OUTAGE CAPACITY AND CODE DESIGN FOR DYING CHANNELS

A Dissertation

by

MENG ZENG

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2011

Major Subject: Electrical Engineering

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Approved by:

Chair of Committee,	Shuguang Cui
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ABSTRACT

Outage Capacity and Code Design for Dying Channels. (August 2011)

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In wireless networks, communication links may be subject to random fatal impacts: for example, sensor networks under sudden power losses or cognitive radio networks with unpredictable primary user spectrum occupancy. Under such circumstances, it is critical to quantify how fast and reliably the information can be collected over attacked links. For a single point-to-point channel subject to a random attack, named as a *dying channel*, we model it as a block-fading (BF) channel with a finite and random channel length. First, we study the outage probability when the coding length K is fixed and uniform power allocation is assumed. Furthermore, we discuss the optimization over K and the power allocation vector $\boldsymbol{P}_K$ to minimize the outage probability. In addition, we extend the single point-to-point dying channel case to the parallel multi-channel case where each sub-channel is a dying channel, and investigate the corresponding asymptotic behavior of the overall outage probability with two different attack models: the independent-attack case and the m-dependent-attack case. It can be shown that the overall outage probability diminishes to zero for both cases as the number of sub-channels increases if the *rate per unit cost* is less than a certain threshold. The outage exponents are also studied to reveal how fast the outage probability improves over the number of sub-channels.

Besides the information-theoretical results, we also study a practical coding scheme for the dying binary erasure channel (DBEC), which is a binary erasure channel (BEC) subject to a random fatal failure. We consider the rateless codes and optimize the degree distribution to maximize the average recovery probability. In particular, we first study the upper bound of the average recovery probability, based on which we define the objective function as the gap between the upper bound and the average recovery probability achieved by a particular degree distribution. We then seek the optimal degree distribution by minimizing the objective function. A simple and heuristic approach is also proposed to provide a suboptimal but good degree distribution.

To my parents and Sisi

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CHAPTER I

INTRODUCTION

Wireless communications is a broad and dynamic field that spurred tremendous excitement and technological advances over the last decades [1]. Wireless communication channels are significantly different from other communication channels, such as wireline communication channels and underwater communication channels. In particular, the wireless radio channel poses a severe challenge as a medium for reliable high-speed communication, not only due to its vulnerability to noise, interference, and other channel impediments, but also the random variation of the above factors as a result of user movement and environmental dynamics. Different fading models have been proposed to capture the dynamics of the wireless radio channels, based on which various transmission schemes have been studied. Typically, in the optimal wireless transmission schemes, the number of channel uses is assumed to be either infinite or deterministically finite, which may not be true in certain scenarios such as cognitive radio communication or stressed military communication. In this dissertation, we will study the fundamental limits of unreliable channel links whose number of channel uses is finite but random. In addition, practical coding schemes over such channels will also be discussed.[1]

A. Overview of Prior Works

Information-theoretic limits of fading channels have been thoroughly studied in the literature and to date many important results are known (see [2] and references therein). Generally speaking, if the transmission delay is not of concern, the classic Shannon

[1]This dissertation follows the style of *IEEE Transactions on Communications.*

capacity for a deterministic additive white Gaussian noise (AWGN) channel can be extended to the ergodic capacity for a fading AWGN channel, which is achievable by a random Gaussian codebook with infinite-length codewords spanning over many fading blocks such that the randomness induced by fading can be averaged out [3] [4]. With the perfect transmitter and receiver channel state information (CSI), the adaptive power allocation serves as an effective method to increase the ergodic capacity. This allocation has the well-known "water-filling" structure [3], where power is allocated over the channel state space. With such an allocation scheme, a user transmits at high power when the channel is good and at low or zero power when the channel is poor. When the CSI is only known at the receiver, the capacity is achievable with special "single-codebook, constant-power" schemes [5].

The validity of the ergodic capacity is based on the assumption that the channel length is infinite. However, many wireless communication applications have certain delay constraints, which limit the practical codeword length to be finite. Thus, the ergodic capacity is no longer a meaningful performance measure. Such situations give rise to the notions of *outage capacity*, *delay-limited capacity*, and *average capacity* [6] [7], each of which provides a more meaningful performance measure than the ergodic capacity. In particular, there usually exists a capacity-versus-outage tradeoff for transmissions over fading channels with finite channel length [8], where a higher target rate results in a larger outage probability. The maximum transmit rate that can be reliably communicated under some prescribed transmit power budget and outage probability constraint is known as the outage capacity. In the extreme case of requiring zero outage probability, the outage capacity then becomes the zero-outage or delay-limited capacity [9]. To study the delay-limited system, the author in [6] adopts a K-block block-fading (BF) AWGN channel model, where K indicates the constraint on transmission delay or the maximum codeword length in blocks. If the CSI for each K-block transmission is known non-causally at the transmitter, transmit power control can significantly improve the outage capacity of the K-block BF channel [5]. When the CSI can be only revealed to the transmitter in a causal manner, a dynamic programming algorithm is developed to achieve the outage capacity of the K-block BF channel in [10].

B. Challenges and Motivation

In the above works, the channel length is either infinite or finite but deterministic. However, there are indeed some practical scenarios where the channel length is finite and random. For example, in a wireless sensor network operating in a hostile environment, sensors may die due to sudden physical attacks such as fire or power losses. Another example may be a cognitive radio network with opportunistic spectrum sharing between the secondary and primary users, where an active secondary link can be corrupted unpredictably when the channel is reoccupied by a primary transmission. How fast and reliably can a piece of information be transmitted over such a channel? This question motivates us to formally define the maximum achievable information rate over a channel with a random and finite channel length, named as a *dying channel*. This type of dying channels has never been thoroughly studied in the traditional information theory, and important theorems are missing to address the fundamental capacity limits.

C. Overview of Contributions

In this dissertation, we investigate such channels by first focusing on a single point-to-point dying link and modelling it as a K-block BF channel subject to a fatal attack that may happen at a random moment within any of the K transmission blocks, or may not happen at all over K blocks. An independent work with similar motivations can be found in [11], where the channel model is a modified binary symmetric channel (BSC). Comparing to their work, we have a different channel model, which is based on the BF-AWGN channel. In addition, we also consider the power allocation over the blocks instead of only considering the optimal coding length. In practice, due to the possible channel death, we cannot assume that there is a reliable feedback channel. As a result, the dynamic programming based method and the automatic repeat request (ARQ) based approach [12] are not considered in this dissertation.

Dying channels also exist in systems with multiple parallel sub-channels (e.g., in a OFDM-based system), where each sub-channel may be under a potential random attack. In such a scenario, we are interested in the overall system outage probability and how the outage probability behaves as the number of sub-channels increases. This leads us to examine the asymptotic outage behavior for the case of a parallel dying channel. We will consider two models of random attacks over the sub-channels: 1) the case of independent random attacks, where the attacks across the sub-channels are independently and identically distributed (i.i.d.); and 2) the case of m-dependent random attacks, where the attacks over m adjacent sub-channels are correlated and the attacks on sub-channels that are m-sub-channels away from each other are independent.

Besides the information theoretical results, we also consider practical coding schemes over the dying channel, especially for the dying binary erasure channel (DBEC). Rateless codes are considered and the degree distribution is optimized to maximize the average recovery probability of the coded bits.

The notations are given as following.

- $\mathbb{R}$ indicates the set of real numbers, $\mathbb{R}_+$ is the set of nonnegative real numbers, and $\mathbb{R}_+^N$ is the set of N-dimensional nonnegative real vectors.

- The Q-function: $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$.
- $\log(x)$ is the natural logarithm.
- $\lceil \cdot \rceil$ is the ceiling operator and $\lfloor \cdot \rfloor$ is the floor operator.

D. Organization

The rest of the dissertation is organized as following. In Chapter II, we study the single dying channel by first introducing the system model and defining the outage probability as performance measure. Then we investigate the outage probability by assuming the uniform power allocation and fixed coding length. Lower and upper bounds for the outage probability are obtained. In particular, the relation between the outage probability and the multiplexing gain is discussed in the high SNR regime. For low and moderate SNR regimes, Gaussian approximation is applied to obtain the approximated outage probability. Furthermore, we consider the optimization over the coding length and the power vector to minimize the outage probability. In Chapter III, we extend the results from single dying channel case to the parallel dying channel case. In this chapter, we consider the independent random attack case and dependent random attack case respectively, where the resulting numbers of survived blocks on each sub-channel are independent or m-dependent. The overall outage probabilities for these two cases are examined. In addition, the outage exponents for these two cases are studied to reveal how fast the outage probability goes to zero as the number of sub-channel increases. After studying the information-theoretical aspects for the dying channel, we then consider the practical code design for the dying channel in Chapter IV. Specifically, we consider the LT codes for the dying binary erasure channel and redesign the degree distribution of the LT codes. First, we review the upper bound of the average recovery probability and then optimize the degree distribution such that the resulting average recovery probability is as close to the upper bound as possible. At last, we conclude our work in Chapter V.

CHAPTER II

OUTAGE CAPACITY OF A SINGLE DYING CHANNEL

We consider a narrowband point-to-point delay-limited fading channel subject to a random fatal attack, where the exact timing of the attack is unknown to the transmitter and the receiver. Only the distribution of the random attack time is known. We further assume that there is no channel state information at the transmitter (CSIT) while there is perfect channel state information at the receiver (CSIR). We build our model of a dying link based on the K-block BF-AWGN channel [6], where the channel remains constant over a block but changes independently from one block to another. The k-th received symbol in the i-th block is given by:

$$y_{k,i} = \sqrt{P_i} h_i x_{k,i} + z_{k,i}, \tag{2.1}$$

where $i = 1, \cdots, K$ indicates the block number, $k = 1, \cdots, B$ indicates the channel uses within a block, P_i is the transmit power in block i, h_i is the channel gain that is circularly symmetric complex Gaussian with unit variance and zero mean, and the noise is assumed to have unit variance throughout this chapter.

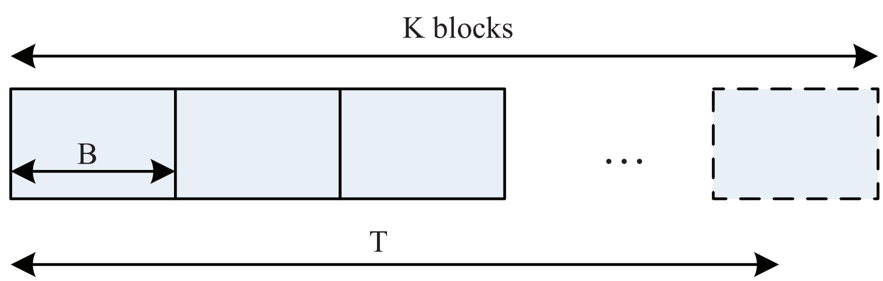

Fig. 1.: Point-to-point dying channel model

As shown in Fig. 1, in our model of the dying channel, the transmitter transmits a codeword over K blocks; and when the fatal attack occurs at time T, which is normalized by the block length B, the communication link is cut off immediately with the current and rest of the blocks lost. The number of survived transmitted blocks is thus a random variable that is less than or equal to K due to the fact that a random attack may happen within any block out of the K blocks or may not happen at all within the K blocks. As we know from the results of BF-AWGN channel [13], we can decode the codeword even if the attack happens within the K blocks as long as the average mutual information of surviving blocks is greater than the code rate R of the transmission, i.e.,

$$\frac{1}{K}\sum_{i=1}^{L}\log(1+\alpha_i P_i) \geq R, \tag{2.2}$$

where the random integer $L = \min(K, \lfloor T \rfloor)$ with $\lfloor \cdot \rfloor$ being the floor operator and α_i's are the fading gains, i.e., $\alpha_i = |h_i|^2, i = 1, \ldots, K$.

The dying channel is apparently non-ergodic and an appropriately defined outage capacity serves as the reasonable performance measure, which is formally defined as follows:

Definition 1. *The outage capacity of a K-block BF-AWGN dying channel with an average transmit power constraint P and a required outage probability η is expressed as*

$$\begin{aligned} C_{\text{out}}(P,\eta) \;\triangleq\; & \max_{K} \sup_{\boldsymbol{P}_K : \sum_{i=1}^{K} P_i \leq KP} \Bigg\{ R : \\ & \mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{L}\log(1+\alpha_i P_i) < R\} < \eta\right]\Bigg\}. \end{aligned} \tag{2.3}$$

Note that the outage probability above is defined over the distributions of the α_i's and T, where we assume that the α_i's and T are independent of each other. In addition, we assume that the transmitter only knows the distributions of the α_i's and T, not their instantaneous values. In this chapter, we frequently assume

Rayleigh fading channels and exponentially distributed random attack time T. Since the life span of the electronic devices usually follows the exponential distribution, the exponentially distributed attack time could accurately model the scenario where the channel death is caused by the device failures. However, many of the findings in the dissertation can also be applied to general distributions of α_i's and T as pointed out later. From the perspective of optimal transmission schemes, the outage capacity maximization problem is equivalent to the outage probability minimization problem [5]. Therefore, we first discuss the outage probability in the following.

A. Outage Probability with Fixed Coding Length and Uniform Power Allocation

In this section, we study the outage probability when the coding length K is fixed and uniform power allocation is adopted. According to the law of total probability, the outage probability can be rewritten as a summation of the probabilities conditioned on different numbers of surviving blocks, i.e.:

$$\begin{aligned} p_{out}(R,P,K) &\triangleq \mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{L}\log(1+\alpha_i P) < R\right] \\ &= w_0 + \sum_{j=1}^{K-1}\mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{j}\log(1+\alpha_i P) < R\right] w_j \\ &\quad + \mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{K}\log(1+\alpha_i P) < R\right] w_K^*, \end{aligned} \tag{2.4}$$

where $w_i = \mathbb{P}[i < T \leq i+1]$ for $i = 0, \cdots, K-1$, and $w_K^* = \mathbb{P}[T > K]$. In general, there are no tractable closed-form expressions for p_{out} given in (2.4). However, we could identify some meaningful bounds of p_{out} and study its asymptotic relationship with SNR and rate.

