GENERIC PROPERTIES OF ACTIONS OF $\mathbb{F}_N$

A Dissertation

by

JAMES MITCHELL HITCHCOCK

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2010

Major Subject: Mathematics

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Approved by:

Chair of Committee,	David Kerr
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ABSTRACT

Generic Properties of Actions of $\mathbb{F}_n$. (August 2010)

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We investigate the genericity of measure-preserving actions of the free group $\mathbb{F}_n$, on possibly countably infinitely many generators, acting on a standard probability space. Specifically, we endow the space of all measure-preserving actions of $\mathbb{F}_n$ acting on a standard probability space with the weak topology and explore what properties may be verified on a comeager set in this topology. In this setting we show an analog of the classical Rokhlin Lemma. From this result we conclude that every action of $\mathbb{F}_n$ may be approximated by actions which factor through a finite group. Using this finite approximation we show the actions of $\mathbb{F}_n$, which are rigid and hence fail to be mixing, are generic. Combined with a recent result of Kerr and Li, we obtain that a generic action of $\mathbb{F}_n$ is weak mixing but not mixing. We also show a generic action of $\mathbb{F}_n$ has Σ-entropy at most zero. With some additional work, we show the finite approximation result may be used to that show for any action of $\mathbb{F}_n$, the crossed product embeds into the tracial ultraproduct of the hyperfinite II_1 factor. We conclude by showing the finite approximation result may be transferred to a subspace of the space of all topological actions of $\mathbb{F}_n$ on the Cantor set. Within this class, we show the set of actions with Σ-entropy at most zero is generic.

To Lela Emmelynn Hitchcock

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1. INTRODUCTION

Classically a dynamical system can be characterized as a group action of the real numbers or integers on a standard probability space. In this context, the concept of a generic dynamical system may be traced back to the early twentieth century. Of these early results, those of Halmos [14] and Rokhlin [30] are probably the most well known. In particular, these results implied the existence of a weak mixing but not mixing dynamical system long before any explicit examples were known.

Throughout the twentieth century many of the fundamental concepts of classical dynamical systems were extended to more general groups such as amenable groups [25,26] and locally compact groups [2]. During this time, interest also arose in actions of the aforementioned groups on compact Hausdorff spaces. Recently there has been renewed interest what properties a generic dynamical system has in these settings.

In the measure-preserving setting Kerr and Pichot [20] showed that for a second countable locally compact group the set of weak mixing actions is generic if and only if the group lacks Kazhdan's property (T). Informally a group is said to have Kazhdan's property (T) if every representation that has nonzero almost invariant vectors admits a nonzero invariant vector. For amenable groups, it is well known that the set of actions which have zero entropy is generic. In the topological setting, most of the known results are for integers actions on the Cantor set, see [1], [13], and [16] for example.

The content of this work focuses on actions of the free group $\mathbb{F}_n$ on n generators for some $n \in \mathbb{N} \cup \{\infty\}$ acting on a nonatomic standard probability space and, to some extent, the Cantor set. In Section 2 we give a brief introduction to both measure-

This dissertation follows the style of *Ergodic Theory and Dynamical Systems.*

preserving and topological dynamical systems as well as establish much of the notation we will use throughout. We also make precise what it means for a property to be generic. In particular, given a group G and measure space or topological space X we topologize the space of all actions of G on X in such a way that it becomes a Polish space. We then define a property to be generic if it can be realized on a set containing a dense G_δ set with respect to this topology. We also introduce the Rokhlin lemma for integer actions of a standard probability space and comment on its significance.

Section 3 is devoted to establishing a result akin to the Rokhlin lemma for actions of $\mathbb{F}_n$ on a nonatomic standard probability space (X, μ). We begin this section by showing the topology defined on the space of all measure-preserving actions of $\mathbb{F}_n$ on X may be simplified. We then define what it means for an action to pointwise permute some finite measurable partition of X. In Proposition 3.2.2 we show this condition implies factoring through a finite group. The culmination of this work is Theorem 3.3.2 which states any action of $\mathbb{F}_n$ can be approximated by one which pointwise permutes some partition.

Sections 4, 5, and 6 are applications of Theorem 3.3.2 and Theorem 3.3.3. In Section 4 we establish that a generic action of $\mathbb{F}_n$ fails to be mixing. Since $\mathbb{F}_n$ is known to lack Kazhdan's property (T), the previously mentioned result of Kerr and Pichot implies a generic action of $\mathbb{F}_n$ is weak mixing but not mixing. Section 5 introduces the notion of Σ-entropy for sofic groups as defined by Bowen [5]. We verify that $\mathbb{F}_n$ is residually finite and thus sofic by giving a rather simple dynamical proof using Bernoulli shifts and Theorem 3.3.3. We then show the set of actions of $\mathbb{F}_n$ whose Σ-entropy is at most zero is generic.

In Section 6 we address a case of Connes' embedding problem. It follows from a result of Brown, Dykema, and Jung [7] on almagamated free products that for any action of $\mathbb{F}_n$ on a nonatomic probability space (X, μ) the crossed product $L^\infty(X, \mu) \rtimes$

$\mathbb{F}_n$ can be embedded into $\mathcal{R}^\omega$, the tracial ultrapower of the hyperfinite II_1-factor. Using the results of Section 3 we modify the argument given by Wassermann [35] showing that the group von Neumann algebra of $\mathbb{F}_2$ embeds into $\mathcal{R}^\omega$ to construct an embedding of $L^\infty(X,\mu) \rtimes \mathbb{F}_n$ into $\mathcal{R}^\omega$.

In Section 7 we extend many of the concepts and results of Section 3 to actions of $\mathbb{F}_n$ on the Cantor set K. The end result is Theorem 7.2.3 which is a topological analog of Theorem 3.3.2. Unlike the measure-preservng case, we note that Theorem 7.2.3 cannot be expected to hold for all actions of $\mathbb{F}_n$ on K. Instead, we give a complete description of those actions for which it does hold. Recently Kerr and Li [19] have extended Bowen's definition of Σ-entropy to actions of sofic groups on compact metric spaces. We show for the set of actions for which Theorem 7.2.3 holds that the set of actions which have entropy at most zero is generic. We conclude the work in Section 8 with possible directions for future study.

2. PRELIMINARIES

In this section we hope to give an adequate introduction to dynamical systems for the reader to follow the theory and results that follow. Unfortunately, the general theory is both vast and quickly developing and we must sacrifice in depth discussions for brevity. For more detailed discussions of ergodic theory and dynamical systems the reader is refered to [4], [12], or [27]. We will also assume a basic knowledge of topology and topological groups such as in [6], and measure theory such as in [11]. Finally, we assume some basic knowledge of von Neumann algebras.

2.1. Introduction to Dynamical Systems

Generally speaking, a dynamical system is a triple (G, X, α) consisting of a group G, a space X, and a group action α of G on X. The study of dynamical systems can then be characterized as the study of the asymptotic behavior of systems. To concretely analyze this asymptotic behavior of a dynamical system (G, X, α) we must make some assumptions on G, X, and α. Classically, G was assumed to be either $\mathbb{R}$ or $\mathbb{Z}$. In the 1980's many fundamental notions of dynamical systems were extended to actions of amenable groups and locally compact groups, see [25, 26] and [2, 12] respectively. In practice the groups we will encounter will be countable groups endowed with the discrete topology. Thus unless otherwise specified, G will be assumed to be a countable discrete group throughout this dissertation.

The assumptions on X and consequently α will be either measure-theoretic or topological in nature. We say (G, X, α) is a *topological dynamical system* or *topological G-system* if X is a compact Hausdorff space and α is a group action of G on X such that $\alpha_s : X \to X$ is a homeomorphism for each $s \in G$. If X is metrizable we say the system (G, X, α) is *metrizable*. Unless otherwise specified, we will assume all topological systems are metrizable. If $G = \mathbb{Z}$ the system $(\mathbb{Z}, X, \alpha)$ is determined by $T = \alpha_1$. Thus we denote topological $\mathbb{Z}$-systems by (X, T) where T is a homeomorphism from X onto itself.

Before defining the measure-theoretic case we must make a brief digression into measure theory. A topological space X is said to be a *Polish space* if it is homeomorphic to complete metric space which is second countable and generates the topology on X. A measurable space $(X, \mathscr{X})$ is said to be a *standard Borel space* if X is a Polish space and $\mathscr{X}$ is the Borel σ-algebra generated by the open subsets of X. A measure space $(X, \mathscr{X}, \mu)$ is said to be a *standard probability space* if $(X, \mathscr{X})$ is a standard Borel space and μ is a probability measure. When there is no risk of confusion, we will omit the σ-algebra $\mathscr{X}$ from $(X, \mathscr{X}, \mu)$ and write (X, μ).

Suppose α is a group action of G on a standard probability space (X, μ). We say α is *measure-preserving* if $\mu(\alpha_s A) = \mu(A)$ for all $A \in \mathscr{X}$ and $s \in G$. Such an action is called a *measure-preserving dynamical system* or *measure-preserving G-system* and we denote it by (G, X, μ, α). As with the topological case, when $G = \mathbb{Z}$ the system is determined by $T = \alpha_1$. Consequently, we denote integer actions by (X, μ, T) where T is a measure-preserving automorphism of X.

The preceding assumptions on α and X give rise to two distinct yet overlapping classes of dynamical systems. In many cases information can be gained by viewing a system as both a topological and measure-preserving dynamical system. Other interesting questions arise from the relationship between topological and measure-preserving systems. It should be noted that other meaningful restrictions can be placed on the triple (G, X, α). For example, we could require X to be a differential manifold and α_s to be a diffeomorphism for each $s \in G$. However, measure-preserving and topological dynamical systems will be of primary interest and we will restrict our attention to them.

The following examples are standard in any introductory text. We give a brief overview for future reference.

Example 2.1.1 (Rotations on the unit circle)**.** Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle with Lebesgue measure λ. Given $w \in \mathbb{T}$ define $T_w : \mathbb{T} \to \mathbb{T}$ by $T_w z = wz$. Then $(\mathbb{T}, \lambda, T_w)$ is a measue preserving $\mathbb{Z}$-system. If $\mathbb{T} \subset \mathbb{C}$ is regarded as compact Hausdorff space then $(\mathbb{T}, T_w)$ is a topological $\mathbb{Z}$-system. The systems $(\mathbb{T}, \lambda, T_w)$ and $(\mathbb{T}, T_w)$ are most interesting when w is not a root of unity.

Example 2.1.2 (Bernoulli shifts)**.** Let $(X, \mathscr{X}, \mu)$ be a standard probability space and G a locally compact group. Let $X^G = \prod_{s\in G} X$ and μ^G be the product measure on the product Borel structure $\mathscr{X}^G$. Then $(X^G, \mathscr{X}^G, \mu^G)$ is again a standard probability space. Viewing elements of X^G as functions $x : G \to X$ we may define $\alpha_s(x)(t) = x(s^{-1}t)$. Then (X^G, μ^G, α) is measure-preserving dynamical system. If we assume X is compact Hausdorff space then the product topology on X^G is again compact and Hausdorff. It can then be verified that (X^G, α) is a topological dynamical system. We refer to the dynamical systems (G, X^G, μ^G, α) and (G, X^G, α) as *Bernoulli shifts.*

Example 2.1.3 (Odometers)**.** Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of integer such that $r_n \geq 2$ for all $n \in \mathbb{N}$. Define $X = \prod_{n=1}^{\infty}\{0, 1, \ldots, r_n - 1\}$. Then X is a Cantor set. Define an operation on X by addition mod r_n on the nth coordinate with carryover to the $(n+1)$th coordinate. Let $\mathbf{1} = (1, 0, 0, \ldots) \in X$. Define $O : X \to X$ by $O(\mathbf{x}) = \mathbf{x} + \mathbf{1}$. Then O is a homeomorphism of X, known as an *odometer* or *adding machine.* When $r_n = 2$ for all $n \in \mathbb{N}$ we call O the *dyadic odometer.*

The original intent of this dissertation was to study generic properties of measure-preserving actions of certain groups. Consequentially, the majority of the content of this dissertation is devoted to measure-preserving dynamical systems. It became evident that some of the more interesting results could be transferred to the topological setting. In Section 7 we will extend the appropriate results to certain topological actions. To further understand measure-preserving actions, we again digress briefly into measure theory.

Let $(X, \mathscr{X}, \mu)$ be a measure space. It is routine to check that $A \sim B$ if and only if $\mu(A \triangle B) = 0$ defines an equivalence relation on $\mathscr{X}$. Given $A \in \mathscr{X}$ let $\widetilde{A}$ be the equivalence class of A under this relation and set $\widetilde{\mathscr{X}} = \{\widetilde{A} : A \in \mathscr{X}\}$. Given $\widetilde{A}, \widetilde{B} \in \widetilde{\mathscr{X}}$ it is easily verified that $\widetilde{A} \cup \widetilde{B} = \widetilde{A \cup B}$, $\widetilde{A} \cap \widetilde{B} = \widetilde{A \cap B}$, and $\widetilde{A}^c = \widetilde{A^c}$ are well defined operations showing $\widetilde{\mathscr{X}}$ is an algebra. In fact, countable unions are also well defined, whence $\widetilde{\mathscr{X}}$ is a σ-algebra. Given $\widetilde{A} \in \widetilde{\mathscr{X}}$ define $\widetilde{\mu}(\widetilde{A}) = \mu(A)$ for some $A \in \widetilde{A}$. Routine calculations show $\widetilde{\mu}$ is a well defined measure. We call the pair $(\widetilde{\mathscr{X}}, \widetilde{\mu})$ the *measure algebra of* $(X, \mathscr{X}, \mu)$.

Suppose $(X, \mathscr{X}, \mu)$ and $(Y, \mathscr{Y}, \nu)$ are measure spaces. A bijection $\phi : X \to Y$ is said to be a *point isomorphism* if $\mu \circ \phi^{-1} = \nu$. We say X and Y are *isomorphic mod 0* if there exist null sets N_μ and N_ν of X and Y respectively and a point isomorphism $\phi : X/N_\mu \to Y/N_\nu$. We say the measure algebras $\widetilde{X}$ and $\widetilde{Y}$ are *isomorphic* if there exists a bijective map $\psi : \widetilde{X} \to \widetilde{Y}$ satisfying:

- $\Phi(\widetilde{A}^c) = \Phi(\widetilde{A})^c$
- $\Phi(\widetilde{A} \cup \widetilde{B}) = \Phi(\widetilde{A}) \cup \Phi(\widetilde{B})$
- $\widetilde{\mu}(\widetilde{A}) = \widetilde{\nu}(\Phi(\widetilde{A}))$.

If X and Y are isomorphic mod 0, it is clear that the respective measure algebras $(\widetilde{X}, \widetilde{\mu})$ and $(\widetilde{Y}, \widetilde{\nu})$ are isomorphic. When $(X, \mathscr{X}, \mu)$ and $(Y, \mathscr{Y}, \nu)$ are nonatomic probability spaces, the converse holds, see Section 6 of Chapter 15 in [31].

Let (G, X, μ, α) be a measure-preserving dynamical system. Then α induces an action $\widetilde{\alpha}$ of G on the measure algebra $\widetilde{\mathscr{X}}$ given by $\widetilde{\alpha}_s(\widetilde{A}) = \alpha_s(A)$ for some $A \in \widetilde{A}$. Similarly, it can be shown every action of the measure algebra $\widetilde{X}$ induces an action of

X when (X, μ) is a nonatomic standard probability space. In particular, this observation allows us to ignore null sets in many instances. Thus we adopt the convention that the underlying standard probability space of (G, X, μ, α) is nonatomic.

2.2. Conjugacy of Dynamical Systems

As in many branches of mathematics, determining when two objects are essentially the same is important in the study of dynamical systems. Specifically, we ask when two dynamical systems are conjugate. In the topological case we say the systems (G, X, α) and (G, Y, β) are *conjugate* if there exists a homeomorphism $\phi : X \to Y$ such that $\phi \circ \alpha = \beta \circ \phi$. We say the measure-preserving systems (G, X, μ, α) and (G, Y, μ, β) are *conjugate* if there exists a measure algebra isomorphism $\phi : \widetilde{\mathscr{X}} \to \widetilde{\mathscr{Y}}$ such that $\phi \circ \widetilde{\alpha} = \widetilde{\beta} \circ \phi$.

In the case of measure-preserving systems we may also speak of systems being isomorphic and spectrally equivalent. We say the systems (G, X, μ, α) and (G, Y, ν, β) are *isomorphic* if there exists nullsets N_μ and N_ν of X and Y respectively and a point isomorphism $\phi : X/N_\mu \to Y/N_\nu$ such that $\phi \circ \alpha = \beta \circ \phi$. It is clear if α and β are isomorphic, then they are conjugate. Under our assumptions that (X, μ) and (Y, ν) are nonatomic standard probability spaces the converse also holds.

A *(unitary) representation* of a topological group G is a pair $(\pi, \mathcal{H})$ consisting of a Hilbert space $\mathcal{H}$ and a homomorphism, π, from G into the group of unitary operators on $\mathcal{H}$ which is weakly continuous. Any measure-preserving dynamical system, (G, X, μ, α), induces a representation π_α of G on $L^2(X, \mu)$ given by $\pi_\alpha(s)f = f \circ \alpha_g^{-1}$. We call π_α the *Koopman representation.* We then say two systems (G, X, μ, α) and (G, Y, ν, β) are *spectrally equivalent* if there exists a unitary map $U : L^2(X, \mu) \to L^2(Y, \nu)$ such that $U \circ \pi_\alpha = \pi_\beta \circ U$.

We conclude this subsection with a few remarks on the relationship among conjugacy, isomorphism, and spectral equivalence. Under the assumption that (X, μ) is a nonatomic standard probability space, conjugacy and isomorphism are equivalent. For technical reasons, the convention is to disregard the latter. It is also clear that conjugacy implies spectral equivalence. In general, the converse need not hold. A discussion of when spectral equivalence implies conjugacy can be found in [15].

2.3. Asymptotic Properties of Dynamical Systems

As previously mentioned, the study of dynamical systems concerns the asymptotic behavior of a given system. In particular, we ask how a system behaves as we approach infinity in the group G. When $G = \mathbb{Z}$ approaching infinity is unambiguous. However, if G is a locally compact we must make explicit what it means to approach infinity. Informally we say a property occurs at infinity if it can be verified off compact subsets. For example, we say a function $\phi : G \to \mathbb{C}$ *vanishes at infinity* if for each $\epsilon > 0$ there exists a compact subset $K \subset G$ such that $|\phi(s)| < \epsilon$ for all $s \notin K$.

We now introduce what it means for both a measure-preserving and topological system to be ergodic, weak mixing and mixing. We refrain from introducing entropy at this time, but will address it in some detail in Sections 5 and 7. We first present the more familiar definitions for measure-preserving $\mathbb{Z}$-systems and then the definitions for measure-preserving actions of locally compact groups and topological actions. We say a $\mathbb{Z}$-system (X, μ, T) is:

- *ergodic* if whenever $A \in \mathscr{X}$ satisfies $TA = A$ then either $\mu(A) = 0$ or $\mu(A^c) = 0$,
- *mixing* if $\lim_{n\to\infty} |\mu(T^k A \cap B) - \mu(A)\mu(B)| = 0$,
- *weak mixing* if $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(T^k A \cap B) - \mu(A)\mu(B)| = 0$.

Suppose (G, X, μ, α) is a G-system where G is a locally compact group. For each

$f, g \in L^2(X, \mu)$ we associate a function, called a *matrix coefficient*, from G into $\mathbb{C}$ defined by $s \mapsto \langle \pi_\alpha(s)f, g\rangle$. We now define ergodicity, mixing and weak mixing for (G, X, μ, α) as follows. We say (G, X, μ, α) is:

- *ergodic* if the only π_α invariant functions in $L^2(X, \mu)$ are the constant functions,
- *mixing* if for all $f, g \in L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$ the matrix coefficient $s \mapsto \langle \pi_\alpha(s)f, g\rangle$ vanishes at infinity,
- *weak mixing* if for all $f, g \in L^2(X, \mu) \ominus \mathbb{C}\mathbf{1}$ the matrix coefficient $s \mapsto \langle \pi_\alpha(s)f, g\rangle$ vanishes at infinity for some sequence of group elements.

Let (G, X, α) be a topological dynamical system. We say (G, X, α) is *transitive* if for all nonempty open sets $U, V \subset X$ there exists $s \in G$ such that $\alpha_s U \cap V \neq \emptyset$. Given another topological dynamical system (G, Y, β) we define the product system $(G, X \times Y, \alpha \times \beta)$ in the natural sense, i.e. $(\alpha \times \beta)_s(x, y) = (\alpha_s x, \beta_s y)$. We then say (G, X, α) is:

- *ergodic* if every proper closed α-invariant subset of X is nowhere dense,
- *mixing* if for all nonempty open sets $U, V \subset X \times X$ there is a compact set $F \subset G$ such that $(\alpha \times \alpha)_s U \cap V \neq \emptyset$ for all $s \in G \setminus F$,
- *weak mixing* if the product system $(G, X \times X, \alpha \times \alpha)$ is transitive.

2.4. The Space of Actions and Generic Properties

It is clear from the definitions that a mixing $\mathbb{Z}$-action must also be weakly mixing. It is then natural to ask if there is an $\mathbb{Z}$-action which is weakly mixing but not mixing. One approach is to attempt to construct such an action directly. Although such examples been exhibited by Katok, Chaccon, and others, they tend to be complicated. An indirect approach is to show a "typical" $\mathbb{Z}$-action is weak mixing but not mixing. This is the approach taken by Halmos [14] and Rohklin [30] in the late 1940's.

The general idea behind the work of Halmos (resp. Rohklin) is to topologize the space of all $\mathbb{Z}$-actions and then show the set of actions which are weak mixing (resp. not mixing) is large in an appropriate sense. By assuring these sets are large enough, their intersection must be nonempty, whence actions which are weak mixing but not mixing must exist. To make this approach more precise we first introduce the relevant topologies on the spaces of measure-preserving and topological actions.

We denote by $\mathrm{Act}(G, X, \mu)$ the space of measure-preserving actions of G on X. Denote by $\mathrm{Aut}(X, \mu)$ the space of measure-preserving automorphisms of X, in which two automorphisms are identified if they agree almost everywhere. Given $\alpha \in \mathrm{Act}(G, X, \mu)$ the map $s \mapsto \alpha_s$ is a homomorphism from G into $\mathrm{Aut}(X, \mu)$. Similarly, given a homomorphism $\phi : G \to \mathrm{Aut}(X, \mu)$ we may define an action $\alpha \in \mathrm{Aut}(G, X, \mu)$ by $\alpha_s x = \phi(s)x$ for almost all $x \in X$. In particular, we may identify $\mathrm{Act}(G, X, \mu)$ with a subset of $\mathrm{Aut}(X, \mu)^G$.

We define the *weak topology* on $\mathrm{Aut}(X, \mu)$ by $T_\gamma \to T$ if and only if $\mu(T_\gamma A \triangle TA) \to 0$ for all $A \in \mathscr{X}$ and the *uniform topology* on $\mathrm{Aut}(X, \mu)$ by $T_\gamma \to T$ if and only if $\sup_{A \in \mathscr{X}} \mu(T_\gamma A \triangle TA) \to 0$. We then define the *weak* (resp. *uniform*) *topology* on $\mathrm{Act}(G, X, \mu)$ as the product topology on $\mathrm{Aut}(X, \mu)^G$ where $\mathrm{Aut}(X, \mu)$ is endowed with the weak (resp. uniform) topology. We denote the weak and uniform topologies by $(\mathrm{Act}(G, X, \mu), w)$ and $(\mathrm{Act}(G, X, \mu), u)$ respectively. Unless otherwise specified, we will assume $\mathrm{Act}(G, X, \mu)$ is endowed with the weak topology.

When G is a countable discrete group $(\mathrm{Act}(G, X, \mu), w)$ is a Polish space, see [17] for a description of a compatible metric. A basic open set for $(\mathrm{Act}(G, X, \mu), w)$ has the form

$$U(\alpha, F, K, \epsilon) = \bigcap_{s \in F} \bigcap_{A \in K} \{\beta \in \mathrm{Act}(G, X, \mu) : \mu(\alpha_s A \triangle \beta_s A) < \epsilon\}$$

where $\epsilon > 0$, F and K are finite subsets of G and $\mathscr{X}$ respectively. Alternatively, we can define a base for the weak topology with respect to the associated Koopman representations. In particular, a basic open set can be described by

$$V(\alpha, F, \Omega, \epsilon) = \bigcap_{s \in F} \bigcap_{\xi \in \Omega} \{\beta \in \operatorname{Act}(G, X, \mu) : \|(\pi_\alpha(s) - \pi_\beta(s))\xi\| < \epsilon\}$$

where $\epsilon > 0$, $F \subset G$ and $\Omega \subset L^2(X, \mu)$ are finite sets. Since the strong and weak operator topologies agree for unitary operators, we could have described a basic open set with respect to the weak operator topology as well.

For topological systems, we define the weak topology in a similar manner. Denote by $\operatorname{Act}(G, X)$ the set of all topological actions of G on X and by $\operatorname{Homeo}(X)$ the space of all homeomorphisms of X onto itself. Then $\operatorname{Act}(G, X)$ may be identified with $\operatorname{Homeo}(X)^G$. As in the measure-preserving case, we define a topology on $\operatorname{Homeo}(X)$ and let $\operatorname{Act}(G, X)$ have the corresponding product topology. We let $\operatorname{Homeo}(X)$ be endowed with the topology of uniform convergence. A basic open set of $\operatorname{Act}(G, X)$ then has the form

$$W(\alpha, F, \Omega, \epsilon) = \bigcap_{s \in F} \bigcap_{f \in \Omega} \{\beta \in \operatorname{Act}(G, X) : \|f \circ \alpha_s - f \circ \beta_s\| < \epsilon\}$$

where $\epsilon > 0$, $F \subset G$ and $\Omega \subset C(X)$ are finite sets. When X is a metric space with metric d we may define a compatible metric d_w on $\operatorname{Homeo}(X)$ by

$$d_w(S, T) = \sup_{x \in X} d(Sx, Tx) + \sup_{x \in X} d(S^{-1}x, T^{-1}x).$$

We can then describe a basic open set of $\operatorname{Act}(G, X)$ by

$$W'(\alpha, F, \epsilon) = \bigcap_{s \in F} \{\beta \in \operatorname{Act}(G, X) : d_w(\alpha_s, \beta_s) < \epsilon\}$$

where $\epsilon > 0$ and $F \subset G$ is finite. It is then clear that $\operatorname{Act}(G, X)$ is a Polish space when endowed with the weak topology.

We now make precise what it means for a set to be "large". Let X be any topological space. We say a subset of X is a G_δ if it is a countable intersection of open subsets of X. A subset is said to be *residual* or *comeager* if it contains a dense G_δ subset. A given property is said to be *generic* if it can be verified on a residual subset. In [14] it is shown weak mixing is generic for $\mathbb{Z}$-actions and in [30] it is shown failure to be mixing is generic for $\mathbb{Z}$-actions.

Since $\mathrm{Act}(G, X, \mu)$ and $\mathrm{Act}(G, X)$ are Polish spaces when endowed with their respective weak topologies, the Baire category theorem holds. In particular, if $\mathscr{O}$ is a countable collection of dense open sets then the Baire category theorem assures $\bigcap \mathscr{O}$ is nonempty and dense. Thus to show a property is generic it suffices to show the set of actions with the prescribed property contains a countable intersection of dense open sets. It also follows from the Baire category theorem that the intersection of two dense G_δ subsets is again a dense G_δ, whence the claim that dense G_δ subsets are "large" is justified. When combibed with the results of Halmos and Rokhlin, the preceding observations show that a generic $\mathbb{Z}$-action is weak mixing but not mixing.

2.5. The Rokhlin Lemma

The Baire category theorem gives a simple criteria for establishing the genericity of a prescribed property provided the existence of dense open sets. In the case of classical measure-preserving dynamics density is often established by means of the Rokhlin Lemma. Given a system (X, μ, T) which is aperiodic, i.e. $\mu(\{x : T^n x = x \text{ for some } n \in \mathbb{N}\}) = 0$, the Rokhlin lemma asserts for every $\epsilon > 0$ and every $n \in \mathbb{N}$ there exists a measurable set B such that $T^i B$, $i = 0, 1, 2, \ldots, n-1$ are pairwise disjoint and $\mu(\bigcup_{i=0}^{n-1} T^i) > 1 - \epsilon$. At first glance the Rokhlin lemma may seem simple. However, it is one of the most useful results in the study of measure-preserving dynamics. In fact, it follows immediately that the periodic $\mathbb{Z}$-actions are dense. The Rokhlin lemma is also key in showing the genericity of weak mixing $\mathbb{Z}$-systems [14] and nonmixing $\mathbb{Z}$-systems [30].

Since its introduction, the Rohklin lemma has been extended in various directions. In [25] and [26] Ornstein and Weiss introduced and proved a version of the Rokhlin lemma for actions of amenable groups. In the topological setting, both Putnum [29] and Bezuglyi, Dooley, and Medynets [3] have given conditions for the existence of a Rokhlin Lemma for homeomorphisms of the Cantor set. Versions of the Rokhlin property have appeared in various other fields of mathematics. One area of note is operator algebras. In [8] Connes introduced a version for von Neumann algebras. In the context of C^*-algebras, versions have appeared in [21] and [28] among other places. One of the goals of this dissertation will be to establish a Rokhlin type result for actions of the free groups.

3. A ROKHLIN TYPE LEMMA FOR ACTIONS OF THE FREE GROUP

3.1. Introduction to Actions of the Free Groups

Let S and S^{-1} be sets such that $S \cap S^{-1} = \emptyset$ and $|S| = |S^{-1}|$. Choose a bijection from S onto S^{-1} and denote the image of s by s^{-1}. We define a *word* on S to be a finite sequence of elements in $S \cup S^{-1}$. If s and s^{-1} are adjacent in a word for some $s \in S$ we may *simplify* the word by omitting them from the sequence. We then say a word is *reduced* if s and s^{-1} are not adjacent for any $s \in S$. We will adopt the standard convention of denoting a sequence $\{s_i\}_{i=1}^{n}$ in $S \cup S^{-1}$ by $s_1 s_2 \ldots s_n$. We denote by e the empty word, i.e. the empty sequence.

