THEORIES ON AUCTIONS WITH PARTICIPATION COSTS

A Dissertation

by

XIAOYONG CAO

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2009

Major Subject: Economics

THEORIES ON AUCTIONS WITH PARTICIPATION COSTS

A Dissertation

by

XIAOYONG CAO

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Approved by:

Chair of Committee,	Guoqiang Tian
Committee Members,	Rajiv Sarin
	Steven Puller
	Ximing Wu
Head of Department,	Larry Oliver

August 2009

Major Subject: Economics

ABSTRACT

Theories on Auctions with Participation Costs. (August 2009)

Xiaoyong Cao, B.S., Huazhong University of Science & Technology;

B.A., Huazhong University of Science & Technology

Chair of Advisory Committee: Dr. Guoqiang Tian

In this dissertation I study theories on auctions with participation costs with various information structure.

Chapter II studies equilibria of second price auctions with differentiated participation costs. We consider equilibria in independent private values environments where bidders' entry costs are common knowledge while valuations are private information. We identify two types of equilibria: monotonic equilibria in which a higher participation cost results in a higher cutoff point for submitting a bid, and neg-monotonic equilibria in which a higher participation cost results in a lower cutoff point. We show that there always exists a monotonic equilibrium, and further, that the equilibrium is unique for concave distribution functions and strictly convex distribution functions with some additional conditions. There exists a neg-monotonic equilibrium when the distribution function is strictly convex and the difference of the participation costs is sufficiently small. We also provide comparative static analysis and study the limit status of equilibria when the difference in bidders' participation costs approaches zero.

Chapter III studies equilibria of second price auctions when values and participation costs are both privation information and are drawn from general distribution functions. We consider the existence and uniqueness of equilibrium. It is shown that there always exists an equilibrium for this general economy, and further there exists a unique symmetric equilibrium when all bidders are ex ante homogenous. Moreover, we identify a sufficient condition under which we have a unique equilibrium in a het-

erogeneous economy with two bidders. Our general framework covers many relevant models in the literature as special cases.

Chapter IV characterizes equilibria of first price auctions with participation costs in the independent private values environment. We focus on the cutoff strategies in which each bidder participates and submits a bid if his value is greater than or equal to a critical value. It is shown that, when bidders are homogenous, there always exists a unique symmetric equilibrium, and further, there is no other equilibrium when valuation distribution functions are concave. However, when distribution functions are elastic at the symmetric equilibrium, there exists an asymmetric equilibrium. We find similar results when bidders are heterogenous.

To My Parents

ACKNOWLEDGMENTS

I would like to thank Dr. Guoqiang Tian, my supervisor, for his valuable advice and for his encouragement to finish this dissertation, and especially for his patience. I also thank Dr. Rajiv Sarin, Dr. Steven Puller, and Dr. Ximing Wu for their helpful advice and comments.

I should mention that my graduate studies were partly supported by the Private Enterprise Research Center (PERC), Texas A&M University.

TABLE OF CONTENTS

CHAPTER Page

LIST OF FIGURES

FIGURE Page

CHAPTER I

INTRODUCTION

An auction is an effective way to extract private information by increasing the competition of potential buyers and thus can increase allocation efficiency from the perspectives of both sellers and the social optimum when we do not have complete information about bidders' types. However, not every auction can be implemented freely. This dissertation studies (Bayesian-Nash) equilibria of sealed-bid auctions for economies with private values and participation costs under various information structure.

A. Motivation

The fundamental structure of auctions with participation costs is one through which an indivisible object is allocated to one of many potential buyers, and in order to participate in the auction, buyers must incur some costs[1]. After the cost is incurred, a bidder can submit a bid. The bidder who submits the highest bid wins the object and the payments to the sellers differ among various auction mechanisms. For example, in first price auctions, the winner pays his own bid and in second price auctions, the winner pays the highest bids among his rivals. If there is only one player in second price auctions, the winner pays zero.

There are many sources for participation costs. For instance, sellers may require that those who submit bids have a certain minimum amount of bidding funds which may compel some bidders to borrow; bidders themselves may have transportation

---

The journal model is *Journal of Economic Theory*.

[1]Related terminology includes participation cost, participation fee, entry cost or opportunity costs of participating in the auction. See Laffont and Green [12], Samuelson [37], McAfee and McMillan [30,31], etc.

costs to go to an auction place; or they need spend some money to learn the rules of the auction and how to submit bids. Bidders even have opportunity costs to attend an auction.

With participation costs, bidders' behavior may change. If a bidder's expected revenue from the auction is less than the participation cost before the auction, he will choose not to participate in the auction. If the expected revenue from the auction is bigger than the costs, the bidder will participate and submit a bid accordingly. Even if a bidder decides to participate in the auction, since he may expect that some other bidders will not participate, his bidding behavior may not be the same as in the standard auction without participation costs. The number of bidders submitting a bid in the auction is less than the number of bidders submitting a bid in the standard auction without participation costs, which may in turn, alter the equilibrium bidding strategy. For example, more bidders can raise coordination costs and will not necessarily improve the revenue of the sellers (cf. Samuelson [37], Harstad, Kagel, and Levin [14], Levin and Smith [21]).

There are some studies on the information acquisition in auctions. A bidder may want to learn how he/she and the others value the item, and thus he/she may incur a cost in information acquisition about their valuations[2]. A main difference between participation costs and information acquisition costs is that information acquisition costs are avoidable while participation costs are not. If a bidder does not want to collect information about her own or others' valuations, she does not incur any cost, but she can still submit bids. Some researchers, such as McAfee and McMillan [30,31], Harstad [13] and Levin and Smith [21], combine the idea of participation costs and

[2]Persico [36] studied the incentives of information acquisition in auctions. He found that bidders have more incentives for information acquisition in first price auctions than in second price auctions.

the idea of information acquisition costs. Compete and Jehiel [8] investigate the advantage of using dynamic auctions in the presence of information acquisition cost only. However, information acquisition costs and participation costs can both be regarded as sunk costs after the bidders submit bids.

Addressing the question of participation costs may have important implications. One can characterize the bidding behavior in an auction with participation costs and see how the equilibria will be different from those without participation costs, and then one can derive the implications to the bidders, to the sellers and to the society which, in turn, may be helpful for the optimal selling mechanism design, see Celic and Yilankaya [6].

B. Literature Review

The study of participation costs in auctions mainly focuses on the second price auction due to its simplicity of bidding behavior. In standard second price auctions, bidding one's own valuation is a weakly dominant strategy. There is also another equilibrium in standard second price auctions as shown in Blume and Heidhues [1]: the bidder with the highest value bids his true value and all others bid zero. This is referred to as the asymmetric bidding equilibrium in the standard second price auction. However, in second price auctions with participation costs, it is still true that if a bidder finds participating optimal, he cannot do better than bidding his true value. Therefore, in this dissertation when we consider second price auctions with participation costs we only consider equilibria in which potential bidders use cutoff strategies; i.e., bid their true values if they are greater than the corresponding cutoff points, do not participate otherwise.[3] All of our results about the uniqueness or multiplicity of the equilibria

[3]There may exist equilibrium in which bidders do not bid their true value when they participate. See the example given in Remark 4 in Chapter II.

should be interpreted accordingly.

Green and Laffont [12] were the first to study second price auctions with participation costs in a general framework where bidders' valuations and participation costs are both private information. However, their proof of the existence and uniqueness of an equilibrium is incomplete additionally having imposed a restrictive assumption of uniform distributions for both values and participation costs. There has been some recent work in the literature on equilibria of the second price auction with participation costs in simplified versions where either only valuations or participation costs are private, while the other is assumed to be common knowledge.

Campbell [5] considered the equilibria in second price auctions in an economic environment with equal participation costs when bidders' values are private information and participation costs are common knowledge. He focused on the coordination of equilibrium choice when multiple equilibria exist. Tan and Yilankaya [40] also studied equilibria of second price auctions in an economic environment with equal participation costs. They proved that the equilibrium is unique and symmetric when bidders' distribution functions for values are concave. They also considered the case in which bidders are asymmetric in the sense that they have different valuation distribution functions while maintaining identical participation costs. Some others, such as Samuelson [37], McAfee and McMillan [30,31], Levin and Smith [21, 22], Stageman [38], and Menezes and Monterio [32] also studies auctions with participation costs. All of these studies assume bidders' participation cost are the same.

The assumption of equal participation costs, however, is stringent and unrealistic in many situations. For instance, bidders may have different transportation costs for traveling to auction spots. Bidders may also have different ability to learn more information about the auctions. Some bidders can easily know valuations of other potential bidders while others do not. Thus, it may be more general for one to study

individuals' behavior in second price auctions with different participation costs which may include the equal participation costs as a special case when the difference of participation costs approach zero.

Differentiated participation costs may also have additional implications. First, they can be used to distinguish bidders. Bidders can use this information to decide whether or not to participate in the auction. One can analyze how a bidder's cutoff point will be affected by others' participation costs. Secondly, while Tan and Yilankaya [40] mainly considered bidding behavior inside the same group, they did not consider how the interaction among the different groups would determine and affect the equilibrium behavior.

Chapter II of this dissertation aims to investigate the equilibria when bidders have different participation costs, an analysis which will be more applicable in reality. By considering bidders with different participation costs, we can investigate how equilibria vary as participation costs change. We can also study the limit behavior of the equilibria as the difference in participation costs approaches zero.

Kaplan and Sela [16] simplified the framework of Laffont and Green [12] in another way. They studied equilibria of second price auctions with participation costs when bidders' participation costs are private information and drawn from the same distribution function, while valuations are common knowledge.

Thus, up to now, the problem considered in Laffont and Green [12]has only been answered in some special settings: either participation costs are commonly known or values are publicly known. However, in reality, it is possible that both the valuations and participation costs are private information. Some participation costs are observable to the seller such as the entry fee; some are unobservable to the seller such as the learning costs. A natural way to deal with this is to allow both valuations and participation costs of bidders are private information and their distribution functions

are general and may be different. Chapter III aims to give an answer to the question raised in Laffont and Green [12] in a general framework[4].

While bidding strategies are very nice and simple in second price auctions, the same cannot be said to first price auctions. Studies of first price auctions in the presence of participation costs, however, have received little attention, although they are used more often in practice[5], like the auctions for tendering, particularly for government contracts and auctions for mining leases. The difficulty partly lies in the fact that in first price auctions, bidding strategies are not so explicit, as compared with the strategies in second price auctions. Bidders in first price auctions no longer bid their true valuations. The degree of shading relies heavily on who others enter the auction and the information inferred from the entrance behavior of those bidders. The effect of the information inferred on the bidding strategy of first price auctions is greater than that on second price auctions. Moreover, when bidders use different thresholds to enter an auction, the valuation distributions updated from their entrance behavior are different so that there may be no explicit bidding function and some bidders may use mixed strategies. As such, it is more technically difficult to solve the cutoff strategy since it is determined by the expected revenue of participating in the auction at the thresholds, which in turn depends on the more complicated bidding functions of bidders who submit bids.

Some studies on equilibrium behavior in economic environments with different

[4]It should be pointed out that the framework considered in Chapter III can be applied to many other participation costs related economic issues. For instance, in order to decide whether or not enter an undeveloped market, one needs to know the possible revenue before he enters the market and compare that with the necessary costs. To do this, one must also consider the possible entrance behavior of other opponents.

[5]Samuelson [37] studies the entrance equilibrium of first price competitive procurement auctions and related welfare problem, focusing on the symmetric cutoff threshold.

valuation distributions have been done which can be used to study the equilibria of first price auctions with participation costs. Kaplan and Zamir [17, 18] discuss the properties of bidding functions when valuations are uniformly distributed with different supports. Martinez-Pardina [25] study the first price auction in which bidders' valuations are common knowledge. They show that in equilibrium bidders whose valuations are common knowledge randomize their bids. Chapter IV of this dissertation investigates Bayesian-Nash equilibria of sealed-bid first price auctions in the independent private values environment with participation costs.

There are a branch of papers dealing with the existence and uniqueness of equilibrium of first price auctions. Many existing work focuses exclusively on the symmetric equilibrium of an auction in which bidders are ex ante the same in the sense that the joint distribution of buyers types is symmetric. With a symmetric distribution of types, it is well known that there is only one symmetric equilibrium (Milgrom and Weber [33], Maskin and Riley [26, 27, 28, 29]study the theory in the absence of symmetry. They show that there can be no asymmetric equilibrium under the assumption that reservation prices are drawn independently from a distribution with finite support and positive mass at the lower endpoint. When drop the symmetry assumption, Maskin and Riley [29] show that the same conditions above can guarantee the uniqueness of equilibrium for two bidders' case. For more than two bidders, the uniqueness of equilibrium requires additional fairly mild assumptions that buyers with the same reservation price have the same preferences, that absolute risk aversion is non-increasing, and that the supports of the different buyers distributions of reservation prices have the same upper endpoint. Lebrun [20] shows that if the value cumulative distribution functions are strictly log-concave at the highest lower extremity of their supports, in the asymmetric independent private values model the uniqueness of the equilibrium of the first-price auction is guaranteed.

C. Main Results of the Dissertation

Chapter II considers economic environments where bidders have private valuations for the object and different participation costs that are common knowledge in second price auctions. We identify two types of equilibria: monotonic equilibria in which a higher participation cost results in a higher cutoff point for entering the auction and submitting a bid, and neg-monotonic equilibria in which a higher participation cost results in a lower cutoff point for entering the auction and submitting a bid. We show that there always exists a monotonic equilibrium, and further that, it is unique for concave distribution functions and strictly convex distribution functions under some additional conditions. Uniqueness of the equilibria can greatly simplify the world. When bidders' distribution functions are strictly convex and the differences among the bidders' participation costs are sufficiently small, there is a neg-monotonic equilibrium. There is no neg-monotonic equilibrium when the difference is sufficiently large. In other words, if the difference in participation costs is sufficiently large, we do not need to consider the existence of neg-monotonic equilibria.

