NONLINEAR ANALYSIS OF BEAMS USING LEAST-SQUARES FINITE ELEMENT MODELS BASED ON THE EULER-BERNOULLI AND TIMOSHENKO BEAM THEORIES

A Thesis

by

AMEETA AMAR RAUT

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

December 2009

Major Subject: Mechanical Engineering

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Approved by:

Chair of Committee,	J. N. Reddy
Committee Members,	Ibrahim Karaman
	Jose Roesset
Head of Department,	Dennis O'Neal

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ABSTRACT

Nonlinear Analysis of Beams Using Least-Squares Finite Element Models Based on the Euler-Bernoulli and Timoshenko Beam Theories.

(December 2009)

Ameeta Amar Raut, B.E., Government College of Engineering, Pune, India

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The conventional finite element models (FEM) of problems in structural mechanics are based on the principles of virtual work and the total potential energy. In these models, the secondary variables, such as the bending moment and shear force, are post-computed and do not yield good accuracy. In addition, in the case of the Timoshenko beam theory, the element with lower-order equal interpolation of the variables suffers from shear locking. In both Euler-Bernoulli and Timoshenko beam theories, the elements based on weak form Galerkin formulation also suffer from membrane locking when applied to geometrically nonlinear problems. In order to alleviate these types of locking, often reduced integration techniques are employed. However, this technique has other disadvantages, such as hour-glass modes or spurious rigid body modes. Hence, it is desirable to develop alternative finite element models that overcome the locking problems. Least-squares finite element models are considered to be better alternatives to the weak form Galerkin finite element models and, therefore, are in this study for investigation. The basic idea behind the least-squares finite element model is to compute the residuals due to the approximation of the variables of each equation being modeled, construct integral statement of the sum of the squares of the residuals (called least-squares functional), and minimize the integral with respect to the unknown parameters (i.e., nodal values) of the approximations. The least-squares formulation helps to retain the generalized displacements and forces (or stress resultants) as independent variables, and also allows the use of equal order interpolation functions for all variables.

In this thesis comparison is made between the solution accuracy of finite element models of the Euler-Bernoulli and Timoshenko beam theories based on two different least-square models with the conventional weak form Galerkin finite element models. The developed models were applied to beam problems with different boundary conditions. The solutions obtained by the least-squares finite element models found to be very accurate for generalized displacements and forces when compared with the exact solutions, and they are more accurate in predicting the forces when compared to the conventional finite element models.

// Shree Ganeshayanamaha //

To my beloved Mumma, Papa, Grandparents, Deepak

and

Dr. Reddy

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NOMENCLATURE

FEM	Finite Element Method
EBT	Euler-Bernoulli beam Theory
TBT	Timoshenko Beam Theory
$V(x)$	Internal Transverse Shear Force
N_{xx}	Internal Axial Force
M_{xx}	Internal Bending Moment
$f(x)$	External Axial Force
$q(x)$	Transverse Distributed Load
A_{xx}^{e}	Extensional Stiffness (*EA*)
B_{xx}^{e}	Extensional-Bending Stiffness
D_{xx}^{e}	Bending Stiffness (*EI*)
Q_{i}^{e}	Nodal Force
Δ_{i}^{e}	Nodal Displacement of the Element
A^{e}	Cross Sectional Area
I^{e}	Second Moment of Area of the Beam
ψ_{j}	Lagrange Interpolation Functions

ϕ_j	Hermite Interpolation Functions
{R}	Residual Vector
{T}	Tangent Matrix
σ_{ij}	Cartesian Component of Stress Tensor
ε_{ij}	Cartesian Component of Strain Tensor
$W_E^{\ e}$	Work Done by External Forces
$W_I^{\ e}$	Work Done by Internal Forces
S_{xx}	Shear Stiffness (GAK_s)
G	Shear Modulus
E	Young's Modulus
K_s	Shear Correction Factor

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1. INTRODUCTION

1.1 Motivation

The finite element method (FEM) is a powerful technique originally developed for numerical solution of complex problems in structural mechanics. The two broad categories into which finite element models can be divided are those based on minimization principles (like in structural mechanics) [1,2] and those based on weighted-residual methods such as the Galerkin method, Petrov-Galerkin method, subdomain method, least-squares method and so on.

There are some numerical challenges that are encountered with conventional finite element models based on the weak form Galerkin formulation, which is the most common in practice. In these models, the secondary variables such as the bending moment and shear force are post-computed, typically at Gauss points and not at the nodes, and do not yield good accuracy. In addition, in the case of the Timoshenko beam theory, the element with lower-order equal interpolation of the generalized displacements suffers from shear locking. In both Euler-Bernoulli and Timoshenko beam theories, the elements based on the weak form Galerkin formulation also suffer from membrane locking [3,4] when applied to geometrically nonlinear problems. Both types of locking are a result of using inconsistent interpolation for the variables involved in the formulation. In order to alleviate these types of locking, often reduced integration techniques are employed. However, such ad-hoc techniques have other disadvantages, such as hour-glass modes or spurious rigid body modes.

This thesis follows the style and format of *Finite Elements in Analysis and Design*.

Thus, it is desirable to develop alternative finite element models that overcome the locking problems and yield good accuracy for stress resultants. Least-squares finite element models are considered to be alternatives to the weak form Galerkin finite element model and thus considered in this study for investigation. The least-squares formulation helps to retain the generalized displacements and forces (or stress resultants) as independent variables, and also allows the use of equal order interpolation functions for all variables.

1.2 Objectives of the Present Study

The purpose of this study is to investigate the effectiveness of the least-squares based finite element models in solving the beam bending problems to overcome shear and membrane locking and predict generalized forces accurately. This study is conducted using the Euler-Bernoulli and Timoshenko beam theories applied to straight beams. The solution accuracy of the least-squares finite element models with conventional finite element models is also assessed.

To achieve the defined objectives, different finite element models of the two beam theories are developed and are applied to beam problems with different boundary conditions. The solution obtained by the least-squares formulation is compared to the solutions obtained from the conventional, weak form Galerkin finite element models.

The following discussion provides the background for the present study.

1.3 Background and Literature Review

A beam is a structural element that has a very large ratio of its length to its cross sectional dimension and is capable of carrying loads by stretching along its length and bending about an axis transverse to its length. When transverse loads are applied on a beam, internal forces are generated which resist the deformation of the beam. If the applied load is large, the magnitude of the internal forces increases. At the same time the deformation of the beam also increases. Consequently, the linear relationship between loads and displacements of the beam is no longer valid.

Depending on the kinematic assumptions, two different theories are often used to model the structural behavior of beams:

1) Euler- Bernoulli beam theory (EBT)
2) Timoshenko beam theory(TBT)

In the Euler Bernoulli beam theory, one neglects the effect of the transverse shear strain whereas in the Timoshenko beam theory it is taken into account.

Both shear and membrane locking in beams are primarily due to the use of inconsistent interpolation of the variables. When equal and lower order interpolation of the displacement and rotation are used in the Timoshenko beam finite element, the element exhibits locking as it is unable to cope with the constraint that the slope should be compatible with the derivative of the deflection in the thin beam limit. The problem of shear locking is often overcome by numerically mimicking different variation (i.e., constant and linear) of the rotation function in shear energy and bending energy through numerical integration [2]. There are several other approaches that have been adopted to eliminate locking [1, 2, 5-10]. The concept of locking was first discussed by Kikuchi and Aizawa [5], and Zienkiewicz and Owen [11] advocated that the reduced integration technique is a means of obtaining accurate solutions. However, such ad-hoc approaches have other disadvantages, such as appearance of hour-glass modes or spurious rigid body modes. Hence, it is desirable to develop alternative finite element models that overcome the locking problems.

In the past few years finite element methods based on least-squares variational principles have drawn considerable attention. It is a general methodology that produces a wide range of algorithms [9]. Given a set of differential equations, the least-squares method allows one to define a convex, unconstrained minimization principle so that the finite element model can be developed in Ritz or weak form Galerkin setting [2]. This model has proved to result in a positive-definite system of equations and significant savings in the computational cost [12].

The least-square approach has been implemented in the finite element context to solve the problems of plate bending, shear-deformable shells, incompressible and compressible fluid flows [1, 13-15] etc. However, there has been no systematic study involving the development of least-squares finite element models of beam theories and their assessment in comparison to the conventional beam finite elements. The present study also accounts for geometric nonlinearity in the von karaman sense.

2. ALTERNATIVE FINITE ELEMENT MODELS

2.1 Introduction

A mathematical model is a set of equations, algebraic as well as differential, which is used to describe the response of a physical system in terms of certain variables. The mathematical models of most mechanical systems are derived using the principles of physics, such as the conservation of mass, conservation of linear momentum, and conservation of energy. The derivation of the governing equations is not as challenging as solving them and computing accurate solution. Numerical methods help to convert these governing differential equations to a set of algebraic equations that can be solved using computers. While solving such equations proper care must be taken to preserve all features of the mathematical model (which reflects the physics of the problem) in the formulation and development of the associated computational model.

There are several methods to obtain numerical solutions of ordinary and partial differential equations. These include the *finite difference method, traditional variational methods (e.g., Ritz and Galerkin methods), the finite element method,* etc. In the finite difference method, the derivatives in the governing differential equations are replaced by discrete values. In a variational approach, the variable(s) of a differential equation are approximated as a linear combination of unknown parameters and known functions, $u(x) \approx U(x) = \sum_{j=1}^{n} c_j \phi_j(x) + \phi_0(x)$, and the parameters c_i are then determined by satisfying the differential equations in a weighted-residual sense (see Reddy [3]). In the finite element method, the domain of the problem is divided into a collection of subdomains (called finite elements), and over each subdomain a variational method is used to set up the discrete problem. The element equations are then put together to obtain a system of algebraic equations for the assemblage of elements. Different types of finite element models are obtained by using different weighted-integral statement. These are discussed in the following section.

2.2 Different Integral Formulations and Finite Element Models

Based on the method used to derive the algebraic equations of a mathematical model, different finite element models of the mathematical model can be developed. These alternative methods are discussed next.

1) ***The Ritz Method***: Here the coefficients of the approximation are determined by minimizing a functional (i.e., first variation of I is equal to zero) equivalent to the governing differential equation $Au - f = 0$,

$$I(u) = \frac{1}{2} B(u,u) - l(u), \quad \delta I = 0 \Rightarrow B(\delta u, u) = l(\delta u) \tag{1}$$

Then the approximations

$$u(x) \approx U_N(x) = \sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x), \;\; \delta u \approx \sum_{j=1}^{N} \delta c_j \phi_j(x) \tag{2}$$

are substituted for u and δu into Eq. (1) to obtain the Ritz finite element model

$$\begin{aligned}
& B\left(\sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x) \right) = l\left(\phi_i(x)\right) \\
& \sum_{j=1}^{N} B\left(\phi_j(x), \phi_i(x)\right) c_j = l\left(\phi_i(x)\right) - B\left(\phi_i(x), \phi_0(x)\right) \\
& or \quad \sum_{j=1}^{N} K_{ij} c_j = F_i \qquad\qquad i = 1,2,3,\ldots.N \\
& K_{ij} = B\left(\phi_j(x), \phi_i(x)\right), \\
& F_i = l\left(\phi_i(x)\right) - B\left(\phi_i(x), \phi_0(x)\right)
\end{aligned}$$

2) ***Weighted Residual Method***: In the weighted residual method, the approximate solution is substituted into the differential equation $Au - f = 0$ and the resulting residual $R = AU - f \neq 0$ is minimized with respect to a weight function. Depending on the choice of the weight function various models can be derived. Various subclasses of the weighted residual method are summarized below. In the general weighted-residual method, we require

$$\int_{\Omega} \psi_i(\mathrm{x}) R(\mathrm{x}, c_j) dxdy = 0 \qquad where(i = 1, 2.....N)$$

where

$$R \equiv A(U_N) - f = A\left(\sum_{j=1}^{N} c_j \phi_j(x) + \phi_0(x)\right) - f \neq 0$$

(a) ***The Petrov-Galerkin Method*** The above weighted residual method is called Petrov-Galerkin method when $\psi_i \neq \phi_i$

$$\sum_{\mathrm{j}=1}^{N} \left[\int_{\Omega} \psi_i A(\phi_j) dx\right] c_j = \int_{\Omega} \psi_i \left[f - A(\phi_0)\right] dx$$

(b) ***The Galerkin Method***: If $\psi_i = \phi_i$ then the weighted residual method is called Galerkin method.

$$A_{ij} = \int_{\Omega} \phi_i A(\phi_j) dx$$
$$F_i = \int_{\Omega} \phi_i \left[f - A(\phi_0)\right] dx$$

The approximation functions used here are of much higher order than the one used in the Ritz method.

(c) ***The Collocation Method***

Here the approximation functions are selected such that the residual will be zero simultaneously. Thus we have $R(x^i, c_j) = 0 \qquad (i = 1, 2....N)$.

(d) ***The Least-Squares Method***

The basic concept behind the least squares method is that it minimizes the square of the residual. The parameter c_j is determined by minimizing the integral of the square of the residual.

$$\frac{\partial}{\partial c_i}\int_{\Omega} R^2\left(\mathrm{x}, c_j\right) dx = 0$$

where

$$R^2 = R_1^2 + R_2^2,\ R_1 = A(u_h) - f,\ R_2 = B(u_h) - g \text{ and}$$

$A(u) = f$ in Ω and $\mathrm{B}(u) = g$ in Γ are the functions.

In the present study, the least squares method is used to formulate the finite element models of the Euler-Bernoulli beam theory (EBT) and the Timoshenko beam theory (TBT).

2.3 Summary

Thus FEM is a numerical method that can be a used to obtain a numerical solution where an analytical solution cannot be developed. FEM was originally developed for analysis of aircraft structures. However due to its general nature it has been applicable in a wide range of problems in structural mechanics, fluid mechanics, electrical engineering etc. This section discusses different types of formulations in finite element analysis. This thesis will discuss more about the theory, formulations and finite element model for least-squares based finite element formulation in details in the subsequent sections. This study will be conducted specifically for beams as they are widely used in many structural applications.

3. THEORETICAL FORMULATION OF EBT AND TBT

3.1 Background

A beam is a structural element that has a very large ratio of its length to its cross section dimension. It can be subjected to a transverse load which includes the normal and the shear stress and the displacements are perpendicular to the normal axis. Beams can be straight or curved. A straight beam is usually modeled by a line segment with vertical displacement and rotations at each end.

When the load is applied on a beam, internal forces are generated which resist the deformation of the beam. If the applied load is large, the magnitude of the internal forces increases. At the same time the deformation of the beam also increases. Thus the linear relationship between load v/s deflection of the beam is no more valid.

The following assumptions are made in the development of linear motion of solid bodies:

1) The displacements are small.
2) The strains developed are very large.
3) The material is linearly elastic.

Due to the small strains the changes in the geometry are ignored. The equilibrium equations are developed for the undeformed configuration. But if the load increases the linear relationships do not hold true. Hence for a general nonlinear formulation of straight or curved beams, the measures of stress and strain consistent with the deformations must be accounted in the formulation.

The following assumptions are made in the study of nonlinear analysis of beams here:

1) The beam is long and thin
2) The transverse displacements are large.
3) The strains developed are very small.
4) The rotations developed are small.

The inplane forces are proportional to the square of the rotation of the transverse normal to the beam axis and are responsible for the nonlinearity.

Depending on the assumptions for transverse shear strain there are two different theories to model the beams:

3) Euler- Bernoulli beam theory (EBT)
4) Timoshenko beam theory(TBT)

The Euler Bernoulli beam theory neglects the effect of the transverse shear strain whereas the Timoshenko beam theory takes into account the effect of transverse shear strain in the formulation.

3.2 Euler-Bernoulli Beam Theory

EBT is the simplest beam theory and is based on displacement field. The following sections will discuss about EBT in detail.

3.2.1 Assumptions

The basic assumptions made in developing the governing equations of EB hypothesis are the plane cross sections perpendicular to the beam axis before deformation remain (a) plane (b) rigid (c) rotate such that they remain perpendicular to the beam axis after deformation.

These assumptions neglect the Poisson's effect and the transverse strain. These two assumptions are taken into account in Timoshenko beam theory.

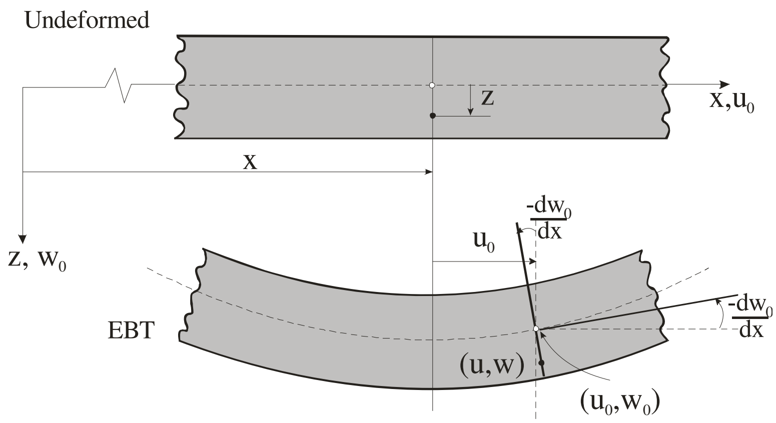


Figure 3.1. Deformation of a beam in Euler-Bernoulli theory

3.2.2 Displacement fields

The displacement field for beams having moderately large rotations but small strains derived from Figure 3.1 is:

$$u_1 = u_0(x) - z\frac{dw_0}{dx},\ u_2 = 0 \text{ and } u_3 = w_0(x) \tag{3.1}$$

where, (u_1, u_2, u_3) are the displacement along (x, y, z) axis and

u_0 is the axial displacement of a point on the neutral axis and

w_0 is the transverse displacement of the point on the neutral axis

3.2.3 Nonlinear strain-displacement relations

The following nonlinear strain-displacement relation is used to calculate the strains

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right) \tag{3.2}$$

Substituting the values of u_1, u_2 and u_3 in the above equations and eliminating the large strain terms but retaining the rotation terms of the transverse normal we get,

$$\varepsilon_{11} = \varepsilon_{xx} = \frac{du_0}{dx} - z\frac{d^2 w_0}{dx^2} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2$$

$$= \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right] - z\left(\frac{d^2 w_0}{dx^2}\right)$$

$$= \varepsilon_{xx}^0 + z\varepsilon_{xx}^1 \tag{3.3}$$

where,

$$\varepsilon_{xx}^0 = \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right], \varepsilon_{xx}^1 = -\left(\frac{d^2 w_0}{dx^2}\right)$$

These strains are known as *von Karman strains*.

3.2.4 Derivation of governing equations

According to the principle of virtual displacement, for a body in equilibrium, the virtual work done by the internal and external forces to move through their virtual displacements is zero. Thus based on this principle the following can be concluded.

$$\delta W^e \equiv \delta W_I^{\ e} + \delta W_E^{\ e} = 0 \tag{3.4}$$

where

$\delta W_I^{\ e}$ is the virtual strain stored in the element due to σ_{ij} (Cartesian component of stress tensor) due to the virtual displacement $\delta\varepsilon_{ij}$ (Cartesian component of strain tensor) and

$\delta W_E^{\ e}$ is the work done by external forces

Thus for a beam element we have,

$$\delta W_I^e = \int_{V_e} \delta\varepsilon_{ij}\sigma_{ij} dV$$

$$\delta W_E^e = \int_{V_e} q\delta w_0 dx + \int_{x_a}^{x_b} f\,\delta u_0 dx + \sum_{i=1}^{6} Q_i^e \delta\Delta_i^e \tag{3.5}$$

where V^e is the elemental volume, q(x) is the distributed transverse load (per unit length), f(x) distributed axial load Q_i^e is the nodal force and $\delta\Delta_i^e$ is the nodal displacement of the element. The nodal displacements and nodal forces in Figure 3.2 are defined by,

$$\Delta_1^e = u_0(x_a),\ \Delta_2^e = w_0(x_a),\ \Delta_3^e = \left(-\frac{dw_0}{dx}\right)_{x_a} \equiv \theta\left(x_a\right)$$

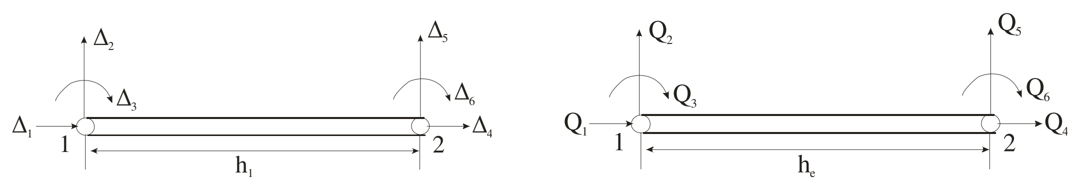
$$\Delta_4^e = u_0(x_b),\ \Delta_5^e = w_0(x_b),\ \Delta_6^e = \left(-\frac{dw_0}{dx}\right)_{x_b} \equiv \theta\left(x_b\right)$$

$$Q_1^e = -N_{xx}\left(x_a\right), \quad Q_4^e = N_{xx}\left(x_b\right)$$

and $$Q_2^e = -\left[\frac{dw_0}{dx}N_{xx} + \frac{dM_{xx}}{dx}\right]_{x_a}, \quad Q_5^e = \left[\frac{dw_0}{dx}N_{xx} + \frac{dM_{xx}}{dx}\right]_{x_b} \tag{3.6}$$

$$Q_3^e = -M_{xx}\left(x_a\right), \quad Q_6^e = M_{xx}\left(x_b\right)$$

The nodal displacements and the nodal forces derived above can be denoted as follows:



(a) (b)

Figure 3.2. (a) Nodal displacements for EBT (b) Nodal forces for EBT

The virtual strain energy equation can be simplified by substituting equation (3.3) in equation (3.5) as follows:

$$\delta W_I^e = \int_{x_a}^{x_b} \int_{A^e} \delta\varepsilon_{xx}\sigma_{xx} dA dx$$

$$= \int_{x_a}^{x_b} \int_{A^e} \left(\delta\varepsilon^0{}_{xx} + z\delta\varepsilon^1{}_{xx}\right)\sigma_{xx} dA dx$$

$$= \int_{x_a}^{x_b} \int_{A^e} \left[\left(\frac{d\delta u_0}{dx} + \frac{dw_0}{dx}\frac{d\delta w_0}{dx}\right) + z\left(\frac{d^2\delta w_0}{dx^2}\right)\right]\sigma_{xx} dA dx$$

$$= \int_{x_a}^{x_b} \int_{A^e} \left[\left(\frac{d\delta u_0}{dx} + \frac{dw_0}{dx}\frac{d\delta w_0}{dx}\right)N_{xx} + M_{xx}\left(\frac{d^2\delta w_0}{dx^2}\right)\right]dx \qquad (3.7)$$

here N_{xx} is the axial force which can be expressed as $N_{xx} = \int_{A^e} \sigma_{xx} dA$ and M_{xx} is the moment which can be expressed as $M_{xx} = \int_{A^e} \sigma_{xx} z dA$

Thus virtual work statement can be written as

$$0 = \int_{x_a}^{x_b} \left[\left(\frac{d\delta u_0}{dx} + \frac{dw_0}{dx}\frac{d\delta w_0}{dx}\right)N_{xx} - M_{xx}\left(\frac{d^2\delta w_0}{dx^2}\right)\right]dx - \int_{V_e} q(x)\,\delta w_0(x)\,dx - \int_{x_a}^{x_b} f(x)\,\delta u_0(x) dx - \sum_{i=1}^{6} Q_i^e \delta\Delta_i^e \qquad (3.8)$$

By separating the two terms involving δu_0 and δw_0 we get the following two equations

$$0 = \int_{x_a}^{x_b} \left[\left(\frac{d\delta u_0}{dx} N_{xx} - f(x)\,\delta u_0(x)\right)dx - Q_1^e \delta\Delta_1^e - Q_4^e \delta\Delta_2^e\right]dx$$

$$0 = \int_{x_a}^{x_b} \left[\frac{d\delta w_0}{dx}\left(\frac{dw_0}{dx} N_{xx}\right) - M_{xx}\frac{d^2\delta w_0}{dx^2} - q(x)\,\delta w_0(x)\right]dx - Q_2^e \delta\Delta_2^e - Q_3^e \delta\Delta_3^e - Q_5^e \delta\Delta_1^e - Q_6^e \delta\Delta_2^e \qquad (3.9)$$

Collecting the terms of δu_0 and δw_0 and simplifying the terms we get,

$$\delta u_0: \qquad -\frac{dN_{xx}}{dx} = f(x)$$

$$\delta w_0: \qquad -\frac{d}{dx}\left(\frac{dw_0}{dx} N_{xx}\right) - \frac{d^2 M_{xx}}{dx^2} = q(x) \qquad (3.10)$$

Thus the boundary conditions are:

$$Q_1^e + N_{xx}(x_a) = 0, \qquad Q_4^e - N_{xx}(x_b) = 0$$
$$Q_2^e + \left[\frac{dw_0}{dx}N_{xx} + \frac{dM_{xx}}{dx}\right]_{x_a} = 0, \qquad Q_5^e - \left[\frac{dw_0}{dx}N_{xx} + \frac{dM_{xx}}{dx}\right]_{x_b} = 0 \qquad (3.11)$$
$$Q_3^e + M_{xx}(x_a) = 0\,, \qquad Q_6^e + M_{xx}(x_b) = 0$$

3.2.5 Vector approach

In this method a beam element of length Δx is analyzed by adding the forces and the moments acting on the beam.

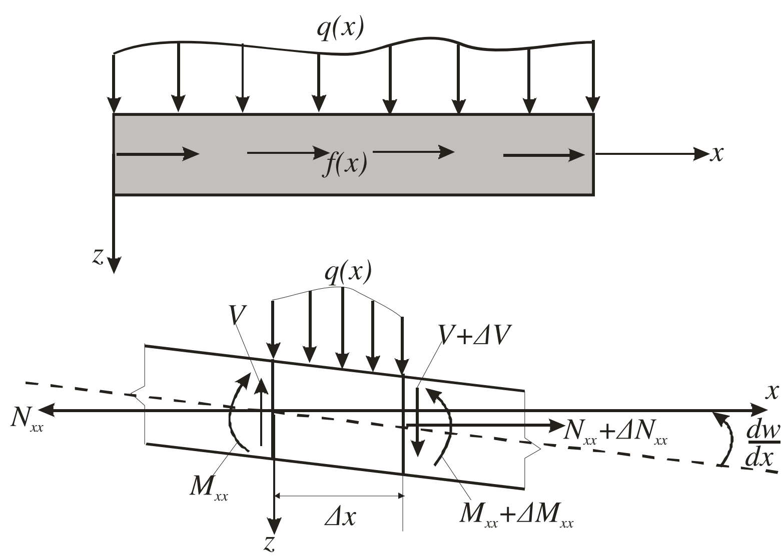


Figure 3.3. A typical beam element with forces and moments under uniformly distributed load

Consider the above beam element with forces and moments under uniformly distributed load is shown in Figure 3.3 where $V(x)$ is the internal vertical shear force,

N_{xx} is the internal axial force,

M_{xx} is the internal bending moment,

$f(x)$ is the external axial force,

$q(x)$ is the distributed load.

Using D Alembert's principle and equating the forces in the X, Y and Z direction we get

$$\sum F_x = 0: \qquad -N_{xx} + (N_{xx} + \Delta N_{xx}) + f(x)\Delta x = 0$$
$$\sum F_y = 0: \qquad -V + (V + \Delta V) + q(x)\Delta x = 0$$
$$\sum F_z = 0: \qquad -M_{xx} + (M_{xx} + \Delta M_{xx}) - V\Delta x + N_{xx}\Delta x \frac{dw_0}{dx} + q(x)\Delta x(c\Delta x) = 0$$

Thus taking the limit as $\Delta x \to 0$ we can conclude

$$\begin{gathered} \frac{dN_{xx}}{dx} + f(x) = 0 \\ \frac{dV}{dx} + q(x) = 0 \\ \frac{dM_{xx}}{dx} - V + N_{xx}\frac{dw_0}{dx} = 0 \end{gathered} \tag{3.12}$$

3.3 Timoshenko Beam Theory

3.3.1 Assumptions

As discussed earlier, basic assumptions made in developing the governing equations of EB hypothesis are the plane cross sections perpendicular to the beam axis before deformation remains (a) plane (b) rigid (c) rotation is independent of the slope of the beam. In TBT the first two assumptions are the same and the third assumption is relaxed by assuming that the rotation of the beam is independent of the slope.