Let $\mathbb{F}_S$ be the set of all reduced words on S. We define a binary operation on $\mathbb{F}_S$ by concatenation followed by simplification if necessary. Then $\mathbb{F}_S$ becomes a group with with the empty word e being the identity. We call $\mathbb{F}_S$ the *free group generated by* S. If S and T are two sets then $\mathbb{F}_S$ and $\mathbb{F}_T$ are isomorphic if and only if $|S| = |T|$. When S is finite we write $\mathbb{F}_n$ where n is the cardinality of S and drop any reference to S. Similarly, we write $\mathbb{F}_\infty$ when S is countable. Throughout, $\mathbb{F}_n$ will denote the free group on n generators for some $n \in \mathbb{N} \cup \{\infty\}$. If we need to reference the generating set of $\mathbb{F}_n$ we will denote it by S.

Suppose for each $s \in S$, $f_s : X \to X$ is a measure-preserving automorphism of X. Let $t \in \mathbb{F}_n$. Then we may represent t uniquely as a reduced word $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n}$ for some $s_1, s_2, \ldots, s_n \in S$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \in \{-1, 1\}$. Define $\alpha_t = f_{s_1}^{\epsilon_1} \circ f_{s_2}^{\epsilon_2} \circ \cdots \circ f_{s_n}^{\epsilon_n}$. Since α_t is a compostion of measurable automorphisms and their inverses for each $t \in \mathbb{F}_n$, we have that α is measure-preserving. That α is a well defined group action is routine to check. In particular, we have verified that $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$.

In Section 2, we gave the description of two bases for the weak topology on $\mathrm{Act}(G, X, \mu)$ when G is a countable discrete group. The first consisted of sets of the form

$$U(\alpha, F, K, \epsilon) = \bigcap_{s\in F}\bigcap_{A\in K}\{\beta \in \mathrm{Act}(G, X, \mu) : \mu(\alpha_s A \triangle \beta_s A) < \epsilon\}$$

where $\epsilon > 0$, F is a finite subset of G and K is a finite collection of Borel subsets of X and the second of sets of the form

$$V(\alpha, F, \Omega, \epsilon) = \bigcap_{s\in F}\bigcap_{\xi\in\Omega}\{\beta \in \mathrm{Act}(G, X, \mu) : \|(\pi_\alpha(s) - \pi_\beta(s))\xi\| < \epsilon\}$$

where $\epsilon > 0$ and $F \subset G$ and $\Omega \subset L^2(X, \mu)$ are finite. For $\mathbb{F}_n$ the following lemmas show that in either case the finite set F can be taken to be a finite subset of generators.

Lemma 3.1.1. *Let $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$. Given a basic open set $V(\alpha, F, \Omega, \epsilon)$ in $\mathrm{Act}(\mathbb{F}_n, X, \mu)$ there exist finite subsets $S' \subset S$ and $\Omega' \subset L^2(X, \mu)$ and a $\delta > 0$ such that $V(\alpha, S', \Omega', \delta) \subset V(\alpha, F, \Omega, \epsilon)$.*

Proof. Let $\epsilon > 0$, Ω be a finite subset of $L^2(X, \mu)$ and F be a finite subset of G. Since F is finite, there exists a finite subset S' of S such that F is contained in the subgroup generated by S'. For each $t \in F$ let m_t be the number of elements of S' and their inverses needed to express t as a reduced word. Set $N = \max_{t\in F}\{m_t\}$ and $\delta = \frac{\epsilon}{N}$. Let

$$\Omega' = \{\pi_\alpha(s_1^{\epsilon_1} s_2^{\epsilon_2} \dots s_n^{\epsilon_n})\xi : \xi \in \Omega,\ s_i \in S',\ \epsilon_i \in \{-1, 1\},\ \text{and } n \leq N\}.$$

Let $\beta \in V(\alpha, S', \Omega', \delta)$, $\xi \in \Omega$ and $t \in F$. Then there exists $n \leq N$, $s_1, s_2, \dots, s_n \in S'$, and $\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\} \in \{-1, 1\}^n$ such that $t = s_1^{\epsilon_1} s_2^{\epsilon_2} \dots s_n^{\epsilon_n}$. Thus

$$\begin{aligned}
&\|(\pi_\beta(t) - \pi_\alpha(t))\xi\| \\
&\quad = \|(\pi_\beta(s_1)^{\epsilon_1}\pi_\beta(s_2)^{\epsilon_2}\dots\pi_\beta(s_n)^{\epsilon_n} - \pi_\alpha(s_1)^{\epsilon_1}\pi_\alpha(s_2)^{\epsilon_2}\dots\pi_\alpha(s_n)^{\epsilon_n})\xi\| \\
&\quad = \|\sum_{k=1}^{n}(\pi_\beta(s_1)^{\epsilon_1}\dots\pi_\beta(s_{k-1})^{\epsilon_{k-1}})(\pi_\beta(s_k)^{\epsilon_k} - \pi_\alpha(s_k)^{\epsilon_k})(\pi_\alpha(s_{k+1})^{\epsilon_{k+1}}\dots\pi_\alpha(s_n)^{\epsilon_n})\xi\|
\end{aligned}$$

$$\leq \sum_{k=1}^{n} \|\pi_\beta(s_1^{\epsilon_1} \ldots s_{k-1}^{\epsilon_{k-1}})(\pi_\beta(s_k)^{\epsilon_k} - \pi_\alpha(s_k)^{\epsilon_k})\pi_\alpha(s_{k+1}^{\epsilon_{k+1}} \ldots s_n^{\epsilon_n})\xi\|$$
$$= \sum_{k=1}^{n} \|(\pi_\beta(s_k)^{\epsilon_k} - \pi_\alpha(s_k)^{\epsilon_k})\pi_\alpha(s_{k+1}^{\epsilon_{k+1}} \ldots s_n^{\epsilon_n})\xi\|.$$

Clearly if $\epsilon_k = 1$ then $\|(\pi_\beta(s_k)^{\epsilon_k} - \pi_\alpha(s_k)^{\epsilon_k})\pi_\alpha(s_{k+1}^{\epsilon_{k+1}} \ldots s_n^{\epsilon_n})\xi\| < \frac{\epsilon}{N}$. If $\epsilon_k = -1$ then

$$\begin{aligned}
&\|(\pi_\alpha(s_k)^{\epsilon_k} - \pi_\beta(s_k)^{\epsilon_k})\pi_\alpha(s_{k+1}^{\epsilon_{k+1}} \ldots s_n^{\epsilon_n})\xi\| \\
&\quad = \|(\pi_\beta(s_k)^{-1}\pi_\beta(s_k)\pi_\alpha(s_k)^{-1} - \pi_\beta(s_k)^{-1}\pi_\alpha(s_k)\pi_\alpha(s_k)^{-1})\pi_\alpha(s_{k+1}^{\epsilon_{k+1}} \ldots s_n^{\epsilon_n})\xi\| \\
&\quad = \|\pi_\beta(s_k)^{-1}(\pi_\beta(s_k) - \pi_\alpha(s_k))\pi_\alpha(s_k)^{-1}\pi_\alpha(s_{k+1}^{\epsilon_{k+1}} \ldots s_n^{\epsilon_n})\xi\| \\
&\quad = \|(\pi_\beta(s_k) - \pi_\alpha(s_k))\pi_\alpha(s_k^{-1}s_{k+1}^{\epsilon_{k+1}} \ldots s_n^{\epsilon_n})\xi\| \\
&\quad < \frac{\epsilon}{N}.
\end{aligned}$$

Therefore,

$$\|(\pi_\beta(t) - \pi_\alpha(t))\xi\| < \sum_{k=1}^{n} \frac{\epsilon}{N} = \frac{n}{N}\epsilon \leq \epsilon$$

whence $\beta \in V(\alpha, F, \Omega, \epsilon)$. □

Lemma 3.1.2. *Let* $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$. *Given a basic open set* $U(\alpha, F, K, \epsilon)$ *in* $\mathrm{Act}(\mathbb{F}_n, X, \mu)$ *there exists a finite subset* $S' \subset S$, *a finite collection* K' *of Borel subsets of* X, *and* $\delta > 0$ *such that* $U(\alpha, S', K', \delta) \subset U(\alpha, F, K, \epsilon)$.

Proof. Let $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$ and $U(\alpha, F, K, \epsilon)$ be a basic open set in $\mathrm{Act}(\mathbb{F}_n, X, \mu)$. Define $\Omega = \{\chi_C\}_{C \in K}$ where χ_C is the characteristic function of C for each $C \in K$. Then Ω is a finite subset of $L^2(X, \mu)$. Thus by Lemma 3.1.1 we may find finite subsets $\Omega' \subset L^2(X, \mu)$ and $S' \subset S$ and $\delta > 0$ such that

$$V(\alpha, S', \Omega', \delta) \subset V(\alpha, F, \Omega, \epsilon) = U(\alpha, F, K, \epsilon).$$

In general elements of Ω' will not be projections. However, for each $f \in \Omega'$ there exists a step function $f' = \sum_{i=1}^{n_f} a_{f,i}\chi_{A_{f,i}}$ such that $\|f - f'\|_2 < \frac{\delta}{3}$. Set $K' = \{A_{f,i} : f \in$

$\Omega',\ 1 \leq i \leq n_f\}$, $N = \max\{n_f : f \in \Omega\}$, and $a = \max\{|a_{f,i}| : f \in \Omega,\ 1 \leq i \leq n_f\}$. Suppose $\beta \in U(\alpha, S', K', \delta')$ where $0 < \delta' < \frac{\delta}{3Na}$. Then for $f \in \Omega'$ and $s \in S'$ we have that

$$\begin{aligned}
\|\alpha_s f - \beta_s f\|_2 &\leq \alpha\|\alpha_s f - \alpha_s f'\|_2 + \|\alpha_s f' - \beta_s f'\|_2 + \|\beta_s f' - \beta_s f\|_2 \\
&< \frac{\delta}{3} + \|\alpha_s(\sum_{i=1}^{n_f} a_{f,i}\chi_{A_{f,i}}) - \beta_s(\sum_{i=1}^{n_f} a_{f,i}\chi_{A_{f,i}})\|_2 + \frac{\delta}{3} \\
&\leq 2\frac{\delta}{3} + \sum_{i=1}^{n_f} |a_{f,i}|\|\alpha_s(\chi_{A_{f,i}}) - \beta_s(\chi_{A_{f,i}})\|_2 \\
&\leq 2\frac{\delta}{3} + \sum_{i=1}^{n_f} a\mu(\alpha_s A_{f,i} \triangle \beta_s A_{f,i}) \\
&< 2\frac{\delta}{3} + \sum_{i=1}^{n_f} a\frac{\delta}{3Na} \\
&\leq 2\frac{\delta}{3} + \frac{n_f\delta}{3N} \\
&\leq 2\frac{\delta}{3} + \frac{\delta}{3} \\
&= \delta.
\end{aligned}$$

Thus $\beta \in V(\alpha, S', \Omega', \delta) \subset U(\alpha, F, K, \epsilon)$ as desired. $\square$

Lemmas 3.1.1 and 3.1.2 show that sets of the form $V(\alpha, S', \Omega, \epsilon)$ and $U(\alpha, S', K, \epsilon)$ where S' is a finite subset of S form bases for the weak topology on $\mathrm{Act}(\mathbb{F}_n, X, \mu)$. If $n < \infty$ then it is clear that S' may be taken to be all of S. When combined with the discussion preceding the lemmas we see that small perturbations of an action on generators of $\mathbb{F}_n$ gives a small perturbation of the action itself.

In Section 2 we stated that the Rokhlin lemma implies periodic approximation for $\mathbb{Z}$-actions. We will now use the preceding observation to prove a slight amplification of this result. Specifically we will apply the Rokhlin lemma to finitely many generators and then perturb the action on each of these generators to be periodic. Although the resulting action is not very useful for many purposes, it does provide a starting point for constructing the analogue of the Rokhlin lemma given at the end of this section.

Proposition 3.1.3. *Let $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$. Then for each finite set $S' \subset S$ and basic open set $U(\alpha, F, K, \epsilon)$ in $\mathrm{Act}(\mathbb{F}_n, X, \mu)$ there exists a natural number M such that for each for each $m > M$ there exist $\hat{\alpha} \in U(\alpha, F, K, \epsilon)$ and Borel subsets $\{B_s\}_{s \in S'}$ such that for each $s \in S'$, $\hat{\alpha}_s^m$ is the identity and $\{\hat{\alpha}_s^i B_s\}_{i=0}^{m-1}$ partitions X. Moreover, M can be chosen to depend only on ϵ.*

Proof. By Lemma 3.1.2 it suffices to assume $F \subset S$. Let $s \in S'$ be fixed but arbitrary. Since the aperiodic automorphisms of $\mathrm{Aut}(X, \mu)$ are dense with respect to the weak topology, there exists $\beta_s \in \mathrm{Aut}(X, \mu)$ such that β_s is aperiodic and $\mu(\beta_s C \triangle \alpha_s C) < \frac{\epsilon}{2}$ for each $C \in K$. Let M be a natural number such that $\frac{1}{M} < \frac{\epsilon}{8}$.

Suppose $m > M$. Then by the Rohklin lemma there exists a Borel subset B_s of X such that the collection $\{\beta_s^i B_s\}_{i=0}^{m-1}$ is pairwise disjoint and $E_s = X \setminus \bigcup_{i=0}^{m-1} \beta_s^i B_s$ has measure less than $\frac{\epsilon}{8}$. Since X is nonatomic there exists pairwise disjoint subsets $\{E_s^i\}_{i=0}^{m-1}$ of E_s such that $\mu(E_s^i) = \frac{\mu(E_s)}{m}$ for each $i \in \{0, 1, \ldots, m-1\}$ and $E_s = \bigcup_{i=0}^{m-1} E_s^i$. For each $i \in \{0, 1, \ldots, m-2\}$ choose an isomorphism $\phi_s^i : E_s^i \to E_s^{i+1}$. Define $\phi_s^{m-1} : E_s^{m-1} \to E_s^0$ by $\phi_s^{m-1} = (\phi_s^{m-2})^{-1} \circ (\phi_s^{m-3})^{-1} \circ \ldots \circ (\phi_s)^{-1}$. Define $\hat{\alpha}_s : X \in X$ by

$$\hat{\alpha}_s x = \begin{cases} \beta_s x & \text{if } x \in \bigcup_{i=1}^{m-2} \beta_s^i B_s \\ \phi_s^i x & \text{if } x \in E_i \text{ for some } i = 0, 1, \ldots, m-1 \\ \beta_s^{-(m-2)} \circ \beta_s^{-(m-3)} \circ \ldots \circ \beta_s^{-1} x & \text{if } x \in \beta_s^{m-1} B_s \end{cases}$$

if $s \in S'$ and $\hat{\alpha}_s x = \alpha_s x$ if $s \in S \setminus S'$. Then $\hat{\alpha}_s \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$ and $\hat{\alpha}_s^m x = x$ for all $s \in S'$ and $x \in X$.

Let $s \in F$ and $C \in K$. If $s \notin S'$ then $\hat{\alpha}_s x = \alpha_s x$ a.e. and thus $\mu(\alpha_s C \triangle \hat{\alpha}_s C) =$

$0 < \epsilon$. Suppose $s \in S'$. If $x \in C \cap \bigcup_{i=0}^{m-2} \beta_s^i B_s$ then $\hat{\alpha}_s x = \beta_s x$, hence $\beta_s C \triangle \hat{\alpha}_s C \subset (\beta_s(\beta_s^{m-1} B_s) \triangle \hat{\alpha}_s(\beta_s^{m-1} B_s)) \cup (\beta_s E_s \triangle \hat{\alpha}_s E_s.)$ Therefore

$$\begin{aligned}
\mu(\beta_s C \triangle \hat{\alpha}_s C) &\leq \mu(\beta_s(\beta_s^{n-1} B_s) \triangle \hat{\alpha}_s(\beta_s^{n-1} B_s)) + \mu(\beta_s E_s \triangle \hat{\alpha}_s E_s) \\
&\leq 2\mu(B_s) + 2\mu(E) \\
&< 2\frac{\epsilon}{8} + 2\frac{\epsilon}{8} \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

It now follows that

$$\begin{aligned}
\mu(\alpha_s C \triangle \hat{\alpha}_s C) &= \mu((\alpha_s C \triangle \beta_s C) \triangle (\beta_s C \triangle \hat{\alpha}_s C)) \\
&\leq \mu(\alpha_s C \triangle \beta_s C) + \mu(\beta_s C \triangle \hat{\alpha}_s C) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore $\hat{\alpha} \in U(\alpha, F, K, \epsilon)$. □

3.2. Permutations of Partitions

Let $\alpha \in \mathrm{Act}(G, X, \mu)$ and $\mathscr{P}$ be a collection of pairwise disjoint Borel subsets of X. Then α is said to *permute* $\mathscr{P}$ if for each $P \in \mathscr{P}$ and $s \in G$ we have $\alpha_s P \in \mathscr{P}$. That is for each $P \in \mathscr{P}$ and $s \in G$ there exists $P' \in \mathscr{P}$ such that $\alpha_s P = P'$, i.e. $\mu(\alpha_s P \triangle P') = 0$. If in addition $\mathscr{P}$ satisfies $\alpha_s P = P$ if and only if $\alpha_s x = x$ a.e. then α is said to *pointwise permute* $\mathscr{P}$. Although this property seems innocent on the surface, it has some surprising consequences. In Theorem 3.3.2 we will show every action of $\mathbb{F}_n$ can be approximated by an action which pointwise permutes some partition. As with the Rokhlin lemma, the density of these actions allows for the genericity of certain properties to be established. We devote the remainder of this subsection to further understanding these actions.

Let $\alpha \in \mathrm{Act}(G, X, \mu)$ and $\mathscr{P}$ be a collection of pairwise disjoint Borel Subsets of X. Suppose $\mathscr{P}'$ is a subset of $\mathscr{P}$ and H is a subgroup of G. We say $\mathscr{P}'$ is (α, H)-*transitive* if given any $P, P' \in \mathscr{P}'$ there exists $h \in H$ such that $\alpha_h P = P'$. If $\alpha_h P \in \mathscr{P}'$ for all $h \in H$ and $P \in \mathscr{P}'$ we say $\mathscr{P}'$ is (α, H)-*invariant*. If $H = G$ we simply say $\mathscr{P}'$ is α-*transitive* or α-*transitive* respectively. The following lemma is elementary but often useful in results that follow.

Lemma 3.2.1. *Let $\alpha \in \mathrm{Act}(G, X, \mu)$. Suppose α permutes some finite partition $\mathscr{P}$. Then there exists disjoint subsets $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_n$ of $\mathscr{P}$ such that $\mathscr{P}_i$ is α-invariant and α-transitive for each $i = 1, 2, \ldots, n$ and $\mathscr{P} = \bigcup_{i=1}^n \mathscr{P}_i$.*

Proof. Fix $P_1 \in \mathscr{P}$ and define $\mathscr{P}_1 = \{\alpha_g P_1 : g \in G\}$. If $\mathscr{P}_1 \neq \mathscr{P}$ choose $P_2 \in \mathscr{P} \setminus \mathscr{P}_1$ and define $\mathscr{P}_2 = \{\alpha_g P_2 : g \in G\}$. Inductively define $\mathscr{P}_i$ by choosing $P_i \in \mathscr{P} \setminus (\mathscr{P}_1 \cup \mathscr{P}_2 \cup \ldots \cup \mathscr{P}_{i-1})$. and setting $\mathscr{P}_i = \{\alpha_g P_i : g \in G\}$. Since $\mathscr{P}$ is finite, this process must end at some finite stage, call it n. Clearly $\mathscr{P}_i$ is α-invariant and α-transitive for each $i = 1, 2, \ldots, n$ and $\mathscr{P} = \bigcup_{i=1}^n \mathscr{P}_i$. ∎

An action $\alpha \in \mathrm{Act}(G, X, \mu)$ is said to *factor through a finite group* if there exists a group homomorphism ϕ from G onto a finite group F and an action $\beta \in \mathrm{Act}(F, X, \mu)$ such that $\alpha_s = \beta_{\phi(s)}$ a.e. for all $s \in G$. If α pointwise permutes some partition $\mathscr{P}$ we now show that α factors through a subgroup of the finite permutation group $S(\mathscr{P})$.

Proposition 3.2.2. *Let $\alpha \in \mathrm{Aut}(G, X, \mu)$. If there exists a finite partition $\mathscr{P}$ of X such that α pointwise permutes $\mathscr{P}$ then α factors through a finite group.*

Proof. Suppose $\mathscr{P}$ is a finite partition of X which is pointwise permuted by α. Given $s \in G$ define $\sigma_s \in \mathscr{P}^{\mathscr{P}}$ by $\sigma_s(P) = \alpha_s P$. Set $F = \{\sigma_s : s \in G\}$. Suppose $\sigma_s(P) =$

$\sigma_s(P')$. Then $\alpha_s P = \alpha_s P'$ from which it follows that $P' = \alpha_{s^{-1}s} P = \alpha_e P = P$. Since $\mathscr{P}$ is finite, σ_s is a permutation of $\mathscr{P}$. If $\sigma_s, \sigma_t \in F$ then $\sigma_s \sigma_t^{-1}(P) = \alpha_s \alpha_t^{-1}(P) = \alpha_{st^{-1}}(P) = \sigma_{st^{-1}}(P)$ for all $P \in \mathscr{P}$. Thus F is a finite subgroup of the permutation group $S(\mathscr{P})$.

Define $\phi : G \to F$ by $\phi(s) = \sigma_s$. Choose $s_1, s_2, \ldots, s_n$ such that $s_i \neq s_j$ if $i \neq j$ and $F = \{\sigma_{s_1}, \sigma_{s_2}, \ldots, \sigma_{s_n}\}$. Define an action β of F on X by setting $\beta_{\sigma_{s_i}} x = \alpha_{s_i} x$ for each $i \in \{1, 2, \ldots, n\}$. Let $s \in G$. Then $\phi(s) = \sigma_s = \sigma_{s_i}$ for some $i \in \{1, 2, \ldots, n\}$. Since $\sigma_s = \sigma_{s_i}$, $\alpha_s P = \alpha_{s_i} P$ for all $P \in \mathscr{P}$ whence $\alpha_s = \alpha_{s_i}$ on each $P \in \mathscr{P}$. Therefore $\alpha_s = \beta_{\phi(s)}$ for all $s \in G$. □

Proposition 3.2.2 shows that if (G, X, μ, α) pointwise permutes some partition then the action is essentially an action of a finite group. One would expect that its Koopman representation would be finite dimensional as well. We show this is indeed the case in Proposition 3.2.3.

Proposition 3.2.3. *Let $\alpha \in \mathrm{Aut}(G, X, \mu)$. Suppose $\mathscr{P}$ is a uniform partition of X pointwise permuted by α. Then there exists Hilbert spaces $\mathcal{H}, \mathcal{K}$ with* $\dim \mathcal{H} < \infty$ *and a representation ρ of G on $\mathcal{H}$ satisfying $\pi_\alpha \cong \rho \otimes id_{\mathcal{K}}$,*

Proof. By Lemma 3.2.1 there exists disjoint subsets $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_n$ of $\mathscr{P}$ such that $\mathscr{P}_i$ is α-invariant and α-transitive for each $i = 1, 2, \ldots, n$ and $\mathscr{P} = \bigcup_{i=1}^n \mathscr{P}_i$. For each $i \in \{1, 2, \ldots, n\}$ enumerate $\mathscr{P}_i$ by $\{P_{i,1}, P_{i,2}, \ldots, P_{i,k_i}\}$ and for each $j \in \{1, 2, \ldots, k_i\}$ choose $s_{i,j} \in G$ such that $\alpha_{s_{i,j}} P_{i,j} = P_{i,1}$. Fix $P' \in \{P_{1,1}, P_{2,1}, \ldots, P_{n,1}\}$. Since $\mathscr{P}$ is uniform there exists a measure-preserving isomorphism $\phi_i : P_{i,1} \to P'$ for each $i = 1, 2, \ldots, k$. Define $U : L^2(X, \mu) \to \ell^2(\mathscr{P}) \otimes L^2(P', \mu)$ by

$$Uf = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \delta_{P_{i,j}} \otimes f_{|P_{i,j}} \circ \alpha_{s_{i,j}^{-1}} \circ \phi_i^{-1}$$

where

$$\delta_{P_{i,j}}(P) = \begin{cases} 1 & \text{if } P = P_{i,j} \\ 0 & \text{otherwise} \end{cases}.$$

The linearity of U is routine to check. To see U is norm preserving observe that for each $f \in L^2(X,\mu)$ we have

$$\begin{aligned}
\|Uf\|^2 &= \|\sum_{i=1}^{n}\sum_{j=1}^{k_i} \delta_{P_{i,j}} \otimes f_{|P_{i,j}} \circ \alpha_{s_{i,j}^{-1}} \circ \phi_i^{-1}\|^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{k_i} \|\delta_{P_{i,j}} \otimes f_{|P_{i,j}} \circ \alpha_{s_{i,j}^{-1}} \circ \phi_i^{-1}\|^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{k_i} \|\delta_{P_{i,j}}\|^2_{\ell^2(\mathscr{P})} \|f_{|P_{i,j}} \circ \alpha_{s_{i,j}^{-1}} \circ \phi_i^{-1}\|^2_{P'} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{k_i} \|f_{|P_{i,j}} \circ \alpha_{s_{i,j}^{-1}} \circ \phi_i^{-1}\|^2_{P'}.
\end{aligned}$$

Since ϕ_i and $\alpha_{s_{i,j}^{-1}}$ are measure-preserving for each $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, k_i\}$ we have

$$\|f_{|P_{i,j}} \circ \alpha_{s_{i,j}^{-1}} \circ \phi_i^{-1}\|^2_{P'} = \|f_{|P_{i,j}} \circ \alpha_{s_{i,j}^{-1}}\|^2_{P_i} = \|f_{|P_{i,j}}\|^2_{P_{i,j}}.$$

As the collection of $P_{i,j}$'s partition X, it follows

$$\|Uf\|^2 = \sum_{i=1}^{n}\sum_{j=1}^{k_i} \|f_{|P_{i,j}} \circ \alpha_{s_{i,j}^{-1}} \circ \phi_i^{-1}\|^2_{P'} = \sum_{i=1}^{n}\sum_{j=1}^{k_i} \|f_{|P_{i,j}}\|^2_{P_{i,j}} = \|f\|^2_X$$

as desired.

Since $\ell^2(\mathscr{P})$ is finite dimensional the Hilbert space tensor product $\ell^2(\mathscr{P}) \otimes L^2(X,\mu)$ coincides with the algebraic tensor product. Thus suppose $\sum_{i=1}^n \sum_{j=1}^{k_i} \delta_{P_{i,j}} \otimes f_{i,j} \in \ell^2(\mathscr{P}) \otimes L^2(P',\mu)$. Define $fx = f_{i,j} \circ \phi_i \circ \alpha_{s_{i,j}} x$ if $x \in P_{i,j}$. Then

$$Uf = \sum_{i=1}^{n}\sum_{j=1}^{k_i} \delta_{P_{i,j}} \otimes f_{|P_{i,j}} \circ \alpha_{s_{i,j}^{-1}} \circ \phi_i^{-1}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{k_i} \delta_{P_{i,j}} \otimes f_{i,j} \circ \phi_i \circ \alpha_{s_{i,j}} \circ \alpha_{s_{i,j}^{-1}} \circ \phi_i^{-1}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{k_i} \delta_{P_{i,j}} \otimes f_{i,j}.$$

Thus U is a unitary map from $L^2(X,\mu)$ onto $\ell^2(\mathscr{P}) \otimes L^2(P',\mu)$.

Let $s \in G$ and $\sum_{i=1}^{n} \sum_{j=1}^{k_i} \delta_{P_{i,j}} \otimes f_{i,j} \in \ell^2(\mathscr{P}) \otimes L^2(P',\mu)$. Let f be defined as in the preceding paragraph. Given $P_{i,j} \in \mathscr{P}$,

$$\alpha_s P_{i,j} = \alpha_s \alpha_{s_{i,j}{}^{-1}} P_{i,1} = \alpha_{ss_{i,j}^{-1}} P_{i,1} \in \mathscr{P}_i.$$

Thus for each $i \in \{1,2,\ldots,n\}$ there exists $\sigma_i \in S(\{1,2,\ldots,k_i\})$ such that $\alpha_s P_{i,j} = P_{i,\sigma_i(j)}$ for all $j \in \{1,2,\ldots,k_i\}$. Note

$$\alpha_{s_{ij}} \alpha_{s^{-1}} P_{i,\sigma_i(j)} = \alpha_{s_{ij}} P_{i,j} = P_{i,1} = \alpha_{s_{i,\sigma_i(j)}} P_{i,\sigma_i(j)}$$

giving $\alpha_{s_{ij}} \alpha_s^{-1} = \alpha_{s_{i,\sigma_i(j)}}$ on $P_{i,\sigma_i(j)}$. Thus if $x \in P_{i,\sigma_i(j)}$ we have $\alpha_s^{-1} x \in P_{i,j}$, whence

$$f \circ \alpha_s^{-1} x = f_{i,j} \circ \phi_i \circ \alpha_{s_{ij}} \alpha_s^{-1} x = f_{i,j} \circ \phi_i \circ \alpha_{s_{i,\sigma_i(j)}} x.$$

Then

$$\begin{aligned}
U\pi_\alpha(s)U^{-1}\Big(\sum_{P_{i,j}\in\mathscr{P}} \delta_{P_{i,j}} \otimes f_{i,j}\Big) &= U\pi_\alpha(s) f \\
&= U(f \circ \alpha_s^{-1}) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{k_i} \delta_{P_{i,\sigma_i(j)}} \otimes f_{i,j} \circ \phi_i \circ \alpha_{s_{i,\sigma_i(j)}} \circ \alpha_{s_{i,\sigma_i(j)}}^{-1} \circ \phi_i^{-1} \\
. &= \sum_{i=1}^{n} \sum_{j=1}^{k_i} \delta_{P_{i,\sigma_i(j)}} \otimes f_{i,j} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{k_i} \delta_{\alpha_g P_{i,j}} \otimes f_{i,j}
\end{aligned}$$

Therefore $\pi_\alpha \cong \rho \otimes id_{L^2(P',\mu)}$ where ρ acts on the finite dimensional Hilbert space $\ell^2(\mathscr{P})$ by $\rho(g)\delta_P = \delta_{\alpha_g P}$. $\square$

If the assumption that $\mathscr{P}$ is uniform is relaxed from Lemma 3.2.3 we may still construct the subsets $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_n$ of $\mathscr{P}$ as in the proof. Since $\mathscr{P}$ is not uniform $\mu(P_{i,1})$ and $\mu(P_{j,1})$ need not be equal if $i \neq j$. Thus there need not exist measure-preserving isomorphisms identifying $P_{i,1}$ and $P_{j,1}$ with a fixed $P' \in \mathscr{P}$ as in the uniform case. However the map

$$f \mapsto \sum_{i=1}^{n} \sum_{j=1}^{k_i} \delta_{P_{i,j}} \otimes f_{|P_{i,j}} \circ \alpha_{s_{i,j}^{-1}}$$

defines a unitary map U from $L^2(X,\mu)$ onto $\bigoplus_{i=1}^n \ell^2(\mathscr{P}_i) \otimes L^2(P_{i,1},\mu)$ such that $U\pi_\alpha U^{-1} = \oplus_{i=1}^n \rho_i \otimes id_{L^2(P_{i,1},\mu)}$ where $\rho_i(g)\delta_P = \delta_{gP}$ for all $P \in \mathscr{P}_i$ and $i \in \{1, 2, \ldots, n\}$.

