MEASURE THEORY OF SELF-SIMILAR GROUPS AND DIGIT TILES

A Dissertation

by

ROSTYSLAV KRAVCHENKO

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2010

Major Subject: Mathematics

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ABSTRACT

Measure Theory of Self-Similar Groups and Digit Tiles. (December 2010)

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This dissertation is devoted to the measure theoretical aspects of the theory of automata and groups generated by them. It consists of two main parts. In the first part we study the action of automata on Bernoulli measures. We describe how a finite-state automorphism of a regular rooted tree changes the Bernoulli measure on the boundary of the tree. It turns out, that a finite-state automorphism of polynomial growth, as defined by Sidki, preserves a measure class of a Bernoulli measure, and we write down the explicit formula for its Radon-Nikodim derivative. On the other hand the image of the Bernoulli measure under the action of a strongly connected finite-state automorphism is singular to the measure itself.

The second part is devoted to introduction of measure into the theory of limit spaces of Nekrashevysh. Let G be a group and $\phi : H \to G$ be a contracting homomorphism from a subgroup $H < G$ of finite index. Nekrashevych associated with the pair (G, ϕ) the limit dynamical system $(\mathcal{J}_G, \mathsf{s})$ and the limit G-space $\mathcal{X}_G$ together with the covering $\cup_{g\in G}\mathcal{T} \cdot g$ by the tile $\mathcal{T}$. We develop the theory of self-similar measures m on these limit spaces. It is shown that $(\mathcal{J}_G, \mathsf{s}, \mathsf{m})$ is conjugate to the one-sided Bernoulli shift. Using sofic subshifts we prove that the tile $\mathcal{T}$ has integer measure and we give an algorithmic way to compute it. In addition we give an algorithm to find the measure of the intersection of tiles $\mathcal{T} \cap (\mathcal{T} \cdot g)$ for $g \in G$. We present applications to the evaluation of the Lebesgue measure of integral self-affine tiles.

Previously the main tools in the theory of self-similar fractals were tools from measure theory and analysis. The methods developed in this disseration provide a new way to investigate self-similar and self-affine fractals, using combinatorics and group theory.

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CHAPTER I

INTRODUCTION

Automata, and especially groups generated by finite automata, play important role in different areas of mathematics, producing examples and counterexamples for many problems, including some famous ones. For instance, the Grigorchuk group ([Gri83]) was the first example of a group of intermediate growth, which gave an answer to a question of Milnor. It is also a particularly simple example of an infinite torsion group. Another example is the solution of the "twisted rabbit" problem of Hubbard by Bartholdi and Nekrashevych ([BN06]).

In [Rya86] Ryabinin computed the so called "stochastic function" of a finite automaton. He applied the automaton to a sequence of 0-s and 1-s with independently chosen entries, with probability of 1 equal to p, and computed frequency $f(p)$ of 1 in the resulting sequence. He called this function f the "stochastic function" of the automaton, and also gave a characterization of the class of all such functions. His treatment was somewhat 'naive', for instance he did not rigorously define the frequency of the resulting sequence; we will make it precise in what follows. More about his result can be found in [KAP85].

The group of all invertible automata over a given alphabet is quite complicated. Thus there were various attempts to single out special classes of automata. One such attempt, which used the structure of the action of an automaton on the tree of words over the alphabet of the automaton, was done by Sidki in [Sid00]. He defined a notion of automata of polynomial growth of degree n, showed that the class of such automata is closed under composition and taking an inverse, and that any group of

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This dissertation follows the style of *Algebra and Discrete Mathematics.*

automata of polynomial growth of degree n does not contain a free group of rank two. He also reformulated his definition of an automaton of polynomial growth in terms of the Moore diagram of the automaton.

Examples of such automata are the Aleshin automaton ([VV07]) which is famous since it was the first automaton such that its states freely generate the free group ([Nek05]), and the Bellaterra automaton ([Nek05]).

The uniform Bernoulli measure on the set of sequences is useful for a variety of questions concerning automata since it is invariant under their action, see [BG00]. Thus it seems natural to consider the more general case of an arbitrary Bernoulli measure and see how it interplays with an action of automata.

In Chapter III we study the push-forward of a Bernoulli measure on a set of infinite words under an action of automaton. The results of Chapter III are published in paper [Kra10]. Firstly, Theorem 1 shows that if an invertible automaton has polynomial growth, then the action defined by any of its states maps a Bernoulli measure to the absolutely continuous one with respect to it. Thus an invertible automaton of polynomial growth preserves the class of Bernoulli measure. Then we study the action of a strongly connected automata. In Theorem 3 we make a rigorous statement of the result of Ryabinin and generalize it for an arbitrary alphabet. Theorem 4 shows that if a Bernoulli measure if not a uniform one, then its push-forward under the action of a strongly connected automata is singular to the Bernoulli measure.

Chapter IV of this dissertation is devoted to the development of measure theory in the setting of limit spaces. The result of Chapter IV are published in paper [BK], written jointly with I. Bondarenko.

Let G be a group and ϕ be a virtual endomorphism of G, which is a homomorphism from a subgroup $H < G$ of finite index to G. Iterative construction involving ϕ (together with some additional data) produces the so called self-similar action (G, X^*) of the group G on the space X^* of finite words over an alphabet X. And conversely, every self-similar action of the group G defines a virtual endomorphism of G, which almost completely describes the action. A rich geometric theory is associated with the pair (G, ϕ) in [Nek05] through the theory of self-similar groups. In Chapter IV we introduce measure to this theory.

Self-similar group is a rather new notion in geometric group theory. Like the self-similar objects in geometry (fractals) are too irregular to be described using the language of classical Euclidean geometry, the self-similar groups possess properties not typical for the traditional group theory. In particular, the class of self-similar groups contains infinite periodic finitely generated groups, just-infinite groups, groups of finite width, etc. (see [Nek05, BGN03, BGZ03] and references therein). At the same time, it was discovered that self-similar groups appear naturally in many areas of mathematics, and have applications to holomorphic dynamics, combinatorics, analysis on fractals, etc. An important class of self-similar groups are contracting groups, which correspond to self-similar actions with contracting virtual endomorphism. A virtual endomorphism ϕ is contracting if it asymptotically contracts the length of group elements with respect to some generating set. The contracting property makes many problems around the group effectively solvable.

V. Nekrashevych in [Nek05] associated a limit dynamical system $(\mathcal{J}_G, \mathsf{s})$ with every contracting self-similar action, where $\mathcal{J}_G$ is a compact metrizable space and $\mathsf{s} : \mathcal{J}_G \to \mathcal{J}_G$ is an expanding continuous map. The limit space $\mathcal{J}_G$ can be defined as the quotient of the space of left-infinite sequences $X^{-\omega} = \{\ldots x_2 x_1 | x_i \in X\}$ by the equivalence relation, which can be recovered from a finite directed labeled graph $\mathcal{N}$, called the nucleus of the action. Another associated geometric object is the limit G-space $\mathcal{X}_G$, which is a metrizable locally compact topological space with a proper co-compact (right) action of G. The limit spaces $\mathcal{J}_G$ and $\mathcal{X}_G$ depend up to homeomorphism only on the pair (G,ϕ). However every self-similar action with the pair (G,ϕ) additionally produces a tile $\mathcal{T}$ of the limit G-space and a covering $\mathcal{X}_G = \cup_{g\in G}\mathcal{T}\cdot g$ (not a tiling in general).

Limit spaces connect self-similar groups with the classical self-similar sets. The self-similar set (the attractor) given by the system of contracting similarities $f_1,\ldots,f_n$ (iterated function system) of a complete metric space is the unique compact set T satisfying $T = \cup_{i=1}^n f_i(T)$. Given a probability vector $p = (p_1,\ldots,p_n)$, Hutchinson [Hut81] showed the existence of a unique probability measure μ supported on T satisfying

$$\mu(A) = \sum_{i=1}^{n} p_i \mu(f_i^{-1}(A)), \text{ for any Borel set } A, \tag{1.1}$$

which is called the self-similar measure. Another way to introduce this measure is to consider the natural coding map $\pi : X^{-\omega} \to T$ given by $\pi(\ldots x_2x_1) = \cap_{m\geq 1} f_{x_1}\circ f_{x_2}\circ \ldots \circ f_{x_m}(T)$. Then the self-similar measure μ is the image of the Bernoulli measure μ_p on $X^{-\omega}$ with weight p (here $\mu_p(x_i) = p_i$) under the projection π. Self-similar measures play an important role in the development of fractal geometry, and have applications in harmonic analysis, conformal dynamics, algebraic number theory, etc. (see [Edg98, Urb03, LNR01, Str94, Ban01] and references therein).

In subsection 1 we study the Bernoulli measure of sofic subshifts and other sets given by a finite directed graph $\Gamma = (V,E)$, whose edges are labeled by elements of X. Consider the set F_v for $v \in V$ of all sequences $\ldots x_2x_1$, which are read along left-infinite paths ending in the vertex v. It is proved in subsection 1 that if the graph Γ is right-resolving (i.e. for every vertex v the outgoing edges at v are labeled distinctly) then the sum $meas(\Gamma) = \sum_{v\in V}\mu_p(F_v)$ is integer, which does not depend on the probability vector p. It can be interpreted as follows: almost every left-infinite sequence belongs to precisely $meas(\Gamma)$ sets F_v. The number $meas(\Gamma)$ we call the measure number of the graph Γ. We propose an algorithmic method to compute the measures $\mu_p(F_v)$ for any graph Γ, in particular its measure number.

The push-forward of the uniform Bernoulli measure on $X^{-\omega}$ provides the self-similar measure m on the limit space $\mathcal{J}_G$. The G-invariant measure μ on the limit G-space $\mathcal{X}_G$ is defined in a similar way. The measure μ restricted to the tile $\mathcal{T}$ satisfies the self-similarity equation (1.1), so it is also self-similar. It is proved in subsection 3 that the measure $\mu(\mathcal{T})$ is equal to the measure number of the nucleus $\mathcal{N}$. In particular it is integer, the fact which generalizes corresponding result for integral self-affine tiles [LW96a]. In addition we give an algorithm to find the measure of intersection of tiles $\mathcal{T} \cap (\mathcal{T} \cdot g)$ for $g \in G$. Then the covering $\mathcal{X}_G = \cup_{g \in G} \mathcal{T} \cdot g$ is a perfect multiple covering of multiplicity $\mu(\mathcal{T})$, i.e. every point of $\mathcal{X}_G$ belongs to at least $\mu(\mathcal{T})$ tiles and almost every point belongs to precisely $\mu(\mathcal{T})$ tiles. This is used to prove that the measures m and μ depend not on the specific self-similar action of G, but only on the pair (G, ϕ) as the limit spaces themselves. Using a criterion from [HR02] we show that the limit dynamical system $(\mathcal{J}_G, \mathsf{s}, \mathsf{m})$ is conjugated to the one-sided Bernoulli shift.

This work is partially motivated by applications presented in subsection 4. If G is a torsion-free nilpotent group with a contractive surjective virtual endomorphism ϕ and a faithful self-similar action, then the measure μ on $\mathcal{X}_G$ can be considered as a Haar measure on the respective nilpotent Lie group, Malcev's completion of G. In the case of self-similar actions of the free abelian group $\mathbb{Z}^n$ the limit G-space $\mathcal{X}_{\mathbb{Z}^n}$ is $\mathbb{R}^n$ and the tile $\mathcal{T}$ is an integral self-affine tile, which are intensively studied for the last two decades (see [LW96b, Vin00, LW96a, LW97, HLR03]). In this case the measure μ is the Lebesgue measure on $\mathbb{R}^n$. One can apply the methods developed in subsection 1 to give an algorithmic way to find the Lebesgue measure of an integral self-affine tile, providing answer to the question in [LW96a] (initially solved in [GY06] without self-similar groups). In addition we have an algorithm to find the Lebesgue measure of the intersection of tiles $\mathcal{T} \cap (\mathcal{T} + a)$ for $a \in \mathbb{Z}^n$ studied in [GY06, EKM09].

CHAPTER II

DEFINITIONS AND PRELIMINARIES*

Let X be a finite set with discrete topology. Denote by $X^* = \{x_1x_2\ldots x_n | x_i \in X, n \geq 0\}$ the set of all finite words over X (including the empty word denoted $\emptyset$). Let X^ω be the set of all right-infinite sequences (words) $x_1x_2\ldots$, $x_i \in X$. Let $X^{-\omega}$ be the set of all left-infinite sequences (words) $\ldots x_2x_1$, $x_i \in X$. We put the product topology on these sets. The length of a word $v = x_1x_2\ldots x_n$ is denoted by $|v| = n$.

The *shift on the space* X^ω is the map $\sigma : X^\omega \to X^\omega$, which deletes the first letter of a word, i.e. $\sigma(x_1x_2x_3\ldots) = x_2x_3\ldots$. The *shift on the space* $X^{-\omega}$ is the map also denoted by σ, which deletes the last letter of a word, i.e. $\sigma(\ldots x_3x_2x_1) = \ldots x_3x_2$. The shifts are continuous $|X|$-to-1 maps. The branches σ_x for $x \in X$ of the inverse of σ are defined by $\sigma_x(x_1x_2\ldots) = xx_1x_2\ldots$ and $\sigma_x(\ldots x_2x_1) = \ldots x_2x_1x$.

We interpret X^* as the set of vertices of a rooted tree. Let $g : X^* \to X^*$ be an endomorphism of X^*. For a word v from X^* let vX^* be the subset of words that have v as a beginning. The endomorphism g maps vX^* to $g(v)X^*$. Identifying vX^* and $g(v)X^*$ with X^* we get an endomorphism of X^*, which we denote by $g|_v$ and call the restriction of g in word v. We have that for each pair of finite words v, w $g(vw) = g(v)g|_v(w)$.

If g is a tree automorphism of X^* we can also extend g to the boundary of X^*, that is, define the action of g on infinite words from X^ω. Indeed, if w is an infinite word from X^ω, and w_m is its beginning of length m, then since g is an endomorphism, the word $g(\overline{w}_n)$ is the beginning of the word $g(w_{n+1})$ for all n. Thus we define $g(w)$

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as an infinite word such that $g(w_n)$ is its beginning for every n.

Definition 1. An automaton $A = (X, S, \pi, \lambda)$ over alphabet X consists of set of states S, transition function $\pi : S \times X \to S$ and output function $\lambda : S \times X \to X$.

