, D_1 \otimes D_2) \tag{IV.7}$$

yields a formal deformation of A, where m denotes the multiplication in A.

B. A Motivational Example

The example described in this section is, to the best of our knowledge, the first known example of a deformation of $S_{\mathbf{q}}(V)\#G$ in which the new relations in the deformed algebra involve not only group elements and the indeterminate, but also elements of V (compare to (I.1) and (I.2)).

Let $q \in \mathbb{C}$ be a primitive nth root of unity for $n \geq 2$. Let $\{w_1, w_2, w_3\}$ denote a basis of a $\mathbb{C}$-vector space V. Recall the definition of $S_{\mathbf{q}}(V)$ introduced in Chapter I. Set $q_{12} = q_{13} = q$ and $q_{23} = 1$. Let $G = \langle \sigma_1, \sigma_2 \mid \sigma_1^n = \sigma_2^n = 1,\ \sigma_1\sigma_2 = \sigma_2\sigma_1 \rangle$ be a group acting on $S_{\mathbf{q}}(V)$ via

$$\begin{aligned} &\sigma_1(w_1) = qw_1, && \sigma_1(w_2) = w_2, && \sigma_1(w_3) = qw_3, \\ &\sigma_2(w_1) = w_1, && \sigma_2(w_2) = qw_2, && \sigma_2(w_3) = qw_3. \end{aligned}$$

Define the functions $\chi_i : G \to \mathbb{C}^\times$ by

$$g(w_i) = \chi_i(g)\, w_i \quad \text{for all } g \in G,\ i = 1, 2, 3.$$

In particular, $\chi_1(\sigma_1) = q$ and $\chi_1(\sigma_2) = 1$.

Define an action of H_q on the generators of $S_{\mathbf{q}}(V)$ and on G by

$$
\begin{aligned}
D_1(w_1) &= \sigma_2, & D_2(w_1) &= 0, & \sigma(w_1) &= q\, w_1,\\
D_1(w_2) &= 0, & D_2(w_2) &= w_3\, \sigma_1\, \sigma_2^{-1}, & \sigma(w_2) &= w_2,\\
D_1(w_3) &= 0, & D_2(w_3) &= 0, & \sigma(w_3) &= w_3,\\
D_1(g) &= 0, & D_2(g) &= 0, & \sigma(g) &= \chi_1(g^{-1})\, g.
\end{aligned}
$$

Then extend this action of H_q to all of $S_{\mathbf{q}}(V)\#G$ under the following conditions:

$$
\begin{aligned}
D_1(ab) &= D_1(a)\, \sigma(b) + a\, D_1(b),\\
D_2(ab) &= D_2(a)\, b + \sigma(a)\, D_2(b),\\
\sigma(ab) &= \sigma(a)\, \sigma(b),
\end{aligned}
$$

for all $a, b \in S_{\mathbf{q}}(V)\#G$.

Remark IV.8. The extension conditions follow from the definition of the coproduct in H_q as stated in Subsection II.B.1. This is done so as to obtain an H_q-module algebra, which is shown to be true in the upcoming discussion.

Recall the definition of the quantum integers introduced in (II.20).

Proposition IV.9. *Let* $\{w_1^i\ w_2^j\ w_3^m\ g \mid i, j, m \in \mathbb{N},\ g \in G\}$ *denote a basis of* $S_{\mathbf{q}}(V)\#G$*. Then assuming it is well-defined, the extension of the action of* H_q *on* $S_{\mathbf{q}}(V)$ *and* G *to all of* $S_{\mathbf{q}}(V)\#G$ *is given by the following formulas:*

$$
\begin{aligned}
D_1(w_1^i\, w_2^j\, w_3^m\, g) &= (i)_q\, q^{j+m}\, \chi_1(g^{-1})\, w_1^{i-1}\, w_2^j\, w_3^m\, \sigma_2\, g,\\
D_2(w_1^i\, w_2^j\, w_3^m\, g) &= (j)_{q^{-1}}\, q^i\, w_1^i\, w_2^{j-1}\, w_3^{m+1}\, \sigma_1\, \sigma_2^{-1}\, g,\\
\sigma(w_1^i\, w_2^j\, w_3^m\, g) &= q^i\, \chi_1(g^{-1})\, w_1^i\, w_2^j\, w_3^m\, g,
\end{aligned}
$$

where a negative exponent of w_ℓ $(\ell = 1, 2, 3)$ *is interpreted to be* 0*.*

Proof. Since

$$\sigma_1(w_1) = qw_1, \quad \sigma_1(w_2) = w_2, \quad \sigma_1(w_3) = qw_3,$$

we get

$$\sigma_1(w_1^i) = q^i\ w_1^i, \quad \sigma_1(w_2^j) = w_2^j, \quad \sigma_1(w_3^m) = q^m\ w_3^m.$$

Thus,

$$\sigma_1(w_1^i\ w_2^j\ w_3^m) = q^{i+m}\ w_1^i\ w_2^j\ w_3^m.$$

Similarly,

$$\sigma_2(w_1) = w_1, \quad \sigma_2(w_2) = qw_2, \quad \sigma_2(w_3) = qw_3,$$

give

$$\sigma_2(w_1^i) = w_1^i, \quad \sigma_2(w_2^j) = q^j\ w_2^j, \quad \sigma_2(w_3^m) = q^m\ w_3^m.$$

Hence,

$$\sigma_2(w_1^i\ w_2^j\ w_3^m) = q^{j+m}\ w_1^i\ w_2^j\ w_3^m.$$

In a similar way,

$$\sigma(w_1) = qw_1, \quad \sigma(w_2) = w_2, \quad \sigma(w_3) = w_3,$$

yield

$$\sigma(w_1^i) = q^i\ w_1^i, \quad \sigma(w_2^j) = w_2^j, \quad \sigma(w_3^m) = w_3^m.$$

Since

$$\sigma(g) = \chi_1(g^{-1})\ g \ \text{ for all } \ g \in G,$$

we have

$$\sigma(w_1^i\ w_2^j\ w_3^m\ g) = q^i\ \chi_1(g^{-1})\ w_1^i\ w_2^j\ w_3^m\ g.$$

Since $D_1(w_1) = \sigma_2$ and $D_1(ab) = D_1(a)\ \sigma(b) + a\ D_1(b)$ for all $a, b \in S_{\mathbf{q}}(V)\#G$, it can be shown by induction that

$$D_1(w_1^i) = (i)_q \, w_1^{i-1} \, \sigma_2.$$

Since $D_1(w_2) = D_1(w_3) = 0$, we have that $D_1(w_2^j) = D_1(w_3^m) = 0$. Thus, we get

$$D_1(w_1^i \, w_2^j) = (i)_q \, q^j \, w_1^{i-1} \, w_2^j \, \sigma_2,$$

and

$$D_1(w_1^i \, w_2^j \, w_3^m) = (i)_q \, q^{j+m} \, w_1^{i-1} \, w_2^j \, w_3^m \, \sigma_2.$$

Since

$$D_1(g) = 0 \ \text{ for all } \ g \in G,$$

we obtain

$$D_1(w_1^i \, w_2^j \, w_3^m \, g) = (i)_q \, q^{j+m} \, \chi_1(g^{-1}) \, w_1^{i-1} \, w_2^j \, w_3^m \, \sigma_2 \, g.$$

Since $D_2(w_2) = w_3 \, \sigma_1 \, \sigma_2^{-1}$ and $D_2(ab) = D_2(a) \, b + \sigma(a) \, D_2(b)$ for all $a, b \in S_{\mathbf{q}}(V)\#G$, it can be shown by induction that

$$D_2(w_2^j) = (j)_{q^{-1}} \, w_2^{j-1} \, w_3 \, \sigma_1 \, \sigma_2^{-1}.$$

Since $D_2(w_1) = D_2(w_3) = 0$, we have that $D_2(w_1^i) = D_2(w_3^m) = 0$. Thus, we get

$$D_2(w_1^i \, w_2^j) = (j)_{q^{-1}} \, q^i \, w_1^i \, w_2^{j-1} \, w_3 \, \sigma_1 \, \sigma_2^{-1},$$

and

$$D_2(w_1^i \, w_2^j \, w_3^m) = (j)_{q^{-1}} \, q^i \, w_1^i \, w_2^{j-1} \, w_3^{m+1} \, \sigma_1 \, \sigma_2^{-1}.$$

Since

$$D_2(g) = 0 \ \text{ for all } \ g \in G,$$

we obtain

$$D_2(w_1^i\, w_2^j\, w_3^m\, g) = (j)_{q^{-1}}\, q^i\, w_1^i\, w_2^{j-1}\, w_3^{m+1}\, \sigma_1\, \sigma_2^{-1}\, g.$$

□

Let us recall that Hopf module algebras were introduced in Definition II.27.

Proposition IV.10. *$S_{\mathbf{q}}(V)\#G$ is an H_q-module algebra.*

Proof. Since the maps σ, D_1 and D_2 are defined on the generators of $S_{\mathbf{q}}(V)$ and on G, and then extended to all of $S_{\mathbf{q}}(V)\#G$, it is necessary to check what happens when an element of $S_{\mathbf{q}}(V)\#G$ has more than one expression in terms of the generators. As long as the relations of $S_{\mathbf{q}}(V)\#G$ are preserved by σ, D_1 and D_2, each map will act as a well-defined linear operator on $S_{\mathbf{q}}(V)\#G$. To get an action of the whole Hopf algebra H_q on $S_{\mathbf{q}}(V)\#G$, we need to check that the relations of H_q are preserved, so that we do get a well-defined action of each element of H_q on $S_{\mathbf{q}}(V)\#G$. Then $S_{\mathbf{q}}(V)\#G$ is an H_q-module. Next, the action of σ, D_1 and D_2 on a product is as it should be, by definition. Finally, $\sigma(1) = 1$ and $D_1(1) = D_2(1) = 0$ follow from the extension conditions. Therefore, it is enough to check the following:

- The relations of H_q are preserved by the generators of $S_{\mathbf{q}}(V)\#G$, that is the relations

  ⋆ $D_1 D_2 = D_2 D_1$,

  ⋆ $q\sigma D_i = D_i \sigma \quad$ for $i = 1, 2$,

  ⋆ $D_1^n = D_2^n = 0 \quad$ if q is a primitive nth root of unity,

  are preserved by each basis element $w_1^i\, w_2^j\, w_3^m\, g$.