3.3. Finite Approximations of Measure-Preserving Actions of $\mathbb{F}_n$

We now present the main technical result of this dissertation. Specifically, we show given an action α of $\mathbb{F}_n$ and a finitely generated subgroup H of $\mathbb{F}_n$, we may approximate α by an action which pointwise permutes some partition when restricted to H. It follows from Proposition 3.1.3 that for a given finite subset $S' \subset S$ any basic open set contains an action α such that α_s pointwise permutes some partition $\mathscr{P}_s$ for each $s \in S'$. If $n = 1$ we have the classical result that the periodic actions are dense. If $n > 1$ then the result is not as clear. For example, if $s, t \in S'$ are distinct then α_{st} need not permute $\mathscr{P}_r$ for any $r \in S'$. We will now show Proposition 3.1.3 can be extended to give the desired approximation. The bulk of this argument is contained in the following lemma.

Lemma 3.3.1. *Let $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$. Suppose there exists finite subsets $R, T \subset S$ and finite uniform partitions $\mathscr{P}$ and $\mathscr{Q}$ of X such that $R \cap T = \emptyset$ and α restricted to* *the subgroups generated by R and T pointwise permutes $\mathscr{P}$ and $\mathscr{Q}$ respectively. Then for any open neighborhood $U(\alpha, F, K, \epsilon)$ and $\delta > 0$, there exist an uniform partition $\mathscr{R}$ and an $\alpha' \in U(\alpha, F, K, \epsilon)$ such that α' restricted to the subgroup generated by $R \cup T$ pointwise permutes $\mathscr{R}$ and the measure of each atom of $\mathscr{R}$ is less than δ. Moreover, α' and $\mathscr{R}$ can be chosen to so that there exists an atom of $\mathscr{R}$ such that α'_t is the identity for all t in the subgroup generated by $R \cup T$.*

Proof. By Lemma 3.1.2 it suffices to find $\alpha' \in U(\alpha, F, K, \epsilon)$ where F is a finite subset of S and K is a finite collection of measurable sets. Let H_R and H_T be the subgroups of $\mathbb{F}_n$ generated by R and T respectively. By Lemma 3.2.1 there exist disjoint subsets $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_{n_p}$ of $\mathscr{P}$ such that $\mathscr{P} = \bigcup_{h=1}^{n_p} \mathscr{P}_h$ and for each $h \in I_p = \{1, 2 \ldots, n_p\}$, $\mathscr{P}_h$ is (α, H_R)-invariant and (α, H_R)-transitive. Similarly, there exists disjoint subsets $\mathscr{Q}_1, \mathscr{Q}_2, \ldots, \mathscr{Q}_{m_q}$ of $\mathscr{Q}$ such that $\mathscr{Q} = \bigcup_{i=1}^{m_q} \mathscr{Q}_i$ and for each $i \in I_q = \{1, 2 \ldots, m_q\}$, $\mathscr{Q}_i$ is (α, H_T)-invariant and (α, H_T)-transitive. For each $h \in I_p$ enumerate $\mathscr{P}_h$ as $\{P_{h,1}, P_{h,2}, \ldots, P_{h,n_{p_h}}\}$ and for each $k \in I_{p_h} = \{1, 2, \ldots, n_{p_h}\}$ choose $r_{h,k} \in H_R$ such that $\alpha_{r_{h,k}} P_{h,1} = P_{h,k}$. Similarly enumerate $\mathscr{Q}_i$ as $\{Q_{i,1}, Q_{i,2}, \ldots, Q_{i,m_{q_i}}\}$ for each $i \in I_q$ and choose $t_{i,j} \in H_T$ such that $\alpha_{t_{i,j}} Q_{i,1} = Q_{i,j}$ for each $j \in I_{q_i} = \{1, 2, \ldots, m_{q_i}\}$.

We think of $\mathscr{P}_h$ and $\mathscr{Q}_i$ as being towers with bases $P_{h,1}$ and $Q_{i,1}$. The action, α, then allows for movement between the levels within a tower. We begin by making a series of refinements of $\mathscr{P}_h$'s and $\mathscr{Q}_i$'s. The general idea of each of these refinements is the same. We first use the action to collapse the levels of each tower to form a partition of the base. We then further refine the base if needed and use α to copy this refinement to each level of the tower creating a partition of X.

For each $i \in I_q$ we create two partitions of $\mathscr{Q}_{i,1}$. The first we form by collecting together the points which visit the same sequence of elements of $\mathscr{P}$ as we move up the levels of $\mathscr{Q}_i$. The second we form by collecting together the points which visit the same sequence of sets in the given finite collection of Borel sets K. We then take the join of these partitions and copy it up to each level of $\mathscr{Q}_i$ to create a refinement of $\mathscr{Q}$. We make this process precise in the following.

Define $\Lambda = \{(h,k) : h \in I_p,\ k \in I_{p_h}\}$. Given $\lambda = (h,k) \in \Lambda$ define P_λ and r_λ as $P_{h,k}$ and $r_{h,k}$ respectively. For $i \in I_q$ set $\Sigma^i = \Lambda^{I_{q_i}}$ and $\sigma \in \Sigma^i$ define

$$A^i_\sigma = \{x \in Q_{i,1} : \alpha_{t_{i,j}} x \in P_{\sigma(j)} \text{ for each } j \in I_{q_i}\}.$$

Let $i \in I_q$. Clearly $A^i_\sigma \subset Q_{i,1}$ for each $\sigma \in \Sigma^i$. Let $x \in Q_{i,1}$. Since $\mathscr{P}$ is a partition of X, for each $j \in I_{q_i}$ there exists $\lambda_j \in \Lambda$ such that $\alpha_{t_{i,j}} x \in P_{\lambda_j}$, whence $x \in A^i_\sigma$ where $\sigma \in \Sigma^i$ is defined by $j \mapsto \lambda_j$. Assume $x \in A^i_\sigma \cap A^i_{\sigma'}$ and $\sigma \neq \sigma'$. Let $j \in I_{q_i}$ be such that $\sigma(j) \neq \sigma'(j)$. Then $\alpha_{t_{i,j}} x \in P_{\sigma(j)} \cap P_{\sigma'(j)} = \emptyset$. Thus $\sigma = \sigma'$ and it follows $\{A^i_\sigma\}_{\sigma \in \Sigma^i}$ partitions $Q_{i,1}$ for each $i \in I_p$.

Enumerate the given finite collection of Borel sets K by $\{C_1, C_2, \ldots, C_m\}$ and set $I_m = \{1, 2, \ldots, m\}$. Let $i \in I_q$ and set $\Gamma^i = \mathscr{P}(I_m)^{I_{p_i}}$. Given $\gamma \in \Gamma^i$ define

$$A^i_\gamma = \{x \in Q_{i,1} : \alpha_{t_{i,j}} x \in (\bigcap_{l \in \gamma(j)} C_l) \cap (\bigcap_{l \in I_m \setminus \gamma(j)} C^c_l) \text{ for all } j \in I_{q_i}\}.$$

Fix $i \in I_q$. Clearly $A^i_\gamma \subset Q_{i,1}$ for all $\gamma \in \Gamma^i$. Let $x \in Q_{i,1}$ and $j \in I_{q_i}$. Define $\gamma_j \in \mathscr{P}(I_m)$ by $l \in \gamma_j$ if and only if $\alpha_{t_{i,j}} x \in C_l$. Then by construction $\alpha_{t_{i,j}} \in C_l$ for all $l \in \gamma_j$ and $\alpha_{t_{i,j}} \notin C_l$ for all $l \in I_m \setminus \gamma_j$. In particular $x \in A^i_\gamma$ where $\gamma \in \Gamma^i$ is defined by $j \mapsto \gamma_j$. Assume $x \in A^i_\gamma \cap A^i_{\gamma'}$ and $\gamma \neq \gamma'$. Let $j \in I_{q_i}$ be such that $\gamma(j) \neq \gamma'(j)$. Then at least one of the sets $\gamma'(j) \setminus \gamma(j)$ and $\gamma(j) \setminus \gamma'(j)$ is nonempty. Without loss of generality suppose there exists $l' \in \gamma(j) \setminus \gamma'(j)$. Then

$$\alpha_{t_{i,k}} x \in \left((\bigcap_{l \in \gamma(k)} C_l) \cap (\bigcap_{l \in I_m \setminus \gamma(k)} C^c_l) \right) \cap \left((\bigcap_{l \in \gamma'(k)} C_l) \cap (\bigcap_{l \in I_m \setminus \gamma'(k)} C^c_l) \right) \subset C_{l'} \cap C^c_{l'} = \emptyset.$$

Therefore $\gamma = \gamma'$ and $\{A^i_\gamma\}_{\gamma \in \Gamma^i}$ partitions $Q_{i,1}$ for each $i \in I_q$.

Given $i \in I_q$ $j \in I_{q_i}$, $\sigma \in \Sigma_i$ and $\gamma \in \Gamma_i$ define $A^i_{\sigma,\gamma,j} = \alpha_{t_{i,j}}(A^i_\sigma \cap A^i_\gamma)$. Let $x \in X$. Then there exists $i \in I_q$ and $j \in I_{q_i}$ such that $x \in Q_{i,j}$. Set $y = \alpha^{-1}_{t_{i,j}} x \in Q_{i,1}$. Since both $\{A^i_\sigma\}_{\sigma\in\Sigma^i}$ and $\{A^i_\gamma\}_{\gamma\in\Gamma^i}$ partition $Q_{i,1}$ there exists $\sigma \in \Sigma^i$ and $\gamma \in \Gamma^i$ such that $y \in A^i_\sigma \cap A^i_\gamma$. Thus $x = \alpha_{t_{i,j}}\alpha^{-1}_{t_{i,j}}x = \alpha_{t_{i,j}} y \in \alpha_{t_{i,j}}(A^i_\sigma \cap A^i_\gamma) = A^i_{\sigma,\gamma,j}$. Suppose $x \in A^i_{\sigma,\gamma,j} \cap A^{i'}_{\theta',\gamma',j'}$. If $i \neq i'$ then $(A^i_\sigma \cap A^i_\gamma) \cap (A^{i'}_{\sigma'} \cap A^{i'}_{\gamma'}) \subset P_{i,1} \cap P_{i',1} = \emptyset$ whence $i = i'$. If $j \neq j'$ then $\alpha_{t_{i,j}}(A^i_\sigma \cap A^i_\gamma) \cap \alpha_{t_{i,j'}}(A^i_{\sigma'} \cap A^i_{\gamma'}) \subset \alpha_{t_{i,j}} P_{i,1} \cap \alpha_{t_{i,j'}} P_{i,1} = \emptyset$ and thus $j = j'$. It then follows $\alpha^{-1}_{t_{i,j}} x \in (A^i_\sigma \cap A^i_\gamma) \cap (A^i_{\sigma'} \cap A^i_{\gamma'})$ which is empty unless $\sigma = \sigma'$ and $\gamma = \gamma'$. Therefore $\mathscr{A} = \{A^i_{\sigma,\gamma,j} : i \in I_q, j \in I_{q_i}, \sigma \in \Sigma, \gamma \in \Gamma\}$ is a partition of X.

We have now created a refinement $\mathscr{A}$ of $\mathscr{Q}$ on which the action restricted to H_T is well behaved, but the action restricted to H_R need not be well behaved. We now create a common refinement of $\mathscr{P}$ and $\mathscr{A}$ in a similar manner as above. This refinement will then be well behaved when the actions is restricted to either H_R or H_T. Let $h \in I_p$. Set $\Theta_h = \mathscr{A}^{I_{p_h}}$. Given $\theta \in \Theta_h$ define

$$A_\theta = \{x \in P_{h,1} : \alpha_{r_{h,k}} x \in \theta(k) \text{ for all } k \in I_{p_h}\}.$$

Let $h \in I_p$ be fixed and set $\Theta'_h = \{\theta \in \Theta_h : \mu(A_\theta) > 0\}$. Suppose $x \in P_{h,1}$. Then for each $k \in I_{p_h}$ there exists $\theta_h \in \mathscr{A}$ such that $\alpha_{r_{h,k}} x \in \theta_k$. Thus $x \in A_\theta$ where $\theta \in \Theta'_h$ is defined by $\theta(k) = \theta_k$. If $x \in A_\theta \cap A_{\theta'}$ then $\alpha_{r_{h,k}} x \in \theta(k) \cap \theta'(k)$ for each $k \in I_{p_h}$. As $\theta(k) \cap \theta'(k) \neq \emptyset$ if and only if $\theta(k) = \theta'(k)$, it follows that $\theta = \theta'$. Thus $\{A_\theta\}_{\theta\in\Theta'_h}$ is a partition of $\mathscr{P}_h$ for each $h \in I_p$.

Define $\Theta' = \bigcup_{i=1}^n \Theta'_i$. Let $i \in I_q$, $\sigma \in \Sigma^i$, $\gamma \in \Gamma^i$, and $j \in I_{q_i}$. Define $\Theta^i_{\sigma,\gamma,j} = \{\theta \in \Theta' : \alpha_{r_{\sigma(j)}} A_\theta \subset A^i_{\sigma,\gamma,j}\}$. Recall, $\sigma(j) = (h,k)$ for some $h \in I_p$ and $k \in I_{p_h}$. Given $\theta \in \Theta^i_{\sigma,\gamma,j}$ it follows $A_\theta \subset \alpha^{-1}_{r_{\sigma(j)}} A^i_{\sigma,\gamma,j} \subset \alpha^{-1}_{r_{\sigma(j)}} P_{\sigma(j)} = \alpha^{-1}_{r_{h,k}} P_{h,k} = P_{h,1}$ whence $\Theta^i_{\sigma,\gamma,j} \subset \Theta'_h$. Suppose $x \in A^i_{\sigma,\gamma,j}$. Then $\alpha^{-1}_{r_{\sigma(j)}} x \in P_{h,1}$ and hence contained in A_θ for some $\theta \in \Theta'_h$. Then $x = \alpha_{r_{\sigma(j)}}(\alpha^{-1}_{r_{\sigma(j)}} x) = \alpha_{r_{h,k}}(\alpha^{-1}_{r_{\sigma(j)}} x) \in \theta(k) \cap A^i_{\sigma,\gamma,j} \neq \emptyset$. In particular it follows that $A^i_{\sigma,\gamma,j} = \theta(k)$. If $y \in A_\theta$ then $\alpha_{g_{\sigma(j)}} y \in \theta(k) = A_{\sigma,\gamma,j}$. It now follows that

$\alpha_{r_{\sigma(j)}} A_\theta \subset A^i_{\sigma,\gamma,j}$ for all $\theta \in \Theta^i_{\sigma,\gamma,j}$ and consequently $A^i_{\sigma,\gamma,j} = \bigcup_{\theta \in \Theta^i_{\sigma,\gamma,j}} \alpha_{r_{\sigma(j)}} A_\theta$.

We could now copy the partition of $P_{h,1}$ to the levels of $\mathscr{P}_h$ for each $h \in I_p$ to create a partition of X. In general, the resulting partition will not be uniform. To create a uniform partition we approximate each element in Θ' by a subset of rational measure. Its then clear that the sum of the measures of the removed sets must also be rational. We may then subdivide $\bigcup_{h \in I_p} P_{h,1}$ into sets of equal measure. This will now give a uniform refinement of both $\mathscr{P}$ and $\mathscr{Q}$. We now show if we make these subdivisions small enough we may perturb α to one which when restricted to the subgroup generated by $R \cup T$ will pointwise permute the resulting partition.

For each $\theta \in \Theta'$ we may find $\epsilon_\theta < \frac{\epsilon}{2|\mathscr{A}||\Theta'|}$ and $A'_\theta \subset A_\theta$ such that $\mu(A'_\theta) = \frac{a_\theta}{b_\theta} \in \mathbb{Q}$ and $\mu(A'_\theta \setminus A_\theta) < \epsilon_\theta$. Let M' be the least common multiple of the collection of integers $\{b_\theta\}_{\theta \in \Theta'}$. Choose $M \in \mathbb{N}$ such that $\frac{1}{M} < \delta$ and $M' \cdot |\mathscr{P}_h|$ divides M for each $h \in I_p$. Define $m_\theta = \mu(A'_\theta) \cdot M$ and $I_\theta = \{1, 2, \ldots, m_\theta\}$. Since X is nonatomic, there exist pairwise disjoint subsets $\{A_{\theta,l}\}_{l \in I_\theta}$ of A'_θ such that $\mu(A_{\theta,l}) = \frac{1}{M}$ and $A'_\theta = \bigcup_{l \in I_\theta} A_{\theta,l}$.

Let $h \in I_p$ and set $E_h = P_{h,1} \setminus \left(\bigcup_{\theta \in \Theta'_h} A'_\theta\right) = \left(\bigcup_{\theta \in \Theta'_h} A_\theta\right) \setminus \left(\bigcup_{\theta \in \Theta'_h} A'_\theta\right) = \bigcup_{\theta \in \Theta'_h} (A_\theta \setminus A_\theta)$. Then

$$\mu(E_h) = \mu(P_{h,1}) - \sum_{\theta \in \Theta'_h} \mu(A'_\theta) = \frac{1}{|\mathscr{P}_h|} - \sum_{\theta \in \Theta'_h} \frac{a_\theta}{b_\theta} = \frac{M_{p_h}}{M} - \sum_{\theta \in \Theta'_h} \frac{a'_\theta}{M}$$

where $a'_\theta \in \mathbb{N}$ satisfies $\frac{a'_\theta}{M} = \frac{a_\theta}{b_\theta}$ for each $\theta \in \Theta'_h$ and $M_{p_h} \in \mathbb{N}$ satisfies $\frac{M_{p_h}}{M} = \frac{1}{|\mathscr{P}_h|}$. Since $\mu(A_\theta \setminus A'_\theta) = \epsilon_\theta > 0$ for each $\theta \in \Theta'_h$, $\mu(E_h) = \sum_{\theta \in \Theta'_h} \mu(A_\theta \setminus A'_\theta) > 0$ and thus $M_h = M_{p_h} - \sum_{\theta \in \Theta'_h} a'_\theta \in \mathbb{N}$. Let $I_{M_h} = \{1, 2, \ldots, M_h\}$ then there exist disjoint subsets $\{E_{h,l}\}_{l \in I_{M_h}}$ of E_h satisfying $\mu(E_{h,l}) = \frac{1}{M}$ for $l \in I_{M_h}$ and $E = \bigcup_{l \in I_{M_h}} E_{h,l}$.

Define $\mathscr{B}_h = \{A_{\theta,l} : \theta \in \Theta'_h,\ l \in I_\theta\}$ and $\mathscr{E}_h = \{E_{h,l} : l \in I_{M_h}\}$ for each $h \in I_p$. Set

$$\mathscr{R} = \{\alpha_{r_{h,k}} B : h \in I_p,\ k \in I_{p_h},\ B \in \mathscr{B}_h\} \cup \{\alpha_{r_{h,k}} E : h \in I_p,\ k \in I_{p_h},\ E \in \mathscr{E}_h\}.$$

Since $P_{h,1} = (\bigcup \mathscr{B}_h) \cup (\bigcup \mathscr{E}_h)$ and $\mathscr{P}_i = \bigcup_{j \in I_{p_i}} \alpha_{r_{i,j}} P_{i,j}$ we see that $\mathscr{R}$ is indeed a partition of X. We now perturb α to pointwise permute this partition. For convenience we pick a reference set $A_0 \in \bigcup_{h \in I_p} \mathscr{B}_h$. Given a set $B \in \bigcup_{h \in I_p} (\mathscr{B}_h \cup \mathscr{E}_h)$ fix a measure preserving isomorphism $\phi_B : A_0 \to B$.

Let $i \in I_q$, $\sigma \in \Sigma^i$ and $\gamma \in \Gamma^i$. Suppose $A_\sigma \cap A_\gamma \neq \emptyset$. Since α is measure-preserving $A^i_{\sigma,\gamma,j} \neq \emptyset$ for all $j \in I_{q_i}$. Define $n^i_{\sigma,\gamma} = \min_{j \in I_{q_i}} \{\sum_{\theta \in \Theta^i_{\sigma,\gamma,j}} |I_\theta|\}$. Fix $\mathscr{B}^i_{\sigma,\gamma,1} \subset \{A_{\theta,l} : \theta \in \Theta^i_{\sigma,\gamma,1},\ l \in I_\theta\}$ such that $|\mathscr{B}^i_{\sigma,\gamma,1}| = n^i_{\sigma,\gamma}$. Choose an injection $\phi^i_{\sigma,\gamma,1} : \mathscr{B}^i_{\sigma,\gamma,1} \to \{A_{\theta,l} : \theta \in \Theta^i_{\sigma,\gamma,2},\ l \in I_\theta\}$ and set $\mathscr{B}^i_{\sigma,\gamma,2} = \phi^i_{\sigma,\gamma,1}(\mathscr{B}_{\sigma,\gamma,1})$. For $j = 2, 3, \ldots, m_{q_i} - 1$, inductively define $\mathscr{B}^i_{\sigma,\gamma,j}$, and $\phi^i_{\sigma,\gamma,j}$ such that $\mathscr{B}^i_{\sigma,\gamma,j} = \phi^i_{\sigma,\gamma,j-1}(\mathscr{B}^i_{\sigma,\gamma,j-1})$, and $\phi^i_{\sigma,\gamma,j}$ is an injection from $\mathscr{B}^i_{\sigma,\gamma,j}$ into $\{A_{\theta,l} : \theta \in \Theta^i_{\sigma,\gamma,j+1},\ l \in I_\theta\}$. Set $\mathscr{B}_{\sigma,\gamma,m_{q_i}} = \phi_{\sigma,\gamma,m_{q_i}-1}(\mathscr{B}_{\sigma,\gamma,m_{q_i}-1})$ and $\phi^i_{\sigma,\gamma,m_{q_i}} = (\phi^i_{\sigma,\gamma,1})^{-1} \circ (\phi^i_{\sigma,\gamma,2})^{-1} \circ \ldots \circ (\phi^i_{\sigma,\gamma,m_{q_i}-1})^{-1}$. Then $\phi^i_{\sigma,\gamma,m_{q_i}}$ is an injection from $\mathscr{B}^i_{\sigma,\gamma,m_{q_i}}$ onto $\mathscr{B}^i_{\sigma,\gamma,0}$. For convenience define $B^i_{\sigma,\gamma,j} = \bigcup_{B \in \mathscr{B}_{\sigma,\gamma,j}} \alpha_{r_{\sigma(j)}} B$ and $E^i_{\sigma,\gamma,j} = A^i_{\sigma,\gamma,j} \setminus B^i_{\sigma,\gamma,j}$.

In this manner we have paired off a large proportion of the elements of $\mathscr{R}$ according to the action of α restricted to H_T. On the remaining elements we will show that the action can be redefined to be the identity. By construction the restriction of α to H_S already pointwise permutes $\mathscr{R}$. However, in the applications it is useful to know there is an atom fixed by all elements of the subgroup generated by $R \cup T$. We show this can be done as well.

Set $\alpha'_s = \alpha_s$ for all $s \in S \setminus (R \cup T)$. For $r \in R$ define

$$\alpha'_r x = \begin{cases} \alpha_r x & \text{if } x \in B^i_{\sigma,\gamma,j} \text{ for some } i \in I_q,\ \sigma \in \Sigma^i,\ \gamma \in \Gamma^i,\ j \in I_{q_i} \\ x & \text{otherwise.} \end{cases}$$

For $t \in T$ we define α'_t as follows. Let $x \in X$. If $x \in E^i_{\sigma,\gamma,j}$ for some $i \in I_q$, $\sigma \in \Sigma^i$, $\gamma \in \Gamma^i$, $j \in I_{q_i}$ we set $\alpha'_t x = x$. Suppose $x \in B^i_{\sigma,\gamma,j}$ for some $i \in I_q$, $\sigma \in \Sigma^i$, $\gamma \in \Gamma^i$, $j \in I_{q_i}$. Since $x \in Q_{i,j}$ it follows that $\alpha_t x \in Q_{i,j'}$ for some $j' \in I_{q_i}$. Then there exists a unique $A_{\theta,l} \in \mathscr{B}^i_{\sigma,\gamma,j}$ such that $x \in \alpha_{r_{\sigma(j)}} A_{\theta,l}$. Moreover, $A_{\theta,l}$ is identified with a unique $A_{\sigma',l'} \in \mathscr{B}^i_{\sigma,\gamma,j'}$. We now define $\alpha'_t x = \alpha_{r_{\sigma(j')}} \circ \phi_{A_{\theta',l'}} \circ \phi^{-1}_{A_{\theta,l}} \circ \alpha^{-1}_{r_{\sigma(j)}} x$. Since $B^i_{\sigma,\gamma,j} \subset A^i_{\sigma,\gamma,j}$ and the latter sets partition X, α_s is a well defined automorphism of X for each $s \in R \cup T$. In particular, α' defines an action of $\mathbb{F}_n$ on X. We now show $\alpha' \in U(\alpha, F, K, \epsilon)$.

Let $s \in F$ and $C \in K$. Suppose $C \cap A^i_{\sigma,\gamma,j} \neq \emptyset$ for some $i \in I_q$, $\sigma \in \Sigma^i$, $\gamma \in \Gamma^i$, and $j \in I_{q_i}$. Then $C \cap \alpha_{t_i,j} A_\gamma = C \cap (\bigcap_{l \in \gamma(j)} C_l) \cap (\bigcap_{l \in I \setminus \gamma(j)} C^c_l) \neq \emptyset$ whence $C = C_{l'}$ for some $l' \in \gamma(j)$. Thus $A_{\sigma,\gamma,j} \subset C$ and hence $C = \cup \mathscr{A}_C$ where $\mathscr{A}_C = \{A \in \mathscr{A} : A \subset C\}$. For $A \in \mathscr{A}$ we have $A = A^i_{\sigma\gamma,j}$ for some $i \in \ I_q$, $\sigma \in \Sigma^i$, $\gamma \in \Gamma^i$, and $j \in I_{q_i}$. Define $\Theta_A = \Theta^i_{\sigma,\gamma,j}$, $B_A = B^i_{\sigma,\gamma,j}$, $\mathscr{B}_A = \mathscr{B}^i_{\sigma,\gamma,j}$, and $E_A = E^i_{\sigma,\gamma,j}$. Set $C' = \bigcup_{A \in \mathscr{A}_C} B_A$ and $E_C = C \setminus C'$.

If $s \in S \setminus (R \cup T)$ then $\alpha_s = \alpha'_s$ and there is nothing to check. Suppose $s \in R$. Then

$$
\begin{aligned}
\mu(\alpha_s C \triangle \alpha'_s C) &= \mu((\alpha_s C' \cup \alpha_s E_C) \triangle (\alpha'_s C' \cup \alpha'_s E_C)) \\
&= \mu((\alpha_s C' \cup \alpha_s E_C) \triangle (\alpha_s C' \cup E_C)) \\
&= \mu(\alpha_s E_C \triangle E_C) \\
&\leq \mu(\alpha_s E_C) + \mu(E_C) \\
&= 2\mu(E_C) \\
&= 2 \sum_{A \in \mathscr{A}_C} \sum_{\theta \in \Theta_A} \epsilon_\theta \\
&< 2|\mathscr{A}_C||\Theta_A| \frac{\epsilon}{2|\mathscr{A}||\Theta'|} \\
&< \epsilon.
\end{aligned}
$$

It remains to check the case when $s \in T$. Let $i \in I_q$, $\sigma \in \Sigma^i$, $\gamma \in \Gamma^i$, and $j \in I_{q_i}$. Choose $j' \in I_{q_i}$ such that $n^i_{\sigma,\gamma} = \sum_{\theta \in \Theta^i_{\sigma,\gamma,j'}} |I_\theta|$. Then $B^i_{\sigma,\gamma,j'} = \bigcup_{\theta \in \Theta^i_{\sigma,\gamma,j'}} \bigcup_{k \in I_\theta} A_{\theta,k} =$

$\bigcup_{\theta\in\Theta^i_{\sigma,\gamma,j'}} A'_\theta$ whence

$$\begin{aligned}
\mu(E^i_{\sigma,\gamma,j'}) &= \mu(A^i_{\sigma,\gamma,j'} \setminus (\bigcup_{\theta\in\Theta^i_{\sigma,\gamma,j'}} A'_\theta)) \\
&= \mu(\bigcup_{\theta\in\Theta^i_{\sigma,\gamma,j'}} (A_\theta \setminus A'_\theta)) \\
&= \sum_{\theta\in\Theta^i_{\sigma,\gamma,j'}} \mu(A_\theta \setminus A'_\theta) \\
&= \sum_{\theta\in\Theta^i_{\sigma,\gamma,j'}} \epsilon_\theta \\
&< \sum_{\theta\in\Theta_{\sigma,\gamma,j'}} \frac{\epsilon}{2|\mathscr{A}||\Theta'|} \\
&= \frac{|\Theta^i_{\sigma,\gamma,j'}|\epsilon}{2|\mathscr{A}||\Theta'|} \\
&\le \frac{\epsilon}{2|\mathscr{A}|}.
\end{aligned}$$

Since α is measure-preserving, it follows $\mu(B^i_{\sigma,\gamma,j}) = \mu(B^i_{\sigma,\gamma,j'})$ and $\mu(E^i_{\sigma,\gamma,j}) < \frac{\epsilon}{2|\mathscr{A}|}$ for all $j \in I_{q_i}$. Then

$$\alpha_s C \triangle \alpha'_s C = \alpha_s C \triangle (\alpha'_s C' \cup \alpha'_s E_C) = \alpha_s C \triangle (\bigcup_{A\in\mathscr{A}_C} \alpha'_s B_A \cup E_C).$$

By construction $\alpha'_s B_A \subset \alpha_s A$ for each $A \in \mathscr{A}$ from which it follows that $\bigcup_{A\in\mathscr{A}_C} \alpha'_s B_A \subset \bigcup_{A\in\mathscr{A}_C} \alpha_s A = \alpha_s(\cup\mathscr{A}_C) = \alpha_s C$. Therefore $\alpha_s C \triangle \alpha'_s C \subset (\alpha_s C \setminus \bigcup_{A\in\mathscr{A}_C} \alpha'_s B_A) \cup E_C$. We now have

$$\begin{aligned}
\mu(\alpha_s C \triangle \alpha'_s C) &\le \mu((\alpha_s C \setminus \bigcup_{A\in\mathscr{A}_C} \alpha'_s B_A) \cup E_C) \\
&\le \mu(\alpha_s C \setminus \bigcup_{A\in\mathscr{A}_C} \alpha'_s B_A) + \mu(E_C) \\
&= \mu(\alpha_s C) - \mu(\bigcup_{A\in\mathscr{A}_C} \alpha'_s B_A) + \mu(E_C)
\end{aligned}$$

$$
\begin{aligned}
&= \mu(C) - \mu(\bigcup_{A \in \mathscr{A}_C} B_A) + \mu(E_C) \\
&= \mu(C \setminus \bigcup_{A \in \mathscr{A}_C} B_A) + \mu(E_C) \\
&= \mu(E_C) + \mu(E_C) \\
&= 2 \sum_{A \in \mathscr{A}_C} \mu(E_A) \\
&< 2 \sum_{A \in \mathscr{A}_C} \frac{\epsilon}{2|\mathscr{A}|} \\
&= 2 \frac{|\mathscr{A}_C|\epsilon}{2|\mathscr{A}|} \\
&< \epsilon.
\end{aligned}
$$

Therefore $\alpha' \in U(\alpha, F, K, \epsilon)$ as desired.