Our study on auction with differentiated participation costs is not only more realistic, but also provides us deeper insight that would help us understand the existence or non-existence of asymmetric equilibria well in auctions with equal participation cost. This can be seen by studying the limit behavior of the monotonic and neg-monotonic equilibria when bidders' participation costs converge to the same value. We show that, when the distribution function of valuation is concave, the monotonic equilibrium converges to the symmetric equilibrium when bidders have the same participation costs. However, when the distribution is strictly convex, the monotonic equilibrium converges to the asymmetric equilibrium. In this case one neg-monotonic equilibrium converges to a symmetric equilibrium, and another neg-monotonic equi-

librium converges to an asymmetric equilibrium.

We also provide some comparative static analysis. It is shown that the cutoff point is increasing in one's own participation costs, but decreasing in opponents' participation costs, and further, as the number of bidders increases, the cutoff points of all bidders will increase.

Chapter III studies equilibria of second price auctions with general distribution functions on valuations and participation costs. The special cases of this general specification includes that either the valuations or participation costs are common knowledge, as those have been investigated in previous literature.

Under a general two-dimensional distribution of the bidders' participation costs and valuations we prove that the equilibria always exist. When bidders have the same distributions, there exists a unique symmetric equilibrium. Moreover, we identify the conditions under which we have a unique equilibrium in a simple two bidder economy. Special cases in which multiple equilibria exist are also discussed. There may exist an equilibrium in which one bidder never participates or an equilibrium in which one bidder always participates.

As compared to the work by Laffont and Green [12], our general framework can not only establish the existence of equilibrium and uniqueness of symmetric equilibrium in the two-dimensional uniform setting, but can also do that in many other two-dimensional settings such as truncated normal distributions, exponential distributions etc. Not restricted to the symmetric equilibrium when all bidders are homogenous, our framework can deal with the asymmetric equilibria which have been seen in literature with one-dimensional private information, like those in Tan and Yilankaya [40].

In Chapter IV, we investigate Bayesian-Nash equilibria of sealed-bid first price auctions in the independent private values environment with participation costs. We

assume bidders know their valuations and participation costs before they make their decisions. Participation costs are assumed to be the same across all the bidders.

When bidders are homogenous, there is a unique symmetric equilibrium. We show that there is no other equilibrium when valuation distribution functions are concave. However, when valuation distribution functions are elastic at the symmetric equilibrium, there always exists an asymmetric equilibrium. It may be remarked that, when a distribution function is strictly convex, it is elastic everywhere, specifically at the symmetric equilibrium, and therefore there exists an asymmetric equilibrium. Moreover, when bidders are in two different groups, the cutoffs used by one group can always be different from those used by the other group.

We also consider the existence of equilibria in an economy with heterogenous bidders in the sense that the distribution functions are different. Specifically, we consider the case where one distribution (called a weak bidder) is first order dominated by another (called a strong bidder). We concentrate on equilibria that the bidders in the same group use the same threshold. We show that there is always an equilibrium in which the strong bidders are more likely to enter the auction by using a smaller cutoff point for valuations. When the distribution functions are concave, the equilibrium is unique. However, when the distribution functions for the weak bidders are strictly convex, and the participation costs are sufficiently large, there exists an equilibrium in which weak bidders are more likely to enter the auction.

In all chapters, the existence of multiple equilibria has important consequences for the strategic behavior of bidders and the efficiency of the auction mechanism. When an auction has a participation cost, a bidder would expect less bidders to submit their bids. When the equilibrium is unique, every bidder has to follow the that equilibrium and has no other choices. However, when multiple equilibria exist, bidders may choose an equilibrium that is more desirable. In this case, some bidders may form a

collusion to cooperate at the entrance stage by choosing a smaller cutoff point that may decrease the probability that other bidders enter the auction, and consequently, may reduce the competition in the bidding stage. An asymmetric equilibrium may become more desirable when an auction can run repeatedly. Tan and Yilankaya [41] investigate the ratifiability of efficient collusive mechanisms in second-price auctions with participation costs. Beside that, an asymmetric equilibrium may be ex-post inefficient. The item being auctioned is not necessarily allocated to the bidder with the highest valuation.

CHAPTER II

SECOND PRICE AUCTIONS WITH DIFFERENTIATED PARTICIPATION COSTS

This chapter studies equilibria of second price auctions with differentiated participation costs. We consider equilibria in independent private values environments where bidders' entry costs are common knowledge while valuations are private information. We identify two types of equilibria: monotonic equilibria in which a higher participation cost results in a higher cutoff point for submitting a bid, and neg-monotonic equilibria in which a higher participation cost results in a lower cutoff point. We show that there always exists a monotonic equilibrium, and further, that the equilibrium is unique for concave distribution functions and strictly convex distribution functions with some additional conditions. There exists a neg-monotonic equilibrium when the distribution function is strictly convex and the difference of the participation costs is sufficiently small. We also provide comparative static analysis and study the limit status of equilibria when the difference in bidders' participation costs approaches zero.

A. Introduction

The study of participation costs in auctions mainly focuses on the second price auction due to its simplicity of bidding behavior. In standard second price auctions, bidding one's own valuation is a weakly dominant strategy. There is also another equilibrium in standard second price auctions as shown in Blume and Heidhues [1]: the bidder with the highest value bids his true value and all others bid zero. This is referred to as the asymmetric bidding equilibrium in the standard second price auction. However, in second price auctions with participation costs, it is still true that if a bidder finds participating optimal, he cannot do better than bidding his true value. Therefore, in

this chapter we only consider equilibria in which potential bidders use cutoff strategies; i.e., bid their true values if they are greater than the corresponding cutoff points, do not participate otherwise.[1] All of our results about the uniqueness or multiplicity of the equilibria should be interpreted accordingly.

Green and Laffont [12] were the first to study the second price auction with participation costs in a general framework where bidders' valuations and participation costs are both private information. However, their proof of the existence and uniqueness of an equilibrium is incomplete additionally having imposed a restrictive assumption of uniform distributions for both values and participation costs. There has been some recent work in the literature on equilibria of the second price auction with participation costs in simplified versions where either only valuations or participation costs are private, while the other is assumed to be common knowledge.

Campbell [5] considered the equilibria in second price auctions in an economic environment with equal participation costs when bidders' values are private information and participation costs are common knowledge. He focused on the coordination of equilibrium choice when multiple equilibria exist. Tan and Yilankaya [40] also studied equilibria of second price auctions in an economic environment with equal participation costs. They proved that the equilibrium is unique and symmetric when bidders' distribution functions for values are concave. They also considered the case in which bidders are asymmetric in the sense that they have different valuation distribution functions while maintaining identical participation costs. SSome others, such as Samuelson [37], McAfee and McMillan [30,31], Levin and Smith [21], Stageman [38], and Menezes and Monterio [32] also studies auctions with participation costs. All of these studies assume bidders' participation cost are the same.

[1]There may exist an equilibrium in which bidders do not bid their true value when they participate. See the example given in Remark 4 below.

The assumption of equal participation costs, however, is stringent and unrealistic in many situations. For instance, bidders may have different transportation costs for traveling to auction spots. Bidders may also have different ability to learn more information about the auctions. Some bidders can easily know valuations of other potential bidders while others do not. Thus, it may be more general for one to study individuals' behavior in second price auctions with different participation costs which may include the equal participation costs as a special case when the difference of participation costs approach zero.

Differentiated participation costs may also have additional implications. First, they can be used to distinguish bidders. Bidders can use this information to decide whether or not to participate in the auction. One can analyze how a bidder's cutoff point will be affected by others' participation costs. Secondly, while Tan and Yilankaya [40] mainly considered bidding behavior inside the same group, they did not consider how the interaction among the different groups would determine and affect the equilibrium behavior.

Kaplan and Sela [16] studied equilibria of the second price auction with participation costs when bidders' participation costs are private information, while valuations are common knowledge. They considered the existence of type-symmetric equilibria.

This chapter aims to investigate the equilibria when bidders have differentiated participation costs, an analysis which will be more applicable in reality. By considering bidders with different participation costs, we can investigate how equilibria vary as participation costs change. We can also study the limit behavior of the equilibria as the difference in participation costs approaches zero.

In this chapter, we identify two types of equilibria: monotonic equilibria in which a higher participation cost results in a higher cutoff point for entering the auction and submitting a bid, and neg-monotonic equilibria in which a higher participation

cost results in a lower cutoff point for entering the auction and submitting a bid. We show that there always exists a monotonic equilibrium, and further that, it is unique for concave distribution functions and strictly convex distribution functions under some additional conditions. Uniqueness of the equilibria can greatly simplify the world. When bidders' distribution functions are strictly convex and the differences among the bidders' participation costs are sufficiently small, there is a neg-monotonic equilibrium. There is no neg-monotonic equilibrium when the difference is sufficiently large. In other words, if the difference in participation costs is sufficiently large, we do not need to consider the existence of neg-monotonic equilibria.

Our study on auction with differentiated participation costs is not only more realistic, but also provides us deeper insight that would help us understand the existence or non-existence of asymmetric equilibria well in auctions with equal participation cost. This can be seen by studying the limit behavior of the monotonic and neg-monotonic equilibria when bidders' participation costs converge to the same value. We show that, when the distribution function of valuation is concave, the monotonic equilibrium converges to the symmetric equilibrium when bidders have the same participation costs. However, when the distribution is strictly convex, the monotonic equilibrium converges to the asymmetric equilibrium. In this case one neg-monotonic equilibrium converges to a symmetric equilibrium, and another neg-monotonic equilibrium converges to an asymmetric equilibrium.

We also provide some comparative static analysis. It is shown that the cutoff point is increasing in one's own participation costs, but decreasing in opponents' participation costs, and further, as the number of bidders increases, the cutoff points of all bidders will increase.

The organization of the chapter is as follows: In Section B, we describe a general setting of economic environments. In Section C, we focus on two bidders with the same

distribution functions and different participation costs to investigate the existence, uniqueness, and limit properties of the equilibria and to make a comparative analysis. In Section D, we extend our basic results to more general economic environments by relaxing the assumptions made in Section C. Concluding remarks are provided in Section E. All the proofs are presented in Section F.

B. The Setup

We consider the independent values economic environment with one seller and $n \geq 2$ potential buyers (bidders). The seller is risk-neutral and has an indivisible object to sell to one of the buyers. The seller values the object as 0. The auction format is the sealed-bid second price auction format (see Vickrey [44]). However, in order to submit a bid, bidder i must pay a participation cost c_i. Buyer i's valuation for the object is v_i, which is private information to the other bidders. It is assumed that v_i is independently distributed with a cumulative distribution function $F_i(v)$ that has continuously differentiable density $f_i(v) > 0$ everywhere with support $[0, 1]$[2]. We will study the equilibrium behavior mainly for the case where bidders have the same distribution functions and then consider the general case of the different distribution functions. The participation costs $c_i \in (0, 1]$ for all i are common knowledge.

Each bidder knows his value, participation cost, and the distributions of the others' valuations. If participating in auction, he is required to pay a non-refundable participation fee. The bidder with the highest bid wins the object and pays the second-highest bid. If there is only one person in the auction, he wins the object and pays 0. If the highest bids are equal for more than one bidder, then he pays his own bid and gains nothing.

[2]Here "0" denotes the value is zero while "1" is a normalization of the highest possible valuation among all bidders.

In this second price auction mechanism with participation costs, the individually rational action set for any type of bidder is $\{No\} \cup [0,1]$, where "$\{No\}$" denotes not participating in the auction. Bidder i incurs the participation cost if and only if his action is different from "$\{No\}$". Let $b_i(v_i, c)$ denote bidder i's strategy where $c = (c_1, ..., c_n)$.

If a bidder finds participating in this second price auction optimal, he cannot do better than bidding his true valuation (i.e., bidding his true valuation is a weakly dominant strategy). Therefore, we can restrict our attention to Bayesian-Nash equilibria in which each bidder uses a cutoff strategy denoted by $v_i^*(c)$, i.e., he bids his valuation if it is greater than or equal to the cutoff point[3] and does not enter otherwise. An equilibrium strategy of each bidder i is then determined by the cutoff point for his valuation, which is the minimum valuation bidder i needs to cover the cost. Thus the bidding decision function of each bidder is characterized by

$$b_i(v_i, c) = \begin{cases} v_i & \text{if } v_i^*(c) \leq v_i \leq 1 \\ \text{No} & \text{otherwise.} \end{cases}$$

For notational convenience, we simply denote $v_i^*(c) = v_i^*$.

Remark 1 When $v_i^* \leq 1$, bidder i will participate in the auction whenever his true value satisfies $v_i^* \leq v_i \leq 1$. However, when bidder i's expected revenue is always less than his participation cost c_i for any $v_i \in [0,1]$, he will never participate in the auction. In this case, his equilibrium strategy (action) is "$\{No\}$". For notional convenience, and also for simplicity of discussion, we use $v_i^* > 1$ to denote the equilibrium strategy of "$\{No\}$". Thus allows us to use a unified notation v_i^* to denote an equilibrium strategy of bidder i, including the equilibrium of "$\{No\}$". The rationale behind using

[3]In Milgrom and Weber [33], the term of "screening level" is used instead of using "cutoff point."

$v_i^* > 1$ to denote the equilibrium strategy of "$\{No\}$" is the following: If we find a value v_i^* such that bidder i's expected revenue is equal to his participation cost c_i by allowing the upper bound of the support to be greater than one, we will end up with a value v_i^* that is greater than one. But the true value is actually less than or equal to one, and thus $v_i^* > 1$ is equivalent to the equilibrium strategy of "$\{No\}$."