3.3.2 Displacement fields

The displacement field for beams having moderately large rotations but small strains as shown in Figure 3.4 is given by

$$u_1 = u_0(x) + z\phi_x(x), \qquad u_2 = 0 \quad \text{and} \qquad u_3 = w_0(x) \tag{3.13}$$

where, (u_1, u_2, u_3) are the displacement along (x, y, z) axis, u_0 is the axial displacement of a point on the neutral axis, and w_0 is the transverse displacement of the point on the neutral axis.

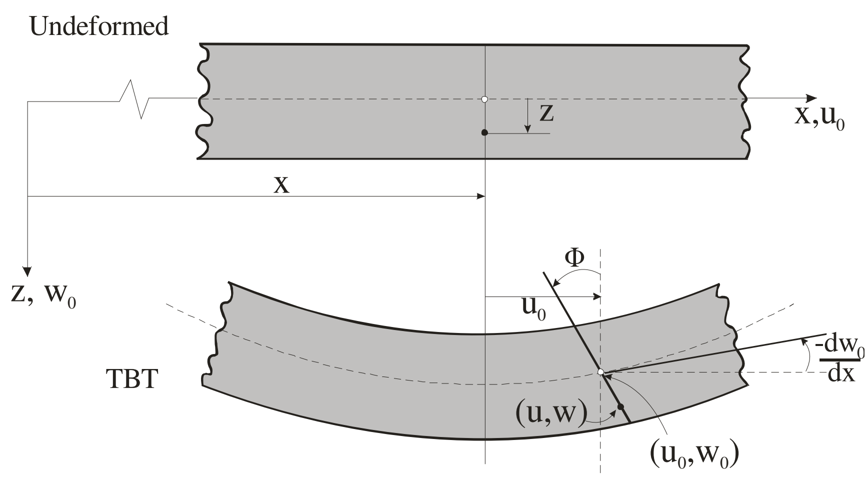


Figure 3.4. Deformation of a beam in Timoshenko theory

3.3.3 Nonlinear strain-displacement relations

The following nonlinear strain-displacement relation is used to calculate the strains as follows

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right) \tag{3.14}$$

Substituting the values of u_1, u_2 and u_3 in the above equations and eliminating the large strain terms but retaining the rotation terms of the transverse normal we get,

$$\varepsilon_{11} = \varepsilon_{xx} = \frac{du_0}{dx} - z\frac{d\phi_x}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2$$

$$= \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right] + z\left(\frac{d\phi_x}{dx}\right)$$

$$= \varepsilon_{xx}^0 + z\varepsilon_{xx}^1 \tag{3.15}$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial x} - \frac{\partial u_3}{\partial x} = \phi_x + \frac{dw_0}{dx} \equiv \gamma_{xz}^0 \tag{3.16}$$

where $\delta\varepsilon_{xx}^0 = \left[\frac{d\delta u_0}{dx} + \frac{dw_0}{dx}\frac{d\delta w_0}{dx}\right]$, $\delta\varepsilon_{xx}^1 = \left(\frac{d\delta\phi_x}{dx}\right)$, $\delta\gamma_{xz}^0 = \delta\phi_x + \frac{d\delta w_0}{dx}$

3.3.4 Derivation of governing equations

As discussed in EBT, the principle of virtual displacement states that for a body in equilibrium, the virtual work done by the internal and external forces to move through their virtual displacements is zero. Thus based on this principle the following can be concluded.

$$\delta W^e \equiv \delta W_I^{\,e} + \delta W_E^{\,e} = 0 \tag{3.17}$$

where $\delta W_I^{\,e}$ the virtual strain is stored in the element due to σ_{ij} (Cartesian component of stress tensor) due to the virtual displacement $\delta\varepsilon_{ij}$ (Cartesian component of strain tensor) and $\delta W_E^{\,e}$ is the work done by external forces.

Thus for a beam element we have,

$$\delta W_I^e = \int_{V_e} \delta\varepsilon_{ij}\sigma_{ij}\,dV$$

$$\delta W_E^e = \int_{V_e} q\delta w_0 dx + \int_{x_a}^{x_b} f\,\delta u_0 dx + \sum_{i=1}^{6} Q_i^e \delta\Delta_i^e \tag{3.18}$$

where V^e is the elemental volume, q(x) is the distributed transverse load (per unit length), f(x) distributed axial load Q_i^e is the nodal force and $\delta\Delta_i^e$ is the nodal displacement of the element.

The virtual strain energy equation can be simplified as follows:

$$\delta W_I^e = \int_{x_a}^{x_b} \int_{A^e} \left(\delta\varepsilon_{xx}\sigma_{xx} + \delta\gamma_{xz}\sigma_{xz}\right) dAdx$$

$$= \int_{x_a}^{x_b} \int_{A^e} \left(\delta\varepsilon^0{}_{xx} + z\delta\varepsilon^1{}_{xx}\right)\sigma_{xx} + \delta\gamma_{xz}^0\sigma_{xz} dAdx$$

$$= \int_{x_a}^{x_b} \int_{A^e} \left[\delta\varepsilon_{xx}^0 N_{xx} + M_{xx}\delta\varepsilon_{xx}^1 + \delta\gamma_{xz}^0 Q_x\right] dx \tag{3.19}$$

where N_{xx} is the axial force which can be expressed as $N_{xx} = \int_{A^e} \sigma_{xx} dA$ and M_{xx} is the moment which can be expressed as $M_{xx} = \int_{A^e} \sigma_{xx} z dA$

Q_x is the element force $Q_x = K_s \int_A \sigma_{xz} dA$

K_s is the shear correction coefficient which takes into account the difference between the shear energy calculated by equilibrium and by Timoshenko beam theory. Solving in the same way as EBT and collecting the terms of δu_0 and δw_0 and simplifying the terms we get,

$$\delta u_0: \quad -\frac{dN_{xx}}{dx} = f(x)$$

$$\delta\phi: \quad -\frac{dM_{xx}}{dx} + Q_{xx} = 0 \tag{3.20}$$

$$\delta w_0: \quad -\frac{d}{dx}\left(\frac{dw_0}{dx} N_{xx}\right) - \frac{d^2 M_{xx}}{dx^2} = q(x)$$

3.4 Summary

This section discusses the introduction to beams and the different assumptions made to derive the beam equation. A more detailed discussion about the two most important theories Euler-Bernoulli and Timoshenko beam theory regarding the derivation of the governing differential equations has been made in this section. The discussion of weak form development and finite element model for EBT and TBT has been done in the next section.

4. FINITE ELEMENT MODEL OF THE EBT

4.1 Weak Form Development

Using the governing equations from equations (3.12) we can develop the weak form as follows:

$$0=\int_{x_a}^{x_b} v_1\left(-\frac{dN_{xx}}{dx}-f\right)dx$$

$$=\int_{x_a}^{x_b}\left(-\frac{dv_1}{dx}N_{xx}-fv_1\right)dx-\left[v_1N_{xx}\right]_{x_a}^{x_b}$$

$$=\int_{x_a}^{x_b}\left(-\frac{dv_1}{dx}N_{xx}-fv_1\right)dx-v_1\left(x_a\right)\left[-N_{xx}\left(x_a\right)\right]-v_1\left(x_b\right)\left[N_{xx}\left(x_b\right)\right]$$

$$0=\int_{x_a}^{x_b} v_2\left[-\frac{d}{dx}\left(\frac{dw_0}{dx}N_{xx}\right)-\frac{d^2M_{xx}}{dx^2}-q\right]dx$$

$$=\int_{x_a}^{x_b}\left[-\frac{dv_2}{dx}\left(\frac{dw_0}{dx}N_{xx}\right)-\frac{d^2v_2}{dx^2}M_{xx}-qv_2\right]dx-\left[v_2\left(\frac{dw_0}{dx}N_{xx}+\frac{dM_{xx}}{dx}\right)\right]_{x_a}^{x_b}-\left[\left(\frac{dv_2}{dx}\right)M_{xx}\right]_{x_a}^{x_b}$$

$$=\int_{x_a}^{x_b}\left[-\frac{dv_2}{dx}\left(\frac{dw_0}{dx}N_{xx}\right)-\frac{d^2v_2}{dx^2}M_{xx}-qv_2\right]dx-v_2\left(x_a\right)\left[-\left(\frac{dw_0}{dx}N_{xx}-\frac{dM_{xx}}{dx}\right)\right]_{x_a}-$$

$$-v_2\left(x_b\right)\left[\left(\frac{dw_0}{dx}N_{xx}+\frac{dM_{xx}}{dx}\right)\right]_{x_b}-\left(\frac{dv_2}{dx}\right)_{x_a}\left[-M_{xx}\left(x_a\right)\right]-\left(-\frac{dv_2}{dx}\right)_{x_b}\left[M_{xx}\left(x_b\right)\right]$$

(4.1)

Here v_1 and v_2 are the weight functions which correspond to δu_0 and δw_0.

As mentioned in the assumptions earlier the EB has small to moderate rotations and the material is assumed to be linearly elastic which results in the following

$$\sigma_{xx}=E^e\varepsilon_{xx} \tag{4.2}$$

The above relationship which defines the relationship between the total stress and the total strain is called as the Hooke's law.

Thus we get

$$\begin{aligned} N_{xx} &= \int_{A^e} \sigma_{xx} dA = \int_{A^e} E^e \varepsilon_{xx} dA \\ &= \int_{A^e} E^e \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 \right] - z\left(\frac{d^2 w_0}{dx^2}\right) dA \\ &= A^e_{xx} \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 \right] - B^e_{xx}\left(\frac{d^2 w_0}{dx^2}\right) \end{aligned} \tag{4.3}$$

$$\begin{aligned} M_{xx} &= \int_{A^e} \sigma_{xx} z dA = \int_{A^e} E^e \varepsilon_{xx} z dA \\ &= \int_{A^e} E^e \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 \right] - z\left(\frac{d^2 w_0}{dx^2}\right) z dA \\ &= B^e_{xx} \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 \right] - D^e_{xx}\left(\frac{d^2 w_0}{dx^2}\right) \end{aligned} \tag{4. 4}$$

where, A^e_{xx} is the extensional stiffness

B^e_{xx} is the extensional-bending stiffness and

D^e_{xx} is the bending stiffness.

For isotropic material we have,

$A^e_{xx} = E^e A^e$, $B^e_{xx} = 0$ and $D^e_{xx} = E^e I^e$ where A^e is the cross section area and I^e is the second moment of inertia of the beam element.

4.2 Finite Element Model

The interpolation functions for the axial and transverse deflection will be

$$u_0(x) = \sum_{j=1}^{2} u_j \psi_j(x) \quad \text{And} \quad w_0(x) = \sum_{j=1}^{4} \bar{\Delta}_j \phi_j(x) \tag{4.5}$$

$$\bar{\Delta}_1 \equiv w_0(x_a), \quad \bar{\Delta}_2 \equiv \theta(x_a), \quad \bar{\Delta}_3 \equiv w_0(x_b), \quad \bar{\Delta}_4 \equiv \theta(x_b) \tag{4.6}$$

In the above equations ψ_j are Lagrange interpolation functions and ϕ_j are Hermite interpolation functions.

Substituting the interpolation function in the weak form equation we get

$$0 = \sum_{j=1}^{2} K_{ij}^{11} u_j + \sum_{J=1}^{4} K_{iJ}^{12} u_J - F_i^1 \qquad (i = 1,2)$$
$$0 = \sum_{j=1}^{2} K_{Ij}^{21} u_j + \sum_{J=1}^{4} K_{IJ}^{22} u_J - F_I^2 \qquad (I = 1,2,3,4) \tag{4.7}$$

where

$$K_{ij}^{11} = \int_{x_a}^{x_b} A_{xx} \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} dx$$

$$K_{iJ}^{12} = \frac{1}{2} \int_{x_a}^{x_b} \left(A_{xx} \frac{dw_0}{dx} \right) \frac{d\psi_i}{dx} \frac{d\phi_J}{dx} dx$$

$$K_{Ij}^{21} = \int_{x_a}^{x_b} A_{xx} \frac{dw_0}{dx} \frac{d\phi_I}{dx} \frac{d\psi_j}{dx} dx, \qquad K_{Ij}^{21} = 2K_{iJ}^{12}$$

$$K_{IJ}^{22} = \int_{x_a}^{x_b} D_{xx} \frac{d^2\phi_I}{dx^2} \frac{d^2\phi_J}{dx^2} dx + \frac{1}{2} \int_{x_a}^{x_b} A_{xx} \left(\frac{dw_0}{dx} \right)^2 \frac{d\phi_I}{dx} \frac{d\phi_J}{dx} dx$$

$$F_i^1 = \int_{x_a}^{x_b} f \psi_i dx + \hat{Q}_i$$
$$F_i^2 = \int_{x_a}^{x_b} q \phi_I dx + \bar{Q}_I \tag{4.8}$$

here

$$\hat{Q}_1 = Q_1, \qquad \hat{Q}_2 = Q_2, \quad \text{and}$$
$$\bar{Q}_1 = Q_3, \qquad \bar{Q}_2 = Q_4, \qquad \bar{Q}_3 = Q_5 \quad \text{and} \quad \bar{Q}_4 = Q_6$$

The stiffness matrix written above is unsymmetric. Hence we will try to linearize the equation by another method as follows,

$$K_{ij}^{11} = \int_{x_a}^{x_b} A_{xx}^e \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$$

$$K_{iJ}^{12} = \frac{1}{2}\int_{x_a}^{x_b} \left(A_{xx}^e \frac{dw_0}{dx} \right)\frac{d\psi_i}{dx}\frac{d\phi_J}{dx}dx$$

$$K_{Ij}^{21} = \int_{x_a}^{x_b} A_{xx}^e \frac{dw_0}{dx}\frac{d\phi_I}{dx}\frac{d\psi_j}{dx}dx, \qquad K_{Ij}^{21}{=}2K_{iJ}^{12}$$

$$K_{IJ}^{22} = \int_{x_a}^{x_b} D_{xx}\frac{d^2\phi_I}{dx^2}\frac{d^2\phi_J}{dx^2}dx + \frac{1}{2}\int_{x_a}^{x_b} A_{xx}^e \left(\frac{dw_0}{dx}\right)^2 \frac{d\phi_I}{dx}\frac{d\phi_J}{dx}dx$$

$$F_i^1 = \int_{x_a}^{x_b} f\psi_i dx + \hat{Q}_i$$

$$F_i^2 = \int_{x_a}^{x_b} q\phi_I dx + \overline{Q}_I$$

$$\sum_{\gamma=1}^{2}\sum_{p=1} K_{ip}^{\alpha\gamma}\Delta_p^\gamma = F_i^\alpha, \qquad \text{or} \qquad \sum_{p=1}^{2} K_{ip}^{\alpha 1}u_p + \sum_{P=1}^{4} K_{iP}^{\alpha 2}\overline{\Delta}_P + = F_i^\alpha \tag{4.9}$$

In matrix form it can be written as

$$\begin{pmatrix} \left[K^{11}\right] & \left[K^{12}\right] \\ \left[K^{21}\right] & \left[K^{22}\right] \end{pmatrix} \begin{Bmatrix} \left\{\Delta^1\right\} \\ \left\{\Delta^2\right\} \end{Bmatrix} = \begin{Bmatrix} \left\{F^1\right\} \\ \left\{F^2\right\} \end{Bmatrix} \tag{4.10}$$

where

$$\Delta_i^1 = u_i, \qquad \text{i=1,2}$$
$$\Delta_i^2 = \overline{\Delta}_i, \qquad \text{i=1,2,3,4}$$

We thus split K_{iJ}^{12} into two parts one of which is taken from the previous solution

$$\int_{x_a}^{x_b}\left\{ A_{xx}^e \frac{d\delta w_0}{dx}\frac{dw_0}{dx}\left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right]\right\}dx$$
$$= \frac{1}{2}\int_{x_a}^{x_b} A_{xx}^e \left\{ \frac{d\delta w_0}{dx}\frac{dw_0}{dx}\frac{du_0}{dx} + \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right]\frac{d\delta w_0}{dx}\frac{dw_0}{dx}\right\}dx \tag{4.11}$$

Thus now we get,

$$\begin{bmatrix} [\bar{K}^{11}] & [\bar{K}^{12}] \\ [\bar{K}^{21}] & [\bar{K}^{22}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{\Delta\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix} \tag{4.12}$$

where

$$\bar{K}_{ij}^{11} = K_{ij}^{11} = \int_{x_a}^{x_b} A_{xx}^{e} \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} dx$$

$$\bar{K}_{iJ}^{12} = K_{iJ}^{12} = \frac{1}{2} \int_{x_a}^{x_b} \left(A_{xx}^{e} \frac{dw_0}{dx} \right) \frac{d\psi_i}{dx} \frac{d\phi_J}{dx} dx$$

$$\bar{K}_{Ij}^{21} = \frac{1}{2} \int_{x_a}^{x_b} A_{xx}^{e} \frac{dw_0}{dx} \frac{d\phi_I}{dx} \frac{d\psi_j}{dx} dx, \qquad K_{Ij}^{21} = \bar{K}_{iJ}^{12}$$

$$\bar{K}_{IJ}^{22} = \int_{x_a}^{x_b} D_{xx} \frac{d^2\phi_I}{dx^2} \frac{d^2\phi_J}{dx^2} dx + \frac{1}{2} \int_{x_a}^{x_b} A_{xx}^{e} \left[\left(\frac{dw_0}{dx} \right)^2 + \frac{du_0}{dx} \right] \frac{d\phi_I}{dx} \frac{d\phi_J}{dx} dx$$

$$F_i^1 = \int_{x_a}^{x_b} f\psi_i dx + \hat{Q}_i \tag{4.13}$$

$$F_i^2 = \int_{x_a}^{x_b} q\phi_I dx + \bar{Q}_I$$

4.3 Membrane Locking

Linearity is one of the assumptions of the EBT. This means that the beam is subjected to bending forces only and there are no axial forces. Thus ideally the beam should not stretch. Thus the axial strain should be zero.

$$\left[\frac{du_0}{dx} + \frac{1}{2} \left(\frac{dw_0}{dx} \right)^2 \right] = 0 \qquad \text{OR} \qquad \frac{du_0}{dx} \approx \left(\frac{dw_0}{dx} \right)^2$$

In bending dominated deformations, the beam undergoes axial displacement along with transverse deflection even when there are no axial forces. In order to develop this transverse deflection the axial strain is developed in the beam. Thus as the load increases the axial stiffness increases. This results in computational difficulties and incorrect solutions. The inaccuracy in the solution is because of the ambiguity between the degree of polynomial variation and the interpolation functions of u_0 and w_0. This phenomenon is called membrane locking. A normal way to solve such problems is to take the minimum interpolation of u_0 and w_0.

4.4 Summary

This section discussed about the conventional weighted residual method for Euler-Bernoulli (EB) beam theory. This part of the research focuses mainly on the weak form development and finite element model. The element coefficients obtained in this finite element model will be assembled to form a global stiffness matrix and the solutions will be obtained by FORTRAN program. A detailed discussion about the solution procedure has been made in this section. A similar discussion about the Timoshenko beam theory (TBT) will be made in the following section.

5. FINITE ELEMENT MODEL OF THE TBT

5.1 Weak Form Development

As mentioned in the assumptions earlier the

$$\sigma_{xx} = E^e \varepsilon_{xx} \quad \text{and} \qquad \sigma_{xz} = G^e \gamma_{xz} \tag{5.1}$$

The above relationship which defines the relationship between the total stress and the total strain is called as the Hooke's law.

From equation (5.1) and (3.20) we get

$$\begin{aligned} N_{xx} &= \int_{A^e} \sigma_{xx} dA = \int_{A^e} E^e \varepsilon_{xx} dA \\ &= \int_{A^e} E^e \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 \right] + z\left(\frac{d\phi_x}{dx}\right) dA \\ &= A^e_{xx} \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 \right] + B^e_{xx}\left(\frac{d\phi_x}{dx}\right) \end{aligned} \tag{5.2}$$

$$\begin{aligned} M_{xx} &= \int_{A^e} \sigma_{xx} z dA = \int_{A^e} E^e \varepsilon_{xx} z dA \\ &= \int_{A^e} E^e \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 \right] + z\left(\frac{d\phi_x}{dx}\right) z dA \\ &= B^e_{xx} \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 \right] + D^e_{xx}\left(\frac{d\phi_x}{dx}\right) \end{aligned} \tag{5.3}$$

$$Q_x = S_{xx}\left(\frac{dw_0}{dx} + \phi_x\right) \tag{5.4}$$

where, A^e_{xx} is the extensional stiffness

B^e_{xx} is the extensional-bending stiffness and

D^e_{xx} is the bending stiffness.

S_{xx} is the shear stiffness and is defined as $S_{xx} = K_s \int_A GdA = K_s GA$ where G is the shear modulus.

For isotropic material we have, $A_{xx}^e = E^e A^e$, $B_{xx}^e = 0$ and $D_{xx}^e = E^e I^e$ where A^e is the cross section area and I^e is the second moment of inertia of the beam element.

Thus, the governing equations for TBT are as follows,

$$
\begin{gathered}
-\frac{d}{dx}\left\{A_{xx}^e\left[\frac{du_0}{dx}+\frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right]\right\}=f \\
-\frac{d}{dx}\left\{A_{xx}^e\frac{dw_0}{dx}\left[\frac{du_0}{dx}+\frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right]\right\}-\frac{d}{dx}\left[S_{xx}\left(\frac{dw_0}{dx}+\phi_x\right)\right]=q \\
-\frac{d}{dx}\left(D_{xx}\frac{d\phi_x}{dx}\right)+S_{xx}\left(\frac{dw_0}{dx}+\phi_x\right)=0
\end{gathered}
\tag{5.5}
$$

5.2 Finite Element Model

For TBT the virtual work statement is equivalent to the following

$$
\begin{aligned}
0&=\int_{x_a}^{x_b}\left[\left(\frac{d\delta u_0}{dx}A_{xx}^e\left[\frac{du_0}{dx}+\frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right]-f(x)\,\delta u_0(x)\right)dx-Q_1^e\delta u_0(x_a)-Q_4^e\delta(x_b)\right] \\
0&=\int_{x_a}^{x_b}\left[\frac{d\delta w_0}{dx}\left\{\left[S_{xx}^e\left(\frac{dw_0}{dx}+\phi_x\right)\right]+A_{xx}^e\frac{dw_0}{dx}\left[\frac{du_0}{dx}+\frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right]\right\}-q(x)\,\delta w_0(x)\right]dx- \\
&\quad Q_2^e\delta w_0(x_a)-Q_5^e\delta w_0(x_b) \\
0&=\int_{x_a}^{x_b}\left[D_{xx}^e\frac{d\delta\phi_x}{dx}\frac{d\phi_x}{dx}+S_{xx}^e\delta\phi_x\left(\frac{dw_0}{dx}+\phi_x\right)\right]dx-Q_3^e\delta\phi_x(x_a)-Q_6^e\delta\phi_x(x_b)
\end{aligned}
\tag{5.6}
$$

Thus the boundary conditions are :

$$
\begin{aligned}
Q_1^e &= -N_{xx}(x_a), & Q_4^e &= N_{xx}(x_b) \\
Q_2^e &= -\left[\frac{dw_0}{dx}N_{xx}+Q_x\right]_{x_a}, & Q_5^e &= \left[\frac{dw_0}{dx}N_{xx}+\frac{dM_{xx}}{dx}\right]_{x_b} \\
Q_3^e &= -M_{xx}(x_a), & Q_6^e &= M_{xx}(x_b)
\end{aligned}
\tag{5.7}
$$

The interpolation functions for the axial and transverse deflection will be

$$u_0\left(x\right)=\sum_{j=1}^{m}u_j\psi_j^{(1)} \quad , \quad w_0\left(x\right)=\sum_{j=1}^{n}w_j\psi_j^{(2)} \text{ and } \phi_x\left(x\right)=\sum_{j=1}^{p}s_j\psi_j^{(3)} \tag{5.8}$$

In the above equations ψ_j are Lagrange interpolation functions substituting the interpolation function in the weak form equation we get

$$\begin{aligned}
0&=\sum_{j=1}^{m}K_{ij}^{11}u_j+\sum_{j=1}^{n}K_{ij}^{12}w_j+\sum_{j=1}^{p}K_{ij}^{13}s_j-F_i^1\\
0&=\sum_{j=1}^{m}K_{ij}^{21}u_j+\sum_{j=1}^{n}K_{ij}^{22}w_j+\sum_{j=1}^{p}K_{ij}^{23}s_j-F_i^2\\
0&=\sum_{j=1}^{m}K_{ij}^{31}u_j+\sum_{j=1}^{n}K_{ij}^{32}w_j+\sum_{j=1}^{p}K_{ij}^{33}s_j-F_i^3
\end{aligned} \tag{5.9}$$

where

$$K_{ij}^{11}=\int_{x_a}^{x_b}A_{xx}\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(1)}}{dx}dx$$

$$K_{ij}^{12}=\frac{1}{2}\int_{x_a}^{x_b}\left(A_{xx}\frac{dw_0}{dx}\right)\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(2)}}{dx}dx$$

$$K_{ij}^{21}=\int_{x_a}^{x_b}A_{xx}\frac{dw_0}{dx}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(1)}}{dx}dx, \qquad K_{ij}^{13}=K_{ij}^{31}=0$$

$$K_{ij}^{22}=\int_{x_a}^{x_b}S_{xx}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx}dx+\frac{1}{2}\int_{x_a}^{x_b}A_{xx}\left(\frac{dw_0}{dx}\right)^2\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx}dx$$

$$K_{ij}^{23}=\int_{x_a}^{x_b}S_{xx}\frac{d\psi_i^{(2)}}{dx}\psi_j^{(3)}dx=K_{ij}^{32}$$

$$K_{ij}^{33}=\int_{x_a}^{x_b}\left(D_{xx}\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(3)}}{dx}+S_{xx}\psi_j^{(3)}\psi_i^{(3)}\right)dx$$

$$F_i^1=\int_{x_a}^{x_b}f\psi_i^{(1)}dx+Q_1\psi_i^{(1)}\left(x_a\right)+Q_4\psi_i^{(1)}\left(x_b\right)$$

$$F_i^2 = \int_{x_a}^{x_b} q\psi_i^{(2)} dx + Q_2\psi_i^{(2)}(x_a) + Q_5\psi_i^{(2)}(x_b)$$
$$F_i^3 = Q_3\psi_i^{(3)}(x_a) + Q_6\psi_i^{(3)}(x_b) \tag{5.10}$$

In matrix form it can be written as

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] \\ [K^{21}] & [K^{22}] & [K^{23}] \\ [K^{31}] & [K^{32}] & [K^{33}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{w\} \\ \{s\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \end{Bmatrix} \tag{5.11}$$

5.3 Shear and Membrane Locking

The simplest Timoshenko element is one which has the linear interpolation of both w_0 and ϕ_x.This means that the slope $\frac{dw_0}{dx}$ should be constant. In this beams the ratio of length to thickness is large and thus the slope will be $-\phi_x$.This contradicts our earlier discussion. Moreover ϕ_x=constant results in zero bending energy while the transverse shear is nonzero. Thus the assumption of linear interpolation function is inconsistent and leads to a stiff thin beam. This phenomenon is called *shear locking*. To overcome this technique reduced integration method is used. In this selective integration technique, the stiffness coefficients associated with the transverse shear strain are evaluated using equal interpolations are used for w_0 and ϕ_x but ϕ_x is treated as constant and other coefficients are derived using full integration method. The shear strain is represented as $\gamma_{xz} = \phi_x + \frac{dw_0}{dx}$ and membrane is given by $\varepsilon_{xx} = \frac{du_0}{dx} - + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2$.The element experiences no stretching which means $\varepsilon_{xx} = \frac{du_0}{dx} - + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 = 0$. In order to satisfy the these constraints we must have

$\phi_x \square \frac{dw_0}{dx}$ and $\frac{du_0}{dx} \square \left(\frac{dw_0}{dx}\right)^2$.Here is ϕ_x is linear and w_0 is quadratic the constraint is satisfied. Similarly when w_0 and u_0 are linear the constraint is automatically satisfied. If quadratic interpolation is used for both w_0 and u_0 then $\frac{du_0}{dx}$ is linear and $\left(\frac{dw_0}{dx}\right)^2$ is quadratic, this creates inconsistency. Here the element again starts experiencing locking. This is called membrane locking.