Let $A \in \mathscr{R}$ and $s \in R \cup T$ and $\epsilon \in \{-1, 1\}$. Suppose $A = \alpha_{r_{h,k}} E$ for some $h \in I_p$, $k \in I_{p_h}$ and $E \in \mathscr{E}_h$. Then $\alpha'_{s^\epsilon} x = x$ for all $x \in A$, i.e. $\alpha'_{s^\epsilon} A = A$. Moreover, α'_{s^ϵ} may be expressed as $\alpha_{r_{h,k}} \circ \phi_E \circ \phi_E^{-1} \circ \alpha_{r_{h,k}}^{-1}$. Suppose $A = \alpha_{r_{h,k}} B$ for some $h \in I_p$, $k \in I_{p_h}$ and $B \in \mathscr{B}_h$. First consider the case when $s \in R$. Then $\alpha'_{s^\epsilon} A = \alpha_{s^\epsilon} A = \alpha_{s^\epsilon} \alpha_{r_{h,k}} B \subset \alpha_{s^\epsilon} P_{h,k} = P_{h,k'}$ for some $k' \in I_{p_h}$, whence $\alpha_{s^\epsilon} P_{h,k} = P_{h,k'}$. Then, $\alpha'_{s^\epsilon} x = \alpha_{r_{h,k'}} \circ \phi_B \circ \phi_B^{-1} \circ \alpha_{r_{h,k}}^{-1} x$ for all $x \in A$ and $\alpha'_{s^\epsilon} A = \alpha_{r_{h,k'}} B \in \mathscr{R}$. Suppose $s \in T$. We have constructed α' such that $\alpha'_{s^\epsilon} = \alpha_{r_{h',k'}} \circ \phi_{B'} \circ \phi_B^{-1} \circ \alpha_{r_{h,k}}^{-1}$ for some $h, h' \in I_p$ $k \in I_{p_h}$, $k' \in I_{p_{h'}}$, $B \in \mathscr{B}_h$, and $B' \in \mathscr{B}_{h'}$ and $\alpha'_{s^\epsilon} A = \alpha_{r_{h',k'}} B' \in \mathscr{R}$. Moreover, it is clear that if $\alpha'_{s^\epsilon} A = A$ then α'_{s^ϵ} is the identity on A.

Since α'_s pointwise permutes $\mathscr{R}$ for each $s \in R \cup T$, we have that α' restricted to the subgroup generated by $R \cup T$ also pointwise permutes $\mathscr{R}$. Moreover, $\mu(A) = \frac{1}{M} < \delta$ for each $A \in \mathscr{R}$ and if $B \in \mathscr{E}_h$ we have $\alpha'_s x = x$ for all $s \in R \cup T$ and consequently for all s' in the subgroup generated by $R \cup T$. $\square$

Theorem 3.3.2. *Let $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$ and $S' \subset S$ be finite. Then given any open* *neighborhood $U(\alpha, F, K, \epsilon)$ and $\delta > 0$ there exists $\alpha' \in U(\alpha, F, K, \epsilon)$ and a uniform partition $\mathscr{P}$ of X such that α' restricted to the subgroup generated by S' pointwise permutes $\mathscr{P}$ and $\mu(P) < \delta$ for all $P \in \mathscr{P}$. Moreover, α' and $\mathscr{P}$ can be chosen so that there exists $P \in \mathscr{P}$ satisfying $\alpha_s P = P$ for all s in the subgroup generated by S'.*

Proof. Let $U(\alpha, F, K, \epsilon)$ be a fixed open neighborhood. Enumerate S' by $\{s_1, s_2, \ldots, s_n\}$. For each $i = 1, 2, \ldots, n$ let $S_i = \{s_1, s_2, \ldots, s_i\}$ and H_i be the subgroup generated by S_i. By Lemma 3.1.3 we may assume there exists a natural number N and Borel subsets $B_1, B_2, \ldots, B_n$ of X such that $\alpha_{s_i}^N x = x$ and $\mathscr{P}_i = \{\alpha_{s_i}^j B_i\}_{j=1}^n$ partitions X for each $i = 1, 2, \ldots, n$.

Set $\alpha^0 = \alpha$. Applying Lemma 3.3.1 to $\mathscr{P}_1$ and $\mathscr{P}_2$ we obtain an action $\alpha^1 \in U(\alpha^0, F, K, \frac{\epsilon}{n})$ and partition $\mathscr{R}_1$ such that α^1 restricted to H_2 pointwise permutes $\mathscr{R}_1$, and $\alpha_s^1 = \alpha_s$ for $s \in S' \setminus S_2$. Inductively applying Lemma 3.3.1 to $\mathscr{R}_i$ and $\mathscr{P}_{i+2}$ for $i = 1, 2, \ldots, n-2$ we obtain an action $\alpha^i \in U(\alpha^{i-1}, F, K, \frac{\epsilon}{n})$ and partition $\mathscr{R}_i$ such that α^i restricted to H_{i+2} pointwise permutes $\mathscr{R}_i$ and $\alpha_s^i = \alpha_s$ for $s \in S \setminus S_{i+2}$. Moreover, we may choose α^{n-2} and $\mathscr{R}_{n-2}$ so that $\mu(R) < \delta$ for all $R \in \mathscr{R}^{n-2}$ and for some $R \in \mathscr{R}_{n-2}$ we have $\alpha_s^{n-2} x = x$ for all $x \in R$ and $s \in H_n$.

Set $\alpha' = \alpha^{n-2}$ and $\mathscr{P} = \mathscr{R}_{n-2}$. Then α' and $\mathscr{P}$ have the desired properties. We need only check that $\alpha' \in U(\alpha, F, K, \epsilon)$. Let $s \in F$ and $C \in K$. Then

$$
\begin{aligned}
\mu(\alpha_s C \triangle \alpha_s' C) &= \mu(\alpha_s C \triangle \alpha_s^{n-2} C) \\
&\leq \mu(\alpha_s C \triangle \alpha_s^{n-3} C) + \mu(\alpha_s^{n-3} C \triangle \alpha_s^{n-2} C) \\
&\vdots \\
&\leq \mu(\alpha_s C \triangle \alpha_s^1 C) + \mu(\alpha_s^2 C \triangle \alpha_s^3 C) + \cdots + \mu(\alpha_s^{n-3} C \triangle \alpha_s^{n-2} C) \\
&< (n-1)\frac{\epsilon}{n} \\
&< \epsilon.
\end{aligned}
$$

□

Using Proposition 3.2.2 we may translate Theorem 3.3.2 to a statement about factoring through finite groups. In particular we have the following Theorem

Theorem 3.3.3. *Let $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$ and $S' \subset S$ be finite. Then given any open neighborhood $U(\alpha, F, K, \epsilon)$ there exists $\alpha' \in U(\alpha, F, K, \epsilon)$ such that α' restricted to the subgroup generated by S' factors through a finite group.*

Theorem 3.3.2 should be compared with Proposition 2.3 of Luboztky and Shalom [22]. In the proof of this proposition the authors approximate a given action of $\mathbb{F}_n$ by a permutation action using Hall's marriage theorem. Although not explicitly stated, it follows easily that the set of actions which pointwise permute some uniform partition is dense. The noticeable difference from Theorem 3.3.2 is we can not assume there exists an atom on which the action acts as the identity for all group elements. We will see in Section 6 that the existence of such an atom allows for some control over what group the approximating action factors through (see Lemma 6.2.1).

A more subtle difference between Theorem 3.3.2 and the result of Luboztky and Shalom is the method of proof. Despite being technically complicated our proof of Theorem 3.3.2 is constructive and elementary in nature. In Section 7 we show with a few obvious changes much of the proof can be applied directly to certain topological actions of $\mathbb{F}_n$ on the Cantor set to obtain an analogous result. In comparison, the argument of Luboztky and Shalom is highly specialized and can not be transferred to the topological setting.

4. RIGIDITY AND THE LACK OF MIXING

It is well known that a generic $\mathbb{Z}$-action is weak mixing but not mixing. In this section we will apply Theorem 3.3.3 to show this holds for actions of the free group as well. It follows from [20] that the weak mixing actions of $\mathbb{F}_n$ are generic. Thus we only need to show the mixing actions fail to be generic.

We say a unitary representation π of a countable group G on a Hilbert space $\mathcal{H}$ is *rigid* if there exists a sequence $\{g_n\} \subset G$ converging to infinity such that $\pi(g_n)$ converges to the identity operator in the strong (equivalently weak) operator topology. We say an action (G, X, μ, α) is *rigid* if its associated representation π_α is rigid. If G is a countable group the definition of rigidity expressed in the following formally weaker way.

Proposition 4.0.1. *Let π be a representation of a countable discrete group G on a separable Hilbert space $\mathcal{H}$. If there exists a sequence $\{g_n\} \subset G$ converging to infinity such that $\{\pi(g_n)\}$ converges in the strong operator topology then π is rigid.*

Proof. Let $\{g_n\} \subset G$ be a sequence converging to infinity and $\{\xi_n\} \subset \mathcal{H}$ be countable and dense. Suppose $\pi(g_n) \to U$ in the strong operator topology. Let $\epsilon > 0$. For each n it is clear there exists $n' > n$ such that $n' > i'$ if $i < n$ and $\|\pi(g_{n'})\xi_n - U\xi_n\| < \epsilon/4$. In addition n' can be chosen such that $g_{n'}^{-1}g_n \notin \{g_{i'}^{-1}g_i\}_{i<n}$. If not, $g_{n'}^{-1}g_n \in \{g_{i'}^{-1}g_i\}_{i<n}$ for all n' then by the pigeonhole principle there exists $s \in \{g_{i'}^{-1}g_i\}_{i<n}$ such that $g_{n'}^{-1}g_n = s$ for infinitely many n'. Thus $g_{n'} = g_n s$ for infinitely many n' contradicting the assumption that $\{g_n\}$ converges to infinity. Given $\xi \in \mathcal{H}$ there exists a subsequence $\{\xi_{n_k}\}$ converging to ξ. Choose K such that $\|\xi_{n_k} - \xi\| < \epsilon/4$ and $\|\pi(g_{n_k})\xi - U\xi\| < \epsilon/4$ for all $k > K$. Then

$$\|\pi(g_{n'_k}^{-1}g_{n_k})\xi - \xi\| = \|\pi(g_{n_k})\xi - \pi(g_{n'_k})\xi\|$$

$$
\begin{aligned}
&\leq \|\pi(g_{n_k})\xi - U\xi\| + \|U\xi - \pi(g_{n'_k})\xi\| \\
&\leq \|\pi(g_{n_k})\xi - U\xi\| + \|U\xi - U\xi_{n'_k}\| + \|U\xi_{n'_k} - \pi(g_{n'_k})\xi\| \\
&\leq \|\pi(g_{n_k})\xi - U\xi\| + \|\xi - \xi_{n'_k}\| + \|U\xi_{n'_k} - \pi(g_{n'_k})\xi_{n'_k}\| + \|\pi(g_{n'_k})(\xi_{n'_k} - \xi)\| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon
\end{aligned}
$$

for all $k > K$. It remains to check that $\{g_{n'_k}^{-1} g_{n_k}\}$ converges to infinity. However this follows from the fact that $g_{n'}^{-1} g_n \notin \{g_{i'}^{-1} g_i\}_{i<n}$. Therefore π is rigid. $\square$

Let $\mathcal{H}$ be a finite-dimensional Hilbert space and G a countable discrete group. Then $\mathcal{H}$ is separable and $\mathscr{U}(\mathcal{H})$ is compact. Let $\{g_n\} \subset G$ be a sequence converging to infinity. Then $\{\pi(g_n)\}$ has a strong operator topology convergent subsequence since $\mathscr{U}(\mathcal{H})$ is compact. An application of Proposition 4.0.1 shows the following:

Corollary 4.0.2. *If $(G, \mathcal{H})$ is a finite-dimensional representation of a countable discrete group G then π is rigid.*

It is clear from the definition that a rigid action fails to be mixing. In particular, to show mixing fails generically it suffices to show the rigid actions are generic. We show now that the rigid actions are in fact a G_δ.

Proposition 4.0.3. *Let G be a countable discrete group and $\mathcal{H}$ a separable Hilbert space. Then set of rigid actions of G is a G_δ subset of* $\mathrm{Act}(G, X, \mu)$*.*

Proof. Let $\{g_n\}_{n=1}^{\infty}$ be an enumeration of G and $\{\xi_n\}_{n=1}^{\infty}$ a dense subset of $L^2(X, \mu)$. For each positive integer n define

$$
\begin{aligned}
A_n &= \left\{\alpha \in \mathrm{Act}(G, X, \mu) : \exists\ m > n \text{ such that } \|\pi_a(g_m)\xi_i - \xi_i\| < \frac{1}{n},\ i = 1 \ldots n\right\} \\
&= \bigcup_{m>n} \bigcap_{i=1}^{n} \left\{\alpha \in \mathrm{Act}(G, X, \mu) : \|\pi_\alpha(g_m)\xi_i - \xi_i\| < \frac{1}{n}\right\}.
\end{aligned}
$$

Note A_n is open and contains the trivial representation for each n. Let $A = \bigcap_{n=1}^{\infty} A_n$. Suppose $\alpha \in \mathrm{Act}(G, X, \mu)$ is rigid. Let $\{s_i\} \subset G$ be a sequence converging to infinity such that $\|\pi_\alpha(s_i)\xi - \xi\| \to 0$ for all $\xi \in \mathcal{H}$. Let n be a fixed but arbitrary positive integer. Then there exists $I > n$ in $\mathbb{N}$ such that $\|\pi_\alpha(s_i)\xi - \xi\| < \frac{1}{n}$ for all $\xi \in \mathcal{H}$ and all $i > I$. By the pigeonhole principle there exists $i > I$ such that $s_i = g_m$ for some $m > n$. Therefore there exists $m > n$ such that$\|\pi_\alpha(g_m)\xi_j - \xi_j\| < \frac{1}{n}$ for all $i = 1 \dots n$ or equivalently $\pi \in A_n$. Thus $\alpha \in A$. Now suppose $\alpha \in A$. Then for each positive integer n there exists $s_n \notin \{g_i\}_{i=1}^n$ such that for $\|\pi_\alpha(s_n)\xi_i - \xi_i\| < \frac{1}{n}$ for all $j = 1, \dots n$. Let F be a finite subset of G. Then $F \subset \{g_i\}_{i=1}^m$ for some m. Therefore $s_n \notin F$ for all $n > m$. That is $\{s_n\}$ converges to infinity. Let $\xi \in \mathcal{H}$ and $\epsilon > 0$. There exists n_0 such that $\|\xi - \xi_{n_0}\| < \frac{\epsilon}{3}$. Choose $N > n_0$ such that $\frac{1}{N} < \frac{\epsilon}{3}$. Then for all $n > N$

$$\begin{aligned} \|\pi_\alpha(s_n)\xi - \xi\| &\leq \|\pi_\alpha(s_n)\xi - \pi_\alpha(s_n)\xi_{n_0}\| + \|\pi_\alpha(s_n)\xi_{n_0} - \xi_{n_0}\| + \|\xi_{n_0} - \xi\| \\ &< 2\|\xi - \xi_{n_0}\| + \frac{1}{N} \\ &< \epsilon. \end{aligned}$$

Thus α is rigid and it follows that the set of rigid actions of $\mathrm{Act}(G, X, \mu)$ is a G_δ. $\square$

We conclude by showing for the free group that rigidity is generic on any finitely generated subgroup of $\mathbb{F}_n$. As noted earlier, this is sufficient to show mixing fails generically on any subgroup of $\mathbb{F}_n$ and ultimately that a generic action of $\mathbb{F}_n$ on X is weak mixing but not mixing. To show the genericity of rigid actions we construct the desired sequence directly using Theorem 3.3.3. However, we could also deduce the result from Theorem 3.3.2, Proposition 3.2.3, and Corollary 4.0.2.

Proposition 4.0.4. *Let $\alpha \in \mathrm{Act}(\mathbb{F}_n, G, \mu)$. For every finite subset $S' \subset S$ $\epsilon > 0$ and finite collection K of Borel sets, there exists $\beta \in U(\alpha, S', K, \epsilon)$ such that β is rigid.*

Proof. Let $H_{S'}$ be the subgroup generated by S'. By Theorem 3.3.3 there exists

$\beta \in U(\alpha, S', K, \epsilon)$ and a finite group F such that β restricted to $H_{S'}$ factors through F. In particular, there exists ϕ mapping $H_{S'}$ onto F and an action $\beta' \in \mathrm{Act}(F, X, \mu)$ such that $\beta_s = \beta'_{\phi(s)}$ for all $s \in H_{S'}$. Let $H \subset H_{S'}$ be countable. By the pigeonhole principle there exists a sequence $\{t_n\} \subset H$ such that $\phi(t_n) = e$ for all $n \in \mathbb{N}$ and $\{t_n\}_{n=1}^{\infty}$ converges to infinity. Let $f \in L^2(X, \mu)$. Then for all $x \in X$ and $n \in \mathbb{N}$ we have

$$f(\beta_{s_n^{-1}} x) = f(\beta'_{\phi(s_n^{-1})} x) = f(\beta'_e x) = f(x).$$

Consequently, $\|\pi_\beta(t_n) f - f\| = 0$ for all $f \in L^2(X, \mu)$ and $n \in \mathbb{N}$. Thus β is rigid as desired. $\square$

For completeness we state the following:

Theorem 4.0.5. *A generic action of $\mathbb{F}_n$ is weakly mixing but not mixing.*

5. MEASURE-THEORETIC ENTROPY OF ACTIONS OF SOFIC GROUPS

Thus far we have encountered three invariants: ergodicity, mixing and weak mixing. We now introduce a fourth, namely entropy. Entropy was first introduced to measure-preserving dynamics by Kolmogorov and Sinai in the late 1950s. It was through their work that the long standing question of whether any two measure-preserving Bernoulli shifts were conjugate was shown to be false. In the late 1960's Ornstein [24] showed entropy is in fact a complete invariant for Bernoulli shifts, i.e. two Bernoulli shifts are conjugate if and only if they have the same entropy. Entropy has since been extended to topological systems as well as actions of more general groups, such as amenable groups. Recently measure-theoretic entropy has been extended to countable sofic groups by Bowen [5]. Kerr and Li [19] have since extended the notion of entropy in [5] to include topological systems.

Generally speaking, entropy is a measure of the average uncertainty of where the action moves the points of the space. Equivalently one may think of entropy as measure of randomness of the dynamical system. Thus if a system has zero entropy we think of the system as being highly deterministic. It is surprising that in many cases zero entropy is generic, e.g. actions of the integers or amenable groups in the measure-preserving case and actions of the integers on a Cantor in the topological case. Our goal for this section is to show the actions with entropy equal to zero or negative infinity in the sense of [5] is generic. To help better understand the notion of entropy, we begin by defining it for classical dynamical systems.

5.1. Measure-Theoretic Entropy of Classical Dynamical Systems

Let (X, μ) be a standard probability space. Let $\mathscr{P}_X$ be the collection of all partitions of X into finitely many measurable sets. Given $\mathscr{P} \in \mathscr{P}_X$ we call an element $P \in \mathscr{P}$

an *atom* of $\mathscr{P}$. Given another partition $\mathscr{R} \in \mathscr{P}_X$ we say that $\mathscr{R}$ *refines* $\mathscr{P}$ if for all $R \in \mathscr{R}$ we have $R \subset P$ for some $P \in \mathscr{P}$. We define the *join* $\mathscr{P} \vee \mathscr{R}$ of $\mathscr{P}$ and $\mathscr{R}$ by $\mathscr{P} \vee \mathscr{R} = \{P \cap R : P \in \mathscr{P},\ R \in \mathscr{R}\}$. Note that $\mathscr{P} \vee \mathscr{R}$ is again a partition of X and is the smallest partition refining both $\mathscr{R}$ and $\mathscr{P}$.

Given a partition $\mathscr{P} \in \mathscr{P}_X$ and a measure-preserving automorphism of X define

$$\begin{aligned} H_\mu(\mathscr{P}) &= -\sum_{P \in \mathscr{P}} \mu(P)\log(\mu(P)), \\ h_\mu(T, \mathscr{P}) &= \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathscr{P} \vee T^{-1}\mathscr{P} \vee \cdots \vee T^{-(n-1)}\mathscr{P}), \\ h_\mu(T) &= \sup_{\mathscr{P} \in \mathscr{P}_X} h_\mu(T, \mathscr{P}). \end{aligned}$$

In the definition of $H_\mu(\mathscr{P})$ we adopt the convention that $\mu(P)\log(\mu(P)) = 0$ when $\mu(P) = 0$. The existence of the limit in the second line is standard in any introductory book on entropy, see [27] for example. We call $H_\mu(\mathscr{P})$ the *entropy of the partition* $\mathscr{P}$, $h_\mu(T, \mathscr{P})$ the *entropy of T with respect to $\mathscr{P}$*, and $h_\mu(T)$ the *entropy* of T. Given $x \in X$ we can think of $H_\mu(\mathscr{P})$ as measuring the uncertainty of which atom of $\mathscr{P}$ contains X, $h_\mu(T, \mathscr{P})$ as the average uncertainty of which atom of $\mathscr{P}$ that T will move x to next given its history, and $h_\mu(T)$ as a measure of the average uncertainty of where T moves points of X.

Computing $h_\mu(T)$ is generally difficult since it involves a supremum over all finite partitions. Fortunately, Kolmogorov showed that $h_\mu(T) = h_\mu(T, \mathscr{P})$ when $\mathscr{P}$ is a generating partition. More precisely, we say a partition $\mathscr{P}$ is *generating* if the smallest T-invariant σ-algebra containing $\mathscr{P}$ is equal to $\mathscr{X}$ modulo null sets. We will discuss generating partition in more detail shortly. Presently we use this fact to compute entropy for rotations on the circle and Bernoulli shifts.

Example 5.1.1. Let $(\mathbb{T}, \lambda, T_w)$ be rotation on the unit circle $\mathbb{T}$ by w. If w is a root of unity then there exists $m \in \mathbb{N}$ such that $T_w^m x = x$ a.e. Let $\mathscr{P}$ be a partition of $\mathbb{T}$

into $r < \infty$ atoms. Then for any $n > 0$ the partition $\mathscr{P} \vee T^{-1}\mathscr{P} \vee \cdots \vee T^{-(n-1)}\mathscr{P}$ has at most r^m atoms. Thus

$$h_\mu(T, \mathscr{P}) = \lim_{n\to\infty} \frac{1}{n} h_\mu(\mathscr{P} \vee T^{-1}\mathscr{P} \vee \cdots \vee T^{-(n-1)}\mathscr{P}) \leq \lim_{n\to\infty} \frac{1}{n}(m \log(r)) = 0.$$

Suppose w is not a root of unity. Define $\mathbb{T}^+ = \{e^{i\theta} : \theta \in [0, \pi)\}$ and $\mathbb{T}^- = \{e^{i\theta} : \theta \in [\pi, 2\pi)\}$. Then the partition $\mathscr{P} = \{\mathbb{T}^+, \mathbb{T}^-\}$ is generating for T_w. Moreover, for any $n > 0$ we have $\mathscr{P} \vee T^{-1}\mathscr{P} \vee \cdots \vee T^{-(n-1)}\mathscr{P}$ has at most $2n$ atoms. Thus

$$h_\mu(T, \mathscr{P}) = \lim_{n\to\infty} \frac{1}{n} h_\mu(\mathscr{P} \vee T^{-1}\mathscr{P} \vee \cdots \vee T^{-(n-1)}\mathscr{P}) \leq \lim_{n\to\infty} \frac{1}{n}(\log(2n)) = 0.$$

Thus $h_\mu(T_w) = 0$ for all $w \in \mathbb{T}$.

Example 5.1.2. Let T be the Bernoulli shift on $(X^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ where $X = \{x_1, x_2, \ldots, x_k\}$ and $\mu(\{x_j\}) = p_j$ for each $j = 1, 2, \ldots, k$ where $\sum_{j=1}^k p_j = 1$. For each $j = 1, 2, \ldots, k$ define $P_j = \{(y_i) \in X^{\mathbb{Z}} : y_0 = x_j\}$. Then the partition $\mathscr{P} = \{P_1, P_2, \ldots, P_k\}$ is generating for T. Then $\mathscr{P} \vee T^{-1}\mathscr{P} \vee \cdots \vee T^{-(n-1)}\mathscr{P} = \{(y_j) \in X^{\mathbb{Z}} : y_0 = x_0',\ y_1 = x_1', \ldots, y_{n-1} = x_{n-1}'\}$ for some $(x_0', x_1', \ldots, x_{n-1}') \in X^n$. Thus

$$\begin{aligned} h_\mu(\mathscr{P} \vee T^{-1}\mathscr{P} \vee \cdots \vee T^{-(n-1)}\mathscr{P}) &= -\sum_{i_1, i_2, \ldots, i_n} p_{i_1} p_{i_2} \cdots p_{i_n} \log(p_{i_1} p_{i_2} \cdots p_{i_n}) \\ &= -n \sum_{j=1}^k p_j \log(p_j). \end{aligned}$$

We now have

$$h_\mu(T) = h_\mu(T, \mathscr{P}) = \lim_{n\to} \frac{1}{n}\left(-n \sum_{j=1}^k p_j \log(p_j)\right) = -\sum_{j=1}^k p_j \log(p_j).$$

In the preceding example suppose $Y = \{y_1, y_2\}$. Set $p = p_1$ then $p_2 = 1 - p$. If T is the Bernoulli shift on Y we have

$$h_\mu(T) = -p \log(p) - (1 - p) \log(1 - p).$$

It is then clear, that we may construct Bernoulli shift such that $h_\mu(T) = h$ for any given $h \in (0, \log(2)]$. Furthermore, by varying the cardinality of Y and the measures of the elements of Y, we may construct a Bernoulli shift with entropy equal to d for any $d \in (0, \infty)$.

5.2. Generating Partitions

We briefly introduced generating partitions in the preceding subsection. We now extend this definition in the natural sense to actions of countable discrete groups. We then give a more analytic description of generating partitions that will be useful in the applications that follow. In particular, we show for a fixed partition $\mathscr{P}$ of (X, μ) the set of actions in $\mathrm{Act}(G, X, \mu)$ for which $\mathscr{P}$ is generating is a G_δ. In the case of the free group $\mathbb{F}_n$ we show this set is dense as well. Thus for a fixed partition $\mathscr{P}$ of (X, μ) the set of actions $(\mathbb{F}_n, X, \mu, \alpha)$ for which $\mathscr{P}$ is generating is generic.

Let $\alpha \in \mathrm{Act}(G, X, \mu)$ and $\mathscr{P}$ be a finite partition of X into measurable sets. Let $\sigma(\mathscr{P})$ be the smallest α-invariant σ-algebra containing the atoms of $\mathscr{P}$. As with the integer case, we say $\mathscr{P}$ is *generating for* α if $\sigma(\mathscr{P}) = \mathscr{X}$ modulo null sets, i.e. for all $A \in \mathscr{X}$ there exists $A' \in \sigma(\mathscr{P})$ such that $\mu(A \triangle A') = 0$. Given a finite subset $F \subset G$ define $\mathscr{P}_\alpha^F = \bigvee_{s \in F} \alpha_s \mathscr{P}$. When the action is clear we simply write $\mathscr{P}^F$. It is straight forward to check that $\mathscr{P}^F$ may be identified with a subset of functions from F into $\mathscr{P}$.

We now give a more analytic description of generating partitions. This description will be useful in showing the set of actions for which a fixed partition is generating is a G_δ.