1. Outage Probability Lower Bound

Notice that the following relationship holds:

$$\begin{aligned}
\mathbb{P}\left[\sum_{i=1}^{j}\log(1+\alpha_i P) < KR\right] &\geq \mathbb{P}\left[j\log(1+\max_{i=1,\cdots,j}\alpha_i P) < KR\right] \\
&\overset{(a)}{=} \prod_{i=1}^{j}\mathbb{P}\left[\log\left(1+\alpha_i P\right) < \frac{KR}{j}\right] \\
&= \left\{F\left(\frac{e^{\frac{KR}{j}}-1}{P}\right)\right\}^{j}, \qquad (2.5)
\end{aligned}$$

where $F(x)$ is the cumulative distribution function (CDF) of the random variable α_i and step (a) comes from the fact that α_i's are i.i.d.. Therefore, with the relationship in (2.5), we have a lower bound for the outage probability in (2.4) as

$$\begin{aligned}
\mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{L}\log(1+\alpha_i P) < R\right] &\geq w_0 + \sum_{i=1}^{K-1}\left\{F\left(\frac{e^{KR/i}-1}{P}\right)\right\}^{i} w_i \\
&\quad + \left\{F\left(\frac{e^{R}-1}{P}\right)\right\}^{K} w_K^{*}. \qquad (2.6)
\end{aligned}$$

2. Outage Probability Upper Bound

On the other hand, there exists an upper bound for the outage probability with a counterpart argument as that for (2.5):

$$\begin{aligned}
&\mathbb{P}\left[\sum_{i=1}^{j}\log(1+\alpha_i P) < KR\right] \leq \mathbb{P}\left[j\log(1+\min_{i=1,\cdots,j}\alpha_i P) < KR\right] \\
&= 1-\prod_{i=1}^{j}\left(1-\mathbb{P}\left[\log(1+\alpha_i P) < \frac{KR}{j}\right]\right) \\
&= 1-\left\{\bar{F}\left(\frac{e^{\frac{KR}{j}}-1}{P}\right)\right\}^{j}, \qquad (2.7)
\end{aligned}$$

where $\bar{F}(x) = 1 - F(x)$. Therefore, an upper bound for the outage probability in (2.4) is given as

$$\begin{aligned} \mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{L}\log(1+\alpha_i P) < R\right] \;\; &\leq \;\; 1 - \sum_{i=1}^{K-1}\left\{\bar{F}\left(\frac{e^{KR/i}-1}{P}\right)\right\}^i w_i \\ &\quad - \left\{\bar{F}\left(\frac{e^{R}-1}{P}\right)\right\}^K w_K^*. \end{aligned} \tag{2.8}$$

In Fig. 2, we numerically evaluate the bounds by assuming a Rayleigh fading channel, an exponentially distributed attack time T with mean equal to 10, a coding length $K = 10$, and a target rate $R = 1$ nat/s/Hz. As shown in Fig. 2, these bounds are tight in high SNR regime but loose in low and moderate SNR regimes[1]. Note that the above lower and upper bounds are valid beyond the Rayleigh fading channel and exponential random attack cases.

3. Gaussian Approximation for Outage Probability

Let $V_i \triangleq \log(1 + \alpha_i P)$. When α_i follows the exponential distribution with parameter $\lambda = 1$, the mean and variance of V_i are given in [14] [15] as following:

$$\mu(P) \;\; = \;\; \mathbb{E}[\log(1+\alpha_i P)] = e^{1/P}\mathrm{E}_1\left(\frac{1}{P}\right), \tag{2.9}$$

$$\sigma^2(P) \;\; = \;\; \frac{2}{P}e^{1/P}G_{3,4}^{4,0}\left(\frac{1}{P}\Big|_{0,-1,-1,-1}^{0,0,0}\right) - \mu^2(P), \tag{2.10}$$

where $\mathrm{E}_1(x) = \int_1^\infty t^{-1}e^{xt}dt$ and $G_{p,q}^{m,n}\left(z\Big|_{b_1,b_2,\cdots,b_q}^{a_1,a_2,\cdots,a_p}\right)$ is the Meijer G-function [16].

Motivated by the central limit theorem (CLT), we approximate the term $\frac{1}{j}\sum_{i=1}^{j} V_i$ by a Gaussian variable with mean $\mu(P)$ and variance $\sigma^2(P)/j$. In our simulation, we

[1]We assume unit noise power, hence SNR=P. By low SNR, we mean that $P \ll 1$; and by high SNR, we mean $P \gg 1$.

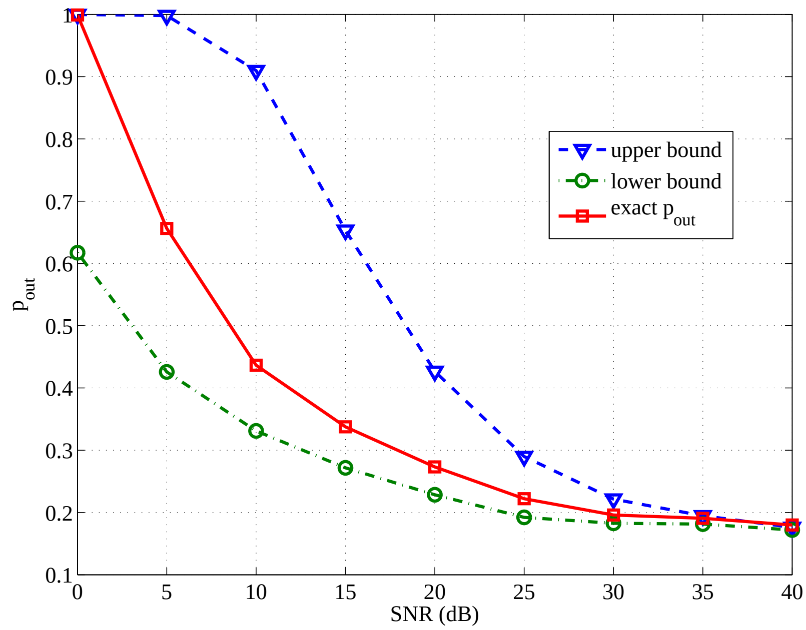

Fig. 2.: Exact p_{out} vs. upper and lower bounds.

find that the accuracy of the approximation is satisfactory even j is small. Therefore,

$$\begin{aligned}\mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{j}\log(1+\alpha_i P)<R\right] &= \mathbb{P}\left[\frac{1}{j}\sum_{i=1}^{j}\log(1+\alpha_i P)<\frac{KR}{j}\right] \\ &\approx Q\left(\frac{\mu(P)-KR/j}{\sigma(P)/\sqrt{j}}\right). \end{aligned} \tag{2.11}$$

As a result, according to (2.4), we have the approximated outage probability as

$$\begin{aligned} p_{out} &= \mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{L}\log(1+\alpha_i P)<R\right] \\ &\approx w_0+\sum_{i=1}^{K-1}Q\left(\frac{\mu(P)-KR/i}{\sigma(P)/\sqrt{i}}\right)w_i+Q\left(\frac{\mu(P)-R}{\sigma(P)/\sqrt{K}}\right)w_K^*. \end{aligned} \tag{2.12}$$

We use the following example to show the accuracy of the Gaussian approximation, where we assume exponential random attack time with mean $1/\lambda = 10$, $K = 10$, and $R = 1$ nat/s/Hz. As we can see from Fig. 3, the Gaussian approximation (GA) is quite accurate for all the SNR regime of interest. Note that the Gaussian approximation approach is also applicable to other random attack time distributions.

4. Asymptotic Outage Probability over SNR

As we see from (2.4), the minimum achievable outage probability is w_0, which is only determined by the statistics of the random attack time. In the Rayleigh fading case, we would like to know the asymptotic relationship among the rate R, the SNR, and the outage probability.

Theorem 1. *Let* $\lim_{P\to\infty}\frac{R}{\log P}=r$ *and* $w_0<w<1$*; then*

$$\lim_{P\to\infty}\mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{L}\log(1+\alpha_i P)<R\right]=\begin{cases} w_0, & \text{if } r<1/K \\ w, & \text{if } 1/K\le r\le 1 \\ 1, & \text{if } r>1. \end{cases}$$

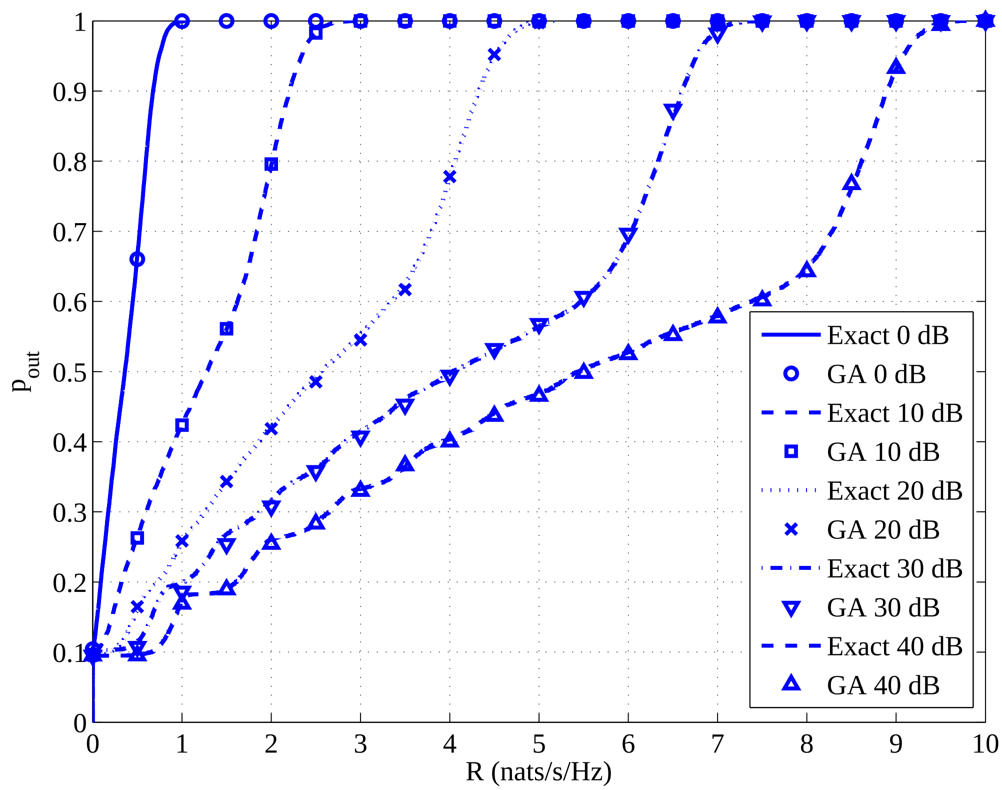

Fig. 3.: Exact and approximated outage probability.

Proof. See Appendix A. □

According to the above theorem, in order to achieve minimum outage probability w_0, the rate R can only increase at the speed of $r \log P$, where the pre-log factor r should be less than $1/K$. If the pre-log factor r is larger than one, the outage probability goes to one as the SNR increases. Any r between $1/K$ and one leads to an outage probability between w_0 and one.

B. Optimization over Coding Length

As shown in (2.3), we can optimize over the coding length K and the power vector $\boldsymbol{P}_K$ to maximize the outage capacity, or equivalently to minimize the outage probability [5]. If a uniform power allocation strategy is adopted, the only thing left for optimization is the coding length K. On one hand, we can have a larger $L = \min(K, \lfloor T \rfloor)$ by increasing K, meaning that we potentially have higher diversity to achieve a lower outage probability. On the other hand, a larger K incurs a higher percentage of blocks being lost after the attack such that the average achievable mutual information per block is lower, which results in a larger outage probability. Since the random attack time determines the number of surviving blocks and K determines the average base, we are interested in finding a proper value of K to "match" the random attack property in the sense that the outage probability is minimized.

1. Low SNR Regime

When SNR is low, we have $\log(1 + \alpha_i P) \approx \alpha_i P$. Thus, when we span a codeword over K blocks, the outage probability conditioned on T (suppose the corresponding number of survived block is j) is given as

$$p_{out|T} = \mathbb{P}\left[\sum_{i=1}^{j} \log(1+\alpha_i P) < KR\right] \approx \mathbb{P}\left[\sum_{i=1}^{j} \alpha_i < KR/P\right]. \qquad (2.13)$$

When using a repetition transmission (over blocks) with maximal-ratio-combining (MRC), the outage probability is given as

$$\begin{aligned} p^{rep}_{out|T} &= \mathbb{P}\left[\log(1+\sum_{i=1}^{j} \alpha_i P) < KR\right] \\ &\approx \mathbb{P}\left[\sum_{i=1}^{j} \alpha_i < KR/P\right]. \end{aligned} \qquad (2.14)$$

Comparing (2.13) and (2.14), we see that the outage performances of these two schemes are the same in the low-SNR regime. This is due to fact that in low SNR regime it is SNR-limited rather than degree-of-freedom-limited such that coding over different blocks does not help to decrease the outage probability. Hence, repetition transmission is approximately optimal for a dying channel in the low SNR regime. Correspondingly, the optimal coding length $K^* = 1$. Note that this result is not limited to the Rayleigh fading and exponential random attack time case.

2. High SNR Regime

For K-block fading channel model in high SNR regime, outage typically occurs when each sub-channel cannot support an evenly-divided rate budget (see Exercise 5.18 in [13]). Thus, conditioned on the attack time T, the outage probability can be written as :

$$\begin{aligned} p_{out|T} &= \mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{j} \log(1+\alpha_i P) < R\right] \\ &\approx \left(\mathbb{P}\left[\log(1+\alpha_i P) < \frac{K}{j}R\right]\right)^j. \end{aligned} \qquad (2.15)$$

For Rayleigh fading, we have $\mathbb{P}[\alpha_i < 1/x] \approx 1/x$ when x is large. Thus, when SNR is high, we can simplify (2.15) as

$$p_{out|T} \approx \frac{e^{KR}}{P^j}. \tag{2.16}$$

With the conditional outage probability given by (2.16), the overall outage probability is

$$p_{out}(K) = w_0 + \sum_{i=1}^{K-1} \frac{e^{KR}}{P^i} w_i + \frac{e^{KR}}{P^K} w_K^*. \tag{2.17}$$

Let $G(t)$ be the CDF of the attack time, which is assumed to be exponentially distributed with parameter λ. Let $w_K^* = 1-G(K)$, $w_i = G(i+1)-G(i) = e^{-\lambda i}(1-e^{-\lambda}) = \beta^i c$ (for $i = 1, \cdots, K-1$) with $c = 1 - e^{-\lambda}$ and $\beta = e^{-\lambda}$. We can rewrite (2.17) as

$$\begin{aligned} p_{out}(K) &= e^{KR} \sum_{i=1}^{K-1} \frac{\beta^i c}{P^i} + \frac{e^{KR}}{P^K}[1 - G(K)] + w_0 \\ &= e^{KR} c \frac{\frac{\beta}{P} - (\frac{\beta}{P})^K}{1 - \frac{\beta}{P}} + \frac{1 - G(K)}{P^K e^{-KR}} + w_0. \end{aligned} \tag{2.18}$$

For high SNR, with $0 < \beta < 1$, β/P is small. Hence, $\frac{\beta/P-(\beta/P)^K}{1-\beta/P} \approx \frac{\beta/P}{1-\beta/P}$ when $K \geq 2$, and (2.18) can be approximated to:

$$p_{out}(K) \approx \xi e^{KR} + \frac{1}{P^K e^{(\lambda-R)K}} + w_0, \tag{2.19}$$

where $\xi = (1 - e^{-\lambda})\frac{\beta/P}{1-\beta/P}$. In order to obtain the optimal K by minimizing $p_{out}(K)$, we first treat (2.19) as a continuous function of K, although K is an integer.