From now on we focus exclusively on cutoff points, since they are sufficient to describe equilibria. We define them with following formal definition:

Definition 1 For the economic environment under consideration, an equilibrium is a cutoff point vector $(v_1^*, v_2^*, ..., v_n^*) \in \mathbb{R}_+^n$ such that each bidder i action's is optimal, given others' cutoff strategies.

We then immediately have the following result:

Lemma 1 $v_i^* \leq 1$ for at least some i.

Since bidders with higher participation costs are less likely to participate in the auction, one may come to the intuition conclusion that bidders with higher participation costs may have higher cutoff points to participate in the auction. One may also perceive that bidders with the same participation costs will use the same cutoff point when their distribution functions are the same. However, as we will show in the chapter, it is possible that a bidder with a higher participation cost may actually have a lower cutoff point to enter the auction. To study these possibilities, we may distinguish two types of equilibria: monotonic equilibria and neg-monotonic equilibria.

Definition 2 An equilibrium $(v_1^*, v_2^*, \ldots, v_n^*) \in \mathbb{R}_+^n$ for the economic environment under consideration is called a *monotonic equilibrium* (*resp. neg-monotonic equilibrium*) if, for any two bidders i and j, $c_i < c_j$ implies $v_i^* < v_j^*$ (resp. $v_i^* \geq v_j^*$).

As usual, when bidders' distribution functions are the same; i.e., $F_1(\cdot) = F_2(\cdot) = \ldots = F_n(\cdot) = F(\cdot)$, we can define the usual symmetric and asymmetric equilibria.

Definition 3 An equilibrium $(v_1^*, v_2^*, \ldots, v_n^*) \in \mathbb{R}_+^n$ is called a *symmetric equilibrium* (*resp. asymmetric equilibrium*) if, for any two bidders i and j, $c_i = c_j$ implies $v_i^* = v_j^*$ (resp. $v_i^* \neq v_j^*$).

Remark 2 Campbell [5] and Tan and Yilankaya [40] studied the existence of symmetric and asymmetric equilibria for the second price auctions with the same participation costs. The terminology of "monotonic" used here means that two variables c and v^* vary in the same direction: a higher participation cost results in a higher cutoff point. When bidders' distribution functions are the same, as one will see in Section D, $v_1^* = v_2^*$ cannot be an equilibrium, provided bidders' participation costs are different. Thus, $c_i < c_j$ implies $v_i^* > v_j^*$ for every neg-monotonic equilibrium, and $c_i < c_j$ implies $v_i^* < v_j^*$ for every monotonic equilibrium. However, when bidders' distribution functions are different, as we will show below, $v_1^* = v_2^*$ may be an equilibrium although bidders' participation costs are different. That is, we have a special neg-monotonic equilibrium with $v_i^* = v_j^*$ even when $c_i < c_j$.

Example 1 We give an simple example to understand the notion of monotonic and neg-monotonic equilibria. Suppose there is one object for sale to two bidders. Both bidders value it at 1. The participation costs are $c_1 < c_2 < 1$. If both bidders enter, they both have negative payoffs. There are two pure strategy equilibria: (bidder 1 enters, bidder 2 stays out) and (bidder 1 stays out, bidder 2 enters), which correspond to two equilibrium cutoff points $(v_1^* = c_1, v_2^* > 1)$ and $(v_1^* > 1, v_2^* = c_2)$. That is, the former one is monotonic and the latter one is neg-monotonic.

Remark 3 When multiple equilibria exist, bidders may also use mixed strategies.

For simplicity, in this chapter we focus only on the pure strategy, not the mixed strategy of using different cutoff points.

C. Two Bidders with Different Participation Costs

In this section we consider an economy with two bidders who have different participation costs c_1 and c_2 with $c_1 < c_2$, and have the same distribution function $F(v)$ on $[0, 1]$, where the costs are common knowledge and valuations are private information.

We first assume, provisionally, that a monotonic equilibrium (v_1^*, v_2^*) exists, i.e., $v_1^* < v_2^*$. By Lemma 1, we must have $v_1^* \leq 1$. When bidder 1's valuation is $v_1 = v_1^*$, his expected revenue is given by $v_1^* F(v_2^*) + 0(1 - F(v_2^*))$, where $F(v_2^*)$ is the probability bidder 2 will not participate in the auction. Indeed, when he participates in the auction and bidder 2 does not participate in the auction, his value is v_1^*. When bidder 2 participates in the auction, it must be the case that $v_2 \geq v_2^*$. Then bidder 1 cannot get the object since $v_2 \geq v_2^* > v_1^* = v_1$, and thus his revenue is zero. Therefore, his expected revenue from the auction is $v_1^* F(v_2^*)$. The zero net-payoff (equilibrium) condition then requires that

$$c_1 = v_1^* F(v_2^*). \tag{2.1}$$

When bidder 2's participation cost is too large, he may never participate in the auction, no matter what his valuation is. In this case, bidder 1 uses $v_1^* = c_1$ as his cutoff point, and bidder 2's expected payoff must satisfy

$$F(c_1) + \int_{c_1}^{1} (1 - v) dF(v) = c_1 F(c_1) + \int_{c_1}^{1} F(v) dv < c_2;$$

i.e., the expected revenue he obtains from participating even when his value is 1 is less than his participation cost, given bidder 1 uses c_1 as the cutoff point. Thus we have $v_2^* > 1$. Then, we may have a monotonic equilibrium with $v_1^* = c_1$ and $v_2^* > 1$.

Now suppose $v_2^* \leq 1$. Then, when bidder 2's valuation is $v_2 = v_2^*$, his expected revenue is

$$v_2^* F(v_1^*) + \int_{v_1^*}^{v_2^*} (v_2^* - v) dF(v),$$

where the first part is the expected revenue when bidder 1 does not enter the auction, and the second part is the expected revenue when both bidders participate in the auction. Note that bidder 2 will lose the object if $v_1 > v_2^*$. The zero expected net-payoff (equilibrium) condition then requires that

$$v_2^* F(v_1^*) + \int_{v_1^*}^{v_2^*} (v_2^* - v) dF(v) = c_2. \tag{2.2}$$

Integrating by parts in the left side of (2.2), we have

$$v_1^* F(v_1^*) + \int_{v_1^*}^{v_2^*} F(v) dv = c_2. \tag{2.3}$$

Note that, from (4.2) and (2.3), one can see the claim in Remark 2 is true: It is impossible for both bidders to use the same cutoff point $v_1^* = v_2^* = v^*$ when their participation costs are different. Indeed, suppose not. Then we must have $c_1 = v^* F(v^*)$ by (4.2) and $c_2 = v^* F(v^*)$ by (2.2). Thus $c_1 = c_2$, which contradicts the fact that $c_2 > c_1$.

Before we proceed to investigate the existence and uniqueness of the monotonic equilibrium, it is necessary to introduce more notation. Campbell [5] and Tan and Yilankaya [40] showed the existence and uniqueness of the symmetric cutoff point $v_i^* = v^s$ when bidders have the same participation cost. In our model, if both bidders have the same participation cost c_1, we have $v_1^s F(v_1^s) = c_1$, and then we can find the symmetric equilibrium cutoff point v_1^s. Now, if both bidders have the participation cost c_2, we can find the symmetric equilibrium cutoff point v_2^s by solving $v_2^s F(v_2^s) = c_2$. Such $v_1^s \leq 1$ and $v_2^s \leq 1$ do exist and are unique since the defined function $m(v) =$

$vF(v)$ is monotonically increasing, $m(0) = 0$, and $m(1) = 1$.

The following lemma shows the relationship between a monotonic equilibrium and symmetric equilibria.

Lemma 2 *Suppose (v_1^*, v_2^*) is a monotonic equilibrium, (v_1^s, v_1^s) and (v_2^s, v_2^s) are symmetric equilibria associated with participation costs $c_1 < c_2$, respectively. Then, we have $v_1^* < v_1^s < v_2^s < v_2^*$.*

This lemma shows that, when bidders have different participation costs, the cutoff point for the bidder with the lower participation cost at the monotonic equilibrium is lower than the cutoff point at the symmetric equilibrium when bidders have the same lower participation cost c_1.

To find a monotonic equilibrium, we define the following two cutoff reaction function equations.

$$xF(y) = c_1 \tag{2.4}$$

$$xF(x) + \int_x^y F(v)dv = c_2 \tag{2.5}$$

with $x < y$, where x corresponds to v_1^*, and y corresponds v_2^*. It can be easily seen that we have $x \geq c_1$ and $y \geq c_2$. They can be regarded as cutoff reaction functions because (2.4) shows how bidder 1 will choose a cutoff point x, given bidder 2's action y. Equation (2.5) shows how bidder 2 will choose a cutoff point y, given bidder 1's a action x. A monotonic equilibrium $(v_1^*, v_2^*) \in [0, 1] \times [0, 1]$ is obtained when x and y satisfy these two equations simultaneously.

From (2.4), we have $x = x(y) = \frac{c_1}{F(y)}$. Then $\frac{dx}{dy} = -\frac{c_1 f(y)}{F^2(y)} < 0$. This implicitly defines y as a decreasing function of x, denoted by $y = y(x)$. We now substitute $y = y(x)$ into the left side of (2.5) and let

$$h(x) = xF(x) + \int_x^{y(x)} F(v)dv - c_2.$$

Substitute $x = x(y)$ into the left side of (2.5) and let

$$\lambda(y) = \frac{c_1}{F(y)} F(\frac{c_1}{F(y)}) + \int_{\frac{c_1}{F(y)}}^{y} F(v)dv.$$

Then $\lambda'(y) = F(y) - \frac{c_1}{F(y)} \frac{c_1}{F^2(y)} f(y) f(\frac{c_1}{F(y)})$. Since $x = \frac{c_1}{F(y)}$, by substitution, we have

$$\lambda'(y) = F(y) - \frac{x^2}{F(y)} f(y) f(x).$$

To consider the existence of neg-monotonic equilibria in which the cutoff points satisfy $v_2^* < v_1^*$ whenever $c_1 < c_2$, we can follow the above process similarly. Also by Lemma 1, we have $v_2^* \leq 1$.

For bidder 2, when $v_2 = v_2^*$, his expected revenue is given by $v_2^* F(v_1^*)$ and the zero profit condition requires that

$$c_2 = v_2^* F(v_1^*). \tag{2.6}$$

For bidder 1, it is possible that $v_1^* > 1$, i.e., bidder 1 will never participate. Again, this requires that

$$F(c_2) + \int_{c_2}^{1} (1 - v) dF(v) = c_2 F(c_2) + \int_{c_2}^{1} F(v)dv < c_1.$$

In this case, we have a neg-monotonic equilibrium with $v_1^* > 1$ and $v_2^* = c_2$.

Now suppose the above inequality cannot be true. Then bidder 1 chooses a cutoff point $v_1^* \in [0, 1]$. When his valuation is $v_1 = v_1^* \leq 1$, he participates in the auction and receives a zero net-payoff so that

$$v_1^* F(v_2^*) + \int_{v_2^*}^{v_1^*} (v_1^* - v) dF(v) - c_1 = 0.$$

Integrating by parts, we get

$$c_1 = v_2^* F(v_2^*) + \int_{v_2^*}^{v_1^*} F(v)dv. \tag{2.7}$$

Since the distribution function $F(v)$ is non-decreasing, we have

$$c_1 > v_1^* F(v_2^*). \tag{2.8}$$

In order for (2.6), (2.8), and $c_2 > c_1$ to be consistent, it requires that

$$v_2^* F(v_1^*) > v_1^* F(v_2^*)$$

or

$$\frac{F(v_1^*)}{v_1^*} > \frac{F(v_2^*)}{v_2^*}. \tag{2.9}$$

To find neg-monotonic equilibrium, through (2.6) and (2.7), we define the two cutoff reaction functions

$$\begin{aligned} y(x) &= c_2/F(x) \\ \phi(x) &= \frac{c_2}{F(x)} F(\frac{c_2}{F(x)}) + \int_{\frac{c_2}{F(x)}}^{x} F(v)dv. \end{aligned}$$

Again, we use x to correspond to v_1^* and y to correspond to v_2^*. Note that we have $x \geq y \geq c_2$.

From Campbell [5] and Tan and Yilankaya [40], we know that when two bidders have the same participation cost c_2 and $F(v)$ is strictly convex, there exists a unique symmetric equilibrium $x = y = v_2^s$ that satisfies $y = x = c_2/F(x)$ and an asymmetric equilibrium (x_0, y_0) with $x_0 > v_2^s$ and $y_0 < v_2^s$, indicating that $\phi(x)$ intersects with c_2 when $x = v_2^s$ and $x = x_0$. Also, by the uniqueness of symmetric equilibrium, $v_1^* \geq v_2^s$ if it exists. Let c_m be the minimum of $\phi(x) = \frac{c_2}{F(x)} F(\frac{c_2}{F(x)}) + \int_{\frac{c_2}{F(x)}}^{x} F(v)dv$ in the interval $[v_2^s, 1]$.

We then have the following proposition on the existence and uniqueness of equilibria:

Proposition 1 (Existence and Uniqueness Theorem) *For the independent pri-*

vate values economic environment with two bidders who have different participation costs $c_2 > c_1$, we have the following conclusions:

(1) There always exists a monotonic equilibrium.

(2) Suppose $F(.)$ is concave. Then the equilibrium is unique and monotonic.

(3) Suppose $F(.)$ is strictly convex. Then

(3.i) the monotonic equilibrium is unique when $\frac{f(v)}{F(v)^2}$ is non-increasing,

(3.ii) the neg-monotonic equilibrium is unique when $c_1 = c_m$,

(3.iii) there is no neg-monotonic equilibrium when $c_1 < c_m$, and

(3.iv) there are at least two neg-monotonic equilibria when $c_m < c_1 < c_2$.