5.4 Summary

In this section a detailed discussion on the derivation of governing equations ,weak form formulations , finite element model and solution procedures has been made. This section also discusses two different types of locking in TBT beams, shear locking and memebrane locking. In order to avoid the inconsistencies observed in EBT ant TBT different methods such as reduced integration method have been implemented in the past. But this method also has its disadvantages of hour-glass modes or spurious rigid body modes. Thus, it is desirable to develop alternative finite element models that overcome the locking problems. An effort has been made to develop models that can use higher order interpolation functions and finite element models were developed using least-squares method. These models will be discussed in the next section.

6. LEAST-SQUARES THEORY & FORMULATION

6.1 Introduction

In order to avoid the locking problems mixed least-squares based finite element models can be considered as an alternative approach to the conventional weighted residual weak form method. A detailed discussion on two different models using least-squares finite element analysis is made in this section.

6.2 Basic Idea

The basic idea behind the least-squares finite element model is to compute the residuals due to the approximation of the variables of each equation being modeled, construct integral statement of the sum of the squares of the residuals (called least-squares functional), and minimize the integral with respect to the unknown parameters of the approximations. To be more explicit, consider an operator equation of the form

$$A(u) = f \quad \text{in} \quad \Omega \text{ and} \quad \mathrm{B}(u) = g \quad \text{in } \Gamma$$

We seek suitable approximation of u as $u_h = \sum_{j=1}^{n} c_j \varphi_j$ .In the least squares method, we seek the minimum of the sum of squares of the residuals in the approximation of equations as follows

$$\frac{\partial}{\partial c_i} \int_{\Omega} R^2 \left(\mathrm{x}, c_j \right) dx = 0$$

where

$$R^2 = R_1^2 + R_2^2, \; R_1 = A(u_h) - f, \; R_2 = B(u_h) - g$$

The necessary condition for the minimum is

$$0 = \delta I\left(u_h\right) = \delta\left\{\int_\Omega \left[A\left(u\right) - f\right]^2 dx + \oint_\Gamma \left[B\left(u\right) - g\right]^2 ds\right\}$$

Thus the variational problem is to seek u_h such that $B\left(\delta u_h, u_h\right) = l(u_h)$ holds for all δu_h. where

$$B\left(\delta u_h, u_h\right) = \int_\Omega \delta\left[A\left(u_h\right)\right] A\left(u_h\right) d\mathrm{x} + \oint_\Gamma \delta\left[B\left(u_h\right)\right] B\left(u_h\right) ds$$
$$l(u_h) = \int_\Omega \delta\left[A\left(u_h\right)\right] f d\mathrm{x} + \oint_\Gamma \delta\left[B\left(u_h\right)\right] g ds$$

Using the above concept, the least-squares finite element models of the Euler-Bernoulli beam theory (EBT) and the Timoshenko beam theory (TBT) are developed as discussed below.

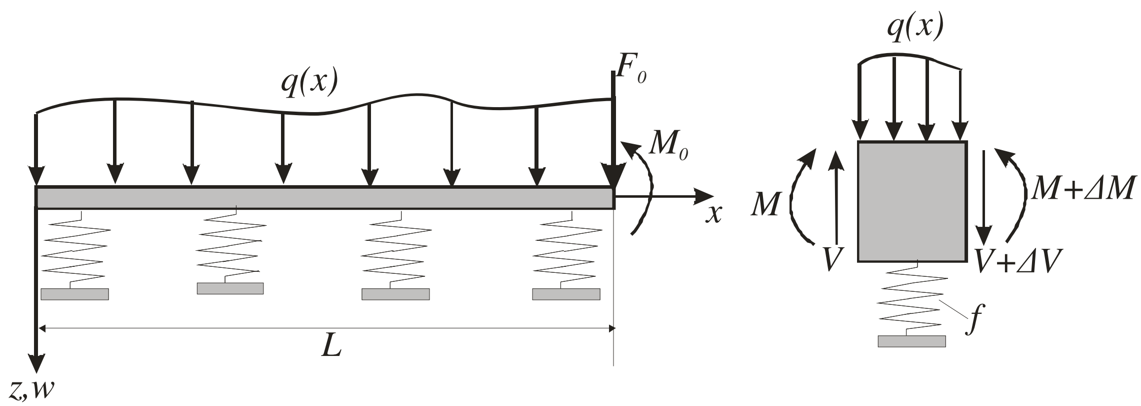


Figure 6.1. A typical beam element with forces and moments under uniformly distributed load

where q is the uniformly distributed load acting on the length L of the beam ,M is the bending moment and V is the shear force.

Hence the governing equations for the beam in Figure 6.1 are

$$\begin{aligned}&\frac{dM}{dx}-V=0\\&\frac{dV}{dx}-c_f w-q=0\\&M+EI\frac{d\theta}{dx}=0\\&\theta+\frac{dw}{dx}=0\end{aligned}\tag{6.1}$$

Or eliminating V we get

$$\begin{aligned}&\frac{d^2M}{dx^2}+c_f w-q=0\\&-\left(\frac{M}{EI}+\frac{d^2w}{dx^2}\right)=0\end{aligned}\tag{6.2}$$

Here we use the approximation

$$w\approx w_h=\sum_{j=1}^{m}\Delta_j^1\phi_j\left(x\right),\quad M\approx M_h=\sum_{j=1}^{n}\Delta_j^2\varphi_j\left(x\right)$$

And the least squares functional will be as follows

$$I\left(w_h,M_h\right)=\int_{x_a}^{x_b}\left[\left(-\frac{d^2M_h}{dx^2}+c_f w-q\right)^2+-\left(\frac{M}{EI}+\frac{d^2w}{dx^2}\right)^2\right]dx\tag{6.3}$$

In matrix form it can be written as

$$\begin{bmatrix}\left[K^{11}\right] & \left[K^{12}\right]\\ \left[K^{21}\right] & \left[K^{22}\right]\end{bmatrix}\begin{Bmatrix}\left\{\Delta^1\right\}\\ \left\{\Delta^2\right\}\end{Bmatrix}=\begin{Bmatrix}\left\{F^1\right\}\\ \left\{F^2\right\}\end{Bmatrix}\tag{6.3}$$

where

$$K_{ij}^{11}=\int_{x_a}^{x_b}\left[c_f^2\phi_i\phi_j+\frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}\right]dx$$

$$K_{iJ}^{12}=\int_{x_a}^{x_b}\left(\frac{1}{EI}\frac{d^2\phi_i}{dx^2}\varphi_j-c_f\phi_i\frac{d^2\varphi_j}{dx^2}\right)dx$$

$$K_{Ij}^{21}=\int_{x_a}^{x_b}\left(\frac{1}{EI}\frac{d^2\phi_j}{dx^2}\varphi_i-c_f\phi_j\frac{d^2\varphi_i}{dx^2}\right)dx$$

$$K_{IJ}^{22}=\int_{x_a}^{x_b}\left[\frac{1}{(EI)^2}\varphi_i\varphi_j+\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}\right]dx \tag{6.4}$$

$$F_i^1=\int_{x_a}^{x_b}c_f\phi_i q(x)dx$$

$$F_i^2=-\int_{x_a}^{x_b}q(x)\frac{d^2\varphi_i}{dx^2}dx$$

6.3 Least-squares Finite Element MODEL 1 for Euler-Bernoulli Beam Theory

This section discusses about the linear and nonlinear formulation of finite element model for EBT.

6.3.1 Linear formulation

Consider the following governing equations,

$$-\frac{dN}{dx}=f$$

$$-\frac{d^2M}{dx^2}-\frac{d}{dx}\left(N\frac{dw}{dx}\right)=q$$

$$M+EI\frac{d^2w}{dx^2}=0 \tag{6.5}$$

where q(x) is the transverse distributed force and N is known in terms of u and as $N=EA\frac{du}{dx}$, $\delta N=EA\frac{d\delta u}{dx}$

The least-squares functional associated with the above set of linearized equations over a typical element is

$$J_L(u_h, w_h, M_h) = \int_{x_b}^{x_a} \left\{ p_1 \left[-\frac{d^2 M_h}{dx^2} - q \right]^2 + \left[-\frac{dN_h}{dx} - f \right]^2 + p_2 \left(M_h + EI \frac{d^2 w_h}{dx^2} \right)^2 \right\} \tag{6.6}$$

where p_1 and p_2 are scaling factors to make the entire residual to have the same physical dimensions and quantities with bar are assumed to be known from the previous iteration and their variations are zero.

The necessary condition for the minimum of J_L is δJ_L=0

$$\begin{aligned} 0 = \int_{x_b}^{x_a} \Bigg[& p_1 \frac{d^2 \delta M_h}{dx^2} \left[\frac{d^2 M_h}{dx^2} + q \right] + \left[EA \frac{d^2 u_h}{dx^2} + f \right] EA \frac{d^2 \delta u_h}{dx^2} + \\ & + p_2 \left(\delta M_h + EI \frac{d^2 \delta w_h}{dx^2} \right) \left(M_h + EI \frac{d^2 w_h}{dx^2} \right) \Bigg] dx \end{aligned} \tag{6.7}$$

Since the physics of the Euler Bernoulli's Beam theory requires the specification of $u, w, \theta = \left(-\frac{dw}{dx} \right), N, MandV = \left(-\frac{dM}{dx} \right)$ we seek Hermite cubic approximations of $u_h . w_h$ and M_h

$$u_h = \sum_{j=1}^{4} \Delta_j^1 \varphi_j(x), \quad w_h = \sum_{j=1}^{4} \Delta_j^2 \varphi_j(x) \text{ and } M_h = \sum_{j=1}^{4} \Delta_j^3 \varphi_j(x)$$

Where $\Delta_j^1, \Delta_j^2 and \Delta_j^3$ denote the nodal values of $\left(u_h, -\frac{du_h}{dx} \right), \left(w_h, -\frac{dw_h}{dx} \right)$ and $\left(M_h, -\frac{dM_h}{dx} \right)$ respectively at the jth node and $\varphi_j(x)$ are the Hermite cubic interpolation functions. Substituting the above equations we get the finite element model as follows.

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] \\ [K^{21}] & [K^{22}] & [K^{23}] \\ [K^{31}] & [K^{32}] & [K^{33}] \end{bmatrix} \begin{Bmatrix} \{\Delta^1\} \\ \{\Delta^2\} \\ \{\Delta^3\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \end{Bmatrix} \tag{6.8}$$

where

$$K_{ij}^{11} = \int_{x_b}^{x_a} (EA) \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx$$

$$K_{ij}^{12} = K_{ij}^{21} = K_{ij}^{13} = K_{ij}^{31} = 0$$

$$K_{ij}^{22} = p_2 \int_{x_b}^{x_a} \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx$$

$$K_{ij}^{23} = \frac{p_2}{EI} \int_{x_b}^{x_a} \frac{d^2\varphi_i}{dx^2} \varphi_j dx = K_{ji}^{32}$$

$$K_{ij}^{33} = \int_{x_b}^{x_a} p_1 \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx + \frac{p_2}{(EI)^2} \int_{x_b}^{x_a} \varphi_i \varphi_j dx$$

$$F_i^1 = -\int_{x_b}^{x_a} \left[f \frac{d^2\varphi_i}{dx^2} \right] dx$$

$$F_i^2 = 0$$

$$F_i^3 = -p_1 \int_{x_b}^{x_a} q \frac{d^2\varphi_i}{dx^2} dx \tag{6.9}$$

6.3.2 Nonlinear formulation

The least-squares finite element model of the following set of nonlinear equations assuming EA and EI as constant was developed as follows:-

$$-\frac{dN}{dx} = f$$

$$-\frac{d^2M}{dx^2} - \frac{d}{dx}\left(N \frac{dw}{dx} \right) = q \tag{6.10}$$

$$M + EI \frac{d^2 w}{dx^2} = 0$$

where q(x) is the transverse distributed force, and N is known in terms of u and was $N = EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx} \right)^2 \right]$

The linearization of the above equations that will be used are

$$-EA\left(\frac{d^2u}{dx^2}+\frac{d\bar{w}}{dx}\frac{d^2w}{dx^2}\right)=f$$

$$-\frac{d^2M}{dx^2}-EA\left(\frac{d^2u}{dx^2}+\frac{d\bar{w}}{dx}\frac{d^2w}{dx^2}\right)\frac{d\bar{w}}{dx}-\bar{N}\frac{d^2w}{dx^2}=q$$

$$M+EI\left(\frac{d^2w}{dx^2}\right)=0 \tag{6.11}$$

where $\bar{N}=EA\left(\frac{d\bar{u}}{dx}+\frac{1}{2}\left(\frac{d^2\bar{w}}{dx^2}\right)^2\right), \delta\bar{N}=0$

The least-squares functional associated with the above set of linearized equations over a typical element is

$$J_L(u_h,w_h,M_h)=\int_{x_b}^{x_a}\left[p_1\left[\frac{d^2M_h}{dx^2}+EA\left(\frac{d^2u_h}{dx^2}+\frac{d\bar{w}_h}{dx}\frac{d^2w_h}{dx^2}\right)\frac{d\bar{w}_h}{dx}+\bar{N}\frac{d^2w_h}{dx^2}+q\right]^2+\right.$$
$$\left.\left[EA\left(\frac{d^2u_h}{dx^2}+\frac{d\bar{w}_h}{dx}\frac{d^2w_h}{dx^2}\right)+f\right]^2+p_2\left(M_h+EI\frac{d^2w_h}{dx^2}\right)^2\right] \tag{6.12}$$

where p_1 and p_2 are scaling factors to make the entire residual to have the same physical dimensions and quantities with bar are assumed to be known from the previous iteration and their variations are zero.

The necessary condition for the minimum of J_L is δJ_L=0

$$0=\int_{x_b}^{x_a}\left[p_1\left[\frac{d^2M_h}{dx^2}+EA\left(\frac{d^2u_h}{dx^2}+\frac{d\bar{w}_h}{dx}\frac{d^2w_h}{dx^2}\right)\frac{d\bar{w}_h}{dx}+\bar{N}\frac{d^2w_h}{dx^2}+q\right]\right.$$
$$\times\left[\frac{d^2\delta M_h}{dx^2}+EA\left(\frac{d^2\delta u_h}{dx^2}+\frac{d\bar{w}_h}{dx}\frac{d^2\delta w_h}{dx^2}\right)\frac{d\bar{w}_h}{dx}+\bar{N}\frac{d^2\delta w_h}{dx^2}\right]+$$
$$EA\left[EA\left(\frac{d^2u_h}{dx^2}+\frac{d\bar{w}_h}{dx}\frac{d^2w_h}{dx^2}\right)+f\right]\left(\frac{d^2\delta u_h}{dx^2}+\frac{d\bar{w}_h}{dx}\frac{d^2\delta w_h}{dx^2}\right)+ \tag{6.13}$$
$$\left.p_2\left(\delta M_h+EI\frac{d^2\delta w_h}{dx^2}\right)\left(M_h+EI\frac{d^2w_h}{dx^2}\right)\right]dx$$

The above statement is equivalent to the following three integral statements:

$$0=\int_{x_b}^{x_a}\left[\frac{d^2\delta u}{dx^2}\left(EAEA\frac{d^2u}{dx^2}+EAEA\frac{d\bar{w}}{dx}\frac{d^2w}{dx^2}+EAf\right)+\right.$$
$$\left.p_1\frac{d\bar{w}}{dx}\frac{d^2\delta u}{dx^2}\left(EA\frac{d^2M}{dx^2}+EAEA\frac{d\bar{w}}{dx}\frac{d^2u}{dx^2}+(EA)^2\hat{N}\frac{d^2w}{dx^2}+EAq\right)\right]dx$$

$$=\int_{x_b}^{x_a}\left[(EA)^2\frac{d^2\delta u}{dx^2}\frac{d^2u}{dx^2}+p_1(EA)^2\left(\frac{d\bar{w}}{dx}\right)^2\frac{d^2\delta u}{dx^2}\frac{d^2u}{dx^2}+(EA)^2\frac{d\bar{w}}{dx}\frac{d^2\delta u}{dx^2}\frac{d^2w}{dx^2}+\right.$$
$$\left.p_1(EA)^2\frac{d\bar{w}}{dx}\hat{N}\frac{d^2\delta u}{dx^2}\frac{d^2w}{dx^2}+p_1EA\frac{d\bar{w}}{dx}\frac{d^2\delta u}{dx^2}\frac{d^2M}{dx^2}+EA\frac{d^2\delta u}{dx^2}\left(f+p_1\frac{d\bar{w}}{dx}q\right)\right]dx$$

(6.13)

$$0=\int_{x_b}^{x_a}\left[EA\frac{d\bar{w}}{dx}\frac{d^2\delta w}{dx^2}\left(EA\frac{d^2u}{dx^2}+EA\frac{d\bar{w}}{dx}\frac{d^2w}{dx^2}+f\right)+p_2EI\frac{d^2\delta w}{dx^2}\left(M+EI\frac{d^2w}{dx^2}\right)+\right.$$
$$\left.p_1EA\hat{N}\frac{d^2\delta w}{dx^2}\left(\frac{d^2M}{dx^2}+EA\frac{d\bar{w}}{dx}\frac{d^2u}{dx^2}+EA\hat{N}\frac{d^2w}{dx^2}+q\right)\right]dx$$

$$= \int_{x_b}^{x_a} \left[(EA)^2 \frac{d\bar{w}}{dx} \frac{d^2\delta w}{dx^2} \frac{d^2 u}{dx^2} + (EA)^2 p_1 \hat{N} \frac{d\bar{w}}{dx} \frac{d^2\delta w}{dx^2} \frac{d^2 u}{dx^2} + p_2 (EI)^2 \frac{d^2\delta w}{dx^2} \frac{d^2 w}{dx^2} + \right.$$

$$(EA)^2 \left(\frac{d\bar{w}}{dx} \right)^2 \frac{d^2\delta w}{dx^2} \frac{d^2 w}{dx^2} + (EA)^2 p_1 \hat{N}^2 \frac{d^2\delta w}{dx^2} \frac{d^2 w}{dx^2} + p_2 EI \frac{d^2\delta w}{dx^2} M + \quad (6.14)$$

$$\left. EAp_1 \hat{N} \frac{d^2\delta w}{dx^2} \frac{d^2 M}{dx^2} + \left(EA \frac{d\bar{w}}{dx} f + EAp_1 \hat{N} q \right) \frac{d^2 \delta w}{dx^2} \right] dx$$

$$0 = \int_{x_b}^{x_a} \left[p_1 \frac{d^2\delta M}{dx^2} \left(\frac{d^2 M}{dx^2} + EA \frac{d\bar{w}}{dx} \frac{d^2 u}{dx^2} + EA\hat{N} \frac{d^2 w}{dx^2} + q \right) + p_2 \delta M \left(M + EI \frac{d^2 w}{dx^2} \right) \right] dx$$

$$= \int_{x_b}^{x_a} \left[EAp_1 \frac{d\bar{w}}{dx} \frac{d^2\delta M}{dx^2} \frac{d^2 u}{dx^2} + p_2 \delta M EI \frac{d^2 w}{dx^2} + p_1 EA\hat{N} \frac{d^2 \delta M}{dx^2} \frac{d^2 w}{dx^2} + p_2 M \delta M + \right.$$

$$\left. p_1 \frac{d^2\delta M}{dx^2} \frac{d^2 M}{dx^2} + \frac{d^2 \delta M}{dx^2} p_1 q \right] dx \quad (6.15)$$

where $N = \left[\frac{d\bar{u}}{dx} + \frac{1}{2} \left(\frac{d\bar{w}}{dx} \right)^2 \right]$, $\quad \hat{N} = \bar{N} + \left(\frac{d\bar{w}}{dx} \right)^2 = \left[\frac{d\bar{u}}{dx} + \frac{3}{2} \left(\frac{d\bar{w}}{dx} \right)^2 \right]$

Since the physics of the Euler Bernoulli's Beam theory requires the specification of $u, w, \theta = \left(-\frac{dw}{dx} \right), N, M$ and $V = \left(-\frac{dM}{dx} \right)$ we seek Hermite cubic approximations of $u_h . w_h$ and M_h

$$u_h = \sum_{j=1}^{4} \Delta_j^1 \varphi_j(x), \quad w_h = \sum_{j=1}^{4} \Delta_j^2 \varphi_j(x) \text{ and } M_h = \sum_{j=1}^{4} \Delta_j^3 \varphi_j(x)$$

where Δ_j^1, Δ_j^2 and Δ_j^3 denote the nodal values of $\left(u_h, -\frac{du_h}{dx} \right), \left(w_h, -\frac{dw_h}{dx} \right)$ and $\left(M_h, -\frac{dM_h}{dx} \right)$ respectively at the jth node and $\varphi_j(x)$ are the Hermite cubic interpolation functions. Substituting the above equations we get the finite element model as follows.

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] \\ [K^{21}] & [K^{22}] & [K^{23}] \\ [K^{31}] & [K^{32}] & [K^{33}] \end{bmatrix} \begin{Bmatrix} \{\Delta^1\} \\ \{\Delta^2\} \\ \{\Delta^3\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \end{Bmatrix} \tag{6.16}$$

where

$$K_{ij}^{11} = \int_{x_b}^{x_a} (EA)^2 \frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx + p_1\int_{x_b}^{x_a} (EA)^2 \left(\frac{d\overline{w}}{dx}\right)^2 \frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx$$

$$K_{ij}^{12} = \int_{x_b}^{x_a} (EA)^2 \left(1+p_1\hat{N}\right)\frac{d\overline{w}}{dx}\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx$$

$$K_{ij}^{13} = p_1\int_{x_b}^{x_a} EA\frac{d\overline{w}}{dx}\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx$$

$$K_{ij}^{21} = \int_{x_b}^{x_a} (EA)^2 \left(1+p_1\hat{N}\right)\frac{d\overline{w}}{dx}\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx$$

$$K_{ij}^{22} = \int_{x_b}^{x_a} (EA)^2 \left[\left(\frac{d\overline{w}}{dx}\right)^2 + p_1\hat{N}^2\right]\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx + p_2(EI)^2\int_{x_b}^{x_a}\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx$$

$$K_{ij}^{23} = \int_{x_b}^{x_a} p_1 EA\hat{N}\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx + p_2 EI\int_{x_b}^{x_a}\frac{d^2\varphi_i}{dx^2}\varphi_j dx$$

$$K_{ij}^{31} = \int_{x_b}^{x_a} p_1 EA\frac{d\overline{w}}{dx}\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx$$

$$K_{ij}^{32} = \int_{x_b}^{x_a} p_1 (EA)^2\hat{N}\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx + p_2 EI\int_{x_b}^{x_a}\frac{d^2\varphi_i}{dx^2}\varphi_j dx$$

$$K_{ij}^{33} = \int_{x_b}^{x_a} p_1\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}dx + p_2\int_{x_b}^{x_a}\varphi_i\varphi_j dx$$

$$F_i^1 = -EA\int_{x_b}^{x_a}\left[f\,\frac{d^2\varphi_i}{dx^2} + qp_1\frac{d\overline{w}}{dx}\frac{d^2\varphi_i}{dx^2}\right]dx$$

$$F_i^2 = -\int_{x_b}^{x_a}\left[EAf\,\frac{d\overline{w}}{dx} + qp_1 EA\hat{N}\right]\frac{d^2\varphi_i}{dx^2}\,dx$$

$$F_i^3 = -p_1\int_{x_b}^{x_a} q\,\frac{d^2\varphi_i}{dx^2}\,dx \qquad (6.17)$$

From the terms of K_{ij}^{33} it is clear that the terms p_1 and p_2 should be taken such that $p_2 = p_1 / h^2$, where h is the element length.

6.4 Least-squares Finite Element MODEL 1 for Timoshenko Beam Theory

6.4.1 Linear formulation

The equations that arise in connection with the Linear Timoshenko beam theory are

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f$$

$$-\frac{d}{dx}\left[GAK_S\left(\phi + \frac{dw}{dx}\right)\right] - \frac{d}{dx}\left(N\frac{dw}{dx}\right) = q$$

$$-\frac{d}{dx}\left(EI\frac{d\phi}{dx}\right) + GAK_S\left(\phi + \frac{dw}{dx}\right) = 0 \qquad (6.18)$$

The least-squares functional associated with the above set of linearized equations over a typical element is

$$J_L(u_h, w_h, \phi_h) = \int_{x_b}^{x_a}\left[p_1\left[-GAK_S\left(\frac{d\phi_h}{dx} + \frac{d^2 w_h}{dx^2}\right) - q\right]^2 + p_2\left[-EI\frac{d^2\phi_h}{dx^2} + GAK_S\left(\phi_h + \frac{dw_h}{dx}\right)\right]^2 + \left[-EA\left(\frac{d^2 u_h}{dx^2}\right) - f\right]^2\right]dx \qquad (6.19)$$

where p_1 and p_2 are scaling factors to make the entire residual to have the same physical dimensions and quantities with bar are assumed to be known from the previous iteration and their variations are zero.

