NON-SMOOTH DYNAMICS USING DIFFERENTIAL-ALGEBRAIC EQUATIONS PERSPECTIVE: MODELING AND NUMERICAL SOLUTIONS

A Thesis

by

PRIYANKA GOTIKA

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

December 2011

Major Subject: Mechanical Engineering

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Approved by:

Chair of Committee,	Kalyana B. Nakshatrala
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Head of Department,	Jerald Caton

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ABSTRACT

Non-smooth Dynamics Using Differential-algebraic Equations perspective: Modeling and Numerical Solutions. (December 2011)

Priyanka Gotika, B.E.; M.S., Birla Institute of Technology & Science-Pilani

Chair of Advisory Committee: Dr. Kalyana B. Nakshatrala

This thesis addressed non-smooth dynamics of lumped parameter systems, and was restricted to Filippov-type systems. The main objective of this thesis was two-fold. Firstly, modeling aspects of Filippov-type non-smooth dynamical systems were addressed with an emphasis on the constitutive assumptions and mathematical structure behind these models. Secondly, robust algorithms were presented to obtain numerical solutions for various Filippov-type lumped parameter systems. Governing equations were written using two different mathematical approaches. The first approach was based on differential inclusions and the second approach was based on differential-algebraic equations. The differential inclusions approach is more amenable to mathematical analysis using existing mathematical tools. On the other hand, the approach based on differential-algebraic equations gives more insight into the constitutive assumptions of a chosen model and easier to obtain numerical solutions.

Bingham-type models in which the force cannot be expressed in terms of kinematic variables but the kinematic variables can be expressed in terms of force were considered. Further, Coulomb friction was considered in which neither the force can be expressed in terms of kinematic variables nor the kinematic variables in terms of force. However, one can write implicit constitutive equations in terms of kinematic quantities and force. A numerical framework was set up to study such systems and the algorithm was devised. Towards the end, representative dynamical systems of practical significance were considered. The devised algorithm was implemented on these systems and the results were obtained. The results show that the setting offered by differential-algebraic equations is appropriate for studying dynamics of lumped parameter systems under implicit constitutive models.

To my mom, dad and sister

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I wish to express my deepest gratitude to my advisor, Dr. Kalyana B. Nakshatrala, who has been supporting and encouraging me towards research in the field of computational mechanics right from the beginning of my Master's study. I am indebted a ton for the innumerable discussions after class and thought-provoking research meetings that made me take the big step of committing myself to this thesis. Without his constant support, this thesis would not have taken its final form.

I thank Dr. J. N. Reddy for setting high standards towards research that helped this thesis to come out efficiently. Also, I thank Dr. Steve Suh for his valuable advice and for serving as a committee member for this thesis.

I would like to thank all my professors who have set examples and laid a strong foundation that forged my thought process which was crucial while working on this thesis. Finally, I acknowledge my parents, sister and friends, for their priceless moral support and love, without which this thesis would not have been realised.

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CHAPTER I

INTRODUCTION AND MOTIVATION

Mechanical systems can be broadly divided into two classes: smooth and non-smooth. This classification is based on the system dynamics observed with the aid of their corresponding mathematical models. Study of smooth systems had been extensively carried out in the past, mostly modeled using ordinary differential equations (ODEs). One classical example is the simple harmonic oscillator which is a continuous model governed by a second-order ordinary differential equation. There are several existence and uniqueness results for the solution of ODEs. For example, Cauchy-Peano theorem [1], Carathéodory theorem [1] and, many others.

Non-smooth systems are significantly more difficult to study in terms of modeling, obtaining analytical solutions, and mathematical analysis [2, 3]. More mathematical aspects of non-smooth systems can be seen from References [4] and [5]. Non-smooth systems can be further divided into three basic classifications: non-smooth continuous systems or piece-wise continuous systems, discontinuous systems of Filippov-type, and impacting systems [6]. Non-smooth continuous systems are studied using the theory of ordinary differential equations under some assumptions. Till date, Filippov systems are studied using differential inclusion formalism, and impacting systems using the theory of complementarity systems. Though large volumes of literature exist for all the three kinds of systems, study of non-smooth systems is still an active area of research. A review paper on bifurcations in non-smooth systems gives the advances of research in this direction [7]. Most physical systems have one or more non-smooth characteristics involved in their dynamics. Some of the popular examples

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The journal model is *IEEE Transactions on Automatic Control.*

involving non-smooth dynamics are friction, granular mechanics simulations, multi-body dynamics applications and robotics. Among these, discontinuous systems of Filippov-type form the main subject of this thesis. Filippov-type systems are informally defined as systems whose mathematical models consist of a set of first-order differential equations with discontinuous right-hand side [6]. For example, systems that involve non-smooth characteristics like dry friction have a discontinuous function in their mathematical models. The mathematical modeling of friction [8, 9, 10] had been revolving as an amusing subject among many physicists and mathematicians for decades now. Most of the existing literature on models of dry friction are based on the Amontons-Coulomb model. However, it should be noted that the model cannot be used for general applications, and has many limitations. Application of this model to dynamical systems with dry friction gives rise to differential equations with a discontinuity in the right-hand side. This is one of the most important motivations behind the study of discontinuous systems of Filippov-type. This motivation for the study of Filippov-type systems drawn from friction models is from mechanics stand point. But Filippov-type differential equations arise in various fields like electrical systems, control systems or for that matter any system which has an on/off switching mechanism [11, 12].

Filippov and others have made enormous contributions towards the development of theories to study such systems [13, 14, 15]. In particular, Filippov formulated the so-called Filippov's convex method using which he extended the concept of discontinuous differential equations to differential inclusions. He also formulated the most celebrated Filippov's theorem describing the conditions for the existence of solution for such differential inclusions. This theorem is discussed in detail in forth-coming chapters. In mathematical literature, Filippov systems are typically studied using differential inclusions approach [16, 17]. Though this theory is rich in its own sense, misnomers like “multi-valued functions” had to be introduced to support the arguments [13].

In this thesis, a different kind of a framework is developed to understand the non-smooth behaviour of Filippov-type systems. This approach lies in modeling the systems to form a set of differential-algebraic equations (DAEs) or in some cases differential-algebraic inequalities (DAIs). In a crude sense with respect to dynamical systems, DAEs can be described as a set containing differential and algebraic equations. These governing equations arise from the system dynamics like linear momentum balance and the constitutive relations of energy storage/dissipating mechanisms [18, 19]. It is important to mention that frequently, these set of DAEs can be simplified into a system of ordinary differential equations (ODEs) [20] making their analysis easier. Occasionally they are needed to be solved as a system of DAEs which is somewhat complex yet straight forward compared to the methods used to solve differential inclusions. For example, if we consider a simple dynamical system with Bingham-type dash-pot, the constitutive equation of dash-pot cannot be written in a way in which force due to dash-pot is expressed as a “standard function” of velocity. This can be seen from Figure 1. Writing the constitutive equations using multi-valued functions is the only way to do that. On the other hand, the same constitutive relation can be written as a standard function if velocity is expressed in terms of force due to dash-pot [21]. Definitions for multi-values function and standard function are discussed in Chapter III. This way of expressing kinematic quantities in terms of dynamic quantities gives rise to a set of differential-algebraic equations. Such system of equations are generally referred to as semi-explicit DAEs. In some systems like those involving Coulomb friction, only implicit constitutive relations are possible [22]. In this thesis, a spring-mass system involving Coulomb friction is considered and this accounts for a broader class of differential-algebraic equations called implicit DAEs.

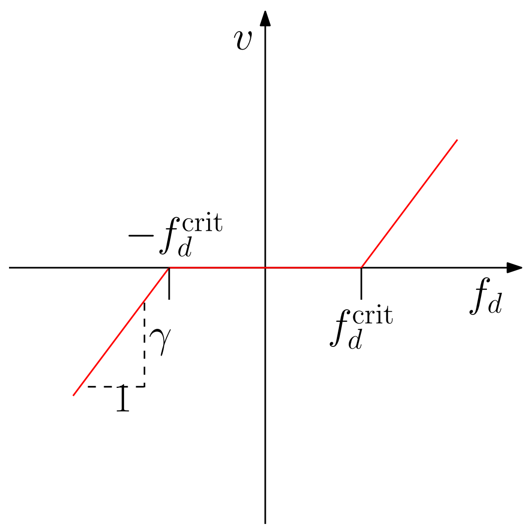


Fig. 1.: Constitutive relation for Bingham-type fluid. Force due to dash-pot cannot be written as a function of kinematic variables (that is, velocity of the mass). On the other hand, as shown in the figure, velocity can be written as a function of the force in the dash-pot.

At this point it should be clear that this way of expressing kinematic and dynamic quantities should not be confused with the causality relationship. This difference is clearly explained in Reference [23]. For example, in many situations in Physics, the relation $F = ma$ is confused as a cause and effect relationship but it is not. Similarly, for the present work even if dynamic quantity is expressed in terms of kinematic quantity or the other way round, either way it need not be a cause-effect relationship.

All these concepts introduced here are discussed in detail in forth-coming chapters. On the whole, it can be summarized that differential inclusions (DIs) are an extension to the concept of discontinuous differential equations with more details from Reference [13], and ordinary differential equations(ODEs) form a special class of differential-algebraic equations (DAEs). Details on this can be seen from Reference [24].

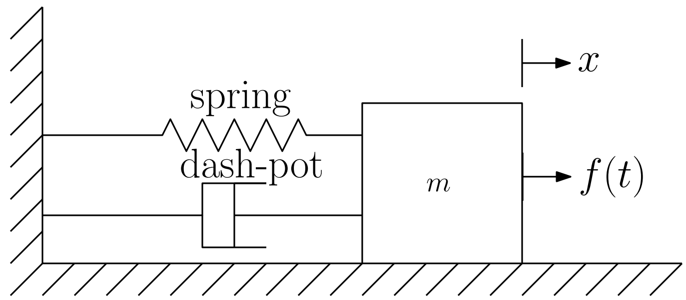


Fig. 2.: Spring-mass-dash-pot system that is represented as a lumped parameter system. m is the mass of the body subject to external force $f(t)$ and x denotes the displacement of the system.

From the present work without introducing misnomers like "multi-valued functions" as in differential inclusions it can be shown that the Filippov systems can be modeled using differential-algebraic equations [21]. At this point it can be claimed that DAEs are mathematically more elegant and numerically more robust. The purpose of the rest of the thesis is to drive home this point. To keep things simple, we considered a lumped parameter system of a spring-mass-dash-pot system as in Figure 2 for analysis. Modeling the spring-mass-dash-pot system as a lumped parameter system makes the study simple. This is discussed in Reference [25].

A. Main contributions

The main contribution of this study is to provide robust numerical techniques to solve the set of differential-algebraic equations obtained from governing equations of lumped parameter systems considered. In order to accommodate for different situations, three cases are considered whose characteristics are different and representative. The cases considered are:

(1) Systems with non-linear constitutive relations governed by semi-explicit DAEs.

(2) Systems with Coulomb friction governed by implicit DAEs.

(3) Systems with unilateral constraints governed by differential-algebraic inequalities.

A generalized numerical framework is developed to accommodate all the cases.

B. Organization of the thesis

The remainder of this thesis is organized as follows. Chapter II discusses various Filippov systems that can be treated using differential-algebraic equations approach. Chapter III is about notation and definitions that are needed for mathematical modeling. In Chapter IV, each of the systems discussed in Chapter II are mathematically modeled using the differential inclusions and the differential-algebraic equations approaches. Numerical algorithms that are developed to treat these problems are discussed in Chapter V. In Chapter VI, representative examples are considered and the numerical algorithms are verified for correctness. Conclusions are drawn in Chapter VII.

CHAPTER II

VARIOUS EXAMPLES: SMOOTH AND NON-SMOOTH DYNAMICAL SYSTEMS

This chapter presents various dynamical systems, which will be further studied in subsequent chapters. These dynamical systems can be mathematically modeled using the framework provided by differential-algebraic equations, which is the main focus of this thesis. An alternate way to model these systems (which is common in the mathematics literature) is using differential inclusions. As mentioned earlier it is not always possible to express dynamic quantities like force in terms of kinematic quantities such as displacement and velocity. Below few examples shall be discussed of how such situations arise in practicality. These examples would serve for a better understanding of the concepts of differential-algebraic equations and differential inclusions in all.

A. Classical simple harmonic oscillator

Firstly, starting off with a smooth system, a classic example is a simple harmonic oscillator. An undamped spring hanging on its own weight as in Figure 3 can be considered as one of the examples of simple harmonic oscillator. The response will be smooth in both kinematic and dynamic quantities (if the forcing function is smooth). That is, the linear spring is incapable of producing non-smooth dynamics.

B. Spring-mass-dash-pot systems

For all the below cases consider a system as in Figure 2. The Filippov-type discontinuity arises in these cases due to the dynamics of fluid in the dash-pot considered.

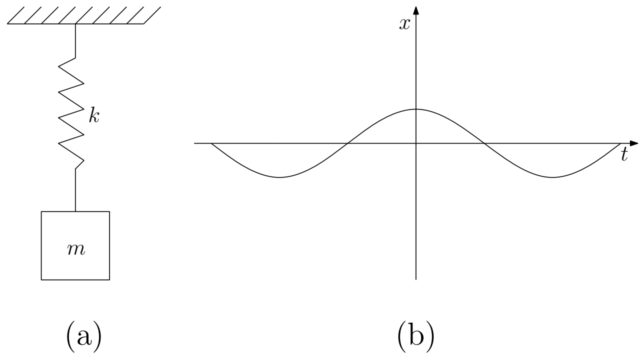
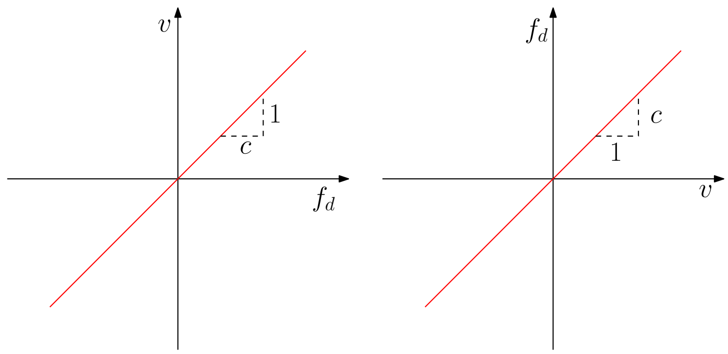


Fig. 3.: A simple harmonic oscillator: (a) Pictorial description (b) A typical displacement versus time response under free vibration (i.e., the external force is zero).

1. For the first case consider a *visco-elastic dash-pot*. The constitutive relation between velocity of the system and force due to dash-pot can be written such that velocity v is written as a function of force due to dash-pot f_{d} and vice-versa. For such systems the governing equations reduce to a set of ODEs. A graphical representation of the constitutive relation can be seen in Figure 4. This system also comes under the category of smooth systems.

