TOWARD A CLASSIFICATION OF THE RANKS AND BORDER RANKS OF ALL (3,3,3) TRILINEAR FORMS

A Junior Scholars Thesis

by

DEREK JAMES ALLUMS

Submitted to the Office of Undergraduate Research
Texas A&M University
in partial fulfillment of the requirements for the designation as

UNDERGRADUATE RESEARCH SCHOLAR

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ABSTRACT

Toward a Classification of the Ranks and Border Ranks of All (3,3,3) Trilinear Forms.
(April 2011)

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The study of the ranks and border ranks of tensors is an active area of research. By the example of determining the complexity of matrix multiplication I introduce the reader to the notion of the rank and border rank of a tensor. Then, after presenting basic preliminary material from algebraic geometry and multilinear algebra, I quantify precisely what it means for some tensor to be of given rank, border rank, symmetric rank or symmetric rank. Objects of a given (symmetric) border rank are then interpreted geometrically as elements of certain secant varieties of Veronese and Segre varieties. Using this, I describe some of the techniques used to arrive at the classification of all (3,3,3) trilinear forms presented by Kok Omn Ng. The main result of this thesis is a classification of all the border ranks and some of the ranks of the 24 normal forms given by Kok Omn Ng in *The classification of (3,3,3) trilinear forms*.

DEDICATION

To Lindsey

ACKNOWLEDGMENTS

I wish to thank my advisor, Dr. J.M. Landsberg, for his invaluable help throughout this project as well as his encouragement to begin it in the first place. Additionally, I am grateful for the continued and unrelenting support of my family and friends.

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CHAPTER I
INTRODUCTION

Given some procedure to be executed, such as computing a bilinear map, one is interested in how efficiently it can be performed. For our purposes, we will formalize the notion of ”efficiency” by looking at the ranks and border ranks of tensors, to be be defined in a later chapter. However, a special case of these more general concepts is that of multiplication of $n \times n$ matrices since it may be understood as a bilinear map. That is, $M_{n,n,n} : M \times M \to M$ where M denotes the vector space of $n \times n$ complex-valued matrices. A particular reason one is interested in the efficiency of matrix multiplication is that the applications are vast since in the world of technology, where matrices are an invaluable tool for organizing data, the fastest execution possible is preferred.

The first non-trivial case to examine is the 2×2 case and the standard algorithm is as follows:

$$\begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix} \times \begin{bmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{bmatrix} = \begin{bmatrix} a_1^1 b_1^1 + a_2^1 b_1^2 & a_1^1 b_2^1 + a_2^1 b_2^2 \\ a_1^2 b_1^1 + a_2^2 b_1^2 & a_1^2 b_2^1 + a_2^2 b_2^2 \end{bmatrix} = \begin{bmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{bmatrix}.$$

Notice there are exactly 8 multiplications required to execute this map and indeed it is the number of multiplications as opposed to the number of additions which are of greatest interest. Though it might seem obvious that this is the best one could do, whether or not it was the best was not known until 1969. V. Strassen showed it is not the best by giving an

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The journal model is *Foundations of Computational Mathematics.*

algorithm which gives the desired product using only 7 multiplications. Let

$$\begin{aligned}
I &= (a_1^1 + a_2^2)(b_1^1 + b_2^2), \\
II &= (a_1^2 + a_2^2)b_1^1, \\
III &= -a_1^1(b_2^1 - b_2^2), \\
IV &= a_2^2(-b_1^1 + b_1^2), \\
V &= (a_1^1 + a_2^1)b_2^2, \\
VI &= (-a_1^1 + a_1^2)(b_1^1 + b_2^1), \\
VII &= (a_2^1 - a_2^2)(b_1^2 + b_2^2).
\end{aligned}$$

Then,

$$\begin{aligned}
c_1^1 &= I + IV - V + VII, \\
c_1^2 &= II + IV, \\
c_2^1 &= III + V, \\
c_2^2 &= I + III - II + VI
\end{aligned}$$

gives the product using 7 multiplications. An immediate consequence of this case is the application to multiplication of $2^n \times 2^n$ matrices since we can multiply in blocks using the same algorithm, thus greatly reducing the number of multiplications needed. Additionally, one can extend $n \times n$ matrices to $2^m \times 2^m$ matrices for some $m \in \mathbb{N}$ with blocks of zeroes. Then, one can multiply $n \times n$ matrices using only $\sim n^{log_2(7)} \approx n^{2.81}$. versus the standard approach which would require n^3 multiplications.

As previously mentioned, matrix multiplication is a bilinear map. Since our vector spaces will be over $\mathbb{C}$ unless otherwise noted, we can write out the map more explicitly (recall V^*

denotes the dual space of the vector space V):

$$M_{n,n,n} : \mathbb{C}^{n^2} \times \mathbb{C}^{n^2} \to \mathbb{C}^{n^2}$$

and after a choice of elements $\alpha^i \in (\mathbb{C}^{n^2})^*, \beta^i \in (\mathbb{C}^{n^2})^*, c_i \in (\mathbb{C}^{n^2}), v, w \in \mathbb{C}^{n^2}$, we can write

$$M_{n,n,n}(v,w) = \sum_{i=1}^{r} \alpha^i(v)\beta^i(w)c_i.$$

Later we will see that $M_{n,n,n} \in (\mathbb{C}^{n^2})^* \otimes (\mathbb{C}^{n^2})^* \otimes (\mathbb{C}^{n^2})$ where $\otimes$ denotes the tensor product. This motivates the following provisional definition of rank.

Definition .1. *The minimal number r over all such presentations of $M_{n,n,n}$ is called the rank of $M_{n,n,n}$ and is denoted $\boldsymbol{R}(M_{n,n,n})$.*

Note that each term of the form $\alpha^i(v)\beta^i(w)c_i$ is one *one* multiplication since it sends the scalar $\alpha^i(v)\beta^i(w)$ to the c_i spot in the product. Thus, $\mathbf{R}(M_{2,2,2}) \leq 7$ by the previous discussion of Strassen's algorithm. It was later proved by [4] that equality holds. One should now be gaining a clearer perspective of the central problem at hand: finding tests for the rank of some multilinear map, that is, finding tests for the efficiency of a given mapping. This problem can be examined through various lenses each of which gives the researcher a different and enlightening perspective.

Toward the goal of the classification of the (symmetric) ranks and (symmetric) border ranks of all trilinear forms, this research aims to accomplish this goal for all $(3,3,3)$ trilinear forms. All $(2,\mathbf{a},\mathbf{b})$ trilinear forms have been classified in this respect, so this is the first open case to be determined [2].

CHAPTER II

METHOD

This research uses both algebra and geometry and thus, one needs some basic background knowledge from both. This first section relates some elementary concepts from multilinear algebra.

Multilinear algebra

Although a substantial of knowledge of group theory is not necessary, two groups will appear several times throughout this thesis and their understanding is crucial.

Definition .2. *Let V be a vector space. Denote by $GL(V)$ the group of invertible linear maps $V \to V$ under composition.*

Note that after fixing a basis of V, the group $GL(V)$ can be realized as the group of changes of bases of V. For our purposes, the most important aspect of this group is the way it acts on $V^*, End(V)$ and the space of homogeneous polynomials of degree d on V for each d. The second group we will need is the group of permutations of n objects and is called the symmetric group on n letters, denoted S_n. Its importance will be seen later.

Let V be a vector space. Recall $V^* := \{\alpha : V \to \mathbb{C} : \alpha \text{ is linear}\}$. That is, V^* is the space of linear functionals on V. Then, for vector spaces V, W, let $V^* \otimes W$ denote the space of linear maps from V to W. Accordingly, $V \otimes W$ denotes the space of linear maps from V^* to W. Note that $dim(V \otimes W) = dim(V^* \otimes W) = dim(V) \cdot dim(W)$. There is a natural generalization to k-linear maps.

Definition .3. *Let $V_1, \ldots, V_k$ be vector spaces. A function $f : V_1 \times \cdots \times V_k \to \mathbb{C}$ is said to be k-linear, or multilinear, if it is linear in each factor V_i. We denote the space of such* *multilinear functions $V_1^*\otimes\cdots\otimes V_k^*$ and call this new space the tensor product of the vector spaces $V_1^*,\ldots,V_k^*$. An element $T\in V_1^*\otimes\cdots\otimes V_k^*$ is called a tensor, or more specifically a k-tensor.*

We can generalize further and define $V_1^*\otimes\cdots\otimes V_k^*\otimes W$ as the space of multilinear maps $f: V_1\times\cdots\times V_k\to W$ for a vector space W. Additionally, it is an easy exercise to see that $V\otimes W$ may be defined as the space of linear maps $V^*\to W$, the space of linear maps $W^*\to V$, the space of bilinear maps $V^*\times W^*\to\mathbb{C}$ or the dual space of $V^*\otimes W^*$. From this, one can realize $V_1\otimes\cdots\otimes V_k$ any number of ways depending on the situation.