Let us first consider the convexity of (2.19) over a real-valued K. By taking the second-order derivative of (2.19) over K, we have the following:

$$\frac{\partial^2 p_{out}(K)}{\partial K^2} = \xi R^2 e^{KR} + \frac{[\lambda + \log P - R]^2}{(P e^{\lambda - R})^K}. \tag{2.20}$$

Since we have $\lambda > 0$ and $1 - \beta/P > 0$ in the high SNR regime, it holds that $\xi > 0$.

Therefore, (2.20) is non-negative in the high SNR regime, which means that (2.19) is convex over real-valued K.

Given the convexity of (2.19), the optimal K can be derived by setting its first-order derivative to zero and finding the root. Consequently, the optimal solution K^* is obtained as follows:

$$K^* = \log\left[\frac{\lambda + \log P - R}{\xi R}\right] \frac{1}{\lambda + \log P}.$$

Obviously, K^* is unique given a set of $\xi, P, R,$ and λ. Since a feasible K for the original problem should be an integer, we need to choose the optimal integer solution from $\lfloor K^* \rfloor$ and $\lceil K^* \rceil$, whichever gives a smaller value of (2.19).

C. Optimization over Power Vector

In the previous section, we investigated the optimal coding length K that minimizes the outage probability by assuming uniform power allocation. We now consider the optimization over both the coding length K and the power vector $\boldsymbol{P}_K$ to minimize the outage probability. Note that optimizing over K is in general a 1-D search over integers, which is not complex. Since the main complexity of minimizing the outage probability lies in the optimization over $\boldsymbol{P}_K$, we first focus on the outage probability minimization problem over $\boldsymbol{P}_K$ for a given fixed K, which is expressed as:

$$\begin{aligned} \min_{\boldsymbol{P}_K} \quad & \mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{L} \log(1+\alpha_i P_i) < R\right] \\ \text{s.t.} \quad & \frac{1}{K}\sum_{i=1}^{K} P_i \le P. \end{aligned} \tag{2.21}$$

After obtaining the optimal outage probabilities conditioned on a range of K values, we choose the minimum one as the global optimal value.

1. Properties of Optimal Power Allocation

We start solving the above optimization problem by investigating the general properties of the optimal power allocation over a dying channel for a given K.

With the law of total probability, we can expand the outage probability in the objective of (2.21) as follows,

$$
\begin{aligned}
& \mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{L}\log(1+\alpha_i P_i) < R\right] \\
= \; & w_0 + \sum_{j=1}^{K-1}\mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{j}\log(1+\alpha_i P) < R\right] w_j \\
& + \mathbb{P}\left[\frac{1}{K}\sum_{i=1}^{K}\log(1+\alpha_i P) < R\right] w_K^*, \qquad (2.22)
\end{aligned}
$$

where w_i's are defined in Section A. With the above result, we then discuss the optimal power allocation for a dying channel under different conditions.

Case 1: i.i.d. Fading Gains

Theorem 2. *When fading gains over blocks are i.i.d., the optimal power allocation profile is non-increasing.*

Proof. The proof is provided in Appendix B. □

This is a general result regardless of the specific distributions of fading gains. That is, the optimal power vector lies in a convex cone $\mathcal{D}_+ = \{\boldsymbol{P}_K \in \mathbb{R}_+^K : P_1 \geq P_2 \geq \cdots \geq P_K\}$, no matter what distribution the fading gain follows, as long as the i.i.d. assumption holds.

Case 2: Identically Fading Gains Now we consider the case where the fading gains over all the blocks are the same, while they are still random. This represents the case where fading gains are highly correlated in time.

Theorem 3. *When the fading gains α_i's are the same, the optimal coding length is $K = 1$ with $P_1 = P$.*

Proof. The proof is provided in Appendix C. □

This result implies that the optimal transmission scheme for a highly correlated dying channel is to simply transmit independent blocks instead of jointly-coded blocks. Note that both Theorem 2 and Theorem 3 can be applied for general channel and attack time distributions.

2. Power Allocations for High SNR Regime

For the case of Rayleigh fading and exponential random attack time, we can further convert the corresponding optimization problem into convex one and derive the optimal power vector efficiently. Given (2.15) and conditioned on the attack time T, the conditional outage probability can be written as:

$$p_{out|T} = \mathbb{P}\left[\sum_{i=1}^{j} \log(1+\alpha_i P_i) < KR\right] \approx \prod_{i=1}^{j} \mathbb{P}\left[\log(1+\alpha_i P_i) < \frac{K}{j}R\right]. \qquad (2.23)$$

For Rayleigh fading, we have $\Pr(\alpha_i < 1/x) \approx 1/x$ when x is large. Thus, when SNR is high, we can simplify (2.23) as

$$p_{out|T} \approx \frac{(e^{KR/j}-1)^j}{\prod_{i=1}^{j} P_i}. \qquad (2.24)$$

The outage probability with Rayleigh fading in high SNR is approximated as below by substituting (2.24) into (2.22):

$$p_{out}(K) \approx w_0 + \frac{e^{KR}-1}{P_1} w_1 + \frac{(e^{KR/2}-1)^2}{P_1 P_2} w_2 + \cdots + \frac{(e^{KR/K}-1)^K}{\prod_{i=1}^{K} P_i} w_K^*. \qquad (2.25)$$

Denoting $c_i = w_i(e^{KR/i} - 1)^i$, we further simplify (2.25) as

$$p_{out}(K) \approx w_0 + \frac{c_1}{P_1} + \frac{c_2}{P_1 P_2} + \cdots + \frac{c_K}{\prod_{i=1}^{K} P_i}.$$

Since the optimal power vector lies in a convex cone as shown in Theorem 2, the problem can be formulated as a convex optimization problem (refer to Appendix D for the convexity proof):

$$\begin{aligned} \min_{\boldsymbol{P}_{K \in D_+}} \quad & w_0 + \frac{c_1}{P_1} + \frac{c_2}{P_1 P_2} + \cdots + \frac{c_K}{\prod_{i=1}^{K} P_i} \\ \text{s.t.} \quad & \sum_{i=1}^{K} P_i \leq KP, \end{aligned} \tag{2.26}$$

where $D_+ = \{\boldsymbol{P} \in \mathbb{R}_+^K : P_1 \geq P_2 \geq \cdots \geq P_K \geq 0\}$ is a convex cone. Thus, the optimal power vector can be efficiently solved with standard convex optimization algorithms such as the interior point method [17].

We set the $R = 0.5$ nats/s/Hz, $1/\lambda = 5$ for the exponential random attack, and average power $P = 10$ dB. As we can see in Fig. 4, the power vector derived by solving problem (2.26) achieves better performance in terms of the outage probability than the uniform power allocation case.

D. Summary

In this chapter, we defined the outage capacity of a single dying channel. Based on the definition, we first investigated the outage probability when uniform power allocation and fixed coding length are given. In high SNR regime, the tradeoff between the multiplexing gain and the outage probability is discussed. Then, we optimized over the coding length and the power vector to minimize the outage probability. Specifically, some general optimal transmission schemes, which are independent of the statistics of the random attack time and the fading gains, are presented.

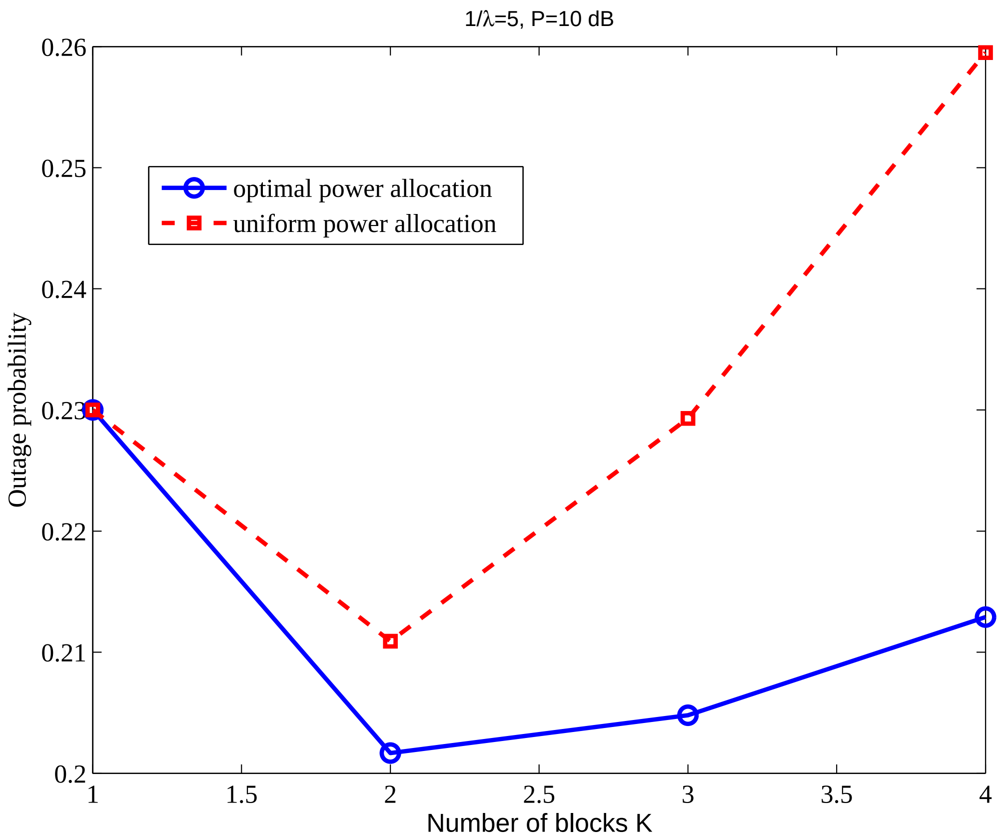

Fig. 4.: Outage probability with non-uniform and uniform power allocation, for exponential random attack time and Rayleigh fading.

CHAPTER III

OUTAGE PROBABILITY OVER PARALLEL DYING CHANNELS

In the dying channel example of cognitive radio networks, secondary users have access to vacant frequency bands that are licensed to primary users. Some primary users may suddenly show up and take over some frequency bands, which results in connection losses if these frequency bands are being used by certain secondary users. Hence, each sub-channel (a frequency band) may have a different random delay constraint for information transmission due to the uncertainty of non-uniform primary user occupancy patterns. Specifically, the above system can be modeled as follows. Given a link with N parallel sub-channels as shown in Fig. 5, the codeword is spanned in the time domain over K blocks and also across all the N sub-channels. In some sub-channels, random attacks terminate the transmission before it is completed such that less than K blocks are delivered. For other sub-channels, K blocks are assumed to be safely transmitted. What is the maximum rate for reliable communication over such a link? For the single channel case, it turns out that there is no way to achieve arbitrarily small outage with a finite transmit power. However, in this chapter, we extend the results of the single dying channel to the parallel multi-channel case and show that an arbitrarily small outage probability is achievable by exploiting the inherent multi-channel diversity.

A. Outage Probability Definition

Definition 2. *The outage probability of the parallel multi-channel case is given as*

$$p_{out}(R, P, N) = \mathbb{P}\left[\sum_{i=1}^{N} \frac{1}{K} \sum_{k=1}^{L_i} \log(1 + \alpha_k^{(i)} P/N) < R\right], \tag{3.1}$$

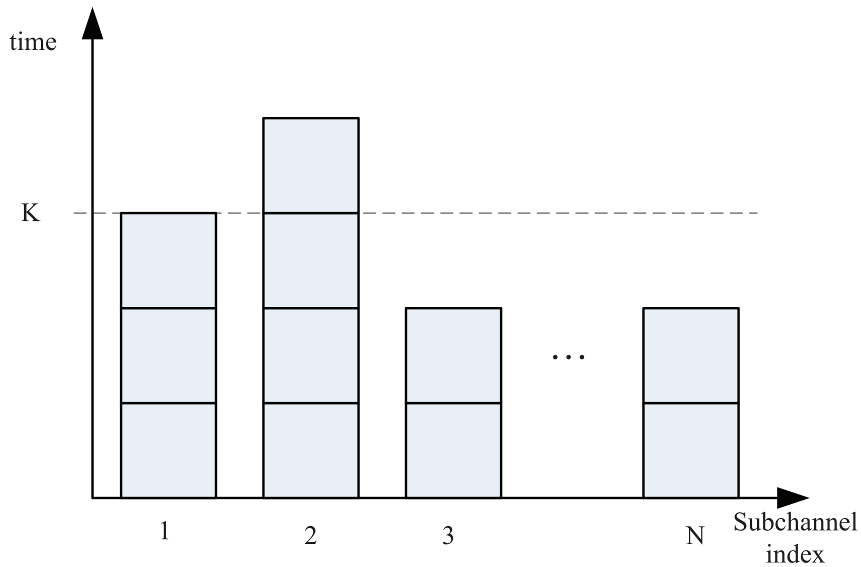

Fig. 5.: Parallel dying channels.

where R is the total rate over N sub-channels, $\alpha_k^{(i)}$ is the fading gain of block k at sub-channel i, N is the number of sub-channels, $L_i = \min\{K, \lfloor T_i \rfloor\}$ is the random number of surviving blocks at sub-channel i, K is the number of blocks over which a codeword is spanned in the time domain, and P is the total average power such that P/N is the average power for each sub-channel. Since the asymptotic behavior is concerned, uniform power allocation is assumed over N sub-channels. According to different attack models, in the next two sections we investigate the asymptotic behavior of the above outage probability in two cases: the independent random attack case and the m-dependent random attack case.

B. Independent Random Attack Case

Let the average power P be finite. Since $\log(1+x) \approx x$ if $|x| \ll 1$, when N is large, we rewrite (3.1) as

$$p_{out}(R, P, N) \approx \mathbb{P}\left[\frac{1}{N}\sum_{i=1}^{N}\frac{1}{K}\sum_{k=1}^{L_i}\alpha_k^{(i)}P < R\right]. \tag{3.2}$$

We assume that the fading gains $\alpha_k^{(i)}$'s are i.i.d., and let the random variable Y_i be

$$Y_i = \frac{1}{K}\sum_{k=1}^{L_i}\alpha_k^{(i)}.$$

For the case of independent random attack, we assume that L_i's are i.i.d., and hence Y_i's are i.i.d..

The outage probability given by (3.2) can be recast as:

$$p_{out}(R, P, N) \approx \mathbb{P}\left[\frac{1}{N}\sum_{i=1}^{N}Y_i < R/P\right]. \tag{3.3}$$

Motivated by the central limit theorem, we approximate $\frac{1}{N}\sum_{i=1}^{N}Y_i$ by a Gaussian random variable with the mean μ_Y and variance as σ_Y^2/N, where the mean and variance of Y_i are given as following.