- The relations of $S_{\mathbf{q}}(V)\#G$ are preserved by the generators of H_q, that is the relations

- $\star$ $w_1w_2 = qw_2w_1,\ w_1w_3 = qw_3w_1,\ w_2w_3 = w_3w_2$,
- $\star$ $gw_i = g(w_i)g \quad \text{for } i = 1, 2, 3,\ g \in G$,

are preserved by σ, D_1 and D_2.

For the sake of brevity, we omit these calculations. □

Remark IV.11. By construction, σ is an automorphism of $S_{\mathbf{q}}(V)\#G$. By Remark II.24 and Example II.33, we may conclude that D_1 acts as a $\sigma, 1$-skew derivation and D_2 acts as a $1, \sigma$-skew derivation of $S_{\mathbf{q}}(V)\#G$.

As a consequence of Proposition IV.10, if we set, for instance, $q = -1$ (and hence $n = 2$), then by (IV.7), we get that

$$m \circ \exp_q(tD_1 \otimes D_2) = m \circ \left(\sum_{i=0}^{n-1} \frac{1}{(i)_q!} \, (tD_1 \otimes D_2)^i \right) = m \circ (\mathrm{id} \otimes \mathrm{id} + tD_1 \otimes D_2) \quad \text{(IV.12)}$$

yields a deformation of $S_{\mathbf{q}}(V)\#G$.

Proposition IV.13. *If $q = -1$, then the deformation of $S_{\mathbf{q}}(V)\#G$ given by* (IV.12) *is*

$$(T(V)\#G)[[t]] \ / \ (w_1w_2 + w_2w_1 + tw_3\sigma_1,\ w_1w_3 + w_3w_1,\ w_2w_3 - w_3w_2).$$

Proof. Let us denote by $\mathcal{D}$ the deformation of $S_{\mathbf{q}}(V)\#G$ obtained by defining the multiplication on $(S_{\mathbf{q}}(V)\#G)[[t]]$ by (IV.12). To find the new relations in the deformed algebra $\mathcal{D}$, consider, for example,

$$\begin{aligned} w_1 * w_2 &= (m \circ (\mathrm{id} \otimes \mathrm{id} + t\ D_1 \otimes D_2))\ (w_1 \otimes w_2) \\ &= m\ (w_1 \otimes w_2) + m\ (t\ (D_1 \otimes D_2)\ (w_1 \otimes w_2)) \\ &= w_1\ w_2 + m\ (t\ (D_1(w_1) \otimes D_2(w_2))) \\ &= w_1\ w_2 + m\ (t\ (\sigma_2 \otimes w_3\ \sigma_1\ \sigma_2^{-1})) \\ &= w_1\ w_2 + t\ \sigma_2\ w_3\ \sigma_1\ \sigma_2^{-1} \\ &= w_1\ w_2 - t\ w_3\ \sigma_1. \end{aligned}$$

Similarly,

$$\begin{aligned} w_2 * w_1 &= (m \circ (\mathrm{id} \otimes \mathrm{id} + t\ D_1 \otimes D_2))\ (w_2 \otimes w_1) \\ &= m\ (w_2 \otimes w_1) + m\ (t\ (D_1 \otimes D_2)\ (w_2 \otimes w_1)) \\ &= w_2\ w_1 + m\ (t\ (D_1(w_2) \otimes D_2(w_1))) \\ &= w_2\ w_1. \end{aligned}$$

Thus,

$$w_1 * w_2 + w_2 * w_1 = w_1\ w_2 + w_2\ w_1 - t\ w_3\ \sigma_1 = -t\ w_3\ \sigma_1,$$

since $w_1 w_2 = -w_2 w_1$ in the original algebra. Dropping the $*$ notation, we get that the new relation in $\mathcal{D}$ is

$$w_1 w_2 + w_2 w_1 = -t w_3 \sigma_1,$$

as desired. Similar calculations show that in the deformed algebra $\mathcal{D}$, the following relations also hold:

$$w_1 w_3 = -w_3 w_1 \quad \text{and} \quad w_2 w_3 = w_3 w_2.$$

Define an algebra homomorphism

$$\varphi : (T(V)\#G)[[t]] \to \mathcal{D}$$
$$w_i \mapsto w_i$$
$$g \mapsto g.$$

By construction, the map φ is surjective since w_i and g are in the image of φ and these elements generate $\mathcal{D}$ by an inductive argument. To see this, recall that by definition, $\mathcal{D}$ is $(S_{\mathbf{q}}(V)\#G)[[t]]$ as a vector space. As a free $\mathbf{k}[[t]]$-module, $\mathcal{D}$ has a free generating set $\{w_1^\alpha\, w_2^\beta\, w_3^\gamma\, g \mid \alpha, \beta, \gamma \text{ are nonnegative integers}\}$. It is possible to show by induction on the degree $\alpha+\beta+\gamma$ of $w_1^\alpha\, w_2^\beta\, w_3^\gamma\, g$ that this element is in the image of φ.

Let I denote the ideal of $(T(V)\#G)[[t]]$ generated by the relations

$$w_1w_2 + w_2w_1 + tw_3\sigma_1, \quad w_1w_3 + w_3w_1, \quad w_2w_3 - w_3w_2.$$

The calculations presented above show that $I \subseteq \ker(\varphi)$. Note that $(T(V)\#G)[[t]] \,/\, I$ is a filtered algebra, due to the nature of the elements of I. To be more precise, recall that in Example II.39, we saw that $T(V)\#G$ is a filtered algebra. Then by setting the degree of the indeterminate t to be 0, $(T(V)\#G)[[t]]$ is also filtered. To show that the filtration passes to the quotient, it suffices to notice that, in the notation of Example II.39, $w_1w_2 + w_2w_1 \in S_2$ and $w_3\sigma_1 \in S_1 \subset S_2$.

By an induction argument on the degree, it is possible to show that the elements of the form $w_1^\alpha\ w_2^\beta\ w_3^\gamma\ g$, where α, β and γ are nonnegative integers, span $(T(V)\#G)[[t]] \,/\, I$ as a free $\mathbf{k}[[t]]$-module. Thus, the dimension of the associated graded algebra of $(T(V)\#G)[[t]] \,/\, I$ in each degree n is at most the number of elements of the form $w_1^\alpha\, w_2^\beta\, w_3^\gamma\, g$ with $\alpha+\beta+\gamma = n$. Recall that the associated graded algebra was introduced in Definition II.42. On the other hand, since $I \subseteq \ker(\varphi)$, the map

$$(T(V)\#G)[[t]] \;/\; I \;\longrightarrow\; (T(V)\#G)[[t]] \;/\; \ker(\varphi)$$

is surjective. Thus, the dimension of the associated graded algebra of $(T(V)\#G)[[t]] \;/\; I$ in each degree n is at least the dimension of the associated graded algebra of $(T(V)\#G)[[t]] \;/\; \ker(\varphi)$ in degree n. Since $(T(V)\#G)[[t]] \;/\; \ker(\varphi)$ is isomorphic to $\mathcal{D}$ and we know that the elements of the form $w_1^{\alpha}\, w_2^{\beta}\, w_3^{\gamma}\, g$ form a basis of $\mathcal{D}$, it follows that the deformation is precisely $(T(V)\#G)[[t]] \;/\; I$. □

As advertised, the vector space element w_3 appears in the new relations multiplied by the indeterminate t and the group element σ_1. Thus, restricting to $(T(V)\#G)[t]$ and specializing to $t = 1$, this deformation involves relations of type

$$w_i w_j - q_{ij} w_j w_i - \sum_{g \in G} w_m\, a_g(w_i, w_j)\, g \qquad \text{for some } m.$$

Notice that these relations differ from those used to define the braided Cherednik algebras (I.2) by the presence of w_m.

Remark IV.14. The arguments presented in the proof of Proposition IV.13 can be generalized to any primitive nth root of unity q for $n \geq 2$. For example, set $n = 3$. Then by (IV.7), we get that

$$m \circ \exp_q(t\; D_1 \otimes D_2) = m \circ \left(\mathrm{id} \otimes \mathrm{id} + t\; D_1 \otimes D_2 + \frac{1}{1+q}\,(t^2\; D_1^2 \otimes D_2^2) \right) \tag{IV.15}$$

yields a deformation of $S_{\mathbf{q}}(V)\#G$. In this case, the deformation can be found to be

$$(T(V)\#G)[[t]] \;/\; (w_1 w_2 + w_2 w_1 - q t w_3 \sigma_1,\; w_1 w_3 + w_3 w_1,\; w_2 w_3 - w_3 w_2). \tag{IV.16}$$

It is sufficient to notice that $D_i^2(w_j) = 0$ for $i, j = 1, 2$.

Remark IV.17. Notice that the deformation (IV.16) of $S_{\mathbf{q}}(V)\#G$ is a graded deformation in the sense of Braverman and Gaitsgory [4]. To see this, we assign degree 1 to the indeterminate t and to the vector space elements w_i, and degree 0 to the group elements g. In this way, each of the relations obtained are homogeneous of degree 2. Therefore, the quotient is graded.