The formal proof can be found in Section F of this chapter. Here we provide some intuition as to why the results are true. To investigate the existence and uniqueness of the equilibria, we first note the extreme case where there may be an equilibrium in which one bidder will never participate in the auction. We then exam how functions $\lambda(y)$ and $\phi(x)$ intersect with c_2 and c_1, respectively. The existence of a monotonic equilibrium can be established by the intermediate value theorem. The uniqueness of the monotonic (neg-monotonic) equilibrium comes from the fact that $\lambda(y)$ and $\phi(x)$ intersect with c_2 and c_1, respectively, at most once on the interval $y \in [v_1^s, 1]$ and $[v_2^s, 1]$. When $F(.)$ is concave, $\lambda(y)$ is a monotonic increasing function, and thus the monotonic equilibrium is unique. When $F(.)$ is strictly convex, we can also show the unique monotonic equilibrium and the existence and uniqueness of neg-monotonic equilibria for some types of convex distribution functions.

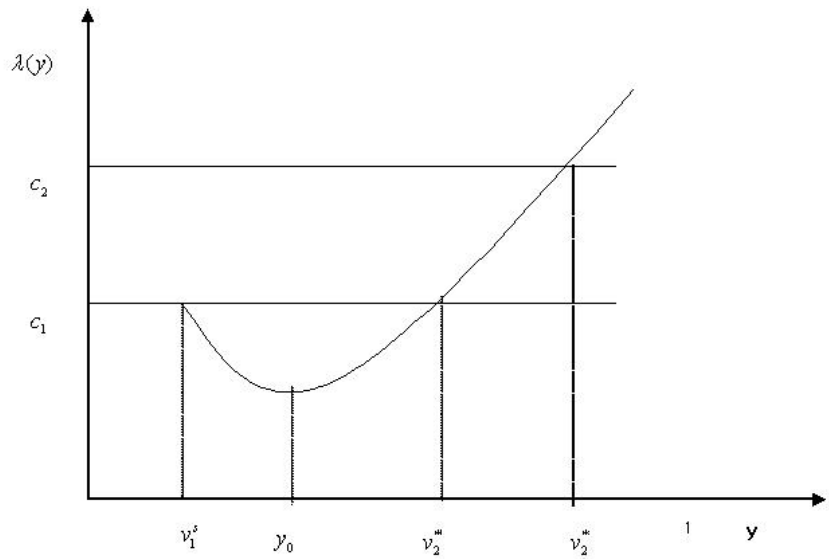

Fig. 1. Uniqueness for Convex Case

Remark 4 There are some facts that may be mentioned for understanding the contents and proof of Proposition 8:

1. For any strictly convex power functions and exponential functions, $\frac{f(v)}{F(v)^2}$ is a non-increasing function of v. Thus, the set of such strictly convex functions is not empty. A figure can be used for understanding why there is a unique monotonic equilibrium for this type of strictly convex distribution. In Figure 1, $\lambda(y)$ starts from v_1^s with negative slope. When $\lambda'(y) = 0$ has at most one solution, $\lambda(y)$ intersects with c_2 at most one time, indicating that the monotonic equilibrium is unique.
2. From the proof in the Section F, one can see that it is always true that $c_2 > c_m$. Then, as long as $c_2 - c_1$ is sufficiently small, we have $c_2 > c_1 > c_m$. Thus, we can conclude that when $c_2 - c_1$ is sufficiently small, there are two neg-monotonic equilibria that are given by (x_1, y_1) and (x_2, y_2)

with $y_1 = y(x_1)$, $y_2 = y(x_2)$, and $y_1 < y_2 < v_2^s < x_1 < x_m < x_2 < x_0$. Thus, when $F(.)$ is not concave, the existence of a neg-monotonic equilibrium depends on the difference of participation costs, $c_2 - c_1$. For instance, when $c_1 = 0.3$, $c_2 = 0.32$, and $F(v) = \frac{v+v^3}{2}$ (which is strictly convex), we have one monotonic equilibrium $(0.3753, 0.8911)$ and two neg-monotonic equilibria $(0.6995, 0.6142)$ and $(0.8301, 0.4564)$. However, when $c_1 = 0.3$, $c_2 = 0.4$, and $F(v) = \frac{v+v^3}{2}$, we only have one monotonic equilibrium $(0.3003, 0.9994)$. Thus, this example demonstrates that there are multiple neg-monotonic equilibria when $c_2 - c_1$ is sufficiently small, and there is no neg-monotonic equilibrium when $c_2 - c_1$ is large enough.

3. Figure 2 can help us to understand the proof in Section F and the points mentioned above. $\phi(x)$ starts from $y = v_2^s$ with negative slope. When $c_2 - c_1$ is small enough, it intersects with c_1; i.e., a neg-monotonic equilibrium exists. When $c_2 - c_1$ is big enough so that $c_1 < c_m$, $\phi(x)$ and c_1 can not intersect; i.e., no neg-monotonic equilibrium exists. From the figure, when c_1 is close to c_2, there are at least two intersection points for $y = \phi(x)$ and $y = c_1$, which means there are at least two neg-monotonic equilibria, say, (x_1, y_1) and (x_2, y_2).
4. Campbell [5] and Tan and Yilankaya [40] showed that there exists an asymmetric equilibrium when distribution functions are strictly convex. However, our result shows that the strict convexity of the distribution function alone is not a sufficient condition for the existence of a neg-monotonic equilibrium, unless the difference $c_2 - c_1$ is small enough. In fact, this result implies that one can refine equilibria and always eliminate non-equilibria by making participation costs for bid-

ders sufficiently different when necessary.

5. In the proof of Proposition 8, the condition that $F(.)$ is concave can be weakened to $F(v) \geq vf(v)$ for all $v \in [c_1, 1]$, and the condition that $F(.)$ is strictly convex can be weakened to $F(v) < vf(v)$ for all $v \in [c_2, 1]$.
6. A non-truth-telling equilibrium may exist when bidders do not use weakly dominant bidding strategies even if they participate. For example, suppose bidder 1 bids zero when he enters and bidder 2 bids 1 when he enters. For bidder 1, he only wins when bidder 2 does not enter, hence in equilibrium $v_1^* F(v_2^*) = c_1$. Now for bidder 2, he always wins once he enters and pays nothing. At equilibrium we have $v_2^* = c_2$. Thus $v_1^* = \frac{c_1}{F(c_2)}$. So if bidders do not use dominant bidding strategy when they enter, we may have other cutoff equilibria.

The intuition for the existence of neg-monotonic equilibria when $F(.)$ is strictly convex and $c_2 - c_1$ is sufficiently small is: when bidder 2 uses a smaller cutoff v_2^* to enter the auction, the expected payoff for bidder 1 with a lower value to enter the auction is small even when he wins the auction. This is true because bidder 1's expected payment to the seller, which is equal to the expected value of bidder 2's valuation, is high as $F(.)$ is strictly convex. In this case, bidder 1 would stay out of the auction by using a larger cutoff point, and thus we have a neg-monotonic equilibrium. This can only happen when $c_2 - c_1$ is sufficiently small which makes none of two bidders has obviously advantage over another. When $c_2 - c_1$ is big enough, bidder 2 will be in a disadvantage as compared to bidder 1, and thus bidder 2 has to use a bigger cutoff than bidder 1. When $F(.)$ is concave, the above argument cannot be applied. Now the expected payment of bidder 1 to the seller when he wins

is small since bidders tend to have small valuations. Bidder 1 with lower value may also benefit from participating in the auction which can prevents bidder 2 enters the auction with a smaller cutoff value.

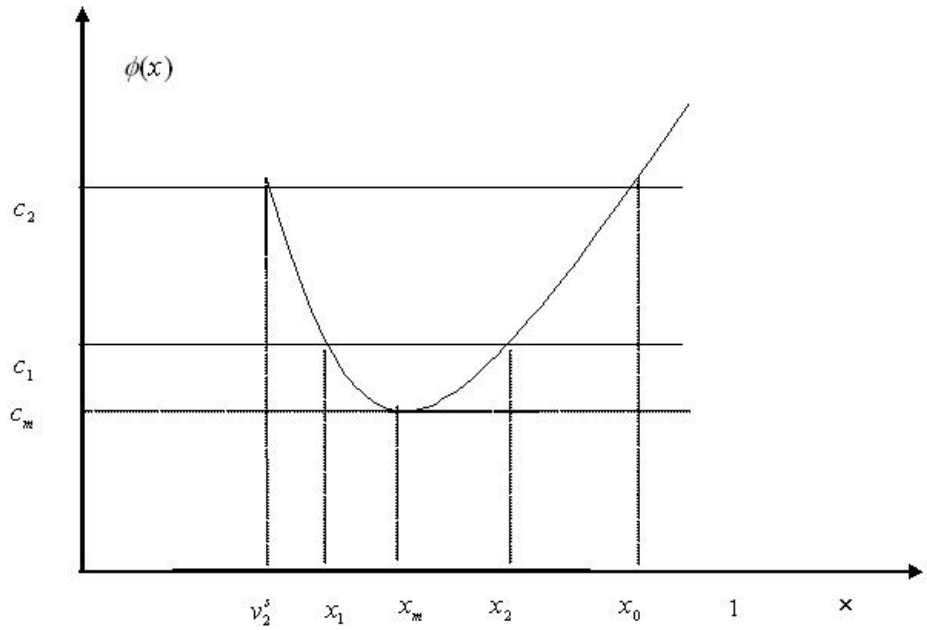

Fig. 2. Existence of Counter-Monotonic Equilibria for Convex Case

One may wonder what would happen at the limits of monotonic and neg-monotonic equilibria as $c_2 - c_1 \to 0$. Should a monotonic equilibrium converge to a symmetric equilibrium or a neg-monotonic equilibrium converge to an asymmetric equilibrium when $c_2 \to c_1$?

For instance, suppose c_1 is constant at, 0.30, and let c_2 decrease from some point until $c_2 = c_1 = 0.3$. Will there be any convergence behavior for monotonic and neg-monotonic equilibria in this case? Do they converge to a symmetric equilibrium or an asymmetric equilibrium (if it exists) for a given distribution function? Some numerical experiments are given in Table I.

Table I. Sequences of Monotonic and Neg-Monotonic Equilibria

c_1	c_2	$F(v) = \sqrt{v}$	$F(v) = v^2$		
0.3	0.3	(0.4481, 0.4481)	(0.6694, 0.6694)	(0.3425, 0.9358)	(0.9358, 0.3425)
0.3	0.31	(0.4387, 0.4675)	(0.6845, 0.6616)	(0.3327, 0.9426)	(0.9318, 0.3570)
0.3	0.32	(0.4303, 0.4861)	(0.7000, 0.6530)	(0.3237, 0.9627)	(0.9271, 0.3723)
0.3	0.33	(0.4226, 0.5038)	(0.7162, 0.6434)	(0.3155, 0.9751)	(0.9216, 0.3886)
0.3	0.34	(0.4156, 0.5210)	(0.7332, 0.6323)	(0.3079, 0.9870)	(0.9150, 0.4061)
0.3	0.35	(0.4091, 0.5376)	(0.7517, 0.6193)	(0.3009, 0.9985)	(0.9068, 0.4256)
0.3	0.36	(0.4032, 0.5537)	(0.7725, 0.6038)	(0.3000, 1.0000)	(0.8963, 0.4418)
0.3	0.37	(0.3976, 0.5694)	(0.7984, 0.5804)	(0.3000, 1.0000)	(0.8805, 0.4773)
0.3	0.38	(0.3923, 0.5847)	NA	(0.3000, 1.0000)	NA

From the table, when $F(v) = \sqrt{v}$, which is concave , we only have the monotonic equilibrium and is unique. Tan and Yilanyaka [40] proved that when $F(.)$ is concave there is only one unique symmetric equilibrium and no asymmetric equilibrium. Then is natural that, when c_2 converges to c_1, the unique monotonic equilibrium will converge to the unique symmetric equilibrium, as can be seen from Table 1.

However, when $F(v) = v^2$, which is a strictly convex distribution function, we can see from the table that when $c_2 - c_1$ is small enough, there exist one monotonic and two neg-monotonic equilibria, but when $c_2 - c_1$ is big enough, there is only one monotonic equilibrium. We can also see from the table, somewhat surprisingly, that unlike the monotonic equilibrium, one sequence of neg-monotonic equilibria converges to the symmetric equilibrium, while the other sequence of monotonic equilibria converges to the asymmetric equilibrium. Thus, the notion of monotonic/neg-monotonic equilibrium is not a trivial generalization of symmetric/asymmetric equilibria.

Actually, these limit relationships among monotonic/neg-monotonic equilibria

and symmetric/asymmetric equilibria are true for general concave and strictly convex functions.

Proposition 2 (Limit Theorem) *For the independent private values economic environment with two bidders with participation costs $c_2 > c_1$, we have the following conclusions:*

(1) Suppose $F(.)$ is concave. The unique monotonic equilibrium (no neg-monotonic equilibrium) converges to the unique symmetric equilibrium as $c_2 - c_1 \to 0$.

(2) Suppose $F(.)$ is strictly convex and $\frac{f(v)}{F(v)^2}$ is a non-increasing function of v. The unique monotonic equilibrium converges to an asymmetric equilibrium as $c_2 - c_1 \to 0$.

(3) Suppose $F(.)$ is strictly convex. When $c_2 - c_1 \to 0$, there are two neg-monotonic of which one converges to the unique symmetric equilibrium, and the other converges to an asymmetric equilibrium.