According to Theorem 7.4 in [18] on the sum of a random number of random variables, we derive the following relations:

$$\begin{aligned} \mu_Y &= \frac{1}{K}\mathbb{E}(L)\mathbb{E}(\alpha) && (3.4)\\ \sigma_Y^2 &= \frac{1}{K^2}[\mathbb{E}(L)Var(\alpha) + Var(L)\mathbb{E}(\alpha)^2], && (3.5)\end{aligned}$$

where α is a nominal random variable denoting the fading gain, L is a nominal integer random variable denoting the number of surviving blocks of each sub-channel, and $\mathbb{E}(\cdot)$ and $Var(\cdot)$ denote the expectation and variance, respectively. As such, according to the Gaussian approximation, the outage probability can be approximated as:

$$p_{out}(R, P, N) \approx Q\left(\frac{\mu_Y - R/P}{\sigma_Y/\sqrt{N}}\right). \tag{3.6}$$

As $N \to \infty$, according to the law of large number, $\frac{1}{N}\sum_{i=1}^{N} Y_i$ converges to μ_Y. The outage probability decreases to 0 over N if R/P is less than μ_Y, or converges to 1 if R/P is larger than μ_Y.[1] That is, even though all sub-channels are subject to fatal attacks, the outage probability can still be made arbitrarily small when N is large enough if the rate per unit cost is set in a conservative fashion, where μ_Y is a key threshold. This is remarkably different from the single dying channel case in which the outage probability is always finite since there are only a finite and random number of blocks to span a codeword.

C. m-dependent Random Attack Case

In the previous section, we discussed the case where L_i's are independent. However, in a practical system, such as cognitive radio networks, the primary users usually use a bunch of adjacent sub-channels instead of dispersive sub-channels. Thus, the L_i's across adjacent sub-channels are possibly correlated; and consequently the achievable rates across adjacent sub-channels are also correlated. On the other hand, if two sub-channels are far away from each other, it is reasonable to treat them as independent. Thus, we assume that Y_i's are strictly stationary [2] and m-dependent[3] with the same

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[1] R/P is interpreted as the rate per unit cost in [19]. It is interesting to see that the quantity of rate per unit cost plays an important role here, which is due to the fact that we operate over both a finite power and a finite coding length.

[2] Call a sequence $\{X_n, n \geq 1\}$ strictly stationary if, for every k, the joint distribution of $(X_{n+1}, \cdots, X_{n+k})$ is independent of n.

[3] Call a sequence $\{X_n, n \geq 1\}$ m-dependent if for any integer t, the σ-fields $\sigma(X_j, j \leq t)$ and $\sigma(X_j, j \geq t+m+1)$ are independent. Simply put, X_i and X_j are independent if $|i-j| > m$.

mean and variance.

1. Central Limit Theorem for m-Dependent Random Variables

We first cite the central limit theorem for stationary and m-dependent summands from [20] (Theorem 9.1 therein).

Theorem 4 (Hoeffding and Robbins)**.** *Suppose $\{X_n, n \geq 1\}$ is a strictly stationary m-dependent sequence with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 < \infty$. Then as $N \to \infty$, we have*

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_i - \mu) \to \mathcal{N}(0, \upsilon_m), \tag{3.7}$$

where $\upsilon_m = \sigma^2 + 2\sum_{i=1}^{m} Cov(X_t, X_{t+i})$ with $Cov(X_t, X_{t+i})$ the covariance of X_t and X_{t+i}.

Proof. The detailed proof can be found in [21]. $\square$

2. Approximate Outage Probability

As assumed, the random sequence $\{Y_1, Y_2, \cdots, Y_N\}$ is stationary and m-dependent, and Y_i's have the same mean and variance. Then the covariance is given as:

$$\mathrm{Cov}(Y_i Y_{i+h}) = \begin{cases} 0 & |h| > m \\ \gamma(h) - \mu_Y^2 & |h| \leq m, \end{cases} \tag{3.8}$$

where μ_Y is the expectation of Y_i given in (3.4) and $\gamma(h) = E(Y_i Y_{i+h})$. Meanwhile,

$$\upsilon_m = \sigma_Y^2 + 2\sum_{h=1}^{m} \left(\gamma(h) - \mu_Y^2\right). \tag{3.9}$$

Due to the fact that the fading gains $\alpha_p^{(i)}$ and $\alpha_q^{(i+h)}$ are independent if $p \neq q$ or

$h \neq 0$, we could easily obtain $\gamma(h)$ for $|h| \leq m, h \neq 0$, as:

$$\begin{aligned}
\gamma(h) &= \frac{1}{K^2} E\left[\sum_{p=1}^{L_i} \alpha_p^{(i)} \sum_{q=1}^{L_{i+h}} \alpha_q^{(i+h)}\right] \\
&= \frac{1}{K^2} E\left[E\left(\sum_{p=1}^{L_i} \alpha_p^{(i)} \sum_{q=1}^{L_{i+h}} \alpha_q^{(i+h)} \middle| L_i L_{i+h}\right)\right] \\
&= \frac{\mu_\alpha^2}{K^2} E(L_i L_{i+h}) \qquad (3.10)
\end{aligned}$$

Assume that L_i and L_j have the same correlation coefficient ρ if $|i - j| \leq m$ and $i \neq j$. Then (3.10) is simplified as

$$\gamma(h) = \frac{\mu_\alpha^2}{K^2}(\rho\sigma_L^2 + \mu_L^2), \qquad (3.11)$$

where ρ is a non-negative correlation coefficient, μ_L and σ_L are the mean and variance of the random variable L, respectively.

Substituting (3.4), (3.5), and (3.11) into (3.9), we have

$$\begin{aligned}
\upsilon_m &= \sigma_Y^2 + 2m\frac{\rho\mu_\alpha^2\sigma_L^2}{K^2} \\
&= \frac{\mu_L\sigma_\alpha^2}{K^2} + \frac{\mu_\alpha^2\sigma_L^2}{K^2}(1 + 2m\rho). \qquad (3.12)
\end{aligned}$$

Motivated by the central limit theorem for m-dependent random sequence, we again approximate $\frac{1}{N}\sum_{i=1}^{N} Y_i$ by a Gaussian random variable with mean μ_Y and the variance υ_m/N, where μ_Y is given in (3.4) and υ_m is given in (3.12). Hence, the outage probability for the m-dependent random attack case can be written as follows,

$$p_{out}(R, P, N) \approx Q\left(\frac{\mu_Y - R/P}{\sqrt{\upsilon_m/N}}\right). \qquad (3.13)$$

As we see from (3.12) that $\upsilon_m \geq \sigma_Y^2$, comparing (3.6) and (3.13), we conclude that the outage probability of the independent attack case is smaller than that of the m-dependent case given the same setting when the rate per unit cost R/P is less than

μ_Y and the number of sub-channels N is large.

D. Outage Exponent for Parallel Dying Channels

As we learn from the previous sections, the outage probability over parallel multiple channels goes to zero as N increases if $R/P < \mu_Y$ for both of the two attack cases. In this section, we investigate how fast the outage probability decreases as N increases for both cases, which is measured by the outage exponent [22] defined as

$$\mathcal{E}(t) = \lim_{N\to\infty} \frac{-\log p_{out}(R,P,N)}{N}, \tag{3.14}$$

where $t = R/P$.

1. Independent Attack Case

According to the results in [22], we could derive the outage exponent for the independent attack case as

$$\mathcal{E}(t) = \sup_{s\le 0} \left\{ st - \Lambda(s) \right\}, \tag{3.15}$$

for $\forall t \le t_0$, where $t_0 = \mu_Y$ and

$$\Lambda(s) \quad := \quad \log \mathbb{E}\left[\exp(sY_i)\right] = \log M_Y(s), \tag{3.16}$$

with $M_Y(s)$ the moment generating function of Y_i. According to Theorem 7.5 in [18], we have $M_Y(s) = h(f(s/K))$ where $h(z)$ and $f(s)$ are the probability generating function of the discrete random variable L_i and the moment generating function of the continuous random variable $\alpha_k^{(i)}$, respectively. [4]

[4] The moment generating function of the sum of a random number of random variables, i.e., $S_L = X_1 + X_2 + \cdots + X_L$, is the compound function $h(f(s))$, where L is a random integer independent of X_i, $h(z)$ is the probability generating function of L, and $f(s)$ is the moment generating functions of X_i.

Example: If Rayleigh fading is assumed, $\alpha_k^{(i)}$ is exponentially distributed; hence the corresponding moment generating function is $f(s) = (1 - s/\lambda_\alpha)^{-1}$, where λ_α is the parameter for the distribution of the $\alpha_k^{(i)}$. Assuming that the random attack time has an exponential distribution, L is an integer random variable with following distribution:

$$w_0 = \mathbb{P}\left[0 \leq T < 1\right], w_1 = \mathbb{P}\left[1 \leq T < 2\right], \cdots, w_{K-1} = \mathbb{P}\left[K-1 \leq T < K\right], w_K = \mathbb{P}\left[T \geq K\right].$$

Thus, we have $h(z) = \sum_{i=0}^{K} w_i z^i$ and $M_Y(s) = \sum_{i=0}^{K} w_i (1 - s/(K\lambda_\alpha))^{-i}$. Then we can derive the outage exponent numerically by solving (3.15) for a given t.

2. m-Dependent Attack Case

For the m-dependent attack case, the large deviation result for the m-dependent random sequence is given as follows [23],

$$\mathcal{E}_{mdp}(t) = \sup_{u \leq 0} \left\{ tu - \Lambda(u) \right\}, \tag{3.17}$$

where

$$\Lambda(u) := \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}\left[\exp(uS_n)\right] \tag{3.18}$$

and $S_N = \sum_{i=1}^{N} Y_i$. As $N \to \infty$, according to Theorem 4, S_N is approximately equal to a Gaussian random variable of distribution $\mathcal{N}(N\mu, N\upsilon_m)$. Thus, we have

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E}[\exp(uS_N)] = u\mu + \frac{1}{2} u^2 \upsilon_m. \tag{3.19}$$

As a result, the outage exponent for the m-dependent attack case is given as

$$\mathcal{E}_{mdp}(t) = \sup_{u \leq 0} \left\{ u(t - \mu_Y) - \frac{1}{2} u^2 \upsilon_m \right\} = \frac{(\mu_Y - t)^2}{2\upsilon_m}, \tag{3.20}$$

where υ_m is given in (3.12).

Numerical results are provided here to validate our analysis for the parallel multi-channel case. We choose the random attack time T to be exponentially distributed with parameter $1/\lambda = 5$ and K is chosen to be 5. Rayleigh fading is assumed and the fading gain $\alpha_k^{(i)}$ is exponentially distributed with parameter 1 and the noise has unit power. First, we demonstrate the convergence of the outage probability for the independent attack case and the m-dependent attack case, where the value of μ_Y according to the above simulation setup is 0.571. For the independent attack case, as shown in Fig. 6, the solid and dashed curves are derived by (3.6) while the circles and crosses are obtained by simulations. We also observe similar convergence for the m-dependent attack case in Fig. 7. In both figures, the outage probability goes to 0 if $R/P < \mu_Y$, or goes to 1 if $R/P > \mu_Y$. We see that the accuracy of Gaussian approximations is acceptable with reasonably large N values.

Second, we compare the outage probability performance between the independent case and the m-dependent case. Here $P = 2$ and $R = 0.5$ nats/s. As shown in Fig. 8, the outage performance of the m-dependent case is worse than that of the independent case even when $m = 1$ and $\rho = 0.8$. This is due to the fact that when $R/P < \mu_Y$, the independent attack case is expected to have a smaller outage probability as we discussed at the end of Section 2. However, the outage probability of the m-dependent case still decreases to 0 but at a slower rate as the number of sub-channels N increases, which is caused by the fact that the m-dependent attack case has a smaller outage exponent.

In Fig. 9, we compare the various outage exponent values between these two cases over the rate per unit cost R/P with the simulation setup as follows: $K = 5$, $m = 1$, and $\rho = 0.8$. First, we see that the outage exponent for the independent attack case is larger than that of the m-dependent attack case when the average attack time $1/\lambda$ is the same. Second, for both of the independent attack case and the

m-dependent attack case, a larger average attack time $1/\lambda$ results in a larger outage exponent.

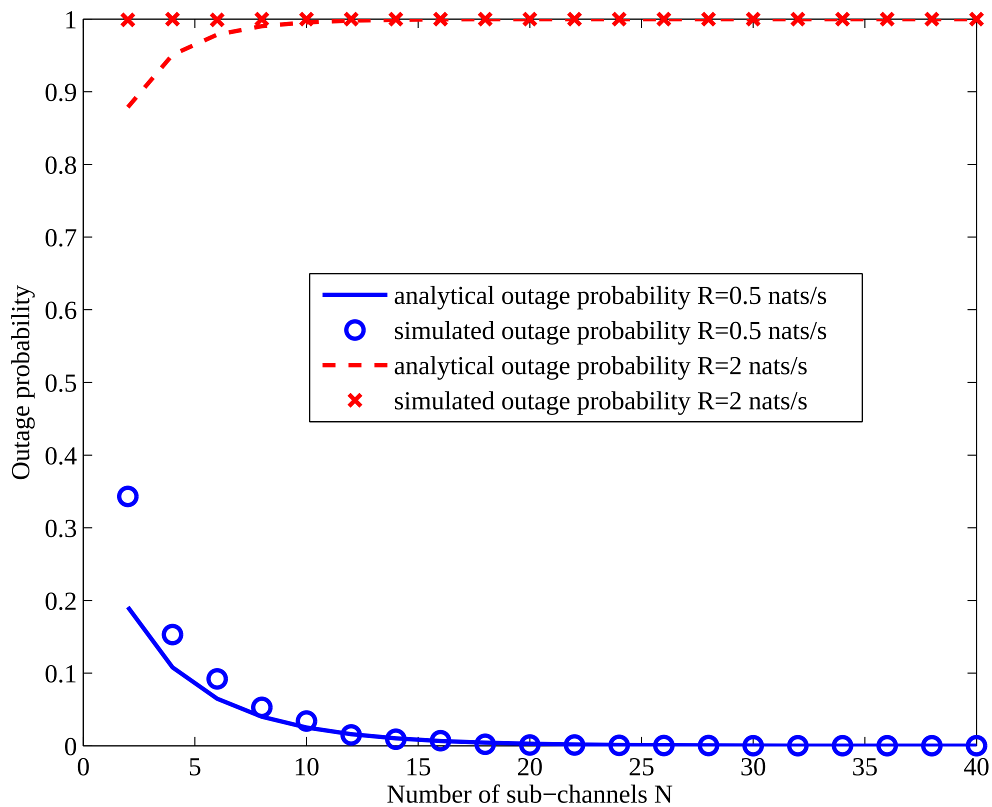

Fig. 6.: Outage probability convergence behavior of the independent case: μ_Y=0.571 and P=2.