The necessary condition for the minimum of J_L is δJ_L=0

$$0=\int_{x_b}^{x_a}\left[p_1\left[-GAK_S\left(\frac{d\delta\phi_h}{dx}+\frac{d^2\delta w_h}{dx^2}\right)\right]\left[-GAK_S\left(\frac{d\phi_h}{dx}+\frac{d^2 w_h}{dx^2}\right)+q\right]+\right.$$
$$\left[EA\frac{d^2u_h}{dx^2}+f\right]EA\frac{d^2\delta u_h}{dx^2}+ \tag{6.20}$$
$$\left.p_2\left(GAK_S\left(\delta\phi_h+\frac{d\delta w_h}{dx}\right)-EI\frac{d^2\delta\phi_h}{dx^2}\right)\left(GAK_S\left(\phi_h+\frac{dw_h}{dx}\right)-EI\frac{d^2\phi_h}{dx^2}\right)\right]$$

Since the physics of the Euler Bernoulli's Beam theory requires the specification of $u, w, \theta=\left(-\frac{dw}{dx}\right), N, M and V=\left(-\frac{dM}{dx}\right)$ we seek Hermite cubic approximations of $u_h . w_h$ and M_h

$$u_h=\sum_{j=1}^{4}\Delta_j^1\varphi_j(x),\quad w_h=\sum_{j=1}^{4}\Delta_j^2\varphi_j(x) \text{ and } M_h=\sum_{j=1}^{4}\Delta_j^3\varphi_j(x)$$

where $\Delta_j^1, \Delta_j^2 and \Delta_j^3$ denote the nodal values of $\left(u_h,-\frac{du_h}{dx}\right)$, $\left(w_h,-\frac{dw_h}{dx}\right)$ and $\left(M_h,-\frac{dM_h}{dx}\right)$ respectively at the jth node and $\varphi_j(x)$ are the Hermite cubic interpolation functions. Substituting the above equations we get the finite element model as follows.

$$\begin{bmatrix}[K^{11}] & [K^{12}] & [K^{13}]\\ [K^{21}] & [K^{22}] & [K^{23}]\\ [K^{31}] & [K^{32}] & [K^{33}]\end{bmatrix}\begin{Bmatrix}\{\Delta^1\}\\ \{\Delta^2\}\\ \{\Delta^3\}\end{Bmatrix}=\begin{Bmatrix}\{F^1\}\\ \{F^2\}\\ \{F^3\}\end{Bmatrix} \tag{6.21}$$

where

$$K_{ij}^{11} = \int_{x_b}^{x_a} (EA) \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dxdx$$

$$K_{ij}^{12} = K_{ij}^{21} = K_{ij}^{13} = K_{ij}^{31} = 0$$

$$K_{ij}^{22} = \int_{x_b}^{x_a} p_1 \left[GAK_S \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx + p_2 \left(GAK_S \right) \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} \right] dx$$

$$K_{ij}^{23} = \int_{x_b}^{x_a} p_1 GAK_S \left[\frac{d^2\varphi_i}{dx^2} \frac{d\varphi_j}{dx} dx + p_2 \frac{d\varphi_i}{dx} \left(-EI \frac{d^2\varphi_j}{dx^2} + GAK_S \varphi_j \right) \right] dx = K_{ji}^{32}$$

$$K_{ij}^{33} = \int_{x_b}^{x_a} p_1 \left[\left(GAK_S \right) \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} + p_2 \int_{x_b}^{x_a} \left(-\gamma \frac{d^2\varphi_i}{dx^2} + \varphi_i \right) \left(-EI \frac{d^2\varphi_j}{dx^2} + GAK_S \varphi_j \right) \right] dx$$

$$F_i^1 = -\int_{x_b}^{x_a} \left[f \frac{d^2\varphi_i}{dx^2} \right] dx$$

$$F_i^2 = -\int_{x_b}^{x_a} q p_1 \frac{d^2\varphi_i}{dx^2} dx$$

$$F_i^3 = -p_1 \int_{x_b}^{x_a} q \frac{d\varphi_i}{dx} dx \qquad (6.22)$$

where $\gamma = \frac{EI}{GAK_S}$

6.4.2 Nonlinear formulation

The least-squares finite element model of the following set of nonlinear equations assuming EA , EI, GAK_s as constant was developed as follows:-

$$-\frac{dN}{dx} = f$$

$$-\frac{d}{dx}\left[GAK_S \left(\phi + \frac{dw}{dx} \right) \right] - \frac{d}{dx}\left(N \frac{dw}{dx} \right) = q \qquad (6.23)$$

$$-\frac{d}{dx}\left(EI \frac{d\phi}{dx} \right) + GAK_S \left(\phi + \frac{dw}{dx} \right) = 0$$

where q(x) is the transverse distributed force, and N is known in terms of u and was

$$N = EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

The linearization of the above equations that will be used are

$$-EA\left(\frac{d^2u}{dx^2} + \frac{d\bar{w}}{dx}\frac{d^2w}{dx^2}\right) = f$$

$$-GAK_S\left(\frac{d\phi}{dx} + \frac{d^2w}{dx^2}\right) - EA\frac{d^2u}{dx^2}\frac{d\bar{w}}{dx} - \bar{N}\frac{d^2w}{dx^2} = q$$

$$-\frac{d\left(EI\dfrac{d\phi}{dx}\right)}{dx} + GAK_S\left(\phi + \frac{dw}{dx}\right) = 0 \qquad (6.24)$$

where $\bar{N} = EA\left(\frac{d\bar{u}}{dx} + \frac{1}{2}\left(\frac{d^2\bar{w}}{dx^2}\right)^2\right)$, $\hat{N} = EA\left(\frac{d\bar{u}}{dx} + \frac{3}{2}\left(\frac{d^2\bar{w}}{dx^2}\right)^2\right)$

The least-squares functional associated with the above set of linearized equations over a typical element is

$$J_L(u_h, w_h, \phi_h) = \int_{x_b}^{x_a}\Bigg[p_1\left[-GAK_S\left(\frac{d\phi_h}{dx} + \frac{d^2w_h}{dx^2}\right) + EA\frac{d^2u_h}{dx^2}\frac{d\bar{w}_h}{dx} + \hat{N}\frac{d^2w_h}{dx^2} + q\right]^2 +$$
$$p_2\left[-EI\frac{d^2\phi_h}{dx^2} + GAK_S\left(\phi_h + \frac{dw_h}{dx}\right)\right]^2 + \left[EA\left(\frac{d^2u_h}{dx^2} + \frac{d^2w_h}{dx^2}\frac{d\bar{w}_h}{dx}\right) + f\right]^2\Bigg]dx$$

(6.25)

where p_1 and p_2 are scaling factors to make the entire residual to have the same physical dimensions and quantities with bar are assumed to be known from the previous iteration and their variations are zero.

The necessary condition for the minimum of J_L is δJ_L=0

$$0=\int_{x_b}^{x_a}\left[p_1\left[-GAK_S\left(\frac{d\delta\phi_h}{dx}+\frac{d^2\delta w_h}{dx^2}\right)+EA\frac{d^2\delta u_h}{dx^2}\frac{d\overline{w}_h}{dx}+\hat{N}\frac{d^2\delta w_h}{dx^2}\right]\right.$$
$$\times\left[-GAK_S\left(\frac{d\phi_h}{dx}+\frac{d^2 w_h}{dx^2}\right)+EA\frac{d^2u_h}{dx^2}\frac{d\overline{w}_h}{dx}+\hat{N}\frac{d^2 w_h}{dx^2}+q\right]+$$
$$EA\left[EA\left(\frac{d^2u_h}{dx^2}+\frac{d\overline{w}_h}{dx}\frac{d^2w_h}{dx^2}\right)+f\right]\left(\frac{d^2\delta u_h}{dx^2}+\frac{d\overline{w}_h}{dx}\frac{d^2\delta w_h}{dx^2}\right)+$$
$$\left.p_2\left(GAK_S\left(\delta\phi_h+\frac{d\delta w_h}{dx}\right)-EI\frac{d^2\delta\phi_h}{dx^2}\right)\left(GAK_S\left(\phi_h+\frac{dw_h}{dx}\right)-EI\frac{d^2\phi_h}{dx^2}\right)\right]dx$$

(6.26)

The above statement is equivalent to the following three integral statement

$$0=\int_{x_b}^{x_a}\left[\frac{d^2\delta u_h}{dx^2}EA\left(EA\frac{d^2u}{dx^2}+EA\frac{d\overline{w}}{dx}\frac{d^2w}{dx^2}+f\right)+\right.$$
$$\left.p_1\frac{d\overline{w}}{dx}EA\frac{d^2\delta u}{dx^2}\left(-GAK_S\left(\frac{d\phi_h}{dx}+\frac{d^2w_h}{dx^2}\right)+EA\frac{d\overline{w}_h}{dx}\frac{d^2u_h}{dx^2}+\hat{N}\frac{d^2w_h}{dx^2}+q\right)\right]dx \tag{6.27}$$

$$0=\int_{x_b}^{x_a}\left[p_2GAK_S\frac{d^2\delta w_h}{dx^2}\left(-EI\frac{d^2\delta\phi_h}{dx^2}+GAK_S\left(\phi_h+\frac{dw_h}{dx}\right)\right)+\right.$$
$$EA\frac{d\overline{w}_h}{dx}\frac{d^2\delta w_h}{dx^2}\left[EA\left(\frac{d^2u_h}{dx^2}+\frac{d\overline{w}_h}{dx}\frac{d^2w_h}{dx^2}\right)+f\right]+\ p_1\left(GAK_S\frac{d^2\delta w_h}{dx^2}+\hat{N}\frac{d^2\delta w_h}{dx^2}\right)*$$
$$\left.\left(GAK_S\left(\frac{d\phi_h}{dx}+\frac{d^2w_h}{dx^2}\right)+EA\frac{d\overline{w}_h}{dx}\frac{d^2u_h}{dx^2}+\hat{N}\frac{d^2w_h}{dx^2}+q\right)\right]dx$$

(6.28)

$$0=\int_{x_b}^{x_a}\left[p_1-GAK_S\frac{d\delta\phi_h}{dx}\left(-GAK_S\left(\frac{d\phi_h}{dx}+\frac{d^2w_h}{dx^2}\right)+EA\frac{d\overline{w}_h}{dx}\frac{d^2u_h}{dx^2}+\hat{N}\frac{d^2w_h}{dx^2}+q\right)+\right.$$
$$\left.p_2\left(-EI\frac{d^2\delta\phi_h}{dx^2}+GAK_S\delta\phi_h\right)\left(GAK_S\left(\phi_h+\frac{dw_h}{dx}\right)-EI\frac{d^2\phi_h}{dx^2}\right)\right]dx \tag{6.29}$$

where $N = \left[\frac{d\bar{u}}{dx} + \frac{1}{2}\left(\frac{d\bar{w}}{dx}\right)^2 \right]$, $\hat{N} = \bar{N} + \left(\frac{d\bar{w}}{dx}\right)^2 = \left[\frac{d\bar{u}}{dx} + \frac{3}{2}\left(\frac{d\bar{w}}{dx}\right)^2 \right]$

Since the physics of the Euler Bernoulli's Beam theory requires the specification of $u, w, \theta = \left(-\frac{dw}{dx}\right), N, M and V = \left(-\frac{dM}{dx}\right)$ we seek Hermite cubic approximations of $u_h . w_h$ and M_h

$$u_h = \sum_{j=1}^{4} \Delta_j^1 \varphi_j(x), \quad w_h = \sum_{j=1}^{4} \Delta_j^2 \varphi_j(x) \text{ and } M_h = \sum_{j=1}^{4} \Delta_j^3 \varphi_j(x)$$

where $\Delta_j^1, \Delta_j^2 and \Delta_j^3$ denote the nodal values of $\left(u_h, -\frac{du_h}{dx}\right), \left(w_h, -\frac{dw_h}{dx}\right)$ and $\left(M_h, -\frac{dM_h}{dx}\right)$ respectively at the jth node and $\varphi_j(x)$ are the Hermite cubic interpolation functions. Substituting the above equations we get the finite element model as follows.

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] \\ [K^{21}] & [K^{22}] & [K^{23}] \\ [K^{31}] & [K^{32}] & [K^{33}] \end{bmatrix} \begin{Bmatrix} \{\Delta^1\} \\ \{\Delta^2\} \\ \{\Delta^3\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \end{Bmatrix} \tag{6.30}$$

where

$$K_{ij}^{11} = \int_{x_b}^{x_a} (EA)^2 \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx + p_1 \int_{x_b}^{x_a} (EA)^2 \left(\frac{d\bar{w}_h}{dx}\right)^2 \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx$$

$$K_{ij}^{12} = \int_{x_b}^{x_a} (EA)\left(EA + p_1\hat{N} + p_1 GAK_S\right) \frac{d\bar{w}_h}{dx} \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx$$

$$K_{ij}^{13} = p_1 GAK_S \int_{x_b}^{x_a} EA \frac{d\bar{w}_h}{dx} \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx$$

$$K_{ij}^{21} = \int_{x_b}^{x_a} (EA)\left(EA + p_1\hat{N} + p_1 GAK_S\right) \frac{d\bar{w}_h}{dx} \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx$$

$$K_{ij}^{22} = \int_{x_b}^{x_a} p_1 \left[\left(GAK_S + \hat{N} \right)^2 \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx + p_2 \left(GAK_S \right)^2 \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} + \left(EA \right)^2 \left(\frac{d\overline{w}_h}{dx} \right)^2 \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} \right] dx$$

$$K_{ij}^{23} = \int_{x_b}^{x_a} p_1 GAK_S \left(GAK_S + \hat{N} \right) \left[\frac{d^2\varphi_i}{dx^2} \frac{d\varphi_j}{dx} dx + p_2 GAK_S \frac{d\varphi_j}{dx} \left(-EI \frac{d^2\varphi_j}{dx^2} + GAK_S \varphi_j \right) \right] dx$$

$$K_{ij}^{31} = \int_{x_b}^{x_a} p_1 \left(EA \right) \left(GAK_S \right) \frac{d\overline{w}_h}{dx} \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} dx$$

$$K_{ij}^{32} = \int_{x_b}^{x_a} p_1 GAK_S \left(GAK_S + \hat{N} \right) \left[\frac{d^2\varphi_j}{dx^2} \frac{d\varphi_i}{dx} dx + p_2 GAK_S \frac{d\varphi_J}{dx} \left(-EI \frac{d^2\varphi_i}{dx^2} + GAK_S \varphi_i \right) \right] dx$$

$$K_{ij}^{33} = \int_{x_b}^{x_a} p_1 \left[\left(GAK_S \right)^2 \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} + p_2 \int_{x_b}^{x_a} \left(-EI \frac{d^2\varphi_i}{dx^2} + GAK_S \varphi_i \right) \left(-EI \frac{d^2\varphi_j}{dx^2} + GAK_S \varphi_j \right) \right] dx$$

$$F_i^1 = -EA \int_{x_b}^{x_a} \left[f \frac{d^2\varphi_i}{dx^2} + q p_1 \frac{d\overline{w}_h}{dx} \frac{d^2\varphi_i}{dx^2} \right] dx$$

$$F_i^2 = -\int_{x_b}^{x_a} \left[EAf \frac{d\overline{w}}{dx} + q p_1 \left(GAK_S + \hat{N} \right) \right] \frac{d^2\varphi_i}{dx^2} dx$$

$$F_i^3 = -p_1 GAK_S \int_{x_b}^{x_a} q \frac{d\varphi_i}{dx} dx \qquad (6.31)$$

From the terms of K_{ij}^{33} it is clear that the terms p_1 and p_2 should be taken such that $p_2 = p_1 / h^2$, where h is the element length.

6.5 Least-squares Finite Element MODEL 2 for Euler-Bernoulli Beam Theory

6.5.1 Linear formulation

Consider the four first-order governing equations

$$-\frac{dN}{dx}-f=0, \qquad \frac{N}{EA}-\frac{du}{dx}=0$$
$$-\frac{dV}{dx}+kw-q=0, \qquad \theta+\frac{dw}{dx}=0 \tag{6.32}$$
$$\frac{M}{b}-\frac{d\theta}{dx}=0, \qquad -V+\frac{dM}{dx}=0$$

here b=EI

The least-squares functional associated with the above six equations over a typical element is

$$J_2(u_h,w_h,\ldots)=\int_{x_a}^{x_b}\left[\left(\frac{dN_h}{dx}+f\right)^2+\left(\frac{N_h}{EA}-\frac{du_h}{dx}\right)^2+\left(-\frac{dV_h}{dx}+kw_h-q\right)^2+\left(\theta_h+\frac{dw_h}{dx}\right)^2+\right.$$
$$\left.\left(\frac{M_h}{b}-\frac{d\theta_h}{dx}\right)^2+\left(-V_h+\frac{dM_h}{dx}\right)^2\right]dx$$

(6.33)

Here *b=EI* and the necessary condition for minimum of J_2 is

$$0=\int_{x_a}^{x_b}\left[\frac{d\delta N_h}{dx}\left(\frac{dN_h}{dx}+f\right)+\left(\frac{\delta N_h}{EA}-\frac{d\delta u_h}{dx}\right)\left(\frac{N_h}{EA}-\frac{du_h}{dx}\right)+\right.$$
$$\left(-\frac{d\delta V_h}{dx}+k\delta w_h\right)\left(-\frac{dV_h}{dx}+kw_h-q\right)+\left(\delta\theta_h+\frac{d\delta w_h}{dx}\right)\left(\theta_h+\frac{dw_h}{dx}\right)+ \tag{6.34}$$
$$\left.\left(\frac{\delta M_h}{EI}-\frac{d\delta\theta_h}{dx}\right)\left(\frac{M_h}{EI}-\frac{d\theta_h}{dx}\right)+\left(-\delta V_h+\frac{d\delta M_h}{dx}\right)\left(-V_h+\frac{dM_h}{dx}\right)\right]dx$$

The four statements associated with the statement in the above equations are:

$$
\begin{aligned}
0 &= \int_{x_a}^{x_b}\left[-\frac{d\delta u_h}{dx}\left(\frac{N_h}{EA}-\frac{du_h}{dx}\right)\right]dx \\
0 &= \int_{x_a}^{x_b}\left[k\delta w_h\left(-\frac{dV_h}{dx}+kw_h-q\right)+\frac{d\delta w_h}{dx}\left(\theta_h+\frac{dw_h}{dx}\right)\right]dx \\
0 &= \int_{x_a}^{x_b}\left[\delta\theta_h\left(\theta_h+\frac{dw_h}{dx}\right)-\frac{d\delta\theta_h}{dx}\left(\frac{M_h}{EI}-\frac{d\theta_h}{dx}\right)\right]dx \\
0 &= \int_{x_a}^{x_b}\left[\frac{d\delta N_h}{dx}\left(\frac{dN_h}{dx}+f\right)+\left(\frac{\delta N_h}{EA}\right)\left(\frac{N_h}{EA}-\frac{du_h}{dx}\right)\right]dx \\
0 &= \int_{x_a}^{x_b}\left[\left(\frac{\delta M_h}{b}\right)\left(\frac{M_h}{b}-\frac{d\theta_h}{dx}\right)+\frac{d\delta M_h}{dx}\left(-V_h+\frac{dM_h}{dx}\right)\right]dx \\
0 &= \int_{x_a}^{x_b}\left[\left(-\frac{d\delta V_h}{dx}\right)\left(-\frac{dV_h}{dx}+kw_h-q\right)-\delta V_h\left(-V_h+\frac{dM_h}{dx}\right)\right]dx
\end{aligned}
\tag{6.35}
$$

In this model, all physical variables that enter the specification of the boundary conditions appear as unknowns. Hence they are all approximated by Lagrange interpolation functions. Let,

$$
u_h=\sum_{j=1}^{m}u_j\psi_j(x),\quad w_h=\sum_{j=1}^{m}w_j\psi_j(x),\qquad \theta_h=\sum_{j=1}^{m}\theta_j\psi_j(x),
$$

$$
N_h=\sum_{j=1}^{m}N_j\psi_j(x),\quad M_h=\sum_{j=1}^{m}M_j\psi_j(x),\qquad V_h=\sum_{j=1}^{m}V_j\psi_j(x)
$$

Where w_j,θ_j,M_j and V_j denote the nodal values of w_h,θ_h,M_h and V_h respectively at the jth node. Thus we obtain the following finite element model

$$
\begin{bmatrix}
[K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] & [K^{15}] & [K^{16}] \\
[K^{21}] & [K^{22}] & [K^{23}] & [K^{24}] & [K^{25}] & [K^{26}] \\
[K^{31}] & [K^{32}] & [K^{33}] & [K^{34}] & [K^{35}] & [K^{36}] \\
[K^{41}] & [K^{42}] & [K^{43}] & [K^{44}] & [K^{45}] & [K^{46}] \\
[K^{51}] & [K^{52}] & [K^{53}] & [K^{54}] & [K^{55}] & [K^{56}] \\
[K^{61}] & [K^{62}] & [K^{63}] & [K^{64}] & [K^{65}] & [K^{66}]
\end{bmatrix}
\begin{pmatrix}\{u\}\\ \{w\}\\ \{\theta\}\\ \{N\}\\ \{M\}\\ \{V\}\end{pmatrix}
=
\begin{pmatrix}\{F^1\}\\ \{F^2\}\\ \{F^3\}\\ \{F^4\}\\ \{F^5\}\\ \{F^6\}\end{pmatrix}
\tag{6.36}
$$

where

$$K_{ij}^{11} = \int_{x_a}^{x_b} \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$$

$$K_{ij}^{14} = -\frac{1}{EA}\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left(k^2 \frac{d\psi_j}{dx}\psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} \right) dx$$

$$K_{ij}^{23} = \int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{26} = -\int_{x_a}^{x_b} k\frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{32} = \int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{33} = \int_{x_a}^{x_b} \left(\psi_j \psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} \right) dx$$

$$K_{ij}^{35} = -\frac{1}{EI}\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{41} = -\frac{1}{EA}\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{44} = \int_{x_a}^{x_b} \left(\frac{1}{(EA)^2}\psi_j \psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} \right) dx$$

$$K_{ij}^{53} = -\frac{1}{EI}\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{55} = \int_{x_a}^{x_b} \left(\frac{1}{(EI)^2}\psi_j \psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} \right) dx$$

$$K_{ij}^{56} = -\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{62} = -\int_{x_a}^{x_b} k\frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{65} = -\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{66} = \int_{x_a}^{x_b} \left(\psi_j \psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} \right) dx$$

$$F_i^2 = \int_{x_a}^{x_b} kq\psi_i dx \tag{6.37}$$

$$F_i^4 = -\int_{x_a}^{x_b} f \frac{d\psi_i}{dx} dx$$

$$F_i^6 = -\int_{x_a}^{x_b} q \frac{d\psi_i}{dx} dx$$

6.5.2 Nonlinear formulation

Here consider the first-order equations

$$
\begin{aligned}
-\frac{dN}{dx} - f = 0, &\qquad \frac{N}{EA} - \left[\frac{du}{dx} + \left(\frac{dw}{dx} \right)^2 \right] = 0 \\
-\frac{dV}{dx} + \frac{d}{dx}(N\theta) - q = 0, &\qquad \theta + \frac{dw}{dx} = 0 \\
\frac{M}{EI} - \frac{d\theta}{dx} = 0, &\qquad -V + \frac{dM}{dx} = 0
\end{aligned}
\tag{6.38}
$$

The least-squares functional associated with the above six equations over a typical element is

$$J_2(u_h, w_h, \ldots) = \int_{x_a}^{x_b} \left[\left(\frac{dN_h}{dx} + f \right)^2 + \left(\frac{N_h}{EA} - \left[\frac{du_h}{dx} + \left(\frac{dw_h}{dx} \right)^2 \right] \right)^2 + \left(-\frac{dV_h}{dx} + \frac{d}{dx}(N\theta) - q \right)^2 + \left(\theta_h + \frac{dw_h}{dx} \right)^2 + \left(\frac{M_h}{EI} - \frac{d\theta_h}{dx} \right)^2 + \left(-V_h + \frac{dM_h}{dx} \right)^2 \right] dx \tag{6.39}$$

The necessary condition for minimum of J_2 is

$$0=\int_{x_a}^{x_b}\left[\frac{d\delta N_h}{dx}\left(\frac{dN_h}{dx}+f\right)+\left(\frac{\delta N_h}{EA}-\left[\frac{d\delta u_h}{dx}+\left(\frac{dw_h}{dx}\frac{d\delta w_h}{dx}\right)\right]\right)\left(\frac{N_h}{EA}-\left[\frac{du_h}{dx}+\left(\frac{dw_h}{dx}\right)^2\right]\right)+\right.$$
$$\left(-\frac{d\delta V_h}{dx}+\frac{d}{dx}\left(\delta N_h\theta_h+N_h\delta\theta_h\right)\right)\left(-\frac{dV_h}{dx}+\frac{d}{dx}\left(N_h\theta_h\right)-q\right)+$$
$$\left(\delta\theta_h+\frac{d\delta w_h}{dx}\right)\left(\theta_h+\frac{dw_h}{dx}\right)+\left(\frac{\delta M_h}{EI}-\frac{d\delta\theta_h}{dx}\right)\left(\frac{M_h}{EI}-\frac{d\theta_h}{dx}\right)+$$
$$\left.\left(-\delta V_h+\frac{d\delta M_h}{dx}\right)\left(-V_h+\frac{dM_h}{dx}\right)\right]dx$$

(6.40)

The statements associated with the statement in the above equations are:

$$0=\int_{x_a}^{x_b}\left[-\frac{d\delta u_h}{dx}\left(\frac{N_h}{EA}-\left(\frac{du_h}{dx}+\frac{1}{2}\left(\frac{dw_h}{dx}\right)^2\right)\right)\right]dx$$

$$0=\int_{x_a}^{x_b}\left[-\frac{d\delta w_h}{dx}\frac{dw_h}{dx}\left[\frac{N_h}{EA}-\left(\frac{du_h}{dx}+\frac{1}{2}\left(\frac{dw_h}{dx}\right)^2\right)\right]+\frac{d\delta w_h}{dx}\left(\theta_h+\frac{dw_h}{dx}\right)\right]dx$$

$$0=\int_{x_a}^{x_b}\left[\delta\theta_h\left(\theta_h+\frac{dw_h}{dx}\right)-\frac{d\delta\theta_h}{dx}\left(\frac{M_h}{EI}-\frac{d\theta_h}{dx}\right)+\left(\frac{d\delta\theta_h}{dx}N_h+\frac{dN_h}{dx}\delta\theta_h\right)*\right.$$
$$\left.\left(-\frac{dV_h}{dx}+\frac{d\theta_h}{dx}N+\frac{dN_h}{dx}\theta_h-q\right)\right]dx$$

$$0=\int_{x_a}^{x_b}\left[\frac{d\delta N_h}{dx}\left(\frac{dN_h}{dx}+f\right)+\left(\frac{\delta N_h}{EA}\right)\left(\frac{N_h}{EA}-\left(\frac{du_h}{dx}+\frac{1}{2}\left(\frac{dw_h}{dx}\right)^2\right)\right)+\right.$$
$$\left.\left(\frac{d\theta_h}{dx}\delta N+\frac{d\delta N}{dx}\delta\theta_h\right)\left(-\frac{dV}{dx}+\frac{d\theta_h}{dx}N_h+\frac{dN_h}{dx}\theta_h-q\right)\right]dx$$

$$0=\int_{x_a}^{x_b}\left[\left(-\frac{\delta M_h}{EI}\right)\left(\frac{M_h}{EI}-\frac{d\theta_h}{dx}\right)+\frac{d\delta M_h}{dx}\left(-V_h+\frac{dM_h}{dx}\right)\right]dx$$

(6.41)

$$0=\int_{x_a}^{x_b}\left[\left(-\frac{d\delta V_h}{dx}\right)\left(-\frac{dV_h}{dx}+kw_h-q\right)-\frac{d\delta V_h}{dx}\left(-\frac{dV_h}{dx}+\frac{d\theta_h}{dx}N_h+\frac{dN_h}{dx}\theta_h-q\right)\right]dx$$

In this model, all physical variables that enter the specification of the boundary conditions appear as unknowns. Hence they are all approximated by Lagrange interpolation functions. Let,

$$u_h = \sum_{j=1}^{m} u_j \psi_j(x), \quad w_h = \sum_{j=1}^{m} w_j \psi_j(x), \qquad \theta_h = \sum_{j=1}^{m} \theta_j \psi_j(x),$$

$$N_h = \sum_{j=1}^{m} N_j \psi_j(x), \quad M_h = \sum_{j=1}^{m} M_j \psi_j(x), \quad V_h = \sum_{j=1}^{m} V_j \psi_j(x)$$

Where w_j, θ_j, M_j and V_j denote the nodal values of w_h, θ_h, M_h and V_h respectively at the jth node. Thus we obtain the following finite element model

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] & [K^{15}] & [K^{16}] \\ [K^{21}] & [K^{22}] & [K^{23}] & [K^{24}] & [K^{25}] & [K^{26}] \\ [K^{31}] & [K^{32}] & [K^{33}] & [K^{34}] & [K^{35}] & [K^{36}] \\ [K^{41}] & [K^{42}] & [K^{43}] & [K^{44}] & [K^{45}] & [K^{46}] \\ [K^{51}] & [K^{52}] & [K^{53}] & [K^{54}] & [K^{55}] & [K^{56}] \\ [K^{61}] & [K^{62}] & [K^{63}] & [K^{64}] & [K^{65}] & [K^{66}] \end{bmatrix} \begin{pmatrix} \{u\} \\ \{w\} \\ \{\theta\} \\ \{N\} \\ \{M\} \\ \{V\} \end{pmatrix} = \begin{pmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \\ \{F^4\} \\ \{F^5\} \\ \{F^6\} \end{pmatrix} \tag{6.42}$$

where

$$K_{ij}^{11} = \int_{x_a}^{x_b} \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx, \qquad K_{ij}^{12} = \int_{x_a}^{x_b} \frac{1}{2}\frac{dw}{dx}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$$

$$K_{ij}^{14} = -\frac{1}{EA}\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx, \quad K_{ij}^{21} = \int_{x_a}^{x_b} \frac{dw}{dx}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left(\frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + \frac{1}{2}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\left(\frac{dw}{dx}\right)^2 \right) dx$$

$$K_{ij}^{23} = \int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx, \quad K_{ij}^{32} = \int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{33} = \int_{x_a}^{x_b} \left(\psi_j \psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + \left(\frac{d\psi_i}{dx} N + \frac{dN}{dx}\psi_i \right)\left(\frac{dN}{dx}\psi_j + N\frac{d\psi_j}{dx} \right) \right) dx$$

$$K_{ij}^{35} = -\frac{1}{EI}\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx, \quad K_{ij}^{36} = -\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\left(\frac{d\psi_i}{dx} N + \frac{dN}{dx}\psi_i \right) dx$$

$$K_{ij}^{41} = -\frac{1}{EA}\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx, \quad K_{ij}^{42} = -\frac{1}{EA}\frac{1}{2}\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i \frac{dw}{dx} dx$$

$$K_{ij}^{43} = \left(\frac{d\psi_i}{dx}\bar{\theta} + \psi_i \frac{d\bar{\theta}}{dx} \right)\left(\frac{dN}{dx}\psi_j + N\frac{d\psi_j}{dx} \right)$$

$$K_{ij}^{44} = \int_{x_a}^{x_b} \left(\frac{1}{(EA)^2}\psi_j \psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} \right) dx$$

$$K_{ij}^{46} = -\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\left(\frac{d\psi_i}{dx}\bar{\theta} + \psi_i \frac{d\bar{\theta}}{dx} \right) dx, \quad K_{ij}^{53} = -\frac{1}{EI}\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{55} = \int_{x_a}^{x_b} \left(\frac{1}{(EI)^2}\psi_j \psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} \right) dx, \quad K_{ij}^{56} = -\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{63} = -\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\left(\frac{d\psi_j}{dx} N + \frac{dN}{dx}\psi_j \right) dx, \quad K_{ij}^{65} = -\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{66} = \int_{x_a}^{x_b} \left(\psi_j \psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} \right) dx, \quad F_i^3 = \int_{x_a}^{x_b} \left[q\left(\frac{d\psi_i}{dx}\bar{N} + \psi_i \frac{d\bar{N}}{dx} \right) \right] dx$$

$$F_i^4 = \int_{x_a}^{x_b} \left[-f\frac{d\psi_i}{dx} + q\left(\frac{d\psi_i}{dx}\bar{\theta} + \psi_i \frac{d\bar{\theta}}{dx} \right) \right] dx, \quad F_i^6 = -\int_{x_a}^{x_b} q\frac{d\psi_i}{dx} dx \qquad (6.43)$$