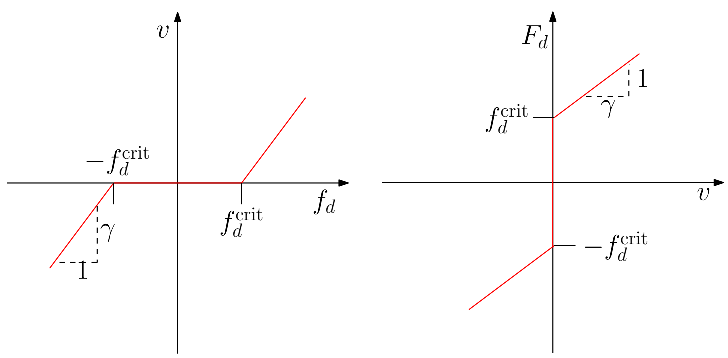
2. Now consider a *Bingham-type dash-pot with linear characteristics*. In this model it is not possible to write force due to dash-pot f_{d} in terms of velocity v but the other way round is possible. This can be seen from Figure 5.

3. A *Bingham-type dash-pot with non-linear but monotonic characteristics* is considered. Even in this case, the force due to dash-pot F_{d} cannot be written as a function of velocity v but v can be written as a function of f_{d}. The only difference is that since it is a non-linear model, obtaining analytical solutions is not straight-forward. For example, a Bingham-type dash-pot model governed by a monotonic function can be seen in Figure 6.



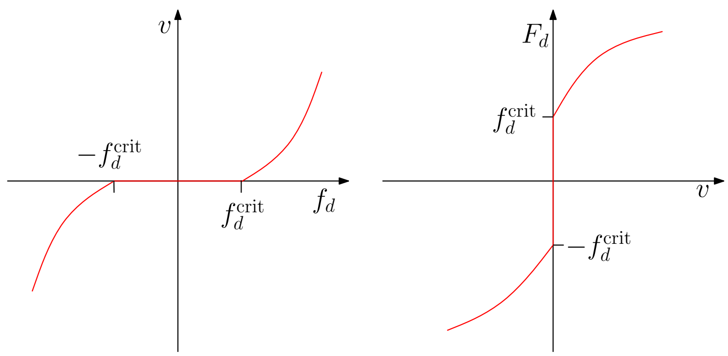
(a) v versus f_{d} (b) f_{d} versus v

Fig. 4.: A pictorial representation of constitutive relation between velocity and force due to dash-pot for a dash-pot governed by visco-elastic model.



(a) v versus f_{d} (b) F_{d} versus v

Fig. 5.: A pictorial representation of constitutive relation between velocity and force due to dash-pot for a dash-pot governed by Bingham-type fluid of linear characteristics. In Figure (a), the notation is lowercase for f_d to represent a standard function whereas in Figure (b), the notation of force due to dashpot is written as F_d in upper case to represent a multi-valued function.



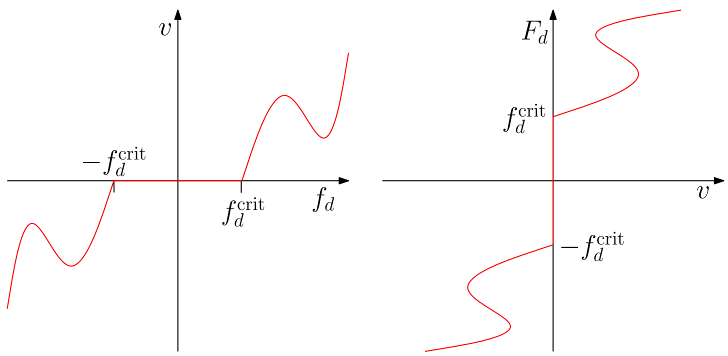
(a) v versus f_{d} (b) F_{d} versus v

Fig. 6.: A pictorial representation of constitutive relation between velocity and force due to dash-pot for a dash-pot governed by Bingham-type fluid of non-linear monotonic characteristics. In Figure (a), the notation is lowercase for f_d to represent a standard function whereas in Figure (b), the notation of force due to dashpot is written as F_d in upper case to represent a multi-valued function.

4. Another special case among non-linear models could be a *Bingham-type dash-pot with non-monotonic characteristics*. This model could be challenging in the sense that it being non-monotonic it may give rise to bifurcations or limit points. Hence, the choice of the appropriate numerical method makes it more interesting case. A representative example of such a non-linear non-monotonic model is shown in Figure 7. The same problem arises when we try to write force in terms of kinematic variables as in the previous case.

C. Coulomb friction model

The next and most important example of Filippov-type system is a dynamical system involving *Coulomb friction*. This type of friction arises due to the movement between solid surfaces. It is not even possible to write force due to friction in terms of



(a) v versus f_{d} (b) F_{d} versus v

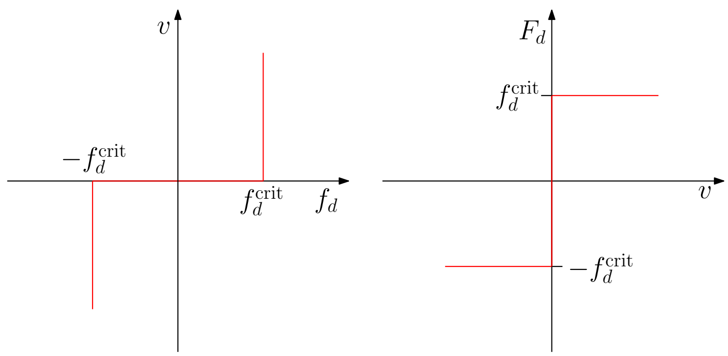
Fig. 7.: A pictorial representation of constitutive relation between velocity and force due to dash-pot for a dash-pot governed by Bingham-type fluid of non-linear non-monotonic characteristics. In Figure (a), the notation is lowercase for f_d to represent a standard function whereas in Figure (b), the notation of force due to dashpot is written as F_d in upper case to represent a multi-valued function.

kinematic quantities. The constitutive relations obtained are only implicit relations. Many of the mechanical systems in industry involve friction, Coulomb friction model is a popular one. Thus, this study is important from both theoretical and practical points of view. Figure 8 gives a pictorial representation between velocity of the system and frictional force, and it is clear that writing explicit expressions between velocity and force is not possible.

Remark C.1 *The non-linearity in Bingham-type models is due to fluid friction. But the non-linearity in Coulomb model is due to dry friction between solid surfaces.*

D. System with unilateral constraint

The study of system dynamics when subjected to *unilateral constraints* is much more interesting. The dynamics of such systems with constraints gives rise to a sys-



(a) v versus $f_{\rm d}$ (b) $f_{\rm d}$ versus v

Fig. 8.: A pictorial representation of constitutive relation between velocity and frictional force for a system with friction governed by Coulomb friction model.

tem of differential-algebraic inequalities. For example, consider a spring-mass system with Bingham-type dash-pot of linear characteristics as in Figure 5, and a unilateral constraint placed at a certain distance such that there is a constraint on the displacement as shown in Figure 9. This forms a special case and does not come under Filippov-type system since at the time of impact the discontinuity in the system of equations is different. A contact-resolution algorithm is used at impact to find the time of impact and dynamics is suitably modified during impact and release.

Thus it can be seen from this chapter that in various practical situations where the constitutive relations cannot be expressed such that dynamic quantities are written as standard functions or explicit expressions of kinematic quantities we may resort to writing the constitutive relations in a way giving rise to DAEs/DAIs. With this motivation we can proceed to the next chapter which gives in detail all the notations, definitions and existence theorems that would be helpful in understanding the point presented in this thesis coherently.

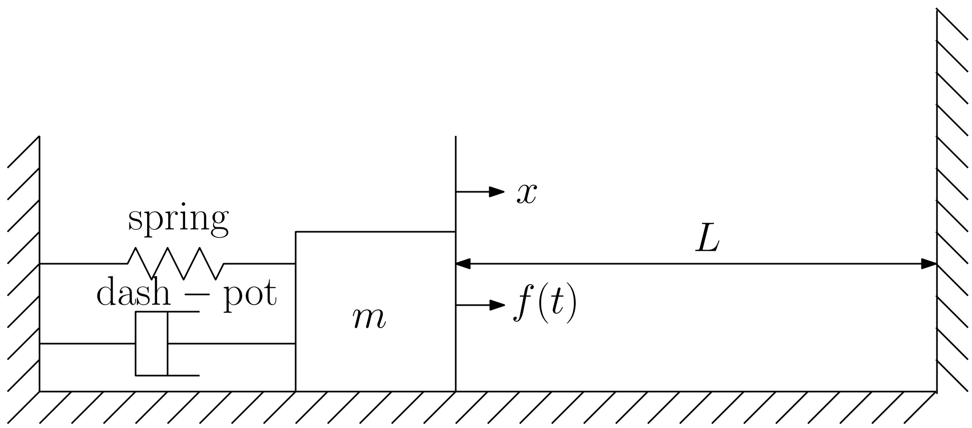


Fig. 9.: A spring-mass-dash-pot system representing a lumped parameter system with unilateral constraint placed at a distance of L units from mass m restricting the displacement x of the system. Here, $f(t)$ is the external force acting on the system.

CHAPTER III

NOTATION AND DEFINITIONS

In this chapter, relevant notations and definitions shall be introduced, which will be used in the remainder of this study.

A. Convex analysis preliminaries

Definition A.1 (Function) *Let $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$ be two sets. A* real-valued function *$f : X \to Y$ associates to each element $x \in X$ a unique value in Y, which is denoted by $f(x)$. The sets X and Y are respectively called the domain and co-domain. The graph of the function f is defined as*

$$\text{graph}[f] := \{(x, f(x)) \mid x \in X\} \tag{3.1}$$

Definition A.2 (Continuous function) *Let f be a function such that $f : \mathbb{R} \to \mathbb{R}$ and let c be an element of the domain. The function f is said to be* continuous *at the point c if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$, $\forall x \in \mathbb{R}$. This is also called the* epsilon-delta definition *for a continuous function.*

Definition A.3 (Lipschitz continuity) *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be* Lipschitz continuous *if there exists a real constant $K \geq 0$ such that for all x_1 and x_2 in $\mathbb{R}$, the condition $|f(x_2) - f(x_1)| \leq K|x_2 - x_1|$ satisfies. The constant K is also called the Lipschitz constant.*

Definition A.4 (Convex set) *A set $C \subset \mathbb{R}$ is* convex set *if for each $\mathbf{x} \in C$ and $\mathbf{y} \in C$ also $(1 - q)\mathbf{x} + q\mathbf{y} \in C$ for arbitrary with $0 \leq q \leq 1$*

Definition A.5 (Open set) *A subset U of $\mathbb{R}$ is called an* open set *if, given any point x in U there exists a real number $\epsilon > 0$ such that given any point y in $\mathbb{R}$ whose Euclidean distance from x is smaller than ϵ, y also belongs to U. Equivalently, a subset U of $\mathbb{R}$ is open if every point in U has a neighborhood in $\mathbb{R}$ contained in U.*

This definition will be useful to define the closed set.

Definition A.6 (Closed set) *A* closed set *is a set whose compliment is an open set*

Definition A.7 (Set-valued function/multi-valued function) *A* set-valued function/multi-valued function *$F : \mathbb{R} \to \mathbb{R}$ is a map from $\mathbb{R}$ to the* subsets of $\mathbb{R}$, *that is for every $x \in \mathbb{R}$, we associate a (potentially empty) set $F(x)$, $x \in \mathbb{R}$. Another definition that can be used is, a* set-valued function/multi-valued function *$F(x)$, $x \in \mathbb{R}$ is a function almost everywhere except at a finite number of isolated points where $F(x)$ forms a subset of $\mathbb{R}$. A set-valued function can therefore contain vertical segments on its graph.*

To distinguish between a function and a set-valued function, lower case shall be used for a function (e.g., $f(x)$), and upper case for a set-valued function (e.g., $F(x)$). By definition, a function is single-valued, which excludes vertical lines, loops and surfaces on its graph. Hence, herein the usage of "multi-valued functions" is not prescribed to. This definition of set-valued function or multi-valued function loses its validity in the sense of the definition of a function.

Definition A.8 (Measurable multi-valued function) *A multi-valued function $F : S \to \mathbb{R}$ is* measurable *if for every open(closed) $C \subseteq \mathbb{R}$, $x \in S : F(x) \cap C \neq \emptyset$ is Lebesgue measurable.*

Definition A.9 (Upper semi-continuous function) *A set-valued function $F(x)$ is* upper semi-continuous *in x if for $y \to x$ implies*

$$\sup_{a\in F(y)}\inf_{b\in F(x)}\|a-b\|\rightarrow 0 \tag{3.2}$$

Having understood the definition of set-valued/multi-valued function, let us give the signum notation here. For the present thesis, two different signum notations were used.

- Matlab and Wolfram Mathworld uses the following notation, where every mapping is one-one satisfying the definition of a function.

$$\text{sign}[x] = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \tag{3.3}$$

- This notation is commonly used to model differential inclusions, which involves set-valued functions

$$\text{Sign}[x] \in \begin{cases} \{-1\} & x < 0 \\ [-1, 1] & x = 0 \\ \{1\} & x > 0 \end{cases} \tag{3.4}$$

B. Ordinary differential equations (ODEs)

With the above definitions and notations, we are almost equipped with everything to understand differential inclusions and differential-algebraic equations. But both the concepts share a different relationship with ODEs. Differential inclusions are an extension to the concept of ordinary discontinuous differential equations where as the ODEs form a special class of DAEs. So, this is the right place to define ODEs and their properties in general.

Before jumping into the definitions of ordinary differential equations and their existence theorems, let us introduce some notation, which is the same as in Reference

[1] as follows:

- I : an open interval on the real line $-\infty < t < \infty$ (or) two real numbers a and b exist such that $a < t < b$
- $C^k(I)$: the set of all complex-valued functions having k-continuous derivatives on I
- $f \in C^k(I)$: If $\frac{d^k f}{dt^k}$ exists and continuous on I
- D : An open connected set (Domain), in the real (t, x) plane
- $C^k(D)$: the set of all complex-valued functions having k-continuous partial-derivatives on D
- $f \in C^k(D)$: If all the k^{th}-order partial derivatives $\frac{\partial^k f}{\partial t^p \partial x^q}$, where $(p + q = k)$ exist and are continuous on D
- $C(I)$ and $C(D)$: the continuous functions $C^0(I)$ and $C^0(D)$ on I and D

Let $f \in C(D)$ be a real-valued function continuous in the domain D, then the definition of the ordinary differential equation can be developed as follows.

Definition B.1 *[1] To find a differentiable function ϕ defined on a real t interval, I such that*

1. $(t, \phi(t)) \in D; \quad (t \in I)$
2. $\phi\prime(t) = f(t, \phi(t)); \quad (t \in I, \prime = \frac{d}{dt})$

This problem is called an ordinary differential equation of first order *and is denoted by*

$$x\prime = f(t, x) \quad (\prime = \frac{d}{dt}) \tag{3.5}$$

If at least one such interval I and function ϕ exists, then ϕ is termed as *solution of the ordinary differential equation* (3.5). It should be noted that there could be many or no solution existing for (3.5) too. There are several existence and uniqueness theorems for the solution of ordinary differential equations developed by numerous mathematicians. In this report, two most important and relevant ones of them are quoted. For the following theorems, the notations as described above will be used.