Given tree endomorphism g we can construct an automaton by considering the set $\{g|_v : v \in X^*\}$ of all restrictions of g as the set of states S, with the transition function $\pi(s, x) = s|_x$, and the output function $\lambda(s, x) = s(x)$, for $s \in S$, $x \in X$. We will call it the automaton of restrictions of g.

An automaton is usually visualized with the help of its Moore diagram. It is a directed labeled graph with the set of states S as vertices, with each state s labeled by permutation $x \mapsto \lambda(s, x)$ and with arrows going from s to $\pi(s, x)$ labeled by x.

A finite automaton is an automaton with finite set of states. A tree endomorphism g is called finite-state if the automaton of restrictions of g is finite.

The output and transition function of an automaton can be extended to the set of words over X, by the inductive rules

$$\lambda(s, xv) = \lambda(s, x)\lambda(\pi(s, x), v), \quad \pi(s, xv) = \pi(\pi(s, x), v),$$

where v is a word and x is a letter from X. In the case of an automaton of restrictions of tree endomorphism, we have that $\lambda(s, v) = s(v)$, and $\pi(s, v) = s|_v$, for any finite word v.

Let now g be a finite-state endomorphism of X^*. We say that g has polynomial growth if the number $\alpha(g, k)$ of words v of length k, such that $g|_v$ is nontrivial endomorphism of a rooted tree, (the trivial endomorphism is such that maps each vertex to itself), grows polynomially with k. Notice that if g has polynomial growth then for some word v $g|_v$ is trivial, since the set of all words of length k grows exponentially with k. Thus the automaton of restrictions of g has a trivial state. The fact that g has polynomial growth is equivalent to the fact that the Moore diagram of the automaton of restrictions of g does not have a vertex with two different nontrivial simple cycles going through it, where by the nontrivial cycle we mean a cycle that does not contain a trivial state, and by a simple cycle we mean a cycle without self-intersections.

If the automaton of restrictions of a endomorphism g has strongly connected Moore diagram, by which we mean that for every two vertices there is a directed path that goes from one vertex to another, then we say that g is strongly connected. In a sense it is the opposite notion to the notion of a endomorphism of polynomial growth: it is easy to see that a strongly connected endomorphism g does not have a trivial state, and through each state of its automaton of restrictions pass at least two different cycles.

Let $p = (p(x))_{x \in X}$ be a probability vector (fixed for the rest of the chapter) and let μ_p be the *Bernoulli measure* on X^ω with weight p, i.e. this measure is defined on cylindrical sets by

$$\mu_p(x_1 x_2 \dots x_n X^\omega) = p(x_1) p(x_2) \dots p(x_n).$$

The measure on $X^{-\omega}$ is defined in the same way. We always suppose that $p_x > 0$ for all $x \in X$ (otherwise we can pass to a smaller alphabet X). In case $p_x = \frac{1}{|X|}$ for all $x \in X$, the measure μ_p is the *uniform Bernoulli measure* denoted μ_u. The dynamical system $(X^\omega, \sigma, \mu_u)$ is called the *one-sided Bernoulli $|X|$-shift*. The measure μ_p is the unique regular Borel probability measure on X^ω that satisfies the self-similarity condition:

$$\mu_p(A) = \sum_{x \in X} p(x) \mu_p(\sigma_x^{-1}(A))$$

for any Borel set $A \subset X^\omega$.

CHAPTER III

THE ACTION OF FINITE STATE TREE AUTOMORPHISMS ON BERNOULLI MEASURES*

In this chapter we will denote the Bernoulli measure μ_p simply by μ, since the vector p is fixed.

1 Tree automorphism of polynomial growth

We want to prove that if g has polynomial growth, then the measure $g_*\mu$ is absolutely continuous with respect to μ. In fact, it is easy to produce the Radon-Nikodim derivative $dg_*\mu/d\mu$, as follows. Denote by V the set of words from X^*, such that the restriction of g^{-1} in every $v \in V$ is the identical transformation. Let V_{max} be the subset of all such words v from V, such that no proper prefix of v belongs to V.

Lemma 1. *Let g have polynomial growth, then $X^\omega - \cup_{v\in V_{max}} vX^\omega$ is at most countable, thus its measure μ is 0.*

Proof. Note that since g has polynomial growth, g^{-1} also has polynomial growth. An infinite word w belongs to $X^\omega - \cup_{v\in V_{max}} vX^\omega$ if and only if $g^{-1}|_{w_n}$ is not trivial, for any natural n, where w_n is prefix of length n of the word w. Consider the Moore diagram of the automaton of the restrictions of g^{-1}. The word w defines an infinite path $g^{-1}|_{w_n}$ in the Moore diagram of this automaton, that consists of nontrivial states. Suppose the vertex s happens infinitely often in the path defined by w. It means that the path contains a cycle that passes through s. Since g^{-1} has polynomial growth there

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is only one such cycle, (see [Ufn82]), and it means that the path, after it has passed through s the first time, stays on the cycle after that. Thus w is eventually periodic. The set of eventually periodic words is countable. $\square$

We have

Theorem 1. *For an automorphism g of polynomial growth and Bernoulli measure μ, the push-forward $g_*\mu$ is absolutely continuous with respect to μ and*

$$\frac{dg_*\mu}{d\mu} = \sum_{v \in V_{max}} \frac{\mu(g^{-1}(vX^\omega))}{\mu(vX^\omega)} \chi_{vX^\omega}.$$

Proof. Let

$$g' = \sum_{v \in V_{max}} \frac{\mu(g^{-1}(vX^\omega))}{\mu(vX^\omega)} \chi_{vX^\omega}.$$

Since cylindrical sets generate the Borel $\sigma-$algebra of X^ω, it suffices to check that the measures $dg_*\mu$ and $g'.d\mu$ agree on the cylindrical sets. Note also that since the set $X^\omega - \cup_{v\in V} vX^\omega = X^\omega - \cup_{v \in V_{max}} vX^\omega$ is at most countable, any cylindrical set not of the form vX^ω for some $v \in V$ can be expressed as a union of cylindrical sets $\{vX^\omega | v \in V\}$ modulo countable subset. Since both measures $dg_*\mu$ and $g'.d\mu$ are continuous, in order to show that they are equal it suffices to check that they agree on all sets of the form $\{vX^\omega | v \in V\}$.

Take any v in V. Then there is a unique v' in V_{max} such that v' is the prefix of v. It follows that $v = v'w$ and we have

$$\int_{vX^\omega} g' d\mu = \frac{\mu(g^{-1}v'X^\omega)}{\mu(v'X^\omega)} \mu(vX^\omega) = \mu(g^{-1}v'X^\omega)\mu(wX^\omega).$$

On the other hand, the fact that v' is in V implies that $g^{-1}(v'w) = g(v')w$, thus $g_*\mu(vX^\omega) = \mu(g^{-1}vX^\omega) = \mu(g^{-1}(v')wX^\omega) = \mu(g^{-1}v'X^\omega)\mu(wX^\omega)$. $\square$

Let us compute the corresponding function for the automorphism a of the binary tree: a swaps the 0 and 1 in the beginning of every word. Then a has polynomial growth, the set V consists of all non-empty words, and $V_{max} = \{0, 1\}$. If μ assigns probability $p(0)$ to 0 and $p(1)$ to 1, we have:

$$\frac{da_*\mu}{d\mu} = \frac{\mu(a(0X^\omega))}{\mu(0X^\omega)}\chi_{0X^\omega} + \frac{\mu(a(1X^\omega))}{\mu(1X^\omega)}\chi_{1X^\omega} = \frac{p(1)}{p(0)}\chi_{0X^\omega} + \frac{p(0)}{p(1)}\chi_{1X^\omega}.$$

Let a, b, c, d be the states of the automaton that defines the Grigorchuk group. Consider the automorphism b. Denote by v_n the word of length $n+1$ with n 1's at the beginning and one 0 at the end. Then the set V_{max} for b consists of all words v_n for $n = 2 \mod 3$ and all words v_n0, v_n1 for $n \neq 2 \mod 3$. Moreover $b(v_n) = v_n$ for any n, and $b(v_n0) = v_n1$, and $b(v_n1) = v_n0$. Thus the Radon-Nikodim derivative for b and the measure μ is

$$\frac{db_*\mu}{d\mu} = \sum_{n=2 \mod 3} \chi_{v_nX^\omega} + \frac{p(1)}{p(0)} \sum_{n\neq 2 \mod 3} \chi_{v_n0X^\omega} + \frac{p(0)}{p(1)} \sum_{n\neq 2 \mod 3} \chi_{v_n1X^\omega}.$$

2 Strongly connected tree automorphism

Let g be the finite-state strongly connected tree endomorphism of X^*, and S be the set of its restrictions, which we also call the set of states of g. Let w be an infinite word and x a letter in X. We are interested in the frequency of x-s in the image of w under the action of the tree endomorphism g. In other words, we seek the existence and the value of the limit when n goes to infinity of a sequence

$$\frac{1}{n}\sum_{k=0}^{n-1} \chi_x(\sigma^k g(w)),$$

where χ_x is a characteristic function of the subset xX^ω.

We start by considering a sequence of random variables $\zeta_n : X^\omega \to S$, such that

ζ_1 is constant and $\zeta_{n+1}(w) = \pi(\zeta_n(w), w_n)$. We have the following lemma:

Lemma 2. *ζ_n is an ergodic Markov chain.*

Proof. Note that $\zeta_n(w)$ depends only on the beginning of w of length $n-1$. We then have that

$$\mu(\zeta_{n+1} = s_{n+1} | \zeta_n = s_n, \ldots, \zeta_1 = s_1) = \mu(\pi(s_n, w_n) = s_{n+1} | \zeta_n = s_n, \ldots, \zeta_1 = s_1) =$$
$$\mu(\pi(s_n, w_n) = s_{n+1}),$$

provided that $\mu(\zeta_n = s_n, \ldots, \zeta_1 = s_1) > 0$, since sets $\{w | \pi(s_n, w_n) = s_{n+1}\}$ and $\{w | \zeta_n = s_n, \ldots, \zeta_1 = s_1\}$ are independent. Thus ζ_n is indeed a Markov chain. The corresponding transition probability from s to s' is equal to $\sum p(x)$ where the sum runs through all such x that $\pi(s, x) = s'$. Since g is strongly connected, it follows that for any s, s' there is a path from s to s', thus the probability to get from s to s' is positive. It follows that ζ_n is ergodic. $\square$

Let q denote the stationary distribution of the chain ζ_n. Consider the sequence $w \mapsto (\zeta_n(w), w_n)$. We have the lemma:

Lemma 3. *(ζ_n, w_n) is an ergodic Markov chain with state space $S \times X$ and stationary distribution $q \otimes p$, $q \otimes p(s, x) = q(s)p(x)$.*

Proof. We have

$$\mu(\zeta_{n+1} = s_{n+1}, w_{n+1} = x_{n+1} | \zeta_n = s_n, w_n = x_n, \ldots, \zeta_1 = s_1, w_1 = x_1) =$$
$$\mu(\pi(s_n, x_n) = s_{n+1}, w_{n+1} = x_{n+1} | \zeta_n = s_n, w_n = x_n, \ldots, \zeta_1 = s_1, w_1 = x_1) =$$
$$\begin{cases} p(x_{n+1}) \text{ if } \pi(s_n, x_n) = s_{n+1} \\ \quad 0 \text{ otherwise} \end{cases},$$

if $\mu(\zeta_n = s_n, w_n = x_n, \ldots, \zeta_1 = s_1, w_1 = x_1) > 0$, thus (ζ_n, w_n) is a Markov chain. Denote the corresponding transition probability by $p((s, x), (s', y))$. To prove that

(ζ_n, w_n) is ergodic, for any two elements (s, x) and (s', y) we have to construct a sequence of elements of $S \times X$, which starts at (s, x) and ends at (s', y), such that the consecutive transition probabilities are non-zero. Since g is strongly connected there exists a sequence of states $s_1 = s, s_2, \ldots, s_m = s'$ and a sequence of elements of X $x_1 = x, \ldots, x_{m-1}$, such that $\pi(s_k, x_k) = s_{k+1}$. Then $p((s_k, x_k), (s_{k+1}, x_{k+1})) = p(x_{k+1}) > 0$, which shows that the Markov chain (ζ_n, w_n) is ergodic.

We have that $\sum_{s \in S} q(s) \sum_{x \in X : \pi(s,x) = s'} p(x) = q(s')$, since $\sum_{x \in X : \pi(s,x) = s'} p(x)$ is the transition probability for the Markov chain ζ_n. It follows

$$\sum_{s \in S, x \in X} q(s)p(x)p((s, x), (s', y)) = \sum_{s,x : \pi(s,x) = s'} q(s)p(x)p(y) =$$

$$p(y) \sum_{s} q(s) \sum_{x \in X : \pi(s,x) = s'} p(x) = q(s')p(y),$$

and so $q \otimes p$ is the stationary distribution for the Markov chain (ζ_n, w_n). □

Let $\mathbf{P}$ be the shift invariant ergodic measure on $(S \times X)^\omega$, corresponding to the stationary distribution $q \otimes p$ and let $\mathbf{P}_g$ be the measure on $(S \times X)^\omega$, corresponding to the initial distribution $\delta_g \otimes p$. Note that $\mathbf{P}_g$ is absolutely continuous with respect to $\mathbf{P}$. Define a map h_g from X^ω to $(S \times X)^\omega$ in the following way. Take any sequence $w \in X^\omega$ and apply the transformation g to it. We then get also a sequence of states $\hat{s} = (g, g|_{w_1}, g|_{w_2}, \ldots)$, where w_n is the prefix of length n of w, which is obtained when reading w by g and we put $h_g(w) := (\hat{s}, w)$. We have the following lemma:

Lemma 4. *The push-forward of the measure μ under the map h_g is the measure $\mathbf{P}_g$.*

Proof. It follows from the fact that

$$h_g^{-1}((s_1, x_1) \ldots (s_m, x_m)(S \times X)^\omega) = \begin{cases} \emptyset \text{ if } s_1 \neq g \text{ or } \pi(s_k, x_k) \neq s_{k+1} \text{ for some } k \\ x_1 \ldots x_m X^\omega \text{ otherwise} \end{cases}$$

□

Consider the map $\tilde{\lambda} : (S \times X)^\omega \to X^\omega$,

$$\tilde{\lambda}((s_n, x_n)_{n\geq 1}) = (\lambda(s_n, x_n))_{n\geq 1}.$$

(note that $\tilde{\lambda} h_g = g$). Let $\mathbf{Q} = \tilde{\lambda}_* \mathbf{P}$. We have the lemma:

Lemma 5. *Let p' be the one-dimensional distribution of $\mathbf{Q}$, $p'(x) = \mathbf{Q}(xX^\omega), x \in X$. Then $p'(x) = \sum_{s\in S} q(s)p(\sigma_s^{-1}(x))$, where $\sigma_s(x) = \lambda(s, x)$.*

Proof. It follows from the equality

$$\tilde{\lambda}^{-1}(xX^\omega) = \coprod_{s\in S} (s, \sigma_s^{-1}(x))(S \times X)^\omega.$$

□

We can now prove the theorem,

Theorem 2. *The limit when n goes to infinity of*

$$\frac{1}{n}\sum_{k=0}^{n-1} \chi_x(\sigma^k g(w))$$

exists and is the same for almost all w with respect to the measure μ, and is equal to $p'(x)$, where p' is the one-dimensional distribution of $\mathbf{Q}$ from Lemma 5.