Lemma 5.2.1. *Let $(X, \mathscr{B}, \mu)$ be a standard probability space and G a countable discrete group. Let $\mathscr{P}$ be a partition of X and $\alpha \in \mathrm{Act}(G, X, \mu)$. Then $\mathscr{P}$ is generating* *for α if and only if for all $A \in \mathscr{B}$ and $\epsilon > 0$ there exist finite subsets $F \subset G$ and $\mathcal{A} \subset \mathscr{P}^F$ such that $\mu(A \triangle (\bigcup \mathcal{A})) < \epsilon$.*

Proof. Suppose $\mathscr{P}$ is generating for α. Let $\mathscr{C}$ be the collection of all $A \in \sigma(\mathscr{P})$ such that for all $\epsilon > 0$ there exists finite subsets $F \subset G$ and $\mathcal{A} \subset \mathscr{P}^F$ satisfying $\mu(A \triangle (\bigcup \mathcal{A})) < \epsilon$. Let $A \in \mathscr{C}$, $s \in G$, and $\epsilon > 0$. Then there exist finite subsets F of G and $\mathcal{A} \subset \mathscr{P}^F$ such that $\mu(A \triangle (\bigcup \mathcal{A})) < \epsilon$. Let $F' = sF$ then

$$\mathscr{P}^{F'} = \bigvee_{t \in F'} \alpha_t \mathscr{P} = \bigvee_{t \in F} \alpha_s \alpha_t \mathscr{P} = \alpha_s \bigvee_{t \in F} \alpha_t \mathscr{P} = \alpha_s \mathscr{P}^F$$

whence $\alpha_s \mathcal{A} \subset \mathscr{P}^{F'}$. Thus

$$\begin{aligned} \mu((\alpha_s A) \triangle (\bigcup \alpha_s \mathcal{A})) &= \mu((\alpha_s A) \triangle (\alpha_s \bigcup \mathcal{A})) \\ &= \mu(\alpha_s (A \triangle (\bigcup \mathcal{A}))) \\ &= \mu(A \triangle (\bigcup \mathcal{A})) \\ &< \epsilon. \end{aligned}$$

Therefore $\mathscr{C}$ is α-invariant. Since $\mathscr{P}^F$ is finite, $\mathcal{B} = \mathscr{P}^F \setminus \mathcal{A}$ is finite as well. Moreover, $(\bigcup \mathcal{A})^c = \bigcup \mathcal{B}$, whence $\mu(A^c \triangle (\bigcup \mathcal{B})) = \mu(A^c \triangle (\bigcup \mathcal{A})^c) = \mu(A \triangle (\bigcup \mathcal{A})) < \epsilon$.

It remains to check that $\mathscr{C}$ is closed under countable unions. Suppose $\{A_n\}_{n=1}^{\infty}$ be a countable collection of subsets of $\mathscr{C}$. Without loss of generality we may assume $\{A_n\}_{n=1}^{\infty}$ is pairwise disjoint. Since X is a probability space, there exists $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} \mu(A_n) < \frac{\epsilon}{3}$. For each $n = 1, 2, \ldots, N$ there exists finite subsets $F_n \subset G$ and $\mathcal{A}_n \subset \mathscr{P}^{F_n}$ such that $\mu(A_n \triangle (\bigcup \mathcal{A}_n)) < \frac{\epsilon}{3N}$. Let $F = \bigcup_{n=1}^{N} F_n$. Since $F_n \subset F$ the partition $\mathscr{P}^F$ refines $\mathscr{P}^{F_n}$. In particular for each $A \in \mathcal{A}_n$ there exists $\mathcal{B}_A \subset \mathscr{P}^F$ such that $A = \bigcup \mathcal{B}_A$. Define $\mathcal{A} = \bigcup_{n=1}^{N} \bigcup_{A \in \mathcal{A}_n} \mathcal{B}_A$. Then

$$\bigcup \mathcal{A} = \bigcup_{n=1}^{N} \bigcup_{A \in \mathcal{A}_n} \mathcal{B}_{A_n} = \bigcup_{n=1}^{N} \bigcup \mathcal{A}_n$$

whence

$$\begin{aligned}
\mu\left(\left(\bigcup_{n=1}^{N} A_n\right) \triangle \left(\bigcup \mathcal{A}\right)\right) &= \mu\left(\left(\bigcup_{n=1}^{N} A_n\right) \triangle \left(\bigcup_{n=1}^{N} \bigcup \mathcal{A}_n\right)\right) \\
&\leq \sum_{n=1}^{N} \mu\left(A_n \triangle \left(\bigcup \mathcal{A}_n\right)\right) \\
&< \sum_{n=1}^{N} \frac{\epsilon}{3N} \\
&= \frac{\epsilon}{3}.
\end{aligned}$$

Let $A_N = \bigcup_{n=1}^{N}$, $A_\infty = \bigcup_{n=N+1}^{\infty}$ and $A = \bigcup \mathcal{A}$. Then

$$\begin{aligned}
\mu\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \triangle \left(\bigcup \mathcal{A}\right)\right) &= \mu((A_N \cup A_\infty) \triangle A) \\
&\leq \mu((A_N \cup A_\infty) \setminus A) + \mu(A \setminus (A_N \cup A_\infty)) \\
&= \mu((A_N \cap A^c) \cup (A_\infty \cap A^c)) + \mu(A \cap A^c \cap B^c) \\
&\leq \mu(A_N \cap A^c) + \mu(A_\infty \cap A^c) + \mu(A \cap A_N^c) \\
&\leq \mu(A_N \setminus A) + \mu(A_\infty) + \mu(A \setminus A_N) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

Thus $\mathscr{C}$ is a α-invariant σ-algebra. As it is clear that $\mathscr{C}$ contains the the atoms of $\mathscr{P}$ we must have $\sigma(\mathscr{P}) \subset \mathscr{C}$.

Conversely, suppose for any $A \in \mathscr{X}$, and $\epsilon > 0$ there exists a finite subset F of G and $A_1, A_2, \ldots, A_m \in \mathscr{P}^F$ such that $\mu(A \triangle (\bigcup_{i=1}^{m} A_i)) < \epsilon$. For each $n > 1$ choose $F_n \subset G$ and $\mathcal{B}_n \subset \mathscr{P}_n^F$ such that $\mu(A \triangle B_n) < \frac{1}{2^{n+1}}$ where $B_n = \bigcup \mathscr{B}_n$. Then for each $k \geq 1$,

$$\mu\left(A \triangle \bigcup_{n=k}^{\infty} B_n\right) = \mu\left(\bigcup_{n=k}^{\infty} (A \triangle B_n)\right) \leq \sum_{n=k}^{\infty} \mu(A \triangle B_n) \leq \sum_{n=k}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2^k}.$$

Then $A' = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n \in \sigma(\mathscr{P})$ satisfies

$$\begin{aligned}
\mu(A \triangle A') &= \mu(A \triangle (\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n)) \\
&= \mu(\bigcap_{k=1}^{\infty} \mu(A \triangle (\bigcup_{n=k}^{\infty} B_n)) \\
&= \lim_{k \to \infty} \mu(A \triangle (\bigcup_{n=k}^{\infty} B_n)) \\
&\leq \lim_{k \to \infty} \frac{1}{2^k} \\
&= 0.
\end{aligned}$$

Therefore $\mathscr{P}$ is generating for α. $\square$

Given a fixed finite partition $\mathscr{P}$ of X define by $\mathrm{Gen}(\mathscr{P}, G, X, \mu)$ the set all $\alpha \in \mathrm{Act}(G, X, \mu)$ such that $\mathscr{P}$ is generating for α. Given $A \in \mathscr{X}$ and $\epsilon > 0$ define $U(A, \epsilon) \subset \mathrm{Act}(G, X, \mu)$ by

$$U(A, \epsilon) = \{\alpha : \mu(A \triangle (\bigcup \mathcal{A})) < \epsilon \text{ for some finite subsets } F \subset G \text{ and } \mathcal{A} \subset \mathscr{P}_{\alpha}^{F}\}.$$

If $U(A, \epsilon)$ is empty, then it is open. Suppose $U(A, \epsilon)$ is nonempty. Let $\alpha \in U(A, \epsilon)$. Then there exists a finite subsets $F \subset G$, and $\mathcal{A} \subset \mathscr{P}_{\alpha}^{F}$, and $\delta > 0$ such that $\mu(A \triangle (\bigcup \mathcal{A})) + \delta < \epsilon$. Let $\beta \in U(\alpha, F, \mathscr{P}, \frac{\delta}{|F||\mathcal{A}|})$. For each $A' \in \mathcal{A}$ let $\phi_{A'} : F \to \mathscr{P}$ be such that $A' = \bigcap_{g \in F} \alpha_g \phi_{A'}(g)$. Define $B_{A'} = \bigcap_{g \in F} \beta_g \phi_{A'}(g)$. Then

$$\begin{aligned}
\mu(A' \triangle B_{A'}) &= \mu(\bigcap_{g \in F} \alpha_g \phi_{A'}(g) \triangle \bigcap_{g \in F} \beta_g \phi_{A'}(g)) \\
&\leq \sum_{g \in F} \mu(\alpha_g \phi_{A'}(g) \triangle \beta_g \phi_{A'}(g)) \\
&< \sum_{g \in F} \frac{\delta}{|F||\mathcal{A}|} \\
&= \frac{\delta}{|\mathcal{A}|}.
\end{aligned}$$

Let $\mathcal{B} = \{B_{A'} : A' \in \mathcal{A}\} \subset \mathscr{P}^F_\beta$. Then

$$\begin{aligned}
\mu(A\triangle(\bigcup\mathcal{B})) &\leq \mu(A\triangle(\bigcup\mathcal{A})) + \mu((\bigcup\mathcal{A})\triangle(\bigcup\mathcal{B})) \\
&= \mu(A\triangle(\bigcup\mathcal{A})) + \mu((\bigcup_{A'\in\mathcal{A}} A')\triangle(\bigcup_{A'\in\mathcal{A}} B_{A'})) \\
&\leq \mu(A\triangle(\bigcup\mathcal{A})) + \sum_{A'\in\mathcal{A}} \mu(A'\triangle B_{A'}) \\
&< \mu(A\triangle(\bigcup\mathcal{A})) + \sum_{A'\in\mathcal{A}} \frac{\delta}{|\mathcal{A}|} \\
&= \mu(A\triangle(\bigcup\mathcal{A})) + \delta \\
&< \epsilon.
\end{aligned}$$

Thus $U(\alpha, F, \mathscr{P}, \frac{\delta}{|F||\mathcal{A}|}) \subset U(A, \epsilon)$, i.e. $U(A, \epsilon)$ is open.

Let $\{A_n\}_{n=1}^\infty \subset \mathscr{X}$ be such that the sequence of equvalence classes $\{\widetilde{A_n}\}_{n=1}^\infty$ is dense in the measure algebra $\widetilde{X}$ of X. Clearly $\mathrm{Gen}(\mathscr{P}, G, X, \mu) \subset \bigcap_{n=1}^\infty \bigcap_{m=1}^\infty U(A_n, \frac{1}{m})$. Suppose $\alpha \in \bigcap_{n=1}^\infty \bigcap_{m=1}^\infty U(A_n, \frac{1}{m})$. Let $A \in \mathscr{X}$ and $\epsilon > 0$. Choose $m, n \in \mathbb{N}$ such that $\mu(A\triangle A_n) < \frac{\epsilon}{2}$ and $\frac{1}{m} < \frac{\epsilon}{2}$. Since $\alpha \in U(A_n, \frac{1}{m})$ there exist finite subsets $F \subset G$ and $\mathcal{A} \subset \mathscr{P}^F$ such that $\mu(A_n\triangle(\bigcup\mathcal{A})) < \frac{1}{m} < \frac{\epsilon}{2}$. Then

$$\mu(A\triangle(\bigcup\mathcal{A})) \leq \mu(A\triangle A_n) + \mu(A_n\triangle(\bigcup\mathcal{A})) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whence it follows from Lemma 5.2.1 that $\alpha \in \mathrm{Gen}(\mathscr{P}, G, X, u)$. Thus we have proved the following

Proposition 5.2.2. *Let G be a countable group and (X, μ) a standard probability space. Let $\mathscr{P}$ be a fixed finite partition of X. Then* $\mathrm{Gen}(\mathscr{P}, G, X, \mu)$ *is a G_δ set.*

In general, given a partition $\mathscr{P}$ there is no guarantee that $\mathrm{Gen}(\mathscr{P}, G, X, \mu)$ is nonempty. However, if G is assumed to be $\mathbb{F}_n$ we will show that $\mathrm{Gen}(\mathscr{P}, G, X, \mu)$ is not only nonempty, but is in fact a dense for all finite partitions $\mathscr{P}$ of X. Unfortunately the proof relies on perturbing the action on a generator to what is known as a prime transformation and does not translate well to other groups.

Let $(X, \mathscr{X}, \mu)$ and $(X', \mathscr{X}', \mu')$ be standard probability spaces, G a countable discrete group and $\alpha \in \mathrm{Act}(G, X, \mu)$. We say $\alpha' \in \mathrm{Act}(G, X', \mu')$ is a *factor* of α if there exists measurable sets $X_0 \subset X$ and $X_0' \subset X'$ of full measure satisfying $\alpha_g X_0 \subset X$ and $\alpha_g' X_0' \subset X'$ for all $g \in G$ and a measurable onto map $\phi : X \to X'$ satisfying $\phi \circ \alpha_g = \alpha_g' \circ \phi$ for all $g \in G$ and $\mu\phi^{-1} = \mu'$. If α has no proper factors, α is said to be *prime*. It can be shown that the factors of α correspond to the α-invariant sub-σ-algebras of $\mathscr{X}$ (see [34] p61 for the case when $G = \mathbb{Z}$). Suppose α is prime and $\mathscr{P}$ is any finite partition of X. Then $\sigma(\mathscr{P})$ is an α-invariant sub-σ-algebra of $\mathscr{X}$ and hence must be equal to $\mathscr{X}$ modulo null sets. That is, if α is prime then every finite partition of X is generating.

For the case of single operators, several examples of prime transformations are known, see [23], [10] or [32] for example. In each of these examples, the prime transformation is ergodic and hence has dense conjugacy class. We can now show $\mathrm{Gen}(\mathscr{P}, \mathbb{F}_n, X, \mu)$ is nonempty and dense for all finite partitions $\mathscr{P}$ of X.

Proposition 5.2.3. *For each finite measurable partion $\mathscr{P}$ of a standard probability space (X, μ),* $\mathrm{Gen}(\mathscr{P}, \mathbb{F}_n, X, \mu)$ *is generic.*

Proof. Let $\mathscr{P}$ be a finite measurable partition of (X, μ). From Proposition 5.2.2 we know $\mathrm{Gen}(\mathscr{P}, \mathbb{F}_n, X, \mu)$ is a G_δ set. Thus is remains to check $\mathrm{Gen}(\mathscr{P}, \mathbb{F}_n, X, \mu)$ is nonempty and dense. Let $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$, $\epsilon > 0$, and $\mathcal{A} \subset \mathscr{X}$ and $S' \subset S$ be finite subsets. Let $s \in S'$ be fixed but arbitrary. By density of the prime transformations for $\mathbb{Z}$-systems there exists a prime transformation T such $\mu(TA \triangle \alpha_s A) < \epsilon$ for all

$A \in \mathcal{A}$. Define an action of $\mathbb{F}_n$ on X by

$$\alpha'_t x = \begin{cases} Tx & \text{if } t = s \\ \alpha_t x & \text{otherwise} \end{cases}$$

for each $t \in S'$. It is clear that $\alpha' \in U(\alpha, S', \mathcal{A}, \epsilon)$ whence it remains to check that $\mathscr{P}$ is generating for α'. Recall $\sigma(\mathscr{P})$ is the smallest α' invariant σ-algebra containing the atoms of $\mathscr{P}$. Thus $\sigma(\mathscr{P})$ is also α'_s invariant and thus equal to $\mathscr{X}$ modulo null sets. That is, $\alpha' \in \mathrm{Gen}(\mathscr{P}, \mathbb{F}_n, X, \mu)$ completing the proof. $\square$

5.3. Measure-Theoretic Entropy of Actions of Sofic Groups

Suppose G is a countable group and $m \geq 1$. Denote by $\mathrm{Sym}(m)$ the full symmetric group on $\{1, 2, \ldots, m\}$. Given a map σ from G into $\mathrm{Sym}(m)$ and a finite subset $F \subset G$ we define $V_\sigma(F)$ to be the set of all $v \in \{1, 2, \ldots, m\}$ such that for all $s, t \in F$ we have $\sigma(s)\sigma(t)v = \sigma(st)v$ and $\sigma(s)v \neq \sigma(t)v$ if $s \neq t$. If $|V_\sigma(F)| \geq (1 - \epsilon)m$ we say σ is an *(F, ϵ)-approximation to G*. A sequence $\{\sigma_i\}_{i=1}^\infty$ of maps $\sigma_i : G \to \mathrm{Sym}(m_i)$ is called a *sofic approximation to G* if each σ_i is an (F_i, ϵ_i)-approximation to G for some sequence $\{F_i\}_{i=1}^\infty$ of finite subsets of G and sequence $\{\epsilon_i\}_{i=1}^\infty$ of positive real numbers satisfying $F_i \subset F_{i+1}$ for all $i \in \mathbb{N}$, $\bigcup_{i \in \mathbb{N}} F_i = G$ and $\epsilon_i \to 0$ as $i \to \infty$. We say G is a *sofic group* if there exists a sofic approximation to G.

We wish to verify that $\mathbb{F}_n$ is in fact a sofic group. Unfortunately, a direct proof is not obvious. Instead we introduce the class of residually finite groups and show every residually finite group is sofic. We say a group G is *residually finite* if one of the following equivalent conditions hold:

- for all $s \in G$ there exists a finite group and a homomorphism $\phi : G \to K$ such that $\phi(s) \neq e$,

- there exists a chain $G = H_1 \supset H_2 \supset H_3 \supset \cdots$ of normal subsets of G such that $\bigcap_{i\in\mathbb{N}} H_i = \{e\}$ and $|G/H_i| < \infty$ for each $i \in \mathbb{N}$.

It is known fhat every residually finite group and amenable group is sofic [36]. Currently, it is unknown if all countable groups are sofic or not.

It is well known that $\mathbb{F}_n$ is residually finite and thus sofic. The standard proof makes use of the fact that subgroups of residually finite group are residually finite and that $\mathbb{F}_n$ embeds into $\mathbb{F}_2$ which embeds into $SL_2(\mathbb{Z})$, the latter of which is residually finite. We use the results of Section 3 to give a dynamical proof that $\mathbb{F}_n$ is residually finite.

Proposition 5.3.1. *Let G be a countable discrete group. Suppose the set of $\alpha \in \mathrm{Act}(G, X, \mu)$ which factor through a finite group are weakly dense in $\mathrm{Act}(G, X, \mu)$ for some standard probability space (X, μ). Then G is residually finite.*

Proof. Let α be the Bernoulli shift on (X^G, μ^G) for some standard probability space (X, μ). Let $s \in G$ have infinite order. Let $A \subset X$ be such that measure $0 < \mu(A) < 1$. Then there exists a positive integer m such that $\mu(\alpha_{s^m} A \triangle A) > \epsilon$ for some $\epsilon > 0$. By assumption, there exists $\alpha' \in U(\alpha, \{A\}, \{s, s^2, \ldots, s^m\}, \epsilon)$ such that α' factors through some finite group F, i.e. there exists $\phi : G \to F$ and $\beta \in \mathrm{Act}(G, X^G, \mu^G)$ such that $\alpha'_s = \beta_{\phi(s)}$ for all $s \in G$. Suppose $\phi(s) = e$. Then $\phi(s^j) = e$ for all $j \in \mathbb{N}$. In partitcular, we have

$$\mu(\alpha_{s^m} A \triangle A) = \mu(\alpha_{s^m} A \triangle \beta_e A) = \mu(\alpha_{s^m} A \triangle \beta_{\phi(s^m)} A) = \mu(\alpha_{s^m} A \triangle \alpha'_{s^m} A) < \epsilon,$$

a contradiction. Thus $\phi(s) \neq e$ completing the proof. $\square$

We now have the following corollary:

Corollary 5.3.2. *$\mathbb{F}_n$ is residually finite and thus sofic.*

Let G be a countable discrete group, (X,μ) a standard probability space and $\alpha \in \mathrm{Act}(G,X,\mu)$. Suppose $\mathscr{P} = \{P_1, P_2, \ldots\}$ is an ordered partition of X. Let $\sigma : G \to \mathrm{Sym}(m)$ be a map, ν be the uniform probability measure on $\{1,2,\ldots,m\}$ and $\mathscr{R} = \{R_1, R_2, \ldots\}$ be a partition of $\{1,2,\ldots,m\}$. Let F be a finite subset of G. Given a function $\phi : F \to \mathbb{N}$, let $P^\alpha_\phi = \bigcap_{g\in F} \alpha_g P_{\phi(g)}$ and $R^\sigma_\phi = \bigcap_{g\in F} \sigma(g) R_\phi(g)$ and define

$$d^\alpha_{F,\sigma}(\mathscr{P},\mathscr{R}) = \sum_{\phi:F\to\mathbb{N}} |\mu(P^\alpha_\phi) - \nu(R^\sigma_\phi)|.$$

If $\mathscr{P}$ is finite, then for $\epsilon > 0$, let $\mathcal{AP}_\sigma(\alpha, \mathscr{P} : F, \epsilon)$ be the set of all ordered partitions $\mathscr{R}$ of $\{1,2,\ldots,m\}$ with the same number of atoms as $\mathscr{P}$ satisfying $d^\alpha_{F,\sigma}(\mathscr{P},\mathscr{R}) < \epsilon$. The set $\mathcal{AP}_\sigma(\alpha, \mathscr{P} : F, \epsilon)$ is called the *set of approximating partitions*.

Suppose G is a sofic group with sofic approximation $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(m_i)\}_{i=1}^\infty$, and $\alpha \in \mathrm{Act}(G,X,\mu)$. Given a finite partition $\mathscr{P}$ of X we define the *mean Σ-entropy of $\mathscr{P}$*, denoted $h_{\Sigma,\mu}(\alpha,\mathscr{P})$, as follows. For every $\epsilon > 0$ and finite subset $F \subset G$ define

$$h_{\Sigma,\mu}(\alpha, \mathscr{P} : F, \epsilon) = \limsup_{i\to\infty} \frac{1}{m_i} \log |\mathcal{AP}_{\sigma_i}(\alpha, \mathscr{P} : F, \epsilon)|.$$

If $\mathcal{AP}(\sigma_i, \mathscr{P} : F, \epsilon)$ is empty then we interpret $\log(0) = -\infty$. We then define

$$h_{\Sigma,\mu}(\alpha, \mathscr{P} : F) = \inf_{\epsilon>0} h_{\Sigma,\mu}(\alpha, \mathscr{P} : F, \epsilon)$$

and

$$h_{\Sigma,\mu}(\alpha, \mathscr{P}) = \inf_{F\subset G} h_{\Sigma,\mu}(\alpha, \mathscr{P} : F)$$

where the latter infimum is taken over all nonempty finite subsets $F \subset G$. Since $\epsilon > \delta$ implies $\mathcal{AP}_\sigma(\alpha, \mathscr{P} : F, \epsilon) \supset \mathcal{AP}_\sigma(\alpha, \mathscr{P} : F, \delta)$, the infimum defining $h_{\Sigma,\mu}(\alpha, \mathscr{P} : F)$ may be replaced by the limit as ϵ decreases to zero. Similarly if $F \subset F'$ it follows

$\mathcal{AP}_\sigma(\alpha, \mathscr{P} : F, \epsilon) \supset \mathcal{AP}_\sigma(\alpha, \mathscr{P} : F', \epsilon)$ whence

$$h_{\Sigma,\mu}(\alpha, \mathscr{P}) = \lim_{n\to\infty} h_{\Sigma,\mu}(\alpha, \mathscr{P} : F_n)$$

where $\{F_n\}_{n=1}^{\infty}$ in an increasing sequence of finite subsets of G which exhausts G. If $\mathcal{AP}_\sigma(\alpha, \mathscr{P} : F, \epsilon)$ is empty for all sufficiently large i we define $h_{\Sigma,\mu}(\alpha, \mathscr{P}) = -\infty$.

When there exists a generating partition $\mathscr{P}$ satisfying $H_\mu(\mathscr{P}) < \infty$ we define the Σ-*entropy*, denoted $h_{\Sigma,\mu}(\alpha)$, by

$$h_{\Sigma,\mu}(\alpha) = h_{\Sigma,\mu}(\alpha, \mathscr{P}).$$

It is shown in [5] that if $\mathscr{P}$ and $\mathscr{R}$ are generating partitions having finite entropy then $h_{\Sigma,\mu}(\alpha, \mathscr{P}) = h_{\Sigma,\mu}(\alpha, \mathscr{R})$. Thus the Σ-entropy of α is well defined if there exists a generating partition with finite entropy. If no such partition exists, $h_{\Sigma,\mu}(\alpha)$ is undefined.

Although $h_{\Sigma,\mu}(\alpha)$ may not be defined for an action, the value $h_{\Sigma,\mu}(\alpha, \mathscr{P})$ is defined for any finite ordered partition. Thus for a fixed ordered partition $\mathscr{P}$ and $\delta > 0$ we may consider the the set

$$H_{\Sigma,\delta}(\mathscr{P}, G, X, \mu) = \{\alpha \in \mathrm{Act}(G, X, \mu) : h_{\Sigma,\mu}(\alpha, \mathscr{P}) < \delta\}.$$

It is shown in [5], that for a Bernoulli shift α, we have $h_{\Sigma,\mu}(\alpha, \mathscr{P}) = H_\mu(\mathscr{P})$, whence $H_\delta(\mathscr{P}, G, X, \mu)$ is nonempty for each $\delta > 0$. Suppose $\alpha \in H_\delta(\mathscr{P}, G, X, \mu)$. Choose a finite subset $F \subset G$ and $\epsilon > 0$ such that $h_{\Sigma,\mu}(\alpha, \mathscr{P} : F, \frac{\epsilon}{2}) < \delta$. Suppose $\beta \in U(\alpha, F, \mathscr{P}, \epsilon')$ where $0 < \epsilon' < \frac{\epsilon}{2|F||\mathscr{P}|^{|F|}}$. If $\mathcal{AP}_{\sigma_i}(\beta, \mathscr{P} : F, \epsilon) = \emptyset$ for all sufficiently large i then $h_{\Sigma,\mu}(\beta, \mathscr{P}) = -\infty$ whence $\beta \in H_{\Sigma,\delta}(\mathscr{P}, G, X, \mu)$. Given $\phi : F \to \mathbb{N}$ we have

$$\mu(P_\phi^\alpha \triangle P_\phi^\beta) = \mu(\bigcap_{s\in F} \alpha_s P_{\phi(s)} \triangle \bigcap_{s\in F} \beta_s P_{\phi(s)})$$

$$
\begin{aligned}
&\leq \mu(\bigcup_{s\in F} \alpha_s P_{\phi(s)} \triangle \alpha_s P_{\phi(s)}) \\
&\leq \sum_{s\in F} \mu(\alpha_s P_{\phi(s)} \triangle \alpha_s P_{\phi(s)}) \\
&< |F|\epsilon'.
\end{aligned}
$$

Thus for $\mathscr{R} \in \mathcal{AP}_{\sigma_i}(\beta, \mathscr{P} : F, \epsilon)$ we have that

$$
\begin{aligned}
d^{\alpha}_{F,\sigma_i}(\mathscr{P}, \mathscr{R}) &= \sum_{\phi: F\to\mathbb{N}} |\mu(P^{\alpha}_{\phi}) - \nu(R^{\sigma_i}_{\phi})| \\
&\leq \sum_{\phi: F\to\mathbb{N}} |\mu(P^{\beta}_{\phi}) - \nu(R^{\sigma_i}_{\phi})| + \sum_{\phi: F\to\mathbb{N}} |\mu(P^{\alpha}_{\phi}) - \mu(P^{\beta}_{\phi})| \\
&\leq d^{\beta}_{F,\sigma_i}(\mathscr{P}, \mathscr{R}) + \sum_{\phi: F\to\mathbb{N}} \mu(P^{\alpha}_{\phi} \triangle P^{\beta}_{\phi}) \\
&< \frac{\epsilon}{2} + |\mathscr{P}|^{|F|}|F|\epsilon' \\
&< \epsilon.
\end{aligned}
$$

Thus we have $\mathcal{AP}_{\sigma_i}(\beta, \mathscr{P} : F, \frac{\epsilon}{2}) \subset \mathcal{AP}_{\sigma_i}(\alpha, \mathscr{P} : F, \epsilon)$ for each $i \in \mathbb{N}$, whence

$$h_{\Sigma,\mu}(\beta, \mathscr{P} : F, \frac{\epsilon}{2}) < h_{\Sigma,\mu}(\alpha, \mathscr{P} : F, \epsilon) < \delta.$$

Since $h_{\Sigma,\mu}(\beta, \mathscr{P} : F, \epsilon)$ is nonincreasing if either ϵ decreases or F increases we have $h_{\Sigma,\mu}(\alpha, \mathscr{P}) < \delta$. In particular, $H_{\Sigma,\delta}(\mathscr{P}, G, X, \mu)$ is nonempty and open for each $\delta > 0$. Define $H_{\Sigma,0}(\mathscr{P}, G, X, \mu) = \{\alpha \in \mathrm{Act}(G, X, \mu) : h_{\Sigma,\mu}(\alpha, \mathscr{P}) \in \{0, -\infty\}\}$. Then for any sequence $\{\delta_i\}_{i=1}^{\infty}$ decreasing to 0 we have $H_{\Sigma,0}(\mathscr{P}, G, X, \mu) = \bigcap_{i=1}^{\infty} H_{\Sigma,\delta_i}(\mathscr{P}, G, X, \mu)$, i.e. $H_{\Sigma,0}(\mathscr{P}, G, X, \mu)$ is a G_δ.

Suppose $\alpha \in \mathrm{Act}(G, X, \mu)$ factors through a finite group F. Let $\phi : G \to F$ and $\beta \in \mathrm{Act}(F, X, \mu)$ be such that $\alpha_s = \beta_{\phi(s)}$ for all $s \in G$. Then by pigeonhole principle there exist $s, t \in G$ such that $s \neq t$ and $\phi(s) = \phi(t)$. Alternatively, there exists a nontrivial $s \in G$ such that $\alpha_s = \alpha_e$. If s has infinite order, we claim this latter condition is sufficient to show $h_{\Sigma,\mu}(\alpha, \mathscr{P}) \in \{0, -\infty\}$.