Theorem B.2 (Cauchy-Peano existence theorem) *If $f \in C$ on the rectangle R, then there exists a solution $\phi \in C^1$ of the ordinary differential equation* (3.5) *on $|t-\tau| \leq \alpha$ for which $\phi(\tau) = \xi$. where, R is a rectangle defined as*

$$R: \ |t-\tau| \leq \alpha \quad |x-\xi| \leq b \quad (a, b > 0) \tag{3.6}$$

Theorem B.3 (Carathéodory theorem) *Let f be defined on $\mathbb{R}$, and suppose it is measurable in t for each fixed x, continuous in x for each fixed t. If there exists a Lebesgue-integrable function m on the interval $|t-\tau| \leq a$ such that*

$$|f(t,x)| \leq m(t) \quad ((t,x) \in \mathbb{R}) \tag{3.7}$$

then there exists a solution ϕ of (3.5) *in the extended sense on some interval $|t-\tau| \leq \beta$, $(\beta > 0)$, satisfying $\phi(\tau) = \xi$.*

Mathematical proofs to the above two theorems can be found in Reference [1].

C. Differential inclusions (DIs)

The study of differential inclusions and its application started almost in the mid-thirties of 20th century. It has gained a mathematical form with the development of control theory. There were several mathematicians who worked on developing existence and uniqueness conditions for the solution of differential inclusions. Filippov along with Weżewski developed the most celebrated Filippov-Ważewski relaxation theorem under some weak assumptions [26]. Filippov [13] extended the concept of discontinuous differential equations to differential inclusions and developed existence and uniqueness of the solutions. He considered a differential inclusion of the form

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = x_0 \tag{3.8}$$

where $\mathbf{F}(\mathbf{x}, t)$ is a set-valued function as defined before. The following theorem states the condition for existence and uniqueness of the solution for such a differential inclusion.

Theorem C.1 (Filippov's theorem [13]) *Assuming that $\mathbf{F}(t, \mathbf{x})$ is an upper semi-continuous function of $\mathbf{x}$, measurable in t, and that $\mathbf{F}(t, \mathbf{x})$ is a closed, convex set for all t and x, existence of the solutions for the initial value problem* (3.8) *for a sufficiently small interval $[t_0, t_0 + \epsilon]$, $\epsilon > 0$ is then followed. If $\mathbf{F}$ does not "blow-up" then global existence can also be shown.*

And the uniqueness is proved with an additional condition that $\mathbf{F}(\mathbf{t}, \mathbf{x})$ satisfies a one-sided Lipschitz condition which states as

$$(x_1 - x_2)^T (\mathbf{F}(t, x_1) - \mathbf{F}(t, x_2)) \leq C \|x_1 - x_2\|^2 \tag{3.9}$$

for some C, $\forall x_1$ and x_2.

As mentioned earlier, it can be seen that solutions to differential inclusions in the sense of Filippov exist when modeled using multi-valued functions. Having said that, in further chapters when specific lumped parameter systems are considered, modeling using differential inclusions becomes clear.

D. Differential-algebraic equations (DAEs)

The system of equations of the form

$$\mathbf{K}(t, \mathbf{x}, \dot{\mathbf{x}}) = 0 \tag{3.10}$$

are called differential-algebraic equations(DAEs). If $\frac{\partial K}{\partial \mathbf{x}}$ is non-singular then the above equation can be written as an explicit ordinary differential equation(ODE) [18]. Thus a generalization can be made here that (ODEs) are a special class of DAEs. An explicit ODE is of the form, $\frac{d\mathbf{x}}{dt} - \mathbf{f}(t, \mathbf{x}) = 0$. This is a special case of a more general system of the form (3.10). The equations of the form (3.10) are called implicit ODEs and popularly DAEs.

Definition D.1 *The system of equations of the form*

$$\dot{\mathbf{x}}_1 = \mathbf{f}(t, \mathbf{x}_1, \mathbf{x}_2) \quad \mathbf{x}_1(t) \in \mathbb{R}^p, \mathbf{x}_2(t) \in \mathbb{R}^q \tag{3.11a}$$

$$\mathbf{0} = \mathbf{g}(t, \mathbf{x}_1, \mathbf{x}_2) \quad \mathbf{g} \in \mathbb{R}^q \tag{3.11b}$$

are called semi-explicit DAEs.

The DAEs given by (3.11a) and (3.11b) can be thought of as a limiting case of the singularly perturbed ODE.

$$\dot{\mathbf{x}}_1 = \mathbf{f}(t, \mathbf{x}_1, \mathbf{x}_2) \tag{3.12a}$$

$$\epsilon\dot{\mathbf{x}}_2 = \mathbf{g}(t, \mathbf{x}_1, \mathbf{x}_2) \tag{3.12b}$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{x}_2}$ is assumed to be non-singular. The limiting case where $\epsilon = 0$ of (3.12a) and (3.12a) is called the reduction problem as in (3.11a) and (3.11b).

Definition D.2 (Differential index of a DAE) Differential index *of a DAE is a non-negative number that denotes the complexity level of a differential-algebraic equa-*

tion to be numerically solved and analyzed. It can be taken as a measure of how close/far from an ordinary differential equation i.e., a smaller index indicates it is quite simpler compared to a higher-index DAE.

For example, if the algebraic equation (3.11b) is differentiated once with respect to t resulting in a set of implicit ODEs. Then the differential index of DAE is said to be one. Another notation for index of a DAE is perturbation index and details on this can be seen from Reference [19].

If the algebraic constraint (3.11b) is an inequality then the governing equations turn out to be a system of differential-algebraic inequalities (DAIs). A model problem that gives rise to DAIs is considered to give more insight. Research on similar lines can be found in Reference [27].

Equipped with these definitions and notations we can now go ahead and define the problem and the governing equations for various cases considered in Chapter II using differential-algebraic equations perspective.

CHAPTER IV

MATHEMATICAL MODELING

In this thesis, only lumped parameter systems are considered for analysis to keep the study simple. A lumped parameter system is a way to approximate the distributed system behavior under certain assumptions. This greatly simplifies the study of the system under consideration [25]. It is mainly useful in the fields of electronics, mechanics and control theory. Idealization of energy storage/dissipation is one of the general applications of lumped parameter systems.

A good example of a lumped parameter model is the representation of an electrical network, represented by a circuit diagram as shown in Figure 10. The parameter resistance is lumped and represented by idealized resistor instead of considering the Maxwell equations of actual system. Another good example is a thermal system where average temperature of the system is considered as a lumped parameter.

A. Problem definition and governing equations

The lumped parameter system considered for all problems in this thesis is a spring-mass-dash-pot system with a linear spring and varying characteristics of dash-pot as

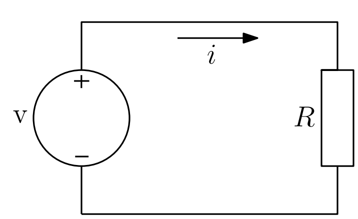


Fig. 10.: A lumped parameter model of an electronic circuit where i denotes current, R denotes resistance and v denotes voltage source.

discussed in Chapter I. For the rest of the thesis, let x denote the displacement and v denote the velocity of the system which are considered to be the kinematic variables of the system. Let f_s be the force due to spring and f_d be the force due to dash-pot on the mass m.

With these notations, let us first see how governing equations look like in differential inclusions approach in the most general form for the same system with linear spring and dash-pot with Bingham-type fluid. The momentum balance equation as a differential inclusion is written as

$$\dot{v}(t) \in \frac{1}{m}[F(t) - F_s(t) - F_d(t)] \tag{4.1}$$

The constitutive relations for the spring and dash-pot are:

$$\alpha(x, F_s) = 0 \tag{4.2a}$$

$$\beta(v, F_d) = 0 \tag{4.2b}$$

For the various examples considered in this thesis, the constitutive relations (4.2a) and (4.2b) are written as inclusions. Inclusions are witten using the signum notation $Sign[\cdot]$ as in Chapter III. Equations (4.1), (4.2a) and (4.2b) form the governing equations in differential inclusions approach. Depending on the characteristics of the dash-pot we wish to consider for analysis $\beta(v, F_d)$ is varied and the system dynamics is studied.

With that insight into the way in which the governing equations in differential inclusion approach would look like, let us now see the general form of governing equations of the same system in DAE approach. Momentum balance gives rise to a differential equation and written as

$$\dot{v}(t) = \frac{1}{m}[f(t) - f_s(t) - f_d(t)] \tag{4.3}$$

The constitutive relations for the energy storage and dissipation mechanisms are written as

$$\alpha(x, f_s) = 0 \tag{4.4a}$$

$$\beta(v, f_d) = 0 \tag{4.4b}$$

where $\alpha(x, f_s) = 0$ is $x(t) = \frac{f_s}{k}$ for linear spring. The relation (4.4b) would turn out to be either explicit or implicit algebraic equations which keeps varying specific to the problem considered from linear to non-linear monotonic and non-linear non-monotonic characteristics. Equations (4.3), (4.4a) and (4.4b) form the set of governing equations in DAE approach. Also, it is to be noted that upper-case notation is used for modeling using differential inclusion approach and lower-case for differential-algebraic equations approach for all problems considered. In order to understand the robustness of the numerical framework developed, we consider different models for analysis to accommodate non-linear constitutive relations which give rise to semi-explicit differential-algebraic equations, lumped parameter system with Coulomb friction which gives rise to implicit differential-algebraic equations, and system with constraint which gives rise to differential-algebraic inequalities. In the following sections, each problem is dealt separately.

B. Lumped parameter system with a Bingham-type dash-pot of non-linear monotonic characteristics

A lumped parameter system as shown in Figure 11 is considered. The spring considered is a linear spring and the dash-pot is considered such that velocity of the system is a function of force due to dash-pot. For the present case, the function is considered to be non-linear monotonic in nature. In Reference [21], a similar problem

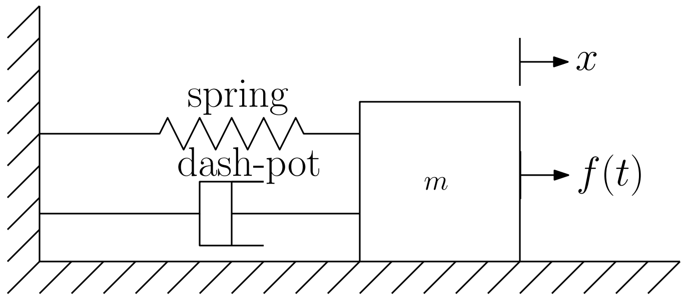


Fig. 11.: Spring-mass-dash-pot system representing a lumped parameter system subjected to an external force $f(t)$. Displacement of the system is given by $x(t)$.

was solved for a dash-pot with Bingham-type fluid where the function was considered to be linear in nature. The governing equations of the system in differential inclusions approach will be similar to the equations (4.1), (4.2a) and (4.2b) with the constitutive relations written as follows:

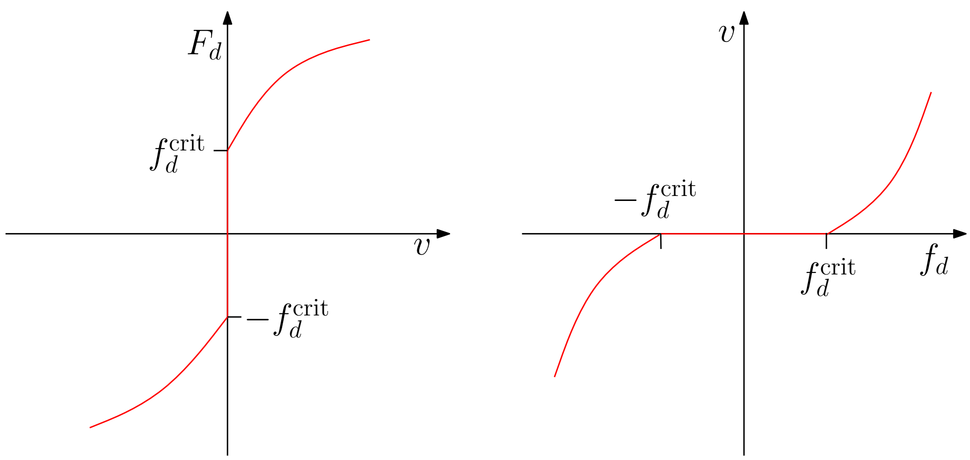
$$F_s(t) = kx(t) \tag{4.5a}$$

$$F_d(t) = \text{Sign}[v(t)]f_d^{\text{crit}} + \eta\beta(v(t)) \tag{4.5b}$$

f_d^{crit} is the critical force for Bingham-type fluids and $\beta(v)$ will be considered as a non-linear monotonic function which is cubic in nature for this case as shown in Figure 12(a). From this figure, it is evident that F_d is expressed in terms of v only as an inclusion and not as a standard function.

The governing equations for this system in differential-algebraic equations approach is as follows. Momentum balance is a differential equation and written as

$$\dot{v}(t) = \frac{1}{m}[f(t) - f_s(t) - f_d(t)] \tag{4.6}$$



(a) v vs. f_d

(b) F_d vs. v

Fig. 12.: A pictorial representation of constitutive relation between velocity and force due to dash-pot for a dash-pot governed by Bingham-type fluid of non-linear monotonic characteristics, a cubic function in this case. In Figure (a), the notation is lower-case for f_d to represent a standard function i.e., velocity can be written as a standard function of force due to dash-pot. In Figure (b), the notation of force due to dash-pot is written as F_d in upper case to represent a multi-valued function i.e., force due to dash-pot can be written in terms of velocity only by using multi-valued function.

and

$$x(t) = \frac{f_s}{k} \tag{4.7a}$$

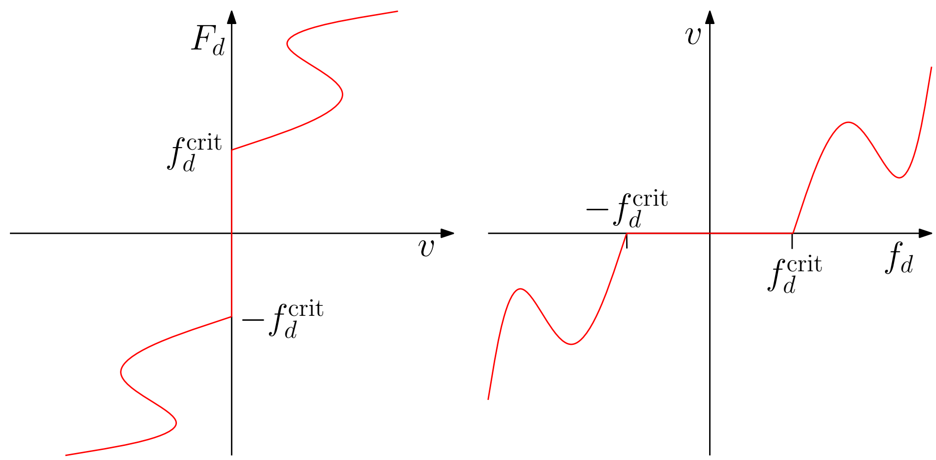
$$v(t) = \begin{cases} 0 & |f_d| \leq f_d^{\text{crit}} \\ \gamma(f_d)^2(f_d - \text{sign}[f_d] f_d^{\text{crit}}) & |f_d| > f_d^{\text{crit}} \end{cases} \tag{4.7b}$$

For the purpose of serving as an example of monotonic nature, the Bingham-type dash-pot is characterized by a function that is cubic in nature as in Figure 12(b). Equations (4.6), (4.7a) and (4.7b) form the governing equations in differential-algebraic equations approach.