Proof. By the Birkgoff pointwise ergodic theorem we have that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_x(\sigma^k(v)) = \int_{v\in X} \chi_x(v) d\mathbf{Q}(v) = p'(x), \tag{3.1}$$

for $\mathbf{Q}$ -almost all v. Since $\mathbf{P}_g$ is absolutely continuous with respect to $\mathbf{P}$, $\tilde{\lambda}_*\mathbf{P}_g = (\tilde{\lambda} h_g)_*\mu = g_*\mu$ is absolutely continuous with respect to $\mathbf{Q}$. Thus the equality (3.1) holds also for almost all v with respect to the measure $g_*\mu$. Putting $v = g(w)$ in (3.1) we get

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_x(\sigma^k g(w)) = p'(x),$$

for μ -almost all w. $\square$

We reformulate 2 in the following more intuitive way:

Theorem 3. *Let g be a strongly connected tree endomorphism, $w \in X^\omega$. Let μ be the Bernoulli measure with the probability of y equal to $p(y)$ for $y \in X$. Then the frequency of x in the sequence $g(w)$ exists and is the same for almost all w with respect to μ and this frequency is equal to $\sum_{s\in S}(\sum_{y\in X}\chi_x(\lambda(s,y))p(y))q(s)$, where S is the set of restrictions of g and $q(s)$ are the stationary probabilities for the ergodic Markov chain on S, $\zeta_{n+1} = \pi(\zeta_n, w_n)$ defined by the transition probabilities $\sum_{y:\pi(s,y)=t} p(y)$.*

Using the ergodic theorem once again, we can derive the following theorem:

Theorem 4. *Suppose that the nontrivial tree automorphism g is strongly connected. If there is x such that $p(x) \neq 1/d$, then μ and the image measure $g_*\mu$ are singular.*

Proof. We first prove that if there is an i such that $p(x) \neq 1/d$, then $p \neq p'$. Indeed, for τ in $S(X)$ let S_τ be the set of states of g such that $\sigma_s(y) = \lambda(s,y) = \tau(y)$ for $y \in X$. ($S(X)$ is the group of all permutations of the set X). Since g is an automorphism, the set of all states S of g is equal to the union of all S_τ. Now,

$$p'(x) = \sum_{s\in S} q(s)p(\sigma_s^{-1}(x)) = \sum_{\tau\in S(X)}\sum_{s\in S_\tau} q(s)p(\tau^{-1}(x)) =$$

$$\sum_{\tau\in S(X)}\sum_{s\in S_\tau} q(s)p(\tau^{-1}(x))$$

Denote $q(\tau^{-1}) = \sum_{s\in S_\tau} q(s)$. Then $p'(x) = \sum_{\tau\in S(X)} p(\tau(x))q(\tau)$, for all x. Choose y such that $p(y)$ is maximal among all $p(x)$. Then

$$p'(y) = \sum_{\tau\in S(X)} p(\tau(y))q(\tau) < p(y)\sum_{\tau\in S(X)} q(\tau) = p(y).$$

Since p, p' are the one-dimensional distributions of shift invariant measures μ and $\mathbf{Q}$ correspondingly, it follows from $p \neq p'$ and the pointwise ergodic theorem that μ

and Q are singular. It is left to note that since $g_*\mu$ is absolutely continuous with respect to $\mathbf{Q}$, then $g_*\mu$ and μ are also singular. $\square$

CHAPTER IV

DIGIT TILES AND LIMIT SPACES OF SELF-SIMILAR GROUPS*

1 Bernoulli measure of sofic subshifts

In this subsection all considered graphs are directed and labeled with X as the set of labels. Let $\Gamma = (V, E)$ be such a graph and take a vertex $v \in V$. We say that a sequence $x_1x_2\ldots \in X^\omega$ (a word $x_1x_2\ldots x_n \in X^*$) *starts in the vertex* v if there exists a right-infinite path $e_1e_2\ldots$ (finite path $e_1e_2\ldots e_n$) in Γ, which starts in v and is labeled by $x_1x_2\ldots$ (respectively $x_1x_2\ldots x_n$). Similarly, we say that a sequence $\ldots x_2x_1 \in X^{-\omega}$ (a word $x_n\ldots x_2x_1 \in X^*$) *ends in the vertex* v if there exists a left-infinite path $\ldots e_2e_1$ (finite path $e_n\ldots e_2e_1$) in Γ, which ends in v and is labeled by $\ldots x_2x_1$ (respectively $x_n\ldots x_2x_1$). For every $w \in X^* \cup X^{-\omega}$ denote by $V_\Gamma(w) = V(w) \subset V$ the set of all vertices $v \in V$ such that the sequence w ends in v. Observe that $V(w'w) \subseteq V(w)$ for arbitrary word w' and finite word w.

For every vertex $v \in V$ denote by B_v the set of all right-infinite sequences that start in v, and denote by F_v the set of all left-infinite sequences that end in v. The sets B_v and F_v are closed correspondingly in X^ω and $X^{-\omega}$, thus compact and measurable. The sets $\mathcal{B} = \cup_{v\in V} B_v$ and $\mathcal{F} = \cup_{v\in V} F_v$ are the one-sided (respectively, right and left) *sofic subshifts* associated with the graph Γ. The sets $F_v, v \in V$, satisfy the recursion

$$F_v = \bigcup_{u \xrightarrow{x} v} \sigma_x(F_u)$$

(here the union is taken over all edges which end in v). Hence, associating the map σ_x

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with every edge e of the graph Γ labeled by x, the collection of sets $\{F_v, v \in V\}$ can be seen as the graph-directed iterated function system on the sofic subshift $\mathcal{F}$ with the underlying graph Γ (see [BGN03]). All the maps σ_x^{-1} are restrictions of the shift σ, and thus $\{F_v x, v \in V\}$ is the Markov partition of the dynamical system $(\mathcal{F}, \sigma)$. Similarly, the collection of sets $\{B_v, v \in V\}$ can be seen as the graph-directed iterated function system on the sofic subshift $\mathcal{B}$.

A labeled graph $\Gamma = (V, E)$ is called *right-resolving* (Shannon graph in some terminology) if for every vertex $v \in V$ the edges starting at v have different labels. Every sofic subshift can be given by a right-resolving graph (see Theorem 3.3.2 in [LM95]). A right-resolving graph is called *strictly right-resolving* if every vertex $v \in V$ has an outgoing edge labeled by x for every $x \in X$.

For a labeled graph $\Gamma = (V, E)$ we use the following notations:

$$\vec{\mu}_p(\mathcal{B}) = (\mu_p(B_v))_{v \in V}, \quad \vec{\mu}_p(\mathcal{F}) = (\mu_p(F_v))_{v \in V}, \quad \text{and} \quad \mu_p(\Gamma) = \sum_{v \in V} \mu_p(F_v).$$

Next we study the properties of these quantities and describe an algorithmic way to find them. First we discuss the problem for right-resolving graphs, and then reduce the general case to the right-resolving one.

Theorem 5. *Let $\Gamma = (V, E)$ be a finite right-resolving graph. Then*

$$\mu_p(\Gamma) = \min_{w \in X^{-\omega}} |V(w)| = \min_{w \in X^*} |V(w)|.$$

In particular, the measure $\mu_p(\Gamma)$ is integer.

Proof. Let $w = \ldots x_2 x_1 \in X^{-\omega}$ and denote $w_n = x_n \ldots x_2 x_1$ for $n \geq 1$. Observe, that $V(w) \subseteq V(w_n)$ and $V(w_n) \subseteq V(w_m)$ for $n \geq m$.

Take a vertex $v \in \cap_{n \geq 1} V(w_n)$. Let P_n be the set of all paths $e_n \ldots e_2 e_1$ labeled by w_n and ending in v. The set P_n is a finite non-empty set for every n, and $e_{n-1} \ldots e_2 e_1 \in$

P_{n-1} for every path $e_n \dots e_2 e_1 \in P_n$. Since the inverse limit of a sequence of finite non-empty sets is non-empty, there exists a left-infinite path $\dots e_2 e_1$ labeled by w and ending in v. Then $v \in V(w)$ and we get

$$V(w) = \bigcap_{n \geq 1} V(w_n). \tag{4.1}$$

From this follows that $|V(w)| = \min_{n\geq 1} |V(w_n)|$ for $w \in X^{-\omega}$. Then

$$\min_{w \in X^{-\omega}} |V(w)| = \min_{w \in X^{-\omega}} \min_{n \geq 1} |V(w_n)| = \min_{w \in X^*} |V(w)|$$

and the second equality of the theorem is proved.

Define the integer $k = \min_{w \in X^{-\omega}} |V(w)|$ and consider the set $\mathcal{O} = \mathcal{O}(\Gamma) \subseteq X^{-\omega}$ of all sequences $w \in X^{-\omega}$ such that $|V(w)| = k$. Define $\mathcal{O}^*$ as the set of finite words that satisfy the same condition. We have the following lemma.

Lemma 6. *The set $\mathcal{O}$ is open and dense in $X^{-\omega}$, and $\mu_p(\mathcal{O}) = 1$. For each $w \in \mathcal{O}$ there is a beginning of w that belongs to $\mathcal{O}^*$. Equivalently, $\mathcal{O} = \cup_{w \in \mathcal{O}^*} X^{-\omega} w$.*

Proof. If $w \in \mathcal{O}$ then $k = |V(w)| = \min_{n \geq 1} |V(w_n)|$ and there exists $N \geq 1$ such that $|V(w_N)| = k$. Then $k \leq |V(\omega w_N)| \leq |V(w_N)| = k$ for all $\omega \in X^{-\omega}$. Hence $w_N \in \mathcal{O}^*$, $X^{-\omega} w_N \subseteq \mathcal{O}$ and so $\mathcal{O}$ is open, and thus measurable.

Let $u \in X^*$ be such that $|V(u)| = k$. Let us show that if $w \in X^{-\omega}$ contains the subword u then $w \in \mathcal{O}$. If $w = w'u$ then $k \leq |V(w'u)| \leq |V(u)| = k$ and $w \in \mathcal{O}$. Observe that $V(ux)$ is the set of those vertices $v \in V$ for which there exists an edge labeled by x which starts in some vertex of $V(u)$ and ends in v. Since the graph Γ is right-resolving there is no more than one such an edge for each vertex of $V(u)$, and thus $|V(ux)| \leq |V(u)|$. It implies that if $w = w'uu'$ than $k \leq |V(w)| \leq |V(u)| = k$ and thus $|V(w)| = k$, so $w \in \mathcal{O}$. The Bernoulli measure of the set of all words $w'uu', u' \in X^*, w' \in X^{-\omega}$, is equal to 1. Thus $\mu_p(\mathcal{O}) = 1$. It follows also that $\mathcal{O}$ is dense in $X^{-\omega}$. $\square$

By construction of the set $\mathcal{O}$ for every $w \in \mathcal{O}$ there exist exactly k vertices v such that $w \in F_v$. Let χ_{F_v} be the characteristic function of the set F_v. Then $\sum_{v\in V} \chi_{F_v} = k$ almost everywhere. Integrating we get $\mu_p(\Gamma) = \sum\limits_{v\in V} \mu_p(F_v) = k$. $\square$

Remark 1. The theorem holds not only for Bernoulli measures. The only property that was used is that for every word $u \in X^*$ the set of all words, which contain u as a subword, has measure 1. It should be pointed out that the number $\mu(\Gamma)$ is independent on the chosen measure μ, while the measures $\mu(F_v)$, $\mu(B_v)$ and $\sum_{v\in V} \mu(B_v)$ depend on μ.

Definition 2. The number $meas(\Gamma) = \mu_p(\Gamma)$ is called the *measure number* of the graph Γ.

Theorem 5 shows that almost every sequence $w \in X^{-\omega}$ ends in precisely $meas(\Gamma)$ vertices of Γ.

We will use the following proposition in the next subsections.

Proposition 1. *Let $\Gamma = (V, E)$ be a finite labeled graph. Then $\mathcal{B} = X^\omega$ if and only if $\mathcal{F} = X^{-\omega}$. In particular, $\mathcal{F} = X^{-\omega}$ for a finite strictly right-resolving graph Γ.*

Proof. Since the inverse limit of nonempty finite sets is nonempty, $\mathcal{B} = X^\omega$ ($\mathcal{F} = X^{-\omega}$) is equivalent to the fact that every finite word $v \in X^*$ labels some path in Γ. $\square$

The matrix $A = (a_{vu})_{v,u\in V}$, where a_{vu} is equal to the number of edges from v to u, is the *adjacency matrix* of the graph Γ. For the probability vector $p = (p_x)_{x\in X}$ define the matrix

$$T_p = (t_{vu})_{v,u\in V}, \text{ where } t_{vu} = \sum_{v \xrightarrow{x} u} p_x$$

(the sum is taken over all edges from v to u). The matrices A and T_p are irreducible if and only if the graph Γ is strongly connected. If the graph Γ is right-resolving, then the matrix T_p is the *transition matrix* of the random walk on the weighted directed graph Γ, where each edge labeled by x has weight p_x. In this case the row sums of the matrix A are $\leq |X|$, and the row sums of the matrix T_p are ≤ 1, hence the spectral radius of A is $\leq |X|$, and the spectral radius of T_p is ≤ 1. If the graph Γ is strictly right-resolving, then the transition matrix T_p is right stochastic.