Lemma 5.3.3. *Let G be a sofic group with sofic approximation Σ, $\mathscr{P} = \{P_1, P_2, \ldots, P_d\}$ be an ordered partition of X and $\alpha \in \mathrm{Act}(G, X, \mu)$. Suppose there exists $s \in G$ of infinite order such that $\alpha_s = \alpha_e$. Then $h_{\Sigma,\mu}(\alpha, \mathscr{P}) \leq 0$.*

Proof. Suppose $h_{\Sigma,\mu}(\alpha, \mathscr{P}) \geq 0$. Let $F = \{s, e\}$ and $0 < \epsilon < \frac{1}{2}$. Fix $n \in \mathbb{N}$ such that $\frac{1}{n}\log(d) < \epsilon$. Let $I \in \mathbb{N}$ be such that $\{e, s^{-1}, s^{-2}, \ldots, s^{-n}\} \subset F_i$ for all $i \geq I$. Let $i \geq I$ be fixed but arbitrary. Let $\mathcal{R}$ be the set of all ordered partitions of $\{1, 2, \ldots, m_i\}$ having d atoms. Define $\psi : \mathcal{R} \to (\{0, 1, 2, \ldots, d\} \cup \{\star\})^{\{1,2,\ldots,m_i\}}$ by

$$\psi(\mathscr{R})(v) = \begin{cases} \star & \text{if } v \in \{1, 2, \ldots, m_i\} \setminus V_{\sigma_i}(F_i), \\ j & \text{if } \sigma_i(s)^{-1}v \in R_j \text{ and } v \notin R_j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f \in \psi(\mathcal{R})$ and suppose $|\bigcup_{j=1}^{d} f^{-1}(\{j\})| > m_i\epsilon$. Then for $\mathscr{R} \in \phi^{-1}(\{f\})$ it follows that

$$\begin{aligned} d_F(\mathscr{P}, \mathscr{R}) &= \sum_{\phi \in \{1,2,\ldots,d\}^F} |\mu(\alpha_e P_{\psi(e)} \cap \alpha_s P_{\psi(s)}) - \nu(\sigma_i(e) R_{\psi(e)} \cap \sigma_i(s) R_{\psi(s)})| \\ &= \sum_{1 \leq j,k \leq d} |\mu(P_j \cap P_k) - \nu(R_j \cap \sigma_i(s) R_k)| \\ &\geq \sum_{1 \leq j \neq k \leq d} \nu(R_j \cap \sigma_i(s) R_k) \\ &= \nu\left(\bigcup_{1 \leq j \neq k \leq d} R_j \cap \sigma_i(s) R_k\right) \\ &\geq \nu\left(\bigcup_{j=1}^{d} f^{-1}(\{j\})\right) \\ &> \frac{1}{m_i}\epsilon m_i \\ &= \epsilon. \end{aligned}$$

Thus $\mathscr{R} \notin \mathcal{AP}_{\sigma_i}(\mathscr{P}, \alpha, F, \epsilon)$. In particular, $\mathcal{AP}_{\sigma_i}(\mathscr{P}, \alpha, F, \epsilon) \subset \bigcup_{f \in \mathcal{F}} \psi^{-1}(f)$ where $\mathcal{F} = \{f \in \psi(\mathcal{R}) : |\bigcup_{j=1}^{d} f^{-1}(\{j\})| \leq m_i\epsilon\}$.

Let $f \in \mathcal{F}$ and define

$$V_\star(f) = \{v : f(v) = \star\},$$
$$V'(f) = \{v \in V_{\sigma_i}(F_i) : f(\sigma_i(s)^{-j}v) = 0 \text{ for all } j \in \mathbb{N}\},$$
$$V''(f) = V_{\sigma_i}(F_i) \setminus V'(f).$$

Suppose $v \in V''(f)$. Then there exists a positive integer j such that $f(\sigma(s)^{-j}v) \neq 0$. Let j_v be the smallest such positive integer. For each $j = 1, 2, \ldots, d$ define $R''_j = \{v \in V''(f) : f(\sigma(s)^{-j_v}v) = j\}$. Suppose $\mathscr{R} \in \psi^{-1}(f)$. Clearly $R''_j \subset R_j \cap V''(f)$. Suppose $v \in R_j \cap V''(f)$. Then $v \in R_j$ and $f(\sigma(s)^{-j_v}v) \neq 0$. If $f(\sigma(s)^{-j_v}v) \neq j$ then $v \notin R_j$. Thus $R_j \cap V''(f) = R''_j$ for all $\mathscr{R} \in \psi^{-1}(f)$. Suppose $v \in V'(f)$. Since $\{e, s^{-1}, s^{-2}, \ldots, s^{-n}\} \subset F_i$, v is contained in a subcycle of length greater than n. Since $\sigma_i(s)^{-j}v = 0$ for all $j \in \mathbb{N}$ this cycle must be entrirely contained in one of the d atoms of $\mathscr{R}$. As there are at most $\frac{m_i}{n}$ subcycles of $\sigma_i(s)$ we have at most $d^{\frac{m_i}{n}}$ ways of partitioning $V'(f)$. Note that $V_\star(f) = \{1, 2, \ldots, m_i\} \setminus V_{\sigma_i}(F_i)$, whence $|V_\star(f)| < m_i\epsilon_i$. Thus allowing for all possibilities on $V_\star(f)$ we have at most $d^{\epsilon_i m_i}$ possible ways of partitioning $V_\star(f)$. Thus we have

$$|\psi^{-1}(f)| < d^{\epsilon_i m_i} d^{\frac{m_i}{n}} = d^{m_i(\epsilon_i + \frac{1}{n})}$$

for each $f \in \mathcal{F}$.

To bound $|\mathcal{F}|$ note that

$$\begin{aligned} |\mathcal{F}| &= |\{f \in \psi(\mathcal{R}) : |\bigcup_{j=1}^{d} f^{-1}(\{j\})| \leq m_i\epsilon\}| \\ &= \sum_{l=i}^{m_i\epsilon} |\{f \in \psi(\mathcal{R}) : |\bigcup_{j=1}^{d} f^{-1}(\{j\})| = l\}| \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=i}^{m_i\epsilon} \binom{m_i}{l} d^l \\
&\leq d^{m_i\epsilon} \sum_{l=i}^{m_i\epsilon} \binom{m_i}{l} \\
&\leq d^{m_i\epsilon}(m_i\epsilon)\binom{m_i}{m_i\epsilon}
\end{aligned}
$$

Where the last inequality holds since $m_i\epsilon < \frac{m_i}{2}$. We now have that

$$\log|\mathcal{AP}_{\sigma_i}(\mathscr{P}, \alpha, F, \epsilon)| \leq \log(m_i\epsilon) + m_i(\epsilon + \epsilon_i + \frac{1}{n})\log(d) + \log\binom{m_i}{m_i\epsilon}$$

Applying Stirling's formula we have for large m_i that

$$
\begin{aligned}
\log\binom{m_i}{m_i\epsilon} &\approx \log\frac{\left(\frac{m_i}{e}\right)^{m_i}\sqrt{2\pi m_i}}{\left(\frac{m_i\epsilon}{e}\right)^{m_i\epsilon}\sqrt{2\pi m_i\epsilon}\left(\frac{(1-\epsilon)m_i}{e}\right)^{(1-\epsilon)m_i}\sqrt{2\pi(1-\epsilon)m_i}} \\
&= \log\left(\sqrt{\frac{1}{2\pi\epsilon(1-\epsilon)m_i}}\left(\frac{1-\epsilon}{\epsilon}\right)^{m_i\epsilon}\frac{1}{1-\epsilon}\right) \\
&= -\frac{1}{2}\log(2\pi\epsilon(1-\epsilon)m_i) + m_i\epsilon\log\left(\frac{1-\epsilon}{\epsilon}\right) + \log\frac{1}{1-\epsilon}.
\end{aligned}
$$

It then follows that

$$h_{\Sigma,\mu}(\alpha, \mathscr{P} : F, \epsilon) = \limsup_{i\to\infty}\frac{1}{m_i}\log|\mathcal{AP}(\sigma_i, \mathscr{P} : F, \epsilon)| \leq \epsilon + \epsilon\log(d) + \epsilon\log\left(\frac{1-\epsilon}{\epsilon}\right)$$

whence

$$0 \leq h_{\Sigma,\mu}(\alpha, \mathscr{P}, F) = \lim_{\epsilon\to 0} h_{\Sigma,\mu}(\alpha, \mathscr{P} : F, \epsilon) \leq \lim_{\epsilon\to 0}\left[\epsilon + \epsilon\log(d) + \epsilon\log\left(\frac{1-\epsilon}{\epsilon}\right)\right] = 0.$$

It then follows directly that $h_{\Sigma,\mu}(\alpha, \mathscr{P}) = 0$, as desired. □

Unfortunately if α factors through a finite group F, then it is not generating for any finite partition $\mathscr{P}$ of X. Indeed, it follows there exists $\epsilon > 0$ such that for any finite subset $F \subset G$ any nonempty element of $\mathscr{P}^F$ has measure greater than ϵ. But then for sets of sufficiently small measure Lemma 5.2.1 fails. Thus Lemma 5.3.3 only shows that for a fixed finite ordered partition the set of actions

$$\{\alpha \in \mathrm{Act}(G, X, \mu) : h_{\Sigma,\mu}(\alpha, \mathscr{P}) \in \{0, -\infty\}\}$$

is generic. In general, this is not sufficient to conclude the set of actions with Σ-entropy either equal to 0 or negative infinity is generic. However, we do have

$$h_{\Sigma,0}(G, X, \mu) \supset \mathrm{Gen}(\mathscr{P}, G, X, \mu) \cap H_{\Sigma,0}(\mathscr{P}, G, X, \mu)$$

for any finite partition $\mathscr{P}$. Thus we have the following theorem:

Theorem 5.3.4. *Let G be a sofic group with sofic approximation Σ. Suppose the set of actions in $\mathrm{Act}(G, X, \mu)$ which factor through a finite group are dense and $\mathrm{Gen}(\mathscr{P}, G, X, \mu)$ is a dense G_δ for some finite partition $\mathscr{P}$ of X. Then $H_{\Sigma,0}(G, X, \mu)$ is generic.*

Corollary 5.3.5. *$H_{\Sigma,0}(\mathbb{F}_n, X, \mu)$ is generic.*

Proof. This follows from Theorems 3.3.3 and 5.3.4 and Proposition 5.2.3. $\square$

6. A CASE OF CONNES' EMBEDDING PROBLEM

6.1. von Neumann Algebras and the Crossed Product

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a subspace of $\mathscr{B}(\mathcal{H})$. Define the *commutant*, $\mathcal{M}'$, of $\mathcal{M}$ by $\mathcal{M}' = \{B \in \mathscr{B}(\mathcal{H}) : AB = BA \text{ for all } A \in \mathcal{M}\}$. If $\mathcal{M}$ satisfies one of the following equivalent conditions

1. $\mathcal{M}$ is closed in the weak topology,
2. $\mathcal{M}$ is closed in the strong topology,
3. $\mathcal{M} = \mathcal{M}'' = (\mathcal{M}')'$

it is said to be *von Neumann algebra (acting on $\mathcal{H}$).* We say a von Neumann algebra is a *factor* if $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}1$. For the remainder of this subsection we fix a von Neuman algebra $\mathcal{M}$ acting on some Hilbert Space $\mathcal{H}$.

A group action α of a countable discrete group G on $\mathcal{M}$ is said to be an *action by automorphisms* on $\mathcal{M}$ if for each $s \in G$ we have α_s is a $*$-automorphishms of $\mathcal{M}$. We say α is *free* if for all $g \neq e$ we have $yx = x\alpha_g(y)$ for all $y \in \mathcal{M}$ implies $x = 0$. If the algebra of fixed points $\mathcal{M}^G$ is equal to $\mathbb{C}\mathbf{1}$ then α is said to be *ergodic*. Given an action α of G on $\mathcal{M}$ we devote the remainder of this subsection to constructing a von Neumann algebra $\mathcal{M} \rtimes_\alpha G$ acting on the Hilbert space $\mathcal{H} \otimes \ell^2(G)$ called the crossed product.

For any locally compact group G, there exists a unique Haar measure μ on the Borel Subsets of G. Denote by $C_c(G)$ the vector space of complex valued functions on G having compact support. We then define $L^2(G,\mu)$ as the completion $C_c(G)$ with respect to the inner product $\langle f, g\rangle = \int f\overline{g}d\mu$. If G is a countable discrete group, as is our case, then μ is the counting measure and $L^2(G,\mu) = \ell^2(G)$.

When defining the crossed product it is be convenient to view elements of $\mathcal{H} \otimes \ell^2(G)$ as functions from G into $\mathcal{H}$. To do so, let $C_c(G, \mathcal{H})$ be the vector space of all functions from G into $\mathcal{H}$ with compact support. Define an inner product on $C_c(G, \mathcal{H})$ by $\langle f, g\rangle = \int \langle f(s), g(s)\rangle_{\mathcal{H}} d\mu$ where $\langle \cdot, \cdot\rangle_{\mathcal{H}}$ is the inner product on $\mathcal{H}$. Let $L^2(G, \mathcal{H})$ be the completion of $C_c(G, \mathcal{H})$ with respect to $\langle \cdot, \cdot\rangle$. Given $\xi \in \mathcal{H}$ and $f \in C_c(G)$, $U(\xi \otimes f)(s) = f(s)\xi$ extends to a unitary operator from $\mathcal{H} \otimes \ell^2(G)$ onto $L^2(G, \mathcal{H})$. In particular, we may view $\xi \otimes f \in \mathcal{H} \otimes \ell^2(G)$ as a function from G into $\mathcal{H}$ given by $\xi \otimes f(s) = f(s)\xi$.

Define maps $\pi_\alpha : \mathcal{M} \to \mathscr{B}(L^2(G, \mathcal{H}))$ and $u : G \to \mathscr{B}(L^2(G, \mathcal{H}))$ by

$$(\pi_\alpha(x)f)(s) = \alpha_{s^{-1}}(x)f(s) \text{ and } (u(s)f)(t) = f(s^{-1}t).$$

Then π_α is a faithful normal *-representation of $\mathcal{M}$ in $L^2(G, \mathcal{H})$ and u is a unitary representation of G in $L^2(G, \mathcal{H})$. Moreover, for each $x \in \mathcal{M}$ and $t \in G$, π_α and u satisfy

$$u(t)\pi_\alpha(x)u(t)^* = \pi_\alpha(\alpha_t(x)).$$

For notational purposes we generally write u_s for $u(s)$ and x for $\pi_\alpha(x)$.

We define the *crossed product*, $\mathcal{M} \rtimes_\alpha G$, of $\mathcal{M}$ by α as the von Neumann algebra generated by $\{\pi_\alpha(x), u(s) : x \in \mathcal{M},\ s \in G\}$. The finite sums $\sum_{s\in F} x_s u_s$ for some $F \subset G$ form $*$-subalgebra of $\mathcal{M} \rtimes_\alpha G$ where adjoints and multiplication are defined as follows

$$\left(\sum_{s\in F} x_s u_s\right)^* = \sum_{s\in F} \alpha_s(x_{s^{-1}})u_s$$

and

$$\left(\sum_{s\in F_1} x_s u_s\right)\left(\sum_{t\in F_2} y_t u_t\right) = \sum_{s\in F_1, t\in F_2} x_s\alpha_t(x_t)u_{st}.$$

When α is free and ergodic $\mathcal{M} \rtimes_\alpha G$ is a factor. There always exists a normal, faithful conditional expectation $\mathbb{E}_{\mathcal{M}}$ from $\mathcal{M} \rtimes_\alpha G$ to $\mathcal{M}$. Furthermore, if there exists a trace τ on $\mathcal{M}$ there exists a trace τ_α on $\mathcal{M} \rtimes_\alpha G$ defined by

$$\tau_\alpha\left(\sum_{s\in G} x_s u_s\right) = \tau(x_e)$$

where e is the identity of G. We conclude this subsection with an example.

Example 6.1.1. Let (X, μ) be a standard probability space and α be a measure preserving action of a countable group G on X. For each $f \in L^\infty(X, \mu)$ define M_f on $L^2(X, \mu)$ by $M_f g = fg$. It is routine to verify the set $\{M_f : f \in L^\infty(X, \mu)\}$ is a self adjoint, unital, weakly closed, subalgebra of $\mathscr{B}(L^2(X, \mu))$ and hence a (commutative) von Neumann algebra. Furthermore, α induces an action by automorphisms $\widehat{\alpha}$ on $L^\infty(X, \mu)$ given by $\widehat{\alpha}_g(f)x = f(\alpha_{g^{-1}}x)$. We may then construct the the crossed product $L^\infty(X, \mu) \rtimes_\alpha G$ acting on $L^2(X, \mu) \otimes \ell^2(G)$. When α is free and ergodic, $\widehat{\alpha}$ is also free and ergodic and it follows $L^\infty(X, \mu) \rtimes_\alpha G$ is a factor.

6.2. Connes' Embedding Problem

It is a well known fact that every von Neumann algebra decomposes as a direct integral of factors. Furthermore, each factor may be classified as being either type I, type II, or type III. We say a factor is type I factor if it contains a minimal nonzero projection, type II if it contains nonzero finite projections but no minimal projections, and type III if it contains no nonzero finite projections.

Type I factors are isomorphic to $\mathscr{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$ whence we may further subdivide them as type I_n where n is the dimension of $\mathcal{K}$. The type III can be further classified into subtypes III_λ for $\lambda \in [0, 1]$, however this is beyond the scope of this dissertation and we refer the reader to [33] for further details. Type II factors are subdivided as type II_1 if the identity operator is finite and type II_∞ otherwise.

Of the type II factors we will be primarily interested in the hyperfinite II_1 factor.

This factor is unique and can be described in several ways. A common description is it is the unique II_1 factor which is approximately finite dimensional, i.e. there exists an increasing sequence of finite dimensional *-subalgebras whose union is σ-weakly dense. Unless otherwise specified, $\mathcal{R}$ will be used to denote the hyperfinite II_1 factor.

Let $\mathbb{I}$ be an infinite set. A subset ω of $\mathscr{P}(\mathbb{I})$ is said to be an *ultrafilter* if $\emptyset \notin \omega$, ω is closed under finite intersections and for all $A \subset \mathbb{I}$ either $A \in \omega$ or $\mathbb{I} \setminus A \in \omega$. If ω satisfies $\bigcap_{A\in\omega} A \neq \emptyset$ then it is clear there exists a unique element $i \in \mathbb{I}$ such that $\omega = \{A \subset \mathbb{I} : i \in A\}$. Such an ultrafilter is called *principal.* If ω is not principal then it is said to be *free.*

Suppose ω is an unltrafilter on $\mathbb{N}$ and X is a topological space. A sequence $\{x_n\} \subset X$ is said to *converge along the ultrafilter* ω to $x \in X$ if for every neighborhood V of x, $\{n \in \mathbb{N} : x_n \in V\} \in \omega$. The point x is said to be an ω-limit of $\{x_n\}$ and denote it by $x = \lim_{n\to\omega} x_n$. If X is Hausdorff, then the limit along an ultrafilter is unique. Also if $\{x_n\}$ is contained in some compact set, then the limit along ω exists. In particular every bounded sequence in $\mathbb{R}^n$ has a unique limit along the ultrafilter ω.

It can be shown that there exists a unique tracial state τ on $\mathcal{R}$. Furthermore, this trace gives rise to a norm on $\mathcal{R}$ given by $\|a\|_2 = \tau(a^*a)^{\frac{1}{2}}$ for all $a \in \mathcal{R}$. Consider $\ell^\infty(\mathcal{R}) = \{(a_k) \in \mathcal{R}^{\mathbb{N}} : \sup_{k\in\mathbb{N}} \|a\|_2 < \infty\}$ endowed with the norm $\|(a_k)\| = \sup_{k\in\mathbb{N}} \|a_k\|_2$. Set $I_\omega = \{(a_k) \in \ell^\infty(\mathcal{R}) : \lim_{k\to\omega} \|a_k\|_2 = 0\}$. Routine calculations show I_ω is an ideal of $\ell^\infty(\mathcal{R})$. We show it is in fact a closed ideal. Suppose (a_k) is in the closure of I_ω. Let $\epsilon > 0$ then there is $(b_k) \in I_\omega$ such that $\|a - b\| < \frac{\epsilon}{2}$. Then $\|b_k\|_2 \leq \|a_k - b_k\|_2 + \|b_k\| < \frac{\epsilon}{2} + \|b_k\|_2$. Thus the set $\{k : \|b_k\|_2 < \frac{\epsilon}{2}\}$ is contained in $\{k : \|a_k\|_2 < \epsilon\}$. Since $(b_k) \in I_\omega$ the former set is in ω the latter set must be as well. In particular $\lim_{k\to\omega} \|a_k\|_2 = 0$ as desired. We call the quotient $\ell^\infty(\mathcal{R})/I_\omega$ the tracial ultrapower of $\mathcal{R}$ and denote it $\mathcal{R}^\omega$. It is again a II_1 factor with natural trace τ_ω defined by $\tau_\omega((a_k) + I_\omega) = \lim_{k\to\omega} \tau(a_k)$. When working with R^ω we will follow convention and omit the I_ω when there is no ambiguity.

In Section V of [9], Alain Connes asked whether every separable II$_1$ factor $\mathcal{F}$ can be embedded into $\mathcal{R}^\omega$ for any free ultrafilter, i.e. does there exists a trace preserving $*$-homomorphism from $\mathcal{F}$ into $\mathcal{R}^\omega$. It follows from a general result of Brown, Dykema, and Jung on free products that the crossed product $L^\infty(X,\mu)\rtimes_\alpha \mathbb{F}_n$ embeds into $\mathcal{R}^\omega$ for any $\alpha\in \mathrm{Act}(\mathbb{F}_n,X,\mu)$, see Corollary 4.5 in [7]. We will use the results of Section 3 to give an explicit combinatorial proof that the crossed product embeds into $\mathcal{R}^\omega$.

In [35], Wassermann showed that the group von Neumann algebra of $\mathbb{F}_2$ embeds into $\mathcal{R}^\omega$. Wassermann's proof relies on the fact that $\mathbb{F}_2$ is residually finite. In particular, let $H_1\supset H_2\supset\cdots$ be a decreasing sequence of normal subgroups of $\mathbb{F}_2$ having trivial intersection and satisfying $|\mathbb{F}_2/H_i| = n_i < \infty$. Then for each i and $g\in\mathbb{F}_2$ composing the left regular representation with the canonical quotient map gives a representation of g in the matrix algebra M_{n_i}. Since M_{n_i} embeds into $\mathcal{R}$ for each i this gives rise to an embedding of the group von Neuman algebra into $\mathcal{R}^\omega$. Furthermore, the condition that the H_i's have trivial intersection implies this embedding is trace preserving.

Given $\alpha\in\mathrm{Act}(\mathbb{F}_n,X,\mu)$ one could then use the separability of the weak topology on $\mathrm{Act}(\mathbb{F}_n,X,\mu)$ and Theorem 3.3.3 to construct a sequence $\{\alpha^{(i)}\}_{i=1}^\infty\subset\mathrm{Act}(\mathbb{F}_n,X,\mu)$ such that $\alpha^{(i)}$ factors through a finite group G_i and $\alpha^{(i)}\to\alpha$. For each i let ϕ_i be a homomorphism from $\mathbb{F}_n$ onto G_i and $\beta^{(i)}\in\mathrm{Act}(G_i,X,\mu)$ be such that $\alpha^{(i)}_s=\beta^{(i)}_{\phi_i(s)}$ for all $s\in\mathbb{F}_n$. Since $\beta^{(i)}$ is an action of a finite group we have

$$L^\infty(X,\mu)\rtimes_{\alpha^{(i)}}\mathbb{F}_n\cong L^\infty(X,\mu)\rtimes_{\beta^{(i)}}G_i\hookrightarrow L^\infty(X,\mu)\otimes M_{|G_i|}\hookrightarrow\mathcal{R}\otimes M_{|G_i|}\cong\mathcal{R}.$$

We can then proceed as in Wassermann's proof to construct a map from $L^\infty(X,\mu)\rtimes_\alpha \mathbb{F}_n$ into $\mathcal{R}^\omega$. We will see this map will not be trace preserving unless $\bigcap_{i=1}^\infty\ker(\phi_i)$ is trivial. It remains to check that this latter condition can be arranged. It follows from

Theorem 3.3.2 that we may approximate α by an action β which pointwise permutes some partition $\mathscr{P}$. Furthermore, we can choose β and $\mathscr{P}$ so that β fixes some atom P of $\mathscr{P}$. We now show in the following lemma that β may be redefined on P to give the desired property.

Lemma 6.2.1. *Let H be a normal subgroup of $\mathbb{F}_n$ such that $\mathbb{F}_n/H$ is finite. Then for any basic open $W(\alpha, F, K, \epsilon)$ there exists $\alpha' \in W(\alpha, F, K, \epsilon)$ factoring through a finite group G such that if $\phi : \mathbb{F}_n \to G$ is the implementing homomorphism we have* $\ker(\phi) \subset H$.

Proof. Let $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$. Given a basic open set $W(\alpha, F, K, \epsilon)$ Theorem 3.3.2 assures the existence of $\alpha' \in W(\alpha, F, K, \epsilon)$ which pointwise permutes some partition $\mathscr{P}$. Furthermore, we may assume $\mu(P) < \frac{\epsilon}{2}$ for all $P \in \mathscr{P}$ and that there exists $P' \in \mathscr{P}$ such that $\alpha_s P' = P'$ for all $s \in F$. Without loss of generality we may also assume $F \subset S$. We also have by Proposition 3.2.2 that α' factors through some finite group G. That is there exists $\phi : \mathbb{F}_n \to G$ and $\beta' \in \mathrm{Act}(G, X, \mu)$ such that $\alpha_s = \beta_{\phi(s)}$ for all $s \in \mathbb{F}_n$.

Since $|\mathbb{F}_n/H| < \infty$ we may identify P' with $Y \times \mathbb{F}_n/H$ where (Y, ν) is a nonatomic probability space such that $\nu(Y) = \frac{\mu(P')}{|\mathbb{F}_n/H|}$ We then have a natural action γ of $\mathbb{F}_n$ on $Y \times \mathbb{F}_n/H$ defined by $\gamma_s(y, tH) = (y, (st)H)$. Let $\psi : P' \to \mathbb{F}_n/H$ be the isomorphism identifying P' and $Y \times \mathbb{F}_n/H$. Define $\widehat{\alpha} \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$ as follows. For each $s \in S$ and $x \in X$ define

$$\widehat{\alpha}_s x = \begin{cases} \alpha_s x & \text{if } x \notin P' \\ \psi^{-1}\gamma_s\psi(x) & \text{if } x \in P' \end{cases}.$$

We claim $\widehat{\alpha} \in W(\alpha, F, K, \epsilon)$. Suppose $s \in F$ and $C \in K$. If $C \cap P' = \emptyset$ then $\alpha_s C = \widehat{\alpha}_s C$ and there is nothing to check. Suppose $P' \cap C \neq \emptyset$. Then

$$\mu(\alpha_s C \triangle \widehat{\alpha}_s C) = \mu(\alpha_s(P' \cap C) \triangle \widehat{\alpha}_s(P' \cap C))$$

$$\leq \mu(\alpha_s(P' \cap C)) + \mu(\widehat{\alpha}_s(P' \cap C))$$
$$\leq \mu(\alpha_s P') + \mu(\widehat{\alpha}_s P)$$
$$= 2\mu(\alpha_s P')$$
$$< \epsilon$$

as desired.

Define an action of $G \times \mathbb{F}_n/H$ on X as follows. Given $(s, tH) \in G \times \mathbb{F}_n/H$ and $x \in X$ set

$$\widehat{\beta}_s x = \begin{cases} \beta_s x & \text{if } x \notin P' \\ \psi^{-1}\gamma_s\psi(x) & \text{if } x \in P' \end{cases}.$$

Define a map $\Phi : \mathbb{F}_n \to G \times \mathbb{F}_n/H$ by $\Phi(s) = (\phi(s), sH)$. Then it is clear that $\widehat{\alpha}_s = \widehat{\beta}_{\Phi(s)}$ for all $s \in \mathbb{F}_n$. That is $\widehat{\alpha}$ factors through the finite group $G' = \Phi(\mathbb{F}_n) \cong \mathbb{F}_n/\ker(\Phi)$. Furthermore, note that $\ker(\Phi) \subset H$. $\square$

Let $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$. Since $\mathbb{F}_n$ is residually finite there exists normal subgroups $\mathbb{F}_n = H_1 \supset H_2 \supset \cdots$ such that $|\mathbb{F}_n/H_i| < \infty$ for each $i = 1, 2, \ldots$ and $\bigcap_{i=1}^{\infty} H_i = \{e\}$. We may then apply Lemma 6.2.1 to construct a sequence $\{\alpha^{(i)}\}_{i=1}^{\infty} \subset \mathrm{Act}(\mathbb{F}_n, X, \mu)$ converging to α such that $\alpha^{(i)}$ fractors through the finite group G_i and the implementing homomorphism $\phi_i : \mathbb{F}_n \to G_i$ satisfies $\ker(\phi) \subset H_i$ for each $i \in \mathbb{N}$. In particular, we have $\bigcap_{i=1}^{\infty} \ker(\phi_i) \subset \bigcap_{i=1}^{\infty} H_i = \{e\}$. Thus we have shown the following corollary:

Corollary 6.2.2. *Let $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$. Then there exists a sequence of actions $\{\alpha_i\}_{i=1}^{\infty} \subset \mathrm{Act}(\mathbb{F}_n, X, \mu)$ converging to α such that each α_i factors through a finite group G_i and the implementing homomorphisms $\{\phi_i : \mathbb{F}_n \to G_i\}_{i=1}^{\infty}$ satisfy $\bigcap_{i=1}^{\infty} \ker(\phi_i) = \{e\}$.*

We now modify the argument in [35] to show $L^{\infty}(X, \mu) \rtimes_{\alpha} \mathbb{F}_n$ embeds into R^{ω} for all $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$.