C. Lumped parameter system with a Bingham-type dash-pot of non-linear non-monotonic characteristics

The second case considered is a similar lumped parameter system but the characteristics of the dash-pot are modified such that a non-linear non-monotonic function characterizes the dash-pot. This problem is considered for analysis because the idea is to develop a generalized framework to treat a wider class of Filippov systems. Some of the numerical techniques that are used to solve the non-linear monotonic functions fail for non-monotonic cases at limit points or bifurcation points. This leads to the challenge of developing such a generalized framework. Hence, the study of such a non-smooth system with a dash-pot of non-linear non-monotonic characteristics bears practical importance.

Let us consider the same lumped parameter system with dash-pot characteristics as shown in Figures 13(a) and 13(b).



(a) v vs. f_d (b) F_d vs. v

Fig. 13.: A pictorial representation of constitutive relation between velocity and force due to dash-pot for a dash-pot governed by Bingham-type fluid of non-linear non-monotonic characteristics, a polynomial function in this case. In Figure (a), the notation is lower-case for f_d to represent a standard function i.e., velocity can be written as a standard function of force due to dash-pot. In Figure (b), the notation of force due to dash-pot is written as F_d in upper case to represent a multi-valued function i.e., force due to dash-pot can be written in terms of velocity only by using multi-valued function.

To write the governing equations in differential inclusions approach, it is the same as in previous case with (4.1), (4.5a) and (4.5b) but with $\beta(v)$ as a non-linear non-monotonic function which is shown in Figure 13(a). The governing equations of this system in differential-algebraic equations approach, is written as follows. The momentum balance equation is the same differential equation

$$\dot{v}(t) = \frac{1}{m}[f(t) - f_s(t) - f_d(t)] \tag{4.8}$$

The constitutive relations are as follows:

$$x = \frac{f_s}{k} \tag{4.9a}$$

$$v = \begin{cases} 0 & |f_d| \leq f_d^{\text{crit}} \\ (1 + 0.1(f_d^2 - 4)(f_d^2 - 9))(f_d - \text{sign}[f_d]f_d^{\text{crit}}) & |f_d| > f_d^{\text{crit}} \end{cases} \tag{4.9b}$$

Equations (4.8), (4.9a), and (4.9b) represent the governing equations of the system in the differential-algebraic equations approach. The relation (4.9b) is only for the purpose of representating a non-monotonic function.

D. Lumped parameter system with Coulomb friction

Models with Coulomb friction are of great interest among Filippov systems as discussed in Chapter I. For the present case, the system considered is shown in Figure 14 involving Coulomb friction. Mechanical systems with Coulomb friction is a classical example of Filippov systems. One important thing to be noted here is that force f_d is due to dry friction unlike fluid friction as in previous cases. Hence, the governing equations needs modification which would affect the system dynamics. The system with such friction is pictorially represented in Figure 14.

Coulomb model is graphically represented by the Figures 15(a) and 15(b).

From Figures 15(a) and 15(b) it is seen that either way it is not possible to express one quantity in terms of the other but one can write implicit relations. Keeping the notations for displacement, velocity and the forces in the system the same the governing equations for the system in differential inclusions approach are written as

$$\dot{v}(t) \in \frac{1}{m}[f(t) - F_s(t) - F_d(t)] \tag{4.10}$$

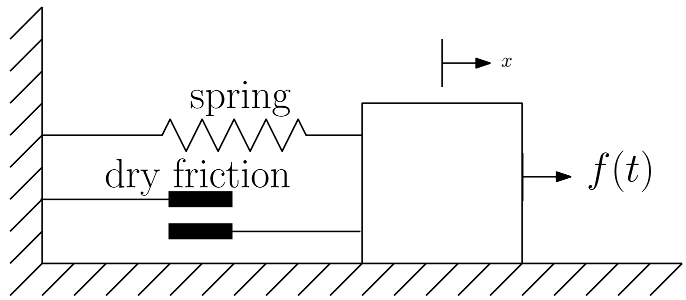
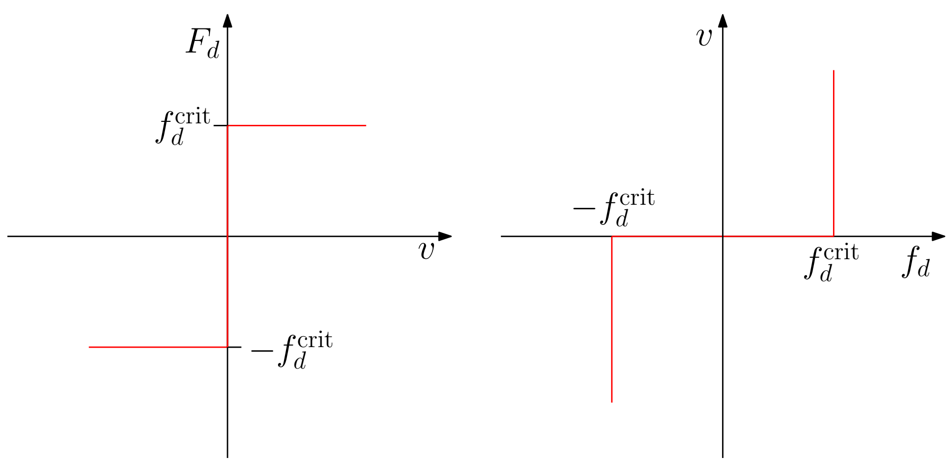


Fig. 14.: A spring-mass-system involving friction governed by Coulomb friction model in this case. Here $f(t)$ denotes the external force x denotes the displacement of the system.



(a) v vs. f_d (b) f_d vs. v

Fig. 15.: A pictorial representation of constitutive relation between velocity and frictional force for a system with friction governed by Coulomb friction model. It should be noted that neither kinematic quantities can be expressed as a function of dynamic quantities nor dynamic quantities as a function of kinematic quantities.

The constitutive relation for the linear spring and Coulomb friction are written as

$$F_s(t) = kx(t) \tag{4.11a}$$

$$F_d(t) = \text{Sign}[v(t)]f_{\text{d}}^{\text{crit}} \tag{4.11b}$$

Equations (4.10), (4.11a) and (4.11b) form the system of governing equations in differential inclusion approach. In DAEs approach the governing equations are written as follows. The momentum balance equation is written as

$$\dot{v}(t) = \frac{1}{m}[f(t) - f_s(t) - f_d(t)] \tag{4.12}$$

The constitutive equations for the linear spring and Coulomb friction for DAE approach is written as

$$x(t) = f_s(t)/k \tag{4.13a}$$

$$\begin{aligned} -f_d^{\text{crit}} \leq f_d \leq f_d^{\text{crit}} & \\ if|f_d| < f_d^{\text{crit}} \qquad & v = 0 \\ if|f_d| = f_d^{\text{crit}} \qquad & \text{Sign}[f_d]v \geq 0 \end{aligned} \tag{4.13b}$$

Equations (4.12), (4.13a) and (4.13b) form the set of governing equations for the lumped parameter system with Coulomb friction in DAE approach. It is interesting to note here that the constitutive relations for Coulomb friction are implicit relations unlike the previous two examples. The essence of this problem is to write a numerical solver for such DAEs which might include implicit relations as in this case.

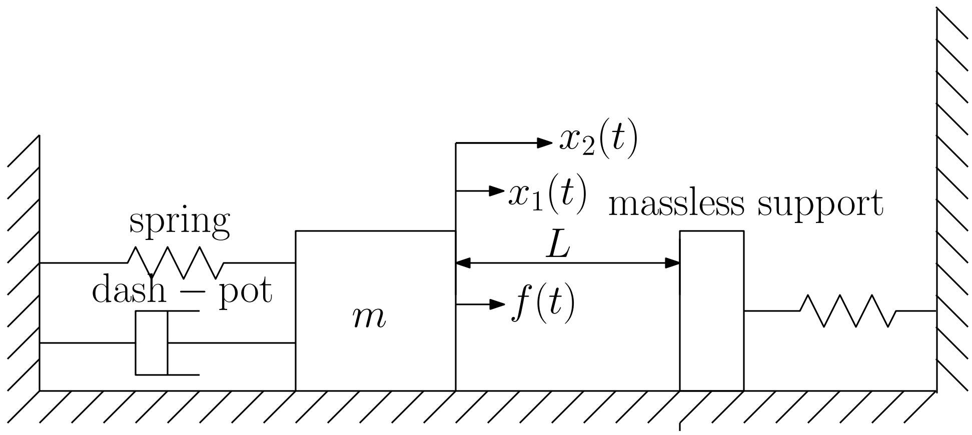
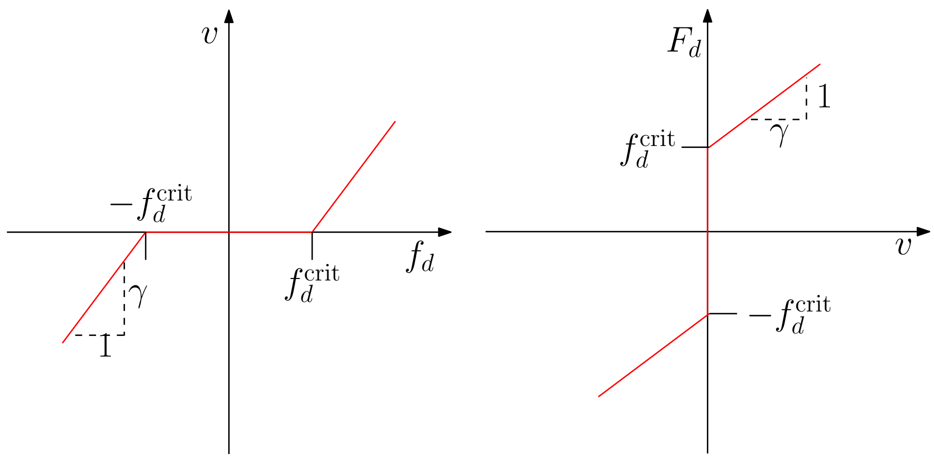


Fig. 16.: A spring-mass-dash-pot system with a massless support placed at a distance of L units such that there is a constraint on the displacement of the system. Here $f(t)$ denotes the external force, x_1 denotes the displacement of mass m and x_2 denotes the displacement of massless support measured from the reference point as shown in the figure.

E. Lumped parameter system with a Bingham-type dash-pot of linear characteristics and a constraint

In this section a lumped parameter system similar to the previous problems is chosen with a Bingham-type dash-pot that is characterized by a linear function. A constraint is placed on the the displacement of the system by placing a massless support at a distance L units. This support is connected to a spring of spring constant k_{support} as shown in Figure 16. and the Bingham-type dash-pot characterized by linear function is as shown in Figure 17. Instead of having a restriction on displacement by placing a rigid wall, we have used a massless support in this case. This is because using a rigid wall as a constraint makes the dynamics more complicated since characteristics of the rigid wall play a very important role. Hence, this modification is made to the system using a massless support connected to rigid wall by a spring. However, the



(a) v vs. f_d (b) F_d vs. v

Fig. 17.: Constitutive relation between velocity and force due to dash-pot of a Bingham-type dash-pot with linear characteristics. In Figure (a), the notation is lower-case for f_d to represent a standard function i.e., velocity can be written as a standard function of force due to dash-pot. In figure (b), the notation of force due to dash-pot is written as F_d in upper case to represent a multi-valued function i.e., force due to dash-pot can be written in terms of velocity only by using multi-valued function.

spring constant of this spring can be set to very high numbers such that the massless support can represent a rigid wall.

To model this system in differential inclusions approach the governing equations are same as in (4.1), (4.5a) and (4.5b) with $\beta(v)$ being linear in nature. This particular dynamical system is considered for analysis to study the mathematical model that gives rise to a set of differential-algebraic inequalities. The inequalities arise due to the constraint placed on displacement x_1. Let us now see the governing equations in both differential inclusion approach and differential-algebraic equations approach. In differential inclusions approach, the momentum balance will be an inclusion and has a new parameter λ added to represent the extra force parameter that takes effect upon contact.

$$\dot{v}(t) \in \frac{1}{m}[f(t) - F_s(t) - F_d(t)] + \lambda(t) \tag{4.14}$$

The constitutive relations for linear spring and Bingham-type dash-pot characterized with linear function are written as

$$F_s(t) = kx(t) \tag{4.15a}$$

$$F_d(t) = \text{Sign}[v(t)]F_d^{\text{crit}} + \frac{v(t)}{\gamma} \tag{4.15b}$$

Pictorially the Bingham-type dash-pot with linear characteristics is shown in Figure 17. When constraints are placed as above, we get additional set of inequalities that govern the behaviour of the system. These are listed below. The difference between displacements of two springs is

$$x_2 - x_1 \geq 0 \tag{4.16}$$

The condition on $\lambda(t)$ is given by

$$\lambda(t)(x_2 - x_1) = 0 \tag{4.17}$$

where,

$$\lambda(t) = \begin{cases} 0 & x_2 - x_1 > 0 \\ -k_{\text{support}}(x_2 - L) & x_2 - x_1 \leq 0 \end{cases} \tag{4.18}$$

Equations (4.16), (4.17) and (4.18) are similar to Karush-Kuhn-Tucker conditions and show that the parameter λ will be equal to zero when there is no contact and is in effect other-wise. λ is related to k_{support} in the above described manner. To write governing equations in differential-algebraic equations perspective, the momentum balance and constitutive relation for the spring remain the same as in equations (4.12) and (4.13a). The constitutive relation the Bingham-type dash-pot characterized by linear function is written as

$$v(t) = \begin{cases} 0 & |f_d(t)| \leq f_d^{\text{crit}} \\ \gamma(f_d(t) - \text{sign}[f_d(t)]f_d^{\text{crit}}) & |f_d(t)| > f_d^{\text{crit}} \end{cases} \tag{4.19}$$

With this example, it can be clearly seen how a constraint gives rise to differential-algebraic inequalities. The numerical frame work will be the same for this problem too but the algorithm has to be changed in that a contact resolution algorithm should be implemented at the point of contact of mass with massless support. This problem is of practical importance in the sense that if the spring constant k_{support} is increased to higher values we can treat it as a unilateral constraint like a rigid wall restricting the movement of the mass m.

CHAPTER V

NUMERICAL ALGORITHMS

For Filippov systems governed by differential-algebraic equations as in previous chapter, finding analytical solutions is not straight forward. In such situations we have to resort to numerical solutions. However, the challenge is choosing an appropriate technique. For many problems involving differential-algebraic equations the backward difference formulae (BDF) are very stable and accurate [28, 29, 30]. A theorem with proof from Reference [18] shows the usefulness of BDF for problems involving DAEs. The theorem is stated as follows: *A k-step BDF with constant step size, applied to the constant coefficient DAE with index v, is convergent with order k after (v-1)k+1 steps.* It is assumed that the DAEs obtained from governing equations are solvable [31]. Backward Euler time stepping scheme was chosen as the appropriate numerical method considering this assumption and the above theorem. This is the simplest of BDF available and numerical framework is developed. In literature others have also developed such solvers for system of equations that can be written as DAEs [29, 32]. Stability proofs for index-2 DAEs are given in Reference [33].