Also important to this research are symmetric tensors. They can be defined formally in several different ways and below are four equivalent definitions for the two factor case. At the end of this section, the skew-symmetric tensors are defined which are a cousin of the symmetric tensors.

Definition .4. *Let $V^{\otimes 2}=V\otimes V$ with basis $\{v_i\otimes v_j|\ 1\le i,j\le n\}$ and $\sigma\in S_2$. Define the space of symmetric 2-tensors*

$$\begin{aligned}S^2V:=&span\{v_i\otimes v_j+v_j\otimes v_i\mid 1\le i,j\le n\}\\ =&span\{v\otimes v\mid v\in V\}\\ =&\{X\in V\otimes V\mid X(\alpha,\beta)=X(\beta,\alpha)\forall\alpha,\beta\in V^*\}\\ =&\{X\in V\otimes V\mid X\circ\sigma=X\}.\end{aligned}$$

Note that for $\phi\in S^2V$ and $g\in GL(V)$, $g\cdot\phi\in S^2V$ and similarly for Λ^2V. Furthermore, there is a natural generalization to S^dV for arbitrary d. But first, we must define the map $\pi_S: V^{\otimes d}\to V^{\otimes d}$ on elements $v_1\otimes\cdots\otimes v_d\in V^{\otimes d}$ by

$$\pi_S(v_1\otimes\cdots\otimes v_d)=\frac{1}{d!}\sum_{\sigma\in S_d}v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(d)}.$$

Then $S^dV := \pi_S(V^{\otimes d})$ is called the d^{th} symmetric power of V. Note that this definition is indeed a generalization of the previous definitions of S^2V since we have

$$\begin{aligned} S^dV =& \{X \in V^{\otimes d} | \pi_S(X) = X\} \\ =& \{X \in V^{\otimes d} | X \circ \sigma = X \ \forall \sigma \in S_d\}. \end{aligned}$$

Additionally, it is not hard to see that S^dV is invariant under the action of $GL(V)$. Furthermore, there is yet another useful way to realize S^dV and one which is exploited constantly in this area of research. Namely, S^dV^* is the space of homogeneous polynomials of degree d on V. To be precise, let $\bar{Q}$ be some multilinear form. Then the map $x \to \bar{Q}(x,\ldots,x)$ is a polynomial mapping of degree d. The process of passing from a homogeneous polynomial to a multilinear form is called polarization. In general, for some arbitrary symmetric multilinear form, the polarization identity is

$$\bar{Q}(x_1,\ldots,x_d) = \frac{1}{d!} \sum_{I \subset [d], I \neq \emptyset} (-1)^{d-|I|} Q\left(\sum_{i \in I} x_i\right)$$

where we denote $[d] = \{1,\ldots,d\}$. It is important to note that $S^dV \subset S^sV \otimes S^{d-s}V$ in a very natural way via polarization. For $\phi \in S^dV$, write $\phi_{s,d-s} \in S^sV \otimes S^{d-s}V$ for it's image. By previous statements, we know $S^sV \otimes S^{d-s}V$ can be thought of as the space of linear maps $S^sV^* \to S^{d-s}V$ and so $\phi_{s,d-s}(\alpha_1 \cdots \alpha_s) = \bar{\phi}(\alpha_1,\ldots,\alpha_s,\cdot,\ldots,\cdot)$. I make one more important definition regarding symmetric tensors before moving to the skew-symmetric case.

Definition .5. *For a vector space V, let $V^{\otimes} := \oplus_{k \geq 0} V^{\otimes k}$. Then, the symmetric tensor algebra is given by $S^\bullet V := \oplus_d S^dV$ where multiplication is defined by $\alpha\beta = \pi_S(\alpha \otimes \beta)$ for $\alpha \in S^sV$ and $\beta \in S^tV$.*

Definition .6. *Let $V^{\otimes 2} = V \otimes V$ with basis $\{v_i \otimes v_j | \ 1 \leq i, j \leq n\}$ and $\sigma \in S_2$. Define the* *space of skew-symmetric 2-tensors*

$$\begin{aligned}\Lambda^2 V :=& span\{v_i \otimes v_j - v_j \otimes v_i \mid 1 \leq i, j, \leq n\}\\ =& span\{v \otimes w - w \otimes v \mid v, w \in V\}\\ =& \{X \in V \otimes V \mid X(\alpha, \beta) = -X(\beta, \alpha) \forall \alpha, \beta \in V^*\}\\ =& \{X \in V \otimes V \mid X \circ \sigma = -X\}.\end{aligned}$$

Now, we make a similar generalization from $\Lambda^2 V$ to $\Lambda^k V$ by defining a map $\pi_\Lambda : V^{\otimes k} \to V^{\otimes k}$ by

$$v_1 \otimes \cdots \otimes v_k \mapsto v_1 \wedge \cdots \wedge v_k := \frac{1}{k!} \sum_{\sigma \in S_k} (sgn(\sigma)) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

where $sgn(\sigma)$ denotes the sign of the permutation $\sigma \in S_k$ as usual. The image of this map is called the space of skew-symmetric (or alternating) k-tensors. Again note that this is a valid generalization from the above case when $k = 2$ since

$$\Lambda^k V = \{X \in V^{\otimes k} | X \circ \sigma = sgn(\sigma) X \ \forall \sigma \in S_k\}.$$

A final important note regarding group actions: the actions of S_d and $GL(V)$ on $V^{\otimes d}$ commute with each other. That is, for $\sigma \in S_d$, $g \in GL(V)$, $T \in V^{\otimes d}$, we have $\sigma \cdot g \cdot T = g \cdot \sigma \cdot T$.

Algebraic geometry: first definitions

As previously mentioned, it is important to consider not just the algebraic perspective, but the geometric one as well. As such, I present some basic terminology and concepts from algebraic geometry. Specialized topics central to this research appear in the last subsection. First, we need the following fundamental definition.

Definition .7. *Define n-dimensional complex projective space to be* $\mathbb{P}^n = \mathbb{P}\mathbb{C}^n := (\mathbb{C}^{n+1} \backslash \{0\}) / \sim$

where $\sim$ is the equivalence relation given by $\mathbb{C}^n \ni (v_1, \ldots, v_n) \sim (\lambda v_1, \ldots, \lambda v_n)$ for some non-zero scalar λ.

Denote the set of equivalence classes of some $v \in V$ by $[v] \in \mathbb{P}V$. Furthermore, for a vector space V, let $\pi : V \backslash \{0\} \to \mathbb{P}V$ denote the projection. For a subset $Z \subset \mathbb{P}V$, let $\hat{Z} := \pi^{-1}(Z)$ denote the cone over Z. Call the image of such a cone in projective space its projectivization.

Definition .8. *An algebraic variety is the projectivization of the set of common zeroes of some collection of homogeneous polynomials on V. The ideal, denoted $I(X) \subset S^{\bullet}V^*$, of some variety $X \subset \mathbb{P}V$ is the set of all polynomials vanishing on $\hat{X}$.*

Definition .9. *Let $Z \subset \mathbb{P}V$ be a subset. Then define*

$$I(Z) := \{P \in S^{\bullet}V^* \mid P|_{\hat{Z}} \equiv 0\}.$$

Additionally, define $\bar{Z}$ to be the set of common zeroes of $I(Z)$. Call $\bar{Z}$ the Zariski closure of Z.

Several basic facts about ideals which are not too difficult to prove but whose qualitative statements are important to keep in mind are that if $X \subset Y$ are varieties, then $I(Y) \subset I(X)$, $I(Y \cup Z) = I(Y) \cap I(Z)$ for varieties Y, Z and $X \cap Y$ is a variety if $X, Y \subset \mathbb{P}V$ are.

Rank, border rank and their symmetric analogs

First, a tensor in $V_1 \otimes \cdots \otimes V_n$ is said to be of rank 1 if it is of the form $v_1 \otimes \cdots \otimes v_n$ where each $v_i \in V_i$.