Proposition 2. *Let $\Gamma = (V, E)$ be a finite right-resolving graph with the transition matrix T_p. If the vector $\vec{\mu}_p(\mathcal{B})$ is nonzero then it is the right eigenvector of T_p for the eigenvalue* 1. *If the vector $\vec{\mu}_p(\mathcal{F})$ is nonzero then it is the left eigenvector of T_p for the eigenvalue* 1.

Proof. By construction, for every vertex $v \in V$ we have

$$B_v = \bigsqcup_{v \xrightarrow{x} u} x B_u$$

(here the union is disjoint because the graph Γ is right-resolving). It implies

$$\mu_p(B_v) = \sum_{v \xrightarrow{x} u} p(x) \mu_p(B_u) = \sum_{u \in V} t_{vu} \mu_p(B_u).$$

Thus the nonzero vector $\vec{\mu}_p(\mathcal{B})$ is the right eigenvector of T_p for the eigenvalue 1.

Similarly,

$$F_v = \bigcup_{u \xrightarrow{x} v} F_u x, \ v \in V,$$

and, since the graph Γ is right-resolving, that implies

$$\mu_p(F_v) \leq \sum_{u \xrightarrow{x} v} p(x) \mu_p(F_u) = \sum_{u \in V} t_{uv} \mu_p(F_u) \quad \Rightarrow \quad \vec{\mu}_p(\mathcal{F}) \leq \vec{\mu}_p(\mathcal{F}) T_p.$$

The standard arguments based on the theory of nonnegative matrices (see for example [HLR03, proof of Theorem 4.5], [GY06, page 197], [Rot06]) end the proof. □

Corollary 1. *Let $\Gamma = (V, E)$ be a finite right-resolving graph. Let $\{\Gamma_i\}$ be the set of all strongly connected components of Γ, which are strictly right-resolving graphs. Then $meas(\Gamma) = \sum_i meas(\Gamma_i)$.*

In particular, if a finite strictly right-resolving graph Γ contains a vertex v_0 such that for each vertex v there is a path in Γ from v to v_0 and for each $x \in X$ there is an edge from v_0 to v_0 labeled by x (the open set condition for graphs), then $meas(\Gamma) = 1$.

Corollary 2. *Let $\Gamma = (V, E)$ be a finite right-resolving graph with the adjacency matrix A. For the uniform Bernoulli measure μ_u the nonzero vectors $\vec{\mu}_u(\mathcal{B})$ and $\vec{\mu}_u(\mathcal{F})$ are respectively the right and left eigenvectors of A for the eigenvalue $|X|$.*

Although Theorem 5 gives a useful characterization of the number $\mu_p(\Gamma)$, it does not present an algorithmic way to find it. It follows from Proposition 2 that the problem of finding $\vec{\mu}_p(\mathcal{F})$ and $\mu_p(\Gamma)$ reduces to the strongly connected components which are strictly right-resolving graphs (for all other vertices $\mu_p(F_v) = 0$). Notice that if Γ is a strongly connected strictly right-resolving graph, then the vector $\vec{\mu}_p(\mathcal{F})/\mu_p(\Gamma)$ is the unique stationary probability distribution of the stochastic matrix T_p.

At the same time Proposition 2 implies the algorithm to find the vector $\vec{\mu}_p(\mathcal{B})$ for a right-resolving graph. Indeed, a left eigenvector of T_p for the eigenvalue 1 is uniquely defined if we know its entries $\mu_p(B_v)$ for vertices v in the strongly connected components of Γ without outgoing edges. For every such a component Γ', we have $B_v = X^\omega$ and $\mu_p(B_v) = 1$ for every vertex $v \in \Gamma'$ if the component Γ' is a strictly right-resolving graph, and $\mu_p(B_v) = 0$ otherwise. In particular, if the matrix T_p is rational then the values $\mu_p(B_v)$ are rational.

Given a finite labeled graph $\Gamma = (V, E)$ the problems of finding the measures $\mu_p(B_v)$ and $\mu_p(F_v)$ are equivalent, and can be reduced to right-resolving graphs. The problem is that the system

$$B_v = \bigcup_{v \xrightarrow{x} u} x B_u = \bigsqcup_{x \in X} x \left(\bigcup_{v \xrightarrow{x} u} B_u \right), \ v \in V$$

is not self-similar in the sense that the above expression involves not only the sets B_v but also their finite unions. Introducing additional terms to this recursion corresponding to these unions we get a system with a right-resolving graph. This procedure is similar to the construction described in the proof of Theorem 3.3.2 in [LM95].

Proposition 3. *For every finite graph $\Gamma = (V, E)$ one can construct a finite right-resolving graph $\Gamma' = (V', E')$ with the property that for every $v \in V$ there exists $v' \in V'$ such that $B_v = B_{v'}$.*

Similarly, one can find the measures of subshifts $\mathcal{B}$ and $\mathcal{F}$ by introducing new vertices corresponding to $\cup_{v \in V} B_v$ and $\cup_{v \in V} F_v$.

Corollary 3. *Let $\Gamma = (V, E)$ be a finite labeled graph. For the uniform Bernoulli measure μ_u all measures $\mu_u(B_v)$, $\mu_u(F_v)$, $\mu_u(\Gamma)$ are rational.*

Consider the question how to find the measure of the intersection $B_v \cap B_u$ for $v, u \in V$. Construct a new graph $\mathcal{G}$ with the set of vertices $V \times V$ and put an edge from (v, u) to (v', u') labeled by $x \in X$ for every edges $v \xrightarrow{x} v'$ and $u \xrightarrow{x} u'$ in the graph Γ (label products of graphs by Definition 3.4.8 in [LM95]). It is easy to see that then $B_{(v,u)} = B_v \cap B_u$ (see Proposition 3.4.10 in [LM95]).

2 Self-similar actions and its limit spaces

We review in this subsection the basic definitions and theorems concerning self-similar groups. For a more detailed account and for the references, see [Nek05].

Self-similar actions. A faithful action of a group G on the set X^* is called *self-similar* if for every $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w)$$

for all $w \in X^*$. The element h is called the *restriction* of g on x and denoted $h = g|_x$. Inductively one defines the restriction $g|_{x_1x_2\ldots x_n} = g|_{x_1}|_{x_2}\ldots|_{x_n}$ for every word $x_1\ldots x_n \in X^n$. Notice that $(g\cdot h)|_v = g|_{h(v)}\cdot h|_v$ (we are using left actions).

Virtual endomorphisms. The study of the self-similar actions of a group is in some sense the study of the *virtual endomorphisms* of this group, which are homomorphisms from a subgroup of finite index to the group. There is a general way to construct a self-similar representation of a group with a given associated virtual endomorphism. Let $\phi : H \to G$ be a virtual endomorphism of the group G, where $H < G$ is a subgroup of index d. Let us choose a left coset transversal $T = \{g_0, g_1, \ldots, g_{d-1}\}$ for the subgroup H, and a sequence $C = \{h_0, h_1, \ldots, h_{d-1}\}$ of elements of G called a *cocycle*. The self-similar action (G, X^*) with the alphabet $X = \{x_0, x_1, \ldots, x_{d-1}\}$ defined by the triple (ϕ, T, C) is given by

$$g(x_i) = x_j, \qquad g|_{x_i} = h_j^{-1}\phi(g_j^{-1}gg_i)h_i,$$

where j is such that $g_j^{-1}gg_i \in H$ (such j is unique). The action may be not faithful, the kernel can be described using Proposition 2.7.5 in [Nek05].

Conversely, every self-similar action can be obtained in this way. Let (G, X^*) be a self-similar action and take a letter $x \in X$. The stabilizer $St_G(x)$ of the letter

x in the group G is a subgroup of index $\leq |X|$ in G. Then the map $\phi_x : g \mapsto g|_x$ is a homomorphism from $St_G(x)$ to G called *the virtual endomorphism associated to the self-similar action*. Choose $T = \{g_y : y \in X\}$ and $C = \{h_y : y \in X\}$ such that $g_y(x) = y$ and $h_y = (g_y|_x)^{-1}$. Then T is a coset transversal for the subgroup $St_G(x)$ and the self-similar action (G, X^*) is defined by the triple (ϕ_x, T, C). Different self-similar actions of the group G with the same associated virtual endomorphism are conjugated by Proposition 2.3.4 in [Nek05].

Contracting self-similar actions. An important class of self-similar actions are contracting actions. A self-similar action of a group G is called *contracting* if there exists a finite set $\mathcal{N}$ such that for every $g \in G$ there exists $k \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all words $v \in X^*$ of length $\geq k$. The smallest set $\mathcal{N}$ with this property is called the *nucleus* of the self-similar action. The nucleus itself is self-similar in the sense that $g|_v \in \mathcal{N}$ for every $g \in \mathcal{N}$ and $v \in X^*$. It can be represented by the *Moore diagram*, which is the directed labeled graph with the set of vertices $\mathcal{N}$, where there is an edge from g to $g|_x$ labeled $(x, g(x))$ for every $x \in X$ and $g \in \mathcal{N}$. We identify the nucleus with its Moore diagram, also denoted by $\mathcal{N}$. The contracting property of the action depends only on the virtual endomorphism but not on the chosen coset transversal and cocycle (see Corollary 2.11.7 in [Nek05]). Notice that every contracting self-similar group is countable.

Self-similar groups are related to self-similar sets through the notion of limit spaces.

Limit G-spaces. Let us fix a contracting self-similar action (G, X^*). Consider the space $X^{-\omega} \times G$ of all sequences $\ldots x_2 x_1 \cdot g$, $x_i \in X$ and $g \in G$, with the product topology of discrete sets X and G. Two elements $\ldots x_2 x_1 \cdot g$ and $\ldots y_2 y_1 \cdot h$ of $X^{-\omega} \times G$ are called *asymptotically equivalent* if there exist a finite set $K \subset G$ and a sequence

$g_n \in K, n \geq 1$, such that

$$g_n(x_n x_{n-1} \dots x_1) = y_n y_{n-1} \dots y_1 \qquad \text{and} \qquad g_n|_{x_n x_{n-1} \dots x_1} \cdot g = h$$

for every $n \geq 1$. This equivalence relation can be recovered from the nucleus $\mathcal{N}$ of the action (Proposition 3.2.6 in [Nek05]).

Proposition 4. *Two elements $\dots x_2 x_1 \cdot g$ and $\dots y_2 y_1 \cdot h$ of $X^{-\omega} \times G$ are asymptotically equivalent if and only if there exists a left-infinite path $\dots e_2 e_1$ in the nucleus $\mathcal{N}$ ending in the vertex hg^{-1} such that the edge e_i is labeled by (x_i, y_i).*

The quotient of the set $X^{-\omega} \times G$ by the asymptotic equivalence relation is called the *limit G-space* of the action and denoted $\mathcal{X}_{(G,X^*)}$. The group G naturally acts on the space $\mathcal{X}_{(G,X^*)}$ by multiplication from the right.

The map τ_x defined by the formula

$$\tau_x(\dots x_2 x_1 \cdot g) = \dots x_2 x_1 g(x) \cdot g|_x$$

is a well-defined continuous map on the limit G-space $\mathcal{X}_G$ for every $x \in X$, which is not a homeomorphism in general. Inductively one defines $\tau_{x_1 x_2 \dots x_n} = \tau_{x_n} \circ \tau_{x_{n-1}} \circ \dots \circ \tau_{x_1}$.

The image of $X^{-\omega} \times 1$ in $\mathcal{X}_G$ is called the *(digit) tile* $\mathcal{T}$ of the action. The image of $X^{-\omega} v \times 1$ for $v \in X^n$ is called the *tile* $\mathcal{T}_v$, equivalently $\mathcal{T}_v = \tau_v(\mathcal{T})$. It follows directly from definition that

$$\mathcal{X}_G = \bigcup_{g \in G} \mathcal{T} \cdot g \qquad \text{and} \qquad \mathcal{T} = \bigcup_{v \in X^n} \mathcal{T}_v.$$

Two tiles $\mathcal{T} \cdot g$ and $\mathcal{T} \cdot h$ intersect if and only if $gh^{-1} \in \mathcal{N}$. A contracting action (G, X^*) satisfies the *open set condition* if for any element g of the nucleus $\mathcal{N}$ there exists a word $v \in X^*$ such that $g|_v = 1$, i.e. in the nucleus $\mathcal{N}$ there is a path from any vertex to the trivial state. If the action satisfies the open set condition then the tile $\mathcal{T}$ is the closure of its interior, and any two different tiles have disjoint interiors; otherwise every tile $\mathcal{T} \cdot g$ is covered by the other tiles (see Proposition 3.3.7 in [Nek05]).

The tile $\mathcal{T}$ and the partition of $\mathcal{X}_G$ on tiles $\mathcal{T} \cdot g$ depend on the specific self-similar action of the group G. However, up to homeomorphism the limit G-space $\mathcal{X}_{(G,X^*)}$ is uniquely defined by the associated virtual endomorphism ϕ of the group, hence we denote it by $\mathcal{X}_G(\phi)$ (or $\mathcal{X}_G$ for short).

Theorem 6. *Let $\phi : H \to G$ be a virtual endomorphism of the group G. Let (G, X^*) and (G, Y^*) be the contracting self-similar actions defined respectively by the triples (ϕ, T, C) and (ϕ, T', C'). Then $\mathcal{X}_{(G,X^*)}$ and $\mathcal{X}_{(G,Y^*)}$ are homeomorphic and the homeomorphism is the map $\alpha : \mathcal{X}_{(G,X^*)} \longrightarrow \mathcal{X}_{(G,Y^*)}$ such that*

$$\alpha(\tau_x(t)) = \tau_y(\alpha(t)) \cdot s_x, \quad \textit{for} \;\; t \in \mathcal{X}_{(G,X^*)},$$

where $s_x = {h'}_y^{-1} \phi({g'}_y^{-1} g_x) h_x$ and y is such that $g'_y g_x^{-1} \in H$.