The backward Euler scheme is an implicit method and used as follows to discretize the first derivative of a quantity at n^{th} time step having known its value at $(n+1)^{\text{th}}$ time step.

$$y(t+dt) = y(t) + dt\ \dot{y}(t+dt) \tag{5.1}$$

where a super posed dot "·" denotes a derivative with respect to time and dt is the time step considered. The following notation is adopted for velocity and acceleration of the lumped parameter system at n^{th} time step where acceleration is first order time derivative of velocity.

$$v^{(n)} = v(t = t_n) \tag{5.2a}$$

$$a^{(n)} = \dot{v}(t = t_n) \tag{5.2b}$$

Therefore, using backward Euler scheme the acceleration at $(n+1)^{\text{th}}$ time step is written as,

$$a^{(n+1)} = \frac{v^{(n+1)} - v^{(n)}}{\Delta t} \tag{5.3}$$

For all the model problems considered for analysis a predictor-corrector algorithm is used. For the problem with constraint, in addition to predictor-corrector algorithm a contact-resolution algorithm is also used. Both of these are treated in detail in this chapter.

Before detailing these algorithms, the numerical framework with a set of algebraic equations that are arrived at upon discretization of the governing equations in DAE approach is developed. By applying discretized form of acceleration as in (5.3) to the momentum balance equation (4.6) we obtain for $(n+1)^{\text{th}}$ time step,

$$\frac{v^{(n+1)} - v^{(n)}}{\Delta t} = \frac{1}{m}[f^{(n+1)} - {f_s}^{(n+1)} - {f_d}^{(n+1)}] \tag{5.4}$$

which implies

$$v^{(n+1)} = v^{(n)} + \frac{\Delta t}{m}[f^{(n+1)} - {f_s}^{(n+1)} - {f_d}^{(n+1)}] \tag{5.5}$$

Now, using the backward Euler scheme to the time derivative of force due to the spring f_s at $(n+1)^{\text{th}}$ time step,

$$\dot{f}_s(t = t_{n+1}) = \frac{f_s^{(n+1)} - f_s^{(n)}}{\Delta t} \tag{5.6}$$

and substituting (5.6) into the constitutive relation of linear spring (4.4a),

$$f_s^{(n+1)} = k\Delta t v^{(n+1)} + f_s^{(n)} \tag{5.7}$$

Now, using (5.6) in (5.5)

$$v^{(n+1)} = v^{(n)} + \frac{\Delta t}{m}[f^{(n+1)} - k\Delta t v^{(n+1)} - f_s^{(n)} - f_d^{(n+1)}] \tag{5.8}$$

which implies,

$$v^{(n+1)} = \frac{\Delta t}{m(1+\frac{k\Delta t^2}{m})}[(\frac{m}{\Delta t}v^{(n)} + f^{(n+1)} - f_s^{(n)}) - f_d^{(n+1)}] \tag{5.9}$$

Equation (5.9) is a straight line if plotted at $(n+1)^{\text{th}}$ time step against velocity $v^{(n+1)}$ versus force due to dash-pot $f_d^{(n+1)}$ at all the time steps. Having constructed the framework to this point, let us now apply it to each problem separately and device an algorithm for each of the three genres of problems discussed in previous chapters.

Solution obtained by solving (5.9) and discretized form of corresponding constitutive relation of the dash-pot gives the values of force due to dash-pot f_d and velocity v at $(n+1)^{\text{th}}$ time step once the values of displacement x, velocity v, force due to the spring f_s and force due to the dash-pot f_d are known at the n^{th} time-step. The initial conditions that needs to be considered for the system with Coulomb friction are slightly different from the other cases.

A. Algorithm for the system with Bingham-type dash-pot of non-linear monotonic characteristics

The essence of this problem lies in using the appropriate numerical method to solve the non-linearity of the dash-pot characteristics that are chosen since the analytical solutions are not readily available like in the case of Bingham-type dash-pot with linear characteristics in Reference [21].

Among many, Newton-Raphson method has proved to be one of the most powerful and potential numerical techniques owing to the precision in finding the roots of non-linear system of equations like the ones we have in the present problem. Initial conditions for velocity $v^{(0)}$ and force due to dash-pot $f_d^{(0)}$ having known, the iterative Newton-Raphson method is used and the results at the end of last time-step are obtained. Results include displacement x, velocity v, force due to spring f_s and force due to dash-pot f_d.

By taking the advantage of in-built Matlab commands like "fsolve", these set of non-linear equations can be solved to required precision. A predictor-corrector algorithm is used and the sequence of steps are listed below which gives a complete insight into the algorithm used while coding in Matlab.

Though the governing equations are formulated earlier for each of the problems separately, just for the sake of compactness the final form of equations upon discretization is listed here to get more insight into the algorithm.

$$v^{(n+1)} = \frac{\Delta t}{m(1+\frac{k\Delta t^2}{m})}[(\frac{m}{\Delta t}v^{(n)} + f^{(n+1)} - f_s^{(n)}) - f_d^{(n+1)}] \tag{5.10a}$$

$$v^{(n+1)} = \begin{cases} 0 & |f_d^{(n+1)}| \leq f_d^{\text{crit}} \\ \gamma(f_d^{(n+1)})^2(f_d^{(n+1)} - \text{sign}[f_d^{(n+1)}]f_d^{\text{crit}}) & |f_d^{(n+1)}| > f_d^{\text{crit}} \end{cases} \tag{5.10b}$$

The whole point is to device an algorithm to solve these equations for f_d and v at $(n+1)^{\text{th}}$ time step having known the quantities at n^{th} time step. The step by step process of the algorithm is discussed here.

1. Choice of initial conditions: Different external forcing conditions can be considered for the lumped parameter system for representation purposes. The initial values of displacement $x^{(0)}$ force due to dash-pot $f_d^{(0)}$ are considered from which

the velocity and force due to spring, $v^{(0)}$ and $f_s^{(0)}$ are obtained.

2. Finding the predictor: The predictor $\tilde{f}^{(n)}$ is calculated at first time step, $n = 1$ using the external forcing function at first time step and initial conditions as

$$\tilde{f}^{(n+1)} = \frac{m}{\Delta t} v^{(n)} + f^{(n+1)} - f_s^{(n)} \tag{5.11}$$

3. Corrector step: The predictor value is checked against the critical force for dash-pot f_d^{crit} at previous time step and necessary steps are taken as follows

   - If $\tilde{f}^{(n+1)} < f_d^{\text{crit}}$, then the solution is trivial where

$$v^{(n+1)} = 0 \tag{5.12a}$$

$$f_d^{(n+1)} = \tilde{f}^{(n+1)} \tag{5.12b}$$

   which implies,

$$f_s^{(n+1)} = f_s^{(n)} + \Delta t k v^{(n+1)} \tag{5.13a}$$

$$x^{(n+1)} = \frac{f_s^{(n+1)}}{k} \tag{5.13b}$$

   - If $\tilde{f}^{(n+1)} \geq f_d^{\text{crit}}$, then we need to employ the iterative Newton-Raphson method.

4. Newton-Raphson method from t_n to t_{n+1}: To calculate the value of $f_d^{(n+1)}$ having known the value of $f_d^{(n)}$ using Newton-Raphson formula

$$f_d^{(n+1)} = f_d^{(n)} - \frac{f(f_d^{(n)})}{f'(f_d^{(n)})} \tag{5.14}$$

where, $f(f_d^{(n+1)})$ is obtained by solving (5.10a) and (5.10b) for the condition,

$f_d(t)| > f_d^{\rm crit}$. The function $f(f_d^{(n+1)}$ is

$$f(f_d^{(n+1)}) = \gamma f_d^{2(n)}(f_d^{(n)} - \text{sign}[f_d^{(n)}]f_d^{\rm crit}) - \frac{\Delta t}{m(1+\frac{k\Delta t^2}{m})}(\tilde{f}^{(n+1)} - f_d^{(n+1)}) \tag{5.15}$$

This step is carried out iteratively until $|\frac{f_d^{(n+1)} - f_d^{(n)}}{f_d^{(n+1)}}| \leq$ tolerance.

B. Algorithm for the system with Bingham-type dash-pot of non-linear non-monotonic characteristics

In this case, the dash-pot characteristics are non-linear non-monotonic in nature. Equation (5.10a) is obtained upon discretization of the governing equations as shown above. The discretized form of the constitutive relation of dash-pot is as follows.

$$v^{(n+1)} = \begin{cases} 0 & |f_d^{(n+1)}| \leq f_{\rm d}^{\rm crit} \\ (1 + 0.1(f_d^{2(n+1)} - 4)(f_d^{2(n+1)} - 9)) & \\ (f_d^{(n+1)} - \text{sign}[f_d^{(n+1)}]f_{\rm d}^{\rm crit}) & |f_d^{(n+1)}| > f_{\rm d}^{\rm crit} \end{cases} \tag{5.16}$$

So, the complication here arises where equation (5.10a) which is a straight line intersects with equation (5.16) which is non-linear and non-monotonic in nature. A unique point of intersection need not exist always as shown in Figure 18 at some time instant. Newton-Raphson method gives the solution as one of the point of intersection randomly or even fails to converge at limit points and bifurcation points. This calls for the need resort to some other techniques such that correct results are obtained. In this thesis, two approaches are implemented to select the correct point of intersection when a unique point of intersection does not exist.

1. Approach to see that there are no sudden jumps in the velocity of the system.

2. Approach such that the rate of dissipation of the dash-pot is maximized which

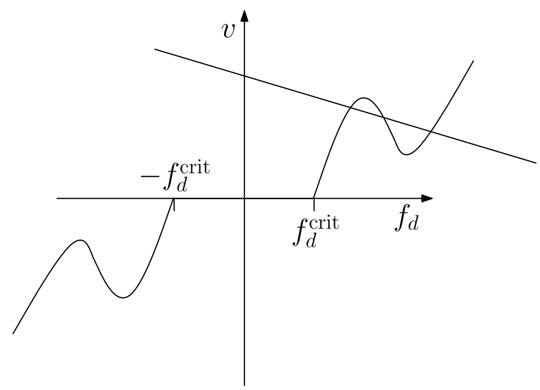


Fig. 18.: Pictorial representation of the intersection of the straight line with constitutive relation of the dash-pot at a particular instant of time.

satisfies the second law of thermodynamics.

For both the approaches, a similar predictor-corrector algorithm is used and the way it would be implemented remains the same for all the time instants where the line (5.10a) intersects with (5.16) at a unique point. Once the situation as in Figure 18 arises the implementation will need a change i.e., either of the above approaches should be used. The step by step algorithm incorporating this is given below.

1. Choice of initial conditions: Different external forcing conditions can be considered for the lumped parameter system for representation purposes. The initial values of displacement $x^{(0)}$ force due to dash-pot $f_d^{(0)}$ are considered from which the velocity and force in the spring, $v^{(0)}$ and $f_s^{(0)}$ are obtained.

2. Finding the predictor: The predictor is again $\tilde{f}^{(n)}$ is calculated for the first time step, $n = 1$ using the external forcing function at first time step and initial conditions as,

$$\tilde{f}^{(n+1)} = \frac{m}{\Delta t} v^{(n)} + f^{(n+1)} - f_s^{(n)} \tag{5.17}$$

3. Corrector step: The predictor value is checked against the critical force for dash-pot f_d^{crit} at previous time step and necessary steps are taken as follows

   - If $\tilde{f}^{(n+1)} < f_d^{\text{crit}}$, then solution is trivial where,

     $$v^{(n+1)} = 0 \tag{5.18a}$$

     $$f_d^{(n+1)} = \tilde{f}^{(n+1)} \tag{5.18b}$$

     which implies that other quantities are obtained as,

     $$f_s^{(n+1)} = f_s^{(n)} + \Delta t k v^{(n+1)} \tag{5.19a}$$

     $$x^{(n+1)} = \frac{f_s^{(n+1)}}{k} \tag{5.19b}$$

   - If $\tilde{f}^{(n+1)} \geq f_d^{\text{crit}}$, then we need to employ one of the approaches dealt above. Firstly, all the possible roots of intersection are calculated. If the intersection happens at a unique point then the point of intersection gives the values of v and f_d at $(n+1)^{\text{th}}$ time step. Otherwise, two approaches are considered as follows:

     – Approach 1:

       The Euclidean distance between each of the velocities at different points of intersection and the velocity at the previous time step is calculated. The velocity for $(n+1)^{\text{th}}$ with least distance is considered.

     – Approach 2:

       Each point of intersection gives v and f_d. That point which maximizes the product $f_d \cdot v$ is chosen as the correct value for $(n+1)^{\text{th}}$ time step.

4. From this value of velocity, other quantities can be obtained at $(n+1)^{\text{th}}$ time step.

In the next chapter where a set of representative examples are considered both the approaches were implemented and obtained interesting results. To obtain all the possible roots, the in-built Matlab command "roots" was used.

C. Algorithm for the system with Coulomb friction

As said earlier the difference of this problem with the previous two cases is that we obtained a set of implicit equations and an effective algorithm has to be devised to address this. The following sequence of steps shows how this particular problem involving Coulomb friction is dealt with.

1. Choice of initial conditions: There will be a slight modification in choosing the initial conditions for this case. Along with $x^{(0)}$, $f_d^{(0)}$ we also choose $v^{(0)}$. Initial value of $f_d^{(0)}$ is taken as $\frac{f_d^{\text{crit}}}{2}$ for all the cases in this thesis. The initial value of force due to $f_s^{(0)}$ is obtained from initial conditions of displacement.

2. Finding the predictor: The predictor $\tilde{f}^{(n)}$ is calculated at first time step, $n = 1$ using the external forcing function at first time step and initial conditions as,

$$\tilde{f}^{(n+1)} = \frac{m}{\Delta t} v^{(n)} + f^{(n+1)} - f_s^{(n)} \tag{5.20}$$

3. Corrector step: The predictor value is checked against critical force for dash-pot f_d^{crit} at previous time step and necessary steps are taken as follows:

   - If $\tilde{f}^{(n+1)} < f_d^{\text{crit}}$, then the solution is trivial where,

$$v^{(n+1)} = 0 \tag{5.21a}$$

$$f_d^{(n+1)} = \tilde{f}^{(n+1)} \tag{5.21b}$$

$$f_s^{(n+1)} = f_s^{(n)} + \Delta t k v^{(n+1)} \tag{5.22a}$$

$$x^{(n+1)} = \frac{f_s^{(n+1)}}{k} \tag{5.22b}$$

- If $\tilde{f}^{(n+1)} \geq f_d^{\text{crit}}$,

$$f_d^{(n+1)} = f_d^{\text{crit}} \tag{5.23}$$

and the velocity at next time step is written as

$$\mathbf{v}^{(\mathbf{n+1})} = \frac{1}{(1 + \Delta t^2 \frac{k}{m})} \frac{\Delta t}{m} (\tilde{f}^{(n+1)} - f_d^{(n+1)}) \tag{5.24}$$

From here, the displacement and force due to spring at the next time step are obtained as,

$$x^{(n+1)} = x^{(n)} + \Delta t v^{(n+1)} \tag{5.25a}$$

$$f_s^{(n+1)} = k x^{(n+1)} \tag{5.25b}$$

D. Algorithm for the system with Bingham-type dash-pot of linear characteristics and a constraint

Algorithm remains the same as that of systems with monotonic or non-monotonic functions for dash-pot until the point of contact i.e., until $x = L$ which is predictor-corrector algorithm. At the point of contact we need to implement the contact-resolution algorithm and the details are discussed below step by step.