Definition .10. *Let $V_1, \ldots, V_n$ be vector spaces and $T \in V_1 \otimes \cdots \otimes V_n$. The rank of T, denoted $\boldsymbol{R}(T)$ is the smallest natural number r such that $T = \sum_{j=1}^{r} T_j$ where each $T_j \in V_1 \otimes \cdots \otimes V_n$ is of rank 1.*

Definition .11. *Using the notation in the previous definition, T is said to be of border rank r, denoted $\underline{\mathbf{R}}(T) = r$, if it is the limit of tensors of rank r but not of any $s < r$.*

Border rank being such a crucial concept, I offer a simple example from [2]. Namely, the 3-tensor

$$T = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1$$

can be shown to be of rank 3 since we have

$$T = a_1 \otimes b_1 \otimes (c_1 + c_2) + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1.$$

However, $\underline{\mathbf{R}}(T) = 2$ since T is the limit as $\varepsilon \to 0$ of the following sequence of rank 2 tensors:

$$T(\varepsilon) = \frac{1}{\varepsilon}[(\varepsilon - 1)a_1 \otimes b_1 \otimes c_1 + (a_1 + \varepsilon a_2) \otimes (b_1 + \varepsilon b_2) \otimes (c_1 + \varepsilon c_2)].$$

Definition .12. *Let $\phi \in S^d V$ for some vector space V. The symmetric rank of ϕ, denoted $\boldsymbol{R}_S(\phi)$, is the smallest number r such that $\phi = x_1^d + \cdots + x_r^d$ where each $x_i \in V$. That is, $\boldsymbol{R}_S(\phi) = r$ if and only if r is the minimal natural number such that ϕ is the sum of r d^{th} powers in V.*

Definition .13. *Using the notation in the previous definition, the symmetric border rank of $\phi \in S^d V$, denoted $\underline{\boldsymbol{R}}_S(\phi)$ is the smallest r such that there exists a sequence of polynomials ϕ_ε, each of rank r, such that ϕ is the limit as $\varepsilon \to 0$ of $\{\phi_\varepsilon\}$.*

Joins and secant varieties

There exists elegant geometric interpretations of the algebraic concept(s) of (the variations of) rank. But first, more background information is needed. To better understand the general idea of secant varieties, it is helpful to begin with a straightforward special case. In particular, I will describe the join of a curve $\mathcal{C} \in \mathbb{P}V$ and a point $x \in \mathbb{P}V$: $J(x,\mathcal{C})$. The cone over $\mathcal{C}$ with vertex x is defined to be

$$J(x,\mathcal{C}) := \bigcup_{y \in \mathcal{C}} \overline{\text{all points on a } \mathbb{P}^1_{xy}}$$

where the bar denotes Zariksi closure as previously defined. Now, I present the following, more general, definition.

Definition .14. *Let $Y,Z \in \mathbb{P}V$ be varieties. The Join of Y,Z is defined to be*

$$J(Y,Z) = \bigcup_{\substack{y \in Y \\ z \in Z}} \overline{\textit{all points on a } \mathbb{P}^1_{yz}}.$$

An important special case is when $Y = Z$. If this occurs, $J(Y,Z) = J(Y,Y) =: \sigma_2(Y)$ is called the secant variety of Y and contains all points on all tangent and secant lines of Y. Inductively, if $Y_1,\dots Y_r \in \mathbb{P}V$ are varieties, then $J(Y_1,\dots,Y_r) = J(J(Y_1,\dots,Y_{r-1}),Y_r)$ or

$$J(Y_1,\dots,Y_r) = \bigcup_{y_j \in Y_j} \overline{\text{points on } \mathbb{P} < y_1,\dots,y_r >}$$

where $< \cdot >$ denotes the linear span as usual. Then, if $Y_j = Y$ for each j, we define $\sigma_r(Y) := J(\underbrace{Y,\dots,Y}_{r})$ to be the r^{th} secant variety of the variety Y.

Two more important varieties (indeed the most important for this thesis) are the n-factor Segre variety and the d^{th} Veronese embedding.

Definition .15. *Let $V_1,\dots,V_n$ be vector spaces and define $V := V_1 \otimes \cdots \otimes V_n$. Then the n-factor Segre is the image of the following map:*

$$Seg : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n \longrightarrow \mathbb{P}V$$

$$([v_1],\dots,[v_n]) \mapsto [v_1 \otimes \cdots \otimes v_n].$$

The importance of this map is not yet clear, but will become so in the next section. For now, note that for fixed $n \in \mathbb{N}$, the Segre is the projectivization of the rank one n-tensors. In particular, for $n = 2$, we have the projectivization of rank one matrices, after choosing bases.

Definition .16. *The d^{th} Veronese embedding of $\mathbb{P}V$, $v_d(\mathbb{P}V) \subset \mathbb{P}S^dV$, is defined as the image of the following map:*

$$v_d : \mathbb{P}V \to \mathbb{P}S^dV$$
$$[v] \mapsto [v^d] = [\underbrace{v \otimes \cdots \otimes v}_{d}].$$

Notice $v_d(\mathbb{P}V) \subset Seg(\mathbb{P}V \times \cdots \times \mathbb{P}V)$. Also, I will omit the underbrace or overbrace when the number of factors is understood.

The concepts of border rank and symmetric border rank each have geometric interpretations. Let $\sigma_r^0 \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$ denote the projectivization of tensors of rank at most r. Then, define $\sigma_r := \bar{\sigma}_r^0$. The fact that I am using notation seen in the previous section on secant varieties is not a coincidence. In particular, $\sigma_r(Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)) = \sigma_r$. A similar statement holds for symmetric rank and border rank. All this information is best summarized in the more general concept of X-rank.

Definition .17. *Let $X \subset \mathbb{P}V$ be a variety not contained in a hyperplane and let $p \in \mathbb{P}V$. Define the X-rank of p, denoted $R_X(p)$, to be the smallest number r such that p is in the linear span of r points of X. Thus, if $\sigma_{r-1}(X) \neq \mathbb{P}V$, then $\sigma_r(X)$ is the Zariski closure of the set of points of X-rank r. Furthermore, the X-border rank of p, denoted $\underline{R}_X(p)$, is the smallest r such that $p \in \sigma_r(X)$.*

So, if $X = Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$ and $p \in V = V_1 \otimes \cdots \otimes V_n$, then $\underline{R}_X([p]) = \underline{\mathbf{R}}(p)$ as usual and $R_X([p]) = \mathbf{R}(p)$. Similarly, if $p \in V = S^dW$ and $X = v_d(\mathbb{P}W)$, we have that $\underline{R}_X([p]) = \underline{\mathbf{R}}_S(p)$ and $R_X([p]) = \mathbf{R}_S(p)$. Such an interpretation is crucial when seeking resolution of the border rank version of a conjecture of P. Comon which states that equality holds in $\underline{\mathbf{R}}_S(\phi) \geq \underline{\mathbf{R}}(\phi)$ for some symmetric tensor ϕ. Namely, confirming the conjecture would amount to showing that (for $\phi \in S^d V$) $[\phi] \in \sigma_r(v_d(\mathbb{P}V)) \backslash \sigma_{r-1}(v_d(\mathbb{P}(V))$ but

$$[\phi] \notin \sigma_{r-1}(Seg(\overbrace{\mathbb{P}V \times \cdots \times \mathbb{P}V}^{d})).$$

In the coming pages I will use these geometric techniques to describe the ideas behind Ng's classification of all $T \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ [3].

CHAPTER III

RESULTS

Ottaviani's version of Strassen's equations

One of the main results of this research was the computation of the border ranks of all $T \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ on Ng's list via Ottaviani's version of Strassen's equations. After a choice of bases of $A, B, C = \mathbb{C}^3$, one can write $T = x_1 \otimes A_1 + x_2 \otimes A_2 + x_3 \otimes A_3$ where in this case each A_i is a 3×3 matrix $B^* \to C$ and $x_i \in A$. Then define

$$T_A^{\wedge} := \begin{pmatrix} 0 & A_3 & -A_2 \\ -A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{pmatrix}.$$

Theorem .18. *[2]. Let $T \in A \otimes B \otimes C$. Assume $3 \leq dim(A) \leq dim(B) \leq dim(C)$.*

(1) *If $\underline{\boldsymbol{R}}(T) \leq r$, then $rank(T_A^{\wedge}) \leq r(dim(A) - 1)$.*

(2) *If $T \in A \otimes B \otimes C$ is generic and $dim(A) = 3, dim(B) = dim(C) \geq 3$, then $rank(T_A^{\wedge}) = 3 \cdot dim(B)$.*

Thus, for k even, the $(k+1) \times (k+1)$ minors of $T_A^{\wedge}$ furnish equations for $\hat{\sigma}_{\frac{k}{2},(A \otimes B \otimes C)}$, the set of 3-tensors of border rank at most $\frac{k}{2}$.

The following example illustrates this method. Consider the following tensor in matrix form:

$$T = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{pmatrix} = a_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= a_1 \otimes A_1 + a_2 \otimes A_2 + a_3 \otimes A_3.$$

Computation reveals that $det(T_A^{\wedge}) = 0$. Equivalently, the 9×9 minor(s) of $T_A^{\wedge}$ are zero so by above $T \in \hat{\sigma}_{4,(A\otimes B\otimes C)}$. To see that $T \notin \hat{\sigma}_{3,(A\otimes B\otimes C)}$, we compute the 7×7 minors to see that some are non-zero. Thus, $\underline{\mathbf{R}}(T) = 4$. This process was repeated for the 24 cases below.