Some intuition can be given here for the convergence results of the equilibria. By the continuity of the reaction function, as the participation costs c_1 and c_2 converge, the set of equilibria will converge to the set of equilibria when $c_1 = c_2$. In particular, if we focus on the equilibrium in which bidder 1 uses the smallest cutoff point among all bidder 1's equilibrium cutoffs (which is necessarily a monotonic equilibrium), this will converge to the equilibrium for $c_1 = c_2$ in which bidder 1 uses the smallest cutoff among all of bidder 1's equilibrium cutoffs. Thus, if the equilibrium is unique when $c_1 = c_2$, and there is a unique monotonic equilibrium for all c_1 and c_2 in the sequence, that equilibrium sequence must converge to the symmetric equilibrium. However, if there are asymmetric equilibria when $c_1 = c_2$, then the equilibrium in which bidder 1 uses the smallest cutoff must converge to the asymmetric equilibrium in which bidder

1 uses the smaller cutoff. Hence, if the monotonic equilibrium is unique, then it will converge, and the equilibrium that converges to the symmetric equilibrium must be neg-monotonic.

From Figures 1 and 2, one can see that, as $c_2 - c_1 \to 0$, any monotonic/non monotonic equilibrium converges along the bidders' reaction curves determined by $\lambda(y)$ and ϕ to the nearest equilibrium, whether it is symmetric or asymmetric.

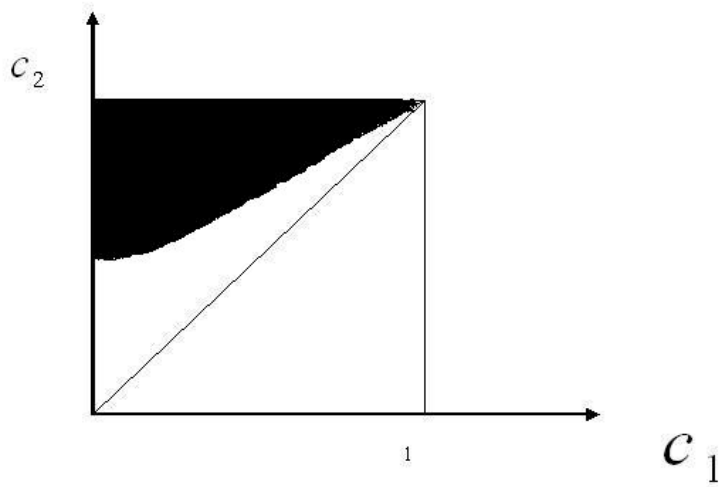

Fig. 3. Uniform Case

Before finishing this section, we examine the effects of changes in participation costs on equilibrium behavior.

Proposition 3 (Comparative Static Theorem) *For the independent private values economic environment with two bidders, suppose the values of bidders are drawn from a concave distribution function $F(.)$ and the participation costs c_1 and c_2 are common knowledge. Then an increase in participation cost c_i increases i's cutoff point v_i^* but decreases the opponent's cutoff point v_j^* for $j \neq i$.*

In fact, when $F(v)$ is uniform, we can derive the unique equilibrium explicitly, and analyze equilibrium behavior directly. The condition for $v_2^* > 1$ implies $c_2 > \frac{1}{2} + \frac{1}{2}c_1^2$. In Figure 3, above the parabola $c_2 = \frac{1}{2} + \frac{1}{2}c_1^2$ and inside the square (the shaded area) is the area where bidder 2 will never participate ($v_2^* > 1$) and bidder 1 uses $v_1^* = c_1$ as his cutoff point. In the area between $c_1 = c_2$ and the parabola, we have $c_1 < c_2 \leq \frac{1}{2} + \frac{1}{2}c_1^2$. In this case, there is a unique monotonic equilibrium with $v_1^* \leq 1$ and $v_2^* \leq 1$ that can be solved explicitly.

Using (4.2) and (2.3) under the uniform distribution, we have

$$(P1) \begin{cases} v_2^* > v_1^* \\ c_1 = v_1^* v_2^* \\ \frac{1}{2}(v_1^{*2} + v_2^{*2}) = c_2. \end{cases}$$

Solving these equations, we have

$$\begin{cases} v_1^* = \frac{1}{2}(\sqrt{2(c_1 + c_2)} - \sqrt{2(c_2 - c_1)}) \\ v_2^* = \frac{1}{2}(\sqrt{2(c_1 + c_2)} + \sqrt{2(c_2 - c_1)}). \end{cases}$$

We can also require here that $v_2^* \leq 1$ to see what conditions should be satisfied. From $\frac{1}{2}(\sqrt{2(c_1 + c_2)} + \sqrt{2(c_2 - c_1)}) \leq 1$, we immediately have $c_2 \leq \frac{1}{2} + \frac{1}{2}c_1^2$, which is exactly the same condition required for $v_2^* \leq 1$.

Since

$$\frac{\partial v_1^*}{\partial c_1} = (2\sqrt{c_1 + c_2})^{-1} + (2\sqrt{c_2 - c_1})^{-1} > 0$$
$$\frac{\partial v_2^*}{\partial c_2} = (2\sqrt{c_1 + c_2})^{-1} + (2\sqrt{c_2 + c_1})^{-1} > 0,$$

the equilibrium cutoff points are increasing functions in their own participation costs; the higher a bidder's own participation cost is, the less likely he will participate in the auction and submit the bid.

Also, since $c_1 + c_2 > c_2 - c_1$, we have

$$\frac{\partial v_1^*}{\partial c_2} = (2\sqrt{c_1 + c_2})^{-1} - (2\sqrt{c_2 - c_1})^{-1} < 0$$

$$\frac{\partial v_2^*}{\partial c_1} = (2\sqrt{c_1 + c_2})^{-1} - (2\sqrt{c_2 - c_1})^{-1} < 0.$$

The cutoff point of each bidder is a decreasing function of the other's participation cost. The intuition behind this is clear. The higher your opponent's participation cost is, the less likely he will participate in the auction. Thus, it is more likely you will win the object, so your expected net-payoff tends to be higher. Consequently, you will be more willing to participate in the auction, so your cutoff point will be lower.

Remark 5 Here we express the comparative statics in terms of c_1 and c_2. We can also express these in terms of v_1^* and v_2^*: $\frac{\partial v_1^*}{\partial c_1} = \frac{v_2^*}{v_2^{*2} - v_1^{*2}} > 0$, $\frac{\partial v_2^*}{\partial c_1} = -\frac{v_1^*}{v_2^{*2} - v_1^{*2}} < 0$, $\frac{\partial v_1^*}{\partial c_2} = -\frac{v_1^*}{v_2^{*2} - v_1^{*2}} < 0$, and $\frac{\partial v_2^*}{\partial c_2} = \frac{v_2^*}{v_2^{*2} - v_1^{*2}} > 0$.

More generally, suppose we have a monotonic equilibrium (v_1^*, v_2^*) for the costs (c_1, c_2). Now choose (c_1', c_2') satisfying $c_1' \geq c_1$ and $c_2' \leq c_2$. Note that bidder 1's best response when bidder 2's cutoff is in $[c_2, v_2^*]$ must lie in $[v_1^*, 1]$ since c_1 has weakly increased and bidder 2's best response when bidder 1's cutoff is in $[v_1^*, 1]$ must lie in $[c_2, v_2^*]$ since c_2 has weakly decreased. Thus for costs (c_1', c_2'), the sets $[v_1^*, 1]$ for bidder 1 and $[c_2, v_2^*]$ for bidder 2 are closed under best response. So there must be an equilibrium in which each bidder uses a cutoff point from his specified set. That is, when one bidder's cost increases and the other's decreases, there is necessarily a new equilibrium in which the former uses a greater cutoff point and the latter uses a smaller cutoff. Of course when $F(.)$ is concave, which gives us a unique equilibrium, we obtain the above result.

D. Extensions

In the previous section, we studied the equilibrium behavior for an economic environment with two bidders and the same continuously differentiable distribution functions defined on the support $[0, 1]$. In this section, we briefly discuss some extensions by relaxing these assumptions.

1. Two Types of Bidders with Different Participation Costs

In this subsection we extend the model in Section 3 to a more general economic environment where there are two types of bidders. Type 1 possesses lower participation costs c_1. The number of bidders in this type is n_1. Type 2 possesses higher participation costs $c_2 > c_1$. There are n_2 bidders in this type. The total number of bidders is $n = n_1 + n_2$. For simplicity, we only consider the type-symmetric equilibrium by assuming that that bidders with the same participation cost use the same cutoff point. We will consider the type-asymmetric equilibrium next subsection.

Again, we first assume, provisionally, that a monotonic equilibrium exists. By Lemma 1, we have $v_1^* \leq 1$. As discussed in Section 3, for each bidder in type 1, when his valuation is $v_1 = v_1^*$, he is indifferent between participating in the auction and not participating in the auction. Thus we have the zero net-payoff equation

$$c_1 = v_1^* F(v_1^*)^{n_1 - 1} F(v_2^*)^{n_2}. \tag{2.10}$$

For each bidder in type 2 who has participation cost c_2 and uses v_2^* as his decision point to participate, if $v_2^* > 1$ (i.e., type 2 bidders never participate) then $v_1^* = v_1'$, where v_1' is determined by $c_1 = v_1' F(v_1')^{n_1 - 1}$. For this to be an equilibrium strategy

of "$\{No\}$", by the same reason as in last section, we need

$$v_1' F(v_1')^{n_1} + \int_{v_1'}^{1} F(v)^{n_1} dv < c_2;$$

i.e., given the strategy of type 1 bidders, the expected revenue of any type 2 bidder participating in the auction is less than his participation cost even when his value is 1. Then, $v_1^* = v_1'$ for each bidder in type 1 and $v_2^* > 1$ for each bidder in type 2 comprise a monotonic equilibrium.

When a type 2 bidder chooses a cutoff point $v_2^* \leq 1$, and $v_2 = v_2^*$, we have

$$c_2 = v_2^* F(v_1^*)^{n_1} F(v_2^*)^{n_2-1} + F(v_2^*)^{n_2-1} \int_{v_1^*}^{v_2^*} (v_2^* - v) d(1 - \int_{v}^{1} dF(v))^{n_1},$$

where the first part on the right side is the expected revenue when he is the only bidder in the auction submitting the bid. The second part is the expected revenue when he is the only type 2 bidder submitting a bid, and there is at least one bidder in type 1 submitting a bid. $(1 - \int_v^1 dF(v))^{n_1}$ is the probability that at least one type 1 bidder participates in the auction and bids at most v. Simplifying the equation, we have

$$c_2 = v_2^* F(v_1^*)^{n_1} F(v_2^*)^{n_2-1} + F(v_2^*)^{n_2-1} \int_{v_1^*}^{v_2^*} (v_2^* - v) dF(v)^{n_1}. \tag{2.11}$$

Note that, when $n_1 = n_2 = 1$, this reduces to equation (2.2). Integrating by parts to equation (2.11), we have

$$c_2 = v_1^* F(v_1^*)^{n_1} F(v_2^*)^{n_2-1} + F(v_2^*)^{n_2-1} \int_{v_1^*}^{v_2^*} F(v)^{n_1} dv. \tag{2.12}$$

When a type-symmetric neg-monotonic equilibrium exists, by Lemma 1, we have $v_2^* \leq 1$. For bidders in type 1, if $v_2' F(v_2')^{n_2} + \int_{v_2'}^{1} F(v)^{n_2} dv < c_1$, where v_2' is determined by $c_2 = v_2' F(v_2')^{n_2-1}$, then we have $v_1^* > 1$. Bidders in type 1 never participate in the auction. Then, in this case, $v_1^* > 1$ for bidders in type 1, and $v_2^* = v_2'$ for bidders in

type 2 compose a type-symmetric neg-monotonic equilibrium.

Suppose that $v_2' F(v_2')^{n_2} + \int_{v_2'}^{1} F(v)^{n_2} dv < c_1$ is not true. Then $v_1^* \leq 1$. Similar to the previous section, the zero net-payoff condition requires that

$$\begin{aligned} c_2 &= v_2^* F(v_2^*)^{n_2-1} F(v_1^*)^{n_1} \\ c_1 &= v_1^* F(v_1^*)^{n_1-1} F(v_2^*)^{n_2} + F(v_1^*)^{n_1-1} \int_{v_2^*}^{v_1^*} (v_1^* - v) d(F(v))^{n_2}. \end{aligned} \tag{2.13}$$

Integrating (2.13) by parts, we have

$$c_1 = v_2^* F(v_2^*)^{n_2} F(v_1^*)^{n_1-1} + F(v_1^*)^{n_1-1} \int_{v_2^*}^{v_1^*} F(v)^{n_2} dv \tag{2.14}$$

and thus

$$c_1 > v_2^* F(v_2^*)^{n_2} F(v_1^*)^{n_1-1} + F(v_1^*)^{n_1-1} (v_1^* - v_2^*) F(v_2^*)^{n_2} = v_1^* F(v_2^*)^{n_2} F(v_1^*)^{n_1-1}.$$

In order for this to be consistent with $c_1 < c_2$, one necessary condition required is

$$c_2 = v_2^* F(v_2^*)^{n_2-1} F(v_1^*)^{n_1} > v_1^* F(v_2^*)^{n_2} F(v_1^*)^{n_1-1},$$

or $\frac{F(v_2^*)}{v_2^*} < \frac{F(v_1^*)}{v_1^*}$, which is the same as in the case of two bidders.

Consider the following 2 equations:

$$\begin{aligned} c_2 &= y F(y)^{n_2-1} F(x)^{n_1} \\ c_1 &= y F(y)^{n_2} F(x)^{n_1-1} + F(x)^{n_1-1} \int_{y}^{x} F(v)^{n_2} dv. \end{aligned}$$

The first equation implicitly defines y as a function of x, denoted by $y(x)$, which has a fixed point v_2^s determined by $c_2 = v_2^s F(v_2^s)^{n_1+n_2-1}$. Then when $x > v_2^s$, we have $y < v_2^s$.