Remark IV.18. This example can be generalized to higher dimensions as follows: Let q be a primitive nth root of unity ($n \geq 2$) and let $\{w_1, \ldots, w_k\}$ denote a basis of a $\mathbb{C}$-vector space V. Set $q_{1j} = q$ for $j = 2, \ldots, k$, and $q_{ij} = 1$ for $i, j = 2, \ldots, k$. Let $G = \langle \sigma_1, \sigma_2 \mid \sigma_1^n = \sigma_2^n = 1,\ \sigma_1\sigma_2 = \sigma_2\sigma_1 \rangle$ be a group acting on $S_{\mathbf{q}}(V)$ via

$$\begin{aligned}
\sigma_1(w_1) &= qw_1, & \sigma_2(w_1) &= w_1,\\
\sigma_1(w_2) &= w_2, & \sigma_2(w_2) &= qw_2,\\
\sigma_1(w_3) &= qw_3, & \sigma_2(w_3) &= qw_3,\\
&\vdots & &\vdots\\
\sigma_1(w_k) &= qw_k, & \sigma_2(w_k) &= qw_k.
\end{aligned}$$

Define the functions $\chi_i : G \to \mathbb{C}^\times$ by

$$g(w_i) = \chi_i(g)\, w_i \quad \text{for all } g \in G,\ i = 1, \ldots, k.$$

Define an action of H_q on the generators of $S_{\mathbf{q}}(V)$ and on G as follows:

Let $\alpha_1, \alpha_2, \ldots, \alpha_{k-2}, \beta_1, \beta_2, \ldots, \beta_{k-2} \in \mathbb{N}$. Then define

$$\begin{aligned}
D_1(w_1) &= w_3^{\alpha_1 n}\, w_4^{\alpha_2 n} \cdots w_k^{\alpha_{k-2}\, n}\, \sigma_2, & D_2(w_1) &= 0,\\
D_1(w_2) &= 0, & D_2(w_2) &= w_3^{\beta_1 n+1}\, w_4^{\beta_2 n} \cdots w_k^{\beta_{k-2}\, n}\, \sigma_1\, \sigma_2^{-1},\\
D_1(w_3) &= 0, & D_2(w_3) &= 0,\\
&\vdots & &\vdots\\
D_1(w_k) &= 0, & D_2(w_k) &= 0,\\
D_1(g) &= 0, & D_2(g) &= 0,
\end{aligned}$$

and

$$\begin{aligned}
\sigma(w_1) &= q w_1,\\
\sigma(w_2) &= w_2,\\
\sigma(w_3) &= w_3,\\
&\vdots\\
\sigma(w_k) &= w_k,\\
\sigma(g) &= \chi_1(g^{-1})\, g.
\end{aligned}$$

If we extend the action of H_q to all of $S_{\mathbf{q}}(V)\#G$ under the same conditions as before and denote a basis of $S_{\mathbf{q}}(V)\#G$ by $\{w_1^{i_1}\, w_2^{i_2} \ldots w_k^{i_k}\, g \mid i_1, \ldots, i_k \in \mathbb{N},\ g \in G\}$, then we obtain the following formulas:

$$\begin{aligned}
D_1(w_1^{i_1}\, w_2^{i_2} \ldots w_k^{i_k}\, g) &= (i_1)_q\, q^{i_2+\cdots+i_k}\, \chi_1(g^{-1})\, w_1^{i_1-1}\, w_2^{i_2}\, w_3^{i_3+\alpha_1 n} \ldots w_k^{i_k+\alpha_{k-2}\, n}\, \sigma_2\, g,\\
D_2(w_1^{i_1}\, w_2^{i_2} \ldots w_k^{i_k}\, g) &= (i_2)_{q^{-1}}\, q^{i_1}\, w_1^{i_1}\, w_2^{i_2-1}\, w_3^{i_3+\beta_1 n+1}\, w_4^{i_4+\beta_2 n} \ldots w_k^{i_k+\beta_{k-2}\, n}\, \sigma_1\, \sigma_2^{-1}\, g,\\
\sigma(w_1^{i_1}\, w_2^{i_2} \ldots w_k^{i_k}\, g) &= q^{i_1}\, \chi_1(g^{-1})\, w_1^{i_1}\, w_2^{i_2} \ldots w_k^{i_k}\, g.
\end{aligned}$$

It is possible to show that $S_{\mathbf{q}}(V)\#G$ is an H_q-module algebra in this case as well.

Thus, again by (IV.7), $m \circ \exp_q(t\ D_1 \otimes D_2)$ gives a deformation of $S_{\mathbf{q}}(V)\#G$.

If $q = -1$, then as before, $m \circ (\mathrm{id} \otimes \mathrm{id} + t\ D_1 \otimes D_2)$ yields a deformation which, in this case, can be found to be the quotient of $(T(V)\#G)[[t]]$ by an ideal that contains the following relations:

$$w_1 w_2 + w_2 w_1 + t\ (-1)^{(\beta_1 + \cdots + \beta_{k-2})n}\ w_3^{(\alpha_1+\beta_1)n+1}\ w_4^{(\alpha_2+\beta_2)n} \cdots w_k^{(\alpha_{k-2}+\beta_{k-2})n}\ \sigma_1, \quad \text{(IV.19)}$$

$$w_1 w_j + w_j w_1 \quad \text{for } j = 3, \ldots, k,$$

$$w_i w_j - w_j w_i \quad \text{for } i, j = 2, \ldots, k.$$

The proof of this statement is analogous to the proof of Proposition IV.13. For the sake of brevity, we skip the details. Notice that in this case we do not obtain a graded deformation in the sense of Braverman and Gaitsgory [4] unless $\alpha_1 = \cdots = \alpha_{k-2} = \beta_1 = \cdots = \beta_{k-2} = 0$.

C. The General Case

Let us present some generalizations of the results obtained in [20] and [37] to the case of $S_{\mathbf{q}}(V)\#G$. We provide the necessary and sufficient conditions for $S_{\mathbf{q}}(V)\#G$ to have the structure of an H_q-module algebra under some assumptions. As a consequence, by applying (IV.7), we obtain more explicit examples of deformations.

1. H_q-module Algebra Structures on Arbitrary Algebras

Let A be a $\mathbf{k}$-algebra and let $\sigma, D_1, D_2 : A \to A$ be arbitrary $\mathbf{k}$-linear maps. Recall that a module was introduced in Definition II.26. By abuse of notation, we identify σ, D_1 and D_2 with the generators of H_q by the same name. Then A can be given the structure of an H_q-module via a (necessarily unique) $\mathbf{k}$-linear map $H_q \otimes A \to A$

if and only if the maps σ, D_1 and D_2 satisfy the following conditions:

$$\sigma \text{ is a bijective map,} \tag{IV.20a}$$

$$D_1 D_2 = D_2 D_1, \tag{IV.20b}$$

$$q\sigma D_i = D_i \sigma \text{ for } i = 1, 2, \tag{IV.20c}$$

$$\text{if } q \text{ is a primitive } n\text{th root of unity, then } D_1^n = D_2^n = 0. \tag{IV.20d}$$

Remark IV.21. To see why conditions (IV.20) are necessary and sufficient, notice that if we are given maps σ, D_1 and D_2 as linear operators from A to A, there is only one possible way that we can extend this action to all the elements of H_q since σ, D_1 and D_2 generate H_q. So if such an extension exists, it is unique. The only question then is whether such an action is well-defined, that is whether it exists. Since we start with actions of σ, D_1 and D_2 on A, we would need to check the relations of H_q, which are precisely conditions (IV.20). Only then would we know that our original choices of linear maps extend to give a well-defined map $H_q \otimes A \to A$.

Given σ, D_1 and D_2 such that (IV.20) holds, the following result determines the conditions that these maps must satisfy so that A becomes an H_q-module algebra via the map $H_q \otimes A \to A$.

Theorem IV.22. *Let $\sigma, D_1, D_2 : A \to A$ be* $\mathbf{k}$*-linear maps satisfying* (IV.20). *Then A is an H_q-module algebra via the map $H_q \otimes A \to A$ if and only if*

$$\sigma(ab) = \sigma(a)\,\sigma(b) \;\; \textit{for all } a, b \in A, \tag{IV.23a}$$

$$D_1(ab) = D_1(a)\,\sigma(b) + a\,D_1(b) \;\; \textit{for all } a, b \in A, \tag{IV.23b}$$

$$D_2(ab) = D_2(a)\,b + \sigma(a)\,D_2(b) \;\; \textit{for all } a, b \in A. \tag{IV.23c}$$

Proof. By definition, to show that A is an H_q-module algebra, we need to check the relations of A, the relations of H_q, and the condition (II.28). The relations of A need not be checked since σ, D_1 and D_2 are well-defined maps from A to A, which means they automatically must preserve the relations of A. The relations of H_q are precisely items (IV.20). The condition (II.28) is equivalent to items (IV.23) since σ, D_1 and D_2 generate H_q and Δ is an algebra homomorphism, so all the elements of H_q satisfy (II.28). □

Remark IV.24. This result is a specialization of Theorem 2.4 in [20] with α being the identity map on A.

2. H_q-module Algebra Structures on Smash Products

Let V be a $\mathbf{k}$-vector space with basis $\{w_1, \dots, w_k\}$. Let G be a group acting linearly on V. Assume that the action of G on V is diagonal with respect to the basis $\{w_1, \dots, w_k\}$. Then there exist maps $\chi_i : G \to \mathbf{k}^\times$ such that

$$g(w_i) = \chi_i(g)\, w_i \quad \text{for all } g \in G,\ i = 1, \dots, k.$$

Extend the action of G on V to $S_{\mathbf{q}}(V)$ by algebra automorphisms.