E. Summary

In this chapter, we extended the results in single dying channel to the parallel dying channel. First, we give the outage probability definition for the parallel dying channel and then investigate asymptotic behavior of the outage probability as the number of sub-channels increases. In particular, we consider the independent random at-

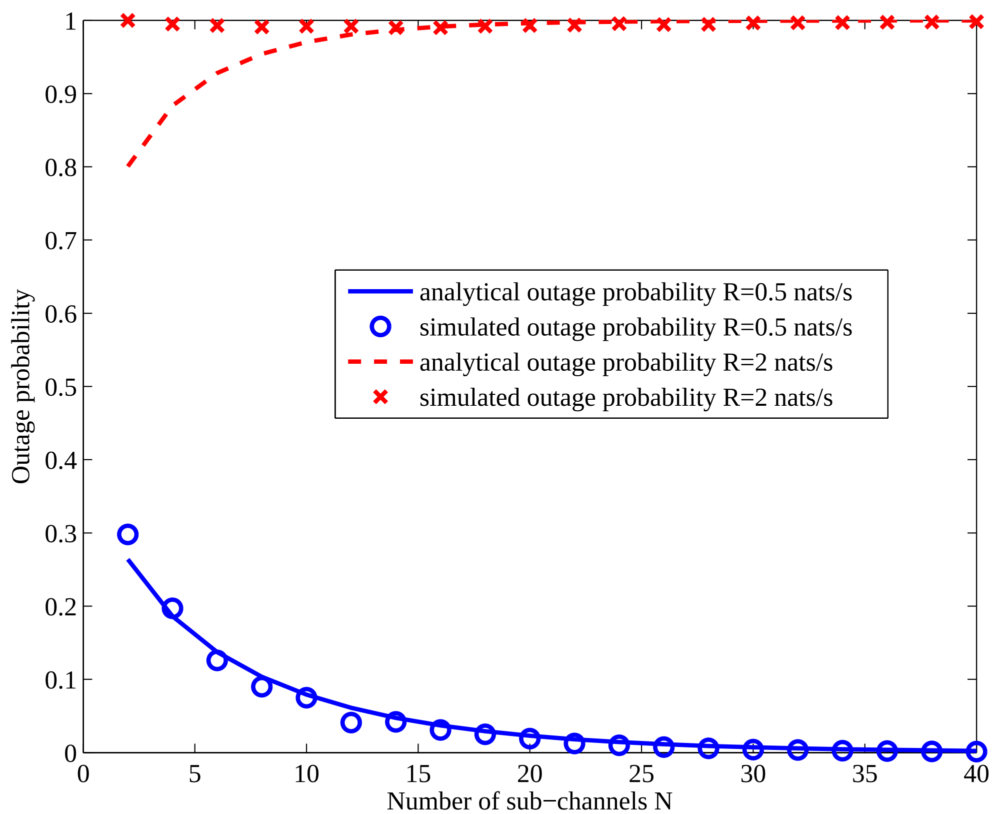

Fig. 7.: Outage probability convergence behavior of the m-dependent case: μ_Y=0.571, m=1, ρ=0.8, and P=2.

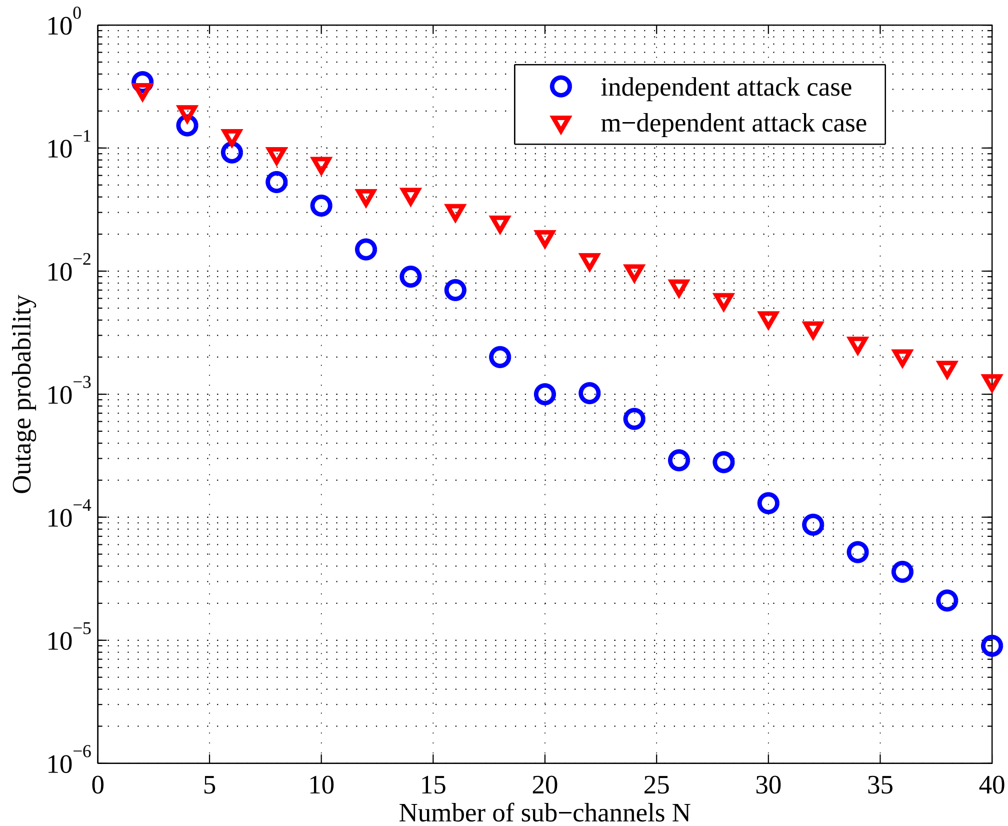

Fig. 8.: Outage probabilities comparison. P=2, R=0.5 nats/s, m=1, and ρ=0.8.

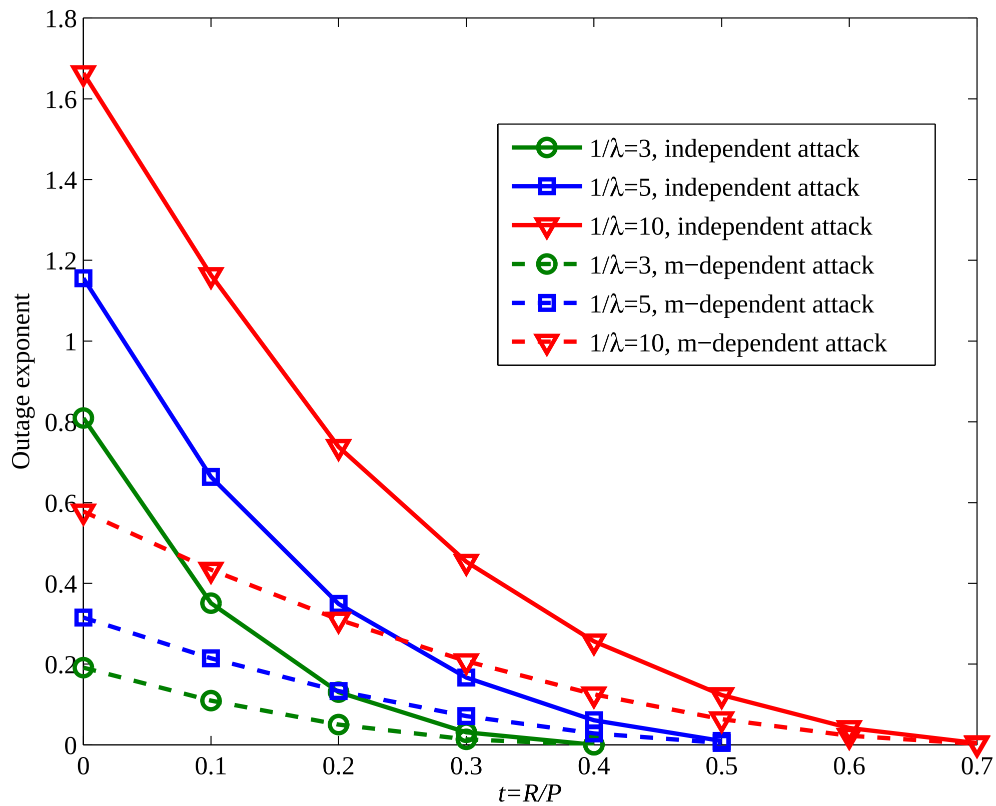

Fig. 9.: Outage exponents for independent and m-dependent random attack cases: m=1, ρ=0.8, and K=5.

tack model and the m-dependent random attack model. For both cases, the outage probability will go to zero if the rate per unit cost is larger than a given threshold. In addition, the outage exponents are studied for both cases to reveal how fast the outage probability goes to zero.

CHAPTER IV

ON DESIGN OF RATELESS CODES OVER DYING BINARY ERASURE CHANNEL

In this chapter, we consider a practical coding scheme over the dying binary erasure channel (DBEC). In particularly, we consider transmitting a trunk of data over a DBEC with a large message length k. Since reliable communication is not achievable over the dying channel, our objective is to convey as many information bits[1] as possible. For a given message length k, the problem is equivalent to maximizing the recovery probability of the received bits. Since the transmitter has no knowledge on the actual channel length before transmitting, it cannot pre-determine the rate and the associated codeword length. Therefore, the fixed-rate codes are not suitable. Instead, the rateless codes [24–26] could serve as good candidates. In 1974, the concept of incremental redundancy (IR) was proposed in [24]. Afterwards, people have been trying to combine the IR concept with error control codes to fully exploit each received symbol at the receiver. The resulting codes are called rateless codes, which are characterized by a continuous stream of coded bits generated from a single fixed-length message. Specifically, the transmitter continuously produces and transmits such coded bits until the receiver collects enough information to reliably recover the original message. In particular, the Luby Transform (LT) codes [25] and Raptor codes [26] are two most well-known rateless codes, where they are both designed to recover the whole frame of message bits with high probability when the coding overhead r, which is defined as the ratio between the number of received bits and the number of message bits k, is just slightly larger than 1. However, when r is less

[1] In this dissertation, we assume the symbol to be binary without loss of generality.

than the required, only a small fraction of message bits can be recovered. As a result, the author of [27] investigated the asymptotic recovery probability when $r < 1$ and provided a tight outer bound on the fraction of recoverable bits. In addition, the authors in [28] also studied the intermediate performance and characterized the Pareto tradeoff. In the above works, the authors are only interested in the region over $r < 1$ and try to maximize the recovery probability for a *fixed* r. In this dissertation, we consider the case where r is a *random variable*, which can be either less or larger than 1 due to the random attack. Our objective is to seek a *single* degree distribution that is tuned to the statistics of the dying channel such that the average recovery probability is maximized.

A. System Model

In general, a DBEC is a special binary erasure channel that has a finite but random number of channel uses, as opposed to the ordinary BEC that has an infinite number of channel uses. We define the life span of a DBEC as its channel length, denoted as T, which is a random variable. We assume that the probability distribution of the life span is known *a priori* by both the transmitter and the receiver. All the channel realizations that have the same life span differ from each other by the number of erasures and the positions where the erasures occur. We now consider rateless codes and define following preliminaries.

Let $\Omega_1, \cdots, \Omega_D$ be a probability distribution such that Ω_i denotes the probability that the integer i is chosen and $\sum_{i=1}^{D} \Omega_i = 1$, where D is the largest degree for the degree distribution used for the rateless codes. We could represent such a distribution by the coefficients of the generator polynomial $\Omega(x) = \sum_{i=1}^{D} \Omega_i x^i$. Suppose we encode k message bits with the degree distribution specified by $\Omega(x)$. Each coded bit is generated as follows.

- Randomly choose a degree realization d from a degree distribution specified by $\Omega(x)$.
- Choose uniformly at random d distinct input symbols as neighbors.
- The value of the encoding symbol is the exclusive-or of the d chosen neighbors.

Repeat the above procedure, we will obtain a string of coded bits.

The relationship between the message bits and coded bits can be represented by a bipartite graph, where the message bits are represented by variable nodes and the generated coded bits are represented by check nodes. The chosen variable nodes for each check node are called the neighbors of that check node. A particular encoding process can be specified by a generator matrix $\mathbf{G}$ with k rows and t columns such that a codeword $\mathbf{c}$ of length t can be represented as $\mathbf{c} = \mathbf{mG}$, where $\mathbf{m}$ is a row vector containing k message bits, the matrix element $G_{i,j} = 1$ if the i-th message bit is chosen as the neighbor of the coded bit j, and otherwise $G_{ij} = 0, i = 1, \cdots, k, j = 1, \cdots, t$. The number of 1's in each column of $\mathbf{G}$ is called weight, and the weight distribution is determined by $\Omega(x)$. Given $\Omega(x)$, we denote $\mathcal{G}_{\Omega(x)}$ as the ensemble of generator matrices $\mathbf{G}$'s whose weight distribution is specified by $\Omega(x)$.

LT codes are decoded by applying the belief-propagation (BP) decoder [29] to N of the output symbols of the encoder. The required value of N depends on the output degree distribution of the LT codes and the desired success probability of the decoder. According to the results in [25], if k message bits are encoded and transmitted over the ordinary BEC, any subset of symbols of size $k + O(\sqrt{k}\ln^2(k/\delta))$ is sufficient to recover the original k message bits successfully with a probability at least $1 - \delta$ (In fact, the actual decoding failure probability is much smaller than δ, which is suggested by Luby's conservative analysis [30].).

We assume that the encoder and decoder share the same random number generator such that the random number d and the indices of the randomly chosen neighboring message bits for each coded bits are known by both the encoder and decoder. The transmitter keeps sending the coded bits and the receiver keeps receiving the bits without feedback. At the moment that the channel dies, the receiver detects the event and starts to decode the message bits. In particular, the actual number of transmitted coded bits before the channel dies, denoted as $\mathbf{N}_1$, is a random variable equal to the life span of the channel realization. In addition, the actual number of received coded bits, denoted as $\mathbf{N}_2$, is also a random variable less than or equal to $\mathbf{N}_1$ due to the erasures that occurred during the transmissions. For a given message length k, the received coding overhead $\mathbf{R}$ is defined as $\mathbf{R} = \mathbf{N}_2/k$. Subsequently, we define the following quantities.

Definition 3. *Let n_d be the number of recoverable bits. The recovery probability conditioned on the degree distribution $\Omega(x)$ and $\mathbf{R} = r$ is defined as*

$$s(r, \Omega(x)) \triangleq \mathop{\mathbb{E}}_{\mathcal{G}_{\Omega(x)}} \left[\frac{n_d}{k} | \mathbf{R} = r, \Omega(x) \right]. \tag{4.1}$$

The degree d for each coded bit is randomly generated according to the degree distribution $\Omega(x)$. In addition, the neighboring message bits for each coded bits are randomly chosen. Therefore, all the realizations of d and random choices of neighboring message bits form the ensemble of random graphs, denoted as $\mathcal{G}_{\Omega(x)}$. Taking expectation over the ensemble, we have the conditional recovery probability $s(r, \Omega(x))$. Furthermore, by taking expectation over $\mathbf{R}$, we have the expected recovery probability as follows.