1. Choice of initial conditions: Different external forcing conditions can be considered for the lumped parameter system for representation purposes. The initial values of displacement $x^{(0)}$ force due to dash-pot $f_d^{(0)}$ are considered from which

the velocity and force due to, $v^{(0)}$ and $f_s^{(0)}$ are obtained.

2. Contact check: Four different cases arises during contact check.

   (a) Case 1: $x_n < L$ "AND" there is no contact in the previous time step

   - Proceed with predictor-corrector algorithm as elaborated in previous cases

   (b) Case 2: $x_n < L$ "AND" there is contact in the previous time step

   - Time instant at which the contact gets released is calculated
   - Value of Δt is changed according to above calculated time instant
   - Proceed to contact-resolution algorithm. Contact-resolution algorithm are discussed in detail in this chapter.

   (c) Case 3: $x_n \geq L$ "AND" there is no contact in the previous time step

   - Time instant where impact occurs is calculated
   - Value of Δt is changed accordingly
   - Proceed to contact-resolution algorithm

   (d) Case 4: $x_n \geq L$ "AND" there is contact in the previous time step

   - The value of Δt is retained
   - Proceed to contact-resolution algorithm

3. The contact-resolution algorithm is separately dealt here where the predictor and corrector are modified as enumerated below.

   (a) Predictor Step:

   $\tilde{f}^{(n+1)} = \frac{m}{\Delta t} v^{(n)} + f^{(n+1)} - kx^{(n)}$

   $f^{(n+1)} = f^{(n+1)} + k_{\text{support}} L$

   $k = k_{\text{spring}} + k_{\text{support}}$

(b) Corrector Step

- If $|\tilde{f}^{(n+1)}| \leq f_d^{\text{crit}}$

  $v^{(n+1)} = 0$

  $f_d^{(n+1)} = \tilde{f}^{(n+1)}$

- Otherwise

  $f_d^{(n+1)} = f_d^{(n)} - \frac{f(f_d^{(n)})}{f'(f_d^{(n)})}$

  (Newton-Raphson formula)

(c) Upon calculation of $f_d^{(n+1)}$, $v^{(n+1)}$ is calculated using the constitutive relation of the dash-pot

(d) The force due to spring, $f_s^{(n+1)}$ and $x^{(n+1)}$ are calculated using

$$f_s^{(n+1)} = f_s^{(n)} + \Delta t k v^{(n+1)} \tag{5.26a}$$

$$x^{(n+1)} = \frac{f_s^{(n+1)}}{k} \tag{5.26b}$$

(e) In-built Matlab commands like "fsolve" help in obtaining the solution using the iterative Newton-Raphson method.

For the purpose of representation, the value of k_{support} is increased gradually to represent a rigid wall and the dynamics is studied. Representative numerical examples are considered and results are analyzed in the next chapter

CHAPTER VI

REPRESENTATIVE NUMERICAL EXAMPLES

In this chapter few numerical examples are considered to verify the correctness of the algorithms devised in the previous chapter. These examples are just for the purpose of representation and hence similar cases that are considered in [21] are considered. Two different scenarios one where in the external forcing function is a non-zero quantity but with initial displacement being zero. Another scenario is one in which the external force is kept zero and initial displacement is given. For all the numerical examples, it is assumed that $\gamma = 1$, $f_d^{\text{crit}} = 1$, $m = 1$, $k = 100$ and $\Delta t = 10^{-3}$ and each problem is detailed below.

- Case 1: Non-zero external forcing function

  For this case it is considered that the initial displacement, $x^{(0)} = 0$ and the initial force due to dash-pot, $f_d^{(0)} = 0$. Considering these conditions it is implied that the initial conditions for velocity and the force due to linear spring are $v^{(0)} = 0$ and $f_s^{(0)} = 0$.

  Two external forcing conditions considered are as follows:

$$f_1(t) = \left\{ \begin{array}{cc} 10\sin(5\pi t) & 0 \leq t \leq 1 \\ 0 & t > 1 \end{array} \right\} \tag{6.1a}$$

$$f_2(t) = \left\{ \begin{array}{cc} 0.5\sin(5\pi t) & 0 \leq t \leq 1 \\ 0 & t > 1 \end{array} \right\} \tag{6.1b}$$

  The above forcing functions are chosen such that the external force $f(t)$ is either greater than or less than the critical force f_d^{crit}.

$$\max[f_1] > f_d^{\text{crit}} \tag{6.2a}$$

$$\max[f_2] < f_d^{\text{crit}} \tag{6.2b}$$

- Case 2: Non-zero initial displacement

  Here the external force, $f(t) = 0$ and consider

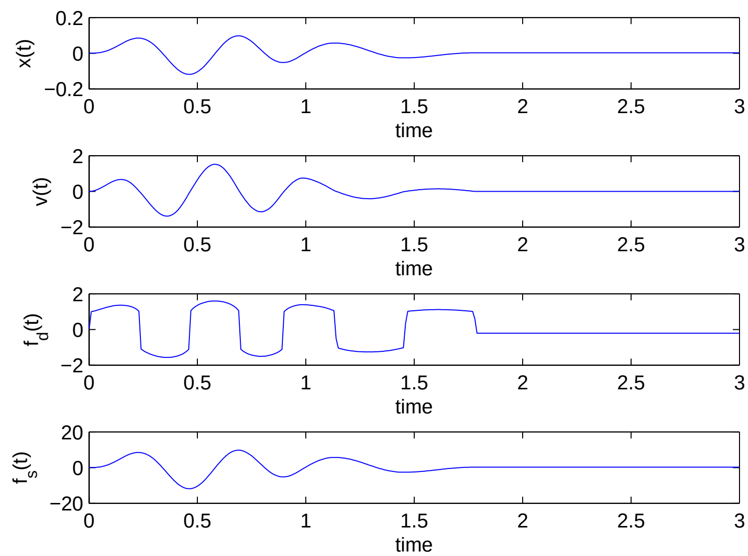
  $f_d^{(0)} = 0$, which implies that the initial condition for velocity becomes, $v^{(0)} = 0$. Two cases are considered with initial displacements $x^{(0)}(t) = 0.005$ and $x^{(0)}(t) = 0.5$ which implies the force due to spring are $f_s^{(0)} = 0.5$ and $f_s^{(0)} = 50$, respectively.

Having described the representative problems, the results for all the model problems considered are presented below. All the results are generated by implementing the algorithms in Matlab.

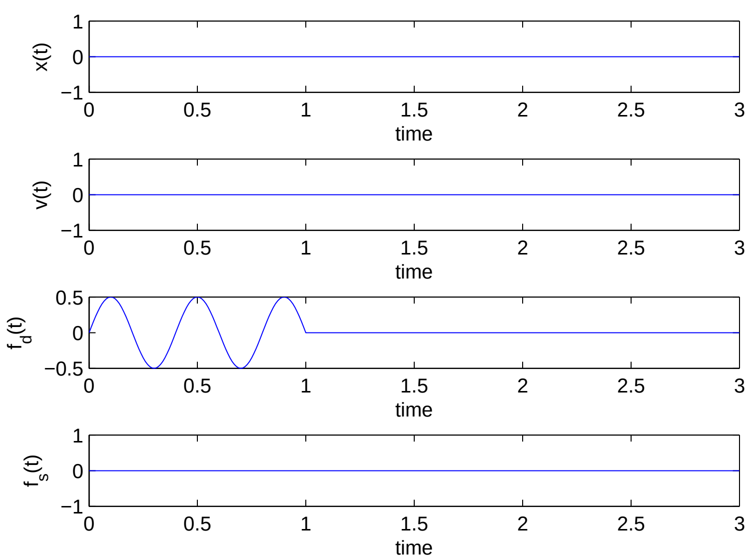
A. Numerical results for the system with Bingham-type dash-pot of non-linear monotonic characteristics

Incorporating the above mentioned initial value problems for a Bingham-type dash-pot with non-linear monotonic characteristics, the algorithm described in the previous chapter is implemented. Figures 19(a) and 19(b) represent the dynamics of the system with dash-pot characterized by non-linear monotonic function which is cubic in nature.

From the results obtained, it can be inferred that when the applied external force was greater than the critical force for dash-pot as in (6.2a), the non-smooth dynamics experienced by the system can be understood by observing the jumps in the f_d vs. time curve in Figure 19(a). But when the applied external force is less than critical



(a) $\max[f] > f_d^{\text{crit}}$



(b) $\max[f] < f_d^{\text{crit}}$

Fig. 19.: Dynamics of a spring-mass-dash-pot system with dash-pot of non-linear monotonic characteristics. The initial value problem is a non-zero external forcing condition with zero initial displacement and initial force due to dash-pot.

force for dash-pot as in (6.2b), then the applied force is not sufficient to provide non-zero velocities. It can be observed from the f_d vs. time curve in Figure 19(b) that though the velocities are zero, there is a non-zero force due to the dash-pot.

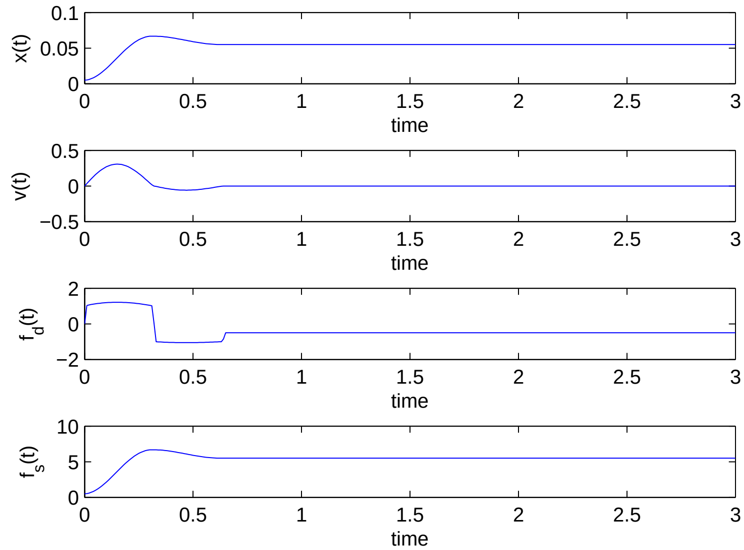
For the second case of initial conditions where in a non-zero initial displacement is considered, the results are as plotted in Figures 20(a) and 20(b). In the first case where displacement is 0.5, the force due to spring is more causing the velocity to grow beyond zero and the non-smooth dynamics is observed from f_d vs. time curve in Figure 20(a). In the second case the initial displacement is 0.005 and the velocity is zero at all time instants. This is the interesting phenomenon to be observed that is a characteristic of Filippov-type systems.

B. Numerical results for the system with Bingham-type dash-pot of non-linear non-monotonic characteristics

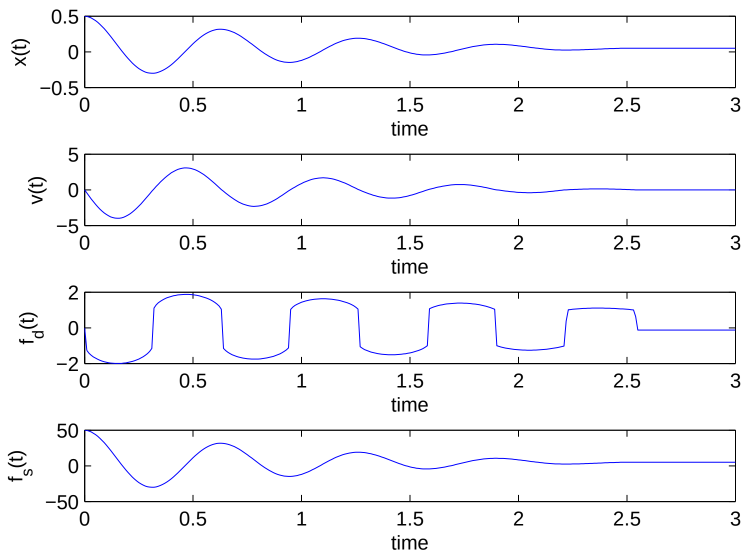
In this chapter, the results for representative initial value problems considered for a system with dash-pot governed by non-linear non-monotonic function described by (4.9b). As discussed before, we can use two approaches here.

- Results for approach 1 where in the velocity jumps are avoided:
  In this approach, the way in which velocities are obtained when the force due to dash-pot exceeds beyond the critical force is different from the second approach. In the first approach according to the algorithm from previous chapter, the velocities are chosen such that there is no jump in velocity profile. The way in which the system velocities change is shown in Figure 21. The results obtained for the initial value problems chosen are shown in Figures 22 and 23

- Results for approach 2 where rate of dissipation is maximized:
  In this approach, the velocities are obtained such that the product $f_d \cdot v$ is



(a) $x^{(0)} = 0.005$



(b) $x^{(0)} = 0.5$

Fig. 20.: Dynamics of a spring-mass-dash-pot system with dash-pot of non-linear monotonic characteristics. The initial value problem is a non-zero initial displacement condition with zero external force $f(t)$ and initial force due to dash-pot.