Insert $y(x)$ into the right side of the second equation and let

$$\phi(x) = y(x) F(y(x))^{n_2} F(x)^{n_1-1} + F(x)^{n_1-1} \int_{y(x)}^{x} F(v)^{n_2} dv.$$

Let c_m be the minimum of $\phi(x) = y(x)F(y(x))^{n_2}F(x)^{n_1-1} + F(x)^{n_1-1}\int_{y(x)}^{x} F(v)^{n_2}dv$ in the interval $[v_2^s, 1]$.

We then have the following proposition:

Proposition 4 (Existence and Uniqueness Theorem) *For the independent private values economic environment with two types of bidders who have different participation costs $c_2 > c_1$, we have the following conclusions for type-symmetric equilibria:*

(1) There is always a type-symmetric monotonic equilibrium.

(2) Suppose $F(.)$ is concave. Then, the type-symmetric equilibrium is unique.

(3) Suppose $F(.)$ is strictly convex. Then,

(3.i) the type-symmetric monotonic equilibrium is unique if $\frac{f(v)}{F^2(v)}$ and $\frac{F(v)}{vf(v)}$ are non-increasing,

(3.ii) the type-symmetric neg-monotonic equilibrium is unique when $c_1 = c_m$,

(3.iii) there is no type-symmetric neg-monotonic equilibrium when $c_1 < c_m$ and

(3.iv) there are at least two type-symmetric neg-monotonic equilibria when $c_m < c_1 < c_2$.

As in the case of two bidders, when $c_2 - c_1 \to 0$, we have similar convergence results: When $F(.)$ is concave, the unique type-symmetric monotonic equilibrium (there is no type-symmetric neg-monotonic equilibrium) converges to the unique type-symmetric equilibrium as $c_2 - c_1 \to 0$. When $F(.)$ is strictly convex, $\frac{f(v)}{F(v)^2}$ and $\frac{F(v)}{vf(v)}$ are non-increasing, the unique type-symmetric monotonic equilibrium converges to a type asymmetric equilibrium as $c_2 - c_1 \to 0$. When $F(.)$ is strictly convex, there are two

type-symmetric neg-monotonic equilibria such that one converges to the unique type-symmetric equilibrium, and the other converges to a type-asymmetric equilibrium as $c_2 - c_1 \to 0$.

We can similarly examine the effects of changes in costs and in numbers of bidders on equilibrium behavior. For simplicity, we only consider the case of the uniform distribution function. As in the last section, the comparative analysis can be obtained for general distribution functions.

From (3.1) and (2.11), we have

$$c_1 = v_1^{*n_1} v_2^{*n_2} \tag{2.15}$$

$$c_2 = v_1^{*n_1+1} v_2^{*n_2-1} + v_2^{*n_2-1} \int_{v_1^*}^{v_2^*} v^{n_1} dv \tag{2.16}$$

which gives us

$$c_2 = \frac{n_1}{n_1+1} v_1^{*n_1+1} v_2^{*n_2-1} + \frac{1}{n_1+1} v_2^{*n_1+n_2}. \tag{2.17}$$

(2.15) and (2.17) provide the conditions that should be satisfied simultaneously for the equilibrium cutoff points.

We then have the following proposition:

Proposition 5 *Suppose the values of bidders are drawn from a uniform distribution function $F(.)$ and the participation costs c_1 and c_2 are publicly known information. Then we have*

1) an increase in participation cost c_i increases i's cutoff point v_i^*, but decreases his opponents' cutoff point v_j^* for $i, j = 1, 2, j \neq i$.

2) the cutoff points for both types of bidders increase when the number of any type of bidder increases.

Proposition 5.(1) gives us results similar to those in previous section. A bidder's cutoff point is an increasing function of his own participation cost and a decreasing

function of others' participation costs. Proposition 5.(2) shows that more bidders in the auction will increase the competitiveness among the potential bidders, and this will reduce the possible payoff to each bidder. Thus, bidders will be less likely to participate in the auction, and their value cutoff points will increase.

2. Type Asymmetric Equilibria

In this subsection we give a brief discussion on allowing asymmetric cutoff points within a group. To allow such a possibility, we consider the simplest economy with three bidders in the two groups. The first two bidders' participation costs are the same so that $c_1 = c_2$, and the third bidder's participation cost is c_3. For simplicity, we assume that distribution functions are the same for all bidders. Let v_1^* and v_2^* be the corresponding cutoff points for the two bidders in type 1 and v_3^* be the cutoff point for type 2 bidder. We assume $c_1 = c_2 < c_3$ and $v_1^* < v_2^*$. There are three cases to be considered.

Case 1: $v_1^* < v_2^* < v_3^*$. Then we have

$$\begin{aligned}
c_1 &= v_1^* F(v_2^*) F(v_3^*), \\
c_2 &\geq v_2^* F(v_1^*) F(v_3^*) + F(v_3^*) \int_{v_1^*}^{v_2^*} (v_2^* - v) dF(v), \\
c_3 &\geq v_3^* F(v_1^*) F(v_2^*) + F(v_2^*) \int_{v_1^*}^{v_2^*} (v_3^* - v) dF(v) + \int_{v_2^*}^{v_3^*} (v_3^* - v) dF(v)^2.
\end{aligned}$$

The above equations hold with equality when $v_i^* \leq 1$. On the right side of the third equation, the first part is the revenue bidder 3 receives when the other two bidders do not participate in the auction. The second part is the revenue he receives when the highest bid of the other two is less than v_2^*. This happens when bidder 2 does not participate in the auction. The third part is the revenue when the others' highest bid is greater than v_2^* and less than v_3^*.

When $F(v)$ is concave, we cannot have such an equilibrium. To see this, from the first two equations, we have $v_1^* F(v_2^*)F(v_3^*) > v_2^* F(v_1^*)F(v_3^*)$; i.e., we have $\frac{F(v_2^*)}{v_2^*} > \frac{F(v_1^*)}{v_1^*}$ with $v_2^* > v_1^*$, which cannot be true when $F(v)$ is concave.

When $F(v)$ is strictly convex, from the first two equations, we treat v_3^* as a constant. Then it seems as if bidder 1 and bidder 2 possess participation costs $\frac{c_1}{F(v_3^*)}$. We know there is an equilibrium in which $v_1^* < v_2^*$ and the equilibrium is a function of v_3^*. Inserting into the third equation, we can get v_3^*. In particular, when $v_3^* > 1$, bidder 3 never participates in the auction.

Case 2: $v_1^* < v_3^* < v_2^*$. Then we have

$$\begin{aligned}
c_1 &= v_1^* F(v_2^*)F(v_3^*), \\
c_3 &\geq v_3^* F(v_1^*)F(v_2^*) + F(v_2^*)\int_{v_1^*}^{v_3^*}(v_3^* - v)dF(v), \\
c_2 &\geq v_2^* F(v_1^*)F(v_3^*) + F(v_3^*)\int_{v_1^*}^{v_3^*}(v_2^* - v)dF(v) + \int_{v_3^*}^{v_2^*}(v_2^* - v)dF(v)^2.
\end{aligned}$$

When $F(v)$ is concave, from the first and third equation above, we have $v_1^* F(v_2^*)F(v_3^*) > v_2^* F(v_1^*)F(v_3^*)$, which again cannot be true for $v_1^* < v_2^*$. So $v_1^* = v_3^*$. The problem can be reduced to the type symmetric equilibrium. When $F(v)$ is strictly convex, we can treat v_1^* in the second and third equation as constant. From the discussion in Section C, we know that when $c_3 - c_1$ is sufficiently small, there exists an equilibrium in which $v_2^* < v_3^*$. A limiting case is when $c_3 = c_1$. As Tan and Yilankaya [40] point out, when $F(v)$ is strictly convex but not log-concave, there may exist equilibria with three or more cutoff points.

Case 3: $v_3^* < v_1^* < v_2^*$. The discussion for this is similar to Case 2.

Summarizing our discussion above and the results we obtain in Sections C and D, we have the following proposition:

Proposition 6 *For the independent private values economy with two groups and*

three bidders, when $F(.)$ is concave, we only have the unique type-symmetric monotonic equilibrium. When $F(.)$ is strictly convex, type-asymmetric equilibria exist.

3. Bidders with Different Valuation Distributions

We consider an economy where bidders have different valuation distributions $F_1(v)$ and $F_2(v)$. Here $F_i(v)$ is the probability that bidder i's valuation is less than or equal to v, and $i = 1, 2$. Tan and Yilankaya [40] considered a similar economic environment where there are two groups of bidders with different valuation distributions but the same participation costs. Miralles [34] studied equilibrium behavior when bidders' valuation distributions can be ordered in a first order stochastic dominance ranking but still retain the same participation cost.

Here, we allow both valuation distribution functions and participation costs of bidders to be different. Again, we assume $c_1 < c_2$ and use x and y to refer the cutoff points used by bidder 1 and 2, respectively. We want to investigate the existence of equilibria and equilibria behavior.

To find a monotonic equilibrium, we need to consider the following two equations:

$$c_1 = xF_2(y)$$

$$c_2 \geq xF_1(x) + \int_x^y F_1(v)dv.$$

Again, the first equation implicitly defines x as a decreasing function of y, denoted by $x(y)$. We then have $\frac{dx}{dy} = -\frac{xf_2(y)}{F_2(y)}$. Also we know $x(y)$ has a fixed point $v_1^s \neq 0$ determined by $c_1 = v_1^s F_2(v_1^s)$. Since $x(y)$ is monotonically decreasing, we have $x < v_1^s$ and $y > v_1^s$.

Inserting $x(y)$ into the second equation and letting $\lambda(y) = xF_1(x) + \int_x^y F_1(v)dv$

with $x < y$, we have

$$\lambda'(y) = F_1(y) + xf_1(x)\frac{dx}{dy} = \frac{F_1(y)F_2(y) - x^2 f_1(x)f_2(y)}{F_2(y)}.$$

When $F_1(v)$ and $F_2(v)$ are both concave, we have

$$\lambda'(y) > \frac{F_1(y)F_2(y) - xyf_1(x)f_2(y)}{F_2(y)} > \frac{F_1(y)F_2(y) - F_1(x)F_2(y)}{F_2(y)} > 0,$$

which indicates that $\lambda(y)$ is a monotonically increasing function.

For the existence of neg-monotonic equilibrium, we consider the following two equations:

$$c_2 = yF_1(x)$$

$$c_1 \geq yF_2(y) + \int_y^x F_2(v)dv.$$

From the first equation we have $y = \frac{c_2}{F_1(x)}$. Inserting it into the right side of the second equation and letting $\phi(x) = yF_2(y) + \int_y^x F_2(v)dv$ with $x \geq y$, by the same reason as before, we have $\phi'(x) > 0$ when both $F_1(v), F_2(v)$ are concave. $y = \frac{c_2}{F_1(x)}$ also has a fixed point v_2^s determined by $c_2 = v_2^s F_1(v_2^s)$. Since $x(y)$ is monotonically decreasing, we have $y < v_2^s$ and $x > v_2^s$.

We then have the following proposition:

Proposition 7 (Existence and Uniqueness Theorem) For a two-bidder economy with different continuously differentiable distribution functions $F_1(v)$ and $F_2(v)$ and different costs $c_1 < c_2$, we have the following results:

(1) There always exists an equilibrium (v_1^*, v_2^*).

(2) Suppose $F_1(.)$ and $F_2(.)$ are both concave and $F_1(v) < F_2(v)$ for all $v \in (0, 1)$. Then there exists a unique equilibrium that is monotonic.

(3) Suppose $F_1(.)$ and $F_2(.)$ are both concave and $F_1(v) > F_2(v)$ for all $v \in (0, 1)$. Let v_1^s and v_2^s satisfy $c_1 = v_1^s F_2(v_1^s)$ and $c_2 = v_2^s F_1(v_2^s)$,

respectively. Then, we have

i) If $v_1^s < v_2^s$, there is a unique equilibrium that is monotonic;

ii) If $v_1^s > v_2^s$, there is a unique equilibrium that is neg-monotonic, satisfying $v_1^* > v_2^*$;

iii) If $v_1^s = v_2^s = v^s$, there is a unique equilibrium that is a special neg-monotonic equilibrium, satisfying $v_1^* = v_2^* = v^s$.

Remark 6 $F_1(v) < F_2(v)$ for all $v \in [0, 1]$ means that bidder 1 is a strong bidder in the sense that there is a high probability that his valuation is higher than bidder 2's valuation. A higher valuation together with a smaller participation cost makes bidder 1 more likely to participate in the auction; i.e., he is more likely to choose a lower cutoff point. However when $F_1(v) > F_2(v)$ for all $v \in (0, 1)$, bidders with higher participation costs may have lower or identical cutoff points even though their participation costs are higher.

When $F_1(v) > F_2(v)$ for all $v \in (0, 1)$, then, for each given value v, the probability that bidder 2 does not participate in the auction is less than that of bidder 1. Thus, bidder 2 has an advantage in winning the bid and a disadvantage in the participation cost. When the advantage can overbid the disadvantage, bidder 2 has a lower cutoff point, rather than a higher one, resulting in the nonexistence of a monotonic equilibrium. We can also interpret this in another way. $F_1(v) > F_2(v)$ implies that $F_1(v)$ is more concave than $F_2(v)$ and that bidder 1 is more risk averse than bidder 2. This reduces his entrance probability by leading him to choose a higher cutoff point.

Remark 7 Unlike the results obtained in Section C, (3.iii) shows that when bidders' distribution functions are different, $v_1^* = v_2^*$ can be an equilibrium although bidders' participation costs are different. That is, we have a special neg-monotonic equilibrium

with $v_1^* = v_2^*$ even when $c_1 < c_2$. When bidders' distribution functions are the same, as in Section C, this is impossible.

Thus, when bidders have different distributions on valuations, some of the previous results no longer hold true. The distributions of valuations have substantial effects on types of equilibria.