Proof. The statement follows from Sections 2.1–2.5 in [Nek05]. □

Limit dynamical system. The factor of the limit G-space $\mathcal{X}_G$ by the action of the group G is called the *limit space* $\mathcal{J}_G = \mathcal{J}_G(\phi)$. It follows from the definition that we may also consider $\mathcal{J}_G$ as a factor of $X^{-\omega}$ by the following equivalence relation: two left-infinite sequences $\ldots x_2 x_1, \ldots y_2 y_1$ are equivalent if and only if there exists a left-infinite path $\ldots e_2 e_1$ in the nucleus $\mathcal{N}$ such that the edge e_i is labeled by (x_i, y_i). The limit space $\mathcal{J}_G$ is compact, metrizable, finite-dimensional space. It is connected if the group G is finitely generated and acts transitively on X^n for all n.

The equivalence relation on $X^{-\omega}$ is invariant under the shift σ, therefore σ induces a continuous surjective map $\mathsf{s} : \mathcal{J}_G \to \mathcal{J}_G$, and every point of $\mathcal{J}_G$ has at most $|X|$ preimages under s. The dynamical system $(\mathcal{J}_G, \mathsf{s})$ is called the *limit dynamical system* of the self-similar action.

The image of $X^{-\omega}v$ for $v \in X^n$ in $\mathcal{J}_G$ is called the *tile* T_v of the n-th level. Clearly

$$\mathsf{T}_v = \bigcup_{x \in X} \mathsf{T}_{xv} \qquad \text{and} \qquad \mathsf{s}(\mathsf{T}_{vx}) = \mathsf{T}_v$$

for every $v \in X^*$ and $x \in X$. Two tiles T_v and T_u of the same level have nonempty intersection if and only if there exists $h \in \mathcal{N}$ such that $h(v) = u$. Under the open set condition, every tile T_v is the closure of its interior, and any two different tiles of the same level have disjoint interiors (Proposition 3.6.5 in [Nek05]). It will be used in the next subsection that

$$\lim_{n\to\infty} \max_{v \in X^n} \operatorname{diam}(\mathsf{T}_\mathrm{v}) = 0 \tag{4.2}$$

for any chosen metric on the limit space $\mathcal{J}_G$ (see Theorem 3.6.9 in [Nek05]).

The inverse limit of the topological spaces $\mathcal{J}_G \xleftarrow{\mathsf{s}} \mathcal{J}_G \xleftarrow{\mathsf{s}} \cdots$ is called the *limit solenoid* $\mathcal{S}_G$. One can consider $\mathcal{S}_G$ as a factor of the space $X^{\mathbb{Z}}$ of two-sided infinite sequences by the equivalence relation, where two sequences ξ, η are equivalent if and only if there exist a two-sided infinite path in the nucleus labeled by the pair (ξ, η). The two-sided shift on $X^{\mathbb{Z}}$ induces a homeomorphism $\mathsf{e} : \mathcal{S}_G \to \mathcal{S}_G$.

3 Self-similar measures on limit spaces

Let us fix a contracting self-similar action (G, X^*).

Invariant measure on the limit G-space $\mathcal{X}_G$. We consider the uniform Bernoulli measure μ_u on the space $X^{-\omega}$ and the counting measure on the group G, and we put the product measure on the space $X^{-\omega} \times G$. The push-forward of this measure under the factor map $\pi_{\mathcal{X}} : X^{-\omega} \times G \to \mathcal{X}_G$ defines the *measure μ on the limit G-space $\mathcal{X}_G$*. The measure μ is a G-invariant σ-finite regular Borel measure on $\mathcal{X}_G$.

Proposition 5. *The measures of tiles have the following properties.*

1. $\mu(\mathcal{T}_v) = |X|^n \cdot \mu(\mathcal{T}_{uv})$ *for every* $v \in X^*$ *and* $u \in X^n$.

2. $\mu(\mathcal{T}_v \cap \mathcal{T}_{v'}) = 0$ *for* $v, v' \in X^n, v \neq v'$.

3. *Let* $\mu|_{\mathcal{T}}$ *be the measure* μ *restricted to the tile* $\mathcal{T}$. *Then*

$$\mu|_{\mathcal{T}}(A) = \sum_{x \in X} \frac{1}{|X|} \mu|_{\mathcal{T}}(\tau_x^{-1}(A))$$

for any Borel set $A \subset \mathcal{T}$.

Proof. Let us show that $\mu(A) \geq |X|\mu(\tau_x(A))$ for any Borel set A. Consider sequences which represent points of the sets A and $\tau_x(A)$:

$$\pi_{\mathcal{X}}^{-1}(A) = \bigcup_{g \in G} T_g \cdot g \quad \Rightarrow \quad \pi_{\mathcal{X}}^{-1}(\tau_x(A)) = \bigcup_{g \in G} T_g g(x) \cdot g|_x.$$

It implies

$$\mu(\tau_x(A)) \leq \sum_{g \in G} \frac{1}{|X|} \mu_u(T_g) = \frac{1}{|X|} \mu(A).$$

By applying this inequality n times we get $|X|^n \mu(\mathcal{T}_v) = |X|^n \mu(\tau_v(\mathcal{T})) \leq \mu(\mathcal{T})$ for $v \in X^n$. Since $\mathcal{T} = \cup_{u \in X^n} \mathcal{T}_u$ we have that

$$\mu(\mathcal{T}) \leq \sum_{u \in X^n} \mu(\mathcal{T}_u) \leq \sum_{u \in X^n} \frac{1}{|X|^n} \mu(\mathcal{T}) = \mu(\mathcal{T}).$$

Hence all the above inequalities are actually equalities, $\mu(\mathcal{T}) = |X|^n \mu(\mathcal{T}_u)$ for every $u \in X^n$, and $\mu(\mathcal{T}_u \cap \mathcal{T}_{u'}) = 0$ for different $u, u' \in X^n$.

Notice that since every Borel set $A \subset \mathcal{T}$ can be approximated by unions of tiles of the same level and using items 1 and 2 we have that if $\mu(\tau_x(A)) < \varepsilon$ then $\mu(A) < \varepsilon |X|$.

It is left to prove item 3. First, let us show that the assertion holds for the tiles $\mathcal{T}_v$. Since the measure $\mu|_{\mathcal{T}}$ is concentrated on the tile $\mathcal{T}$, up to sets of measure zero the set $\tau_x^{-1}(\mathcal{T}_v)$ is equal $\mathcal{T}_u$ if $v = ux$, and is empty if the last letter of v is not x. Really, if $t \in \tau_x^{-1}(\mathcal{T}_{ux})$ and $t \in \mathcal{T}_v$ with $v \neq u, |v| = |u|$, then $\tau_x(t) \in \mathcal{T}_{vx} \cap \mathcal{T}_{ux}$ and the measure of such points is zero. Hence

$$\mu|_{\mathcal{T}}(\mathcal{T}_v) = \frac{1}{|X|}\mu|_{\mathcal{T}}(\mathcal{T}_u) = \sum_{x\in X}\frac{1}{|X|}\mu|_{\mathcal{T}}(\tau_x^{-1}(\mathcal{T}_v)).$$

Now we can approximate any Borel set by unions of tiles and pass to the limit. □

The tile $\mathcal{T}$ of the limit G-space $\mathcal{X}_G$ can be considered as the attractor of the iterated function system τ_x, $x \in X$, i.e.

$$\mathcal{T} = \bigcup_{x\in X} \tau_x(\mathcal{T}).$$

Hence Proposition 5 item 3 implies that $\mu|_{\mathcal{T}}$ is the self-similar measure on $\mathcal{T}$ by the standard definition of Hutchinson (1.1). The measure μ is the G-invariant extension of the self-similar measure $\mu|_{\mathcal{T}}$ to the limit G-space $\mathcal{X}_G$.

Let us show how to find the measure of the tile $\mathcal{T}$. Let $\mathcal{N}$ be the nucleus of the action (G, X^*) identified with its Moore diagram. Replacing each label (x, y) by label x in the nucleus $\mathcal{N}$ we get a strictly right-resolving graph denoted $\Gamma_{\mathcal{N}}$ labeled by elements of X, so that we can apply the methods developed in subsection 1.

Theorem 7. *The measure $\mu(\mathcal{T})$ is equal to the measure number $meas(\Gamma_{\mathcal{N}})$ of the nucleus, in particular it is always integer. Moreover, $\mu(\mathcal{T}) = 1$ if and only if the action satisfies the open set condition.*

Proof. By Proposition 4 we have

$$\pi_{\mathcal{X}}^{-1}(\mathcal{T}) = \bigcup_{g\in\mathcal{N}} F_g \cdot g^{-1},$$

where the sets F_g are defined using the graph $\Gamma_{\mathcal{N}}$ (see subsection 1). Thus

$$\mu(\mathcal{T}) = \sum_{g\in\mathcal{N}} \mu_u(F_g) = meas(\mathcal{N}),$$

which is integer by Theorem 5. Observe that $\mu(\mathcal{T}) \geq 1$ because $F_g = X^{-\omega}$ for $g = 1 \in \mathcal{N}$.

If the action satisfies the open set condition then $\mu(\mathcal{T}) = meas(\Gamma_{\mathcal{N}}) = 1$ by Corollary 1.

Suppose now that the action does not satisfy the open set condition. Then there exists an element h of the nucleus, whose all restrictions are non-trivial. Let $\mathcal{N}_1$ be the set of all restrictions of h, and $\Gamma_{\mathcal{N}_1}$ be the corresponding graph. Then by Proposition 1

$$\sum_{g\in\mathcal{N}_1} \mu_u(F_g) = meas(\Gamma_{\mathcal{N}_1}) \geq 1.$$

Thus $\mu(\mathcal{T}) = \sum_{g\in\mathcal{N}} \mu_u(F_g) \geq meas(\Gamma_{\mathcal{N}_1}) + \mu_u(F_{g=1}) \geq 2.$ □

Remark 2. The measure $\mu(\mathcal{T}) = meas(\Gamma_{\mathcal{N}})$ can be found algorithmically using the remarks after Proposition 2.

The next proposition shows that the covering $\mathcal{X}_G = \cup_{g\in G}\mathcal{T}\cdot g$ is a perfect multiple covering of multiplicity $\mu(\mathcal{T})$.

Proposition 6. *Every point $x \in \mathcal{X}_G$ is covered by at least $\mu(\mathcal{T})$ tiles. The set $\dot{\mathcal{X}}_G$ of all points $x \in \mathcal{X}_G$, which are covered by exactly $\mu(\mathcal{T})$ tiles, is open and dense in $\mathcal{X}_G$, and its complement has measure* 0.

Proof. For each $x \in \mathcal{X}_G$ we define the number n_x of such $g \in G$ that the tile $\mathcal{T} \cdot g$ contains x.

First we prove the inequality. Let $x \in \mathcal{X}_G$ be represented by the pair $w \cdot g$ from $X^{-\omega} \times G$. Then $|V(w)| \geq meas(\Gamma_{\mathcal{N}})$ by Theorem 5. If $h \in V(w)$ it means that the sequence w ends in h, which by Proposition 4 means that there is a sequence $u_h \in X^{-\omega}$ such that w is asymptotically equivalent to $u_h \cdot h$. It follows that $w \cdot g$ is asymptotically equivalent to $u_h \cdot hg$. It means, in turn, that x belongs to the tile $\mathcal{T} \cdot hg$ for every $h \in V(w)$. It follows that $n_x \geq meas(\Gamma_{\mathcal{N}})$.

Consider the set $\mathcal{O} = \mathcal{O}(\Gamma_{\mathcal{N}})$ defined in Lemma 6 using the graph $\Gamma_{\mathcal{N}}$, in other words $\mathcal{O}$ is the set of all $w \in X^{-\omega}$ that end in precisely $meas(\Gamma_{\mathcal{N}})$ elements of $\mathcal{N}$. By the same considerations as above we see that if $w \cdot g$ represents a point x with $n_x = meas(\Gamma_{\mathcal{N}})$ then $|V(w)| = meas(\Gamma_{\mathcal{N}})$, that is, w belongs to $\mathcal{O}$. In other words, the set $\mathcal{O} \times G$ is closed under the asymptotic equivalence relation on $X^{-\omega} \times G$, and it is the inverse image of the set $\dot{\mathcal{X}}_G$ under the factor map $\pi_{\mathcal{X}}$. Since the set $\mathcal{O} \times G$ is open and dense in $X^{-\omega} \times G$ by Lemma 6, the same hold for $\dot{\mathcal{X}}_G$ in $\mathcal{X}_G$. The complement of $\dot{\mathcal{X}}_G$ has measure 0, because the complement of the set $\mathcal{O}$ has measure 0 by Lemma 6. □

Remark 3. Two tiles $\mathcal{T} \cdot g_1$ and $\mathcal{T} \cdot g_2$ for $g_1, g_2 \in G$ have nonempty intersection if and only if $g_1 g_2^{-1} \in \mathcal{N}$. Let us show how to find the measure of this intersection. By Proposition 4 we have

$$\pi_{\mathcal{X}}^{-1}((\mathcal{T} \cdot g_1) \cap (\mathcal{T} \cdot g_2)) = \bigcup_{\substack{h_1, h_2 \in \mathcal{N} \\ g_1 g_2^{-1} = h_1 h_2^{-1}}} (F_{h_1} \cap F_{h_2}) \cdot h_1^{-1} g_1,$$

where the sets F_g are defined using the nucleus $\mathcal{N}$. The word problem in contracting self-similar groups is solvable in polynomial time [Nek05, Proposition 2.13.10] (one can use the program package [MS08]). The measures of intersections $F_{h_1} \cap F_{h_2}$ can be found for example using method described after Corollary 3.

The measure $\mu = \mu_{(G,X^*)}$ on the limit G-space $\mathcal{X}_G$ was defined using the specific self-similar action (G, X^*) of the group G. Let us show that actually this measure depends only on the associated virtual endomorphism ϕ, as the limit G-space itself. It allows us to consider the measure space $(\mathcal{X}_G(\phi), \mu)$ independently of the self-similar action. At the same time, the measure $\mu(\mathcal{T})$ may vary for different self-similar actions as the nucleus does. It is an interesting open question in what cases we can always choose a self-similar action which satisfies the open set condition (see [GM92, LW95] for the abelian case and applications to wavelets).

Theorem 8. *Let $\phi : H \to G$ be a virtual endomorphism of the group G. Let (G, X^*) and (G, Y^*) be the contracting self-similar actions defined respectively by the triples (ϕ, T, C) and (ϕ, T', C'). Then the homeomorphism $\alpha : \mathcal{X}_{(G,X^*)} \longrightarrow \mathcal{X}_{(G,Y^*)}$ from Theorem 6 preserves measure, i.e.*

$$\mu_{(G,Y^*)}(\alpha(A)) = \mu_{(G,X^*)}(A)$$

for any Borel set A.