6.6 Least-squares Finite Element MODEL 2 for Timoshenko Beam Theory

6.6.1 Linear formulation

Consider the first-order governing equations

$$
\begin{aligned}
&-\frac{dN}{dx}-f=0, && \frac{N}{EA}-\frac{du}{dx}=0\\
&-\frac{dV}{dx}-q=0, && V=GAK\left(\frac{dw}{dx}+\phi\right)\\
&\frac{M}{EI}-\frac{d\theta}{dx}=0, && -V+\frac{dM}{dx}=0
\end{aligned}
\tag{6.44}
$$

The least-squares functional associated with the above six equations over a typical element is

$$
J_2(u_h,w_h,\ldots)=\int_{x_a}^{x_b}\left[\left(\frac{dN_h}{dx}+f\right)^2+\left(\frac{N_h}{EA}-\frac{du_h}{dx}\right)^2+\left(-\frac{dV_h}{dx}-q\right)^2+\left(V_h-GAK\left(\frac{dw_h}{dx}+\phi_h\right)\right)^2+\right.
$$
$$
\left.\left(\frac{M_h}{EI}-\frac{d\theta_h}{dx}\right)^2+\left(-V_h+\frac{dM_h}{dx}\right)^2\right]dx
\tag{6.45}
$$

The necessary condition for minimum of J_2 is

$$
0=\int_{x_a}^{x_b}\left[\frac{d\delta N_h}{dx}\left(\frac{dN_h}{dx}+f\right)+\left(\frac{\delta N_h}{EA}-\frac{d\delta u_h}{dx}\right)\left(\frac{N_h}{EA}-\frac{du_h}{dx}\right)+\right.
$$
$$
\left(\frac{d\delta V_h}{dx}\right)\left(\frac{dV_h}{dx}+q\right)+\left(\delta V_h-GAK\left(\frac{d\delta w_h}{dx}+\delta\phi_h\right)\right)\left(V_h-GAK\left(\frac{dw_h}{dx}+\phi_h\right)\right)+
$$
$$
\left.\left(\frac{\delta M_h}{EI}-\frac{d\delta\theta_h}{dx}\right)\left(\frac{M_h}{EI}-\frac{d\theta_h}{dx}\right)+\left(-\delta V_h+\frac{d\delta M_h}{dx}\right)\left(-V_h+\frac{dM_h}{dx}\right)\right]dx
\tag{6.46}
$$

The statements associated with the statement in the above equations are:

$$\begin{aligned}
0 &= \int_{x_a}^{x_b}\left[-\frac{d\delta u_h}{dx}\left(\frac{N_h}{EA}-\frac{du_h}{dx}\right)\right]dx \\
0 &= \int_{x_a}^{x_b}\left[\frac{d\delta w_h}{dx}\left(\frac{V_h}{GAK}-\frac{dw_h}{dx}-\phi_h\right)\right]dx \\
0 &= \int_{x_a}^{x_b}\left[\delta\theta_h\left(\frac{V_h}{GAK}-\frac{dw_h}{dx}-\phi_h\right)-\frac{d\delta\theta_h}{dx}\left(\frac{M_h}{EI}-\frac{d\theta_h}{dx}\right)\right]dx \\
0 &= \int_{x_a}^{x_b}\left[\frac{d\delta N_h}{dx}\left(\frac{dN_h}{dx}+f\right)+\left(\frac{\delta N_h}{EA}\right)\left(\frac{N_h}{EA}-\frac{du_h}{dx}\right)\right]dx \\
0 &= \int_{x_a}^{x_b}\left[\left(\frac{\delta M_h}{EI}\right)\left(\frac{M_h}{EI}-\frac{d\theta_h}{dx}\right)+\frac{d\delta M_h}{dx}\left(-V_h+\frac{dM_h}{dx}\right)\right]dx \\
0 &= \int_{x_a}^{x_b}\left[\left(\frac{d\delta V_h}{dx}\right)\left(\frac{dV_h}{dx}+q\right)-\delta V_h\left(-V_h+\frac{dM_h}{dx}\right)+\frac{\delta V_h}{GAK}\left(\frac{V_h}{GAK}-\frac{dw_h}{dx}-\phi_h\right)\right]dx
\end{aligned} \tag{6.47}$$

In this model, all physical variables that enter the specification of the boundary conditions appear as unknowns. Hence they are all approximated by Lagrange interpolation functions. Let,

$$u_h=\sum_{j=1}^{m}u_j\psi_j(x),\quad w_h=\sum_{j=1}^{m}w_j\psi_j(x),\qquad \phi_h=\sum_{j=1}^{m}\phi_j\psi_j(x),$$

$$N_h=\sum_{j=1}^{m}N_j\psi_j(x),\quad M_h=\sum_{j=1}^{m}M_j\psi_j(x),\qquad V_h=\sum_{j=1}^{m}V_j\psi_j(x)$$

where w_j,θ_j,M_j and V_j denote the nodal values of w_h,θ_h,M_h and V_h respectively at the jth node. Thus we obtain the following finite element model

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] & [K^{15}] & [K^{16}] \\ [K^{21}] & [K^{22}] & [K^{23}] & [K^{24}] & [K^{25}] & [K^{26}] \\ [K^{31}] & [K^{32}] & [K^{33}] & [K^{34}] & [K^{35}] & [K^{36}] \\ [K^{41}] & [K^{42}] & [K^{43}] & [K^{44}] & [K^{45}] & [K^{46}] \\ [K^{51}] & [K^{52}] & [K^{53}] & [K^{54}] & [K^{55}] & [K^{56}] \\ [K^{61}] & [K^{62}] & [K^{63}] & [K^{64}] & [K^{65}] & [K^{66}] \end{bmatrix} \begin{pmatrix} \{u\} \\ \{w\} \\ \{\phi\} \\ \{N\} \\ \{M\} \\ \{V\} \end{pmatrix} = \begin{pmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \\ \{F^4\} \\ \{F^5\} \\ \{F^6\} \end{pmatrix} \tag{6.48}$$

where

$$\begin{aligned}
&K_{ij}^{11} = \int_{x_a}^{x_b} \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx,\ K_{ij}^{14} = -\frac{1}{EA}\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx,\ K_{ij}^{22} = \int_{x_a}^{x_b}\left(\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\right)dx \\
&K_{ij}^{23} = \int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx,\ K_{ij}^{26} = \int_{x_a}^{x_b} \frac{1}{GAK}\frac{d\psi_j}{dx}\psi_i dx,\ K_{ij}^{32} = \int_{x_a}^{x_b} -\frac{d\psi_j}{dx}\psi_i dx \\
&K_{ij}^{33} = \int_{x_a}^{x_b}\left(-\psi_j\psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\right)dx,\ K_{ij}^{35} = -\frac{1}{EI}\int_{x_a}^{x_b}\frac{d\psi_i}{dx}\psi_j dx \\
&K_{ij}^{41} = -\frac{1}{EA}\int_{x_a}^{x_b}\frac{d\psi_j}{dx}\psi_i dx,\quad K_{ij}^{44} = \int_{x_a}^{x_b}\left(\frac{1}{(EA)^2}\psi_j\psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\right)dx \\
&K_{ij}^{53} = -\frac{1}{EI}\int_{x_a}^{x_b}\frac{d\psi_j}{dx}\psi_i dx,\quad K_{ij}^{55} = \int_{x_a}^{x_b}\left(\frac{1}{(EI)^2}\psi_j\psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\right)dx \\
&K_{ij}^{56} = -\int_{x_a}^{x_b}\frac{d\psi_i}{dx}\psi_j dx,\quad K_{ij}^{62} = -\int_{x_a}^{x_b}\frac{d\psi_i}{dx}\psi_j dx,\quad K_{ij}^{63} = -\int_{x_a}^{x_b}\psi_i\psi_j dx \\
&K_{ij}^{65} = -\int_{x_a}^{x_b}\frac{d\psi_j}{dx}\psi_i dx,\ K_{ij}^{66} = \int_{x_a}^{x_b}\left(\psi_j\psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\right)dx \\
&F_i^4 = -\int_{x_a}^{x_b} f\frac{d\psi_i}{dx}dx,\quad F_i^6 = -\int_{x_a}^{x_b} q\frac{d\psi_i}{dx}dx
\end{aligned} \tag{6.49}$$

6.6.2 Nonlinear formulation

Here consider the first-order equations

$$
\begin{aligned}
&-\frac{dN}{dx}-f=0, && \frac{N}{EA}-\left[\frac{du}{dx}+\left(\frac{dw}{dx}\right)^2\right]=0\\
&-\frac{dV}{dx}+\frac{d}{dx}\left(N\left(\frac{V}{GAK}-\phi\right)\right)-q=0, && \frac{V}{GAK}-\left(\frac{dw}{dx}+\phi\right)=0\\
&\frac{M}{EI}-\frac{d\theta}{dx}=0, && -V+\frac{dM}{dx}=0
\end{aligned} \tag{6.50}
$$

The least-squares functional associated with the above six equations over a typical element is

$$
\begin{aligned}
J_2(u_h,w_h,\ldots)=\int_{x_a}^{x_b}\Bigg[&\left(\frac{dN_h}{dx}+f\right)^2+\left(\frac{N_h}{EA}-\left[\frac{du_h}{dx}+\left(\frac{dw_h}{dx}\right)^2\right]\right)^2+\\
&\left[-\frac{dV}{dx}+\frac{d}{dx}\left(N\left(\frac{V}{GAK}-\phi\right)\right)-q\right]^2+\\
&\left(\frac{V}{GAK}-\left(\frac{dw}{dx}+\phi\right)\right)^2+\left(\frac{M_h}{EI}-\frac{d\theta_h}{dx}\right)^2+\left(-V_h+\frac{dM_h}{dx}\right)^2\Bigg]dx
\end{aligned} \tag{6.51}
$$

The necessary condition for minimum of J_2 is

$$
\begin{aligned}
0=\int_{x_a}^{x_b}\Bigg[&\frac{d\delta N_h}{dx}\left(\frac{dN_h}{dx}+f\right)+\left(\frac{\delta N_h}{EA}-\left[\frac{d\delta u_h}{dx}+\left(\frac{dw_h}{dx}\frac{d\delta w_h}{dx}\right)\right]\right)\left(\frac{N_h}{EA}-\left[\frac{du_h}{dx}+\left(\frac{dw_h}{dx}\right)^2\right]\right)+\\
&\left[-\frac{d\delta V}{dx}+\frac{d}{dx}\delta\left(N\left(\frac{V}{GAK}-\phi\right)\right)\right]\left[-\frac{dV}{dx}+\frac{d}{dx}\left(N\left(\frac{V}{GAK}-\phi\right)\right)-q\right]+\\
&\left(\frac{\delta V}{GAK}-\left(\delta\phi+\frac{d\delta w_h}{dx}\right)\right)\left(\frac{V}{GAK}-\left(\phi+\frac{dw_h}{dx}\right)\right)+\left(\frac{\delta M_h}{EI}-\frac{d\delta\theta_h}{dx}\right)\left(\frac{M_h}{EI}-\frac{d\theta_h}{dx}\right)+\\
&\left(-\delta V_h+\frac{d\delta M_h}{dx}\right)\left(-V_h+\frac{dM_h}{dx}\right)\Bigg]dx
\end{aligned} \tag{6.52}
$$

The statements associated with the statement in the above equations are:

$$0=\int_{x_a}^{x_b}\left[-\frac{d\delta u_h}{dx}\left(\frac{N_h}{EA}-\left(\frac{du_h}{dx}+\frac{1}{2}\left(\frac{dw_h}{dx}\right)^2\right)\right)\right]dx$$

$$0=\int_{x_a}^{x_b}\left[-\frac{d\delta w_h}{dx}\frac{dw_h}{dx}\left[\frac{N_h}{EA}-\left(\frac{du_h}{dx}+\frac{1}{2}\left(\frac{dw_h}{dx}\right)^2\right)\right]+\frac{d\delta w_h}{dx}\left(\frac{V_h}{GAK}-\frac{dw_h}{dx}-\phi\right)\right]dx$$

$$0=\int_{x_a}^{x_b}\left[\begin{array}{l}\delta\phi_h\left(\frac{V_h}{GAK}-\phi_h-\frac{dw_h}{dx}\right)-\frac{d\delta\theta_h}{dx}\left(\frac{M_h}{EI}-\frac{d\theta_h}{dx}\right)-\\ \left(\frac{d\delta\theta_h}{dx}N_h+\frac{dN_h}{dx}\delta\theta_h\right)\left(-\frac{dV_h}{dx}+\frac{dN}{dx}\frac{V_h}{GAK}+\frac{N_h}{GAK}\frac{dV_h}{dx}-N\frac{d\phi}{dx}-\phi\frac{dN}{dx}-q\right)\end{array}\right]dx$$

$$\begin{aligned}0=\int_{x_a}^{x_b}\Bigg[&\frac{d\delta N_h}{dx}\left(\frac{dN_h}{dx}+f\right)+\left(\frac{\delta N_h}{EA}\right)\left(\frac{N_h}{EA}-\left(\frac{du_h}{dx}+\frac{1}{2}\left(\frac{dw_h}{dx}\right)^2\right)\right)+\\ &\left(\frac{d\delta N_h}{dx}\frac{V_h}{GAK}+\frac{\delta N_h}{GAK}\frac{dV_h}{dx}-\frac{d\delta N_h}{dx}\phi-\frac{d\phi_h}{dx}\delta N_h\right)*\\ &\left(\frac{dN_h}{dx}\frac{V_h}{GAK}+\frac{N_h}{GAK}\frac{dV_h}{dx}-\frac{dN_h}{dx}\phi-\frac{d\phi_h}{dx}N_h-\frac{dV_h}{dx}-q\right)\Bigg]dx\end{aligned}$$

$$0=\int_{x_a}^{x_b}\left[\left(\frac{\delta M_h}{EI}\right)\left(\frac{M_h}{EI}-\frac{d\theta_h}{dx}\right)+\frac{d\delta M_h}{dx}\left(-V_h+\frac{dM_h}{dx}\right)\right]dx$$

$$\begin{aligned}0=\int_{x_a}^{x_b}\Bigg[&\frac{\delta V_h}{GAK}\left(\frac{V_h}{GAK}-\phi_h-\frac{dw_h}{dx}\right)-\delta V_h\left(-V_h+\frac{dM_h}{dx}\right)+\left(-\frac{d\delta V_h}{dx}+\frac{dN_h}{dx}\frac{\delta V_h}{GAK}+\frac{N_h}{GAK}\frac{d\delta V_h}{dx}\right)\\ &*\left(-\frac{dV_h}{dx}+\frac{dN_h}{dx}\frac{V_h}{GAK}+\frac{N_h}{GAK}\frac{dV_h}{dx}-\frac{dN_h}{dx}\phi-\frac{d\phi_h}{dx}N_h\right)\Bigg]dx\end{aligned}$$

(6.53)

In this model, all physical variables that enter the specification of the boundary conditions appear as unknowns. Hence they are all approximated by Lagrange interpolation functions. Let,

$$u_h=\sum_{j=1}^{m}u_j\psi_j(x),\quad w_h=\sum_{j=1}^{m}w_j\psi_j(x),\quad \theta_h=\sum_{j=1}^{m}\theta_j\psi_j(x),$$

$$N_h=\sum_{j=1}^{m}N_j\psi_j(x),\quad M_h=\sum_{j=1}^{m}M_j\psi_j(x),\quad V_h=\sum_{j=1}^{m}V_j\psi_j(x)$$

where w_j, θ_j, M_j and V_j denote the nodal values of w_h, θ_h, M_h and V_h respectively at the jth node. Thus we obtain the following finite element model

$$\begin{bmatrix} [K^{11}] & [K^{12}] & [K^{13}] & [K^{14}] & [K^{15}] & [K^{16}] \\ [K^{21}] & [K^{22}] & [K^{23}] & [K^{24}] & [K^{25}] & [K^{26}] \\ [K^{31}] & [K^{32}] & [K^{33}] & [K^{34}] & [K^{35}] & [K^{36}] \\ [K^{41}] & [K^{42}] & [K^{43}] & [K^{44}] & [K^{45}] & [K^{46}] \\ [K^{51}] & [K^{52}] & [K^{53}] & [K^{54}] & [K^{55}] & [K^{56}] \\ [K^{61}] & [K^{62}] & [K^{63}] & [K^{64}] & [K^{65}] & [K^{66}] \end{bmatrix} \begin{pmatrix} \{u\} \\ \{w\} \\ \{\phi\} \\ \{N\} \\ \{M\} \\ \{V\} \end{pmatrix} = \begin{pmatrix} \{F^1\} \\ \{F^2\} \\ \{F^3\} \\ \{F^4\} \\ \{F^5\} \\ \{F^6\} \end{pmatrix} \tag{6.54}$$

where

$$K_{ij}^{11} = \int_{x_a}^{x_b} \frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$$

$$K_{ij}^{12} = \int_{x_a}^{x_b} -\frac{1}{2}\frac{dw}{dx}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$$

$$K_{ij}^{14} = -\frac{1}{EA}\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{21} = \int_{x_a}^{x_b} \frac{dw}{dx}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}dx$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} \left(\frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + \frac{1}{2}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\left(\frac{dw}{dx}\right)^2 \right) dx$$

$$K_{ij}^{23} = \int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{24} = -\int_{x_a}^{x_b} \frac{dw}{dx}\frac{1}{EA}\frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{26} = -\int_{x_a}^{x_b} \frac{1}{GAK}\frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{32} = \int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{33} = \int_{x_a}^{x_b} \left(\psi_j\psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + \left(\frac{d\psi_i}{dx}N + \frac{dN}{dx}\psi_i \right)\left(\frac{dN}{dx}\psi_j + N\frac{d\psi_j}{dx} \right) \right) dx$$

$$K_{ij}^{35} = -\frac{1}{EI}\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{36} = \int_{x_a}^{x_b} \left(-\frac{d\psi_j}{dx} + \frac{\psi_j}{GAK}\frac{dN}{dx} + \frac{N}{GAK}\frac{d\psi_j}{dx} \right)\left(\frac{d\psi_i}{dx}N + \frac{dN}{dx}\psi_i \right) + \frac{\psi_j}{GAK}\psi_i dx$$

$$K_{ij}^{41} = -\frac{1}{EA}\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{42} = -\frac{1}{EA}\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i \frac{1}{2}\frac{dw}{dx} dx$$

$$K_{ij}^{44} = \int_{x_a}^{x_b} \left(\begin{aligned} &\frac{1}{(EA)^2}\psi_j\psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + \left(\frac{\bar{V}}{GAK}\frac{d\psi_i}{dx} + \frac{\psi_i}{GAK}\frac{d\bar{V}}{dx} - \frac{d\psi_i}{dx}\phi - \frac{d\phi}{dx}\psi_i \right) \\ &\left(\frac{\bar{V}}{GAK}\frac{d\psi_j}{dx} + \frac{\psi_j}{GAK}\frac{d\bar{V}}{dx} - \frac{d\psi_j}{dx}\phi - \frac{d\phi}{dx}\psi_j \right) \end{aligned} \right) dx$$

$$K_{ij}^{46} = -\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\left(\frac{\bar{V}}{GAK}\frac{d\psi_i}{dx} + \frac{\psi_i}{GAK}\frac{d\bar{V}}{dx} - \frac{d\psi_i}{dx}\phi - \frac{d\phi}{dx}\psi_i \right) dx$$

$$K_{ij}^{53} = -\frac{1}{EI}\int_{x_a}^{x_b} \frac{d\psi_j}{dx}\psi_i dx$$

$$K_{ij}^{55} = \int_{x_a}^{x_b} \left(\frac{1}{(EI)^2}\psi_j\psi_i + \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} \right) dx$$

$$K_{ij}^{56} = -\int_{x_a}^{x_b} \frac{d\psi_i}{dx}\psi_j dx$$

$$K_{ij}^{62} = -\int_{x_a}^{x_b} \frac{d\psi_j}{dx} \frac{\psi_i}{GAK}$$

$$K_{ij}^{63} = \int_{x_a}^{x_b} -\frac{\psi_i \psi_j}{GAK} + \left(-\frac{d\psi_i}{dx} + \frac{dN}{dx}\frac{d\psi_i}{dx} + \frac{N}{GAK}\frac{d\psi_i}{dx} \right)\left(\psi_j \frac{dN}{dx} + N\frac{d\psi_j}{dx} \right) dx$$

$$K_{ij}^{65} = -\int_{x_a}^{x_b} \frac{d\psi_j}{dx} \psi_i dx$$

$$K_{ij}^{66} = \int_{x_a}^{x_b} \left[\left(\frac{1}{(GAK)^2} \psi_j \psi_i + \psi_j \psi_i \right) + \left(-\frac{d\psi_i}{dx} + \frac{dN}{dx}\frac{d\psi_i}{dx} + \frac{N}{GAK}\frac{d\psi_i}{dx} \right) \left(-\frac{d\psi_j}{dx} + \frac{dN}{dx}\frac{\psi_j}{GAK} + \frac{N}{GAK}\frac{d\psi_j}{dx} \right) \right] dx$$

$$F_i^4 = \int_{x_a}^{x_b} \left[-f\frac{d\psi_i}{dx} + q\left(\frac{\bar{V}}{GAK}\frac{d\psi_i}{dx} + \frac{\psi_i}{GAK}\frac{d\bar{V}}{dx} - \frac{d\psi_i}{dx}\phi - \frac{d\phi}{dx}\psi_i \right) \right] dx$$

$$F_i^3 = \int_{x_a}^{x_b} \left[-q\left(\frac{d\psi_i}{dx} N + \frac{dN}{dx}\psi_i \right) \right] dx$$

$$F_i^6 = -\int_{x_a}^{x_b} q\frac{d\psi_i}{dx} dx \tag{6.55}$$

7. SOLUTION APPROACH

Different methods to develop the finite element model have been discussed so far. An interaction of local and global coordinates is used to obtain the results. The element coefficient matrices are assembled. During assembly the stiffness contributed by the adjacent element to the common coordinates will be doubled. Different boundary conditions are imposed and the value of {F} and {u} are computed. The elemental values of primary variables will be considered during the next cycle of iteration. Convergence is reached when the error is less than the tolerance value. For the practical purposes the absolute error should be small to at lower computational expense. The rate at which certain results approach the exact solution is very important.

7.1 Solution Procedures

There are two different iterative methods discussed here

(1) Direct iteration procedure (2) Newton Raphson iteration procedure

7.1.1 Direct iteration procedure

Here the solution of the coefficient matrix is computed using the known value from the previous solution of the (r-1) th iteration. The solution for the *r*th iteration can be determined from the following equation

$$\left[K\left(\{\Delta\}^{(r-1)}\right)\right]\{\Delta\}^{r}=\{F\} \qquad \text{or} \qquad \left[\bar{K}\left(\{\Delta\}^{(r-1)}\right)\right]\{\Delta\}^{r}=\{F\}$$

Thus the initial guess vector should satisfy the boundary conditions.

7.2.2 Newton-Raphson iteration procedure

Consider the following equation,

$$\{R\}\equiv[K]\{U\}-\{F\}=\{0\} \tag{7.1}$$

where {R} is the residual vector. We expand {R} in the Taylor's series as

$$\left\{R\left(\{U\}\right)\right\} = \left\{R\left(\{U\}^{(r-1)}\right)\right\} + \left(\frac{\partial\{R\}}{\partial\{U\}}\right)^{(r-1)}.\{\delta U\} + ...$$

$$\left(\frac{\partial\{R\}}{\partial\{U\}}\right)^{(r-1)}.\{\delta U\} = -\left\{R\left(\{U\}^{(r-1)}\right)\right\} \tag{7.2}$$

$$\left\{T\left(\{U\}^{(r-1)}\right)\right\}.\{\delta U\} = -\left\{R\left(\{U\}^{(r-1)}\right)\right\}$$

Here {T} is the tangent matrix which is equal to,

$$\left\{T\left(\{U\}^{(r-1)}\right)\right\} = \left(\frac{\partial\{R\}}{\partial\{U\}}\right)^{(r-1)} \qquad or \qquad \left\{T\left(\{U\}^{(r-1)}\right)\right\} = \left(\frac{\partial R_i^e}{\partial U_i^e}\right)^{(r-1)} \tag{7.3}$$

For rth iteration we have, $\{\Delta\}^r = \{\Delta\}^{(r-1)} + \{\delta\Delta\}$

Thus we can write,

$$\begin{aligned} T_{ij}^{\alpha\beta} &= \left(\frac{\partial R_i^\alpha}{\partial \Delta_j^\beta}\right)^{(r-1)} \\ R_i^\alpha &= \sum_{\gamma=1}^{2}\sum_{p} K_{ip}^{\alpha\gamma}\Delta_p^\gamma - F_i^\alpha \\ &= \sum_{p=1}^{2} K_{ip}^{\alpha 1}\Delta_p^1 + \sum_{P=1}^{4} K_{iP}^{\alpha 2}\Delta_P^2 - F_i^\alpha \\ &= \sum_{p=1}^{2} K_{ip}^{\alpha 1}u_p + \sum_{P=1}^{4} K_{iP}^{\alpha 2}\overline{\Delta}_P - F_i^\alpha \end{aligned} \tag{7.4}$$

$$\begin{aligned} T_{ij}^{\alpha\beta} = \left(\frac{\partial R_i^\alpha}{\partial \Delta_j^\beta}\right)^{(r-1)} &= \frac{\partial}{\partial \Delta_j^\beta}\left(\sum_{\gamma=1}^{2}\sum_{p} K_{ip}^{\alpha\gamma}\Delta_p^\gamma - F_i^\alpha\right) \\ &= K_{ij}^{\alpha\beta} + \sum_{p=1}^{2}\frac{\partial}{\partial \Delta_j^\beta}\left(K_{ip}^{\alpha 1}\right)u_p + \sum_{P=1}^{4}\frac{\partial}{\partial \Delta_j^\beta}\left(K_{iP}^{\alpha 2}\right)u_P \end{aligned} \tag{7.5}$$

The coefficients of tangent matrices for EBT conventional weighted residual method can be found by substituting the stiffness coefficients in from equations (4.13) in equation (7.5).

$$
\begin{aligned}
&T_{ij}^{11} = K_{ij}^{11} \\
&T_{Ji}^{21} = K_{Ji}^{21} \\
&T_{iJ}^{12} = 2K_{iJ}^{12} \\
&T_{IJ}^{22} = K_{IJ}^{22} + \frac{1}{2}\int_{x_a}^{x_b} A_{xx}\left[\left(\frac{dw_0}{dx}\right)^2 + \frac{du_0}{dx}\right]\frac{d\phi_J}{dx}\frac{d\phi_I}{dx}\,dx
\end{aligned}
\tag{7.6}
$$

Similarly for TBT conventional weighted residual method we get the following coefficients of tangent matrices.

$$
\begin{aligned}
&T_{ij}^{11} = K_{ij}^{11} \\
&T_{ij}^{12} = K_{ij}^{12} + \frac{1}{2}\int_{x_a}^{x_b}\left(A_{xx}\frac{dw_0}{dx}\right)\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(2)}}{dx}\,dx = 2K_{ij}^{12} \\
&T_{ij}^{13} = K_{ij}^{13} \\
&T_{ij}^{21} = K_{ij}^{21} \\
&T_{ij}^{22} = K_{ij}^{22} + \frac{1}{2}\int_{x_a}^{x_b} A_{xx}\left[\left(\frac{dw_0}{dx}\right)^2 + \frac{du_0}{dx}\right]\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx}\,dx \\
&T_{ij}^{23} = K_{ij}^{23} \\
&T_{ij}^{31} = K_{ij}^{31} \\
&T_{ij}^{32} = K_{ij}^{32} \\
&T_{ij}^{33} = K_{ij}^{33}
\end{aligned}
\tag{7.7}
$$

A flowchart to explain the logic behind the computer implementation is shown in Figure 7.1.