Definition 4. *The average recovery probability $\eta(\Omega(x))$ for a given degree distribution*

$\Omega(x)$ is defined as

$$\eta(\Omega(x)) \triangleq \mathbb{E}\left[s(\mathbf{R}, \Omega(x))\right] = \int_r h(r)s(r, \Omega(x))dr, \tag{4.2}$$

where $h(r)$ is probability density function of $\mathbf{R}$.

Note that the underlying message length is k, such that different choices of k lead to different probability distributions of r and thus yield different results. In this dissertation, we would like to find a degree distribution $\Omega(x)$ to maximize the average recovery probability at the decoder for a fixed message length k, i.e.,

$$\begin{aligned} \max_{\Omega(x)} \quad & \eta(\Omega(x)) \\ \text{s.t.} \quad & \sum_{i=1}^{D} \Omega_i = 1. \end{aligned} \tag{4.3}$$

B. Degree Distribution Optimization

In order to maximize $\eta(\Omega(x))$, we first find an upper bound and then minimize the gap between the upper bound and the achievable $\eta(\Omega(x))$. The upper bound can be obtained by individually maximizing the conditional recovery probability $s(r, \Omega(x))$ for each given r and then taking expectation over $\mathbf{R}$.

1. Upper Bound of Recovery Probability

Here we consider the asymptotic case with $k \to \infty$ for the ease of analysis, since the message length k in consideration is assumed to be large. Such an asymptotic approximation leads to acceptable performance as later shown in Section C.

Definition 5. *Let the asymptotical maximum recovery probability for a given r be*

$$z = \lim_{k\to\infty} \max_{\Omega(x)} s(r, \Omega(x)), \tag{4.4}$$

where the corresponding optimal $\Omega(x)$ is denoted as $\Omega_C(x)$.

As we see that z is the limit [2] of the maximum value of $s(r, \Omega(x))$ for a given r as $k \to \infty$. According to [27], the relationship between r, z, and the achieving degree distribution $\Omega_C(x)$ is given as follows.

Lemma 1. *1. When $z \in [0, 1/2)$, $r = -\log(1-z)$ and $\Omega_C(x) = x$, where $\log$ is the natural logarithm.*

2. When $z \in [1/2, 2/3]$, $r = \frac{-\log(1-z)}{2z}$ and $\Omega_C(x) = x^2$.

3. When $z \in (2/3, 1)$, let m be an integer such that $\frac{m-1}{m} < z < \frac{m}{m+1}$, then

$$r = \frac{m-1}{m} + \frac{1}{mz^{m-1}} \sum_{i \geq m} \frac{z^i}{i}. \tag{4.5}$$

The coefficients of $\Omega_C(x)$ are given as

$$\Omega_i = \begin{cases} \frac{1}{ri(i-1)} & 2 \leq i \leq m-1 \\ 1 - \frac{m-2}{r(m-1)} & i = m \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

Proof. The proof is given in [27]. □

Proposition 1. *$r(z)$ given in Lemma 1 is a montonically increasing function of z for $0 \leq z < 1$.*

Proof. The proof is given in Appendix E. □

According to Proposition 1, there exists a bijective mapping between r and z. Therefore, we can instead view z as a monotonically increasing function $z = u(r)$ for $r \in [0, 1)$. In addition, when $r \geq 1$, we have $z = 1$ and the corresponding $\Omega_C(x)$ is

[2] It is shown in [31] that the limit is well-defined, i.e., the limit exists.

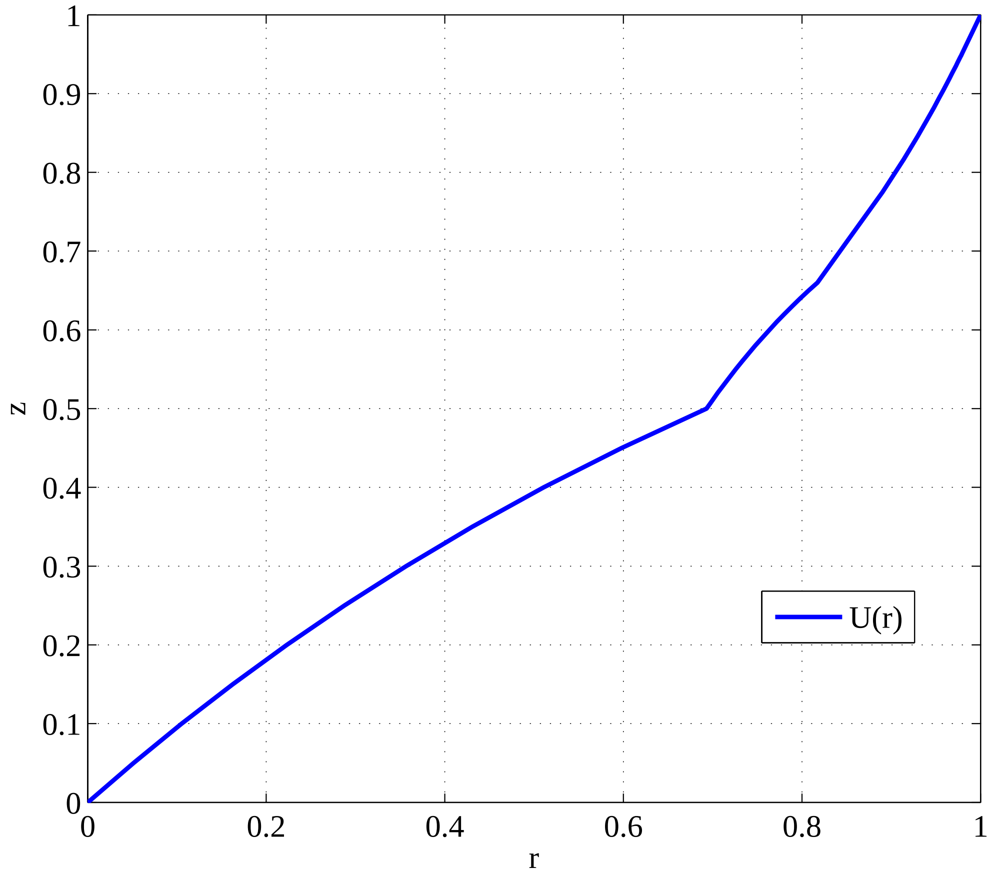

Fig. 10.: Demonstration of Lemma 1 for $0 \leq r \leq 1$.

the soliton distribution [25], which is given as following.

$$\Omega_i = \begin{cases} 1/k & i = 1 \\ \frac{1}{i(i-1)} & i = 2, \cdots k \end{cases} .$$

Therefore, the overall function is given as

$$z = U(r) = \begin{cases} u(r) & r \in [0, 1) \\ 1 & r \geq 1 \end{cases} , \tag{4.7}$$

which is given in Fig. 10.

As a result, for a given $\mathbf{R} = r$, there is a unique maximum recovery probability z and a corresponding optimal degree distribution $\Omega_C(x)$ to achieve z. If we know r non-causally before transmission, we can determine the optimal degree distribution according to Lemma 1; Hence such an non-causal assumption gives us the upper bound for $\eta(\Omega(x))$ as

$$g_1 = \int_r h(r)U(r)\mathrm{d}r, \tag{4.8}$$

where $h(r)$ is probability density function of $\mathbf{R}$. In practice, it is impossible to know r before transmission over the DBEC. In addition, no single degree distribution can achieve $U(r)$ over the whole region. Thus g_1 is not achievable with a single degree distribution in general except for some special cases where $\mathbf{R}$ follows some special probability distributions. For example, if $\Pr\{0 \leq r \leq \log 2\} = 1$, $\Omega(x) = x$ achieves g_1.

2. Optimizing Degree Distribution

The achievable average recovery probability for the rateless codes with a given degree distribution $\Omega(x)$ is

$$g_2 = \int_r h(r)s(r, \Omega(x))\mathrm{d}r. \tag{4.9}$$

Consequently, by taking (4.8) and (4.9) into account, the gap to the upper bound is given as

$$g(\Omega(x)) = \int_r h(r)\Big(U(r) - s(r, \Omega(x))\Big)\mathrm{d}r. \tag{4.10}$$

Therefore, maximizing $\eta(\Omega(x))$ can be transformed into the following optimization problem:

$$\begin{aligned} \min_{\Omega(x)} \quad & g(\Omega(x)) \\ \text{s.t.} \quad & \sum_{i=1}^{D} \Omega_i = 1, \Omega_i \geq 0, i = 1, \cdots, D. \end{aligned} \tag{4.11}$$

It is easy to see in (4.10) that the distance $U(r) - s(r, \Omega(x))$ at r has a large weight if $h(r)$ is large and vice versa. As such, in order to minimize (4.10), we choose $\Omega(x)$ to make the $s(r, \Omega(x))$ close to and even achieving the corresponding upper bound $U(r)$ in the regions with high $h(r)$ values and allow it to be relatively far away from $U(r)$ in other regions. In order to solve (4.11) efficiently, we need to evaluate $s(r, \Omega(x))$ without extensive Monte-Carlo simulations. According to the results on AND-OR tree analysis in [32] and [33], we have the following lemma.

Lemma 2. *Let α be the convergent value of the following iteration:*

$$y_l = \delta(1 - \beta(1 - y_{l-1})), \tag{4.12}$$

where $\beta(x) = \frac{\Omega'(x)}{\Omega'(1)}$ and $\delta(x) = e^{\mu r(x-1)}$ with $\mu = \Omega'(1)$ and $y_0 = 1$. Then $s(r, \Omega(x)) = 1 - \alpha$.

Proof. Given the proof in [33], this result is staightfoward. $\square$

It has been shown in [33] that the sequence $\{y_l\}$ is convergent with respect to the number of decoding iterations l. Note that the AND-OR tree analysis is also based on the assumption that $k \to \infty$. However, in our simulations, it is shown that the recovery probability obtained by the AND-OR tree analysis accurately matches the recovery probability obtained by the belief-propagation (BP) decoder for finite but large enough k.

Remark 1

Since $s(r, \Omega(x))$ does not have a closed-form expression and can only be evaluated numerically as in (4.12), in order to compute the integral in (4.10), we adopt the Riemann sum to approximate the integral. Specifically, we first limit the range of r to $[0, R]$ such that $\Pr(r > R)$ is negligible, and then partition the range of $[0, R]$ into N equal intervals of length Δr. Let $p_i = h(\hat{r}_i)\Delta r$ for $i = 1, \cdots, N$, where $\hat{r}_i$ is the mid-point of interval i. After determining p_i's, we normalize them and obtain the weight $w_i = p_i / \sum_i p_i, i = 1, \cdots, N$.

Now (4.10) can be rewritten as

$$\hat{g}(\Omega(x)) = \sum_{i=1}^{N} w_i \Big(U(\hat{r}_i) - S(\hat{r}_i, \Omega(x)) \Big), \tag{4.13}$$

where the upper-bound value of $U(\hat{r}_i)$'s can be determined beforehand. Subsequently, (4.11) is recast as:

$$\begin{aligned} \min_{\Omega(x)} \quad & \hat{g}(\Omega(x)) \\ \text{s.t.} \quad & \sum_{i=1}^{D} \Omega_i = 1, \Omega_i \geq 0, i = 1, \cdots, D. \end{aligned} \tag{4.14}$$

It is easy to see that $s(\hat{r}_i, \Omega(x))$ is a nonlinear function over Ω_i's since $\delta(x) = e^{\mu r(x-1)}$ in (4.12). The problem (4.14) is thus a constrained nonlinear optimization problem and generally non-convex, where the sequential quadratic programming (SQP) technique [34] can be applied to find a locally optimal solution. The numerical results in Section C show that a significant performance improvement can be achieved over the conventional LT codes even with the locally optimal solutions.

Remark 2

In the case that it is difficult to find the exact probability distribution of r, we could make the following simplifications. Suppose that the channel life span is N_1, and the number of received bits N_2 follows a binomial distribution given N_1. Then we have $\Pr\{N_2 = n|N_1\} = \binom{N_1}{n}(1-\epsilon)^n \epsilon^{N_1-n}$, where the ϵ is the erasure probability. When N_1 is large, the binomial distribution can be approximated as a Gaussian distribution [35]: $\mathcal{N}(N_1(1-\epsilon), N_1(1-\epsilon)\epsilon)$. Therefore, N_2 falls out of the region $[N_1(1-\epsilon) - 3\sqrt{N_1(1-\epsilon)\epsilon}, N_1(1-\epsilon) + 3\sqrt{N_1(1-\epsilon)\epsilon}]$ with a very small probability. In addition, when N_1 is large, we have $\sqrt{N_1(1-\epsilon)\epsilon} \ll N_1(1-\epsilon)$, which implies that the number of received bits concentrates around the value $N_1(1-\epsilon)$. Therefore, we could approximate the number of received bits as $N_2 = (1-\epsilon)N_1$; and consequently the distribution of $r = N_1(1-\epsilon)/k$ can be easily derived from the distribution of N_1 by scaling. In addition, although r is just of rational values since the number of received bits and the message length k are both integers, when k is large, we could approximately treat r as a continuous random variable.

3. Linearly Mixed Degree Distributions

Since exactly solving (4.14) involves nonlinear optimization, which could be complicated, we now seek a simple heuristic approach that can exploit the key feature of the dying channel, which is the randomness of the coding overhead r. Since different regions of r lead to different optimal degree distributions that optimize the conditional recovery probability, we propose the following simple approach to linearly mix three basic probability-weighted degree distributions out of three non-overlapping regions that divide the domain of r.

Let $w_1 = \Pr\{0 \le r < \log 2\}$, $w_2 = \Pr\{\log 2 \le r < 1\}$, and $w_3 = \Pr\{r \ge 1\}$. Correspondingly, we consider three degree distributions $\Omega^{(1)}(x) = x$, $\Omega^{(2)}(x) = x^2$, and $\Omega^{(3)}$ being the soliton distribution. Then we linearly combine $\Omega^{(1)}(x)$, $\Omega^{(2)}(x)$, and $\Omega^{(3)}(x)$ with weight vector $\mathbf{w} = (w_1, w_2, w_3)$, i.e.,

$$\Omega_L(x) = \sum_{i=1}^{3} w_i \Omega^{(i)}(x). \tag{4.15}$$

Note that when $3/4 \log 3 < r < 1$, the optimal degree distribution is not $\Omega(x) = x^2$ but in a complicated form as given in (4.6). However, as shown in [27], the conditional recovery probability $s(\Omega(x), r)$ with $\Omega(x) = x^2$ is at most 20% smaller than the corresponding upper bound in the region $3/4 \log 3 < r < 1$. Hence, for the sake of simplicity, we use $\Omega(x) = x^2$ as the degree distribution for the whole region $\log 2 \leq r < 1$.