Proof. Let $\mathcal{N}$ be the nucleus and $\mathfrak{T}$ be the tile of the action (G, X^*). By Theorem 7 and Theorem 5

$$\mu_{(G,X^*)}(\mathfrak{T}) = meas(\Gamma_{\mathcal{N}}) = \min_{w \in X^{-\omega}} |V(w)| = k \in \mathbb{N}, \tag{4.3}$$

where $V(w)$ is defined using the graph $\Gamma_{\mathcal{N}}$. Consider

$$\pi_Y^{-1}(\alpha(\mathfrak{T})) = \bigcup_{g \in G} T_g \cdot g \quad \text{(here } T_g \subseteq Y^{-\omega}\text{)},$$

where $\pi_Y : Y^{-\omega} \times G \to \mathcal{X}_G$ is the canonical projection.

Take $w \in Y^{-\omega}$ and let us prove that there exist at least k elements $g \in G$ such that $w \in T_g$. The tiles $\mathfrak{T} \cdot g$ cover the limit G-space $\mathcal{X}_{(G,X^*)}$, and we can find $g \in G$ such that if $x = \alpha^{-1}(y)$ for $y = \pi_Y(w \cdot g) \in \mathcal{X}_{(G,Y^*)}$ then x belongs to the tile $\mathfrak{T}$. Then x is represented by the sequence $u \cdot 1$ for $u \in X^{-\omega}$. Equation (4.3) implies $|V(u)| \geq k$ and so there exist k elements $h_1, \ldots, h_k \in \mathcal{N}$ and sequences $u_1, \ldots, u_k \in X^{-\omega}$ such that for every i there exists a left-infinite path in the nucleus $\mathcal{N}$ which ends in h_i and is labeled by (u, u_i). By Proposition 4

$$x = \pi_X(u \cdot 1) = \pi_X(u_i \cdot h_i) \quad \text{and} \quad x \cdot h_i^{-1} \in \mathfrak{T}$$

for every $i = 1, \ldots, k$. Then

$$\pi_Y(w \cdot gh_i^{-1}) = \pi_Y(w \cdot g)h_i^{-1} = y \cdot h_i^{-1} = \alpha(x \cdot h_i^{-1}) \in \alpha(\mathcal{T})$$

and thus $w \in T_{gh_i^{-1}}$ for all $i = 1, \ldots, k$.

Let χ_{T_g} be the characteristic function of the set T_g. Then $\sum_{g\in G} \chi_{T_g}(x) \geq k$ for almost all x. Integrating we get $\mu_{(G,Y^*)}(\alpha(\mathcal{T})) = \sum_{g\in G} \mu_u(T_g) \geq k = \mu_{(G,X^*)}(\mathcal{T})$.

Let us now show that $\mu_{(G,Y^*)}(\alpha(\mathcal{T}_v)) \leq \mu_{(G,X^*)}(\mathcal{T}_v)$ for any $v \in X^*$. Indeed, let $n = |v|$, then it follows from Theorem 6 that $\alpha(\mathcal{T}_v) = \alpha(\tau_v(\mathcal{T})) = \tau_u(\alpha(\mathcal{T}))g$ for some word $u \in Y^n$ and $g \in G$. Thus

$$\begin{aligned}\mu_{(G,Y^*)}(\alpha(\mathcal{T}_v)) &= \mu_{(G,Y^*)}(\tau_u(\alpha(\mathcal{T}))) \leq \frac{1}{|X|^n}\mu_{(G,Y^*)}(\alpha(\mathcal{T})) = \\ &= \frac{1}{|X|^n}\mu_{(G,X^*)}(\mathcal{T}) = \mu_{(G,X^*)}(\mathcal{T}_v).\end{aligned}$$

Let us prove that $\mu_{(G,Y^*)} \circ \alpha$ is absolutely continuous with respect to $\mu_{(G,X^*)}$. Indeed, let $\mu_{(G,X^*)}(A) < \varepsilon$, $\pi_X^{-1}(A) = \cup_g T_g \cdot g$. Then $\sum_g \mu_u(T_g) < \varepsilon$. It follows that there exist $v_{i,g} \in X^*$ such that $T_g \subset \cup_i X^{-\omega} v_{i,g}$ and $\sum_{i,g} |X|^{-|v_{i,g}|} < \varepsilon$. Then $A \subset \cup_{i,g} \mathcal{T}_{v_{i,g}} \cdot g$, and we have that $\sum_{i,g} \mu_{(G,X^*)}(\mathcal{T}_{v_{i,g}}) < \varepsilon \mu_{(G,X^*)}(\mathcal{T})$. Then $\mu_{(G,Y^*)}(\alpha(A)) \leq \sum \mu_{(G,Y^*)}(\alpha(\mathcal{T}_{v_{i,g}})) \leq \sum \mu_{(G,X^*)}(\mathcal{T}_{v_{i,g}}) < \varepsilon \mu_{(G,X^*)}(\mathcal{T})$. Since ε is arbitrary, we are done.

We will now prove that $\mu_{(G,Y^*)} \circ \alpha \leq \mu_{(G,X^*)}$. Since both $\mu_{(G,Y^*)} \circ \alpha$ and $\mu_{(G,X^*)}$ are invariant under multiplication by $g \in G$ it suffices to prove this inequality for sets $A \subset \mathcal{T}$. Since any Borel A is a union of a closed set and a set of arbitrarily small measure, it suffices to prove the inequality for closed sets, as $\mu_{(G,Y^*)} \circ \alpha$ is absolutely continuous with respect to $\mu_{(G,X^*)}$ by above.

So let $A \subset \mathcal{T}$ be a closed set. For each n, let A_n be the union of all tiles $\mathcal{T}_v, v \in X^n$, that have non-empty intersection with A. Let us show that $A = \cap_n A_n$.

Suppose $x \in \cap A_n$, and x is not in A. Then for each n there is $v_n \in X^n$ such that $\mathcal{T}_{v_n}$ has nonempty intersection with A and $x \in \mathcal{T}_{v_n}$. It follows that x has some representation $u_n v_n \in X^{-\omega}$. Since the number of such representations is finite, we may choose subsequence n_k such that v_{n_k} is the beginning of some word $v \in X^{-\omega}$. Since A is compact it follows that A has nonempty intersection with $\cap_k \mathcal{T}_{v_{n_k}}$, thus $\cap_k \mathcal{T}_{v_{n_k}}$ contains at least two points, which is impossible.

Since A_n is the union of some tiles of level n, which are disjoint up to sets of measure 0, the inequality also holds for all A_n. Going to the limit, we get the inequality for A.

By interchanging X and Y we get the reverse inequality, and we are done. □

Remark 4. It is important in the theorem that we take the uniform Bernoulli measure on $X^{-\omega}$. The problem is that the homeomorphism α may change the Bernoulli measure with a non-uniform weight to a measure that is not Bernoulli.

Self-similar measure on the limit space $\mathcal{J}_G$. The push-forward of the uniform Bernoulli measure μ_u under the factor map $\pi_{\mathcal{J}} : X^{-\omega} \to \mathcal{J}_G$ defines the *self-similar measure* m *on the limit space* $\mathcal{J}_G$. The measure m is a regular Borel probability measure on $\mathcal{J}_G$. The shift s is a measure-preserving transformation of $\mathcal{J}_G$.

Consider the set $\mathcal{U} = \mathcal{U}(\mathcal{N})$ of all sequences $w \in X^{-\omega}$ with the property that every left-infinite path in the nucleus $\mathcal{N}$ labeled by (w, w) ends in 1. Define $\mathcal{U}^*$ as the set of finite words that satisfy the same condition.

Lemma 7. *The set $\mathcal{U}$ is open and dense in $X^{-\omega}$, and $\mu_p(\mathcal{U}) = 1$. For each $w \in \mathcal{U}$ there is a beginning of w that belongs to $\mathcal{U}^*$, and $\mathcal{U} = \cup_{w \in \mathcal{U}^*} X^{-\omega} w$.*

The sets $\mathcal{U}$ and $\mathcal{U} \times G$ are closed under the asymptotic equivalence relation on $X^{-\omega}$ and $X^{-\omega} \times G$ respectively.

Proof. Construct the graph Γ with the elements of $\mathcal{N}$ as vertices, for each edge labeled by (x, x) in $\mathcal{N}$ we have the edge in Γ with the same starting and end vertices, and labeled by x; and for each edge in $\mathcal{N}$ labeled by (x, y) for $x \neq y$ there is an edge in Γ with the same starting vertex that ends in the trivial element and labeled by x. Then the set $\mathcal{U}$ coincides with the set $\mathcal{O}(\Gamma)$. Indeed, $h \in V_\Gamma(w)$ if and only if there is a path in $\mathcal{N}$ labeled by (w, w) that ends in h, thus $w \in \mathcal{U}$ if and only if $V_\Gamma(w) = \{1\}$. For every nontrivial element $h \in \mathcal{N}$ there exists a word $v \in X^*$ such that $h(v) \neq v$. It follows that there exists a path in the graph Γ from h to 1. Hence the component $\{1\}$ is the only strongly connected component of the graph Γ without outgoing edges. By Corollary 1 the measure number of Γ is 1, and $\mathcal{U} = \mathcal{O}(\Gamma)$. The first statement of the lemma now follows from Lemma 6.

Let us show that the set $\mathcal{U}$ is closed under the asymptotic equivalence relation (then the set $\mathcal{U} \times G$ is also closed). It is sufficient to show that if there is a path in $\mathcal{N}$ labeled by (u, v) and $u \in \mathcal{U}$ ($v \in \mathcal{U}$) then $v \in \mathcal{U}$ ($u \in \mathcal{U}$). Let the path in $\mathcal{N}$ labeled by (u, v) end in h. It follows that u is asymptotically equivalent to $v \cdot h$. Suppose there is a path in $\mathcal{N}$ labeled by (v, v) that ends in g. It follows that v is asymptotically equivalent to $v \cdot g$. Thus u is asymptotically equivalent to $v \cdot gh$ which is asymptotically equivalent to $u \cdot h^{-1}gh$. By definition, there is a path in $\mathcal{N}$ labeled by (u, u) which ends in $h^{-1}gh$. Since $u \in \mathcal{U}$ we get $h^{-1}gh = 1$, thus $g = 1$. $\square$

Proposition 7. *Almost every point of $\mathcal{J}_G$ has precisely $|X|$ preimages under* s.

Proof. Since every point of $\mathcal{J}_G$ has at most $|X|$ preimages under s, it is enough to show that for almost every $w \in X^{-\omega}$ the map $\pi_{\mathcal{J}} : \sigma^{-1}(w) \to \mathsf{s}^{-1}(\pi_{\mathcal{J}}(w))$ is one-to-one. Suppose that for some $w \in X^{-\omega}$ and $x \neq y$ in X we have $\pi_{\mathcal{J}}(wx) = \pi_{\mathcal{J}}(wy)$. It follows that wx, wy are asymptotically equivalent, thus there is a left-infinite path in $\mathcal{N}$ labeled by (wx, wy). It follows that the prefix of this path labeled by (w, w)

must end in the nontrivial element, so $w \notin \mathcal{U}$. Since $\mu_u(\mathcal{U}) = 1$ by Lemma 7, we are done. □

The definition of the measure m uses encoding of $\mathfrak{J}_G$ by sequences $X^{-\omega}$. At the same time the space $\mathfrak{J}_G$ is defined as the space of obits $\mathcal{X}_G/G$. Let us show that we can recover $(\mathfrak{J}_G, \mathsf{m})$ from the measure space $(\mathcal{X}_G, \mu)$.

Proposition 8. *Let ρ be the factor map $\mathcal{X}_G \to \mathfrak{J}_G$. Then for any $u \in \mathcal{U}^*$ the restriction $\rho|_{\mathfrak{T}_u} : \mathfrak{T}_u \to \mathsf{T}_u$ is a homeomorphism, $\rho^{-1}(\mathsf{T}_u) = \sqcup_{g\in G}\mathfrak{T}_u \cdot g$.*

Proof. By definition, the map $\rho|_{\mathfrak{T}_u} : \mathfrak{T}_u \to \mathsf{T}_u$ is surjective, and $\mathfrak{T}_u$ is compact. Hence in order to prove that ρ is a homeomorphism it is left to show that ρ is injective on $\mathfrak{T}_u$. Take $x, y \in \mathfrak{T}_u$, and let $wu \in X^{-\omega}$ represent x and $vu \in X^{-\omega}$ represent y. Suppose that $\rho(x) = \rho(y)$. By Proposition 4 it means that there is a left-infinite path in $\mathcal{N}$ labeled by (wu, vu). Since $u \in \mathcal{U}^*$, this path must end in 1. It follows, that wu and vu represent the same point of the tile $\mathfrak{T}_u$, and $x = y$.

To prove the second claim, take $x \in \mathfrak{T}_u \cdot g \cap \mathfrak{T}_u \cdot g'$. Then x is represented by two asymptotically equivalent sequences $wu \cdot g$ and $w'u \cdot g'$. It follows that there is a path in the nucleus $\mathcal{N}$ labeled by (u, u) which ends in $g'g^{-1}$. Then $g = g'$ since $u \in \mathcal{U}^*$. □

Theorem 9. *The projection $\mathcal{X}_G \to \mathfrak{J}_G$ is a covering map up to sets of measure zero.*

Proof. Consider the sets $\tilde{\mathcal{X}}_G = \pi_{\mathcal{X}}(\mathcal{U}\times G)$ and $\tilde{\mathfrak{J}}_G = \pi_{\mathfrak{J}}(\mathcal{U})$. It follows that $\tilde{\mathcal{X}}_G/G = \tilde{\mathfrak{J}}_G$. Since the set $\mathcal{U}$ has measure 1 by Lemma 7, the complements of $\mathcal{U} \times G$, of $\tilde{\mathcal{X}}_G$, and of $\tilde{\mathfrak{J}}_G$ have measure 0.