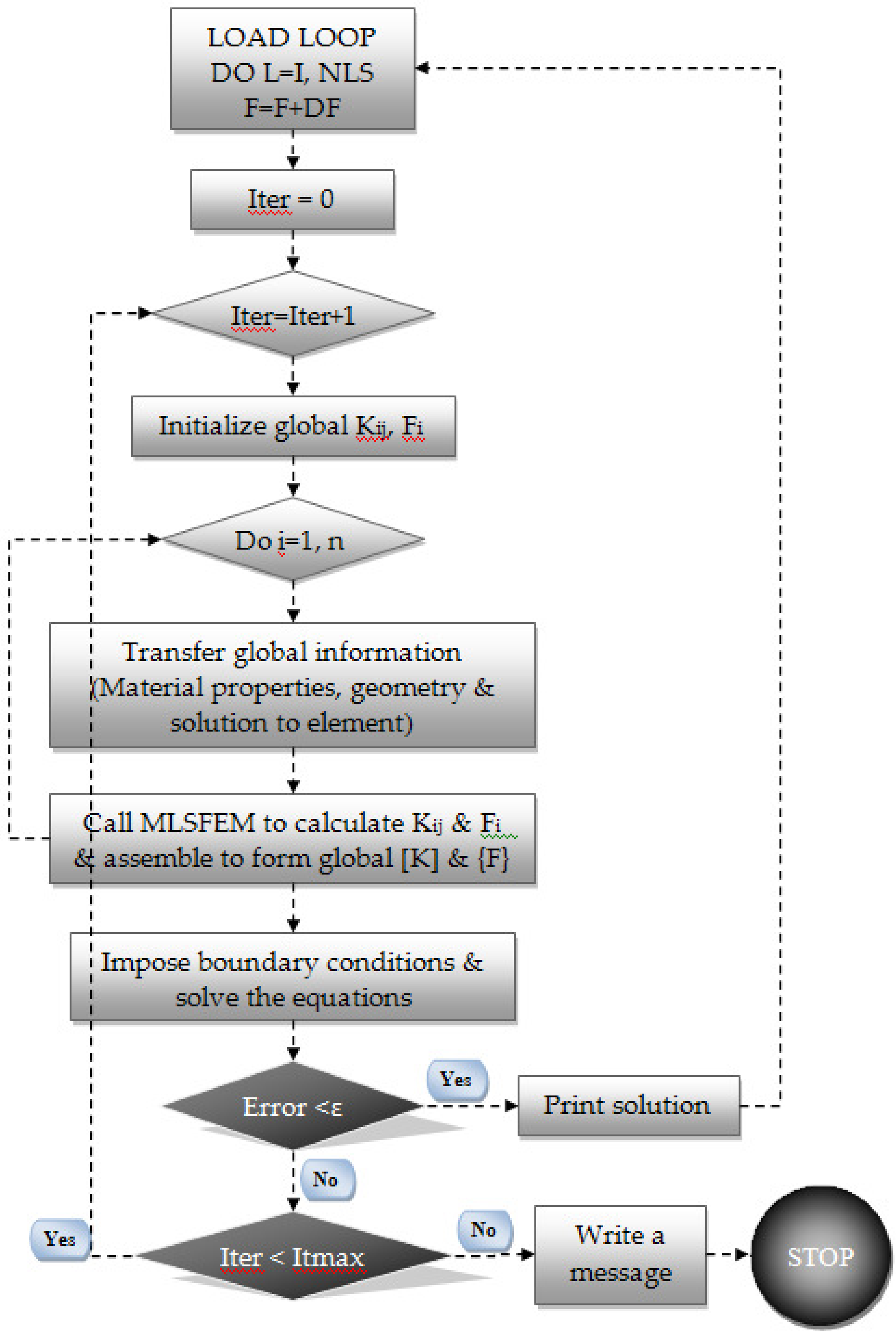


Figure 7.1. A computer implementation flowchart

8. DISCUSSION OF NUMERICAL RESULTS

The following example is considered for EBT and TBT model for conventional weak form and least-squares MODEL1 and MODEL2.

8.1. Example

Consider a linear elastic column which is subjected to a uniformly distributed load q= 10 in 10 load steps .Let L denote the length of the column, EI denote the flexural rigidity of the beam where E is the Young modulus and I is the moment of inertia of the cross section of the column, w(x) denote the deflection function, M denote the bending moment and V denote the shear force. Here calculations have been made with the following data E=30msi,

L=100 in, area (A) =1x1 in^2. tolerance =0.001 maximum number of iterations=30

Boundary conditions: 1) both ends hinged 2) both ends clamped 3) both ends pinned (see Reddy [3]). The beam is analyzed for 4,8, and 32 elements.

8.2. Results

For beam with both ends hinged.

1) 4 ELEMENTS

(A) Conventional weighted residual method (Table 8.1)

Table 8.1: Comparison of displacements in EBT and TBT for hinged-hinged beam

EBT					TBT		
NODE	X	U	W	DW/DX	U	W	PHI
1	0.000	0.000	0.000	-0.017	0.000	0.000	-0.016
2	12.500	0.000	0.202	-0.015	0.000	0.197	-0.015

Table 8.1 continued.

EBT					TBT		
NODE	X	U	W	DW/DX	U	W	PHI
3	25.000	0.000	0.371	-0.011	0.000	0.361	-0.011
4	37.500	0.000	0.482	-0.006	0.000	0.470	-0.006
5	50.000	0.000	0.521	0.000	0.000	0.508	0.000

(B) a) Least-squares finite element MODEL1 –EBT (Table 8.2)

Table 8.2: Comparison of displacements and forces in EBT for hinged-hinged beam

NODE	X	U	DU/DX	W	DW/DX	M	DM/DX
1	0.0000	0.0000	0.0000	0.0000	-0.0166	0.0000	-49.9310
2	12.5000	0.0000	0.0000	0.2020	-0.0152	546.1200	-37.4480
3	25.0000	0.0000	0.0000	0.3706	-0.0114	936.2000	-24.9650
4	37.5000	0.0000	0.0000	0.4815	-0.0061	1170.2000	-12.4830
5	50.0000	0.0000	0.0000	0.5201	0.0000	1248.3000	0.0000

(B) b) Least-squares finite element MODEL1 –TBT (Table 8.3)

Table 8.3: Comparison of displacements and forces in TBT for hinged-hinged beam

NODE	X	U	DU/DX	W	DW/DX	M	DM/DX
1	0.0000	0.0000	0.0000	0.0000	-0.0152	-0.0152	0.0000
2	12.5000	0.0000	0.0000	0.1844	-0.0139	-0.0139	-0.0002

Table 8.3 continued.

NODE	X	U	DU/DX	W	DW/DX	M	DM/DX
3	25.0000	0.0000	0.0000	0.3386	-0.0105	-0.0105	-0.0003
4	37.5000	0.0000	0.0000	0.4402	-0.0056	-0.0056	-0.0004
5	50.0000	0.0000	0.0000	0.4756	0.0000	0.0000	-0.0005

(C) a) Least-squares finite element MODEL2 –EBT (Table 8.4)

Table 8.4: Comparison of displacements and forces in EBT for hinged-hinged beam

NODE	X	U	W	THETA	N	M	V
1	0.0000	0.0000	0.0000	-0.0167	0.0000	0.0000	50.0000
2	3.1250	0.0000	0.0520	-0.0166	0.0000	151.3700	46.8750
3	6.2500	0.0000	0.1034	-0.0163	0.0000	292.9700	43.7500
4	9.3750	0.0000	0.1536	-0.0158	0.0000	424.8000	40.6250
5	12.5000	0.0000	0.2022	-0.0152	0.0000	546.8800	37.5000
6	15.6250	0.0000	0.2487	-0.0145	0.0000	659.1800	34.3750
7	18.7500	0.0000	0.2926	-0.0136	0.0000	761.7200	31.2500
8	21.8750	0.0000	0.3335	-0.0126	0.0000	854.4900	28.1250
9	25.0000	0.0000	0.3711	-0.0115	0.0000	937.5000	25.0000
10	28.1250	0.0000	0.4050	-0.0102	0.0000	1010.7000	21.8750
11	31.2500	0.0000	0.4350	-0.0089	0.0000	1074.2000	18.7500
12	34.3750	0.0000	0.4608	-0.0076	0.0000	1127.9000	15.6250
13	37.5000	0.0000	0.4822	-0.0061	0.0000	1171.9000	12.5000
14	40.6250	0.0000	0.4990	-0.0046	0.0000	1206.1000	9.3750
15	43.7500	0.0000	0.5111	-0.0031	0.0000	1230.5000	6.2500
16	46.8750	0.0000	0.5184	-0.0016	0.0000	1245.1000	3.1250
17	50.0000	0.0000	0.5208	0.0000	0.0000	1250.0000	0.0000

(C) b) Least-squares finite element MODEL2 –TBT (Table 8.5)

Table 8.5: Comparison of displacements and forces in TBT for hinged-hinged beam

NODE	X	U	W	ϕ	N	M	V
1	0.0000	0.0000	0.0000	-0.0167	0.0000	0.0000	50.0000
2	3.1250	0.0000	0.0520	-0.0166	0.0000	151.3700	46.8750
3	6.2500	0.0000	0.1034	-0.0163	0.0000	292.9700	43.7500
4	9.3750	0.0000	0.1536	-0.0158	0.0000	424.8000	40.6250
5	12.5000	0.0000	0.2022	-0.0152	0.0000	546.8800	37.5000
6	15.6250	0.0000	0.2487	-0.0145	0.0000	659.1800	34.3750
7	18.7500	0.0000	0.2926	-0.0136	0.0000	761.7200	31.2500
8	21.8750	0.0000	0.3335	-0.0126	0.0000	854.4900	28.1250
9	25.0000	0.0000	0.3711	-0.0115	0.0000	937.5000	25.0000
10	28.1250	0.0000	0.4050	-0.0102	0.0000	1010.7000	21.8750
11	31.2500	0.0000	0.4350	-0.0089	0.0000	1074.2000	18.7500
12	34.3750	0.0000	0.4608	-0.0076	0.0000	1127.9000	15.6250
13	37.5000	0.0000	0.4822	-0.0061	0.0000	1171.9000	12.5000
14	40.6250	0.0000	0.4990	-0.0046	0.0000	1206.1000	9.3750
15	43.7500	0.0000	0.5111	-0.0031	0.0000	1230.5000	6.2500
16	46.8750	0.0000	0.5184	-0.0016	0.0000	1245.1000	3.1250
17	50.0000	0.0000	0.5208	0.0000	0.0000	1250.0000	0.0000

For beam with both ends clamped.

1) 4 ELEMENTS

(A)Conventional weighted residual method (Table 8.6)

Table 8.6: Comparison of displacements in EBT and TBT for clamped-clamped beam.

EBT					TBT		
NODE	X	U	W	DW/DX	U	W	PHI
1	0.0000	0.0000	0.0000	-0.0167	0.0000	0.0000	-0.0167
2	12.5000	0.0000	0.0199	-0.0152	0.0000	0.0199	-0.0150
3	25.0000	0.0000	0.0585	-0.0115	0.0000	0.0585	-0.0113
4	37.5000	0.0000	0.0914	-0.0061	0.0000	0.0914	-0.0061
5	50.0000	0.0000	0.1040	0.0000	0.0000	0.1040	0.0000

(B)a) Least-squares finite element MODEL1 –EBT (Table 8.7)

Table 8.7: Comparison of displacements and forces in EBT for clamped-clamped beam.

NODE	X	U	DU/DX	W	DW/DX	M	DM/DX
1	0.0000	0.0000	0.0000	0.0000	0.0000	-832.1800	-49.9310
2	12.5000	0.0000	0.0000	0.0199	-0.0027	-286.0600	-37.4480
3	25.0000	0.0000	0.0000	0.0585	-0.0031	104.0200	-24.9650
4	37.5000	0.0000	0.0000	0.0914	-0.0020	338.0700	-12.4830
5	50.0000	0.0000	0.0000	0.1040	0.0000	416.0900	0.0000

(B)b) Least-squares finite element MODEL1 –TBT (Table 8.8)

Table 8.8: Comparison of displacements and forces in TBT for clamped-clamped beam

NODE	X	U	DU/DX	W	DW/DX	M	DM/DX
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003
2	12.5000	0.0000	0.0000	0.0185	-0.0025	-0.0025	0.0001
3	25.0000	0.0000	0.0000	0.0543	-0.0029	-0.0029	0.0000
4	37.5000	0.0000	0.0000	0.0847	-0.0018	-0.0018	-0.0001
5	50.0000	0.0000	0.0000	0.0964	0.0000	0.0000	-0.0002

(C) a) Least-squares finite element MODEL2 –EBT (Table 8.9)

Table 8.9: Comparison of displacements and forces in EBT for clamped-clamped beam

NODE	X	U	W	THETA	N	M	V
1	0.0000	0.0000	0.0000	0.0000	0.0000	-833.2800	50.0000
2	3.1250	0.0000	0.0015	-0.0009	0.0000	-681.9100	46.8750
3	6.2500	0.0000	0.0057	-0.0017	0.0000	-540.3100	43.7500
4	9.3750	0.0000	0.0120	-0.0023	0.0000	-408.4700	40.6250
5	12.5000	0.0000	0.0199	-0.0027	0.0000	-286.4000	37.5000
6	15.6250	0.0000	0.0290	-0.0030	0.0000	-174.1000	34.3750
7	18.7500	0.0000	0.0387	-0.0032	0.0000	-71.5610	31.2500
8	21.8750	0.0000	0.0487	-0.0032	0.0000	21.2130	28.1250
9	25.0000	0.0000	0.0586	-0.0031	0.0000	104.2200	25.0000
10	28.1250	0.0000	0.0681	-0.0029	0.0000	177.4600	21.8750
11	31.2500	0.0000	0.0769	-0.0027	0.0000	240.9400	18.7500
12	34.3750	0.0000	0.0848	-0.0023	0.0000	294.6500	15.6250

Table 8.9 continued.

NODE	X	U	W	THETA	N	M	V
13	37.5000	0.0000	0.0916	-0.0020	0.0000	338.6000	12.5000
14	40.6250	0.0000	0.0970	-0.0015	0.0000	372.7800	9.3750
15	43.7500	0.0000	0.1009	-0.0010	0.0000	397.1900	6.2500
16	46.8750	0.0000	0.1034	-0.0005	0.0000	411.8400	3.1250
17	50.0000	0.0000	0.1042	0.0000	0.0000	416.7200	0.0000

(C) b) Least-squares finite element MODEL2 –TBT (Table 8.10)

Table 8.10: Comparison of displacements and forces in TBT for clamped-clamped beam

NODE	X	U	W	ϕ	N	M	V
1	0.0000	0.0000	0.0000	0.0000	0.0000	-833.1500	49.9960
2	3.1250	0.0000	0.0015	-0.0009	0.0000	-681.7900	46.8710
3	6.2500	0.0000	0.0057	-0.0017	0.0000	-540.2000	43.7460
4	9.3750	0.0000	0.0120	-0.0023	0.0000	-408.3800	40.6210
5	12.5000	0.0000	0.0199	-0.0027	0.0000	-286.3200	37.4960
6	15.6250	0.0000	0.0289	-0.0030	0.0000	-174.0300	34.3710
7	18.7500	0.0000	0.0386	-0.0032	0.0000	-71.5020	31.2470
8	21.8750	0.0000	0.0486	-0.0032	0.0000	21.2620	28.1220
9	25.0000	0.0000	0.0585	-0.0031	0.0000	104.2600	24.9970
10	28.1250	0.0000	0.0680	-0.0029	0.0000	177.4900	21.8720
11	31.2500	0.0000	0.0768	-0.0027	0.0000	240.9600	18.7480
12	34.3750	0.0000	0.0847	-0.0023	0.0000	294.6700	15.6230
13	37.5000	0.0000	0.0914	-0.0020	0.0000	338.6100	12.4980
14	40.6250	0.0000	0.0968	-0.0015	0.0000	372.7800	9.3738
15	43.7500	0.0000	0.1008	-0.0010	0.0000	397.1900	6.2492
16	46.8750	0.0000	0.1032	-0.0005	0.0000	411.8400	3.1246
17	50.0000	0.0000	0.1040	0.0000	0.0000	416.7200	0.0000

For beam with both ends pinned.

1) 4 ELEMENTS

(A)Conventional weighted residual method (Table 8.11)

Table 8.11: Comparison of displacements in EBT and TBT for pinned-pinned beam

EBT					TBT		
NODE	X	U	W	DW/DX	U	W	PHI
1	0.000	0.000	0.000	-0.017	0.000	0.000	-0.016
2	12.500	0.000	0.202	-0.015	0.000	0.197	-0.015
3	25.000	0.000	0.371	-0.011	0.000	0.361	-0.011
4	37.500	0.000	0.482	-0.006	0.000	0.470	-0.006
5	50.000	0.000	0.521	0.000	0.000	0.508	0.000

(B)a) Least-squares finite element MODEL1 –EBT (Table 8.12)

Table 8.12: Comparison of displacements and forces in EBT for pinned-pinned beam

NODE	X	U	DU/DX	W	DW/DX	M	DM/DX
1	0.0000	0.0000	0.0000	0.0000	-0.0166	0.0000	-49.9310
2	12.5000	0.0000	0.0000	0.2020	-0.0152	546.1200	-37.4480
3	25.0000	0.0000	0.0000	0.3706	-0.0114	936.2000	-24.9650
4	37.5000	0.0000	0.0000	0.4815	-0.0061	1170.2000	-12.4830
5	50.0000	0.0000	0.0000	0.5201	0.0000	1248.3000	0.0000

(B)b) Least-squares finite element MODEL1 –TBT (Table 8.13)

Table 8.13: Comparison of displacements and forces in TBT for pinned-pinned beam

NODE	X	U	DU/DX	W	DW/DX	M	DM/DX
1	0.0000	0.0000	0.0000	0.0000	-0.0152	-0.0152	0.0000
2	12.5000	0.0000	0.0000	0.1844	-0.0139	-0.0139	-0.0002
3	25.0000	0.0000	0.0000	0.3386	-0.0105	-0.0105	-0.0003
4	37.5000	0.0000	0.0000	0.4402	-0.0056	-0.0056	-0.0004
5	50.0000	0.0000	0.0000	0.4756	0.0000	0.0000	-0.0005

(C)a) Least-squares finite element MODEL2 –EBT (Table 8.14)

Table 8.14: Comparison of displacements and forces in EBT for pinned-pinned beam

NODE	X	U	W	THETA	N	M	V
1	0.0000	0.0000	0.0000	-0.0167	0.0000	0.0000	50.0000
2	3.1250	0.0000	0.0520	-0.0166	0.0000	151.3700	46.8750
3	6.2500	0.0000	0.1034	-0.0163	0.0000	292.9700	43.7500
4	9.3750	0.0000	0.1536	-0.0158	0.0000	424.8000	40.6250
5	12.5000	0.0000	0.2022	-0.0152	0.0000	546.8800	37.5000
6	15.6250	0.0000	0.2487	-0.0145	0.0000	659.1800	34.3750
7	18.7500	0.0000	0.2926	-0.0136	0.0000	761.7200	31.2500
8	21.8750	0.0000	0.3335	-0.0126	0.0000	854.4900	28.1250
9	25.0000	0.0000	0.3711	-0.0115	0.0000	937.5000	25.0000
10	28.1250	0.0000	0.4050	-0.0102	0.0000	1010.7000	21.8750

Table 8.14 continued.

NODE	X	U	W	THETA	N	M	V
11	31.2500	0.0000	0.4350	-0.0089	0.0000	1074.2000	18.7500
12	34.3750	0.0000	0.4608	-0.0076	0.0000	1127.9000	15.6250
13	37.5000	0.0000	0.4822	-0.0061	0.0000	1171.9000	12.5000
14	40.6250	0.0000	0.4990	-0.0046	0.0000	1206.1000	9.3750
15	43.7500	0.0000	0.5111	-0.0031	0.0000	1230.5000	6.2500
16	46.8750	0.0000	0.5184	-0.0016	0.0000	1245.1000	3.1250
17	50.0000	0.0000	0.5208	0.0000	0.0000	1250.0000	0.0000

(C) b) Least-squares finite element MODEL2 –TBT (Table 8.15)

Table 8.15: Comparison of displacements and forces in TBT for pinned-pinned beam

NODE	X	U	W	ϕ	N	M	$d\phi/dx$
1	0.0000	0.0000	0.0000	-0.0167	0.0000	0.0000	49.9960
2	3.1250	0.0000	0.0520	-0.0166	0.0000	151.3500	46.8710
3	6.2500	0.0000	0.1033	-0.0163	0.0000	292.9400	43.7460
4	9.3750	0.0000	0.1536	-0.0158	0.0000	424.7700	40.6210
5	12.5000	0.0000	0.2022	-0.0152	0.0000	546.8200	37.4960
6	15.6250	0.0000	0.2486	-0.0145	0.0000	659.1200	34.3710
7	18.7500	0.0000	0.2925	-0.0136	0.0000	761.6500	31.2470
8	21.8750	0.0000	0.3334	-0.0126	0.0000	854.4100	28.1220
9	25.0000	0.0000	0.3710	-0.0115	0.0000	937.4100	24.9970
10	28.1250	0.0000	0.4049	-0.0102	0.0000	1010.6000	21.8720
11	31.2500	0.0000	0.4349	-0.0089	0.0000	1074.1000	18.7480
12	34.3750	0.0000	0.4606	-0.0076	0.0000	1127.8000	15.6230

Table 8.15 continued.

NODE	X	U	W	ϕ	N	M	$d\phi/dx$
13	37.5000	0.0000	0.4820	-0.0061	0.0000	1171.8000	12.4980
14	40.6250	0.0000	0.4988	-0.0046	0.0000	1205.9000	9.3738
15	43.7500	0.0000	0.5109	-0.0031	0.0000	1230.3000	6.2492
16	46.8750	0.0000	0.5182	-0.0016	0.0000	1245.0000	3.1246
17	50.0000	0.0000	0.5207	0.0000	0.0000	1249.9000	0.0000

A comparison of finite element results for deflection of beams with pinned-pinned boundary conditions under uniformly distributed load for EBT is shown below in Table 8.16.

Table 8.16: A comparison of results for deflection of beams with pinned-pinned boundary conditions under uniformly distributed load for EBT

q	4 elements	8 elements	32 elements
1	0.30146	0.30146	0.30146
2	0.54802	0.54802	0.54802
3	0.73099	0.73099	0.73099
4	0.86628	0.86628	0.86628
5	0.96642	0.96642	0.96642
6	1.03840	1.03840	1.03840
7	1.08530	1.08530	1.08530
8	1.10720	1.10720	1.10720

Table 8.16 continued.

q	4 elements	8 elements	32 elements
9	1.10090	1.10090	1.10090
10	1.05980	1.05980	1.05980

A comparison of finite element results for deflection of beams with pinned-pinned boundary conditions under uniformly distributed load for TBT is shown below in Table 8.17 .

Table 8.17: A comparison of results for deflection of beams with pinned-pinned boundary conditions under uniformly distributed load for TBT

q	4 elements	8 elements	32 elements
1	0.30134	0.30134	0.30134
2	0.54781	0.54781	0.54781
3	0.73069	0.73069	0.73069
4	0.86590	0.86590	0.86590
5	0.96597	0.96597	0.96597
6	1.03790	1.03790	1.03790
7	1.08470	1.08470	1.08470
8	1.10650	1.10650	1.10650
9	1.10020	1.10020	1.10020
10	1.05900	1.05900	1.05900

A comparison of finite element results for deflection of beams with hinged-hinged boundary conditions under uniformly distributed load for EBT is shown below in Table 8.18.

Table 8.18: A comparison of results for deflection of beams with hinged-hinged boundary conditions under uniformly distributed load for EBT

q	4 elements	8 elements	32 elements
1	0.52083	0.52083	0.52083
2	1.04170	1.04170	1.04170
3	1.56250	1.56250	1.56250
4	2.08330	2.08330	2.08330
5	2.60420	2.60420	2.60420
6	3.12500	3.12500	3.12500
7	3.64580	3.64580	3.64580
8	4.16670	4.16670	4.16670
9	4.68750	4.68750	4.68750
10	5.20830	5.20830	5.20830

A comparison of finite element results for deflection of beams with hinged-hinged boundary conditions under uniformly distributed load for TBT is shown below in Table 8.19.

Table 8.19: A comparison of results for deflection of beams with hinged-hinged boundary conditions under uniformly distributed load for TBT

q	4 elements	8 elements	32 elements
1	0.52096	0.52096	0.52096
2	1.04190	1.04190	1.04190
3	1.56290	1.56290	1.56290
4	2.08380	2.08380	2.08380
5	2.60480	2.60480	2.60480
6	3.12570	3.12580	3.12570
7	3.64670	3.64670	3.64670
8	4.16770	4.16770	4.16770
9	4.68860	4.68860	4.68860
10	5.20960	5.20960	5.20960

A comparison of finite element results for deflection of beams with clamped-clamped boundary conditions under uniformly distributed load for EBT is shown below in Table 8.20.

Table 8.20: A comparison of results for deflection of beams with clamped-clamped boundary conditions under uniformly distributed load for EBT

q	4 elements	8 elements	32 elements
1	0.10410	0.10410	0.10410
2	0.20778	0.20778	0.20778
3	0.31065	0.31065	0.31067

Table 8.20 continued.

q	4 elements	8 elements	32 elements
4	0.41234	0.41234	0.41234
5	0.51250	0.51250	0.51259
6	0.61086	0.61086	0.61104
7	0.70718	0.70718	0.70750
8	0.80128	0.80128	0.80180
9	0.89305	0.89305	0.89390
10	0.98239	0.98239	0.98349

A comparison of finite element results for deflection of beams with clamped-clamped boundary conditions under uniformly distributed load for TBT is shown below in Table 8.21.

Table 8.21: A comparison of results for deflection of beams with clamped-clamped boundary conditions under uniformly distributed load for TBT

q	4 elements	8 elements	32 elements
1	0.10422	0.10422	0.10422
2	0.20803	0.20803	0.20803
3	0.31102	0.31102	0.31103
4	0.41282	0.41282	0.41286
5	0.51310	0.51320	0.51319
6	0.61157	0.61157	0.61175
7	0.70799	0.70799	0.70831

Table 8.21 continued.

q	4 elements	8 elements	32 elements
8	0.80219	0.80219	0.80271
9	0.89405	0.89405	0.89483
10	0.98348	0.98348	0.98458

8.3. Plots

The plot of x vs deflection (w) for different formulations and different elements is shown below in Figures 8.1 and 8.2.

The following are the plots for comparison of a beam clamped at both ends and divided in 4 equal elements.