Theorem IV.25. *Let $\sigma, D_1, D_2 : V \oplus \mathbf{k}G \to S_{\mathbf{q}}(V)\#G$ be $\mathbf{k}$-linear maps. Suppose there exists a group homomorphism $\xi : G \to \mathbf{k}^\times$ such that*

$$\sigma(g) = \xi(g)\, g \quad \textit{for all } g \in G.$$

Then σ, D_1, D_2 extend to give $S_{\mathbf{q}}(V)\#G$ the structure of an H_q-module algebra if and only if for all $g \in G$, $\ell = 1, 2$, $i, j = 1, \dots, k$, the following conditions hold:

$$\sigma : V \to V \textit{ is a bijective } \mathbf{k}G\textit{-linear map,} \tag{IV.26a}$$

$$D_1 D_2(w_i) = D_2 D_1(w_i), \tag{IV.26b}$$

$$D_1D_2(g) = D_2D_1(g), \tag{IV.26c}$$

$$q\sigma D_\ell(w_i) = D_\ell\sigma(w_i), \tag{IV.26d}$$

$$q\sigma D_\ell(g) = \xi(g)\ D_\ell(g), \tag{IV.26e}$$

$$\textit{if } q \textit{ is a primitive nth root of unity, then } D_1^n = D_2^n = 0, \tag{IV.26f}$$

$$D_\ell(w_iw_j) = D_\ell(q_{ij}w_jw_i), \tag{IV.26g}$$

$$\sigma(w_iw_j) = \sigma(q_{ij}w_jw_i), \tag{IV.26h}$$

$$D_\ell(g(w_i)g) = D_\ell(gw_i). \tag{IV.26i}$$

Proof. Notice that $V \oplus \mathbf{k}G$ embeds into $S_{\mathbf{q}}(V)\#G$, so in order to apply Theorem IV.22, first we need to extend the maps σ, D_1 and D_2 from $V \oplus \mathbf{k}G$ to all of $S_{\mathbf{q}}(V)\#G$ by requiring conditions IV.23 to hold. In order to obtain well-defined $\mathbf{k}$-linear maps from $S_{\mathbf{q}}(V)\#G$ to $S_{\mathbf{q}}(V)\#G$, we need to check that the relations of $S_{\mathbf{q}}(V)\#G$ are satisfied by the generators of H_q. These are precisely conditions (IV.26g), (IV.26h) and (IV.26i). Notice that σ automatically preserves the relation $g(w_i)g = gw_i$ since by assumption,

$$g(w_i) = \chi_i(g)\ w_i \quad \text{and} \quad \sigma(g) = \xi(g)\ g \quad \text{ for all } g \in G,\ i = 1, \dots, k,$$

and by (IV.26a), $\sigma : V \to V$ commutes with the action of G, that is

$$\sigma(g(w_i)) = g(\sigma(w_i)) \quad \text{ for all } g \in G,\ i = 1, \dots, k.$$

To be more precise, consider

$$\begin{aligned}\sigma(g(w_i)g) &= \sigma(\chi_i(g)\ w_i\ g)\\ &= \chi_i(g)\ \sigma(w_i\ g)\\ &= \chi_i(g)\ \sigma(w_i)\ \sigma(g)\\ &= \chi_i(g)\ \sigma(w_i)\ \xi(g)\ g\\ &= \chi_i(g)\ \xi(g)\ \sigma(w_i)\ g,\end{aligned}$$

and

$$\begin{aligned}\sigma(gw_i) &= \sigma(g)\ \sigma(w_i)\\ &= \xi(g)\ g\ \sigma(w_i)\\ &= \xi(g)\ g(\sigma(w_i))\ g\\ &= \xi(g)\ \sigma(g(w_i))\ g\\ &= \xi(g)\ \sigma(\chi_i(g)\ w_i)\ g\\ &= \xi(g)\ \chi_i(g)\ \sigma(w_i)\ g.\end{aligned}$$

Next, we need to make sure that the relations of H_q hold, that is items (IV.20). Notice that (IV.20a) is equivalent to (IV.26a) since σ is bijective on $\mathbf{k}G$ by its definition. Moreover, (IV.20d) is precisely (IV.26f). Since it is enough to show that the relations of H_q are satisfied by the generators of $S_{\mathbf{q}}(V)\#G$, it follows that (IV.20b) is equivalent to (IV.26b) and (IV.26c), and (IV.20c) is equivalent to (IV.26d) and (IV.26e). $\square$

Remark IV.27. This result was obtained for the case of $S(V)\#G$ in [20]. Thus, Theorem IV.25 is a generalization of Theorem 3.5 in [20] to $S_{\mathbf{q}}(V)\#G$ with α being the identity map on $S_{\mathbf{q}}(V)\#G$, s the twist map, $\chi_\alpha(g) = 1$ and $\chi_\varsigma(g) = \chi_1(g^{-1})$ for all $g \in G$.

3. A Special Case of H_q-module Algebra Structures on Smash Products

Let V be a $\mathbf{k}$-vector space with basis $\{w_1, \dots, w_k\}$ and let G be a group acting diagonally on V. As before, the maps $\chi_i : G \to \mathbf{k}^\times$ are given by

$$g(w_i) = \chi_i(g)\, w_i \;\text{ for all } g \in G,\; i = 1, \dots, k.$$

Theorem IV.28. *Let $\sigma, D_1, D_2 : V \oplus \mathbf{k}G \to S_{\mathbf{q}}(V)\#G$ be $\mathbf{k}$-linear maps. Assume that the action of σ on V is diagonal with respect to the basis $\{w_1, \dots, w_k\}$. Let $\lambda_i(\sigma) \in \mathbf{k}^\times$ be defined by*

$$\sigma(w_i) = \lambda_i(\sigma)\, w_i \quad \textit{for all } i = 1, \dots, k.$$

Suppose there exists a group homomorphism $\xi : G \to \mathbf{k}^\times$ such that

$$\sigma(g) = \xi(g)\, g \;\; \textit{for all } g \in G.$$

Choose $P_1, P_2 \in S_{\mathbf{q}}(V)$ such that there are scalars q_{P_i,w_j} satisfying the following equation:

$$P_i\, w_j = q_{P_i,w_j}\, w_j\, P_i \;\; \textit{for all } i \neq j.$$

Assume that

$$D_1(w_1) = P_1\, g_1, \quad D_1(w_i) = 0 \;\; \textit{for all } i \neq 1, \quad D_1(g) = 0 \;\; \textit{for all } g \in G,$$

$$D_2(w_2) = P_2\, g_2, \quad D_2(w_i) = 0 \;\; \textit{for all } i \neq 2, \quad D_2(g) = 0 \;\; \textit{for all } g \in G,$$

with $g_1, g_2 \in G$. Then there is an H_q-module algebra structure on $S_{\mathbf{q}}(V)\#G$, for which σ, D_1, D_2 act as the above chosen maps, if and only if

$$q_{P_1,w_i} = q_{1i}\, \lambda_i^{-1}(\sigma)\, \chi_i(g_1^{-1}) \;\; \textit{for all } i \neq 1, \tag{IV.29a}$$

$$q_{P_2,w_i} = q_{2i}\, \lambda_i(\sigma)\, \chi_i(g_2^{-1}) \;\; \textit{for all } i \neq 2, \tag{IV.29b}$$

$$g_1 \textit{ and } g_2 \textit{ belong to the center of } G, \tag{IV.29c}$$

$$g(P_1) = \chi_1(g)\ \xi(g)\ P_1 \textit{ for all } g \in G, \tag{IV.29d}$$

$$g(P_2) = \chi_2(g)\ \xi(g^{-1})\ P_2 \textit{ for all } g \in G, \tag{IV.29e}$$

$$P_1 \in \ker(D_2) \textit{ and } P_2 \in \ker(D_1), \tag{IV.29f}$$

$$\sigma(P_i) = q^{-1}\ \lambda_i(\sigma)\ \xi(g_i^{-1})\ P_i \textit{ for } i = 1, 2, \tag{IV.29g}$$

$$\textit{if } q \textit{ is a primitive } n\textit{th root of unity, then } D_1^n = D_2^n = 0. \tag{IV.29h}$$

Proof. Notice that condition (IV.29h) is exactly item (IV.26f). Since $\sigma(w_i) = \lambda_i(\sigma)w_i$ for all $i = 1, \ldots, k$, where $\lambda_i(\sigma) \in \mathbf{k}^\times$, it follows that $\sigma : V \to V$ is bijective. Since, in addition, $g(w_i) = \chi_i(g)w_i$ for all $g \in G,\ i = 1, \ldots, k$, we have that $\sigma : V \to V$ is a $\mathbf{k}G$-linear map. Thus, item (IV.26a) is satisfied. Also note that condition (IV.26h) is satisfied by the assumption that $\sigma(w_i) = \lambda_i(\sigma)w_i$ for all $i = 1, \ldots, k$. Since by assumption $D_1(g) = D_2(g) = 0$ for all $g \in G$, items (IV.26c) and (IV.26e) hold.

We claim that conditions (IV.29a) and (IV.29b) are equivalent to item (IV.26g). For $i \neq 1$,

$$\begin{aligned} D_1(w_1\ w_i) &= D_1(w_1)\ \sigma(w_i) + w_1\ D_1(w_i) \\ &= D_1(w_1)\ \sigma(w_i) \\ &= P_1\ g_1\ \lambda_i(\sigma)\ w_i \\ &= \lambda_i(\sigma)\ P_1\ g_1(w_i)\ g_1 \\ &= \lambda_i(\sigma)\ P_1\ \chi_i(g_1)\ w_i\ g_1 \\ &= \lambda_i(\sigma)\ \chi_i(g_1)\ P_1\ w_i\ g_1 \end{aligned}$$

and

$$\begin{aligned} D_1(q_{1i}\ w_i\ w_1) &= q_{1i}\ D_1(w_i)\ \sigma(w_1) + q_{1i}\ w_i\ D_1(w_1) \\ &= q_{1i}\ w_i\ D_1(w_1) \\ &= q_{1i}\ w_i\ P_1\ g_1 \\ &= q_{1i}\ q_{P_1,w_i}^{-1}\ P_1\ w_i\ g_1 \end{aligned}$$

Therefore, $D_1(w_1\ w_i) = D_1(q_{1i}\ w_i\ w_1)$ is equivalent to

$$q_{P_1,w_i} = q_{1i}\ \lambda_i^{-1}(\sigma)\ \chi_i(g_1^{-1}) \ \text{ for all } i \neq 1.$$