Since the group G acts properly on $\mathcal{X}_G$, the same holds for $\tilde{\mathcal{X}}_G$. It is left to prove the freeness. Suppose $x \cdot g = x$ for $x \in \tilde{\mathcal{X}}_G$ and $g \in G$. Let $u \cdot h$ be a representative of x in $X^{-\omega} \times G$. It follows that $u \cdot hg$ is asymptotically equivalent to $u \cdot h$, thus there is a path in $\mathcal{N}$ labeled by (u, u) that ends in $h^{-1}gh$. Since $u \in \mathcal{U}$ we have $g = 1$. Hence the projection $\tilde{\mathcal{X}}_G \to \tilde{\mathfrak{J}}_G$ is a covering map, and the statement follows. □

Let $(\mathcal{X}, \nu)$ be a locally compact measure space, the group G acts freely and properly discontinuously on $\mathcal{X}$ by homeomorphisms, and the measure ν is G-invariant. There is a unique measure ν_* on the quotient space $\mathcal{X}/G$, called the *quotient measure*, with the property that if U is an open subset of $\mathcal{X}$ such that $Ug \cap Uh = \emptyset$ for all $g, h \in G$, $g \neq h$, then $\nu_*(U/G) = \nu(U)$.

Proposition 9. *The quotient measure μ_* of the limit G-space $(\mathcal{X}_G, \mu)$ coincides with the measure m on the limit space $\mathfrak{J}_G$.*

Proof. Consider the sets $\tilde{\mathfrak{J}_G}$ and $\tilde{\mathcal{X}_G}$ of full measure from the previous theorem. For every $u \in \mathcal{U}^*$ we have $\mathsf{T}_u \subset \tilde{\mathfrak{J}_G}$, $\mathcal{T}_u \cdot g \subset \tilde{\mathcal{X}_G}$ for every $g \in G$, and $\rho^{-1}(\mathsf{T}_u) = \bigsqcup_{g \in G} \mathcal{T}_u \cdot g$ by Proposition 8. Since $\tilde{\mathfrak{J}_G} = \cup_{u \in \mathcal{U}^*} \mathsf{T}_u$ by Lemma 7 it suffices to show that for any $u \in \mathcal{U}^*$ the restriction of μ_* on T_u is equal to the restriction of m. Take a Borel set $A \subset \mathcal{T}_u$ and consider its preimage $\pi_{\mathcal{X}}^{-1}(A) = \sqcup_{g \in \mathcal{N}} A_g \cdot g$. Then $\pi_{\mathfrak{J}}^{-1}(\rho(A)) = \sqcup_{g \in \mathcal{N}} A_g$. Here the union is disjoint because if $w \in A_g \cap A_h$ then $w \cdot g$ and $w \cdot h$ are asymptotically equivalent to vu and $v'u$ respectively, for some $v, v' \in X^{-\omega}$. It follows that $vu \cdot g^{-1}$ is equivalent to $v'u \cdot h^{-1}$. Since $u \in \mathcal{U}$, $g = h$. Hence

$$\mathsf{m}(\rho(A)) = \mu_u(\pi_{\mathfrak{J}}^{-1}(\rho(A))) = \mu_u(\sqcup_{g \in \mathcal{N}} A_g) = \sum_{g \in \mathcal{N}} \mu_u(A_g).$$

By the property of the quotient measure

$$\mu_*(\rho(A)) = \mu(A) = \sum_{g \in \mathcal{N}} \mu_u(A_g) = \mathsf{m}(\rho(A)).$$

□

Corollary 4. *Theorem 8 holds for the limit space $\mathfrak{J}_G$, i.e. the measure space $(\mathfrak{J}_G, \mathsf{m})$ depends only on the associated virtual endomorphism.*

Corollary 5. $\mathsf{m}(\mathsf{T}_u) = \mu(\mathcal{T}_u) = (1/|X|^{|u|}) meas(\mathcal{T})$ *for $u \in \mathcal{U}^*$.*

Theorem 10. $(\mathcal{J}_G, \mathsf{s}, \mathsf{m})$ *is conjugate to the one-sided Bernoulli* $|X|$*-shift.*

Proof. We will use notions and results from [HR02]. First recall that a measure preserving map with entropy $\log d$ is called *uniformly* d*-to-one endomorphism* if it is almost everywhere d-to-one and the conditional expectation of each preimage is $1/d$. The standard example is $(X^{-\omega}, \sigma, \mu_u)$, which is uniformly $|X|$-to-one. Next we want to show that $(\mathcal{J}_G, \mathsf{s}, \mathsf{m})$ is also uniformly $|X|$-to-one. By Proposition 7 the map $\pi : X^{-\omega} \to \mathcal{J}_G$ is injective on the preimages $\sigma^{-1}(w)$ for almost all $w \in X^{-\omega}$, that is, π is *tree adapted* in terminology of [HR02]. We can apply Lemma 2.3 from [HR02], which says that a tree adapted factor of a uniform d-to-one endomorphism is again uniform d-to-one endomorphism. In particular, the shift s is the map of maximal entropy $\log |X|$.

To prove the theorem, we use the following Theorem 5.5 in [HR02].

Theorem 11. *A uniform* d*-to-one endomorphism* (Y, S, μ) *is one-sidedly conjugated to the one-sided Bernoulli* d*-shift if and only if there exists a generating function* f *so that* (Y, S, μ) *and* f *are tree very weak Bernoulli.*

Recall the definition of tree very weak Bernoulli and generating function. Let (Y, S, μ) be uniformly d-to-one and $f : Y \to R$ be a tree adapted function to a compact metric space R with metric D. The function f is called *generating* if the σ-algebra on Y is generated by $S^{-i}f^{-1}(\mathcal{B}), i \geq 0$, where $\mathcal{B}$ is the σ-algebra of Borel sets of the space R.

Informally, "tree very weak Bernoulli" means that for almost all pairs of points in Y their trees of preimages are close. To give a formal definition note that since S is uniformly d-to-one, for almost all points $y \in Y$ the set $S^{-k}(y)$ contains exactly d^k points, i.e. the tree of preimages is a d-regular rooted tree. The set $\{1, \ldots, d\}^*$ of finite words over $\{1, \ldots, d\}$ can be considered as a d-regular rooted tree, where every word

v is connected with vx for $x \in \{1, \ldots, d\}$, and the root is the empty word $\emptyset$. We can use the tree $\{1, \ldots, d\}^*$ to label the trees of preimages. For almost all points $y \in Y$ there is a map $T_y : \{1, \ldots, d\}^* \to Y$ such that $T_y(\emptyset) = y$ and $T_y(\sigma(v)) = S(T_y(v))$ for all nonempty words $v \in \{1, \ldots, d\}^*$. Every map T_y is tree adapted, and it is uniquely defined up to an automorphism of the tree $\{1, \ldots, d\}^*$. Then (Y, S, μ) and f are called *tree very weak Bernoulli* if for any $\varepsilon > 0$ and all sufficiently large n there is a set $W = W(\varepsilon, n) \subset Y$ with $\mu(W) > 1 - \varepsilon$ such that for any $y, y' \in W$

$$t_n(y, y') = \min_{\psi} \frac{1}{n} \sum_{v \in \{1,\ldots,d\}^*, |v| \leq n} d^{-|v|} D(f(T_y(v)), f(T_{y'}(\psi(v)))) < \varepsilon, \tag{4.4}$$

where the minimum is taken over all automorphisms ψ of the tree $\{1, \ldots, d\}^*$. Notice that the definition of t_n does not depend on the choice of T_y.

Let us show that $(\mathcal{J}_G, \mathsf{s}, \mu_{\mathsf{s}})$ is tree very weak Bernoulli for the identity map $id : \mathcal{J}_G \to \mathcal{J}_G$. It immediately follows that id is tree adapted and generating. Take a point $x \in \mathcal{J}_G$ and let x be represented by some $w \in X^{-\omega}$. Define the map $T_x : X^* \to \mathcal{J}_G$ by the rule $T_x(v) = \pi(wv)$. It is enough to show that $t_n(x, x') \to 0, n \to \infty$, for almost all $x, x' \in \mathcal{J}_G$. Using (4.2) we can find n_1 such that $\max_{v \in X^n} \mathrm{diam}(\mathsf{T}_{\mathrm{v}}) < \epsilon/2$ for all $n \geq n_1$. It means that $D(\pi(wv), \pi(w'v)) < \epsilon/2$ for all $v, |v| \geq n_1$ (here D is a fixed metric on the limit space $\mathcal{J}_G$). Thus, taking ψ to be the identical tree automorphism, we have that

$$t_n(x, x') < \frac{n_1}{n} \mathrm{diam}(\mathcal{J}_G) + \frac{n - n_1}{n} \frac{\epsilon}{2} < \frac{n_1}{n} \mathrm{diam}(\mathcal{J}_{\mathrm{G}}) + \frac{\epsilon}{2} < \epsilon$$

for $n > 2n_1 \mathrm{diam}(\mathcal{J}_{\mathrm{G}})/\epsilon$. Hence $(\mathcal{J}_G, \mathsf{s}, \mathsf{m})$ and id are tree very weak Bernoulli, which finishes the proof. $\square$

In the same way we introduce the measure m_{e} on the limit solenoid $\mathcal{S}_G$ as the push-forward of the uniform Bernoulli measure on $X^{\mathbb{Z}}$. It is easy to see that $(\mathcal{S}_G, \mathsf{e}, \mathsf{m}_{\mathsf{e}})$

is the inverse limit of dynamical systems $(\mathcal{J}_G, \mathsf{s}, \mathsf{m})$ (see [SG89, page 27]). In particular, we get

Corollary 6. $(\mathcal{S}_G, \mathsf{e}, \mathsf{m}_\mathsf{e})$ *is conjugate to the two-sided Bernoulli* $|X|$*-shift.*

4 Applications and examples

Invariant measures on nilpotent Lie groups. Let G be a finitely generated torsion-free nilpotent group. Let $\phi : H \to G$ be a contracting surjective virtual endomorphism such that the associated self-similar action is faithful (i.e. ϕ-core(H) is trivial in the terminology of [BS07]). Then ϕ is also injective by Theorem 1 in [BS07], thus ϕ is an isomorphism and we can apply Theorem 6.1.6 from [Nek05]. The group G and its subgroup H are uniform lattices of a simply connected nilpotent Lie group L by Malcev's completion theorem. The isomorphism $\phi : H \to G$ extends to a contracting automorphism ϕ_L of the Lie group L. There exists a G-equivariant homeomorphism $\Phi : \mathcal{X}_G \to L$ such that $\phi_L(\Phi(t)) = \Phi(\tau_{x_0}(t) \cdot g_0)$ for every $t \in \mathcal{X}_G$ and fixed $x_0 \in X$ and $g_0 \in G$.

Proposition 10. *The push-forward* $\Phi_*\mu$ *of the measure* μ *on the limit* G*-space* $\mathcal{X}_G$ *is the (right) Haar measure on the Lie group* L*.*

Proof. The measure $\Phi_*\mu$ is a non-zero regular Borel measure on L. It is left to prove that it is translation invariant. Since the measure μ is G-invariant and the map Φ is G-equivariant, the measure $\Phi_*\mu$ is G-invariant. By the property of the map Φ we have $\phi_L(\Phi(A)) = \Phi(\tau_{x_0}(A))g_0$ for every Borel set $A \subset \mathcal{X}_G$. Notice that since the map ϕ is injective, we get $\mu(A) = |X|\mu(\tau_x(A))$ (see the proof of Proposition 5) and hence $\Phi_*\mu(B) = |X|\Phi_*\mu(\phi_L(B))$ for every Borel set $B \subset L$. It follows that the measure $\Phi_*\mu$ is $\cup_n \phi_L^n(G)$-invariant. Since ϕ_L is contracting, the set $\cup_n \phi_L^n(G)$ is dense in the Lie group L. Hence $\Phi_*\mu$ is L-invariant. $\square$

The same observation holds in a more general setting of finitely generated virtually nilpotent groups (or under the conditions of Theorem 6.1.6).

Lebesgue measure of self-affine tiles. Let A be an $n \times n$ integer expanding matrix, where expanding means that every eigenvalue has modulus > 1. The lattice $\mathbb{Z}^n$ is invariant under A, and we can choose a coset transversal $D = \{d_1, \ldots, d_m\}$ for $\mathbb{Z}^n/A(\mathbb{Z}^n)$, where $m = |det(A)|$. There exists a unique nonempty compact set $T = T(A, D) \subset \mathbb{R}^n$, called (standard) *integral self-affine tile*, satisfying

$$A(T) = \bigcup_{d \in D} T + d.$$

The tile T has positive Lebesgue measure, is the closure of its interior, and the union above is nonoverlapping (the sets have disjoint interiors) [LW96b]. It is well-known [LW96a] that the tile T has integer Lebesgue measure. The question how to find this measure is studied in [LW96a, HLR03, DH08], and finally answered in [GY06]. A related question how to find the measure of intersection $T \cap (T + a)$ for $a \in \mathbb{Z}^n$ is studied in [GY06, EKM09]. Let us show how to answer these questions using the theory of self-similar groups.

The inverse of the matrix A can be considered as the contracting virtual endomorphism $A^{-1} : A(\mathbb{Z}^n) \to \mathbb{Z}^n$ of the group $\mathbb{Z}^n$, which is actually an isomorphism so that we can apply the previous example of this subsection. Put $X = \{x_1, \ldots, x_m\}$ and let $(\mathbb{Z}^n, X^*)$ be the self-similar contracting action defined by the virtual endomorphism A^{-1}, the coset transversal D, and the trivial cocycle $C = \{1, \ldots, 1\}$ (see Section 3). The group $\mathbb{Z}^n$ is the uniform lattice in the Lie group $\mathbb{R}^n$. Hence by Theorem 6.1.6 in [Nek05] (see also Section 6.2 there) there exists a $\mathbb{Z}^n$-equivariant homeomorphism $\Phi : \mathcal{X}_{\mathbb{Z}^n} \to \mathbb{R}^n$ given by

$$\Phi(\ldots x_{i_2} x_{i_1} \cdot g) = g + A^{-1} d_{i_1} + A^{-2} d_{i_2} + \ldots$$

for $i_j \in \{1, \ldots, m\}$ and $g \in \mathbb{Z}^n$. The image of the tile $\mathcal{T}$ is the self-affine tile T.

Proposition 11. *The push-forward $\Phi_*\mu$ of the measure μ on the limit G-space $\mathcal{X}_{\mathbb{Z}^n}$ is the Lebesgue measure θ on $\mathbb{R}^n$.*

Proof. The measure $\Phi_*\mu$ is the Haar measure on $\mathbb{R}^n$ by the above example. Since the Haar measure is unique up to multiplicative constant, we have that $\Phi_*\mu = c\theta$ for some constant $c > 0$ and we need to prove that $c = 1$.