4. Positive Lower Bound of Supports

The support of valuations also affects the existence of equilibria. When the lower bound of the support of the valuation is not zero, there may be an equilibrium in which one bidder always participates in the auction and the other never participates in the auction.

Suppose the support of the distribution function $F(v)$ is $[v_l, v_h]$. There are six cases for consideration in studying equilibrium behavior of bidders:

Case 1. $v_h < c_1 < c_2$. It is clear both bidders never participate in the auction.

Case 2. $v_l < c_1 < v_h < c_2$. Bidder 2 never participates in the auction. Bidder 1 participates in the auction if $v_1 \geq c_1$ and does not otherwise.

Case 3. $c_1 < v_l < v_h < c_2$. Bidder 2 never participates, and bidder 1 always participates.

Case 4. $v_l < c_1 < c_2 < v_h$. The analysis and results are the same as those in Section C that deals with the special case where $v_l = 0$ and $v_h = 1$.

Case 5. $c_1 < v_l < c_2 < v_h$. We can have the following equilibrium: Bidder 1 always enters, and bidder 2 never enters. For this to be an equilibrium, we need $v_h - v_l < c_2$; that is, the maximum revenue bidder 2 gets from participating in the auction must be smaller than his participation costs. When $c_2 \leq v_h - v_l$, bidder 2 will choose a cutoff point $v_2^* \in [c_2, v_h]$. If there is an equilibrium in which bidder 1

never participates, then bidder 2 uses $v_2^* = c_2$. To have such an equilibrium, we need

$$v_h F(c_2) + \int_{c_2}^{v_h} (v_h - v) dF(v) = c_2 F(c_2) + \int_{c_2}^{v_h} F(v) dv < c_1.$$

A sufficient condition for this is $v_h + c_2 F(c_2) < c_1 + c_2$.

Case 6. $c_1 < c_2 < v_l < v_h$. We can have the following equilibrium: Bidder 1 always participates in the auction, and bidder 2 never participates in the auction. For this to be an equilibrium, we need $v_h - v_l < c_2$. Bidder 2 always participates in the auction, and bidder 1 never participates in the auction. For this to be an equilibrium, we need $v_h - v_l < c_1$. When both bidders choose a cutoff point inside the support of valuations, we can use the same analysis as in Section C to investigate the equilibrium behavior.

E. Conclusion

This chapter investigates equilibria of second price auctions when bidders have private valuations and different participation costs that are common knowledge. We identify two types of equilibria: monotonic and neg-monotonic equilibria. We show that there always exists an equilibrium that is monotonic, and further that, it is unique when $F(.)$ is concave or when $F(.)$ is strictly convex with additional restrictions.

We also consider the existence of neg-monotonic equilibria. We show that when the distribution function of valuation is strictly convex and when the difference of participation costs is sufficiently small, there is a neg-monotonic equilibrium. One policy implication is that one may solve multiple equilibria by eliminating neg-monotonic equilibria through differentiating participation costs significantly. We also show that when the difference in participation costs goes to zero, the monotonic equilibria of concave valuation distribution converge to the symmetric equilibrium, while the

monotonic equilibrium of convex valuation distributions converges to the asymmetric equilibrium. This is contradictory to our common intuition.

We provide some comparative static analysis. we show that the cutoff point is increasing in one's own participation cost but is decreasing in the opponents' participation costs. We also show that as the number of bidders increases, the cutoff points for all bidders will increase. This is consistent with the idea that more potential bidders will increase competition among bidders and will thus reduce the expected payoff of each buyer, with the natural consequence of reduced buyer participation.

We also consider some extensions of our basic model. We discuss equilibrium behavior for the economic environment with two types of bidders, and get similar results. However, when bidders are allowed to have different valuation distribution functions, some of the results for the basic model are no longer true. We also extend the basic model to the one with a positive lower bound of the support. In this case, we may have an equilibrium in which some bidders always enter the auction.

F. Proofs of the Main Results

Proof of Lemma 1:

Suppose not. All bidders never participate in the auction (i.e., $v_i^* > 1$ for all bidders i). When bidder n knows the other $n-1$ bidders will not participate in the auction regardless of their valuations, then bidder n participates in the auction when his value is greater than or equal to his participation cost. Then we have $v_n^* = c_n \leq 1$, a contradiction.

Proof of Lemma 2:

First note that $v_1^s < v_2^s$ by monotonicity of $vF(v)$. When bidder 2 chooses never to participate, then $v_1^* = c_1 < v_1^s$ and $v_2^* > 1$. The above lemma holds obviously.

Now suppose $v_2^* \leq 1$. We have

$$v_1^* F(v_1^*) + \int_{v_1^*}^{v_2^*} F(v)dv = c_2 = v_2^s F(v_2^s).$$

Since $v_1^* F(v_1^*) + \int_{v_1^*}^{v_2^*} F(v)dv = v_2^* F(v_2^*) - \int_{v_1^*}^{v_2^*} vf(v)dv$, we have

$$v_2^* F(v_2^*) - \int_{v_1^*}^{v_2^*} vf(v)dv = v_2^s F(v_2^s).$$

Then $v_2^s F(v_2^s) < v_2^* F(v_2^*)$. We must have $v_2^s < v_2^*$ by the monotonicity of $vf(v)$. Also, since we have $v_2^* > v_1^s$ and $c_1 = v_1^* F(v_2^*) = v_1^s F(v_1^s)$, for this equation to be true, we must have $v_1^* < v_1^s$. Otherwise we have $v_1^* F(v_2^*) > v_1^s F(v_1^s)$, a contradiction. So $v_1^* < v_1^s$. Thus, we prove $v_2^* > v_2^s > v_1^s > v_1^*$.

Proof of Proposition 8:

The proof of Proposition 8 is based on the following fives lemmas (from Lemma 3 to Lemma 7).

Lemma 3 *For the economic environment with two bidders, there always exists an equilibrium that is monotonic; i.e., for $c_2 > c_1$, there exists a cutoff point vector (v_1^*, v_2^*) such that $v_2^* > v_1^*$.*

Proof. When $c_1 F(c_1) + \int_{c_1}^1 F(v)dv < c_2$, as we discussed above, bidder 2 will never participate in the auction and thus $v_1^* = c_1$ and $v_2^* > 1$ constitute a monotonic equilibrium. Now we consider the case of $c_1 F(c_1) + \int_{c_1}^1 F(v)dv \geq c_2$.

Given that the point v_1^s determined by $c_1 = v_1^s F(v_1^s)$, we have $x < v_1^s$ and $y > v_1^s$ by noting that $y = y(x)$ is a decreasing function. Since $h(c_1) = c_1 F(c_1) + \int_{c_1}^1 F(v)dv - c_2 \geq 0$ and $h(v_1^s) = c_1 - c_2 < 0$, there exists a $v_1^* \in [c_1, v_1^s)$ such that $h(v_1^*) = 0$. Thus, $v_1^* < v_1^s$ and $v_2^* = y(v_1^*) > v_1^s$ constitute a monotonic equilibrium. ■

Lemma 4 *If $F(.)$ is concave, there is a unique monotonic equilibrium.*

Proof. Since $F(.)$ is concave, we have $F(v) \geq vF'(v) = vf(v)$ for any point $v \in [0, 1]$, and by noting $y > x$, we have

$$\lambda'(y) = F(y) - \frac{x^2}{F(y)} f(y)f(x) > F(y) - \frac{F(x)xf(y)}{F(y)} > F(y) - \frac{F(x)yf(y)}{F(y)} \geq F(y) - F(x) > 0,$$

which indicates that $\lambda(y)$ is monotonically increasing. First consider the case where $\lambda(1) = c_1 F(c_1) + \int_{c_1}^1 F(v)dv \geq c_2$. Since $\lambda(v_1^s) - c_2 = c_1 - c_2 < 0$, then, by monotonicity and continuity of λ and $x(y)$, $y = v_2^* \in (v_1^s, 1]$ is uniquely determined by $\lambda(y) - c_2 = 0$, as is $x = v_1^* < v_1^s$. Thus, the monotonic equilibrium is unique. Now suppose $\lambda(1) < c_2$. Then bidder 2 will never participate in the auction; thus $x = v_1^* = c_1$ and $v_2^* > 1$ will again be the unique monotonic equilibrium. ■

Lemma 5 *If $F(.)$ is concave, there is no neg-monotonic equilibrium, and thus the equilibrium is unique and monotonic.*

Proof. We first prove there is no neg-monotonic equilibrium in which $v_1^* > 1$. To see this, notice that $v_1^* > 1$ requires $c_1 > c_2 F(c_2) + \int_{c_2}^1 F(v)dv$. However when $F(v)$ is concave, we have

$$c_1 > c_2 F(c_2) + \int_{c_2}^1 F(v)dv \geq c_2 F(c_2) + (1 - c_2)F(c_2) = F(c_2) \geq c_2$$

by noting that $F(c_2) \geq c_2$ since $F(c) = F(c \times 1 + (1 - c)0) \geq cF(1) + (1 - c)F(0) = c$. But this contradicts the fact that $c_1 < c_2$.

We now show that there does not exist any neg-monotonic equilibrium with $v_1^* \leq 1$ either. Suppose not. We then have $v_2^* < v_1^*$ and $\frac{F(v_1^*)}{v_1^*} > \frac{F(v_2^*)}{v_2^*}$, which contradicts the fact that $\frac{F(v)}{v}$ is a non-increasing function when $F(.)$ is a concave function. Thus, there does not exist any neg-monotonic equilibrium in either case. Consequently, by Lemma 4, the equilibrium is unique, which is monotonic. ■

Lemma 6 *Suppose $F(.)$ is strictly convex and $\frac{f(v)}{F(v)^2}$ is non-increasing. Then, there is a unique monotonic equilibrium.*

Proof. Notice that $\lambda'(y)$ can be written as

$$\lambda'(y) = F(y) - \frac{x^2}{F(y)} f(y) f(x) = F(y)[1 - \frac{x^2 f(x) f(y)}{F(y)^2}].$$

Since $\frac{x^2 f(x) f(y)}{F(y)^2}$ is a decreasing function in y by noting $f(v)$ is an increasing function by strict convexity of $F(v)$ and $x = \frac{c_1}{F(y)}$, $1 - \frac{x^2 f(x) f(y)}{F(y)^2}$ is an increasing function in y, as is $\lambda'(y)$. Thus, there is at most one $y = y_0$, if any, satisfying $\lambda'(y_0) = 0$. Also, notice that, when $x = y = v_1^s$,

$$\lambda'(v_1^s) = F(v_1^s) - \frac{v_1^{s2}}{F(v_1^s)} f(v_1^s) f(v_1^s) < 0$$

by the strict convexity of $F(v)$. Then $\lambda(y)$ either decreases over the entire interval $[v_1^s, 1]$ (in this case $y_0 > 1$) or decreases first over $[v_1^s, y_0]$ and then increases over $[y_0, 1]$ if $y_0 \leq 1$. If $\lambda(y)$ decreases over the entire interval $[v_1^s, 1]$, then $\lambda(y) < c_2$ for all $y \in [v_1^s, 1]$, which means bidder 2 never participates in the auction. Thus we have a unique monotonic equilibrium with $v_1^* = c_1$ and $v_2^* > 1$. On the other hand, if $y_0 \leq 1$, $\lambda(y)$ decreases first over $[v_1^s, y_0]$ and then increases over $[y_0, 1]$. Thus $\lambda(y) = c_2 > c_1$ has at most one solution v_2^*. If the solution exists, we have a unique monotonic equilibrium with $v_1^* \leq 1$ and $v_2^* \leq 1$; otherwise the unique monotonic equilibrium is given by $v_1^* = c_1$ and $v_2^* > 1$. ■

Lemma 7 *Suppose $F(.)$ is strictly convex. There exists a neg-monotonic equilibrium when $c_1 = c_m$ and at least two neg-monotonic equilibria when $c_1 > c_1$. There is no neg-monotonic equilibrium when $c_1 < c_m$.*

Proof. Since

$$\phi'(x) = F(x) + y(x)f(y(x))y'(x)$$

and

$$y'(x) = -\frac{yf(x)}{F(x)},$$

we have

$$\phi'(v_2^s) = F(v_2^s) - v_2^s f(v_2^s)\frac{v_2^s f(v_2^s)}{F(v_2^s)} = \frac{F^2(v_2^s) - (v_2^s f(v_2^s))^2}{F(v_2^s)} < 0$$

by noting that $v_2^s f(v_2^s) > F(v_2^s)$ by $F(v) < vf(v)$ for all $v \in [c_2, 1]$ and $v_2^s \geq c_2$, which indicates that $\phi(x)$ is decreasing at $x = x_2^s$. Then $\phi(x)$ has a minimum value $c_m < c_2$ in the interval $[v_2^s, 1]$ since $\phi(v_2^s) = c_2$. Let $\phi(x_m) = c_m$.

When $c_1 < c_m$, we have $\phi(x) > c_1$ in the interval $[v_2^s, 1]$. Thus, there is no neg-monotonic equilibrium with $v_1^* \leq 1$ since the set $\{x|\phi(x) = c_1, v_2^s \leq x \leq 1\}$ is empty. On the other hand, since $\phi(1) = c_2F(c_2) + \int_{c_2}^1 F(v)dv \geq c_m > c_1$, we do not have a neg-monotonic equilibrium at which bidder 1 never participates so that $v_1^* > 1$ is not an equilibrium strategy for bidder 1.

When $c_1 = c_m$, since $\phi(x_m) = c_m$, then $x = x_m, y = c_2/F(x_m)$ is the unique neg-monotonic equilibrium. Note that when $c_1 = c_m$ we do not have an equilibrium in which bidder 1 never participates since $\phi(1) \geq c_m = c_1$.