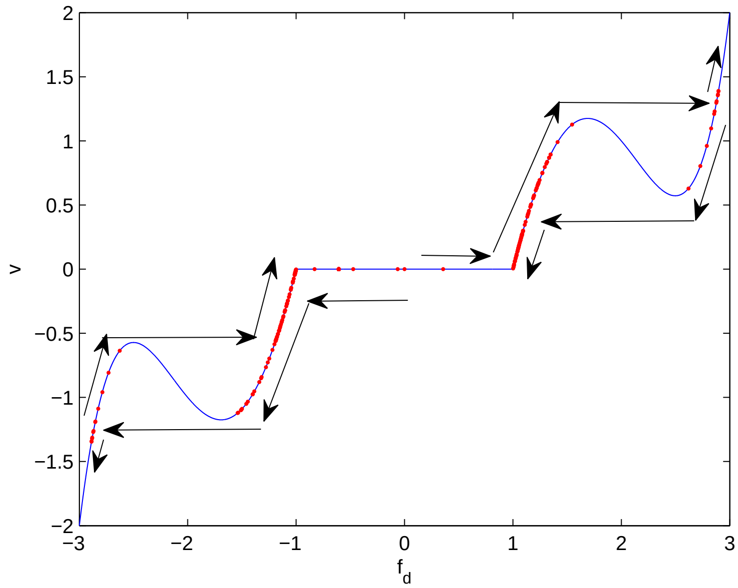
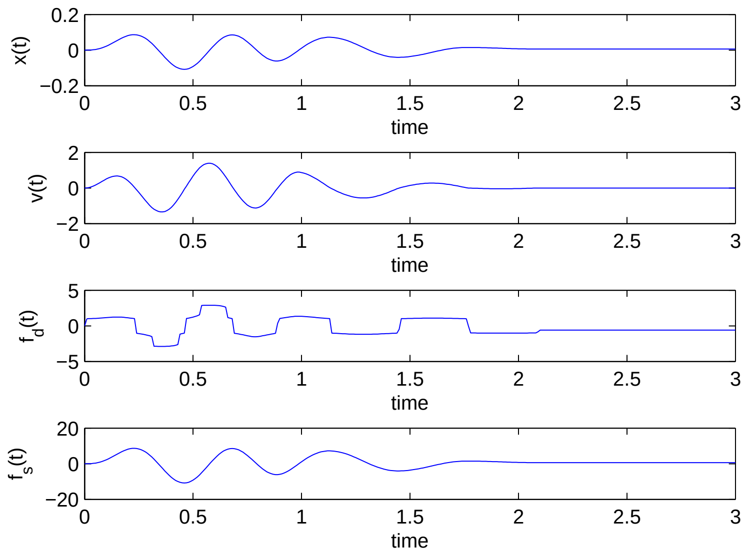
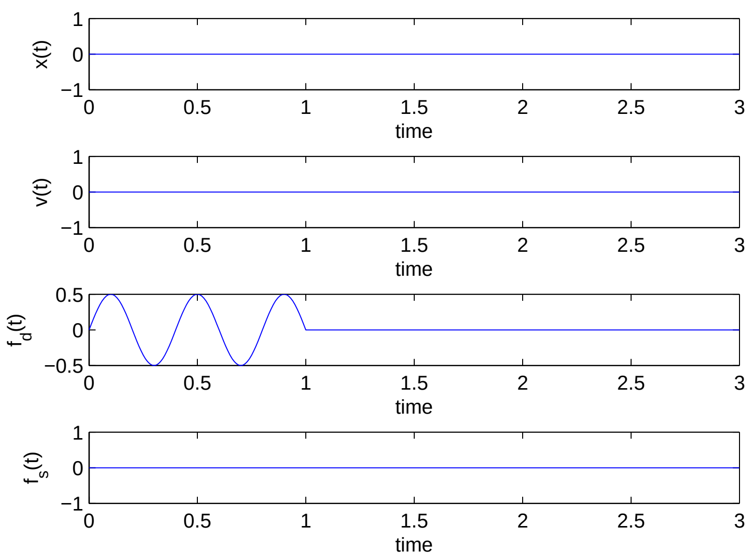


Fig. 21.: A plot velocity versus force due to dash-pot. The arrowed lines show the path traversed when solved by using the approach in which jumps in velocity profile are avoided.

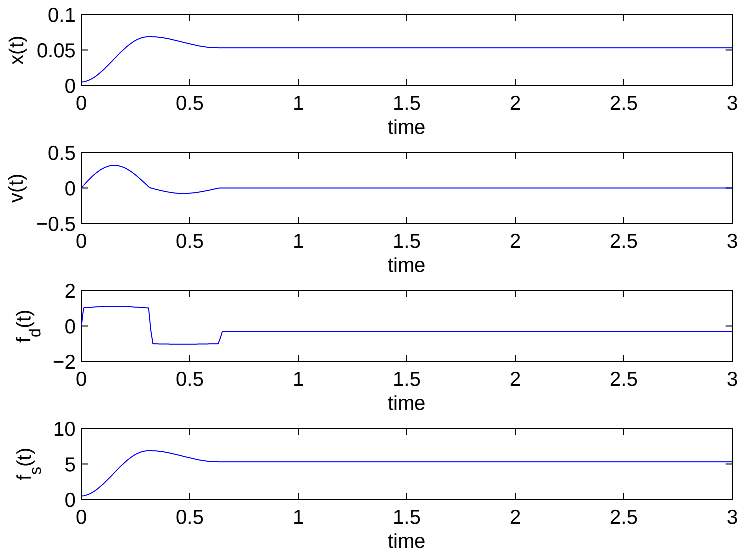


(a) $\max[f] > f_d^{\text{crit}}$

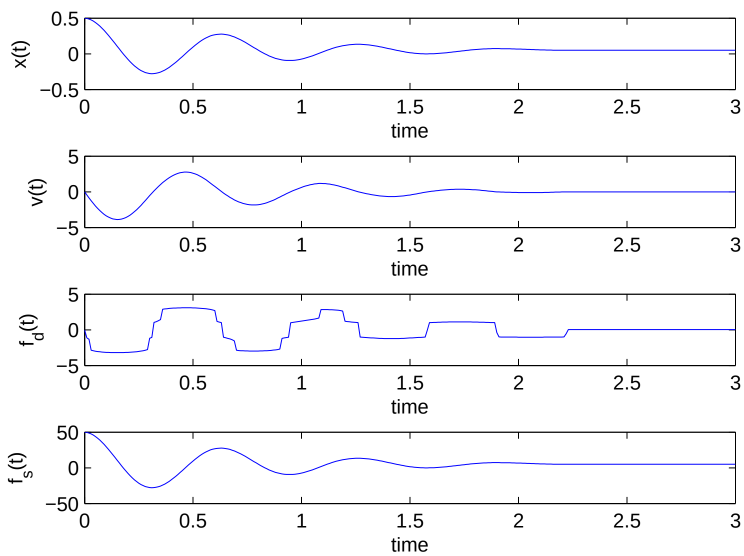


(b) $\max[f] < f_d^{\text{crit}}$

Fig. 22.: Dynamics (by Approach 1) of a spring-mass-dash-pot system with dash-pot of non-linear non-monotonic characteristics. The initial value problem is a non-zero external forcing condition with zero initial displacement and force due to dash-pot.



(a) $x^{(0)} = 0.005$



(b) $x^{(0)} = 0.5$

Fig. 23.: Dynamics (by Approach 1) of a spring-mass-dash-pot system with dash-pot of non-linear monotonic characteristics. The initial value problem is a non-zero initial displacement condition with zero external force $f(t)$ and initial force due to dash-pot.

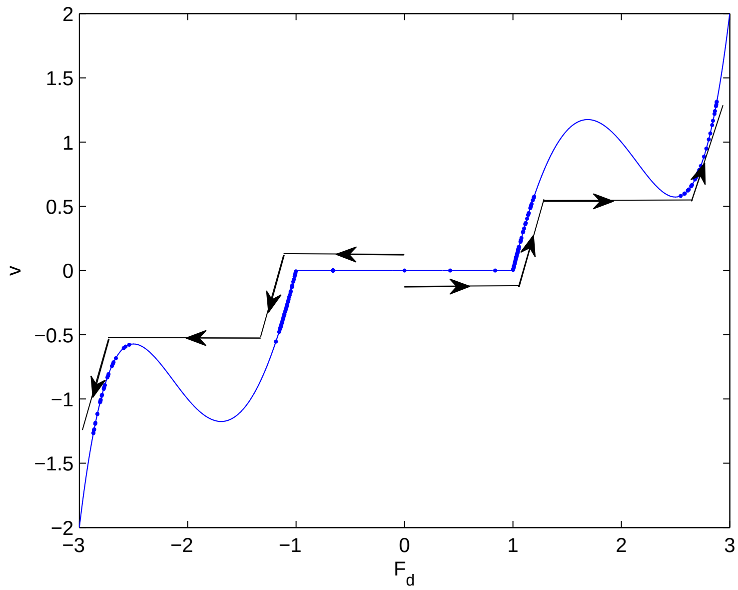
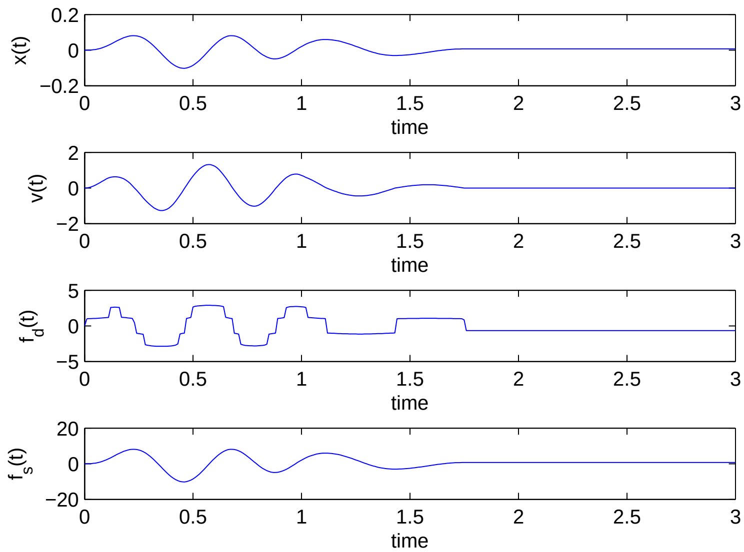


Fig. 24.: A plot for velocity versus force due to dash-pot. The arrowed lines show the path traversed when solved by using the approach in which rate of dissipation is maximized.

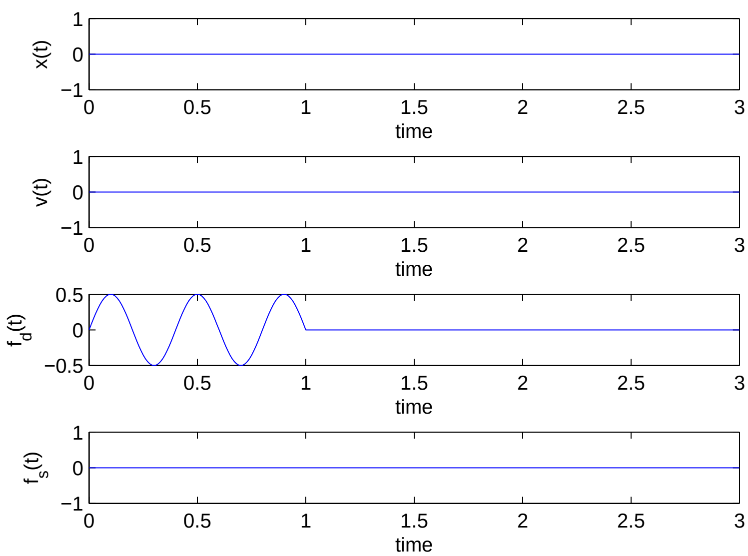
maximized. The velocities obtained take a different path and can be visualized from Figure 24. Even in this case, there are no jumps in the velocity if the v vs. time in Figures 25 and 26.

C. Numerical results for the system with Bingham-type dash-pot representing Coulomb friction

There is a slight variation in the way the initial conditions are chosen for the system with Coulomb friction compared to previous two cases The force due the dash-pot f_d at initial state is assumed and chosen according to the external force applied. For the first case of initial value problem as in (6.1a), the initial displacement is assumed as $x^{(0)} = 0$ and the initial force due to dash-pot, $f_d^{(0)} = f_d^{\text{crit}}/2$. Along with these values,

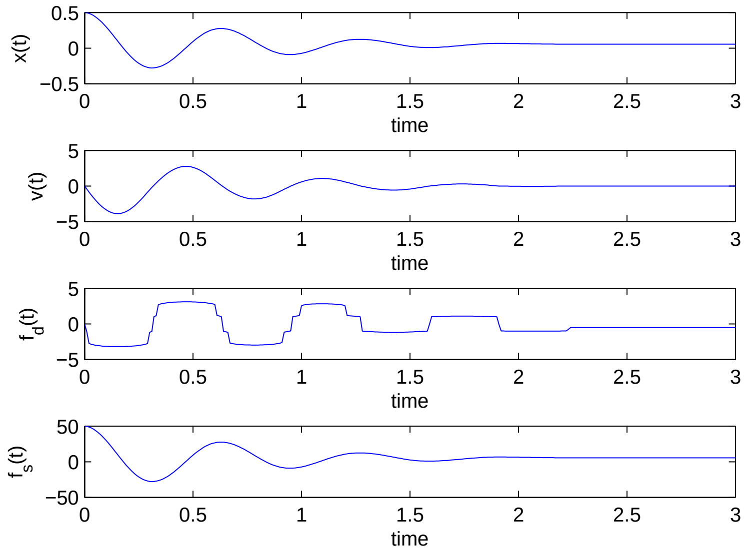


(a) $\max[f] > f_d^{\text{crit}}$

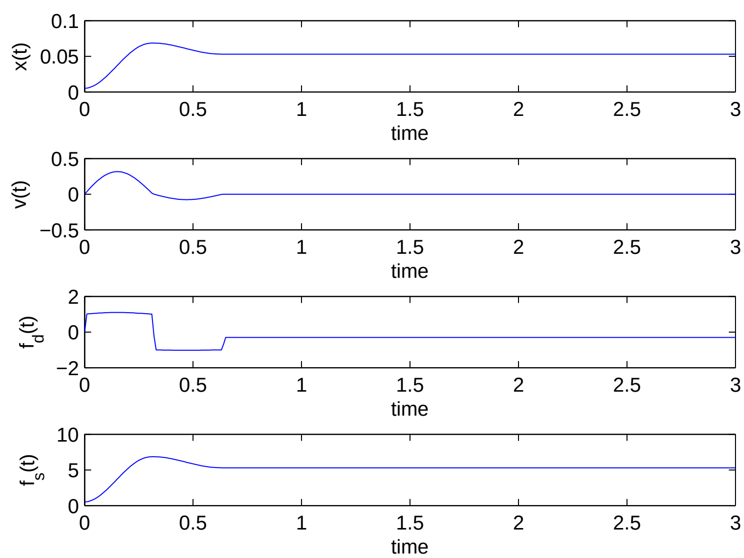


(b) $\max[f] < f_d^{\text{crit}}$

Fig. 25.: Dynamics (by Approach 2) of a spring-mass-dash-pot system with dash-pot of non-linear non-monotonic characteristics. The initial value problem is a non-zero external forcing condition with zero initial displacement and force due to dash-pot.



(a) $x^{(0)} = 0.005$



(b) $x^{(0)} = 0.5$

Fig. 26.: Dynamics (by Approach 2) of a spring-mass-dash-pot system with dash-pot of non-linear monotonic characteristics. The initial value problem is a non-zero initial displacement condition with zero external force $f(t)$ and initial force due to dash-pot.

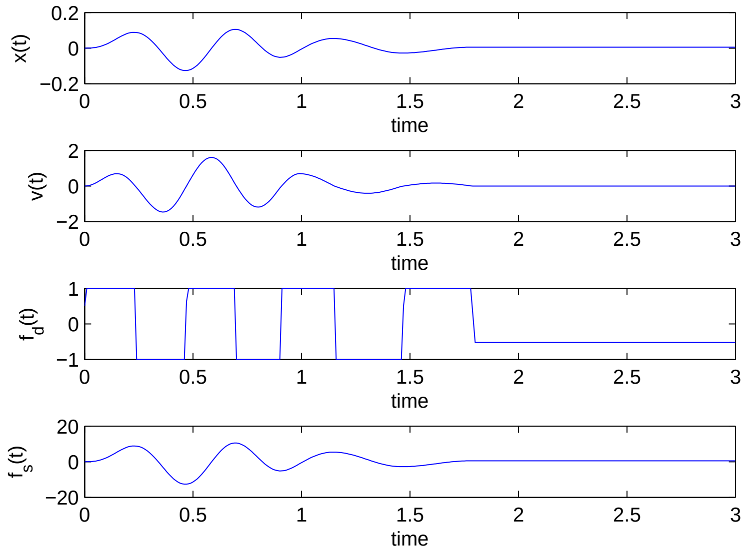
initial velocity is also assumed to be $v^{(0)} = 0$. Similarly, for the second case of initial value problem as in (6.1b) the initial force due to dash-pot $f_d = 0$.