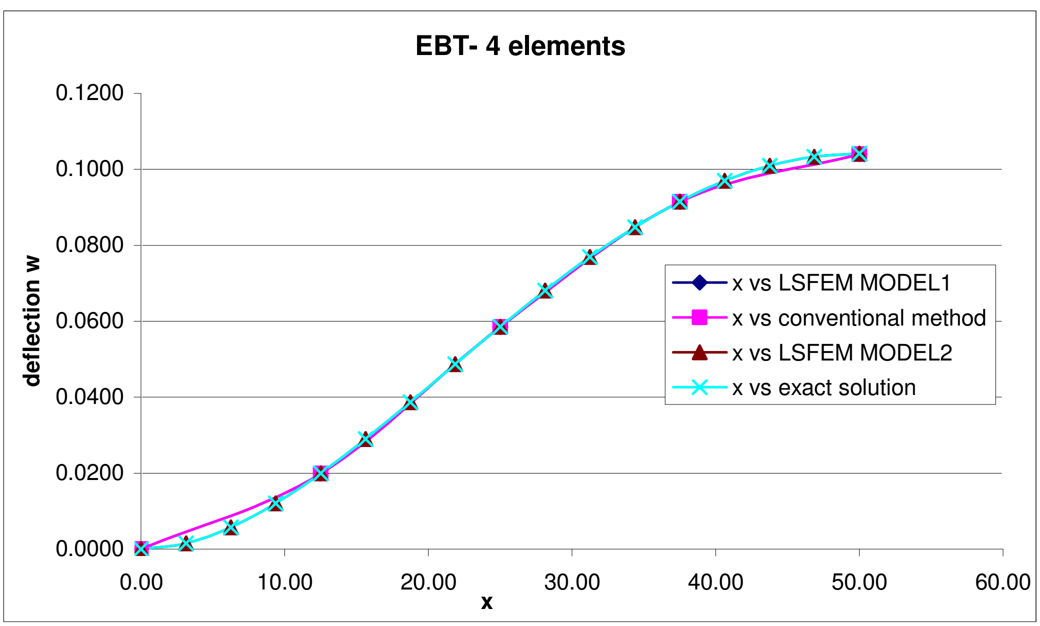


Figure 8.1. Comparison of x vs. deflection in different models for EBT, clamped-clamped, 4 elements

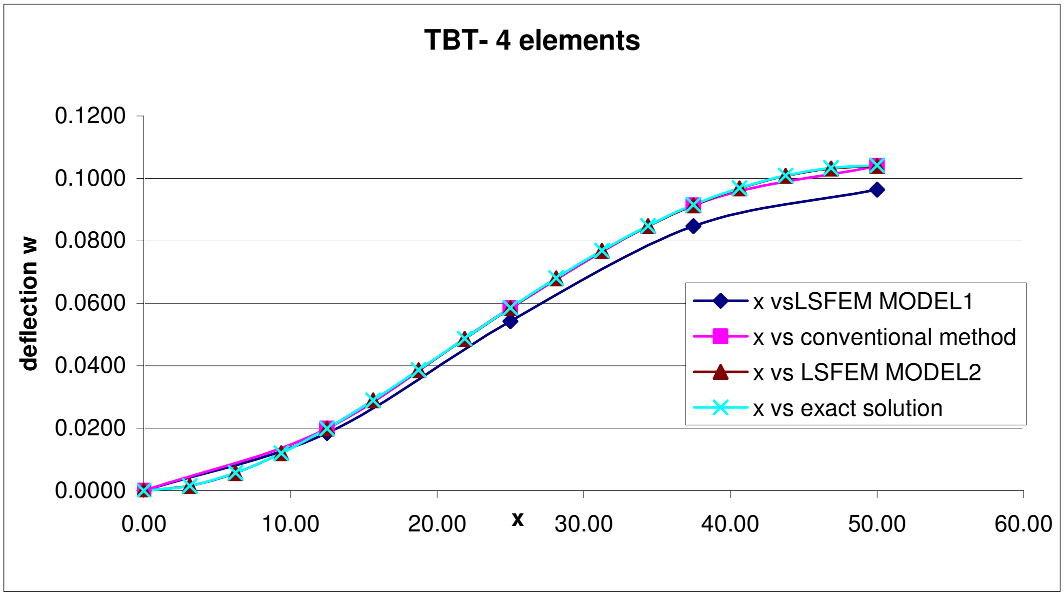


Figure 8.2. Comparison of x vs. deflection in different models for TBT, clamped-clamped ,4 elements

The following are the plots for comparison of a beam clamped at both ends and divided in 8 equal elements in Figures 8.3 and 8.4.

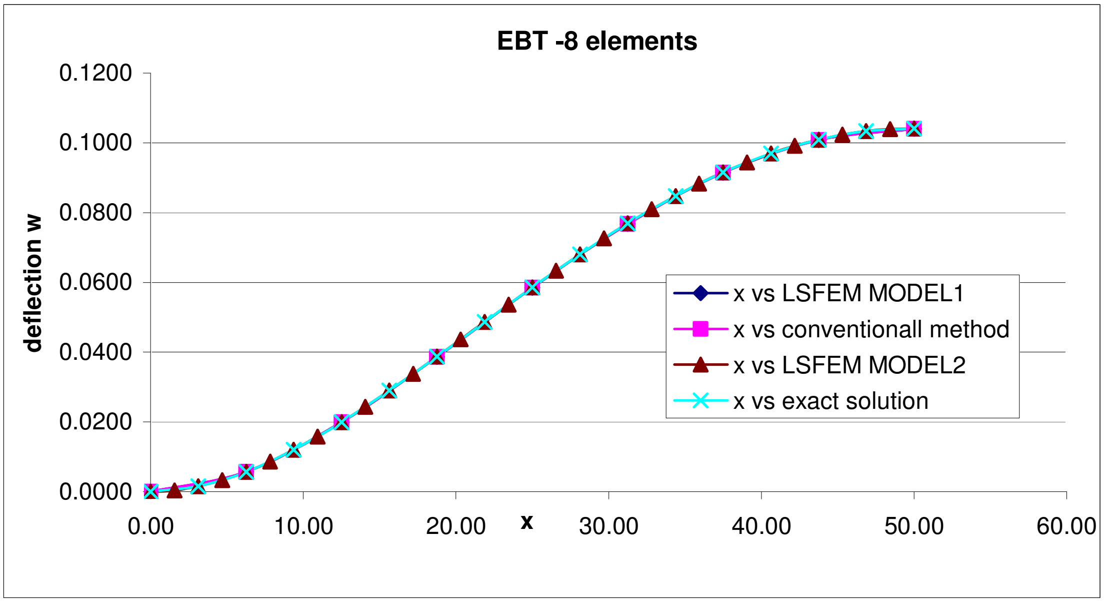


Figure 8.3. Comparison of x vs. deflection in different models for EBT, clamped-clamped, 8 elements

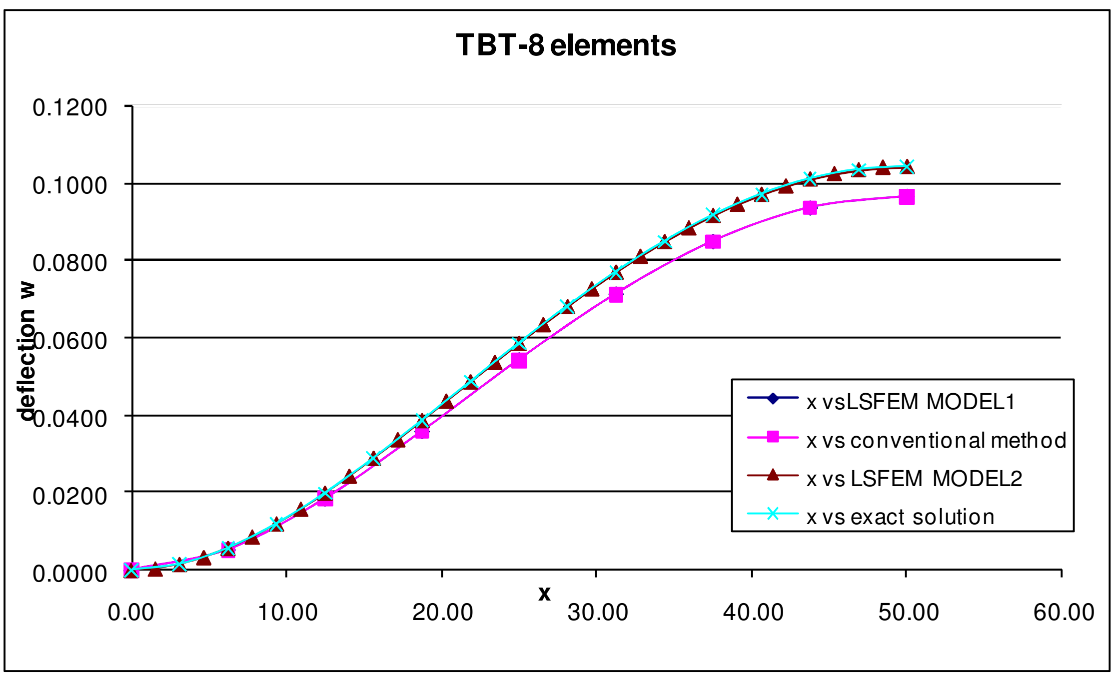


Figure 8.4. Comparison of x vs. deflection in different models for TBT, clamped-clamped, 8 elements

The following are the plots for comparison of a beam clamped at both ends and divided in 32 equal elements in Figures 8.5 and 8.6.

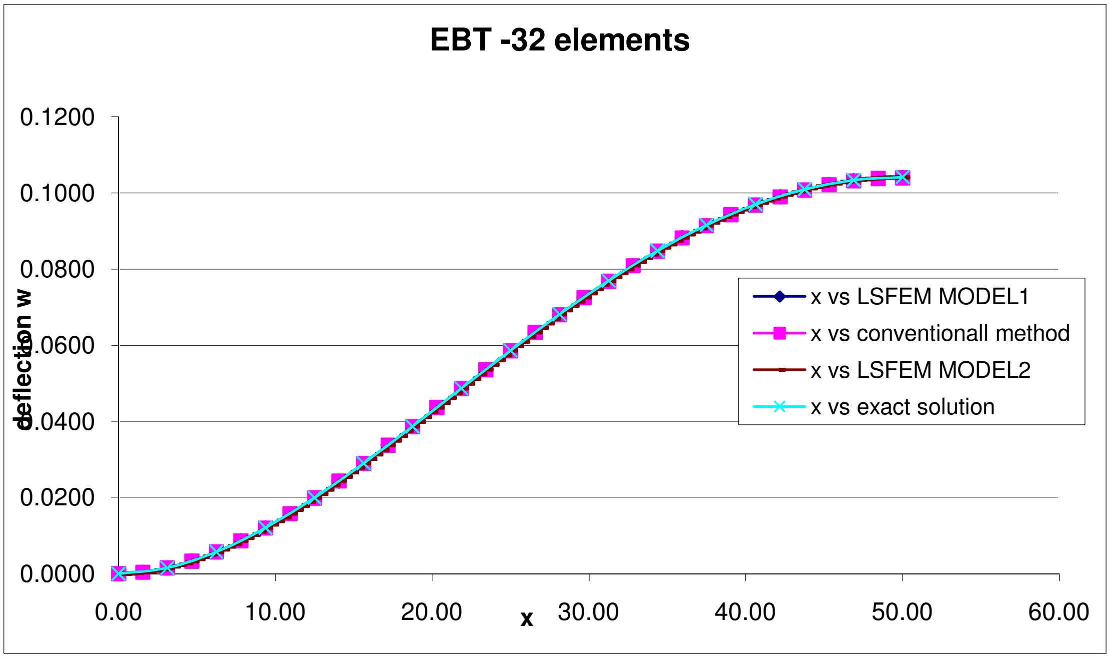


Figure 8.5. Comparison of x vs. deflection in different models for EBT, clamped-clamped 32 elements

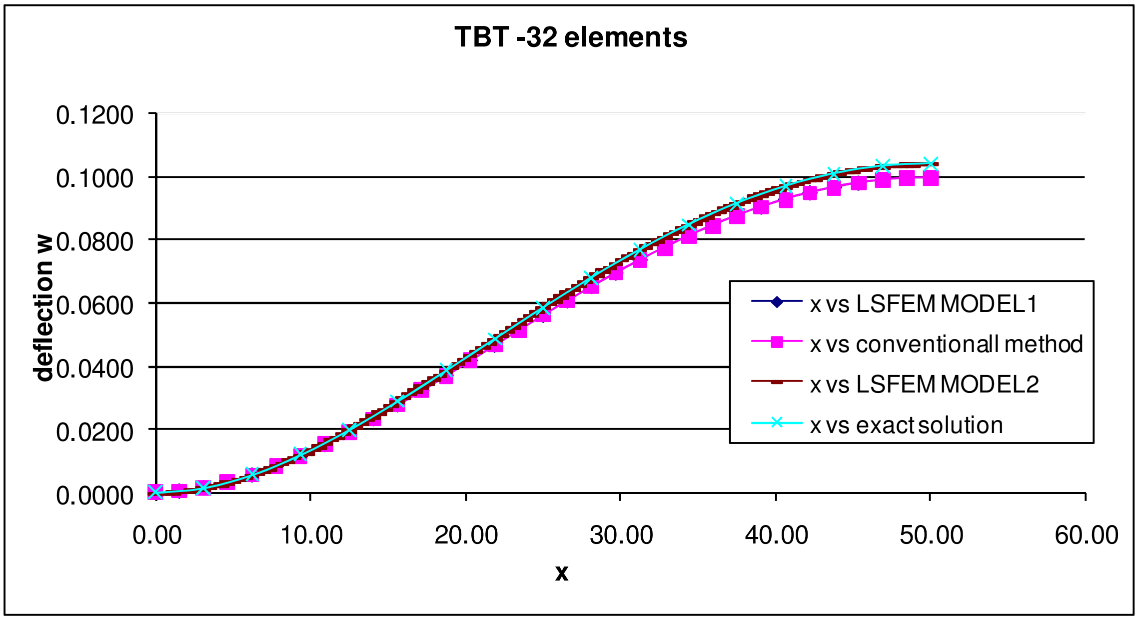


Figure 8.6. Comparison of x vs. deflection in different models for TBT, clamped-clamped 32 elements

The following are the plots for comparison of a beam hinged at both ends and divided in 4 equal elements in Figures 8.7and 8.8.

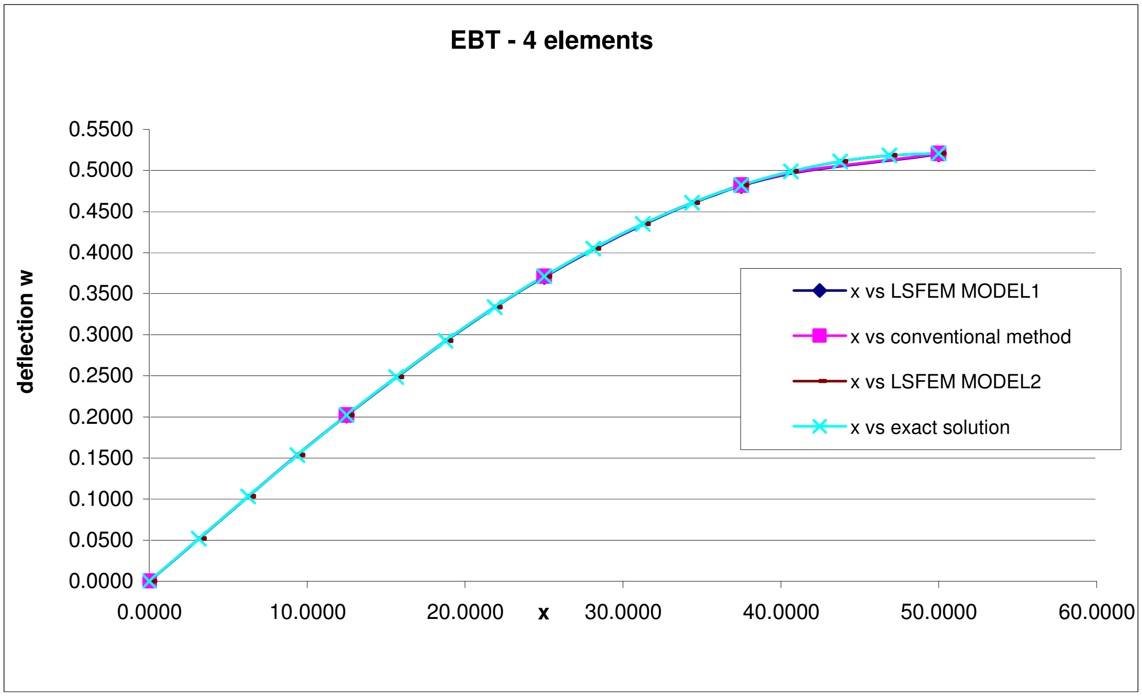


Figure 8.7. Comparison of x vs. deflection in different models for EBT, hinged-hinged ,4 elements

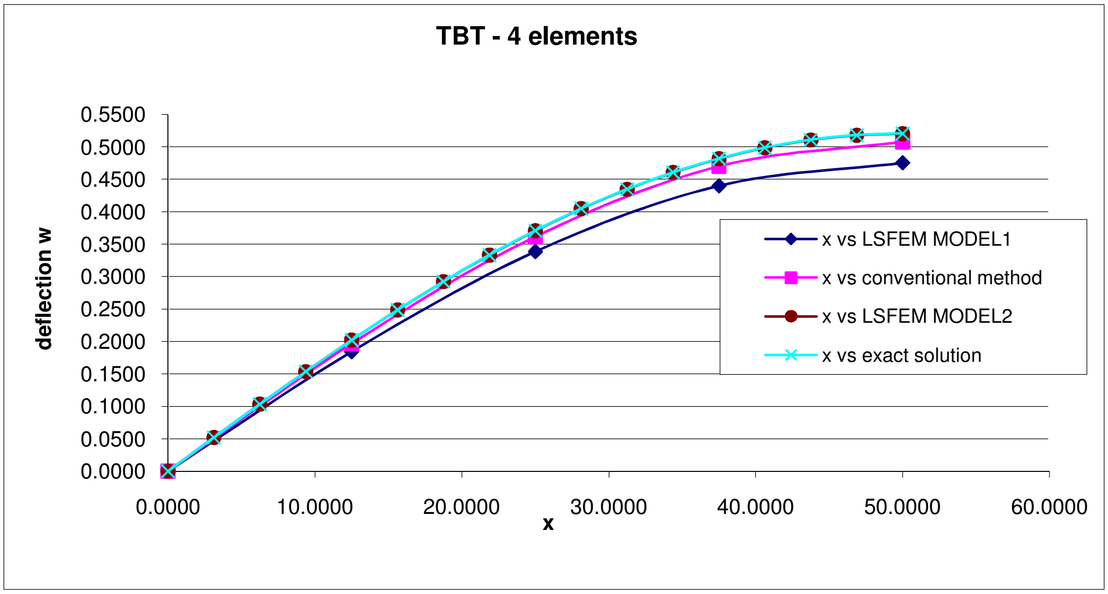


Figure 8.8. Comparison of x vs. deflection in different models for TBT, hinged-hinged, 4 elements

The following are the plots for comparison of a beam hinged at both ends and divided in 8 equal elements in Figures 8.9and 8.10.

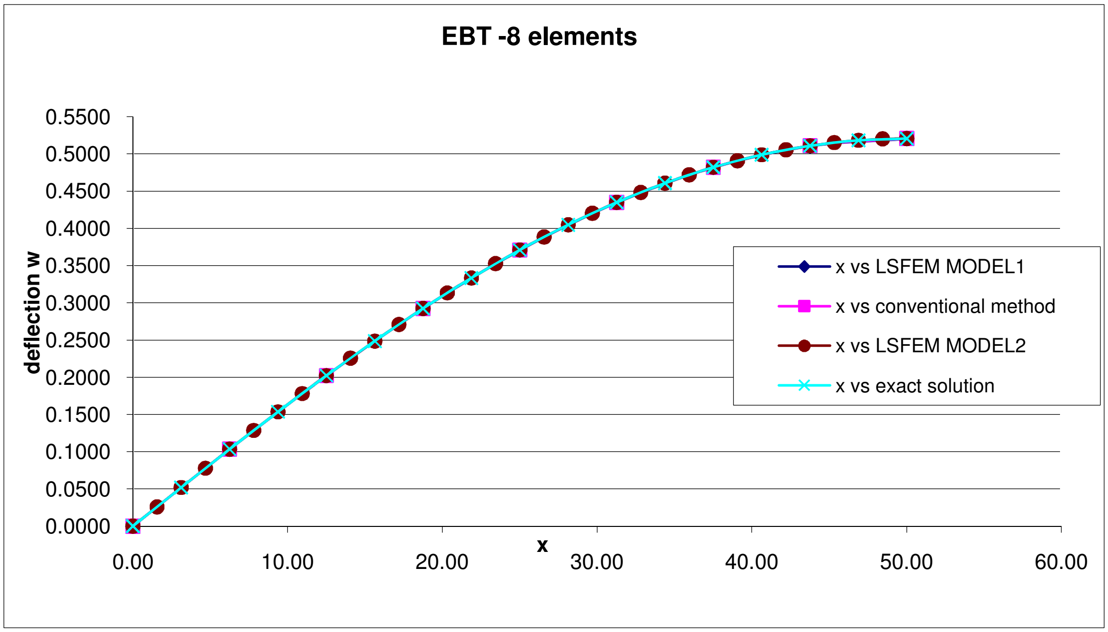


Figure 8.9. Comparison of x vs. deflection in different models for EBT, hinged-hinged,8 elements

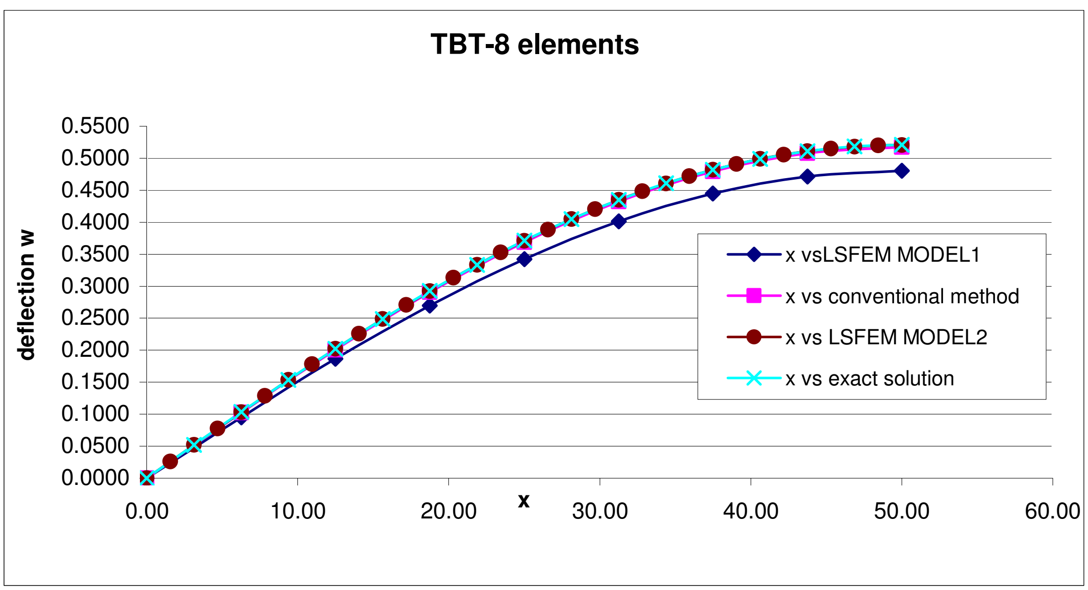


Figure 8.10. Comparison of x vs. deflection in different models for TBT, hinged-hinged,8 elements

The following are the plots for comparison of a beam hinged at both ends and divided in 32 equal elements in Figures 8.11and 8.12.

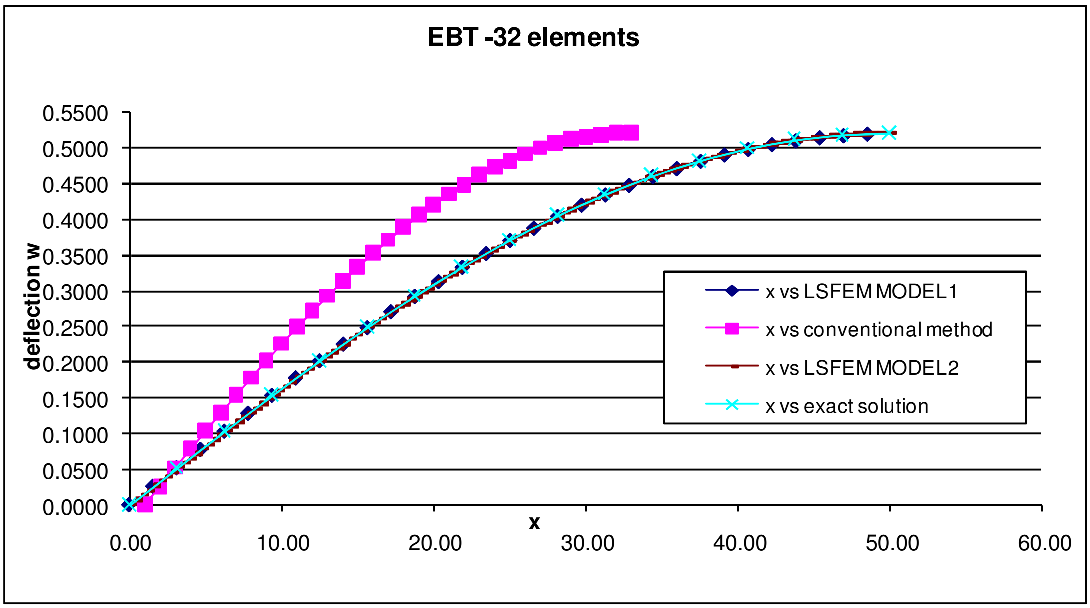


Figure 8.11. Comparison of x vs. deflection in different models for EBT, hinged-hinged 32 elements

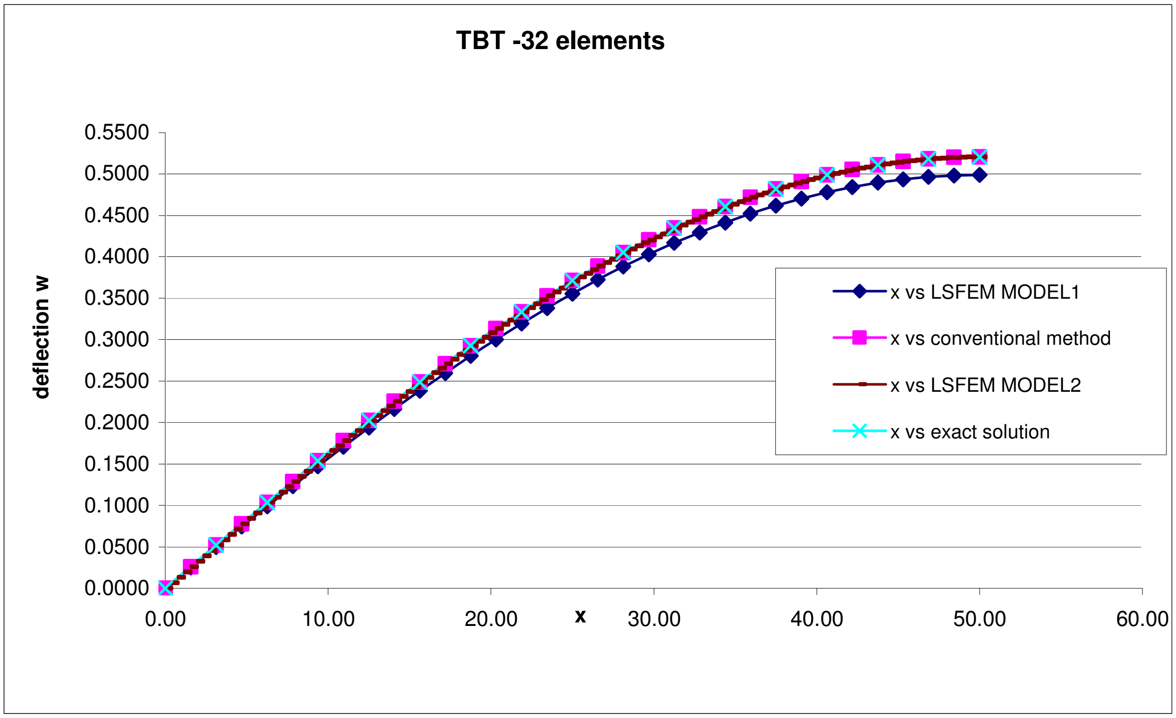


Figure 8.12. Comparison of x vs. deflection in different models for TBT, hinged-hinged 32 elements

The following are the plots for comparison of a beam pinned at both ends and divided in 4 equal elements in Figures 8.13and 8.14.