On the techniques used to classify all $T \in \mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$

First recall the projective linear group $PGL(V) = GL(V)\,/\,Z(V)$ where $Z(V)$ is the group of scalar transformations on V and quotients out naturally since we're in projective space. Define $G := PGL(A)\times PGL(B)\times PGL(C)$. Then note that if some $\phi \in A\otimes B\otimes C = \mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$ is represented as a matrix in

$$\mathrm{Hom}(A^*, B\otimes C) \backsimeq \mathrm{Hom}(A^*, \mathrm{Hom}(C^*, B)),$$

that is, as a 3×3 matrix of linear forms on A^*, the action of G is realized as row and column transformations. Ng's classification is up to this action. The first step is to move the problem of classification into projective space so we consider $\mathbb{P}(A\otimes B\otimes C)$. Note that we can also have $\phi \in \mathrm{Hom}(B^*, A\otimes C)$ or $\phi \in \mathrm{Hom}(C^*, A\otimes B)$ so without loss of generality only the realization $A^* \to B\otimes C$ is considered. Now consider the

$$\mathbb{P}^2 = \mathbb{P}(\phi(A^*)) \subset \mathbb{P}(B\otimes C) = \mathbb{P}^8$$

and note $Seg(\mathbb{P}^2\times\mathbb{P}^2), \sigma(Seg(\mathbb{P}^2\times\mathbb{P}^2) \subset \mathbb{P}(B\otimes C)$ where these varieties are of dimension 4 and 7 respectively. Then one looks at the curve in $\mathbb{P}^2 \cap \sigma(Seg(\mathbb{P}^2\times\mathbb{P}^2))$. If ϕ is generic, it is a known fact to the specialist that the curve is smooth. If ϕ is not generic, it has some singularities. Since the degree of the curve is 3 there is a natural upper bound on the number of potential singularities and the bulk of the paper is a study of these 24 cases.

It is also important to realize exactly how one represents a tensor $\phi \in A\otimes B\otimes C$ as a matrix. Choose bases $\{a_1^*, a_2^*, a_3^*\}, \{b_1^*, b_2^*, b_3^*\}, \{c_1^*, c_2^*, c_3^*\}$ of A^*, B^*, C^* respectively and say

$$a^* = \sum_i a_i a_i^*,\ b^* = \sum_j b_j b_j^*,\ c^* = \sum_k c_k c_k^*,\ \phi = \sum_{i,j,k} \phi_{ijk} x_i \otimes y_j \otimes z_k$$

where the x_i, y_j, z_k are elements of A, B, C respectively and a_i, b_j, c_k are linear forms on A^*, B^*, C^* respectively. Then the three associated matrices (parameterized by a^*, b^*, c^*, respectively) are given by

$$(\phi \lrcorner a^*) = \left(\sum_i \phi_{ijk} a_i\right)_{j,k}, (\phi \lrcorner b^*) = \left(\sum_j \phi_{ijk} b_j\right)_{i,k}, (\phi \lrcorner c^*) = \left(\sum_k \phi_{ijk} c_k\right)_{i,j}.$$

Ng considers the first representation in his paper but for the purposes of classifying the tensors with respect to rank it is often useful to look at the other representations. These have been calculated for each of the 24 cases below.

Classification of ranks, border ranks of all $T \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

In what follows, I present a classification of the border ranks and ranks of these tensors. Various methods were used to calculate the rank and the proofs can differ significantly. In general, the goal is to use the following theorem [2]:

Theorem .19. *Let $T \in A \otimes B \otimes C$. Then $\boldsymbol{R}(T)$ equals the number of rank one matrices needed to span (a space containing) $T(A^*) \subset B \otimes C$ (and similarly for permuted statements).*

Note that only one tensor , (1), is symmetric and both it's symmetric rank and symmetric border rank are trivially 3.

(1) $(\phi \lrcorner a^*) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, (\phi \lrcorner b^*) = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}, (\phi \lrcorner c^*) = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}.$

$T = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3.$

Proposition .20. $\boldsymbol{R}(T) = \underline{\boldsymbol{R}}(T) = 3.$

Proof. The naive approach gives $\mathbf{R}(T) \leq 3$ and since $\underline{\mathbf{R}}(T) = 3$, equality holds. $\square$

(2) $(\phi \lrcorner a^*) = \begin{pmatrix} a_1 & 0 & 0 \\ a_3 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, (\phi \lrcorner b^*) = \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & b_1 & 0 \\ 0 & 0 & b_3 \end{pmatrix}, (\phi \lrcorner c^*) = \begin{pmatrix} c_1 & 0 & c_2 \\ 0 & c_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}.$
$T = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_1 \otimes c_2 + a_3 \otimes b_3 \otimes c_3.$

Proposition .21. $\boldsymbol{R}(T) = \underline{\boldsymbol{R}}(T) = 4.$

Proof. Immediate by using same arguments as in (3). $\square$

(3)$(\phi \lrcorner a^*) = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{pmatrix}, (\phi \lrcorner b^*) = \begin{pmatrix} b_1 & b_3 & 0 \\ 0 & b_2 & 0 \\ b_2 & 0 & b_3 \end{pmatrix}, (\phi \lrcorner c^*) = \begin{pmatrix} c_1 & 0 & 0 \\ c_3 & c_2 & 0 \\ 0 & c_1 & c_3 \end{pmatrix}.$
$T = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_3 + a_2 \otimes b_3 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3.$

Proposition .22. $\boldsymbol{R}(T) \leq 5, \underline{\boldsymbol{R}}(T) = 4.$

Proof. There are 5 entries in $(\phi \lrcorner a^*)$ with no immediate way to combine any, thus giving an upper bound on the rank of 5. $\square$

(4) $(\phi \lrcorner a^*) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & a_1 \\ 0 & a_1 & a_3 \end{pmatrix}, (\phi \lrcorner b^*) = \begin{pmatrix} b_1 & 0 & 0 \\ b_3 & b_2 & 0 \\ b_2 & 0 & b_3 \end{pmatrix}, (\phi \lrcorner c^*) = \begin{pmatrix} c_1 & 0 & 0 \\ c_3 & c_2 & 0 \\ c_2 & 0 & c_3 \end{pmatrix}.$
$T = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_3 + a_1 \otimes b_3 \otimes c_2 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3.$

Proposition .23. $\boldsymbol{R}(T) = \underline{\boldsymbol{R}}(T) = 4$.

Proof. Notice that the space of 2×2 symmetric matrices is spanned by 3 rank one matrices:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, $\mathbf{R}(T) = 3 + 1 = 4$. □

(5) $$(\phi \lrcorner a^*) = \begin{pmatrix} a_1 & a_3 & a_2 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, (\phi \lrcorner b^*) = \begin{pmatrix} b_1 & b_3 & b_2 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}, (\phi \lrcorner c^*) = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & c_1 \\ 0 & c_1 & c_3 \end{pmatrix}.$$

$T = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_3 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_2 \otimes c_1 + a_3 \otimes b_3 \otimes c_3$.

Proposition .24. $\boldsymbol{R}(T) = \underline{\boldsymbol{R}}(T) = 4$.

Proof. As a map $C^* \to A \otimes B$, (5) is the same as (4) after relabeling. Thus, $\mathbf{R}(T) = 4$. □

(6) $$(\phi \lrcorner a^*) = \begin{pmatrix} a_1 & 0 & 0 \\ a_3 & a_2 & 0 \\ a_2 & 0 & a_3 \end{pmatrix}, (\phi \lrcorner b^*) = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & b_1 \\ 0 & b_1 & b_3 \end{pmatrix}, (\phi \lrcorner c^*) = \begin{pmatrix} c_1 & c_3 & c_2 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}.$$

$T = a_1 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_2 \otimes b_1 \otimes c_3 + a_3 \otimes b_1 \otimes c_2 + a_3 \otimes b_3 \otimes c_3$.

Proposition .25. $\boldsymbol{R}(T) = \underline{\boldsymbol{R}}(T) = 4$.

Proof. As a map $B^* \to A \otimes C$, (6) is the same as (4) after relabeling. Thus, $\mathbf{R}(T) = 4$. □

(7) $(\phi\lrcorner a^*) = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -\lambda a_3 & 0 \end{pmatrix}, (\phi\lrcorner b^*) = \begin{pmatrix} b_2 & b_3 & 0 \\ -b_1 & 0 & b_3 \\ 0 & -b_1 & -\lambda b_2 \end{pmatrix}, (\phi\lrcorner c^*) =$
$\begin{pmatrix} -c_2 & -c_3 & 0 \\ c_1 & 0 & -\lambda c_3 \\ 0 & c_1 & c_2 \end{pmatrix}$. $T =$
$a_1\otimes b_2\otimes c_1 + a_1\otimes(-b_1)\otimes c_2 + a_2\otimes b_3\otimes c_1 + a_2\otimes(-b_1)\otimes c_3 + a_3\otimes b_3\otimes c_2 + a_3\otimes(-\lambda b_2)\otimes c_3$.