Intuitively, by using the corresponding probability as weight, we assign a larger weight to the degree distribution $\Omega^{(i)}(x)$ that is close to the optimal one. Consequently, we can take advantage of the knowledge over the statistics of the DBEC. This method does not require solving any optimization problem and hence enjoys a high simplicity. Although this approach is heuristic, the numerical results in Section C show that it could achieve a significant performance improvement over the conventional LT codes.

C. Simulation Results

In the simulation, we evaluate the average recovery probabilities for rateless codes with the optimized degree distribution $\Omega(x)^*$ obtained by solving (4.14) via SQP (labeled as "Optimized $\Omega^*(x)$"), the linearly mixed degree distribution $\Omega_L(x)$ obtained by (4.15) (labeled as "Linearly mixed $\Omega_L(x)$"), and the robust soliton distribution (i.e., the conventional LT codes, labeled as "LT codes"). For each case, we simulate the average recovery probabilities by using the AND-OR tree analysis and the BP decoder, respectively.

We start with the case where the channel life span N_1 follows an exponential distribution with a mean value of 1×10^4 and the channel erasure probability ϵ is 0.2. Since the life span of the electrical devices typically follows the exponential distribution, the exponential life span of DBEC models the scenario where the channel death is caused by the device failures. First, we present the average recovery probabilities $\eta(\Omega(x))$ for the rateless codes with $\Omega^*(x)$'s and $\Omega_L(x)$'s for different k's over 200 channel realizations. As shown in Fig. 11, the average recovery probabilities given by the AND-OR analysis and the BP decoder match very well, which shows that our previous analysis, which is based on the assumption of asymptotically large k, is valid. Second, the performance of the rateless codes with $\Omega^*(x)$ and $\Omega_L(x)$ are significantly better than that of the conventional LT codes. As k increases, the performance gain becomes more and more significant. In addition, the average recovery probabilities with $\Omega^*(x)$ and $\Omega_L(x)$ are close to the upper bound, which is obtained by non-causally knowing the received coding overhead r. At last, we plot the conditional recovery probabilities $s(r, \Omega^*(x))$ in Fig. 12 over different message length k's. In addition, the optimal $\Omega^*(x)$'s are reported in Table I. As we see, when $k = 1 \times 10^3$, we have $\Pr\{r \geq 1\} = 0.8825$. Thus, $s(r, \Omega^*(x))$ is very close to $U(r)$ for $r \geq 1$ and $\Omega^*(x)$ is similar to the soliton distribution, which is the optimal degree distribution for $r \geq 1$. On the other hand, when $k = 1.8 \times 10^4$, $\Omega^*(x)$ changes such that the conditional recovery probability $s(r, \Omega^*(x))$ is very close to $U(r)$ in the region $0 \leq r \leq \log 2$, since $\Pr\{0 \leq r \leq \log 2\} = 0.7883$. From the above results, we see that the optimal degree distribution of the rateless codes has been indeed tuned to the statistics of the DBEC.

In addition, we also examine the case where the channel life span follows a truncated Gaussian distribution, which models the scenario where the channel death is caused by a random attack. Note that we restrict the channel life span to be nonnegative integers and hence it only approximately follows a truncated and quantized Gaussian distribution. We assume the channel life span $N_1 \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 1 \times 10^4, \sigma = 4 \times 10^3$. We also plot the average recovery probabilities for the conventional LT codes and the rateless codes with $\Omega^*(x)$ and $\Omega_L(x)$, respectively. As we see from Fig. 13, significant performance gains are achieved over the conventional LT codes. In addition, the performances with $\Omega^*(x)$ and $\Omega_{L(x)}$ are both close to the upper bound. The $\Omega^*(x)$'s with different k's are reported in Table II and the corresponding $s(r, \Omega^*(x))$ is plotted in Fig. 14. The results are similar to the case of exponentially distributed channel life span.

D. Summary

In this chapter, a practical code was proposed to deliver a large trunk of data over the dying binary erasure channel. The rateless coding scheme was adopted and the degree distribution $\Omega(x)$ was optimized to maximize the average recovery probability given the statistics of the DBEC. We first defined the objective function as the gap between the average recovery probability upper bound and the average recovery probability induced by a particular degree distribution $\Omega(x)$. Then the optimal degree distribution was sought by minimizing the objective function. An efficient heuristic approach was also proposed to reduce the implementation complexity by linearly mixing three basic degree distributions. Simulation results were presented to show that we can achieve a significant performance gain over the conventional LT codes in terms of the average recovery probability.

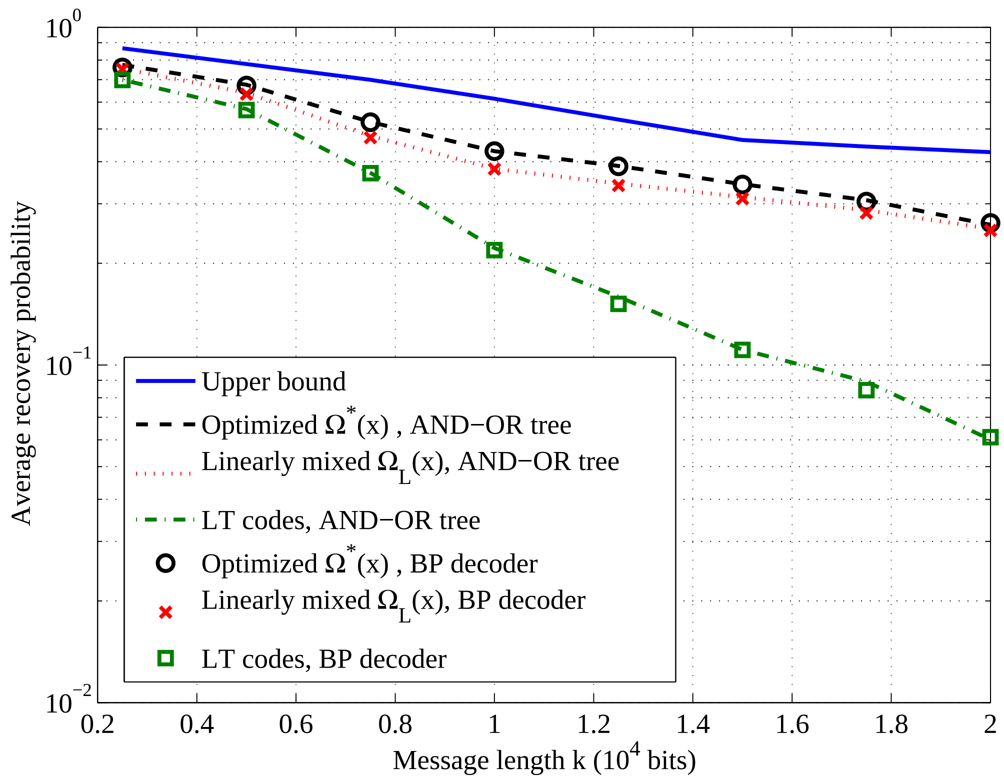

Fig. 11.: Average recovery probability: optimized degree distribution, linearly mixed degree distribution, and conventional LT codes with robust soliton distribution; channel life span follows an exponential distribution with mean 1×10^4.

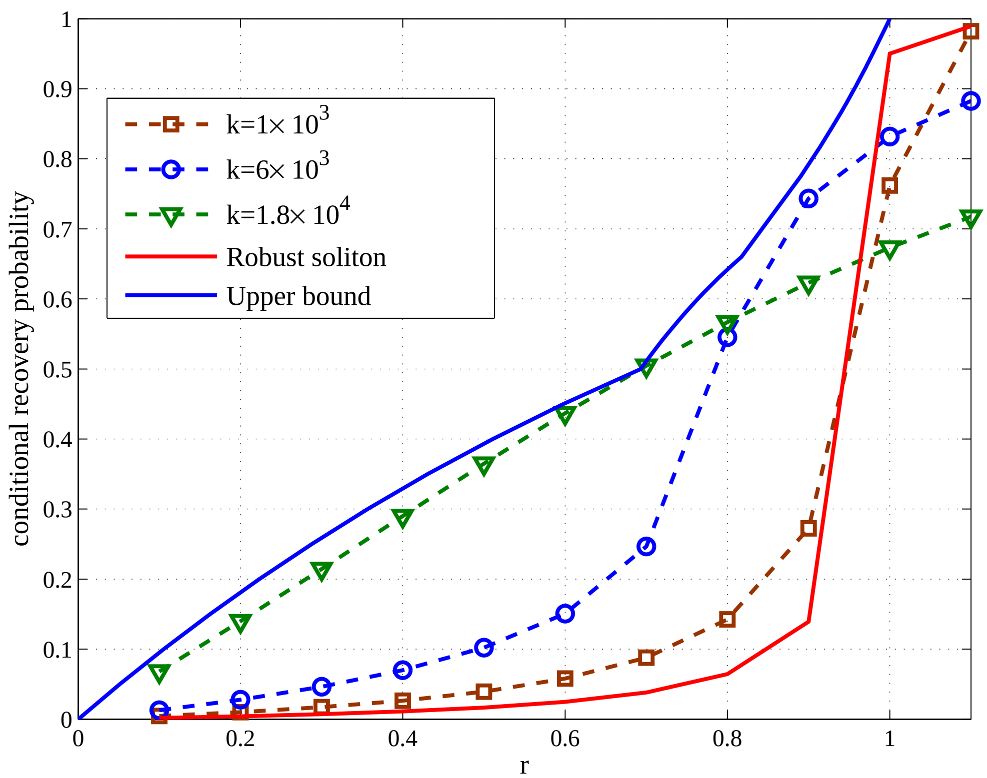

Fig. 12.: Conditional recovery probabilities induced by the optimized degree distributions for message lengthes $k = 1 \times 10^3, 6 \times 10^3,$ and 1.8×10^4; channel life span follows an exponential distribution with mean 1×10^4.

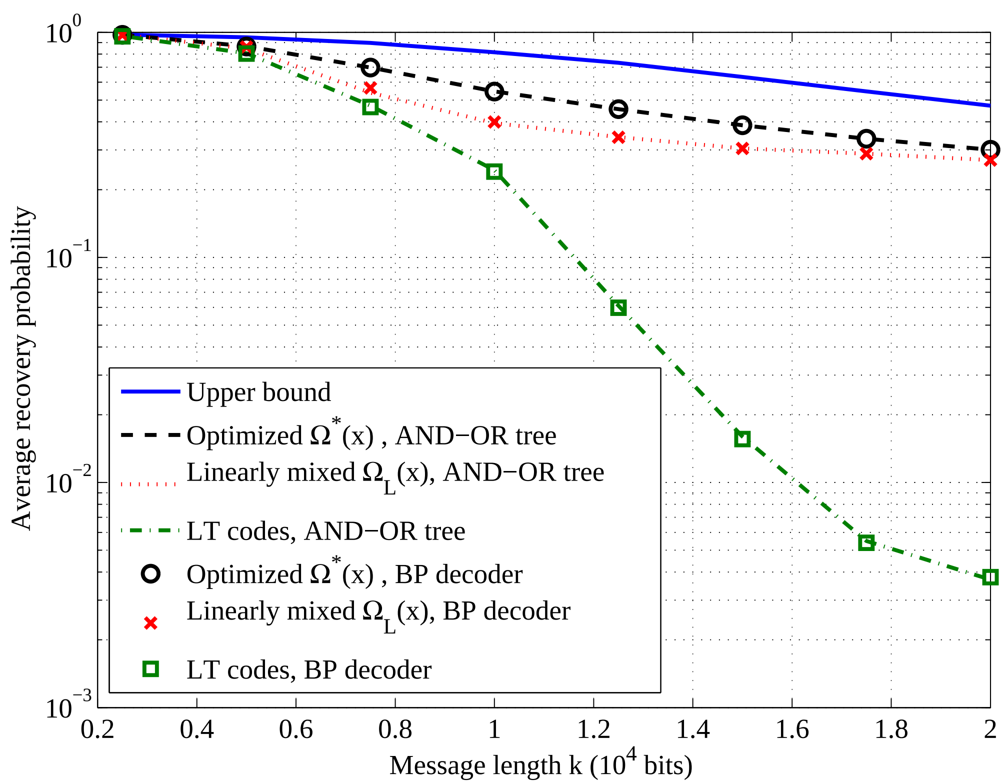

Fig. 13.: Average recovery probability: optimized degree distribution, linearly mixed degree distribution, and conventional LT codes with robust soliton distribution; channel life span follows a Gaussian distribution $\mathcal{N}(1 \times 10^4, (4 \times 10^3)^2)$.

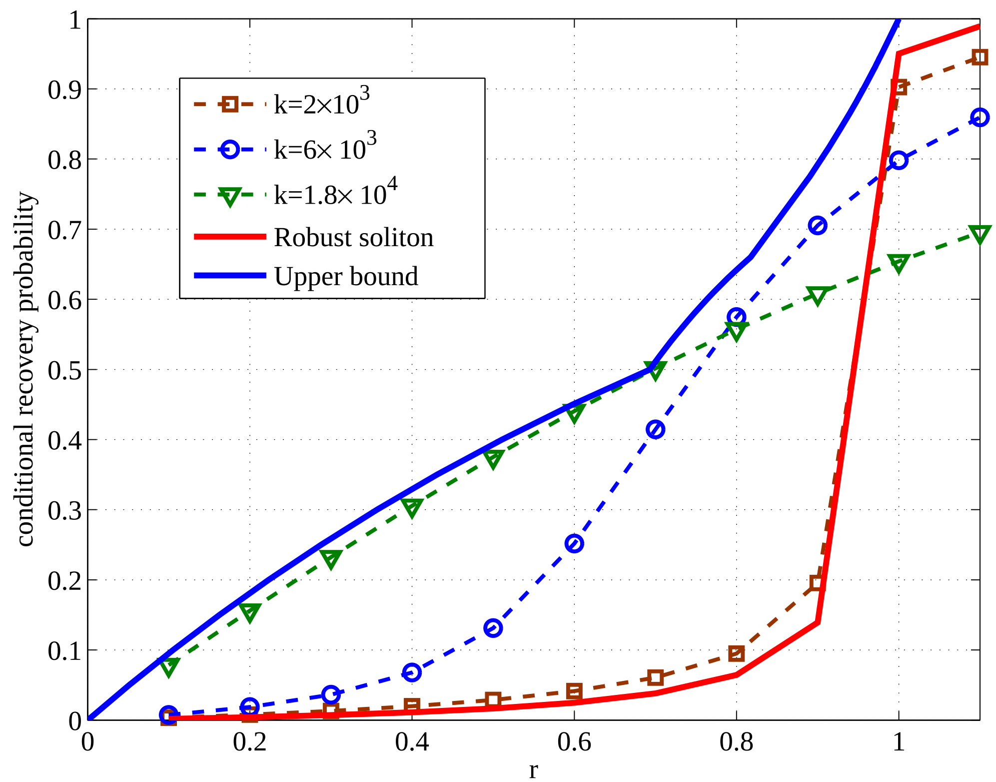

Fig. 14.: Conditional recovery probabilities induced by the optimized degree distributions for message lengthes $k = 1 \times 10^3, 6 \times 10^3,$ and 1.8×10^4; channel life span follows a Gaussian distribution $\mathcal{N}(1 \times 10^4, (4 \times 10^3)^2)$.