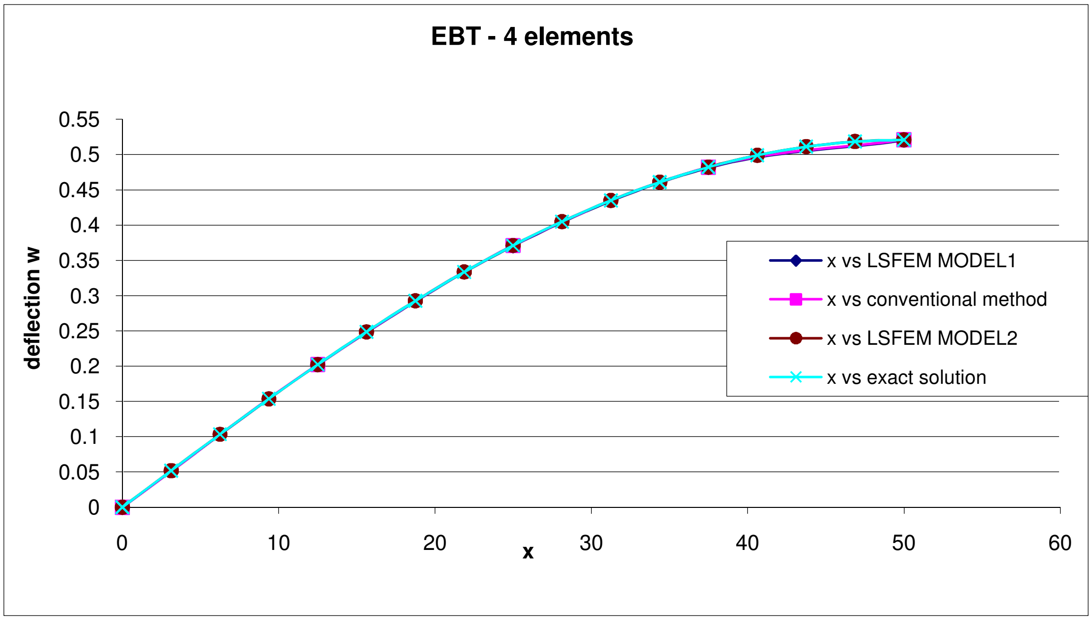


Figure 8.13. Comparison of x vs. deflection in different models for EBT, pinned-pinned ,4 elements

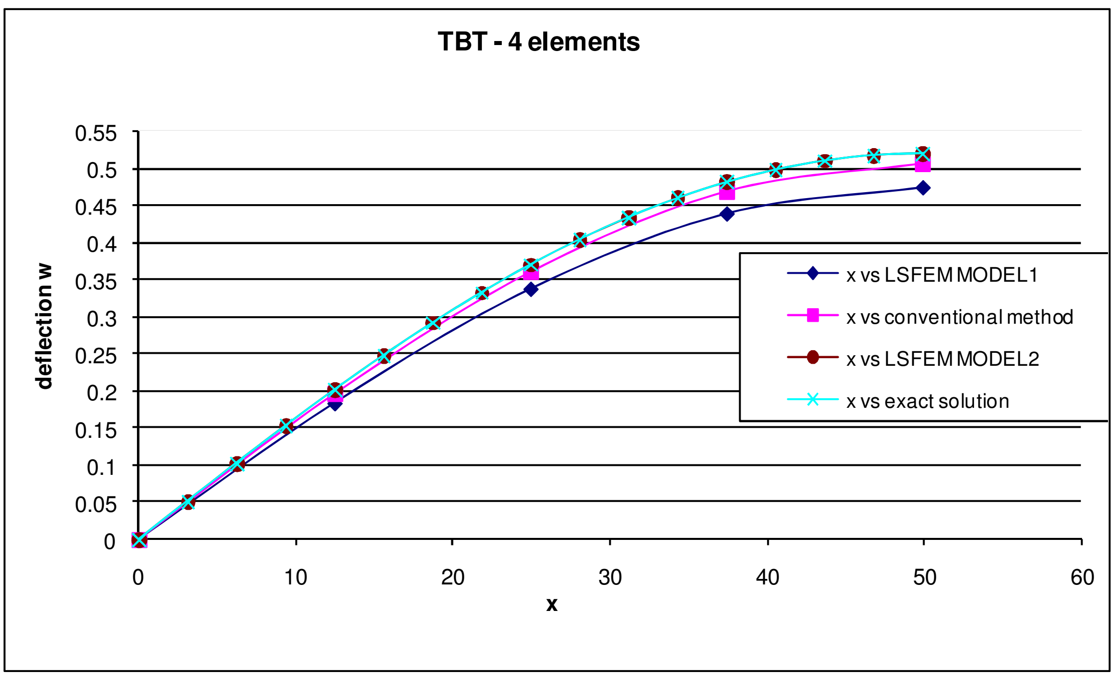


Figure 8.14. Comparison of x vs. deflection in different models for TBT, pinned-pinned, 4 elements

The following are the plots for comparison of a beam pinned at both ends and divided in 8 equal elements in Figures 8.15and 8.16.

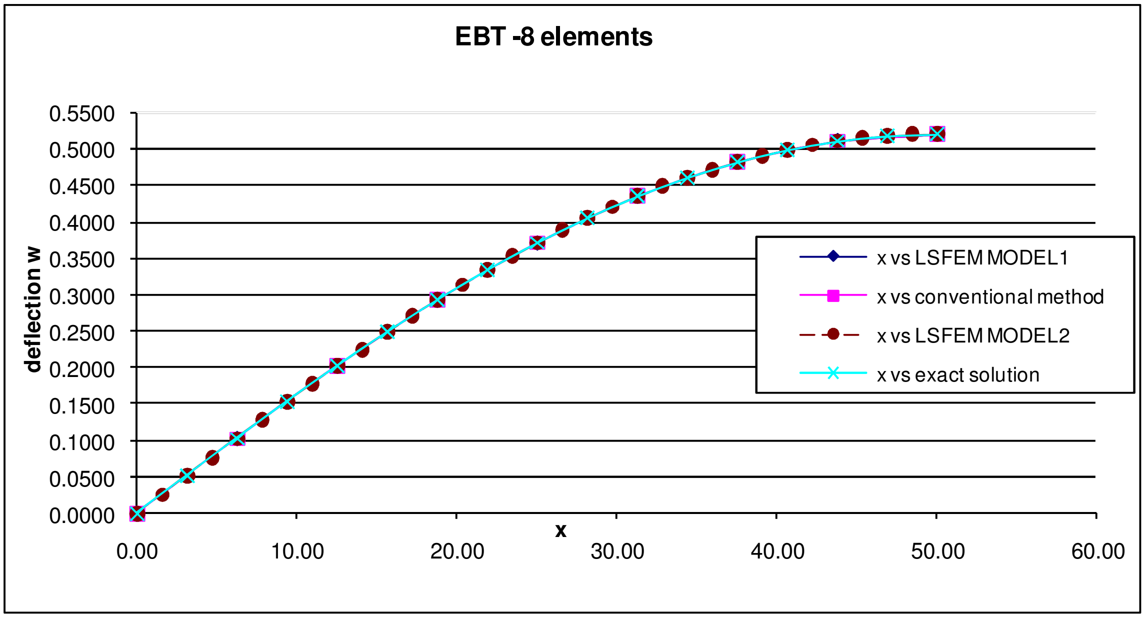


Figure 8.15. Comparison of x vs. deflection in different models for EBT, pinned-pinned, 8 elements

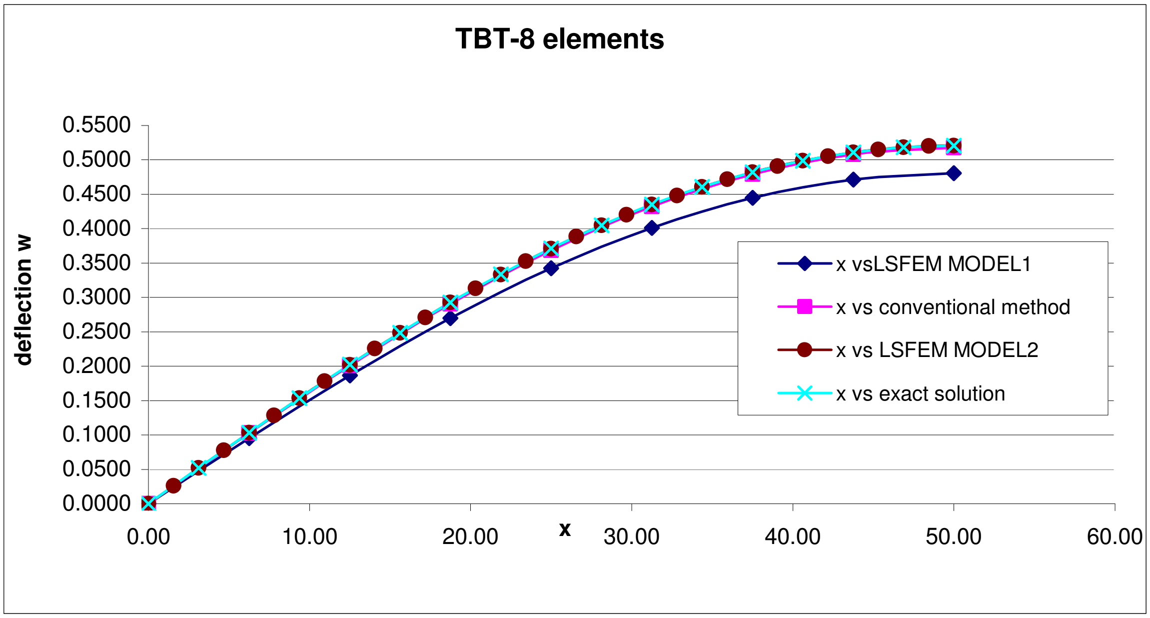


Figure 8.16. Comparison of x vs. deflection in different models for TBT, pinned-pinned, 8 elements

The following are the plots for comparison of a beam pinned at both ends and divided in 4 equal elements in Figures 8.17and 8.18.

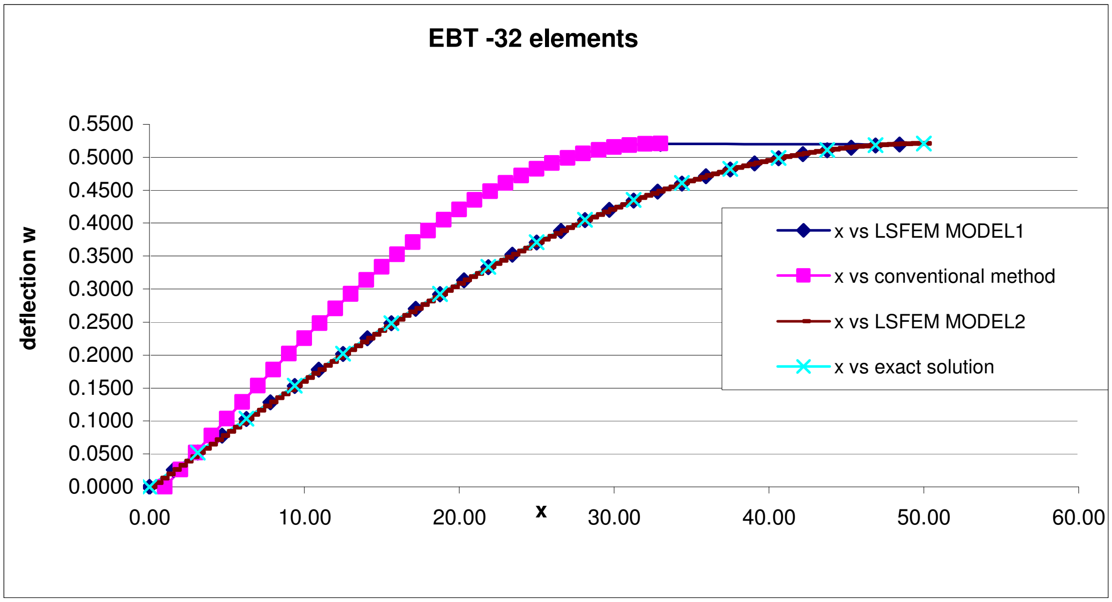


Figure 8.17. Comparison of x vs. deflection in different models for EBT, pinned-pinned, 32 elements

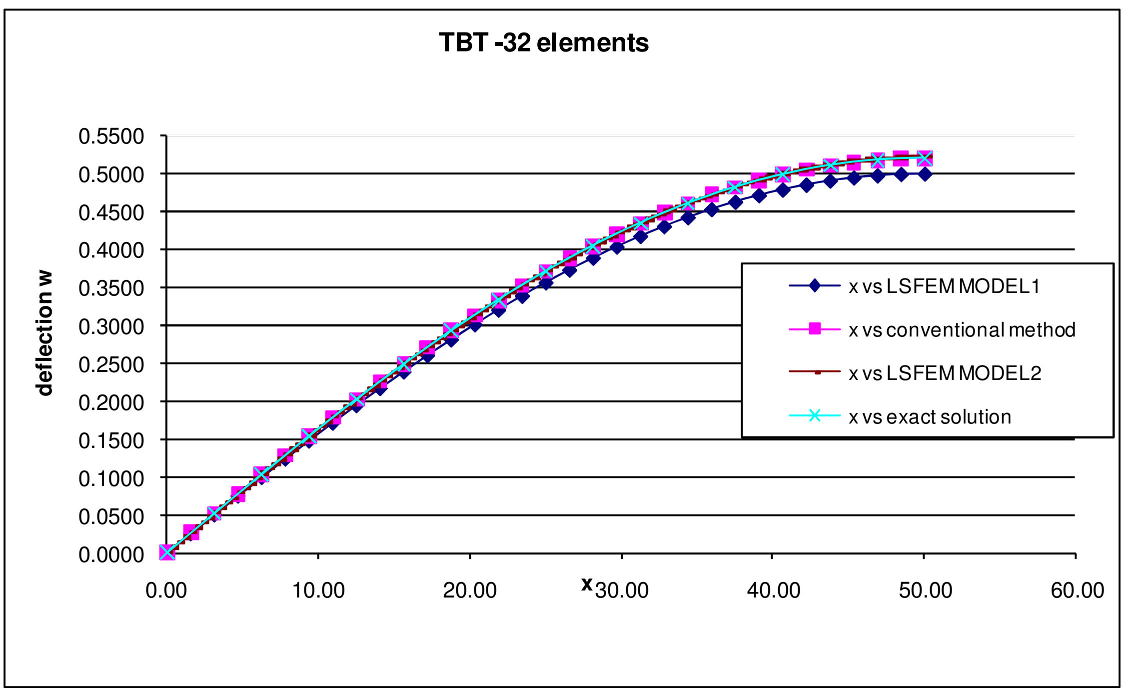


Figure 8.18. Comparison of x vs. deflection in different models for TBT, pinned-pinned 32 elements

A comparison of q vs. maximum deflection for the EBT and TBT using the nonlinear formulation is shown below in Figures 8.19 ,8.20 and 8.21 for different boundary conditions .

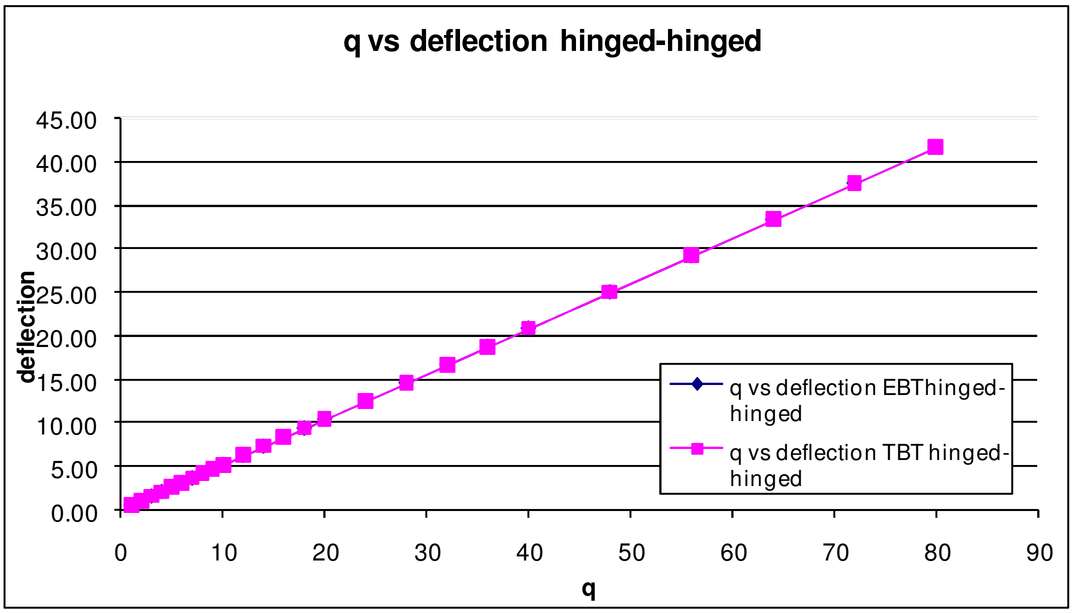


Figure 8.19. Comparison of q vs. maximum deflection for EBT and TBT for hinged-hinged boundary conditions.

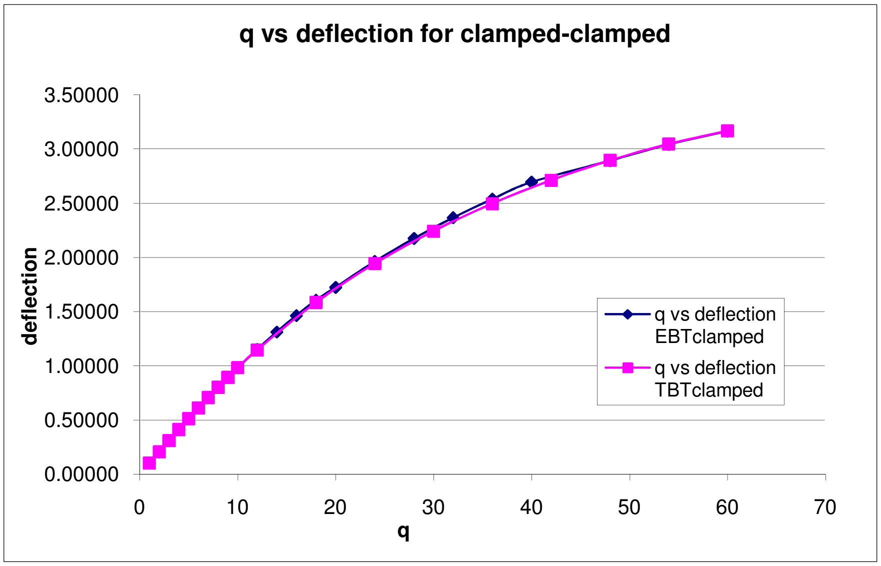


Figure 8.20. Comparison of q vs. maximum deflection for EBT and TBT for clamped-clamped boundary conditions.

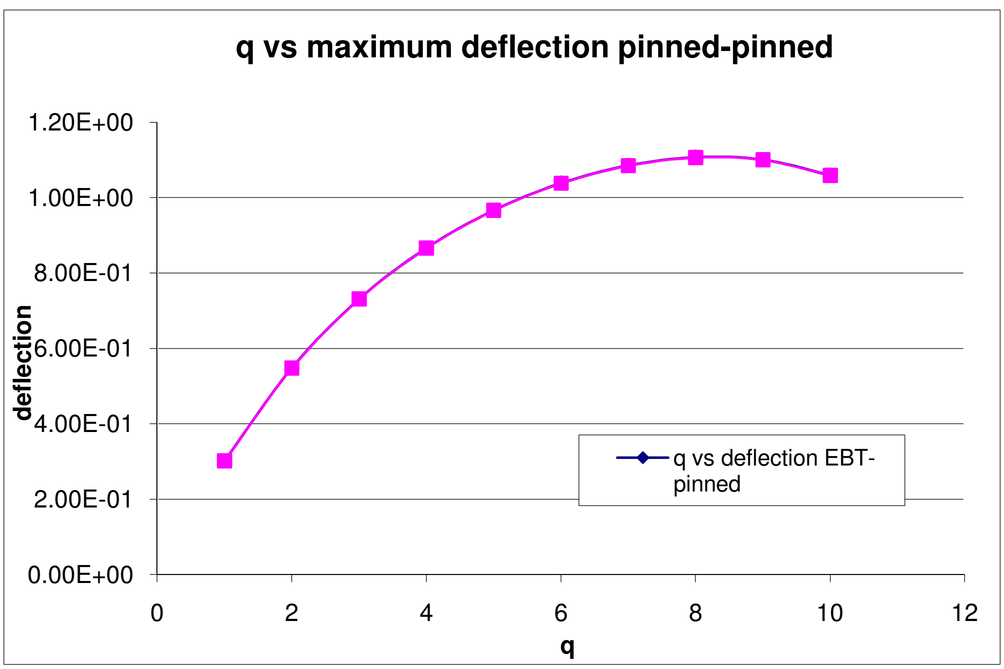


Figure 8.21. Comparison of q vs. maximum deflection for EBT and TBT for pinned-pinned boundary conditions.

A comparison of x vs. shear force and bending moment for LSFEM MODEL2 and conventional method is shown below in Figure 8.22 and Figure 8.23. The shear forces obtained by LSFEM MODEL2 follow a smooth curve where with the conventional method it gives two different values at common points.

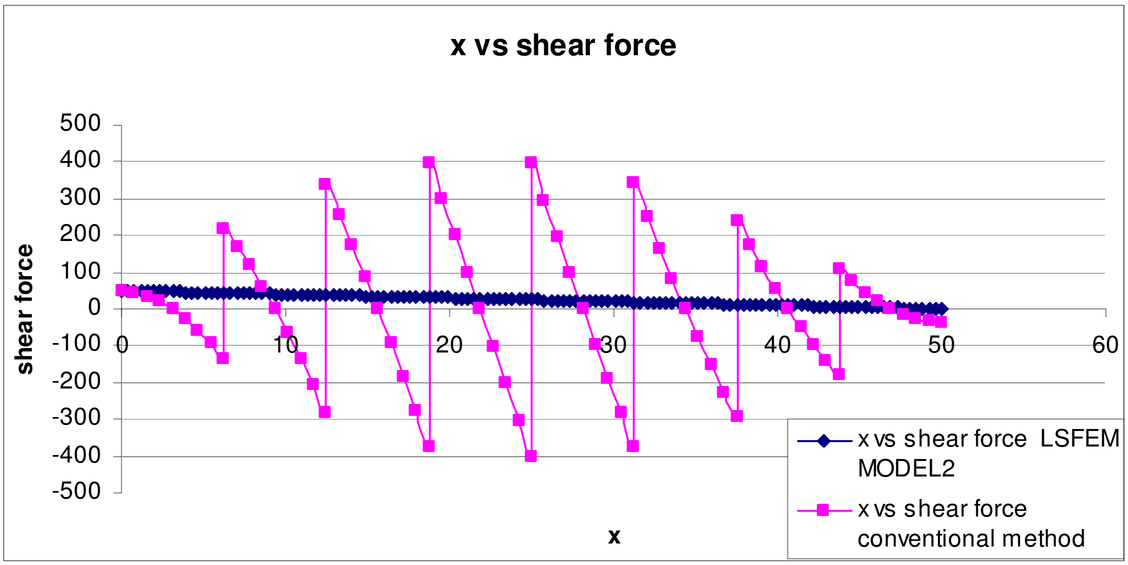


Figure 8.22. Comparison of x vs. Shear force for LSFEM MODEL2 and conventional method

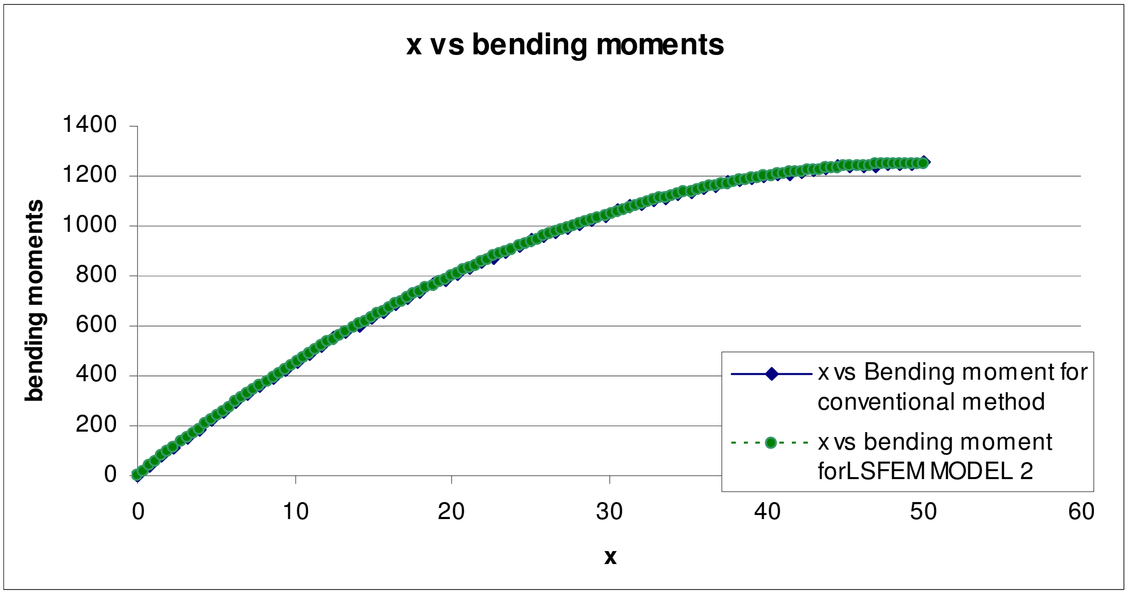


Figure 8.23. Comparison of x vs. Bending Moments for LSFEM MODEL2 and conventional method.

9. SUMMARY AND CONCLUSIONS

From the results presented in section 8, the following observations and conclusions can be made:

1) The plots of x vs. deflection for LSFEM MODEL 1, LSFEM MODEL 2, and conventional method closely fit the exact solution curve. A good solution accuracy for deflection of LSFEM MODEL2 can be observed even for lesser number of elements for various boundary conditions.
2) As the number of elements increases, the plots of x vs. deflection for LSFEM MODEL 1, LSFEM MODEL 2, and conventional method coincide with the exact solution curve for different boundary conditions.
3) The least-squares method helps introducing forces and moments as primary variables and helps increasing the accuracy of the solution.
4) Another salient feature of least-squares method is that once the boundary conditions are imposed the discretization always leads to a positive-definite system of equations which allow the use of fast iterative methods for solution.
5) Thus the theoretical and computational advantages of using the least-squares finite element model were discussed and verified using numerical examples with different boundary conditions and number of elements.
6) Since the internal forces and bending moments serve as independent variables, they can be obtained simultaneously unlike the conventional weighted residual method.

9.1 Future Work

Based on the present study, a systematic and fair comparison of weak form Galerkin models with least-squares models for problems involving plates and shells as well as fluid dynamics can be done further.

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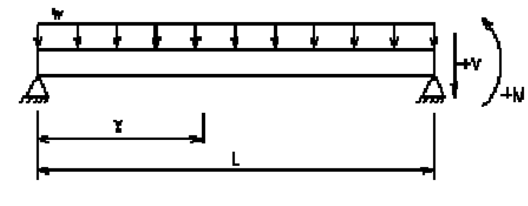
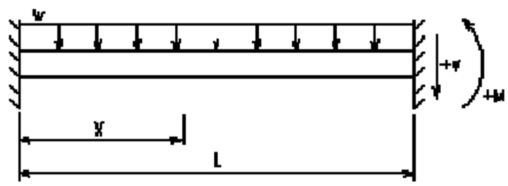
APPENDIX A

COMMON BEAM FORMULAE

(http://structsource.com/analysis/types/beam.htm)

 PINNED-PINNED BEAM WITH UNIFORM LOAD	$V = w\left(\frac{L}{2} - x\right)$ $M = \frac{wx}{2}(L - x)$ $\delta = \frac{wx}{24EI}\left(L^3 - 2Lx^2 + x^3\right)$
 FIXED-FIXED BEAM WITH UNIFORM LOAD	$V = w\left(\frac{L}{2} - x\right)$ $M = \frac{w}{12}\left(6Lx - L^2 - 6x^2\right)$ $\delta = \frac{wx^2}{24EI}(L - x)^2$

VITA

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