Proposition .26. $\boldsymbol{R}(T) \leq 6, \underline{\boldsymbol{R}}(T) = 5$.

Proof. There are 6 entries with no immediate way to combine any, providing an upper bound on the rank of 6. □

(8) $(\phi\lrcorner a^*) = \begin{pmatrix} a_1 & a_3 & a_2 \\ 0 & a_2 & a_1 \\ 0 & 0 & a_3 \end{pmatrix}, (\phi\lrcorner b^*) = \begin{pmatrix} b_1 & b_3 & b_2 \\ b_3 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}, (\phi\lrcorner c^*) = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & c_1 \\ c_2 & c_1 & c_3 \end{pmatrix}$.
$T = a_1\otimes b_1\otimes c_1 + a_1\otimes b_3\otimes c_2 + a_2\otimes b_3\otimes c_1 + a_2\otimes b_2\otimes c_2 + a_3\otimes b_2\otimes c_1 + a_3\otimes b_3\otimes c_3$.

Proposition .27. $\boldsymbol{R}(T) = 5, \underline{\boldsymbol{R}}(T) = 4$.

Proof. With (8) as a map $A^* \to B\otimes C$ and (9) as a map $C^* \to A\otimes B$, $(8)^T = (9)$. Thus, $\mathbf{R}(T) = 5$. □

(9) $(\phi\lrcorner a^*) = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & a_2 & a_1 \\ 0 & a_1 & a_3 \end{pmatrix}, (\phi\lrcorner b^*) = \begin{pmatrix} b_1 & b_3 & 0 \\ b_3 & b_2 & 0 \\ b_2 & 0 & b_3 \end{pmatrix}, (\phi\lrcorner c^*) = \begin{pmatrix} c_1 & 0 & 0 \\ c_3 & c_2 & 0 \\ c_2 & c_1 & c_3 \end{pmatrix}$.
$T = a_1\otimes b_1\otimes c_1 + a_1\otimes b_3\otimes c_2 + a_1\otimes b_2\otimes c_3 + a_2\otimes b_3\otimes c_1 + a_2\otimes b_2\otimes c_2 + a_3\otimes b_3\otimes c_3$.

Proposition .28. $\boldsymbol{R}(T)=5, \underline{\boldsymbol{R}}(T)=4$.

Proof. Consider $T: C^* \to A\otimes B$. Since $\underline{\mathbf{R}}(T)=4$, we may assume $\mathbf{R}(T)=4$ to arrive at a contradiction, proving $\mathbf{R}(T)=5$. Make the identification

$$S_1=\begin{pmatrix}1&0&0\\0&0&0\\0&1&0\end{pmatrix}, S_2=\begin{pmatrix}0&0&0\\0&1&0\\1&0&0\end{pmatrix}, S_3=\begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}, P=\begin{pmatrix}s_1t_1&s_1t_2&s_1t_3\\s_2t_1&s_2t_2&s_2t_3\\s_3t_1&s_3t_2&s_3t_3\end{pmatrix}.$$

If $\mathbf{R}(T)=4$, each $X_i \in< P, X_1, X_2, X_3 >$ where each X_i is of rank one. Then, in particular, S_1 would be in the span of S_2, S_3, P, X_i for some i and we would be able to find constants α, β such that

$$X_i := \begin{pmatrix}1&0&0\\\beta&\alpha&0\\\alpha&1&\beta\end{pmatrix}+\begin{pmatrix}s_1t_1&s_1t_2&s_1t_3\\s_2t_1&s_2t_2&s_2t_3\\s_3t_1&s_3t_2&s_3t_3\end{pmatrix}$$

would be of rank one.

I will first show that $\alpha, \beta \neq 0$. First assume $\alpha=0$. Then we have $S_1+\beta S_3+P=X_1$. Thus,

$$P=\begin{pmatrix}-1&0&0\\-\beta&0&0\\\delta&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\\delta&-1&-\beta\end{pmatrix}, \begin{pmatrix}-1&r_1&r_2\\-\beta&r_1\beta&r_2\beta\\0&0&0\end{pmatrix} \text{ or } \begin{pmatrix}0&r_1&r_1\beta\\0&r_2&r_2\beta\\0&-1&-\beta\end{pmatrix}$$

which implies

$$X_1=\begin{pmatrix}0&0&0\\0&0&0\\\delta&1&\beta\end{pmatrix}, \begin{pmatrix}1&0&0\\\beta&0&0\\\delta&0&0\end{pmatrix}, \begin{pmatrix}0&r_1&r_2\\0&r_1\beta&r_2\beta\\0&1&\beta\end{pmatrix} \text{ or } \begin{pmatrix}1&r_1&r_1\beta\\\beta&r_2&r_2\beta\\0&0&0\end{pmatrix},$$

respectively for $r_i \neq 0$ and δ a constant. Note that the first two cases are equivalent and the last two cases are equivalent and for the last two cases, X_i is rank one only if $\beta=\pm 1$.

Now, if the rank of T is indeed four, we would only need 2 more matrices of rank one to span (a space containing) $< S_1, S_2, S_3 >$. It is not hard to see that this is not possible. In particular, we would to be able to obtain the subspace

$$c_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and in either case we could only do this setting

$$X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

However, (in either case) we have $< S_1, S_2, S_3 > \not\subset < P, X_1, X_2, X_3 >$ so it cannot be the case that $\alpha = 0$. The case for $\beta \neq 0$ is similar. In particular, we could have $S_1 + \alpha S_2 + P = X_1$. Thus

$$P = \begin{pmatrix} -1 & r_1 & 0 \\ 0 & 0 & 0 \\ -\alpha & \alpha r_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\alpha} & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 & 0 \\ \alpha^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so we have

$$X_1 = \begin{pmatrix} 0 & r_1 & 0 \\ 0 & \alpha & 0 \\ 0 & 1+\alpha r_1 & - \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 \\ \alpha & 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 \\ \alpha^2 & \alpha & 0 \\ \alpha & 1 & 0 \end{pmatrix}$$

respectively where $r_1 \neq 0$. As before, the claim is that with these matrices taking up two spots in the spanning set, it is impossible to contain all of $< S_1, S_2, S_3 >$ using only two more rank one matrices. In particular, to obtain the subspace

$$c_3\begin{pmatrix}0&0&0\\1&0&0\\0&0&1\end{pmatrix}$$

we would need at least one of the following two matrices

$$\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}.$$

Note that if $s_1 = 0$ then it is immediate that X_i has rank at least 2.

Now, if one subtracts $\frac{s_2}{s_1}$ times the first row from the second row, $\frac{s_3}{s_1}$ times the first row from the third row and $\frac{1}{\alpha}$ times the second row from the third row in that order, one obtains

$$\begin{pmatrix}1+s_1t_1 & s_1t_2 & s_1t_3\\ * & \alpha & 0\\ * & 0 & \beta\end{pmatrix}$$

which is of rank at least 2 as long as $\alpha, \beta \neq 0$, which is true. Thus, $\mathbf{R}(T) = 5$. □

(10) $(\phi\lrcorner a^*) = \begin{pmatrix}a_1&0&0\\0&a_2&a_1\\a_2&a_1&a_3\end{pmatrix}, (\phi\lrcorner b^*) = \begin{pmatrix}b_1&0&0\\b_3&b_2&0\\b_2&b_1&b_3\end{pmatrix}, (\phi\lrcorner c^*) = \begin{pmatrix}c_1&c_3&0\\c_3&c_2&0\\c_2&0&c_3\end{pmatrix}.$

$T = a_1\otimes b_1\otimes c_1 + a_1\otimes b_3\otimes c_2 + a_1\otimes b_2\otimes c_3 + a_2\otimes b_1\otimes c_3 + a_2\otimes b_2\otimes c_2 + a_3\otimes b_3\otimes c_3.$

Proposition .29. $\boldsymbol{R}(T) = 5, \underline{\boldsymbol{R}}(T) = 4.$

Proof. With (10) as a map $C^* \to A\otimes B$ and (9) as a map $B^* \to A\otimes C$, (10) = (9). Thus, $\mathbf{R}(T) = 5$. □

(11) $(\phi \lrcorner a^*) = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 - a_2 - a_3 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}, (\phi \lrcorner b^*) =$
$\begin{pmatrix} b_2 & b_3 & 0 \\ -b_1 & -b_1 & -b_1 + b_3 \\ 0 & b_1 & -b_2 \end{pmatrix}, (\phi \lrcorner c^*) = \begin{pmatrix} -c_2 & -c_2 - c_3 & -c_2 \\ c_1 & 0 & c_3 \\ 0 & c_1 & c_2 \end{pmatrix}.$