Table I.: $\Omega(x)^*$ for different message length k over exponentially distributed channel life span

	$\Omega^*(x)$
$k = 1 \times 10^3$	$0.0412x + 0.4937x^2 + 0.1610x^3 + 0.0784x^4 + 0.0457x^5$ $+0.0297x^6 + 0.0208x^7 + 0.0157x^8 + 0.0128x^9 + 0.0108x^{10} + 0.0096x^{11}$ $+0.0088x^{12} + 0.0084x^{13} + 0.0083x^{14} + 0.0083x^{15} + 0.0085x^{1}6 + 0.0089x^{17}$ $+0.0093x^{18} + 0.0098x^{19} + 0.0103x^{20}$
$k = 2 \times 10^3$	$0.1177x + 0.4050x^2 + 0.4773x^3$
$k = 6 \times 10^3$	$0.3727x + 0.6273x^2$
$k = 1.0 \times 10^4$	$0.6596x + 0.2343x^2 + 0.0542x^3 + 0.0518x^4$
$k = 1.8 \times 10^4$	$0.6582x + 0.3418x^2$

Table II.: $\Omega(x)^*$ for different message length k over Gaussian distributed channel life span

	$\Omega^*(x)$
$k = 2 \times 10^3$	$0.032x + 0.4471x^2 + 0.2738x^3 + 0.1111x^4 + 0.0028x^5$ $0.0728x^6 + 0.0538x^7 + 0.2738x^3 + 0.0002x^8$
$k = 6 \times 10^3$	$0.06x + 0.8323x^2 + 0.0577x^5 + 0.0472x^6$
$k = 1.0 \times 10^4$	$0.2204x + 0.7796x^2$
$k = 1.8 \times 10^4$	$0.7838x + 0.2087x^2 + 0.0043x^6 + 0.0032x^7$

CHAPTER V

CONCLUSION

In this dissertation, we proposed the dying channel and studied its outage probability. In Chapter II, we started with the single dying channel. We first introduced the system model and defined the outage probability as performance measure. Then we investigated the outage probability by assuming the uniform power allocation and fixed coding length. Lower and upper bounds for the outage probability were obtained. For high SNR regime, the relation between the outage probability and the multiplexing gain was discussed, where it was shown that the outage probability will converged to a non-zero value if the multiplexing gain is smaller than $1/K$. For low and moderate SNR regimes, although no close-form formula was found for the outage probability, Gaussian approximation has been applied to obtain the approximated outage probability. Furthermore, we considered the optimization over the coding length and the power vector to minimize the outage probability. In Chapter III, we extended the results from single dying channel case to the parallel dying channel case. In this chapter, we considered the independent random attack case and dependent random attack case respectively, where the corresponding numbers of survived blocks on each sub-channel were modeled as independent or m-dependent random sequence. The overall outage probabilities for these two cases were examined and analysis showed that the outage probability can go to zero as the number of sub-channels increases, as long as the rate per unit cost is smaller than a given threshold. In addition, the outage exponents for these two cases were studied to reveal how fast the outage probability goes to zero as the number of sub-channel increases. After studying the information-theoretical aspects for the dying channel, we then considered the practical code design for the dying channel in Chapter IV.

Specifically, we consider the LT codes for the dying binary erasure channel. We first reviewed the upper bound of the recovery probability for each bit to be decodable. Then we redesigned the degree distribution of the LT codes such that the resulting average recovery probability is as close to the upper bound as possible. A simple but effective suboptimal degree distribution was also presented.

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APPENDIX A

PROOF OF THEOREM 1

1. According to (2.7), for Rayleigh fading, let $R = r\log P$

$$\begin{aligned}\mathbb{P}\left[\sum_{i=1}^{j}\log(1+\alpha_i P) < KR\right] &\leq 1-\exp\left(-j(e^{KR/j-\log P}-1/P)\right)\\ &= 1-\exp\left(-j(e^{(Kr/j-1)\log P}-1/P)\right)\end{aligned}$$

As $P\to\infty$, when $r < j/K$, $\mathbb{P}\left[\sum_{i=1}^{j}\log(1+\alpha_i P) < KR\right]\to 0$ since the exponent of the exponential function in the upper bound goes to 0. As p_{out} can be written as:

$$\begin{aligned}p_{out} &= w_0+\sum_{j=1}^{K-1}\mathbb{P}\left[\sum_{i=1}^{j}\log(1+\alpha_i P)\leq KR\right]w_j\\ &\quad+\mathbb{P}\left[\sum_{i=1}^{K}\log(1+\alpha_i P)\leq KR\right]w_K^*,\end{aligned}$$

if $r < 1/K$, all terms except w_0 go to 0; and hence p_{out} goes to w_0.

2. Similarly, according to (2.5), we have

$$\begin{aligned}\mathbb{P}\left[\sum_{i=1}^{j}\log(1+\alpha_i P) < KR\right] &\geq \left[1-\exp\left(-e^{KR/j-\log P}+1/P\right)\right]^j,\\ &= \left[1-\exp\left(-e^{(Kr/j-1)\log P}+1/P\right)\right]^j.\end{aligned}$$

As $P\to\infty$, when $r > j/K$, $\mathbb{P}\left[\sum_{i=1}^{j}\log(1+\alpha_i P) < KR\right]\to 1$ since the exponent of the exponential function goes to $-\infty$. As p_{out} can be written as:

$$\begin{aligned}p_{out} &= w_0+\sum_{j=1}^{K-1}\mathbb{P}\left[\sum_{i=1}^{j}\log_2(1+\alpha_i P)\leq KR\right]w_j\\ &\quad+\mathbb{P}\left[\sum_{i=1}^{K}\log_2(1+\alpha_i P)\leq KR\right]w_K^*,\end{aligned}$$

if $r > 1$, $\mathbb{P}\left[\sum_{i=1}^{j} \log_2(1+\alpha_i P) < KR\right]$ goes to 1 for $j = 1\cdots,K$. Thus, p_{out} goes to 1.

3. If $1/K \leq r \leq 1$, $w_0 < p_{out} < 1$.

APPENDIX B

PROOF OF THEOREM 2

Let us consider minimizing the outage probability given by (2.22). When $K = 1$, the proof is trivial.

When $K = 2$, the outage probability is

$$p_{out}(2) \;=\; w_0 + \mathbb{P}\left[\log(1+\alpha_1 P_1) < 2R\right] w_1 + \mathbb{P}\left[\log(1+\alpha_1 P_1) + \log(1+\alpha_2 P_2) < 2R\right] w_2^*.$$

As we see from the above equation, if $P_1 < P_2$, we have $\mathbb{P}\left[\log(1+\alpha_1 P_2) < 2R\right] < \mathbb{P}\left[\log(1+\alpha_1 P_1) < 2R\right]$. Hence, we can achieve a smaller $p_{out}(2)$ by swapping P_1 and P_2, since the last term in $p_{out}(2)$ is not affected by such a swapping while the second term is decreased.

When $K \geq 3$, for any $j > i$, $(i, j \in \{1, \cdots, K\})$, if $P_i < P_j$, by swapping P_i and P_j, all the terms containing both P_i and P_j, i.e., all the probability terms in the form of $\mathbb{P}\big[\cdots + \log(1+\alpha_i P_i) + \cdots + \log(1+\alpha_j P_j) + \cdots < KR\big]$ will not be affected. However, the probability terms containing P_i but not P_j can be decreased by such a swapping. Thus, we could achieve a smaller outage probability in total.

Therefore, the optimal power allocation profile over i.i.d. fading is always non-increasing, i.e., $P_1 \geq P_2 \geq \cdots \geq P_K \geq 0$.

APPENDIX C

PROOF OF THEOREM 3

When the coding length $K = 1$, the outage probability is

$$p_{out}(1) = \mathbb{P}\left[\log(1+\alpha P) < R\right]\mathbb{P}\left[T > 1\right] + w_0.$$

When we choose any other arbitrary values for K, i.e., $K = M$ and $M \neq 1$, according to (2.22), the outage probability is

$$\begin{aligned} p_{out}(M) \quad &= \quad \mathbb{P}\left[\frac{1}{M}\sum_{i=1}^{M}\log(1+\alpha P_i) < R\right]\mathbb{P}\left[T > M\right] + w_0 \\ &\quad + \sum_{i=1}^{M-1}\mathbb{P}\left[\frac{1}{M}\sum_{l=1}^{i}\log(1+\alpha P_l) < R\right]w_i. \end{aligned}$$

Due to the concavity of the log function, we have $\frac{1}{M}\sum_{i=1}^{M}\log(1+\alpha P_i) \leq \log(1+\alpha P)$. Hence,

$$\mathbb{P}\left[\frac{1}{M}\sum_{l=1}^{M}\log(1+\alpha P_l) < R\right] \geq \mathbb{P}\left[\log(1+\alpha P) < R\right]. \tag{C.1}$$

Moreover, it is obvious that summing over only a portion of the M blocks yields an even smaller value, i.e., $\frac{1}{M}\sum_{l=1}^{i}\log(1+\alpha P_l) \leq \log(1+\alpha P)$, with $1 \leq i \leq M-1$. If $\exists P_j > 0$, for $i < j \leq M$, the strong inequality holds. Therefore, we have

$$\mathbb{P}\left[\frac{1}{M}\sum_{l=1}^{i}\log(1+\alpha P_l) < R\right] \geq \mathbb{P}\left[\log(1+\alpha P) < R\right]. \tag{C.2}$$

Noting that $\sum_{i=1}^{M-1} w_i = \mathbb{P}\left[1 < T \leq M\right]$, and considering (C.1) and (C.2), the following inequality can be derived for (C.1):

$$p_{out}(M) \geq w_0 + \mathbb{P}\left[\log(1+\alpha P) < R\right]\left(\mathbb{P}\left[T > M\right] + \mathbb{P}\left[1 < T \leq M\right]\right) = p_{out}(1). \tag{C.3}$$

From (C.3), we see that $p_{out}(1)$ has the smallest outage probability when fading gains are the same, which means that the optimal coding length is $K = 1$ with $P_1 = P$.

APPENDIX D

CONVEXITY OF THE OPTIMIZATION PROBLEM

We first check the Hessian matrix of the objective function in terms of P_i.

$$\nabla^2 p_{out} = \nabla^2 \frac{c_1}{P_1} + \cdots + \nabla^2 \frac{c_K}{\prod_{i=1}^{K} P_i}. \tag{D.1}$$

The jth term is:

$$\nabla^2 \left(\frac{c_j}{\prod_{i=1}^{j} P_i} \right) = c_j \begin{pmatrix} \frac{2}{P_1^3 \prod_{i=2}^{j} P_i} & \frac{1}{P_1^2 P_2^2 \prod_{i=3}^{j} P_i} & \cdots & \frac{1}{P_1^2 P_j^2 \prod_{i=2}^{j-1} P_i} & \mathbf{0} \\ \frac{1}{P_1^2 P_2^2 \prod_{i=3}^{j} P_i} & \frac{2}{P_1 P_2^3 \prod_{i=3}^{j} P_i} & \cdots & \frac{1}{P_2^2 P_j^2 \prod_{i=1, i \neq 2}^{j-1} P_i} & \mathbf{0} \\ \cdots & \cdots & \ddots & \cdots & \\ & \mathbf{0} & & & 0 \end{pmatrix} \tag{D.2}$$

Let $\mathbf{z} \in \mathbb{R}^K$, then

$$\mathbf{z}^T \nabla^2 \left(\frac{c_j}{\prod_{i=1}^{j} P_i} \right) \mathbf{z} = \frac{1}{\prod_{i=1}^{j} P_i} \mathbf{z}^T \boldsymbol{P}^{(j)} (\boldsymbol{P}^{(j)})^T \mathbf{z} + \mathbf{z}^T \boldsymbol{M} \mathbf{z} \geq 0,$$

where $\boldsymbol{P}^{(j)} = (1/P_1, 1/P_2, \cdots, 1/P_j, 0, \cdots, 0)^T$, and $\boldsymbol{M} = \mathrm{diag}\left(\frac{1}{P_1^2}, \frac{1}{P_2^2}, \cdots, \frac{1}{P_j^2}, 0, \cdots, 0 \right)$. Therefore, (D.1) as the summation of all the K terms is positive semi-definite. Hence p_{out} is a convex function in terms of $\boldsymbol{P}_K$. In addition, $\boldsymbol{P}_K$ lies in a convex cone as shown in Theorem. 2. Hence the problem is a convex problem.

APPENDIX E

PROOF OF PROPOSITION 1

1. When $0 \leq z < 1/2$, it is easy to see that $r = -\log(1-z)$ is a monotonically increasing function of z.

2. When $1/2 \leq z \leq 2/3$, $\frac{dr}{dz} = \frac{2z}{1-z} + 2\log(1-z)$ and $\frac{d^2r}{dz^2} = \frac{-2z}{(1-z)^2} < 0$. Since $\frac{dr}{dz}\big|_{z=2/3} > 0$, $\frac{dr}{dz} > 0$ for $1/2 \leq z \leq 2/3$. Therefore, $r = -\frac{\log(1-z)}{2z}$ is a monotonically increasing function over $1/2 \leq z \leq 2/3$.

3. When $2/3 < z < 1$, if $\frac{m-1}{m} < z_1 < z_2 < \frac{m}{m+1}$, it is easy to see that $r(z_1) < r(z_2)$. If $\frac{m-1}{m} < z_1 < \frac{m}{m+1} < z_2$, we have

$$\begin{aligned} r(z_2) - r(z_1) &= \frac{1}{m(m+1)} + \frac{1}{m+1}\sum_{i\geq m+1}\frac{z_2^{i-m}}{i} - \frac{1}{m}\sum_{i\geq m}\frac{z_1^{i-m+1}}{i} \\ &> \frac{1}{m(m+1)} + \frac{1}{(m+1)m}\Bigg(\sum_{i\geq m+1}\frac{m(\frac{m}{m+1})^{i-m}}{i} \\ &\quad -\sum_{i\geq m}\frac{(m+1)(\frac{m}{m+1})^{i-m+1}}{i}\Bigg) = 0 \end{aligned}$$

   Therefore, $r(z)$ is also a monotonically increasing function over $z \in (2/3, 1)$.

As a result, $r(z)$ is a monotonically increasing function over $z \in [0, 1)$.

VITA

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