Theorem 6.2.3. *Let $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$. Then $L^\infty(X,\mu) \rtimes_\alpha \mathbb{F}_n$ embeds into the tracial ultraproduct of the hyperfinite II_1 factor for any free ultrapower ω.*

Proof. Let $\alpha \in \mathrm{Act}(\mathbb{F}_n, X, \mu)$. For convenience we identify X with the unit circle $\mathbb{T} \subset \mathbb{C}$ with Lebesgue measure λ. Furthermore we assume $\mathcal{R}^\omega$ is in its standard representation on the Hilbert space $\mathcal{H}$ with cyclic and separating vector ξ. By Corollary 6.2.2 there exists $\alpha^{(i)} \to \alpha$ such that each $\alpha^{(i)}$ factors through a finite group and the kernels of the implementing homomorphisms $\{\phi_i\}_{i=1}^\infty$ have trivial intersection. Denote by Φ the canonical quotient map from $\ell^\infty(\mathcal{R})$ onto $\ell^\infty(\mathcal{R})/I_w$ where $I_\omega = \{(a_k) \in \ell^\infty(\mathcal{R}) : \lim_{k\to\omega} \tau(a_k^* a_k) = 0\}$. Given $\sum_{s\in F} f_s u_s \in L^\infty(\mathbb{T},\lambda) \rtimes_\alpha \mathbb{F}_n$, for each $i \in \mathbb{N}$ we define a map into $L^\infty(\mathbb{T},\lambda) \rtimes_{\alpha^{(i)}} \mathbb{F}_n$ by

$$\sum_{s\in F} f_s u_s \mapsto \sum_{s\in F} f_s u_{\phi_i(s)}.$$

We have already seen that $L^\infty(\mathbb{T},\lambda) \rtimes_{\alpha^{(i)}} \mathbb{F}_n$ embeds into $\mathcal{R}$. Its then routine to check

$$\sum_{s\in F} f_s u_s \mapsto (\sum_{s\in F} f_s u_{\phi_i(s)})$$

gives a map from $L^\infty(\mathbb{T},\lambda) \rtimes_\alpha \mathbb{F}_n$ into $\ell^\infty(\mathcal{R})$. Denote by $\overline{\sum_{s\in F} f_s u_s}$ the image of $(\sum_{s\in F} f_s u_{\phi_i(s)})$ under Φ. Then $\sum_{s\in F} f_s u_s \mapsto \overline{\sum_{s\in F} f_s u_s}$ gives a map from $L^\infty(\mathbb{T},\lambda) \rtimes_\alpha \mathbb{F}_n$ into $\mathcal{R}^\omega$. We claim this map is trace preserving. Indeed, there exists $i_0 \in \mathbb{N}$ such that $F \cap \ker(\phi_i) = \{e\}$ for all $i > i_0$. Thus

$$\begin{aligned}
\tau_\omega(\overline{\sum_{s\in F} f_s u_s}) &= \lim_{i\to\omega} \tau_{\alpha^{(i)}}(\sum_{s\in F} f_s u_{\phi_i(s)}) \\
&= \lim_{i\to\omega}(\sum_{s\in F\cap\ker(\phi_i)} \tau(f_s)) \\
&= \tau(f_e) \\
&= \tau_\alpha(\sum_{s\in F} f_s u_s).
\end{aligned}$$

Denote by $\mathcal{B}$ the von Neuman subalgebra of $\mathcal{R}^\omega$ generated by

$$\{\overline{\sum_{s\in F} f_s u_s} : \sum_{s\in F} f_s u_s \in L^\infty(\mathbb{T},\lambda) \rtimes_\alpha \mathbb{F}_n\}$$

and by $\mathcal{K}$ the closure of in $\mathcal{H}$ of

$$\{\overline{\sum_{s\in F} f_s u_s}\xi : \sum_{s\in F} f_s u_s \in L^\infty(\mathbb{T},\lambda) \rtimes_\alpha \mathbb{F}_n\}.$$

Note that elements of the form $z^l u_s$ form an orthonormal basis for $L^\infty(\mathbb{T},\lambda)\rtimes_\alpha\mathbb{F}_n$. It is clear that such elements span $L^\infty(\mathbb{T},\lambda)$. To see they are orthonormal note that,

$$\tau((z^n u_s)^*(z^m u_t)) = \tau(u_{s^{-1}} z^{-l} z^m u_t) = \tau(z^{m-l} u_{ts^{-1}}) = \tau(\mathbb{E}_{L^\infty(X,\mu)}[z^{m-l} u_{ts^{-1}}]).$$

If $ts^{-1} \neq e$ then $\tau((z^l u_s)^*(z^m u_t)) = 0$. If $ts^{-1} = e$ then $\tau((z^l u_s)^*(z^m u_t)) = \tau(z^{m-l})$ which equals 1 if $m = l$ and 0 otherwise. That is $\{z^l u_s : l \in \mathbb{Z},\ s \in \mathbb{F}_n\}$ forms an orthonormal basis for $L^\infty(\mathbb{T},\lambda) \rtimes_\alpha \mathbb{F}_n$. Furthermore, note that

$$\begin{aligned}
\langle \overline{z^l u_s}\xi, \overline{z^m u_t}\xi\rangle &= \tau_\omega((z^l u_{\phi_i(s)})^*(z^m u_{\phi_i(t)})) \\
&= \lim_{k\to\omega} \tau((z^l u_{\phi_i(s)})^* z^m u_{\phi_i(t)}) \\
&= \lim_{k\to\omega} \tau(z^{m-l} u_{\phi_i(ts^{-1})}) \\
&= \lim_{k\to\omega} \tau(\mathbb{E}_{L^\infty(X,\mu)}[z^{m-l} u_{\phi_i(ts^{-1})}]).
\end{aligned}$$

Since $\bigcap_{i=1}^\infty \ker(\phi_i) = \{e\}$ we have

$$\langle \overline{z^l u_s}\xi, \overline{z^m u_t}\xi\rangle = \begin{cases} 0 & \text{if } s \neq t \\ \lim_{k\to\omega} \tau(z^{m-l}) & \text{if } s = t \end{cases}$$

from which it follows

$$\langle \overline{z^l u_s}\xi, \overline{z^m u_t}\xi\rangle = \begin{cases} 1 & \text{if } s=t, \ ,l=m \\ 0 & \text{otherwise} \end{cases}.$$

Thus we have $\{\overline{z^l u_s}\xi\}$ is orthonormal in $\mathcal{K}$. It then follows that

$$\sum_{s\in F} f_s u_s \mapsto \overline{\sum_{s\in F} f_s u_s}\xi$$

defines a unitary map $U : L^\infty(\mathbb{T},\lambda) \rtimes_\alpha \mathbb{F}_n \to \mathcal{K}$.

Let P be the orthonormal projection of $\mathcal{H}$ onto $\mathcal{K}$ and let $\mathcal{B}_P$ be the image of $\mathcal{B}$ under P. We claim that $\overline{\sum_{s\in F} f_s u_s} \mapsto \overline{\sum_{s\in F} f_s u_s}P$ is an isomorphism. Indeeed, if $\overline{\sum_{s\in F} f_s u_s}P = 0$ then

$$\|\overline{\sum_{s\in F} f_s u_s}\xi\| = \|\overline{\sum_{s\in F} f_s u_s}P\xi\| = 0$$

from which is follows $\overline{\sum_{s\in F} f_s u_s} = 0$ since ξ is separating. Then for $\overline{\sum_{s\in F} f_s u_s}$ we have

$$\begin{aligned}
U^{-1}(\overline{\sum_{s\in F} f_s u_s})U(\sum_{t\in F'} f_t u_t) &= U^{-1}(\overline{\sum_{s\in F} f_s u_s})(\overline{\sum_{t\in F'} f_t u_t}\xi) \\
&= U^{-1}(\sum_{s\in F} f_s u_{\phi_i(s)})(\sum_{t\in F'} f_t u_{\phi_i(t)})\xi \\
&= U^{-1}(\sum_{s\in F,\ t\in F'} f_s\alpha_t(f_t)u_{\phi_i(st)})\xi \\
&= (\sum_{s\in F,\ t\in F'} f_s\alpha_t(f_t)u_{\phi_i(st)}) \\
&= (\sum_{s\in F} f_s u_s)(\sum_{t\in F'} f_t u_t).
\end{aligned}$$

Thus we have$\sum_{s\in F} f_s u_s \mapsto U^{-1}\overline{\sum_{s\in F} f_s u_s}U$ defines an isomorphism from $L^\infty(\mathbb{T},\lambda) \rtimes_\alpha \mathbb{F}_n$ onto $\mathcal{B}_P$. In particular $L^\infty(\mathbb{T},\lambda) \rtimes_\alpha \mathbb{F}_n \cong \mathcal{B}_P \cong \mathcal{B} \subset \mathcal{R}^\omega$. $\square$

7. TOPOLOGICAL ACTIONS ON THE CANTOR SET

7.1. Introduction to Actions on the Cantor Set

In this section we let K denote the Cantor set and d be a fixed metric on K. Intuitively the paradoxical nature of the Cantor set, e.g. an uncountable, totally disconnected, zero-dimensional, compact metric space, seems to suggest it would be an unnatural space to consider. However, actions on the Cantor set arise naturally in topological dynamics, e.g. Bernoulli shifts, odometers, and the action of $\mathbb{F}_n$ on its Gromov boundary. Furthermore, K exhibits several properties analogous to those of nonatomic standard probability spaces. For example, K is unique up to homeomorphism, any nonempty clopen subset of K is homeomorphic to K, and there exist clopen partitions of K.

In the classical setting, much is known about actions of $\mathbb{Z}$ on K. For example, the set of zero entropy actions is generic [13] and the Rokhlin lemma exists for certain classes of actions, see [3] and [29]. Kechris and Rosendal [18] showed that there exists an action of $\mathbb{Z}$ on K whose conjugacy class is generic. A description of such an action has since been given by Akin, Glasner and Weiss [1]. As this example is well understood, questions about genericity in $\mathrm{Act}(\mathbb{Z}, K)$ are in some sense boring. In particular, we only have to check if this example has the prescribed property.

Recently focus has been placed on studying the genericity of properties in certain subspaces of $\mathrm{Act}(\mathbb{Z}, K)$. For example, Hochman [16] consider the subspaces of transitive actions and totally transitive actions, i.e. transitive for each power. In the latter case, he showed the genericity of actions differs quite drastically from the measure-preserving case. For example, a generic totally transitive action is both mixing and weak mixing.

7.2. Finite Approximations of Actions on the Cantor Set

Our goal for this susection will be to prove a topological analog of Theorem 3.3.2 for actions of $\mathbb{F}_n$ on the Cantor set. The clopen partitions of K provide a natural analog of measurable partitions in the measure-preserving case. Consequently, we say an action α of $\mathbb{F}_n$ on K *pointwise permutes* a clopen partition $\mathscr{P}$ if $\alpha_s P \in \mathscr{P}$ for all $P \in \mathscr{P}$ and $s \in \mathbb{F}_n$ and $\alpha_s P = \alpha_t P$ if and only if $\alpha_s x = \alpha_t x$ for all $x \in P$. If α pointwise permutes a clopen partition $\mathscr{P}$ we may construct a subgroup of the permutation group $S(\mathscr{P})$ through which α factors as in Proposition 3.2.2.

Given a clopen partition $\mathscr{P}$ of K define $C(\mathscr{P}) = \mathrm{span}\{\chi_P : P \in \mathscr{P}\}$. Then for each $P \in \mathscr{P}$ the characteristic function χ_P is continuous since P is clopen, whence $C(\mathscr{P}) \subset C(K)$. In fact, $C(\mathscr{P})$ is a dense subset of $C(K)$. This latter observation allows for a simpler description of the weak topology on $\mathrm{Homeo}(K)$ and thus $\mathrm{Act}(G, K)$. Denote by $\mathrm{Clo}(K)$ the set of all clopen subets of K. Then sets of the form

$$W(g, \mathcal{C}) = \{f \in \mathrm{Homeo}(K) : fC = gC \text{ for all } C \in \mathcal{C}\}$$

where $\mathcal{C}$ is a finite subset of $\mathrm{Clo}(K)$ form a basis for the weak topology on $\mathrm{Homeo}(K)$. Consequently, a basis for the weak topology on $\mathrm{Act}(G, K)$ is given by

$$W(\alpha, F, \mathcal{C}) = \bigcap_{s \in F} \bigcap_{C \in \mathcal{C}} \{\beta \in \mathrm{Act}(G, K) : \alpha_s C = \beta_s C\}$$

where $F \subset G$ and $\mathscr{C} \subset \mathrm{Clo}(K)$ are finite subsets.

As in the measure-preserving case, for $\mathbb{F}_n$ we claim F may be taken to be a subset of the generating set S. Let $S' \subset S$ and $N \in \mathbb{N}$ be defined as in Lemma 3.1.1. That is F is contained in the subgroup generated by S' and every element of F can be represented as reduced word of length at most N in this subgroup. Define

$$\mathcal{C}' = \{\alpha_{s_1^{\epsilon_1} s_2^{\epsilon_2} \ldots s_N^{\epsilon_N}} C : C \in \mathcal{C},\ s_i \in S',\ \epsilon_i \in \{-1, 0, 1\} \text{ for all } i = 1, 2, \ldots N\}.$$

Then $\mathcal{C}'$ is a finite collection of clopen sets of K. Let $\beta \in W(\alpha, S', \mathcal{C}')$ and $t \in F$. Then there exist $n \leq N$, $s_1, s_2, \ldots, s_n \in S'$, and $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\} \in \{-1, 1\}^n$ such that $t = s_1^{\epsilon_1} s_2^{\epsilon_2} \ldots s_n^{\epsilon_n}$. Thus

$$\begin{aligned}
\beta_t C &= \beta_{s_1^{\epsilon_1}} \beta_{s_2^{\epsilon_2}} \ldots \beta_{s_n^{\epsilon_n}} C \\
&= \beta_{s_1^{\epsilon_1}} \beta_{s_2^{\epsilon_2}} \ldots (\alpha_{s_n^{\epsilon_n}} C) \\
&\vdots \\
&= \beta_{s_1^{\epsilon_1}} (\alpha_{s_2^{\epsilon_2}} \ldots \alpha_{s_n^{\epsilon_n}} C) \\
&= \alpha_{s_1^{\epsilon_1}} \alpha_{s_2^{\epsilon_2}} \ldots \alpha_{s_n^{\epsilon_n}} C \\
&= \alpha_t C.
\end{aligned}$$

In particular, we have shown sets of the form $W(\alpha, S', \mathcal{C}')$ where $S' \subset S$ and $\mathcal{C} \subset \mathrm{Clo}(K)$ are finite form a basis for the weak topology on $\mathrm{Act}(\mathbb{F}_n, K)$.

As in the measure-preserving case, we have shown a perturbation of the action on a generator results in a perturbation of the action itself. For measure-preserving actions the Rokhlin lemma then allowed for an arbitrary action to be perturbed to one which when restricted to a generator pointwise permutes some partition. In the topological setting such a perturbation is problematic since the generic homeomorphism of K given in [1] fails to pointwise permute any finite clopen partition of K. Thus it is unlikely that Theorem 3.3.2 will hold for all actions in $\mathrm{Act}(\mathbb{F}_n, K)$. To compensate, we will fix a subspace of $\mathrm{Act}(\mathbb{F}_n, K)$ and work in the closure of that space.

Denote by $\mathcal{PP}(\mathbb{F}_n, K)$ the set of actions in $\mathrm{Act}(\mathbb{F}_n, K)$ which pointwise permutes some finite clopen partition. Let $\alpha \in \mathcal{PP}(\mathbb{F}_n, K)$ and $\mathscr{P}$ be a clopen partition pointwise permuted by α. It is clear that Lemma 3.2.1 holds in the topological case. That is, there exists a partition $\{\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_n\}$ of $\mathscr{P}$ such that each atom is α-invariant and α-transitive. For each $i = 1, 2, \ldots, n$ enumerate $\mathscr{P}_i$ as $\{P_{i,1}, P_{i,2}, \ldots, P_{i,n_i}\}$ and choose $s_{i,j}$ such that $\alpha_{s_{i,j}} P_{i,1} = P_{i,j}$. For each $i = 1, 2, \ldots, n$ fix a nonatomic Borel measure ν_i on $P_{i,1}$. It is clear that

$$\mu(A) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \frac{1}{n_i \nu_i(P_{i,1})} \nu_i(\alpha_{s_{i,j}^{-1}}(A \cap P_{i,j}))$$

defines a nonatomic probability measure of full support on K.

We claim μ is α-invariant. Let $s \in \mathbb{F}_n$. For each $P_{i,j}$ there exists $j' \in \{1, 2, \ldots, n_i\}$ such that $\alpha_s P_{i,j} = P_{i,j'}$. Thus for any Borel set $A \subset K$ we have $\alpha_s(A \cap P_{i,j}) = \alpha_{s_{i,j'} s_{i,j}^{-1}} A \cap P_{i,j'}$. In particular $\alpha_s A = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} \alpha_{s_{i,j'} s_{i,j}^{-1}} A \cap P_{i,j'}$. We now have that

$$\begin{aligned}
\mu(\alpha_s A) &= \sum_{i=1}^{n} \sum_{j=1}^{n_i} \frac{1}{n_i \nu_i(P_{i,1})} \nu_i(\alpha_{s_{i,j'}^{-1}}(\alpha_{s_{i,j'} s_{i,j}^{-1}} A \cap P_{i,j'})) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n_i} \frac{1}{n_i \nu_i(P_{i,1})} \nu_i((\alpha_{s_{i,j}^{-1}} A) \cap P_{i,1}) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n_i} \frac{1}{n_i \nu_i(P_{i,1})} \nu_i(\alpha_{s_{i,j}^{-1}}(A \cap P_{i,j})) \\
&= \mu(A)
\end{aligned}$$

as desired. If we denote by $\mathrm{Act}_{\mathrm{inv}}(\mathbb{F}_n, K)$, of actions of $\mathbb{F}_n$ on K which admit an invariant nonatomic probability measure of full support on K we have then shown the following:

Proposition 7.2.1. $\mathcal{PP}(\mathbb{F}_n, K) \subset \mathrm{Act}_{\mathrm{inv}}(\mathbb{F}_n, K)$.

In fact, we may make a stronger statement about elements of $\mathcal{PP}(\mathbb{F}_n, K)$. A topological dynamical systems is said to be *minimal* is there exists no proper closed invariant subsets. If a dynamical system can be decomposed into minimal subsystems we say the system is a *union of minimal systems*. Define $\mathcal{UM}(\mathbb{F}_n, K)$ to be the set of $\beta \in \mathrm{Act}_{\mathrm{inv}}(\mathbb{F}_n, K)$ such that β_s is a union of minimal systems for each generator s.

We now show the following:

Proposition 7.2.2. *$\mathcal{PP}(\mathbb{F}_n, K) \subset \overline{\mathcal{UM}(\mathbb{F}_n, K)}$.*

Proof. Let $\alpha \in \mathcal{PP}(\mathbb{F}_n, K)$ and $\mathscr{P}$ be finite clopen partition of K which is pointwise permuted by α. Since each $P \in \mathscr{P}$ is nonempty and clopen we may identify P with $\{0,1\}^{\mathbb{N}}$. Let ν_P be the product measure on $\{0,1\}^{\mathbb{N}}$ scaled such that $\nu_P(P) = \frac{1}{|\mathscr{P}|}$. We may then define a nonatomic probability measure ν of full support on K as above. That is, for a Borel set $C \subset K$ we define

$$\nu(C) = \sum_{P \in \mathscr{P}} \nu_P(C \cap P)$$

Fix a basic open neighborhood $W(\alpha, F, \mathcal{C})$ of α. Let $s \in S$. Since α pointwise permutes $\mathscr{P}$ it follows that α_s also pointwise permutes $\mathscr{P}$. Thus there exists disjoint subcollections $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_k$ of $\mathscr{P}$ such $\mathscr{P} = \bigcup_{i=1}^{k} \mathscr{P}_i$ and each $\mathscr{P}_i$ is α_s-transitive. Furthermore, each $\mathscr{P}_i$ may be written as $\{\alpha_{s^l} P_i\}_{l=0}^{n_i-1}$ for some $P_i \in \mathscr{P}_i$. For each $i = 1, 2, \ldots, k$ partition P_i into clopen sets $P_{i,1}, P_{i,2}, \ldots, P_{i,m_i}$ by collecting together the points which visit the same sequence of clopen sets in $\mathcal{C}$ as they move up the levels of $\mathscr{P}_i$.

Since the cylinder sets generate the topology on $\{0,1\}^{\mathbb{N}}$, for each $i = 1, 2, \ldots, k$ there exists a positive integer d_i such that the cylinder sets $\{0,1\}^{d_i} \times \{0,1\}^{\mathbb{N} \setminus \{1,2,\ldots,d_i\}}$ refine $\{P_{i,1}, P_{i,2}, \ldots, P_{i,m_i}\}$. It is clear that there exist measure preserving homeomorphisms from each of these cylinder sets onto $\{0,1\}^{\mathbb{N}}$. Twisting the action on each of these cylinder sets by the dyadic odometer gives rise to a homeomorphism $\alpha'_s \in W(\alpha_s, \mathcal{C})$ which preserves ν. Since the dyadic odometer is minimal we have then shown α'_s may be decomposed into subsystems which are minimal. Furthermore, since each α'_s preserves ν we have defined an action $\alpha' \in W(\alpha, F, \mathcal{C}) \cap \mathcal{UM}(\mathbb{F}_n.K)$. That is, $\mathcal{PP}(\mathbb{F}_n, K) \subset \overline{\mathcal{UM}(\mathbb{F}_n, K)}$ as desired. $\square$

In [29] Putnam showed the existence of a Rokhlin lemma for minimal $\mathbb{Z}$-systems on the Cantor set. In particular, given $\alpha \in \mathcal{UM}(\mathbb{F}_n, K)$ for each $s \in S$ we may apply Putnam's result to each minimal subsystem of α_s to create a finite collection of towers. As in the Rokhlin lemma, the action must be perturbed to obtain an action which pointwise permutes these towers. We give a sketch of Putnum's argument and indicate how we may perturb the action to get the desired result.

Let (K, T) be a minimal system. Given a nonempty clopen set $C \subset K$ define $\lambda : C \to \mathbb{Z}$ by $\lambda(x) = \inf\{n \in \mathbb{Z} : T^n x \in C\}$. The minimality of T ensures λ is well defined. The function λ can also be shown to be continuous, whence its image must be finite. It is then clear that there exist finitely many disjoint clopen subsets $C_1, C_2, \ldots, C_k$ of C and positive integers $n_1, n_2, \ldots, n_k$ such that $\bigcup_{i=1}^k C_i = C$ and $T^{n_i} C_i \in C$ for all $i = 1, 2, \ldots, k$. The minimality of T implies $\mathscr{P} = \{T^i C_j : 1 \leq i \leq k, 1 \leq j \leq n_i\}$ is a partition of K.

In general, $\mathscr{P}$ will not be pointwise permuted by T since $T^{n_i} C_i$ need not equal C_i for each $i = 1, 2, \ldots, k$. However, given any basic neighborhood $W(T, \mathcal{C})$ we may choose C to be contained in some element of $\mathcal{C}$. Then is is clear that T may be perturbed so that it does pointwise permute $\mathscr{P}$. Specifically, for each $i = 1, 2, \ldots, k$ we define $T^{n_i} x = x$ for all $x \in C_i$. Moreover, if T preserves some measure supported on K then the perturbation will as well.

We now state the topological analog of Theorem 3.3.2 as follows:

Theorem 7.2.3. *Let $\alpha \in \mathcal{UM}(\mathbb{F}_n, K)$ and $\mathcal{C} \subset \mathrm{Clo}(K)$ be a finite set. Then for any $\delta > 0$ and finite subset $F \subset \mathbb{F}_n$ there exist a finite clopen partition $\mathscr{P}$ and an $\alpha' \in W(\alpha, F, \mathcal{C})$ such that α' pointwise permutes $\mathscr{P}$ and $\mathrm{Diam}(P) < \delta$ for all $P \in \mathscr{P}$. Moreover, α' factors through a finite group.*

As in the measure-preserving case, Theorem 7.2.3 is proved by inductively refining the partitions which are pointwise permuted by each $s \in S'$ to obtain a single partition which is pointwise permuted by all $s \in S'$ and thus the subgroup generated S'. In other words, the proof relies on showing a topological analog of Lemma 3.3.1. Fortunately, much of the proof of Lemma 3.3.1 can be applied directly to the topological setting.

The notable exception is the way in which we make the final refinement of the partition. In the measure-preserving case we could rationalize each element and then divide into small subsets of equal measure. Provided we controlled the error appropriately we could create a perturbation by matching up a large proportion of these sets according to the given action and redefining the action to be the identity on the remaining sets.

In the topological case it clear that such an approach will not result in a topological perturbation. In particular, we must be more precise as to how we make this subdivision if we are to obtain a perturbation. In fact, we show for a finite clopen partition of K there exists arbitrarily large integers n such that each atom may be subdivided into sets of measure approximately $\frac{1}{n}$. Surprisingly, the proof that such subdivisions can be made follows from the Birkhoff recurrence theorem.

Suppose (X, T) is any topological $\mathbb{Z}$-action. A point $x \in X$ is said be *recurrent* if for each $\epsilon > 0$ the set $\{n \in \mathbb{Z} : d(x, T^n x) < \epsilon\}$ is infinite. Alternatively, we can define recurrence as the existence of a sequence $\{n_i\}_{i=1}^{\infty} \subset \mathbb{N}$ converging to infinity such that $T^{n_i}x$ converges to x. The Birkhoff recurrence theorem assures the existence of at least one recurrent point for any topological system (X, T).

In particular, the rotation on the n-torus $\mathbb{T}^n \cong [0, 1)^n$ given by $T(x_1, x_2, \ldots, x_n) = (x_1 + \lambda_1, x_2 + \lambda_2, \ldots, x_n + \lambda_n)$ for some $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in (0, 1)^n$ has a recurrent point $y = (y_1, y_2, \ldots, y_n)$. Let $\{n_i\}_{i=1}^{\infty} \subset \mathbb{N}$ be a sequence converging to infinity such that $T^{n_i}y \to y$. Define S to be the rotation given by $S(x_1, x_2, \ldots, x_n) =$

$(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$. It is clear that S and T commute. Thus by continuity we have

$$T^{n_i}(0,0,\ldots,0) = T^{n_i}S(y) = S(T^{n_i}y) \to S(y) = (0,0,\ldots,0).$$

That is, $(0,0,\ldots,0)$ is recurrent for T. Thus given ϵ greater than zero we can find arbitrarily large n and some positive integers $m_1, m_2, \ldots, m_n$ such that $|n\lambda_i - m_i| < \epsilon$ for each $i = 1, 2, \ldots, n$. We summarize this result in the following lemma:

Lemma 7.2.4. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in (0,1)$ and $\epsilon, \delta > 0$. Then there exists positive integers $n, m_1, m_2, \ldots, m_n$ such that $\frac{1}{n} < \delta$ and $|n\lambda_i - m_i| < \epsilon$ for each $i = 1, 2, \ldots, n$.*

We now use Lemma 7.2.4 to prove the topological analog of Lemma 3.3.1. Note that Lemma 7.2.4 could be applied in a similar manner as below to create the subdivisions subdivisions in Lemma 3.3.1. The choice not to was made to keep the proof as elementary as possible. The majority of the proof is unchanged from the measure-preserving case. Consequently, we explicitly prove the only the parts that differ and refer the reader to Lemma 3.3.1 for the remainder.

Lemma 7.2.5. *Let $\alpha \in \mathrm{Act}_{inv}(\mathbb{F}_n, K)$. Further suppose there exist finite subsets $R, T \subset S$ and finite clopen partitions $\mathscr{P}$ and $\mathscr{Q}$ of X such that $R \cap T = \emptyset$ and α restricted to the subgroups generated by R and T pointwise permutes $\mathscr{P}$ and $\mathscr{Q}$ respectively. Then for any $\delta > 0$ and finite subsets $F \subset \mathbb{F}_n$ and $\mathcal{C} \subset \mathrm{Clo}(K)$ there exist a finite clopen partition $\mathscr{R}$ and an $\alpha' \in W(\alpha, F, \mathcal{C})$ such that α' restricted to the subgroup generated by $R \cup T$ pointwise permutes $\mathscr{R}$ and each atom of $\mathscr{R}$ has diameter less than δ.*

Proof. First note that it suffices to assume $F \subset S$. Let H_R and H_T be the subgroups of $\mathbb{F}_n$ generated by R and T respectively. As in the measure-preserving case, there exist partitions $\{\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_{n_p}\}$ of $\mathscr{P}$ and $\{\mathscr{Q}_1, \mathscr{Q}_2, \ldots, \mathscr{Q}_{n_q}\}$ of $\mathscr{Q}$ such that each $\mathscr{P}_h$ is (α, H_R)-invariant and (α, H_R)-transitive and each $\mathscr{Q}_i$ is (α, H_T)-invariant and (α, H_T)-transitive. Enumerate each $\mathscr{P}_h$ and $\mathscr{Q}_i$ as $\{P_{h,1}, P_{h,2}, \ldots, P_{h,n_{p_h}}\}$ and $\{Q_{i,1}, Q_{i,2}, \ldots, Q_{i,n_{q_i}}\}$ respectively. Let I_p, I_{p_h}, I_q, and I_{q_i} be defined as in Lemma 3.3.1. For each $h \in I_p$ and $k \in I_{p_h}$ choose choose $r_{h,k} \in H_R$ such that $\alpha_{r_{h,k}} P_{h,1} = P_{h,k}$. Similarly, for each $i \in I_q$ and $j \in I_{q_i}$ choose $t_{i,j} \in H_T$ such that $\alpha_{t_{i,j}} Q_{i,1} = Q_{i,j}$.