$T = a_1 \otimes b_2 \otimes c_1 + a_1 \otimes (-b_1) \otimes c_2 + a_2 \otimes b_3 \otimes c_1 + a_2 \otimes (-b_1) \otimes c_2 + a_2 \otimes (-b_1) \otimes c_3 + a_3 \otimes (-b_1) \otimes c_2 + a_3 \otimes b_3 \otimes c_2 + a_3 \otimes (-b_2) \otimes c_3.$

Proposition .30. $\boldsymbol{R}(T) \leq 6, \underline{\boldsymbol{R}}(T) = 5.$

Proof. Consider the tensor as a map $A^* \to B \otimes C$. Then notice that

$$\begin{pmatrix} 0 & 0 & 0 \\ -a_1 - a_2 - a_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is rank one. Thus, the naive approach yields $\mathbf{R}(T) \leq 6$. □

(12) $(\phi \lrcorner a^*) = \begin{pmatrix} a_1 & 0 & a_3 \\ 0 & a_1 + a_2 & 0 \\ 0 & a_3 & a_2 \end{pmatrix}, (\phi \lrcorner b^*) = \begin{pmatrix} b_1 & 0 & b_3 \\ b_2 & b_2 & 0 \\ 0 & b_3 & b_2 \end{pmatrix}, (\phi \lrcorner c^*) = \begin{pmatrix} c_1 & 0 & 0 \\ c_2 & c_2 & c_3 \\ 0 & c_3 & c_1 \end{pmatrix}.$

$T = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_2 + a_2 \otimes b_2 \otimes c_2 + a_2 \otimes b_3 \otimes c_3 + a_3 \otimes b_3 \otimes c_1 + a_3 \otimes b_2 \otimes c_3.$

Proposition .31. $\boldsymbol{R}(T) = ?, \underline{\boldsymbol{R}}(T) = 4.$

Proof. As a map $C^* \to A \otimes B$, (12) is the same as the transpose of (14) as a map $A^* \to B \otimes C$ after relabeling. Thus, $\mathbf{R}(T) = 5$. □

$$\textbf{(13)}\ (\phi\lrcorner a^*) = \begin{pmatrix} a_1 & a_2 & 0 \\ 0 & a_2 & a_3 \\ 0 & a_3 & a_1 \end{pmatrix}, (\phi\lrcorner b^*) = \begin{pmatrix} b_1 & b_2 & 0 \\ 0 & b_2 & b_3 \\ b_3 & 0 & b_2 \end{pmatrix}, (\phi\lrcorner c^*) = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_1+c_2 & c_3 \\ c_3 & 0 & c_2 \end{pmatrix}.$$

$T = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_3 \otimes c_3 + a_2 \otimes b_2 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_2 + a_3 \otimes b_2 \otimes c_3$.

Proposition .32. $\boldsymbol{R}(T) = ?, \underline{\boldsymbol{R}}(T) = 4$.

Proof. As a map $B^* \to A \otimes C$, (13) is the same as (14) as a map $C^* \to A \otimes B$ after relabeling. Thus, $\mathbf{R}(T) = 5$. ☐

$$\textbf{(14)}\ (\phi\lrcorner a^*) = \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & a_2 & a_3 \\ 0 & a_3 & a_1 \end{pmatrix}, (\phi\lrcorner b^*) = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_1+b_2 & b_3 \\ b_3 & 0 & b_2 \end{pmatrix}, (\phi\lrcorner c^*) = \begin{pmatrix} c_1 & c_2 & 0 \\ 0 & c_2 & c_3 \\ c_3 & 0 & c_2 \end{pmatrix}.$$

$T = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_3 \otimes c_3 + a_2 \otimes b_1 \otimes c_2 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_2 + a_3 \otimes b_2 \otimes c_3$.

Proposition .33. $\boldsymbol{R}(T) = 5, \underline{\boldsymbol{R}}(T) = 4$.

Proof. Consider $T : A^* \to B \otimes C$ and make the identification

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

If $\mathbf{R}(T) = 4$, each $S_i \in < P, X_1, X_2, X_3 >$ where each X_i, P is of rank one. Then, in particular, S_1 would be in the span of S_2, S_3, P, X_i for some i and we would be able to find constants α, β such that

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & \alpha & \beta \\ 0 & \beta & 1 \end{pmatrix} + \begin{pmatrix} s_1t_1 & s_1t_2 & s_1t_3 \\ s_2t_1 & s_2t_2 & s_2t_3 \\ s_3t_1 & s_3t_2 & s_3t_3 \end{pmatrix} =: X_i$$

is rank 1. Note that if $\alpha = 0$ or $\beta = 0$ then it is immediate that X_i has rank at least 2. Furthermore, if $s_1 = 0$, then

$$P = \begin{pmatrix} 0 & 0 & 0 \\ r\beta & -\alpha & -\beta \\ r & -\beta & -1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 & \frac{\beta}{\alpha} \\ 0 & 0 & 0 \\ 0 & -\beta & -1 \end{pmatrix}$$

which implies

$$X_i = \begin{pmatrix} 1 & 0 & 0 \\ \alpha + r\beta & 0 & 0 \\ r & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 & \frac{\beta}{\alpha} \\ \alpha & \alpha & \beta \\ 0 & 0 & 0 \end{pmatrix}$$

for some constant r, respectively. Per the logic in the proof of (9), these matrices do not allow for a choice of two additional matrices of rank one which contain $< S_1, S_2, S_3 >$. In particular, to obtain the subspace

$$a_2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

one would need to annihilate the other entries using only two rank one matrices. It is not hard to see this is impossible. So we now assumed $s_1 \neq 0$.

In this case, subtract $\frac{s_2}{s_1}$ times the first row from the second and $\frac{s_3}{s_1}$ times the first row from the third in that order to obtain :

$$\begin{pmatrix} 1+s_1t_1 & s_1t_2 & s_1t_3 \\ \alpha-\frac{s_2}{s_1} & \alpha & \beta \\ -\frac{s_3}{s_1} & \beta & 0 \end{pmatrix}$$

which is of rank at least 2 as long as $\beta \neq 0$. If $\beta = 0$, then

$$X_i = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} + P$$

has rank one only if $\alpha = 0$. I now show that it is not possible for both $\alpha, \beta = 0$ which in turns shoes that X_i has rank at least 2, proving the rank of T is 5 as desired.

So assume $\alpha, \beta =$. Then, $S_1 + \alpha S_2 + \beta S_3 + P = X_i$ reduces to $S_1 + P = X_i$. Without loss of generality, set $X_i = X_1$. Then, since each $S_i \in < X_1, X_2, X_3, P >$ we have

$$S_1 = a_1X_1 + a_2X_2 + a_3X_3 + a_4P$$

$$S_2 = b_1X_1 + b_2X_2 + b_3X_3 + b_4P$$

$$S_3 = c_1X_1 + c_2X_2 + c_3X_3 + c_4P$$

which then reduces to

$$S_1 = a_1S_1 + a_2X_2 + a_3X_3 + (a_1 + a_4)P$$

$$S_2 = b_1S_1 + b_2X_2 + b_3X_3 + (b_4 + b_1)P$$

$$S_3 = c_1S_1 + c_2X_2 + c_3X_3 + (c_4 + c_1)P$$

where $S_1 \in < X_2, X_3, P >$ implies $S_2, S_3 \in < X_2, X_3, P >$. That is, this implies $\mathbf{R}(T) = 3$ which contradicts the fact that $\underline{\mathbf{R}}(T) = 4$. Thus, not both $\alpha, \beta = 0$ and the proof is complete.

□

(15) $(\phi\lrcorner a^*) = \begin{pmatrix} (\lambda-1)a_3 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -\lambda a_3 & 0 \end{pmatrix}, (\phi\lrcorner b^*) = \begin{pmatrix} b_2 & b_3 & (\lambda-1)b_1 \\ -b_1 & 0 & b_3 \\ 0 & -b_1 & -\lambda b_2 \end{pmatrix}, (\phi\lrcorner c^*) = \begin{pmatrix} -c_2 & -c_3 & (\lambda-1)c_1 \\ c_1 & 0 & -\lambda c_3 \\ 0 & c_1 & c_2 \end{pmatrix}$. $T = a_1\otimes b_2\otimes c_1 + a_1\otimes(-b_1)\otimes c_2 + a_2\otimes b_3\otimes c_1 + a_2\otimes(-b_1)\otimes c_3 + a_3\otimes(\lambda-1)b_1\otimes c_1 + a_3\otimes(-\lambda b_2)\otimes c_3 + a_3\otimes b_3\otimes c_2$.