Recall that $\mu(\mathcal{T})$ is integer by Proposition 7, and almost every point of $\mathcal{X}_G$ is covered by $\mu(\mathcal{T})$ tiles $\mathcal{T} \cdot g$ by Proposition 6. Hence $\Phi_*\mu(T) = \Phi_*\mu(\Phi(\mathcal{T})) = \mu(\mathcal{T})$ is integer and almost every point of $\mathbb{R}^n$ is covered by $\mu(\mathcal{T})$ tiles $T + g$, $g \in \mathbb{Z}^n$, with respect to the measure $\Phi_*\mu$, and thus with respect to the Lebesgue measure θ. It follows that, if χ_{T+g} is the characteristic function of $T + g$, then $\sum_{g\in\mathbb{Z}^n} \chi_{T+g} = \mu(\mathcal{T})$ almost everywhere with respect to both measures. Hence

$$\Phi_*\mu(T) = \mu(\mathcal{T}) = \int_I \sum_{g\in\mathbb{Z}^n} \chi_{T+g} d\theta = \sum_{g\in\mathbb{Z}^n} \int_{I+g} \chi_T d\theta = \int_{\mathbb{R}^n} \chi_T d\theta = \theta(T),$$

where I is the unit cube in $\mathbb{R}^n$. Since $\mu(\mathcal{T})$ is positive, $c = 1$. $\square$

Corollary 7. *The Lebesgue measure of the self-affine tile T is equal to the measure number meas$(\mathcal{N})$ of the nucleus $\mathcal{N}$ of the associated self-similar action $(\mathbb{Z}^n, X^*)$.*

The nucleus of a contracting self-similar action can be found algorithmically using the program package [MS08], and the number $meas(\mathcal{N})$ can be found using the remarks after Proposition 2. The measures of sets $T \cap (T + a)$ for $a \in \mathbb{Z}^n$ can be found by Remark 3.

The methods developed in [GY06] to find the Lebesgue measure of integral self-affine tiles are related to the discussion above. Take the complete automaton [Nek05, page 11] of the self-similar action $(\mathbb{Z}^n, X^*)$ (it actually coincides with the graph $B(\mathbb{Z}^n)$

from [ST02]), revert the direction of every edge, identify edges with the same starting and end vertices labeled by (d_1, d_2) with the same difference $r = d_2 - d_1$, and put the new label r on this edge. We get the graph $\mathcal{G}(\mathbb{Z}^n)$ constructed in [GY06] and [MTT01]. The set W constructed using $\mathcal{G}(\mathbb{Z}^n)$ [GY06, page 195] is precisely the nucleus $\mathcal{N}$. Hence the theory of self-similar groups provides a nice explanation to the ideas in [GY06, Section 3] and the methods developed in Sections 2,4 can be seen as its non-abelian generalization.

It is shown in [LW97] that integral self-affine tile T gives a lattice tiling of $\mathbb{R}^n$ with some lattice $L \subset \mathbb{Z}^n$. An interesting open question is whether this holds for any (self-replicating) contracting self-similar action (G, X^*) (or at least for self-similar actions of torsion-free nilpotent groups), i.e. the tile $\mathcal{T}$ gives a tiling of $\mathcal{X}_G$ with some subgroup $H < G$.

Self-affine sets. Let us now drop the condition that D is a coset transversal, so let it be any finite subset of $\mathbb{Z}^n$. There still exists a unique nonempty compact set $T = T(A, D) \subset \mathbb{R}^n$, called (integral) *self-affine set*, satisfying $A(T) = \cup_{d \in D}(T + d)$. We will show that Proposition 11 provides a method to compute the Lebesgue measure of T for any set D.

If the set D does not contain all coset representatives of $\mathbb{Z}^n/A(\mathbb{Z}^n)$, we extend it to the set $K \supset D$ which does, and choose a coset transversal $X \subset K$.

Construct a directed labeled graph (automaton) $\Gamma = \Gamma(A, K)$ with the set of vertices $\mathbb{Z}^n$ and we put a directed edge from u to v for $u, v \in \mathbb{Z}^n$ labeled by the pair (x, y) for $x, y \in K$ if $u + x = y + Av$. We slightly generalize the definition of the nucleus in the following way. Let the *nucleus* of the graph Γ be the subgraph (subautomaton) $\mathcal{N}$ spanned by all cycles of Γ and all vertices that can be reached following directed paths from the cycles. It is easy to see that since the matrix A is expanding the nucleus $\mathcal{N}$ is a finite graph (this also follows from the proof below). It follows from $u + x = y + Av$ that whenever $\|u\| > (1 - \|A^{-1}\|)^{-1}\max_{x,y\in K} \|A^{-1}(x-y)\|$ then $\|v\| < \|u\|$. It is then easy to check that if $u \in \mathbb{Z}^n$ belongs to the nucleus, then $\|u\| \leq 1 - \|A^{-1}\|^{-1}\max_{x,y\in K} \|A^{-1}(x-y)\|$, thus nucleus is a finite set. Remove every edge in $\mathcal{N}$ whose label is not in $X \times D$, and replace every label (a,b) by a. We get some finite graph $\mathcal{N}_D$ whose edges are labeled by elements of X.

Theorem 12. *The Lebesgue measure of the self-affine set T is equal*

$$\lambda(T) = \sum_{v\in\mathcal{N}_D} \mu(F_v),$$

where F_v is the set of left-infinite sequences that label the paths in $\mathcal{N}_D$ that end in the vertex v.

Proof. Consider the map $\Psi : K^{-\omega} \times \mathbb{Z}^n \to \mathbb{R}^n$ given by the rule

$$\Psi(\ldots x_2 x_1 \cdot v) = v + A^{-1}x_1 + A^{-2}x_2 + \ldots,$$

where $x_i \in K$ and $v \in \mathbb{Z}^n$. Since $\mathbb{Z}^n = E + A(\mathbb{Z}^n)$ the map Ψ is onto. Two elements $\xi = (\ldots x_2x_1, v)$ and $\zeta = (\ldots y_2y_1, u)$ for $x_i, y_i \in K$ and $v, u \in \mathbb{Z}^n$ represent the same point $\Psi(\xi) = \Psi(\zeta)$ in $\mathbb{R}^n$ if and only if there is a finite subset $B \subset \mathbb{Z}^n$ and a sequence $\{v_m\}_{m\geq 1} \in B$ such that there exists the path

$$v_m \xrightarrow{(x_m,y_m)} v_{m-1} \xrightarrow{(x_{m-1},y_{m-1})} \ldots \xrightarrow{(x_2,y_2)} v_1 \xrightarrow{(x_1,y_1)} u - v \tag{4.5}$$

in the graph Γ for every $m \geq 1$. Indeed, this path implies that

$$v_m + x_m + Ax_{m-1} + \ldots + A^{m-1}x_1 + A^m v = y_m + Ay_{m-1} + \ldots + A^{m-1}y_1 + A^m u. \tag{4.6}$$

Applying A^{-m} and using the facts that A^{-1} is contracting and the sequence $\{v_m\}_{m\geq 1}$ attains a finite number of values, we get the equality $\Psi(\xi) = \Psi(\zeta)$. For the converse, we choose v_m such that (4.6) holds, and using equality $\Psi(\xi) = \Psi(\zeta)$ we get that $\{v_m\}_{m\geq 1}$ attains a finite number of values. Notice that since the set B is assumed finite, every element v_m lies either on a cycle or there is a directed path from a cycle to v_m. In particular, all elements v_m should belong to the nucleus $\mathcal{N}$, and we have that the elements ξ and ζ represent the same point in $\mathbb{R}^n$ if and only if there exists a left-infinite path in $\mathcal{N}$ labeled by $(\ldots x_2x_1, \ldots y_2y_1)$ and ending in $u-v$.

Take the restriction $\Phi : X^{-\omega}\times\mathbb{Z}^n \to \mathbb{R}^n$ of the map Ψ on the subset $X^{-\omega}\times\mathbb{Z}^n$. The push-forward of the product measure μ and the counting measure on $\mathbb{Z}^n$ under Φ is the Lebesgue measure on $\mathbb{R}^n$ by Proposition 11. Hence to find the Lebesgue measure of the self-affine set T it is sufficient to find the measure of its preimage in $X^{-\omega}\times\mathbb{Z}^n$. However, T is equal to $\Psi(D^{-\omega}\times 0)$, and hence the sequence $(\ldots x_2x_1, v)$ for $x_i \in X$ and $v \in \mathbb{Z}^n$ represents a point in T if and only if there exists a left-infinite path in the nucleus $\mathcal{N}$, which ends in $-v$ and is labeled by $(\ldots x_2x_1, \ldots y_2y_1)$ for some $y_i \in D$. Hence

$$\Phi^{-1}(\Psi(D^{-\omega}\times 0)) = \bigcup_{v\in\mathcal{N}_D} (F_v, -v), \tag{4.7}$$

and the statement follows. $\square$

CHAPTER V

CONCLUSIONS

In this dissertation we study possible applications of measure theory in the theory of automata and groups generated by automata. Below we outline major results.

The first part of the dissertation is devoted to the action of finite automata on Bernoulli measures, that are not necessarily uniform. The results are contained in Chapter III and are published in [Kra10]. We establish that the result depend on the structure of the automaton. Namely, if the automaton is of polynomial growth, which is equivalent to the fact that in its Moore diagram no point belongs to two different cycles, then image of the Bernoulli measure is absolutely continuous to the measure itself. We are also able to write the Radon-Nikodim derivative of the image.

Theorem 13. *For an automorphism g of polynomial growth and Bernoulli measure μ, the push-forward $g_*\mu$ is absolutely continuous with respect to μ and*

$$\frac{dg_*\mu}{d\mu} = \sum_{v\in V_{max}} \frac{\mu(g^{-1}(vX^\infty))}{\mu(vX^\infty)}\chi_{vX^\infty}.$$

On the other hand, if the automaton is strongly connected, that is, in its Moore diagram each two states are connected by a path, then we prove that if it is moreover invertible, it maps a nonuniform Bernoulli measure to a sigular one. This is connected to an earlier result of Ryabinin in [Rya86], which calculates the frequency of 1 in the output sequnce of the automata on the binary alphabet. We generalize this result to arbitrary alphabet.

Theorem 14. *Let g be a strongly connected tree endomorphism, $w \in X^\infty$. Let μ be the Bernoulli measure with the probability of y equal to $p(y)$ for $y \in X$. Then the frequency of x in the sequence $g(w)$ exists and is the same for almost all w with respect* *to μ and this frequency is equal to $\sum_{s\in S}(\sum_{y\in X}\chi_i(\lambda(s,y))p(y))q(s)$, where S is the set of restrictions of g and $q(s)$ are the stationary probabilities for the ergodic Markov chain on S, $\zeta_{n+1}=\pi(\zeta_n,w_n)$ defined by the transition probabilities $\sum_{z:\pi(s,z)=t}p(z)$.*

Using theorem 14 we prove

Theorem 15. *Suppose that the nontrivial tree automorphism g is strongly connected. If there is i such that $p(x)\neq 1/d$, then μ and the image measure $g_*\mu$ are singular.*

The second part of this dissertation is devoted to introduction of measure in the setting of limit spaces of contracting self-similar groups defined by Nekrashevych in [Nek05]. The results, which are contained in Chapter IV are published in [BK]. We start by establishing some measure theoretical properties of labeled graphs. In particular we prove that if there are no vertices with several outgoing edges labeled by the same letter, then the measure of the left-infinite sequences read along the paths of the graph, is integer.

Theorem 16. *Let $\Gamma=(V,E)$ be a finite right-resolving graph. Then*

$$\mu_p(\Gamma)=\min_{w\in X^{-\omega}}|V(w)|=\min_{w\in X^*}|V(w)|.$$

In particular, the measure $\mu_p(\Gamma)$ is integer.

The limit space $\mathcal{X}_G$ of a self-similar group is defined as factor of a product of the set of left-infinite sequences $X^{-\omega}$ and group G. Therefore, we define the measure on the limit space $\mathcal{X}_G$ as a push-down of the product of the Bernoulli measure on $X^{-\omega}$ and countable measure on G. We prove that the measure we get depends only on the virtual endomorphism.

Theorem 17. *Let $\phi: H\to G$ be a virtual endomorphism of the group G. Let (G,X^*) and (G,Y^*) be the contracting self-similar actions defined respectively by the triples*

(ϕ, T, C) *and* (ϕ, T', C'). *Then the homeomorphism* $\alpha : \mathcal{X}_{(G,X^*)} \longrightarrow \mathcal{X}_{(G,Y^*)}$ *from Theorem* 6 *preserves measure, i.e.*

$$\mu_{(G,Y^*)}(\alpha(A)) = \mu_{(G,X^*)}(A)$$

for any Borel set A.

There is a subset $\mathcal{T}$ of the limit space $\mathcal{X}_G$, which is called the tile and which is defined as an image of $X^{-\omega} \times \{1\}$. Nekrashevych in [Nek05] studied various topological properties of tiles. We compute its measure, using the result above about labeled graphs, which we apply to the nucleus $\mathcal{N}$ of the self-similar contracting group G.

Theorem 18. *The measure* $\mu(\mathcal{T})$ *is equal to the measure number meas*$(\Gamma_{\mathcal{N}})$ *of the nucleus, in particular it is always integer.*

It is well known that when the self-similar group G is free abelian of finite rank, then its limit space $\mathcal{J}_G$ can be identified with $\mathbb{R}^n$. In this case, there is Lebesgue measure on defined on it, along with the measure we introduced above. We prove that these two measures are equal.

Proposition 12. *The push-forward* $\Phi_*\mu$ *of the measure* μ *on the limit* G*-space* $\mathcal{X}_{\mathbb{Z}^n}$ *is the Lebesgue measure* θ *on* $\mathbb{R}^n$.

We also establish several facts concerning measure on the limit space $\mathcal{J}_G$ of a self-similar group G. In particular we prove that $\mathcal{J}_G$ considered as a dynamical system is conjugate to the one-sided Bernoulli shift.

Theorem 19. $(\mathcal{J}_G, \mathsf{s}, \mathsf{m})$ *is conjugate to the one-sided Bernoulli* $|X|$*-shift.*

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