We now refine $\mathscr{Q}$ as in Lemma 3.3.1. That is, enumerate $\mathcal{C}$ by $\{C_1, C_2, \ldots, C_m\}$ and define $I_m = \{1, 2, \ldots, m\}$ and $\Lambda = \{(h,k) : h \in I_p,\ k \in I_{p_h}\}$. For each $i \in I_q$ we set $\Sigma^i = \Lambda^{I_{q_i}}$ and $\Gamma^i = \mathscr{P}(I_m)^{I_{q_i}}$. We now define

$$\begin{aligned} A^i_\sigma &= \{x \in Q_{i,1} : \alpha_{t_{i,j}} x \in P_{\sigma(j)} \text{ for each } j \in I_{q_i}\}, \\ A^i_\gamma &= \{x \in Q_{i,1} : \alpha_{t_{i,j}} x \in (\cap_{l \in \gamma(j)} C_l) \cap (\cap_{l \in I_m \setminus \gamma(j)} C_l^c) \text{ for all } j \in I_{q_i}\}, \\ A^i_{\sigma,\gamma,j} &= \alpha_{t_{i,j}} (A^i_\sigma \cap A^i_\gamma). \end{aligned}$$

It then follows that $\mathscr{A} = \{A^i_{\sigma,\gamma,j} : i \in I_q, j \in I_{q_i}, \sigma \in \Sigma, \gamma \in \Gamma\}$ is a clopen partition of K. For $h \in I_p$ set $\Theta_h = \mathscr{A}^{I_{p_h}}$. Given $\theta \in \Theta_h$ define

$$A_\theta = \{x \in P_{h,1} : \alpha_{r_{h,k}} x \in \theta(k) \text{ for all } k \in I_{p_h}\}.$$

Let $h \in I_p$ be fixed and set $\Theta'_h = \{\theta \in \Theta_h : A_\theta \neq \emptyset\}$. It then follows that $\{A_\theta\}_{\theta \in \Theta'_h}$ is a clopen partition of $\mathscr{P}_h$ for each $h \in I_p$. Define $\Theta' = \bigcup_{i=1}^n \Theta'_i$. For $i \in I_q$, $\sigma \in \Sigma^i$, $\gamma \in \Gamma^i$, and $j \in I_{q_i}$ define $\Theta^i_{\sigma,\gamma,j} = \{\theta \in \Theta' : \alpha_{r_{\sigma(j)}} A_\theta \subset A^i_{\sigma,\gamma,j}\}$. As in the measure-preserving case we $A^i_{\sigma,\gamma,j} = \bigcup_{\theta \in \Theta^i_{\sigma,\gamma,j}} \alpha_{r_{\sigma(j)}} A_\theta$.

Let $\epsilon < \frac{1}{2|\Theta'|}$. Note that if $\theta \in \Theta'$ then A_θ is a nonempty clopen set and thus has positive measure. Thus by Lemma 7.2.4 there exist a positive integer $n > \frac{1}{\delta}$ and positive integers $\{m_\theta\}_{\theta \in \Theta'}$ such that $|n\mu(A_\theta) - m_\theta| < \epsilon$. Thus for each $\theta \in \Theta'$ we may find a measurable partition $\{A_{\theta,l}\}_{l=1}^{m_\theta}$ of A_θ such that $\mu(A_{\theta,l}) = \frac{1}{n}$ for $i = 1, 2, \ldots, m_\theta - 1$ and $|\mu(A_{\theta,m_\theta}) - \frac{1}{n}| < \frac{\epsilon}{n}$.

Let $i \in I_q$, $\sigma \in \Sigma^i$ and $\gamma \in \Gamma^i$ be such that $A_\sigma \cap A_\gamma \neq \emptyset$. For each $j \in I_{q_i}$ define $n^i_{\sigma,\gamma,j} = \sum_{\theta \in \Theta^i_{\sigma,\gamma,j}} m_\theta$. Let $j \in I_{q_i}$. Then

$$\begin{aligned}
\mu(A^i_{\sigma,\gamma,j}) &= \mu(\bigcup_{\theta \in \Theta^i_{\sigma,\gamma,j}} \bigcup_{l=1}^{m_\theta} \alpha_{r_{\sigma(j)}} A_{\theta,l}) \\
&= \sum_{\theta \in \Theta^i_{\sigma,\gamma,j}} \sum_{l=1}^{m_\theta - 1} \frac{1}{n} + \sum_{\theta \in \Theta^i_{\sigma,\gamma,j}} \mu(A_{\theta,m_\theta}) \\
&= (n^i_{\sigma,\gamma,j} - |\Theta^i_{\sigma,\gamma,j}|)\frac{1}{n} + \sum_{\theta \in \Theta^i_{\sigma,\gamma,j}} \mu(A_{\theta,m_\theta}).
\end{aligned}$$

Since $|\mu(A_{\theta,m_\theta}) - \frac{1}{n}| < \frac{\epsilon}{n}$ for each $\theta \in \Theta'$ we have

$$|\Theta^i_{\sigma,\gamma,j}|(\frac{1}{n} - \frac{\epsilon}{n}) \leq \sum_{\theta \in \Theta^i_{\sigma,\gamma,j}} \mu(A_{\theta,m_\theta}) \leq |\Theta^i_{\sigma,\gamma,j}|(\frac{1}{n} + \frac{\epsilon}{n}).$$

In particular, we have

$$|\mu(A^i_{\sigma,\gamma,j}) - \frac{n^i_{\sigma,\gamma,j}}{n}| \leq |\Theta^i_{\sigma,\gamma,j}|\frac{\epsilon}{n} < |\Theta^i_{\sigma,\gamma,j}|\frac{1}{n}\frac{1}{2|\Theta'|} < \frac{1}{2n}$$

for all $j \in I_{q_i}$. Suppose $j, j' \in I_{q_1}$. Since μ is α-invariant we have $\mu(A^i_{\sigma,\gamma,j}) = \mu(A^i_{\sigma,\gamma,j'})$. It then follows that $-1 < n^i_{\sigma,\gamma,j} - n^i_{\sigma,\gamma,j'} < 1$ whence $n^i_{\sigma,\gamma,j} = n^i_{\sigma,\gamma,j'}$ for all $j, j' \in I_{q_i}$.

For each $\theta \in \Theta'$ we choose an arbitrary clopen partition of A_θ consisting of m_θ nonempty atoms. Denote this clopen partition by $\{A'_{\theta,l}\}_{l=1}^{m_\theta}$. Let $I_\theta = \{1, 2, \dots, m_\theta\}$. Given $j \in I_{q_i}$ define $B^i_{\sigma,\gamma,j} = \{A'_{\theta,l} : \theta \in \Theta^i_{\sigma,\gamma,j},\ l \in I_\theta\}$. Since $n^i_{\sigma,\gamma,j} = n^i_{\sigma,\gamma,j'}$ for all $j, j' \in I_{q_i}$ we may choose a bijection $\phi^i_{\sigma,\gamma,j}$ from $B^i_{\sigma,\gamma,j}$ onto $B^i_{\sigma,\gamma,j+1}$ for all $j \in I_{q_i} \setminus \{m_{q_i}\}$. We then define a bijection $\phi^i_{\sigma,\gamma,m_{q_i}}$ from $B^i_{\sigma,\gamma,m_{q_i}}$ onto $B^i_{\sigma,\gamma,1}$ by $\phi^i_{\sigma,\gamma,m_{q_i}} = (\phi^i_{\sigma,\gamma,1})^{-1} \circ (\phi^i_{\sigma,\gamma,2})^{-1} \circ \dots \circ (\phi^i_{\sigma,\gamma,m_{q_i}-1})^{-1}$.

We now define an action $\alpha \in \mathrm{Act}(\mathbb{F}_n, K)$ in a similar manner as in Lemma 3.3.1. Let $\mathscr{B}_h = \{A'_{\theta,l} : \theta \in \Theta'_h,\ l \in I_\theta\}$ for each $h \in I_p$ and set $\mathscr{B} = \bigcup_{h \in I_p} \mathscr{B}_h$. Fix a reference set $A_0 \in \mathscr{B}$ and choose a homeomorphism $\phi_B : A_0 \to B$ for each $B \in \mathscr{B}$.

Set $\alpha'_s = \alpha_s$ for all $s \in S \setminus T$. Let $t \in T$ and $x \in X$. Then $x \in B^i_{\sigma,\gamma,j}$ for some $i \in I_q$, $\sigma \in \Sigma^i$, $\gamma \in \Gamma^i$, $j \in I_{q_i}$. Since $x \in Q_{i,j}$ it follows $\alpha_t x \in Q_{i,j'}$ for some $j' \in I_{q_i}$. Thus there exists a unique $A'_{\theta,l} \in B^i_{\sigma,\gamma,j}$ such that $x \in \alpha_{r_{\sigma(j)}} A'_{\theta,l}$. Moreover, $A'_{\theta,l}$ is identified with a unique $A'_{\theta',l'} \in B^i_{\sigma,\gamma,j'}$. We now define $\alpha'_t x = \alpha_{r_{\sigma(j')}} \circ \phi_{A'_{\theta',l'}} \circ \phi^{-1}_{A'_{\theta,l}} \circ \alpha^{-1}_{r_{\sigma(j)}} x$. It is easily verified that α' is an action for $\mathbb{F}_n$ on K. We claim $\alpha' \in W(\alpha, F, \mathcal{C})$.

Suppose $s \in F$ and $C \in \mathcal{C}$. If $s \in S \setminus T$ then $\alpha_s = \alpha'_s$ and there is nothing to check. Thus we need only consider the case that $s \in T$. As in the measure-preserving case we have $C = \bigcup \mathscr{A}_C$ where $\mathscr{A}_C = \{A \in \mathscr{A} : A \subset C\}$. Thus,

$$\alpha'_s C = \alpha'_s(\bigcup \mathscr{A}_C) = \bigcup_{A \in \mathscr{A}_C} \alpha'_s A = \bigcup_{A \in \mathscr{A}_C} \alpha_s A = \alpha_s(\bigcup \mathscr{A}_C) = \alpha_s C.$$

Therefore $\alpha' \in W(\alpha, F, \mathcal{C})$ as desired.

It can then be verified as in Lemma 3.3.1 that

$$\mathscr{R} = \{\alpha_{r_{h,k}} A'_{\theta,l} : h \in I_p,\ k \in I_{p_h},\ \theta \in \Theta'_h,\ l \in I_\theta\}$$

is a clopen partition of K which is pointwise permuted by α'. $\square$

Since we replaced the measurable partition in Lemma 7.2.5 with an arbitrary clopen partition with the same number of atoms, the invariant measure for the perturbed action will in general be quite different from the original measure. However, we may use the regularity of the original measure to choose the clopen partitions such that the perturbed measure only differs by a small amount on each Borel set of K. That such partitions exist can be seen from the following lemma.

Lemma 7.2.6. *Let μ be a nonatomic finite measure of full support on K and $\delta > 0$ be sufficiently small. Given $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ such that $\sum_{i=1}^n \lambda_i = \mu(X)$, there exists a clopen partition $\mathscr{P} = \{P_1, P_2, \ldots, P_n\}$ such that $|\mu(P_i) - \lambda_i| < \delta$ for all $i = 1, 2, \ldots, n$:*

Proof. Since μ is nonatomic we choose $R_1 \subset K$ such that $\mu(P_1) = \lambda_1$. Since μ is regular, there exist an open set O_1 containing R_1 and a compact set K_1 contained in R_1 such that $\mu(O_1) - \mu(R_1) < \frac{\delta}{n}$ and $\mu(R_1) - \mu(K_1) < \frac{\delta}{n}$. For each $x \in K_1$ choose a clopen set C_x such that $x \in C_x \subset O_1$. Then by compactness there exists $x_1, x_2, \ldots, x_m \in K_1$ such that $K_1 \subset \bigcup_{i=1}^m C_{x_i} \subset O_1$. Let $P_1 = \bigcup_{i=1}^m C_{x_i}$. Then P_1 is clopen and

$$\lambda_1 - \frac{\delta}{n} < \mu(K_1) < \mu(P_1) < \mu(O_1) < \lambda_1 + \frac{\delta}{n}.$$

That is $|\mu(P_1) - \lambda_1| < \frac{\delta}{n}$. Then $K \setminus P_1$ is clopen and we may repeat the preceding argument to find $P_2 \subset K \setminus P_1$ such that $|\mu(P_2) - \lambda_2| < \frac{\delta}{n}$. Continuing to repeat this process we obtain pairwise disjoint clopen sets $P_1, P_2, \ldots, P_{n-1}$ such that $|\mu(P_i) - \lambda_i| < \frac{\delta}{n}$ for each $i = 1, 2, \ldots, n-1$. Set $P_n = K \setminus (\bigcup_{i=1}^n P_i)$. Then P_n is clopen and

$$\begin{aligned}
|\mu(P_n) - \lambda_n| &= |\mu(K) - \mu(\bigcup_{i=1}^{n-1} P_i) - \lambda_n| \\
&= |\sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n-1} \mu(P_i) - \lambda_n| \\
&= |\sum_{i=1}^{n-1} \mu(P_i) - \lambda_i| \\
&= \sum_{i=1}^{n-1} |\mu(P_i) - \lambda_i| \\
&< (n-1)\frac{\delta}{n} \\
&< \delta.
\end{aligned}$$

Thus $\mathscr{P} = \{P_1, P_2, \ldots, P_n\}$ is a clopen partition with the desired property. $\square$

Theorem 7.2.3 can now be proved by inductively Lemma 7.2.5. Furthermore, we have shown the following:

Theorem 7.2.7. $\overline{\mathcal{UM}(\mathbb{F}_n, K)} = \overline{\mathcal{PP}(\mathbb{F}_n, K)}$.

In particular, Theorem 7.2.3 holds for $\alpha \in \mathrm{Act}(\mathbb{F}_n, K)$ if and only if $\alpha \in$

$\overline{\mathcal{UM}(\mathbb{F}_n, K)}$. We conclude this subsection with a few remarks on $\mathcal{UM}(\mathbb{F}_n, K)$. If $n = 1$ then $\mathcal{UM}(\mathbb{F}_1, K)$ contains all minimal homeomorphisms of K since any such homeomorphism admits an invariant measure of full support. If $n \geq 2$ then the action of $\mathbb{F}_n$ of its Gromov boundary is a minimal action on K for which there is no invariant measure. Thus $\mathcal{UM}(\mathbb{F}_n, K)$ does not contain all actions which are minimal on each generator. Finally, we note the Bernoulli shifts are contained in $\overline{\mathcal{UM}(\mathbb{F}_n, K)}$.

7.3. Topological Entropy of Actions of the Cantor Set

In Section 5 we introduced the notion of entropy for measure-preserving actions of the integers and sofic groups. In this section, we define analogous notions for topological actions of these groups. It should be noted that for classical systems there are multiple approaches to defining entropy. When the underlying compact space is metrizable these methods are equivalent. The definition we give below is most suited to the computations of entropy that follow.

Suppose (X, T) is a topological $\mathbb{Z}$-system, where (X, d) is a compact metric space. Let $\epsilon > 0$ and n be a positive integer and $A \subset X$. We say A is an (n, ϵ)-*spanning set* for X if for all $x \in X$ there exists $y \in A$ such that $d(T^i x, T^i y) < \epsilon$ for all $i = 1, 2, \ldots, n-1$. We say A is an (n, ϵ)-*separated set* if for all distinct $x, y \in E$, there exists $i \in \{1, 2, \ldots, n-1\}$ such that $d(T^i x, T^i y) > \epsilon$. Denote by $span(n, \epsilon)$ the smallest cardinality of an (n, ϵ)-spanning set and by $sep(n, \epsilon)$ the maximal cardinality of all (n, ϵ)-separated sets. We then define the *topological entropy*, denoted $h_{\text{top}}(T)$ by

$$h_{\text{top}}(T) = \lim_{\epsilon \to 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log span(n, \epsilon) \right) = \lim_{\epsilon \to 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log sep(n, \epsilon) \right).$$

In Section 5 we computed the measure-theoretic entropy of rotations of the circle and Bernoulli shifts. For comparative purposes, we compute the topological entropy of these systems as well.

Example 7.3.1. Consider the rotation of the circle $(\mathbb{T}, T_w)$. Given $x, y \in \mathbb{T}$ define $d(x, y)$ to be the length of the shortest arc between x and y. Then d is a metric on $\mathbb{T}$. Given $\epsilon > 0$ there exists a finite set $\Theta \subset [0, 2\pi)$ such that the sets $\mathbb{T} = \bigcup_{\theta\in\Theta}^{N}\{e^{i\theta'} : \theta \in (\theta - \frac{\epsilon}{2}, \theta + \frac{\epsilon}{2})\}$. It is then clear that $\{e^{i\theta} : \theta \in \Theta\}$ is $(1, \epsilon)$-spanning. Moreover, since T_w is isometric we have $\{e^{i\theta} : \theta \in \Theta\}$ is (n, ϵ)-spanning for each positive integer n. Thus $span(n, \epsilon) \leq |\Theta| < \infty$. Thus

$$h_{\text{top}}(T_w) = \lim_{\epsilon\to 0}\left(\limsup_{n\to\infty}\frac{1}{n}\log sep(n,\epsilon)\right) \leq \lim_{\epsilon\to 0}\left(\limsup_{n\to\infty}\frac{1}{n}\log|\Theta|\right) = 0.$$

Example 7.3.2. Let T be the Bernoulli shift on $X^{\mathbb{Z}}$ where $X = \{0, 1, 2, \ldots, k-1\}$ is endowed with the discrete topology and $X^{\mathbb{Z}}$ the product topology. Define $d((x_j), (y_j)) = 2^{-l}$ if $l = \min\{|j| : x_j \neq y_j\} < \infty$ and $d((x_j), (y_j) = 0$ otherwise. Then d is a metric on $X^{\mathbb{Z}}$ and generates the topology. Let $0 < \epsilon < 1$ and n be a positive integer. Define $X_n = \{(x_j) : x_j = 0 \text{ for all } j \geq n\}$. Then X_n is (n, ϵ)-separated. Since $|X_n| = k^n$ we have $sep(n, \epsilon) \geq k^n$ and thus

$$h_{\text{top}}(T) = \lim_{\epsilon\to 0}\left(\limsup_{n\to\infty}\frac{1}{n}\log sep(n,\epsilon)\right) \geq \lim_{\epsilon\to 0}\left(\limsup_{n\to\infty}\frac{1}{n}\log k^n\right) = \log k.$$

Choose l such that $2^{-l} < \epsilon$. Then X_{n+l} is an (n, ϵ)-spanning set since for a given element of X there is an element of X_{n+l} which agrees on the first $n + l$ coodinates.

$$h_{\text{top}}(T) = \lim_{\epsilon\to 0}\left(\limsup_{n\to\infty}\frac{1}{n}\log span(n,\epsilon)\right) \leq \lim_{\epsilon\to 0}\left(\limsup_{n\to\infty}\frac{1}{n}\log k^{n+l}\right) = \log k.$$

Thus $h_{\text{top}}(T) = \log k$.

We have seen for the Bernoulli shift T on $\{0, 1, \ldots, k-1\}^{\mathbb{Z}}$ that $h_\mu(T)$ can take any value in $(0, \log k]$ depending on the measure assigned to X. In particular, we have

$h_\mu(T) \leq h_{\text{top}}(T)$. In general, a stronger statement actually holds. For a topological action (X, T) let $\mathcal{M}(X)$ be the set of all T invariant probability measures on X. We then have

$$h_{\text{top}}(T) = \sup_{\mu \in \mathcal{M}(X)} h_\mu(T).$$

This relationship is known as the *variational principle*.

We now turn our attention to entropy of actions of sofic groups. The definition we give here is due to Kerr and Li [19]. Unless otherwise specified, G will be a sofic group with fixed sofic approximation $\Sigma = \{\sigma_i : G \to \text{Sym}(m_i)\}_{i=1}^{\infty}$. Let $\alpha \in \text{Act}(G, X)$ and $\mathcal{P}$ be a finite partition of unity in $C(X)$. Suppose σ is a map from G into the symmetric group $\text{Sym}(d)$ for some positive integer d, $F \subset G$ is finite, and $\delta > 0$. Define $\text{Hom}_\sigma(\alpha, \mathcal{P}, F, \delta)$ to be the set of all unital homomorphisms $\psi : C(X) \to \mathbb{C}^d$ such that

$$\max_{p \in \mathcal{P}, s \in F} \|\psi \circ \alpha_s(p) - \sigma(s) \circ \psi(p)\|_2 < \delta.$$

Given $\epsilon \geq 0$ denote by $N_\epsilon(\text{Hom}_\sigma(\alpha, \mathcal{P}, F, \delta), \rho)$ the maximal cardinality of an ϵ-separated set with respect to the pseudometric

$$\rho(\phi, \psi) = \max_{p \in \mathcal{P}} \|\phi(p) - \psi(p)\|_2.$$

We then define

$$\begin{aligned}
h_\Sigma^\epsilon(\alpha, \mathcal{P}, F, \delta) &= \limsup_{i \to \infty} \frac{1}{m_i} N_\epsilon(\text{Hom}_{\sigma_i}(\alpha, \mathcal{P}, F, \delta), \rho), \\
h_\Sigma^\epsilon(\alpha, \mathcal{P}, F) &= \inf_{\delta > 0} h_\Sigma^\epsilon(\alpha, \mathcal{P}, F, \delta), \\
h_\Sigma^\epsilon(\alpha, \mathcal{P}) &= \inf_{F \subset G} h_\Sigma^\epsilon(\alpha, \mathcal{P}, F), \\
h_\Sigma(\alpha, \mathcal{P}) &= \inf_{\epsilon > 0} h_\Sigma^\epsilon(\alpha, \mathcal{P}).
\end{aligned}$$

As in the measure-preserving case, the infimum in the second to last line is over all nonempty finite subsets of G and $h_\Sigma(\mathcal{P}) = -\infty$ if $\mathrm{Hom}(\mathcal{P}, F, \delta, \sigma_i)$ is empty for all sufficiently large i. When $\mathcal{P}$ and $\mathcal{R}$ are finite generating partitions of unity we have $h_\Sigma(\alpha, \mathcal{P}) = h_\Sigma(\alpha, \mathcal{R})$. We define the Σ-*entropy* of α, denoted $h_\Sigma(\alpha, G)$, to be the common value $h_\Sigma(\alpha, \mathcal{P})$ over all generating partitions of unity of $C(X)$.

It is shown in [19] that variational principle is valid for topological entropy of sofic groups. The consequence of this is that if the Σ-entropy of an action is greater than or equal to zero, then the set of invariant probability measures is nonempty. Thus, our assumption of an invariant measure of full support in Theorem 7.2.3 is natural in this setting. We now turn our attention to showing set of actions with entropy equal to zero or negative infinity is a G_δ set.

Given $\kappa > 0$ denote by $H_{\Sigma,\kappa}(G, X)$ the set $\{\alpha \in \mathrm{Act}(G, X) : h_\Sigma(\alpha, G) < \kappa\}$. Let β be the Bernoulli shift on $\{1, 2, \ldots, k\}^G$. It is shown in [19] that $h_\Sigma(\beta, G) = \log k$ as expected. In particular, $H_{\Sigma,\kappa}(G, X)$ is nonempty for each $\kappa > 0$. Let $\mathcal{P}$ be a finite generating partition of unity of $C(X)$. Suppose $\alpha \in H_{\Sigma,\kappa}(G, X)$. Then we may find a finite set $F \subset G$ and $\epsilon, \delta > 0$ such that $h^\epsilon_\Sigma(\alpha, \mathcal{P}, F, \delta) < \kappa$. Suppose $\beta \in W(\alpha, F, \mathcal{P}, \frac{\delta}{2})$. Let $I \in \mathbb{N}$ be such that $\frac{1}{m_i} N_\epsilon(\mathrm{Hom}_{\sigma_i}(\alpha, \mathcal{P}, F, \delta), \rho) < \kappa$ if $i \geq I$. If $\phi \in \mathrm{Hom}_{\sigma_i}(\beta, \mathcal{P}, F, \frac{\delta}{2})$ for $i \geq I$ then for $p \in \mathcal{P}$ and $s \in F$ we have

$$\begin{aligned}
\|\psi \circ \alpha_s(p) - \sigma_i(s) \circ \psi(p)\|_2 &\leq \|\psi \circ \alpha_s(p) - \psi \circ \beta_s(p)\|_2 + \|\psi \circ \beta_s(p) - \sigma_i(s) \circ \psi(p)\|_2 \\
&\leq \|\psi\| \|\alpha_s(p) - \beta_s(p)\|_2 + \|\psi \circ \alpha_s(p) - \sigma_i(s) \circ \psi(p)\|_2 \\
&< \|\psi\| \frac{\delta}{2} + \frac{\delta}{2} \\
&< \delta.
\end{aligned}$$

In particular, for each $i \geq I$ we have $\mathrm{Hom}_{\sigma_i}(\beta, \mathcal{P}, F, \frac{\delta}{2}) \subset \mathrm{Hom}_{\sigma_i}(\alpha, \mathcal{P}, F, \delta)$ and consequently $N_\epsilon(\mathrm{Hom}_{\sigma_i}(\beta, \mathcal{P}, F, \frac{\delta}{2}), \rho) \leq N_\epsilon(\mathrm{Hom}_{\sigma_i}(\alpha, \mathcal{P}, F, \delta), \rho)$. It now follows directly that $h_\Sigma(\beta, G) < \kappa$, whence $H_{\Sigma,\kappa}(G, X)$ is open. Define

$$H_{\Sigma,0}(G, X) = \{\alpha \in \mathrm{Act}(G, X) : h_\Sigma(\alpha, G) \in \{0, -\infty\}\}.$$

It clear that $H_{\Sigma,0}(G, X) = \cap_{i=1}^{\infty} H_{\Sigma,\kappa_i}(G, X)$ where $\{\kappa_i\}_{i=1}^{\infty}$ is a sequence of real numbers decreasing to zero. In particular, $H_{\Sigma,0}(G, X)$ is a G_δ.

For actions of the Cantor set we now prove a topological analog of Lemma 5.3.3.

Lemma 7.3.3. *Let $\alpha \in \mathrm{Act}(G, K)$. Suppose there exists $s \in G$ of infinite order such that $\alpha_s = \alpha_e$. Then $h_\Sigma(\alpha, G) \in \{0, -\infty\}$.*

Proof. Consider K as a subset of the unit interval $[0, 1]$. Let f be the identity function on $[0, 1]$ restricted to K. Define $\mathcal{P} = \{f, 1 - f\}$. Then $\mathcal{P}$ is a a partition of unity which generates $C(K)$. In particular we have $h_\Sigma(\alpha, G) = h_\Sigma(\alpha, \mathcal{P})$. Let $\epsilon > 0$. Then we may take a clopen partition $\mathscr{P} = \{P_1, P_2, \ldots, P_n\}$ of K such that $\mathrm{Diam}(f(P_i)) < \epsilon$ for each $i = 1, 2, \ldots, n$ and for all $i \neq j$ we have $d(f(B_i), f(B_j)) > \eta$ for some $\eta > 0$.

Let $F = \{s, e\}$ and $\delta > 0$ be such that $\sqrt{\delta} < \eta$. Given $i \in \mathbb{Z}$ and $\phi \in \mathrm{Hom}_{\sigma_i}(\alpha, \mathcal{P}, F, \delta)$ define a parition $\mathscr{C}_\phi$ of $\{1, 2, \ldots, m_i\}$ into sets $C_{\phi,1}, C_{\phi,2}, \ldots, C_{\phi,n}$ such that $a \in C_{\phi,k}$ if and only if $\phi(f)(a) \in f(P_k)$. Since $\alpha_s = \alpha_e$ we have,

$$\begin{aligned}\|\sigma_i(s) \circ \phi(f) - \phi(f)\|_2 &= \|\sigma_i(s) \circ \phi(f) - \alpha_e \circ \phi(f)\|_2 \\ &= \|\sigma_i(s) \circ \phi(f) - \alpha_s \circ \phi(f)\|_2 \\ &< \delta.\end{aligned}$$

Thus for a large proportion of $a \in \{1, 2, \ldots, m_i\}$ it follows that

$$\|\sigma_i(s) \circ \phi(f)(a) - \phi(f)(a)\| < \sqrt{\delta} < \kappa.$$

Moreover this porportion tends to 1 as $\delta \to 0$. In particular we have for all $k = 1, 2, \ldots, n$ we have $\nu(\sigma_i(s)C_k \triangle C_k) < \delta'$ for some $\delta' \to 0$ as $\delta \to 0$.

Suppose $\phi, \psi \in \mathrm{Hom}_{\sigma_i}(\alpha, \mathcal{P}, F, \delta)$ are distinct. If the associated partition $\mathscr{C}_\phi$ and $\mathscr{C}_\psi$ are equal then we have $\|\phi(f) - \psi(f)\|_\infty < \epsilon$. Since ϕ and ψ are unital we have $\|\phi(1-f) - \psi(1-f)\|_\infty = \|\phi(1) - \phi(f) - \psi(1) - \psi(f)\|_\infty = \|\phi(f) - \psi(f)\|_\infty < \epsilon$. Thus $\rho(\phi, \psi) < \epsilon$. In particular the cardinality of an ϵ-separated set in $\mathrm{Hom}_{\sigma_i}(\alpha, \mathcal{P}, F, \delta)$ is bounded by the number of distinct partition of $\{1, 2, \ldots, m_i\}$ such that $\nu(\sigma_i(s)C_k \triangle C_k) < \delta'$. In the proof of Lemma 5.3.3 we showed the number of such partitions is bounded by $e^{m_i\delta''}$ where $\delta'' \to 0$ as $\delta' \to 0$. In particular we have have that

$$h^\epsilon_\Sigma(\alpha, \mathcal{P}, F, \delta) = \limsup_{i\to\infty} \frac{1}{m_i} N_\epsilon(\mathrm{Hom}_{\sigma_i}(\alpha, \mathcal{P}, F, \delta), \rho) \leq \limsup_{i\to\infty} \frac{1}{m_i} \log e^{m_i\delta''} = \delta''$$

and thus

$$h^\epsilon_\Sigma(\alpha, \mathcal{P}, F) = \inf_{\delta>0} h^\epsilon_\Sigma(\alpha, \mathcal{P}, F, \delta) \leq 0.$$

In particular, we now have $h_\Sigma(\alpha, G) \in \{0, \infty\}$ as desired. $\square$

It now follows that:

Theorem 7.3.4. *$H_{\Sigma,0}(\mathbb{F}_n, K)$ is generic in $\overline{\mathcal{UM}(\mathbb{F}_n, K)}$.*

8. CONCLUSION

In the classical setting the Rokhlin lemma is an important tool for many applications. It seems probable that the results of Section 3 should have applications beyond those presented in Sections 4, 5, and 6. In recent years several results on generic properties of homeomorphisms of the Cantor set have appeared in the literature. However, to the best of our knowledge, the results in Section 7 are the first for topological actions of more general groups on the Cantor set. Upon further investigation, the developments in Section 7 should allow for the establishment of other genericity results in $\overline{\mathcal{UM}(\mathbb{F}_n, K)}$.

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