Proposition .34. *$\boldsymbol{R}(T) \leq 7, \underline{\boldsymbol{R}}(T) = 5$ for $\lambda \neq -1$ and $\underline{\boldsymbol{R}}(T) = 4$ for $\lambda = -1$.*

Proof. The naive approach yields an upper bound on the rank of 7. □

(16)

$(\phi\lrcorner a^*) = \begin{pmatrix} a_1 & \lambda a_3 & -a_2 \\ a_3 & a_2 & 0 \\ a_2 & 0 & a_3 \end{pmatrix}, (\phi\lrcorner b^*) = \begin{pmatrix} b_1 & -b_3 & \lambda b_2 \\ 0 & b_2 & b_1 \\ 0 & b_1 & b_3 \end{pmatrix}, (\phi\lrcorner c^*) = \begin{pmatrix} c_1 & c_3 & c_2 \\ 0 & c_2 & \lambda c_1 \\ 0 & -c_1 & c_3 \end{pmatrix}$.

$T = a_1\otimes b_1\otimes c_1 + a_2\otimes(-b_3)\otimes c_1 + a_2\otimes b_2\otimes c_2 + a_2\otimes b_1\otimes c_3 + a_3\otimes b_1\otimes c_2 + a_3\otimes b_3\otimes c_3 + a_3\otimes(\lambda b_2)\otimes c_1$.

Proposition .35. *$\boldsymbol{R}(T) \leq 6, \underline{\boldsymbol{R}}(T) = 4$.*

Proof. The naive approach yields an upper bound on the rank of 6. □

(17) $(\phi\lrcorner a^*) = \begin{pmatrix} a_1 & 0 & 0 \\ a_3 & a_2 & -a_1 \\ a_2 & \lambda a_1 & a_3 \end{pmatrix}, (\phi\lrcorner b^*) = \begin{pmatrix} b_1 & 0 & 0 \\ -b_3 & b_2 & b_1 \\ \lambda b_2 & b_1 & b_3 \end{pmatrix}, (\phi\lrcorner c^*) = \begin{pmatrix} c_1 & c_3 & c_2 \\ \lambda c_3 & c_2 & 0 \\ -c_2 & 0 & c_3 \end{pmatrix}.$

$T = a_1\otimes b_1\otimes c_1 + a_1\otimes(-b_3)\otimes c_2 + a_1\otimes(\lambda b_2)\otimes c_3 + a_2\otimes b_2\otimes c_2 + a_2\otimes b_1\otimes c_3 + a_3\otimes b_1\otimes c_2 + a_3\otimes b_3\otimes c_3.$

Proposition .36. $\boldsymbol{R}(T) \leq 6, \underline{\boldsymbol{R}}(T) = 4.$

Proof. The naive approach yields an upper bound on the rank of 6. □

(18) $(\phi\lrcorner a^*) = \begin{pmatrix} a_1 & a_3 & a_2 \\ 0 & a_2 & \lambda a_1 \\ 0 & -a_1 & a_3 \end{pmatrix}, (\phi\lrcorner b^*) = \begin{pmatrix} b_1 & b_3 & b_2 \\ \lambda b_3 & b_2 & 0 \\ -b_2 & 0 & b_3 \end{pmatrix}, (\phi\lrcorner c^*) = \begin{pmatrix} c_1 & 0 & 0 \\ -c_3 & c_2 & c_1 \\ \lambda c_2 & c_1 & c_3 \end{pmatrix}.$

$T = a_1\otimes b_1\otimes c_1 + a_1\otimes(\lambda b_3)\otimes c_2 + a_1\otimes(-b_2)\otimes c_3 + a_2\otimes b_3\otimes c_1 + a_2\otimes b_2\otimes c_2 + a_3\otimes b_2\otimes c_1 + a_3\otimes b_3\otimes c_3.$

Proposition .37. $\boldsymbol{R}(T) \leq 6, \underline{\boldsymbol{R}}(T) = 4.$

Proof. The naive approach yields an upper bound on the rank of 6. □

(19) $(\phi\lrcorner a^*) = \begin{pmatrix} a_2 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & a_2 - a_3 & 0 \end{pmatrix}, (\phi\lrcorner b^*) = \begin{pmatrix} b_2 & b_1 + b_3 & 0 \\ -b_1 & 0 & b_3 \\ 0 & -b_1 + b_2 & -b_2 \end{pmatrix}, (\phi\lrcorner c^*) =$

$$\begin{pmatrix} -c_2 & c_1 - c_3 & 0 \\ c_1 & c_3 & -c_3 \\ 0 & c_1 & c_2 \end{pmatrix}. \ T = a_1 \otimes b_2 \otimes c_1 + a_1 \otimes (-b_1) \otimes c_2 + a_2 \otimes b_1 \otimes c_1 + a_2 \otimes b_3 \otimes c_1 +$$
$$a_2 \otimes (-b_1) \otimes c_3 + a_2 \otimes b_2 \otimes c_3 + a_3 \otimes b_3 \otimes c_2 + a_3 \otimes (-b_2) \otimes c_3.$$

Proposition .38. $\boldsymbol{R}(T) \leq 6, \underline{\boldsymbol{R}}(T) = 5.$

Proof. The naive approach yields and upper bound on the rank of 6. ☐

(20) $(\phi \lrcorner a^*) = \begin{pmatrix} 0 & a_1 & a_2 + \mu a_3 \\ a_2 & 0 & a_3 \\ a_3 & a_2 & 0 \end{pmatrix}, (\phi \lrcorner b^*) = \begin{pmatrix} b_2 & b_3 & \mu b_3 \\ 0 & b_1 & b_3 \\ 0 & b_2 & b_1 \end{pmatrix}, (\phi \lrcorner c^*) =$

$$\begin{pmatrix} 0 & c_2 & c_3 \\ c_1 & c_3 & 0 \\ 0 & c_1 & \mu c_1 + c_3 \end{pmatrix}. \ T = a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_3 \otimes c_1 + a_2 \otimes b_1 \otimes c_2 + a_2 \otimes b_2 \otimes c_3 +$$
$$a_3 \otimes (\mu b_3) \otimes c_1 + a_3 \otimes b_3 \otimes c_2 + a_3 \otimes b_1 \otimes c_3.$$

Proposition .39. $\boldsymbol{R}(T) = 5, \underline{\boldsymbol{R}}(T) = 4.$

Proof. Notice that the transpose of (20) as a linear map $B^* \to A \otimes C$, (21) is equivalent to (21) (as a linear map $B^* \to A \otimes C$). Thus, $\mathbf{R}(T) = 5$. ☐

(21) $(\phi \lrcorner a^*) = \begin{pmatrix} 0 & a_1 & 0 \\ a_2 & a_3 & a_1 \\ a_3 & 0 & \mu a_1 + a_2 \end{pmatrix}, (\phi \lrcorner b^*) = \begin{pmatrix} b_2 & 0 & 0 \\ b_3 & b_1 & b_2 \\ \mu b_3 & b_3 & b_1 \end{pmatrix}, (\phi \lrcorner c^*) =$

$\begin{pmatrix} 0 & c_2 & c_3 \\ c_1 & 0 & c_2 \\ c_2+\mu c_3 & c_3 & 0 \end{pmatrix}$. $T = a_1{\otimes}b_2{\otimes}c_1 + a_1{\otimes}b_3{\otimes}c_2 + a_1{\otimes}(\mu b_3){\otimes}c_3 + a_2{\otimes}b_3{\otimes}c_3 + a_2{\otimes}b_1{\otimes}c_2 + a_3{\otimes}b_2{\otimes}c_2 + a_3{\otimes}b_1{\otimes}c_3$.

Proposition .40. $\boldsymbol{R}(T) = 5, \underline{\boldsymbol{R}}(T) = 4$.

Proof. Notice that as a linear map $B^* \to A{\otimes}C$, (21) is equivalent to (22) (as a linear map $C^* \to A{\otimes}B$). Thus, $\mathbf{R}(T) = 5$. □

(22) $(\phi\lrcorner a^*) = \begin{pmatrix} 0 & a_2 & a_3 \\ a_1 & a_3 & 0 \\ 0 & a_1 & \mu a_1 + a_2 \end{pmatrix}, (\phi\lrcorner b^*) = \begin{pmatrix} 0 & b_2 & b_3 \\ b_1 & 0 & b_2 \\ b_2+\mu b_3 & b_3 & 0 \end{pmatrix}, (\phi\lrcorner c^*) = \begin{pmatrix} c_2 & 0 & 0 \\ c_3 & c_1 & c_2 \\ \mu c_3 & c_3 & c_1 \end{pmatrix}$. $T = a_1{\otimes}b_1{\otimes}c_2 + a_1{\otimes}b_2{\otimes}c_3 + a_1{\otimes}(\mu b_3){\otimes}c_3 + a_2{\otimes}b_2{\otimes}c_1 + a_2{\otimes}b_3{\otimes}c_3 + a_3{\otimes}b_3{\otimes}c_1 + a_3{\otimes}b_2{\otimes}c_2$.