Similarly, for $i \neq 2$,

$$\begin{aligned} D_2(w_2\ w_i) &= D_2(w_2)\ w_i + \sigma(w_2)\ D_2(w_i) \\ &= D_2(w_2)\ w_i \\ &= P_2\ g_2\ w_i \\ &= P_2\ g_2(w_i)\ g_2 \\ &= P_2\ \chi_i(g_2)\ w_i\ g_2 \\ &= \chi_i(g_2)\ P_2\ w_i\ g_2 \end{aligned}$$

and

$$\begin{aligned} D_2(q_{2i}\ w_i\ w_2) &= q_{2i}\ D_2(w_i)\ w_2 + q_{2i}\ \sigma(w_i)\ D_2(w_2) \\ &= q_{2i}\ \sigma(w_i)\ D_2(w_2) \\ &= q_{2i}\ \lambda_i(\sigma)\ w_i\ P_2\ g_2 \\ &= q_{2i}\ \lambda_i(\sigma)\ q_{P_2,w_i}^{-1}\ P_2\ w_i\ g_2 \end{aligned}$$

Therefore, $D_2(w_2\ w_i) = D_2(q_{2i}\ w_i\ w_2)$ is equivalent to

$$q_{P_2,w_i} = q_{2i}\ \lambda_i(\sigma)\ \chi_i(g_2^{-1}) \ \text{ for all } i \neq 2.$$

We claim that conditions (IV.29c), (IV.29d) and (IV.29e) are equivalent to item (IV.26i). Since $D_1(g) = 0$ for all $g \in G$, we have

$$\begin{aligned} D_1(g(w_i)\ g) &= D_1(g(w_i))\ \sigma(g) + g(w_i)\ D_1(g) \\ &= D_1(g(w_i))\ \sigma(g) \\ &= D_1(\chi_i(g)\ w_i)\ \xi(g)\ g \\ &= \chi_i(g)\ \xi(g)\ D_1(w_i)\ g \end{aligned}$$

and

$$\begin{aligned} D_1(g\ w_i) &= D_1(g)\ \sigma(w_i) + g\ D_1(w_i) \\ &= g\ D_1(w_i) \end{aligned}$$

Thus, it is enough to show that

$$\chi_i(g)\ \xi(g)\ D_1(w_i)\ g = g\ D_1(w_i)$$

holds for all $g \in G,\ i = 1, \ldots, k$, if and only if conditions (IV.29c) and (IV.29d) are satisfied.

There are two possible cases, namely $i = 1$ or $i \neq 1$. If $i = 1$, then

$$\chi_1(g)\ \xi(g)\ D_1(w_1)\ g = g\ D_1(w_1)$$

simplifies to

$$\chi_1(g)\ \xi(g)\ P_1\ g_1\ g = g\ P_1\ g_1.$$

This is equivalent to

$$\chi_1(g)\ \xi(g)\ P_1\ g_1\ g = g(P_1)\ g\ g_1,$$

which holds if and only if g_1 belongs to the center of G and $g(P_1) = \chi_1(g)\ \xi(g)\ P_1$ for all $g \in G$. If $i \neq 1$, then $D_1(w_i) = 0$ so both sides of the equation are equal to zero.

Similarly, since $D_2(g) = 0$ for all $g \in G$, we have

$$\begin{aligned} D_2(g(w_i)\ g) &= D_2(g(w_i))\ g + \sigma(g(w_i))\ D_2(g) \\ &= D_2(g(w_i))\ g \\ &= D_2(\chi_i(g)\ w_i)\ g \\ &= \chi_i(g)\ D_2(w_i)\ g \end{aligned}$$

and

$$\begin{aligned} D_2(g\ w_i) &= D_2(g)\ w_i + \sigma(g)\ D_2(w_i) \\ &= \sigma(g)\ D_2(w_i) \\ &= \xi(g)\ g\ D_2(w_i) \end{aligned}$$

Thus, it is enough to show that

$$\chi_i(g)\ D_2(w_i)\ g = \xi(g)\ g\ D_2(w_i)$$

holds for all $g \in G,\ i = 1, \ldots, k$, if and only if conditions (IV.29c) and (IV.29e) are satisfied.

Again, there are two possible cases, namely $i = 2$ or $i \neq 2$. If $i = 2$, then

$$\chi_2(g)\ D_2(w_2)\ g = \xi(g)\ g\ D_2(w_2)$$

simplifies to

$$\chi_2(g)\ P_2\ g_2\ g = \xi(g)\ g\ P_2\ g_2.$$

This is equivalent to

$$\xi(g^{-1})\ \chi_2(g)\ P_2\ g_2\ g = g(P_2)\ g\ g_2,$$

which holds if and only if g_2 belongs to the center of G and $g(P_2) = \xi(g^{-1})\ \chi_2(g)\ P_2$ for all $g \in G$. If $i \neq 2$, then $D_2(w_i) = 0$ so both sides of the equation are equal to zero.

We claim that condition (IV.29f) is equivalent to item (IV.26b). To see this, notice that $D_1D_2(w_i) = D_2D_1(w_i)$ is trivially satisfied for $w_i \in \ker(D_1) \cap \ker(D_2)$. Since $D_1D_2(w_1) = 0$ and

$$D_2D_1(w_1) = D_2(P_1\ g_1) = D_2(P_1)\ g_1 + \sigma(P_1)\ D_2(g_1) = D_2(P_1)\ g_1,$$

we have that $D_1D_2(w_1) = D_2D_1(w_1)$ holds if and only if $P_1 \in \ker(D_2)$.
Similarly, since $D_2D_1(w_2) = 0$ and

$$D_1D_2(w_2) = D_1(P_2\ g_2) = D_1(P_2)\ \sigma(g_2) + P_2\ D_1(g_2) = D_1(P_2)\ \sigma(g_2),$$

we have that $D_1D_2(w_2) = D_2D_1(w_2)$ holds if and only if $P_2 \in \ker(D_1)$.

We claim that condition (IV.29g) is equivalent to item (IV.26d). To see this, consider the following: For $i = 1, 2$, we have

$$\begin{aligned} D_i\sigma(w_i) &= D_i(\lambda_i(\sigma)\ w_i) \\ &= \lambda_i(\sigma)\ D_i(w_i) \\ &= \lambda_i(\sigma)\ P_i\ g_i \end{aligned}$$

and

$$\begin{aligned} q\sigma D_i(w_i) &= q\ \sigma(P_i\ g_i) \\ &= q\ \sigma(P_i)\ \sigma(g_i) \\ &= q\ \sigma(P_i)\ \xi(g_i)\ g_i \\ &= q\ \xi(g_i)\ \sigma(P_i)\ g_i \end{aligned}$$

Therefore, $D_i\sigma(w_i) = q\sigma D_i(w_i)$ holds if and only if $\sigma(P_i) = q^{-1}\ \lambda_i(\sigma)\ \xi(g_i^{-1})\ P_i$ for $i = 1, 2$. On the other hand, for $i \neq 1, 2$, we have $D_1(w_i) = 0 = D_2(w_i)$ so both sides of the equation

$$D_j\sigma(w_i) = q\sigma D_j(w_i) \ \text{ for } j = 1, 2,\ i = 1, \ldots, k,$$

are zero. □

Remark IV.30. The idea behind this result comes from [20], which deals with the case of $S(V)\#G$. Thus, Theorem IV.29 is a generalization of Theorem 3.6 in [20] to $S_{\mathbf{q}}(V)\#G$ with $f(g, h) = 1$ for all $g, h \in G$, $\lambda_{1g} = \chi_1(g)$, $\lambda_{2g} = \chi_2(g)$ for all $g \in G$, $\nu_1 = 1$, and $\nu_2 = 1$, but with new assumptions on σ, P_1 and P_2.

Remark IV.31. If we set

$$k = 3,\ P_1 = 1,\ P_2 = w_3,\ g_1 = \sigma_2,\ g_2 = \sigma_1\ \sigma_2^{-1},$$

$$\lambda_1(\sigma) = q,\ \lambda_2(\sigma) = \lambda_3(\sigma) = 1,$$

$$q_{P_1,w_2} = q_{P_1,w_3} = q_{P_2,w_3} = 1,\ q_{P_2,w_1} = q^{-1},$$

then we can see that we can apply Theorem IV.28 to the motivational example presented in Section IV.B. We may also apply Theorem IV.28 to the generalization of the motivational example constructed in Remark IV.18 if we set

$$P_1 = w_3^{\alpha_1 n}\, w_4^{\alpha_2 n} \cdots w_k^{\alpha_{k-2} n},\ P_2 = w_3^{\beta_1 n+1}\, w_4^{\beta_2 n} \cdots w_k^{\beta_{k-2} n},\ g_1 = \sigma_2,\ g_2 = \sigma_1\, \sigma_2^{-1},$$

$$\lambda_1(\sigma) = q,\ \lambda_i(\sigma) = 1 \ \text{ for } i = 2, \ldots, k,$$

$$q_{P_1, w_i} = 1 \ \text{ for } i = 2, \ldots, k,$$

$$q_{P_2, w_1} = q^{-(\beta_1 n + 1 + \beta_2 n + \cdots + \beta_{k-2} n)},\ q_{P_2, w_i} = 1 \ \text{ for } i = 3, \ldots, k.$$

Recall the multiplication in the deformed algebra given in Definition IV.1. If, in the setting of Theorem IV.28, we want to find, for instance, an expression for the map μ_1, then we can proceed as follows: By (IV.7), we have that $m \circ \exp_q(t\, D_1 \otimes D_2)$ gives a deformation of $S_{\mathbf{q}}(V)\#G$, where m is the multiplication in $S_{\mathbf{q}}(V)\#G$. As a consequence,

$$\mu_1 = m \circ (D_1 \otimes D_2). \tag{IV.32}$$

Remark IV.33. The assumptions made in Theorem IV.25 and Theorem IV.28 are mainly for simplifying purposes. We believe that it might be possible to prove these same results in a more general setting. This is a subject for future research.