From Figures 27 and 28, it can be observed that whenever velocity of the system is non-zero, force due to the dash-pot is equal to critical force which is one unit in this thesis. Two problems considered with different initial condition scenarios verify the correctness of the algorithm.

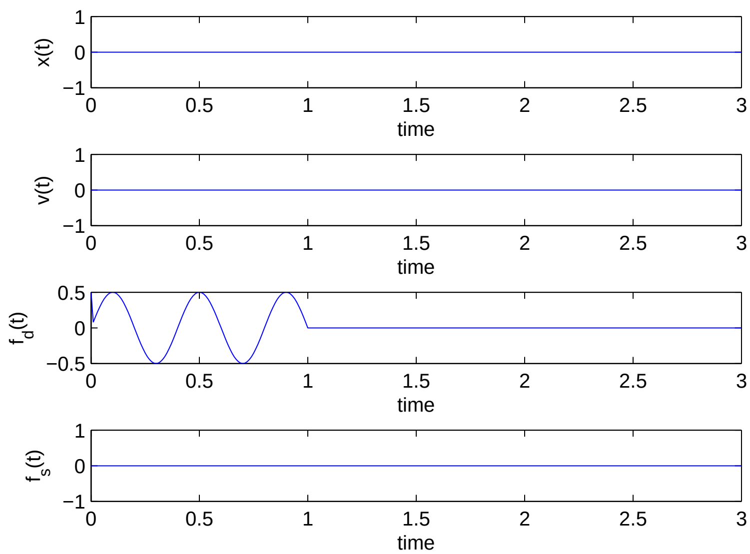
D. Numerical results for system with Bingham-type dash-pot of linear characteristics and a constraint

This system derives interesting results when the value of k_{support} is very high as to represent a rigid wall. For the purpose of representation one scenario as in (6.1a) was considered and kept on increasing the value of k_{support}. The plots are given below. The main observation to be done is that as k_{support} value is increased the points of contact and release can be clearly seen.

It can be observed in these plots that the smoothness in the velocity profile reduces and the jump effect increases with increasing k_{support} values. This is particularly useful when one would want to represent a unilateral constraint using rigid wall by assigning very high values to k_{support}. In this thesis, an array of values were assigned to k_{support} ranging from as low as 10 to as high as 10000 to observe this behaviour and can be observed from Figures 29, 30, 31 and 32.

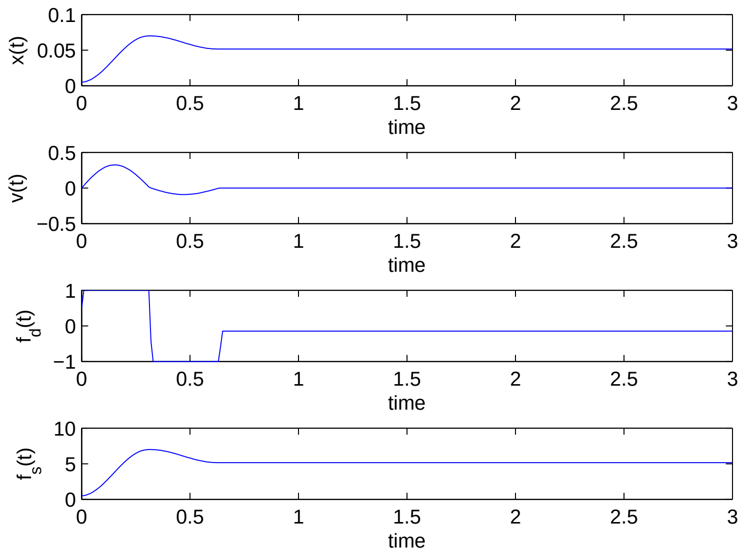


(a) $\max[f] > f_d^{\text{crit}}$

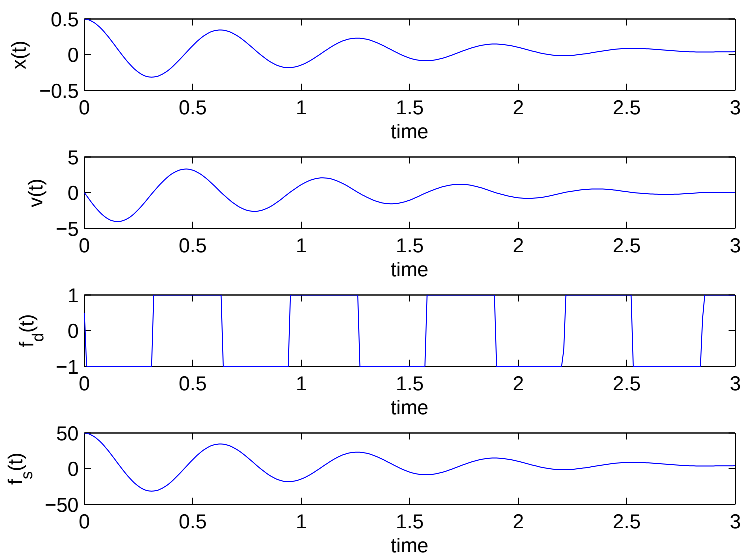


(b) $\max[f] < f_d^{\text{crit}}$

Fig. 27.: Dynamics of a spring-mass system involving Coulomb friction. The initial value is a non-zero external forcing condition with zero initial displacement and initial velocity, initial force due to dash-pot $f_d^{\text{crit}}/2$.



(a) $x^{(0)} = 0.005$



(b) $x^{(0)} = 0.5$

Fig. 28.: Dynamics of a spring-mass system involving Coulomb friction. The initial value problem is a non-zero initial displacement condition with zero external force and initial velocity being, initial force due to dash-pot as $f_d^{\text{crit}}/2$.

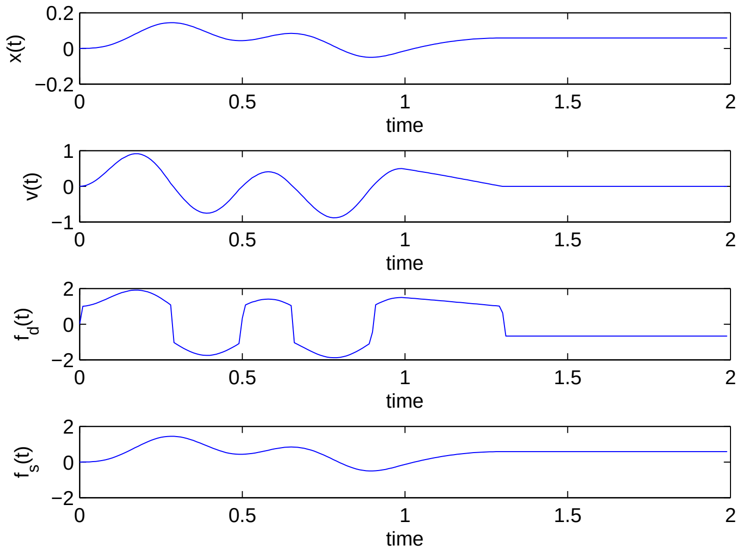


Fig. 29.: Dynamics of system spring-mass-dash-pot system with a massless support connected to a rigid wall by a linear spring of spring constant $k_{\text{support}} = 10$ placed at a distance of L units from mass m.

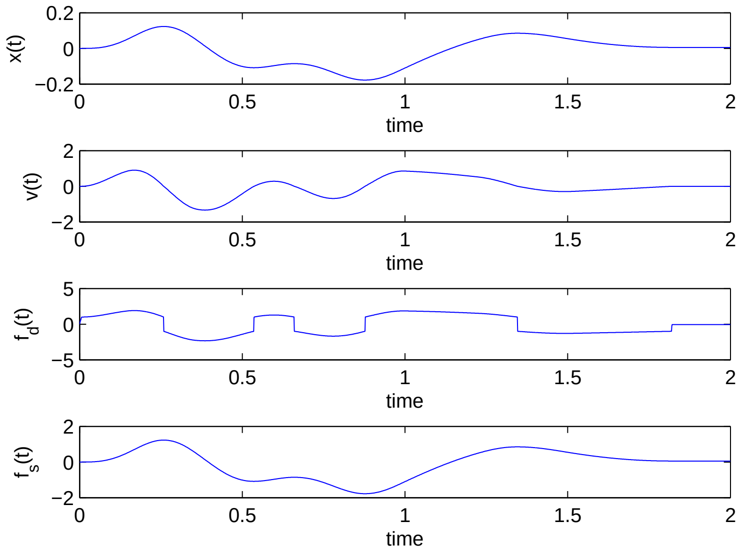


Fig. 30.: Dynamics of system spring-mass-dash-pot system with a massless support connected to a rigid wall by a linear spring of spring constant $k_{\text{support}} = 100$ placed at a distance of L units from mass m.

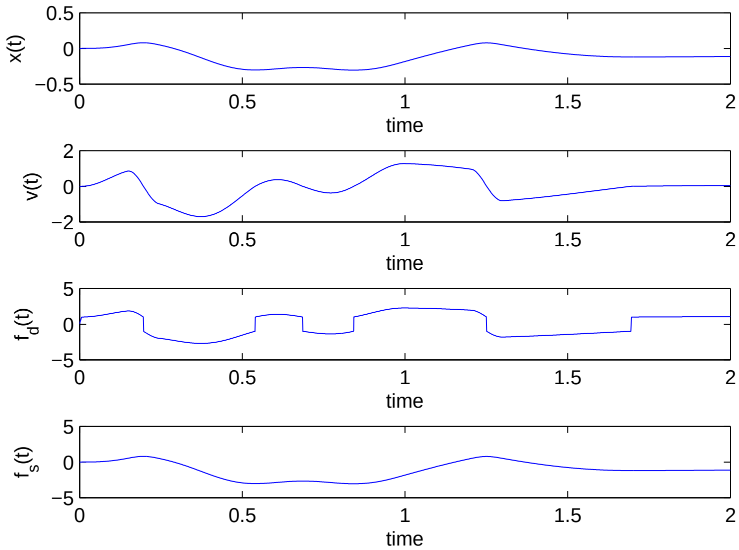


Fig. 31.: Dynamics of system spring-mass-dash-pot system with a massless support connected to a rigid wall by a linear spring of spring constant $k_{\text{support}} = 1000$ placed at a distance of L units from mass m.

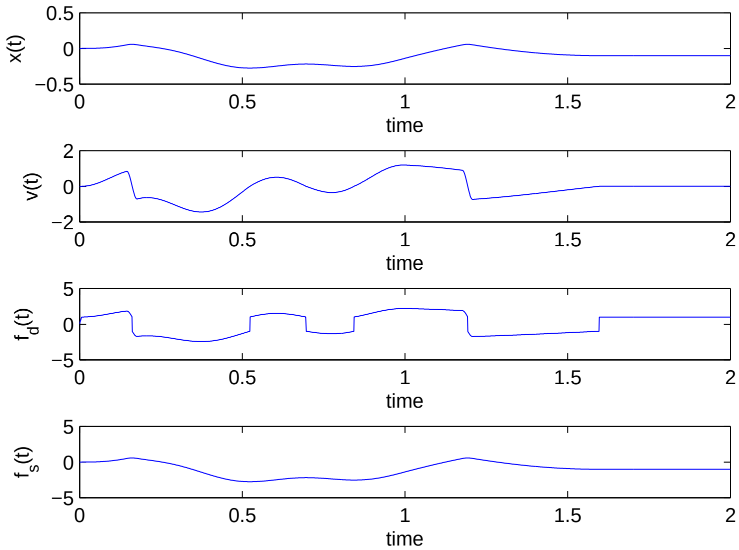


Fig. 32.: Dynamics of system spring-mass-dash-pot system with a massless support connected to a rigid wall by a linear spring of spring constant $k_{\text{support}} = 10000$ placed at a distance of L units from mass m.

CHAPTER VII

CONCLUSIONS

In this thesis, the algorithms are devised for lumped parameter systems, a spring-mass-dash-pot system in this case by making variations to the dash-pot characteristics so that three different problems are addressed. They include one with non-linear characteristics of Bingham-type dash-pot giving rise to explicit constitutive relations, Coulomb friction model giving rise to implicit constitutive relations, and system with constraints giving rise to differential-algebraic inequalities. The heart of this thesis lies in modeling the lumped parameter systems giving rise to differential-algebraic equations by using the unconventional way of writing the constitutive relations such that the kinematic quantities are expressed in terms of dynamic quantities whenever possible. Representative examples are considered at the end to check the correctness of using the differential-algebraic equations approach. The challenges of making the appropriate choice of numerical methods to be used to ensure the consistency of the solutions obtained are met. In future, the study can be extended towards devising algorithms for continuum models using analogies of dynamic variables to stress and kinematic variables to strain. Modeling may be done using various combinations of springs and dash-pots that represent such continuum models by differential-algebraic equations and hence, efficient algorithms can be devised.

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APPENDIX A

BIOGRAPHIES

Carathéodory

Constantin Carathéodory (1873 - 1950) was born in Berlin on September 13, 1873. His contributions extend towards many diverse fields. To list a few: Calculus of Variations, Real Analysis, Measure Theory, Theory of Functions, Thermodynamics, Optics. The highly celebrated Carathéodory theorem takes a special place in the field of differential equations. He is also famous for Carathéodory conjecture which states *"a closed convex surface admits at least two umbilic points"* and still remains unproven. He is a renowned author of several books and many publications of high importance.

Cauchy

Baron Augustin-Louis Cauchy (1789 - 1857) was a French mathematician born on April 21, 1789. He was endowed with "grand pix d'humanitiés" by the emperor of France at the age of fifteen. He dedicated himself to mathematics and laid foundations to great discoveries in a variety of fields. His major creation was the concept of stress which gained enormous attention at the time and became a building block of the entire subject of continuum mechanics.

Filippov

Aleksei Fedorovich Filippov was born in Moscow on September 29, 1923. His main research interest was in the field of differential equations. He was particularly interested in differential equations with non-smooth solutions, which he referred to as differential equations with discontinuous right-hand side. In 1996, he was honored with the title “Honorary Professor of Moscow State University.”

VITA

Priyanka Gotika graduated with a B.E. (Hons) in mechanical engineering and a M.S. (Hons) in mathematics from Birla Institute of Technology and Science in June 2009. Upon graduation she enrolled at Texas A&M University to pursue her Master's degree in mechanical engineering. She received her M.S. degree in December 2011 under the guidance of Dr. Kalyana Babu Nakshatrala. She can be reached through Dr. Kalyana Babu Nakshatrala at: Department of Mechanical Engineering, Texas A&M University, College Station, TX 77843-3123.

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