Proposition .41. $\underline{\boldsymbol{R}}(T) = 4, \boldsymbol{R}(T) = 5$.

Proof. Consider the following (row-equivalent) subspace of $T(C^*)$:

$$\begin{pmatrix} c_2 & 0 & 0 \\ 0 & c_3 & 0 \\ c_3 & c_1 & c_2 \end{pmatrix}.$$

I will show that the number of rank one matrices needed to span (a space containing) this space is at least 5. Since rank is non-decreasing, this proves the rank of our original tensor is at least 5. Note that the naive approach yields a rank of 5 so this would prove the claim.

As before, assume $\mathbf{R}(T) = 4$. Then make the identification

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, S_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, P = \begin{pmatrix} s_1t_1 & s_1t_2 & s_1t_3 \\ s_2t_1 & s_2t_2 & s_2t_3 \\ s_3t_1 & s_3t_2 & s_3t_3 \end{pmatrix}$$

so that $< S_1, S_2, S_3 > \in < X_1, X_2, X_3, P >$ for matrices of rank one X_i. In particular, $S_1 \in < S_2, S_3, P, X_i >$ for some i. That is, we would be able to find constant α, β such that

$$X_i := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ \alpha & \beta & 1 \end{pmatrix} + \begin{pmatrix} s_1t_1 & s_1t_2 & s_1t_3 \\ s_2t_1 & s_2t_2 & s_2t_3 \\ s_3t_1 & s_3t_2 & s_3t_3 \end{pmatrix}$$

would be of rank one. I first show $\alpha \neq 0$. Suppose the contrary. Then

$$< P, X_i > = \left\langle \begin{pmatrix} 0 & \beta r_2 & r_2 \\ 0 & 0 & 0 \\ 0 & -\beta & -1 \end{pmatrix}, \begin{pmatrix} -1 & \beta r_2 & r_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle$$

or

$$< P, X_i > = \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r & -\beta & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ r & 0 & 0 \end{pmatrix} \right\rangle$$

as per the logic used in the proof of (9). In the first case, notice that to generate the subspace

$$c_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

one would either add this matrix to the spanning set as some X_i, which would mean it would be impossible to generated the subspace

$$c_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

or one would need at least two rank one matrices to eliminate other entries. Thus this case is impossible. The situation is nearly identical in the second case, thus $\alpha \neq 0$ and I now assume this.

Note that if $s_1 = 0$ then X_i is visibly of rank at least 2. So assume $s_1 \neq 0$ and subtract $\frac{s_2}{s_1}$ times row one from row two and $\frac{s_3}{s_1}$ times row one from row three in that order to obtain

$$\begin{pmatrix} 1+s_1t_1 & s_1t_2 & s_1t_3 \\ -\frac{s_2}{s_1} & \alpha & 0 \\ \alpha - \frac{s_3}{s_1} & \beta & 1 \end{pmatrix}.$$

The bottom two rows are independent so the matrix has rank one, in particular, only if $\alpha = 0$, which I have shown to not be the case. Thus, X_i is always of rank at least 2 and equivalently, $\mathbf{R}(T) \geq 5$. $\square$

(23) $$(\phi \lrcorner a^*) = \begin{pmatrix} \begin{array}{c}[(\lambda-1)a_1 \\ +(\lambda-1)^2(\lambda^2+\lambda+1)a_2 \\ +(\lambda^2-1)(\lambda^2+\lambda+1)a_3]\end{array} & (\lambda+1)x_1 & (\lambda^2+1)a_2+(\lambda^2-1)a_3 \\ -(\lambda+1)a_1 & -(\lambda-1)a_1 & (\lambda^2-1)a_2+(\lambda^2+1)a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}$$

$(\phi \lrcorner b^*) =$

$$\begin{pmatrix} (\lambda-1)b_1+(\lambda+1)b_2 & [(\lambda-1)^2(\lambda^2+\lambda+1)b_1 & [(\lambda^2-1)(\lambda^2+\lambda+1)b_1 \\ & +(\lambda^2+1)b_3] & +(\lambda^2-1)b_3] \\ -(\lambda+1)b_1-(\lambda-1)b_2 & (\lambda^2-1)b_3 & (\lambda^2+1)b_3 \\ 0 & -b_1 & -b_2 \end{pmatrix}$$

$$(\phi\lrcorner c^*) = \begin{pmatrix} (\lambda-1)c_1-(\lambda+1)c_2 & (\lambda-1)^2(\lambda^2+\lambda+1)c_1 & (\lambda^2-1)(\lambda^2+\lambda_1)c_1 \\ (\lambda+1)c_1-(\lambda-1)c_2 & 0 & -c_3 \\ 0 & (\lambda^2+1)c_1+(\lambda^2-1)c_2 & (\lambda^2-1)c_1+(\lambda^2+1)c_2 \end{pmatrix}.$$

Proposition .42. $\boldsymbol{R}(T) \leq 7, \underline{\boldsymbol{R}}(T) = 4$ *for* $\lambda = -1, \underline{\boldsymbol{R}}(T) = 5$ *for* $\lambda \neq -1$.

Proof. The naive approach gives an upper bound on the rank of 7. ☐

(24) $(\phi\lrcorner a^*) = \begin{pmatrix} \mu^2 a_2 - \mu a_3 & a_1 & a_2 \\ -a_1 & \mu a_1 & -2\mu a_2 + a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}, (\phi\lrcorner b^*) =$

$$\begin{pmatrix} b_2 & \mu^2 b_1 + b_3 & -\mu b_1 \\ -b_1+\mu b_2 & -2\mu b_3 & b_3 \\ 0 & -b_1 & -b_2 \end{pmatrix}, (\phi\lrcorner c^*) = \begin{pmatrix} -c_2 & \mu^2 c_1 - c_3 & -\mu c_1 \\ c_1+\mu c_2 & 0 & -c_3 \\ 0 & c_1 - 2\mu c_2 & c_2 \end{pmatrix}.$$

$T = a_1 \otimes b_2 \otimes c_1 + (-a_1) \otimes b_1 \otimes c_2 + (\mu a_1) \otimes b_2 \otimes c_2 + (\mu^2 a_2) \otimes b_1 \otimes c_1 + a_2 \otimes b_3 \otimes c_1 +$
$(-2\mu a_2) \otimes b_3 \otimes c_2 + (-a_2) \otimes b_1 \otimes c_3 + (-\mu a_3) \otimes b_1 \otimes c_1 + a_3 \otimes b_3 \otimes c_2 + (-a_3) \otimes b_2 \otimes c_3.$

Proposition .43. $\boldsymbol{R}(T) \leq 7, \underline{\boldsymbol{R}}(T) = 5.$

Proof. The naive approach yields an upper bound of 7 for the rank. ☐

CHAPTER IV
SUMMARY AND CONCLUSIONS

Building on Ng's classification of all $\phi \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ up to the action of $PGL(\mathbb{C}^3) \times PGL(\mathbb{C}^3) \times PGL(\mathbb{C}^3)$, I have presented a partial classification of these tensors with respect to their rank and border rank. In particular, upper bounds were found for each of the 24 normal forms given in Ng's paper and the bounds were proved to be tight in 15 cases. The methods of proof varied but were largely algebraic and far from a routine task. At times a suitable upper bound was visibly immediate but in other cases one used more advanced techniques such as considering the tensor as a direct sum of a pencil (two dimensional space of matrices) and a one dimensional space and putting the pencil in Kronecker normal form then invoking a theorem of Grigoriev and Ja'Ja' to deduce it's rank [1]. This technique is not discussed because in the cases I was able to classify, it was always possible to achieve this same upper bound using more basic techniques, however it mya be the case that this will be useful in the completion of the classification.

The border ranks were calculated using Ottaviani's version of Strassen's equations. I implemented these equations in the program Wolfram Mathematica 8 and recorded the results.

The main results are best summarized in Table 1 below.

Table 1: Summary of main results.

#	$\underline{\mathbf{R}}(T)$	$\mathbf{R}(T)$
(1)	3	3
(2)	4	4
(3)	4	4
(4)	4	4
(5)	4	5
(6)	4	4
(7)	5	≤ 6
(8)	4	5
(9)	4	5
(10)	4	5
(11)	5	≤ 6
(12)	4	5

#	$\underline{\mathbf{R}}(T)$	$\mathbf{R}(T)$
(13)	4	5
(14)	4	5
(15)	4 for $\lambda = -1, 5$ for $\lambda \neq -1$	≤ 7
(16)	4	≤ 6
(17)	4	≤ 6
(18)	4	≤ 6
(19)	5	≤ 6
(20)	4	5
(21)	4	5
(22)	4	5
(23)	4 for $\lambda = -1, 5$ for $\lambda \neq -1$	≤ 7
(24)	5	≤ 7

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