CHAPTER V

NONTRIVIALITY OF THE DEFORMATIONS

Once we have found a formal deformation of an algebra, it may turn out that this deformation is trivial, in the sense that it is isomorphic to the formal power series ring with coefficients in the original algebra. To verify that we have indeed obtained a new object, we must prove that there does not exist such an isomorphism, which may be difficult to do directly. Using Hochschild cohomology makes this easier to show.

The main goal of this chapter is to prove that the deformations of $S_{\mathbf{q}}(V)\#G$ obtained in Chapter IV are not isomorphic to $(S_{\mathbf{q}}(V)\#G)[[t]]$. We begin by studying the connection between algebraic deformation theory and Hochschild cohomology in Section A. In Section B, we give the precise characterization of the infinitesimal of the deformations that result from Theorem IV.28 and (IV.7). As we will see, this characterization suffices to prove the nontriviality of the resulting deformations.

A. Deformations and Hochschild Cohomology

The deformations of any algebra are intimately related to its Hochschild cohomology. According to [17], Gerstenhaber's works [13] and [14] marked the beginning of the study of the connection between Hochschild cohomology and algebraic deformation theory. He showed that for an algebra A, the space $\mathrm{HH}^i(A)$ with $i \leq 3$ has a natural interpretation related to the maps μ_i and the obstructions γ_i. As we have seen in Section IV.A, the map μ_1 is a Hochschild 2-cocycle and the obstructions γ_i to the existence of the rest of the maps μ_i are Hochschild 3-cocycles.

This topic is an area of active research. For instance, the relation between (quantum) Drinfeld Hecke algebras and Hochschild cohomology is discussed in [34] and [32].

Roughly speaking, quantum Drinfeld Hecke algebras are generalizations of Drinfeld Hecke algebras in which polynomial rings are replaced by quantum polynomial rings. Recall that Drinfeld Hecke algebras were introduced in Chapter I. For a precise definition of a quantum Drinfeld Hecke algebra, we refer the reader to [32] and [28].

Recall that in Section IV.A, we concluded that the map μ_1, used to define a deformation, is a Hochschild 2-cocycle. If we can prove that the Hochschild cocycle μ_1 represents a nonzero element in the Hochschild cohomology ring, i.e. it is not a coboundary (see Remark III.18), then this automatically implies that the deformation is nontrivial. Let us explain this in more detail. Assume that the deformation of an algebra A is trivial. Then we will show that μ_1 is a coboundary. Denote by $\mathcal{D}$ the deformed algebra. If the deformation is trivial, then there exists a $\mathbf{k}[[t]]$-algebra isomorphism $\varphi : \mathcal{D} \to A[[t]]$ given by

$$\varphi(a) = a + \varphi_1(a)\; t + \varphi_2(a)\; t^2 + \dots$$

for each $a \in A$ (see Section 4 in [17] for further details). Then

$$\varphi(a * b) = \varphi(a)\; \varphi(b) \quad \text{for all } a, b \in \mathcal{D}.$$

The left hand side gives

$$\begin{aligned} \varphi(a * b) &= \varphi(ab + \mu_1(a,b)\; t + \mu_2(a,b)\; t^2 + \dots) \\ &= \varphi(ab) + \varphi(\mu_1(a,b))\; t + \varphi(\mu_2(a,b))\; t^2 + \dots \\ &= ab + \varphi_1(ab)\; t + \varphi_2(ab)\; t^2 + \dots + \mu_1(a,b)\; t + \varphi_1(\mu_1(a,b))\; t^2 + \dots \end{aligned}$$

The right hand side simplifies to

$$\varphi(a)\,\varphi(b) = (a + \varphi_1(a)\,t + \dots)\,(b + \varphi_1(b)\,t + \dots)$$
$$= ab + \varphi_1(a)\,b\,t + a\,\varphi_1(b)\,t + \dots$$

If we consider the coefficients of t, we obtain

$$\varphi_1(ab) + \mu_1(a,b) = \varphi_1(a)\,b + a\,\varphi_1(b),$$

which can be rewritten as

$$\mu_1(a,b) = \varphi_1(a)\,b - \varphi_1(ab) + a\,\varphi_1(b).$$

That is, μ_1 is a coboundary. Therefore, we have shown that if μ_1 is not a coboundary, then the deformation is nontrivial.

B. The Precise Characterization of the Infinitesimal

For the rest of this chapter, we will work in the setting of Subsection IV.C.3. We will assume that all the conditions necessary for Theorem IV.28 to hold are fulfilled. We will also assume that the group G is finite and that P_1 and P_2 are elements of the subalgebra of $S_{\mathbf{q}}(V)$ generated by $\{w_3, \dots, w_k\}$. Notice that the examples presented in Section IV B satisfy these assumptions.

Recall that at the end of Chapter IV, we found μ_1 in the setting of Theorem IV.28 (see (IV.32)). In this section, we will give a direct verification that in this case μ_1 is indeed a Hochschild 2-cocycle by identifying which Hochschild 2-cocycle μ_1 is in relation to a known calculation of the Hochschild cohomology ring of $S_{\mathbf{q}}(V)\#G$. Finally, we will prove that this Hochschild 2-cocycle is nonzero in the Hochschild cohomology ring. As a consequence, the deformations found using Theorem IV.28

and (IV.7) are nontrivial.

In Remark IV.11, we showed that D_1 is a $\sigma, 1$-skew derivation, and D_2 is a $1, \sigma$-skew derivation of $S_{\mathbf{q}}(V)\#G$. Thus, by the discussion in Section III.C, D_1 and D_2 are Hochschild 1-cocycles for $(S_{\mathbf{q}}(V)\#G)_\sigma$ and ${}_\sigma(S_{\mathbf{q}}(V)\#G)$, respectively. Let us show that the infinitesimal μ_1, defined in (IV.32), is a Hochschild 2-cocycle for $S_{\mathbf{q}}(V)\#G$ by verifying that μ_1 satisfies (III.13), that is

$$a\ \mu_1(b, c) + \mu_1(a, bc) = \mu_1(ab, c) + \mu_1(a, b)\ c \quad \text{for all } a, b, c \in S_{\mathbf{q}}(V)\#G.$$

The left hand side is given by

$$\begin{aligned} a\ \mu_1(b, c) + \mu_1(a, bc) &= a\ D_1(b)\ D_2(c) + D_1(a)\ D_2(bc) \\ &= a\ D_1(b)\ D_2(c) + D_1(a)\ D_2(b)\ c + D_1(a)\ \sigma(b)\ D_2(c) \end{aligned}$$

and the right hand side is

$$\begin{aligned} \mu_1(ab, c) + \mu_1(a, b)\ c &= D_1(ab)\ D_2(c) + D_1(a)\ D_2(b)\ c \\ &= D_1(a)\ \sigma(b)\ D_2(c) + a\ D_1(b)\ D_2(c) + D_1(a)\ D_2(b)\ c \end{aligned}$$

Thus, this proves the claim.

Let us introduce some notation. First, recall that we are working on a **k**-vector space V with basis $\{w_1, \dots, w_k\}$. Denote by V^* its dual vector space and by $\{w_1^*, \dots, w_k^*\}$ the corresponding dual basis, i.e.

$$w_i^*(w_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbb{N}^k$ denote the set of all k-tuples of elements from $\mathbb{N}$. For any $\alpha \in \mathbb{N}^k$, the length

$|\alpha|$ of α is defined as

$$|\alpha| = \sum_{i=1}^{k} \alpha_i.$$

Define $w^\alpha = w_1^{\alpha_1} \cdots w_k^{\alpha_k}$ for all $\alpha \in \mathbb{N}^k$. Finally, whenever A is a set with an action of G, we will denote by A^G the elements of A that are invariant under the action of G.

Let $\mathbf{q} = (q_{ij})$ as in Chapter I. The *quantum exterior algebra* of V is defined as

$$\textstyle\bigwedge_{\mathbf{q}}(V) = T(V) \;/\; (w_i w_j + q_{ij} w_j w_i \mid 1 \le i, j \le k),$$

where $T(V)$ denotes the tensor algebra of V. If $q_{ij} = 1$ for all i, j, then we obtain the exterior algebra $\bigwedge(V)$. Denote the multiplication in $\bigwedge_{\mathbf{q}}(V)$ by $\wedge$. For any $\beta \in \{0,1\}^k$, let $w^{\wedge\beta}$ denote the vector $w_{j_1} \wedge \cdots \wedge w_{j_m} \in \bigwedge^m(V)$, which is defined by $m = |\beta|$, $\beta_{j_\ell} = 1$ for all $\ell = 1, \dots, m$, and $j_1 < \cdots < j_m$.

The following Hochschild cohomology was computed in [31]:

$$\mathrm{HH}^2(S_{\mathbf{q}}(V), S_{\mathbf{q}}(V)\#G) \cong \bigoplus_{g \in G} \bigoplus_{\substack{\beta \in \{0,1\}^k \\ |\beta| = 2}} \bigoplus_{\substack{\alpha \in \mathbb{N}^k \\ \alpha - \beta \in C_g}} \mathrm{span}_{\mathbf{k}}\{(w^\alpha \# g) \otimes (w^*)^{\wedge\beta}\} \tag{V.1}$$

as a subspace of $S_{\mathbf{q}}(V)\#G \otimes \bigwedge_{\mathbf{q}^{-1}}(V^*)$, where C_g is defined to be

$$C_g = \left\{ \gamma \in (\mathbb{N} \cup \{-1\})^k \;\middle|\; \text{for every } i = 1, \dots, k, \; \prod_{j=1}^{k} q_{ij}^{\gamma_j} = \chi_i(g) \;\text{ or }\; \gamma_i = -1 \right\} \tag{V.2}$$

for $g \in G$. By Remark III.18, the following result shows that $P_1 \; g_1 \; P_2 \; g_2 \otimes w_1^* \wedge w_2^*$ is a Hochschild 2-cocycle. Later we will show that in fact $P_1 \; g_1 \; P_2 \; g_2 \otimes w_1^* \wedge w_2^*$ may be identified with our Hochschild 2-cocycle μ_1 defined in (IV.32).

Proposition V.3. *The element $P_1 \; g_1 \; P_2 \; g_2 \otimes w_1^* \wedge w_2^*$ is a representative of an element of* $\mathrm{HH}^2(S_{\mathbf{q}}(V)\#G)$.

Proof. By Theorem 4.7 in [31], it suffices to show that $P_1\ g_1\ P_2\ g_2 \otimes w_1^* \wedge w_2^*$ is a representative of an element of $\mathrm{HH}^2(S_{\mathbf{q}}(V), S_{\mathbf{q}}(V)\#G)$ and is invariant under the action of G. For the first part, consider the following simple calculation: By (IV.29e), we have that

$$\begin{aligned} P_1\ g_1\ P_2\ g_2 &= (P_1\#g_1)\ (P_2\#g_2) \\ &= P_1\ g_1(P_2)\,\#\,g_1\ g_2 \\ &= P_1\ \xi(g_1^{-1})\ \chi_2(g_1)\ P_2\,\#\,g_1\ g_2 \\ &= \xi(g_1^{-1})\ \chi_2(g_1)\ P_1\ P_2\,\#\,g_1\ g_2. \end{aligned}$$

Thus, set $g = g_1\ g_2$ in (V.2). The elements P_1 and P_2 are linear combinations of monomials of the form $w_1^{\rho_1}\ w_2^{\rho_2} \cdots w_k^{\rho_k}$ and $w_1^{\delta_1}\ w_2^{\delta_2} \cdots w_k^{\delta_k}$, respectively, for some $\rho_\ell, \delta_\ell \in \mathbb{N}$, $\ell = 1, \ldots, k$. However, since any calculations can be done term-by-term, it suffices to work with just the monomials. Thus, we have

$$P_1\ P_2 = w_1^{\rho_1} \cdots w_k^{\rho_k}\ w_1^{\delta_1} \cdots w_k^{\delta_k}.$$

Set $\alpha = (\rho_1 + \delta_1, \ldots, \rho_k + \delta_k)$ and $\beta = (1, 1, 0, \ldots, 0)$ in (V.1). To show that $P_1\ g_1\ P_2\ g_2 \otimes w_1^* \wedge w_2^* \in \mathrm{HH}^2(S_{\mathbf{q}}(V), S_{\mathbf{q}}(V)\#G)$, we need to prove that $\alpha - \beta \in C_g$ with $g = g_1\ g_2$, where

$$\begin{aligned} \alpha - \beta &= (\rho_1 + \delta_1, \ldots, \rho_k + \delta_k) - (1, 1, 0, \ldots, 0) \\ &= (\rho_1 + \delta_1 - 1, \rho_2 + \delta_2 - 1, \rho_3 + \delta_3, \ldots, \rho_k + \delta_k). \end{aligned}$$

That is, we want to show that for every $i = 1, \ldots, k$, the following holds:

$$q_{i1}^{\rho_1+\delta_1-1}\ q_{i2}^{\rho_2+\delta_2-1}\ q_{i3}^{\rho_3+\delta_3} \cdots q_{ik}^{\rho_k+\delta_k} = \chi_i(g_1)\ \chi_i(g_2) \quad \text{or} \quad \gamma_i = -1.$$

Notice that when $i = 1, 2$, $\rho_i = \delta_i = 0$, and therefore, $\gamma_i = -1$. Otherwise, consider the following:

$$\begin{aligned}
P_1\ P_2\ w_i &= w_1^{\rho_1} \cdots w_k^{\rho_k}\ w_1^{\delta_1} \cdots w_k^{\delta_k}\ w_i \\
&= \left(\prod_{j=1}^{k} q_{ji}^{\delta_j} \right) w_1^{\rho_1} \cdots w_k^{\rho_k}\ w_i\ w_1^{\delta_1} \cdots w_k^{\delta_k} \\
&= \left(\prod_{j=1}^{k} q_{ji}^{\rho_j} \right) \left(\prod_{j=1}^{k} q_{ji}^{\delta_j} \right) w_i\ w_1^{\rho_1} \cdots w_k^{\rho_k}\ w_1^{\delta_1} \cdots w_k^{\delta_k} \\
&= \left(\prod_{j=1}^{k} q_{ji}^{\rho_j+\delta_j} \right) w_i\ P_1\ P_2.
\end{aligned}$$

By (IV.29a) and (IV.29b), we also have that for $i \neq 1, 2$,

$$\begin{aligned}
P_1\ P_2\ w_i &= P_1\ q_{P_2,w_i}\ w_i\ P_2 \\
&= P_1\ q_{2i}\ \lambda_i(\sigma)\ \chi_i(g_2^{-1})\ w_i\ P_2 \\
&= q_{2i}\ \lambda_i(\sigma)\ \chi_i(g_2^{-1})\ P_1\ w_i\ P_2 \\
&= q_{2i}\ \lambda_i(\sigma)\ \chi_i(g_2^{-1})\ q_{P_1,w_i}\ w_i\ P_1\ P_2 \\
&= q_{2i}\ \lambda_i(\sigma)\ \chi_i(g_2^{-1})\ q_{1i}\ \lambda_i^{-1}(\sigma)\ \chi_i(g_1^{-1})\ w_i\ P_1\ P_2 \\
&= q_{1i}\ q_{2i}\ \chi_i(g_1^{-1})\ \chi_i(g_2^{-1})\ w_i\ P_1\ P_2.
\end{aligned}$$

Therefore,

$$\prod_{j=1}^{k} q_{ji}^{\rho_j+\delta_j} = q_{1i}\ q_{2i}\ \chi_i(g_1^{-1})\ \chi_i(g_2^{-1}).$$

Or equivalently,

$$q_{i1}^{-1}\ q_{i2}^{-1} \left(\prod_{j=1}^{k} q_{ij}^{\rho_j+\delta_j} \right) = \chi_i(g_1)\ \chi_i(g_2) \quad \text{for } i \neq 1, 2,$$

which is exactly what we wanted to show.

For the second part, to prove that $P_1\ g_1\ P_2\ g_2 \otimes w_1^* \wedge w_2^*$ is invariant under the action of G, consider the following calculation:

$$g(P_1\ g_1\ P_2\ g_2 \otimes w_1^* \wedge w_2^*) =$$

$$g(P_1)\ g_1\ g(P_2)\ g_2 \otimes g(w_1^*) \wedge g(w_2^*) =$$

$$\chi_1(g)\ \xi(g)\ P_1\ g_1\ \chi_2(g)\ \xi(g^{-1})\ P_2\ g_2 \otimes \chi_1^{-1}(g)\ w_1^* \wedge \chi_2^{-1}(g)\ w_2^* =$$

$$P_1\ g_1\ P_2\ g_2 \otimes w_1^* \wedge w_2^*$$

for all $g \in G$, where we used (IV.29d), (IV.29e) and

$$g(w_i^*) = \chi_i^{-1}(g)\ w_i^*.$$

□

Let us introduce the quantum Koszul resolution of $S_{\mathbf{q}}(V)$, which we will need in what follows. For each $g \in G$, $(S_{\mathbf{q}}(V))_g$ is a left $(S_{\mathbf{q}}(V))^e$-module via the action

$$(a \otimes b)(cg) = ac\ g(b)\ g \qquad \text{for all } a, b, c \in S_{\mathbf{q}}(V),\ g \in G.$$

The following is a free $(S_{\mathbf{q}}(V))^e$-resolution of $S_{\mathbf{q}}(V)$:

$$\cdots \longrightarrow (S_{\mathbf{q}}(V))^e \otimes {\textstyle\bigwedge}_{\mathbf{q}}^2(V) \xrightarrow{d_2} (S_{\mathbf{q}}(V))^e \otimes {\textstyle\bigwedge}_{\mathbf{q}}^1(V) \xrightarrow{d_1} (S_{\mathbf{q}}(V))^e \xrightarrow{mult} S_{\mathbf{q}}(V) \longrightarrow 0,$$

where

$$d_m(1^{\otimes 2} \otimes w_{j_1} \wedge \cdots \wedge w_{j_m}) =$$

$$\sum_{i=1}^{m}(-1)^{i+1}\left[\left(\prod_{s=1}^{i} q_{j_s,j_i}\right) w_{j_i} \otimes 1 - \left(\prod_{s=i}^{m} q_{j_i,j_s}\right) \otimes w_{j_i}\right] \otimes w_{j_1} \wedge \cdots \wedge w_{j_{i-1}} \wedge w_{j_{i+1}} \wedge \cdots \wedge w_{j_m}$$

whenever $1 \leq j_1 < \cdots < j_m \leq k$. We refer the reader to [36] for more details on this construction.

In [32], chain maps Ψ_i are introduced between the bar resolution and the quantum Koszul resolution of $S_{\mathbf{q}}(V)$. Recall that the bar resolution was introduced in

Section III.B. In particular, we have the map

$$\Psi_2 : (S_{\mathbf{q}}(V))^{\otimes 4} \longrightarrow (S_{\mathbf{q}}(V))^e \otimes \textstyle\bigwedge_{\mathbf{q}}^2(V)$$

such that

$$\Psi_2(1 \otimes w_i \otimes w_j \otimes 1) = \begin{cases} 1 \otimes 1 \otimes w_i \wedge w_j, & \text{for } 1 \leq i < j \leq k, \\ 0, & \text{for } i \geq j. \end{cases} \tag{V.4}$$

We will also need the following two maps:

$$R_2 : \mathrm{Hom}_{\mathbf{k}}\big((S_{\mathbf{q}}(V))^{\otimes 2}, S_{\mathbf{q}}(V)\#G\big) \longrightarrow \mathrm{Hom}_{\mathbf{k}}\big((S_{\mathbf{q}}(V))^{\otimes 2}, S_{\mathbf{q}}(V)\#G\big)^G$$
$$R_2(\gamma) = \frac{1}{|G|}\sum_{g\in G} g(\gamma)$$

and

$$\theta_2^* : \mathrm{Hom}_{\mathbf{k}}\big((S_{\mathbf{q}}(V))^{\otimes 2}, S_{\mathbf{q}}(V)\#G\big)^G \longrightarrow \mathrm{Hom}_{\mathbf{k}}\big((S_{\mathbf{q}}(V)\#G)^{\otimes 2}, S_{\mathbf{q}}(V)\#G\big)$$
$$\theta_2^*(\gamma)(ag \otimes bh) = \gamma(a \otimes g(b))\, gh.$$

As discussed in [32], since Ψ_2 may not preserve the action of G, the map R_2 ensures G-invariance of the image. The map θ_2^* extends a function defined on $(S_{\mathbf{q}}(V))^{\otimes 2}$ to a function defined on $(S_{\mathbf{q}}(V)\#G)^{\otimes 2}$.

By Theorem 3.5 in [32], the composition $\theta_2^*\, R_2\, \Psi_2^*$ induces an isomorphism

$$\left(\bigoplus_{g\in G} \mathrm{HH}^2(S_{\mathbf{q}}(V), (S_{\mathbf{q}}(V))_g)\right)^G \longrightarrow \mathrm{HH}^2(S_{\mathbf{q}}(V)\#G).$$

As a consequence, we get that if $\kappa \in (S_{\mathbf{q}}(V)\#G) \otimes \bigwedge_{\mathbf{q}^{-1}}^2(V^*)$, then

$$[\theta_2^* R_2 \Psi_2^*(\kappa)](w_i \otimes w_j) = \frac{1}{|G|}\sum_{g\in G} g(\kappa(\Psi_2(1 \otimes g^{-1}(w_i) \otimes g^{-1}(w_j) \otimes 1))) \text{ for } i < j. \tag{V.5}$$

The following proposition is an explicit description of μ_1.

Proposition V.6. *The map μ_1 can be identified with $P_1\ g_1\ P_2\ g_2 \otimes w_1^* \wedge w_2^*$.*

Proof. Set $\kappa = P_1 g_1 P_2 g_2 \otimes w_1^* \wedge w_2^*$ in (V.5). Then we have that

$$[\theta_2^* R_2 \Psi_2^*(P_1 g_1 P_2 g_2 \otimes w_1^* \wedge w_2^*)](w_i \otimes w_j) =$$

$$\frac{1}{|G|} \sum_{g \in G} g \left(P_1 g_1 P_2 g_2 (w_1^* \wedge w_2^*) \left(\Psi_2 \left(1 \otimes g^{-1}(w_i) \otimes g^{-1}(w_j) \otimes 1\right)\right)\right).$$

By (V.4), we may simplify this expression as follows:

$$\frac{1}{|G|} \sum_{g \in G} g \left(P_1 g_1 P_2 g_2 (w_1^* \wedge w_2^*) \left(1 \otimes 1 \otimes g^{-1}(w_i) \wedge g^{-1}(w_j)\right)\right).$$

Since $g^{-1}(w_i) = \chi_i^{-1}(g)\ w_i$ and applying (IV.29e), this becomes

$$\frac{1}{|G|} \sum_{g \in G} \xi(g_1^{-1})\ \chi_2(g_1)\ g \left(P_1 P_2 \# g_1 g_2 (w_1^* \wedge w_2^*) \left(1 \otimes 1 \otimes \chi_i^{-1}(g)\ w_i \wedge \chi_j^{-1}(g)\ w_j\right)\right).$$

By linearity, we get

$$\frac{1}{|G|}\ \xi(g_1^{-1})\ \chi_2(g_1) \sum_{g \in G} \chi_i^{-1}(g)\ \chi_j^{-1}(g)\ g \left(P_1 P_2 \# g_1 g_2 (w_1^* \wedge w_2^*) \left(1 \otimes 1 \otimes w_i \wedge w_j\right)\right).$$

If we assume that $i < j$, then

$$(w_1^* \wedge w_2^*)(w_i \wedge w_j) = \begin{cases} 1, & \text{if } i = 1 \text{ and } j = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, letting $i = 1$ and $j = 2$, the above expression becomes

$$\frac{1}{|G|}\ \xi(g_1^{-1})\ \chi_2(g_1) \sum_{g \in G} \chi_1^{-1}(g)\ \chi_2^{-1}(g)\ g(P_1 P_2 \# g_1 g_2) =$$

$$\frac{1}{|G|}\ \xi(g_1^{-1})\ \chi_2(g_1) \sum_{g \in G} \chi_1^{-1}(g)\ \chi_2^{-1}(g)\ g(P_1)\ g(P_2) \# g_1 g_2 =$$

$$\frac{1}{|G|}\ \xi(g_1^{-1})\ \chi_2(g_1) \sum_{g \in G} \chi_1^{-1}(g)\ \chi_2^{-1}(g)\ \chi_1(g)\ \xi(g)\ P_1\ \chi_2(g)\ \xi(g^{-1})\ P_2 \# g_1 g_2$$

by (IV.29d) and (IV.29e). Simplifying, we obtain

$$\xi(g_1^{-1})\ \chi_2(g_1)\ P_1\ P_2\#g_1g_2 =$$

$$P_1\ g_1\ P_2\ g_2.$$

Therefore, we have shown that

$$[\theta_2^* R_2 \Psi_2^*(P_1 g_1 P_2 g_2 \otimes w_1^* \wedge w_2^*)](w_i \otimes w_j) = \begin{cases} P_1\ g_1\ P_2\ g_2, & \text{if } i = 1 \text{ and } j = 2, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, since $\mu_1 = m \circ (D_1 \otimes D_2)$, we have that

$$\mu_1(w_i \otimes w_j) = D_1(w_i)\ D_2(w_j) = \begin{cases} P_1\ g_1\ P_2\ g_2, & \text{if } i = 1 \text{ and } j = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, μ_1 can be identified with $P_1\ g_1\ P_2\ g_2 \otimes w_1^* \wedge w_2^*$. □

Since P_1, g_1, P_2, g_2, w_1 and w_2 are nonzero, we know that $P_1\ g_1\ P_2\ g_2 \otimes w_1^* \wedge w_2^*$ is a nonzero Hochschild 2-cocycle. Therefore, we have obtained the following:

Theorem V.7. *When G is finite and P_1 and P_2 are elements of the subalgebra of $S_{\mathbf{q}}(V)$ generated by $\{w_3, \ldots, w_k\}$, all the deformations that result from Theorem IV.28 and* (IV.7) *are nontrivial.*

CHAPTER VI

CONCLUSION

In this dissertation, we have discussed deformations of an algebra. For the case when the deformations arise from an action of the Hopf algebra H_q, introduced in Chapter II, we were able to provide necessary and sufficient conditions for $S_{\mathbf{q}}(V)\#G$ to have the structure of an H_q-module algebra under some assumptions (Chapter IV). Using Hochschild cohomology, we showed in Chapter V that a large class of the previously obtained deformations are nontrivial. A particularly relevant example of this theory was presented in Section IV.B, where we constructed a class of deformations of $S_{\mathbf{q}}(V)\#G$ in which the new relations in the deformed algebra involve elements of the original vector space.

The techniques and ideas developed in this dissertation have enabled us to not only find new examples of deformations but also show that there exist deformations that are not graded in the sense of Braverman and Gaitsgory [4] (see Remark IV.18). This is particularly relevant since much has been done in geometric settings, such as deformations of functions on manifolds [27] and algebraic varieties [26]. However, less is known on deformations of noncommutative algebras, such as smash product algebras.

A dissertation cannot be complete until it provides an overview of possible extensions of the presented work. Some future directions of research include:

- It would be interesting to extend the obtained results to a wider variety of objects. For instance, we could try to obtain the same results for a different Hopf algebra. Our first approach to solve this question is to construct a larger Hopf algebra that contains H_q as a Hopf subalgebra. Then the same universal deformation formula (IV.7) would still apply since the expression only involves D_1 and

D_2. For example, under the additional assumption that $I = (D_1^n, D_2^n, \sigma^n - 1)$, H_q sits inside the larger finite dimensional quantum group $u_q(sl_2) \otimes u_q(sl_2)$.

- Another interesting extension of these results might be the following: Besides $S(V)\#G$ and $S_{\mathbf{q}}(V)\#G$, are there any other objects that could be given an H_q-module algebra structure? In this case, we would probably start by looking at quotients of $S_{\mathbf{q}}(V)$ of the following form: Fix positive integers $N_1, \ldots, N_k$ and consider the finite dimensional algebra $S_{\mathbf{q}}(V) \;/\; (w_1^{N_1}, \ldots, w_k^{N_k})$. Then is it possible to give $(S_{\mathbf{q}}(V) \;/\; (w_1^{N_1}, \ldots, w_k^{N_k})) \# G$ the structure of an H_q-module algebra? The quotient $S_{\mathbf{q}}(V) \;/\; (w_1^{N_1}, \ldots, w_k^{N_k})$ has been studied for instance in [29].

- One important open question in algebraic deformation theory is the following: For a given algebra, does every Hochschild 2-cocycle that is unobstructed (i.e. the corresponding Hochschild 3-cocycle is a coboundary) lift to a deformation? A positive answer for the case of a polynomial algebra follows from [26]. However, the answer is still unknown for the case of $S(V)\#G$ and, more generally, $S_{\mathbf{q}}(V)\#G$. Constructing more examples of such deformations will give some insight on how to answer this question.

- It is known that for any algebra A, if $\mathrm{HH}^2(A) = 0$ then A has no nontrivial deformations (see, for example, [17]). The converse is known to be false in the case when A is the universal enveloping algebra of a Lie algebra [33] and when A is any algebra and $\mathrm{char}(\mathbf{k}) = p$ with p a prime [16]. However, the answer is still unknown for the case when A is any algebra and $\mathrm{char}(\mathbf{k}) = 0$.

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