When $c_m < c_1 < c_2$, we have at least two neg-monotonic equilibria. To see this, first notice that there exists an $x_1 \in (v_2^s, x_m)$ such that $\phi(x_1) = c_1$ by the continuity of $\phi(x)$ and $\phi(x_m) = c_m < c_1$, $\phi(v_2^s) = c_2 > c_1$. If $\phi(1) < c_1$, we have a neg-monotonic equilibrium at which bidder 1 never participates and bidder 2's equilibrium strategy is $v_2^* = c_2$. Otherwise if we have $\phi(1) \geq c_1$, we can also find an $x_2 \in (x_m, 1]$ such that

$\phi(x_2) = c_1$ by the continuity of $\phi(x)$ on $x_2 \in (x_m, 1]$, $\phi(1) > c_1$ and $\phi(x_m) = c_m < c_1$. Then $(x_1, c_2/F(x_1))$ and $(x_2, c_2/F(x_2))$ will be two neg-monotonic equilibria. ■

Proof of Proposition 2:

2.(1): When $F(.)$ is concave, the monotonic equilibrium and symmetric equilibrium are both unique, so we have the result.

2.(2) From the proof of Lemma 6, we know that, when $F(.)$ is strictly convex and $\frac{f(v)}{F(v)^2}$ is a non-increasing function of v, there is at most one y_0 such that $\lambda'(y_0) = 0$; $\lambda(y)$ either decreases over the entire interval $[v_1^s, 1]$ or decreases first over $[v_1^s, y_0]$ and then increases over $[y_0, 1]$ if $y_0 \leq 1$. Thus, there is a unique monotonic equilibrium, which is either given by $v_1^* = c_1$ and $v_2^* > 1$ when $\lambda(y)$ and c_2 have no intersection, or given by (v_1^*, v_2^*) with $v_1^* < v_1^s < y_0 < v_2^* \leq 1$ when $\lambda(y)$ and c_2 have an intersection. Here v_2^* is determined by $\lambda(v_2^*) = c_2$ and $v_1^* = c_1/F(v_2^*)$. Thus, from Figure 1, one can see that, when $c_2 \to c_1$, we have an equilibrium given by an asymmetric equilibrium $(v_1^{*\prime}, v_2^{*\prime})$ with $v_1^{*\prime} < v_1^s < y_0 < v_2^{*\prime} \leq 1$, where $v_2^{*\prime}$ is determined by $\lambda(v_2^{*\prime}) = c_1$ and $v_1^{*\prime} = c_1/F(v_2^{*\prime})$. So the unique monotonic equilibrium converges to an asymmetric equilibrium.

2.(3) When $F(.)$ is strictly convex and $c_2 - c_1$ is sufficiently small, there are two neg-monotonic equilibria (x_1, y_1) and (x_2, y_2) with $y_1 = y(x_1)$, $y_2 = y(x_2)$, and $y_1 < y_2 < v_2^s < x_1 < x_m < x_2 < x_0$ as we showed in Lemma 7. Thus, from Figure 2, as $c_1 \to c_2$, the neg-monotonic equilibrium (x_1, y_1) converges to the symmetric equilibrium (v_2^s, v_2^s), and the other neg-monotonic equilibrium (x_2, y_2) converges to the asymmetric equilibrium (x_0, y_0).

Proof of Proposition 3:

First consider a change in c_1. Taking derivatives with respect to c_1 on both sides of (4.2) and (2.3), we have

$$v_1^* f(v_2^*)\frac{\partial v_2^*}{\partial c_1} + F(v_2^*)\frac{\partial v_1^*}{\partial c_1} = 1,$$

$$F(v_2^*)\frac{\partial v_2^*}{\partial c_1} + v_1^* f(v_1^*)\frac{\partial v_1^*}{\partial c_1} = 0.$$

Solving for $\frac{\partial v_1^*}{\partial c_1}$ and $\frac{\partial v_2^*}{\partial c_1}$, we have

$$\frac{\partial v_1^*}{\partial c_1} = \frac{F(v_2^*)}{F(v_2^*)^2 - v_1^{*2} f(v_1^*)f(v_2^*)},$$
$$\frac{\partial v_2^*}{\partial c_1} = -\frac{v_1^* f(v_1^*)}{F(v_2^*)^2 - v_1^{*2} f(v_1^*)f(v_2^*)}.$$

Note that, since $v_1^* < v_2^*$ and $F(v)$ is concave, we have

$$F(v_2^*)^2 - v_1^{*2} f(v_1^*)f(v_2^*) > F(v_2^*)^2 - v_1^* v_2^* f(v_1^*)f(v_2^*) > 0,$$

and thus we have $\frac{\partial v_1^*}{\partial c_1} > 0$ and $\frac{\partial v_2^*}{\partial c_1} < 0$.

We now consider the change in c_2. Taking derivatives with respect to c_2 on both sides of (4.2) and (2.3) and solving for $\frac{\partial v_1^*}{\partial c_2}$, $\frac{\partial v_2^*}{\partial c_2}$, we have

$$\frac{\partial v_1^*}{\partial c_2} = -\frac{v_1^* f(v_2^*)}{F(v_2^*)^2 - v_1^{*2} f(v_1^*)f(v_2^*)} < 0,$$

$$\frac{\partial v_2^*}{\partial c_2} = \frac{F(v_2^*)}{F(v_2^*)^2 - v_1^{*2} f(v_1^*)f(v_2^*)} > 0$$

by noting that $F(v_2^*)^2 - v_1^{*2} f(v_1^*)f(v_2^*) > 0$.

Proof of Proposition 4:

The proof of Proposition 4 consists of the following lemmas:

Lemma 8 *For the economic environment with with two types of bidders, there always exists a type-symmetric equilibrium that is monotonic; i.e., for $c_2 > c_1$, there exists a cutoff point vector (v_1^*, v_2^*) such that $v_2^* > v_1^*$.*

Proof. Consider the following cutoff point reaction equations

$$c_1 = xF(x)^{n_1-1}F(y)^{n_2} \tag{2.18}$$

$$c_2 = xF(x)^{n_1}F(y)^{n_2-1} + F(y)^{n_2-1}\int_x^y F(v)^{n_1}dv \tag{2.19}$$

with $x < y$. From (2.18),

$$\frac{dx}{dy} = -\frac{n_2 x f(y)F(x)}{F(y)[F(x) + (n_1 - 1)xf(x)]} < 0,$$

which indicates that x is a decreasing function of y.

Given v_1' determined by $c_1 = v_1'F(v_1')^{n_1}$, when $v_1'F(v_1')^{n_1} + \int_{v_1'}^1 F(v)^{n_1}dv < c_2$, bidder 2 will never participate in the auction and thus $v_1^* = c_1$ and $v_2^* > 1$ constitute a monotonic equilibrium. So we only need to consider the case of $v_2^* \leq 1$.

From (2.18), given v_1^s determined by $c_1 = v_1^s F(v_1^s)^{n_1-1}F(v_1^s)^{n_2}$, we have $x < v_1^s$ and $y > v_1^s$ by noting that $y = y(x)$ is a decreasing function. Also , by definition, we have $v_1' < v_1^s$.

Let

$$h(x) = xF(x)^{n_1}F(y(x))^{n_2-1} + F(y(x))^{n_2-1}\int_x^{y(x)} F(v)^{n_1}dv - c_2.$$

Since $h(v_1') = v_1'F(v_1')^{n_1} + \int_{v_1'}^1 F(v)^{n_1}dv - c_2 \geq 0$ and $h(v_1^s) = c_1 - c_2 < 0$, there exists a $v_1^* \in [v_1', v_1^s)$ such that $h(v_1^*) = 0$. Thus, $v_1^* < v_1^s$ and $v_2^* = y(v_1^*) > v_1^s$ constitute a monotonic equilibrium. ■

Lemma 9 *When $F(.)$ is a concave distribution function, there is a unique type-symmetric monotonic equilibrium.*

Proof. Let

$$\begin{aligned}\lambda(y) &= xF(x)^{n_1}F(y)^{n_2-1} + F(y)^{n_2-1}\int_x^y F(v)^{n_1}dv \\ &= F(y)^{n_2-1}(xF(x)^{n_1} + \int_x^y F(v)^{n_1}dv).\end{aligned}$$

We have

$$\begin{aligned}\lambda'(y) &= (n_2-1)F(y)^{n_2-2}f(y)(xF(x)^{n_1} + \int_x^y F(v)^{n_1}dv) \\ &\quad +F(y)^{n_2-1}[F(y)^{n_1} + n_1xf(x)F(x)^{n_1-1}\frac{dx}{dy}], \\ &= F(y)^{n_2-2}[(n_2-1)f(y)\int_x^y F(v)^{n_1}dv + F(y)^{n_1+1} \\ &\quad +(n_2-1)f(y)xF(x)^{n_1} + n_1F(y)xF(x)^{n_1-1}f(x)\frac{dx}{dy}].\end{aligned}$$

Inserting $\frac{dx}{dy}$ into $\lambda'(y)$ and rearranging the terms, we have

$$\begin{aligned}\lambda'(y) &= F(y)^{n_2-2}\{(n_2-1)f(y)\int_x^y F(v)^{n_1}dv + F(y)^{n_1+1} \\ &\quad -xf(y)F(x)^{n_1}[\frac{n_1n_2}{(n_1-1)+\frac{F(x)}{xf(x)}} - (n_2-1)]\} \\ &= F(y)^{n_2-2}f(y)\{(n_2-1)\int_x^y F(v)^{n_1}dv + \frac{F(y)^{n_1+1}}{f(y)} \\ &\quad -xF(x)^{n_1}[\frac{n_1n_2}{(n_1-1)+\frac{F(x)}{xf(x)}} - (n_2-1)]\}.\end{aligned}$$

For (2.18), when $y = x = v_1^s$ determined by $c_1 = v_1^sF(v_1^s)^{n_1-1}F(v_1^s)^{n_2}$, we have

$$\lambda'(v_1^s) = F(v_1^s)^{n_2-2}\{F(v_1^s)^{n_1+1} - v_1^sf(v_1^s)F(v_1^s)^{n_1}[\frac{n_1n_2}{(n_1-1)+\frac{F(v_1^s)}{v_1^sf(v_1^s)}} - (n_2-1)]\}.$$

When $F(.)$ is a concave distribution function, we have $xf(x) \leq F(x)$ and $\frac{n_1n_2}{(n_1-1)+\frac{F(x)}{xf(x)}} - (n_2-1) < 1$ for all x. Thus, for $y > x$

$$\lambda'(y) > F(y)^{n_2-2}[(n_2-1)f(y)\int_x^y F(v)^{n_1}dv + F(y)^{n_1+1} - yf(y)F(y)^{n_1}] > 0.$$

So $\lambda(y)$ is an increasing function of y when $y > v_1^s$. Then $y = v_2^* > v_1^s$ can be uniquely determined by $\lambda(y) = c_2$. This together with $v_2^* = x(v_2^*) < v_1^s$ constitutes a monotonic equilibrium. If for all $y \in (v_1^s, 1]$ we have $\lambda(y) < c_2$, then bidder 2 will never participate; i.e., $v_2^* > 1$ and $v_2^* < 1$ as determined by $c_1 = v_1^* F(v_1^*)^{n_1-1}$ compose a unique monotonic equilibrium. In either case we only have one monotonic equilibrium. ■

Lemma 10 *When $F(.)$ is concave, there is no type-symmetric neg-monotonic equilibrium.*

Proof. We only need to show that there is no neg-monotonic equilibrium in which $v_1^* > 1$. The case where there is no neg-monotonic equilibrium with $v_1^* \leq 1$ is the same as in Lemma 5. Suppose not. We then have

$$c_1 > v_2' F(v_2')^{n_2} + \int_{v_2'}^{1} F(v)^{n_2} dv \geq F(v_2')^{n_2} = \frac{c_2 F(v_2')}{v_2'} \geq c_2$$

by noting that $F(v_2') \geq v_2'$ since $F(v) = F(v \times 1 + (1-v)0) \geq vF(1) + (1-v)F(0) = v$. But this contradicts the fact that $c_1 < c_2$. ■

Lemma 11 *Suppose $F(.)$ is strictly convex. If $\frac{f(v)}{F(v)^2}$ and $\frac{F(v)}{vf(v)}$ are non-increasing functions, then there is a unique type-symmetric monotonic equilibrium.*

Proof. When $F(.)$ is strictly convex, $f(v)$ is an increasing function and $vf(v) > F(v)$ for all v. Then we have $\lambda'(v_1^s) < 0$. Let

$$\begin{aligned} \lambda'(y) &= F(y)^{n_2-2} f(y)\{(n_2-1)\int_x^y F(v)^{n_1} dv + \frac{F(y)^{n_1+1}}{f(y)} \\ &- xF(x)^{n_1}[\frac{n_1 n_2}{(n_1-1) + \frac{F(x)}{xf(x)}} - (n_2-1)]\} = 0. \end{aligned}$$

Then we have

$$\begin{aligned} & F(y)^{n_2-2}f(y)(n_2-1)\int_x^y F(v)^{n_1}dv \\ + \quad & F(y)^{n_1+n_2-1}[1-\frac{f(y)xF(x)^{n_1}}{F(y)^{n_1+1}}(\frac{n_1n_2}{(n_1-1)+\frac{F(x)}{xf(x)}}-(n_2-1))]=0. \end{aligned}$$

When $\frac{f(v)}{F(v)^2}$ is non-increasing, then $\frac{f(y)}{F(y)^{n_1+1}}$ is decreasing in y. Since $\frac{F(x)}{xf(x)}$ is non-increasing and $x(y)$ is decreasing, $\frac{F(x)}{xf(x)}$ is non-decreasing in y, and thus $xF(x)^{n_1}[\frac{n_1n_2}{(n_1-1)+\frac{F(x)}{xf(x)}}-(n_2-1)]$ is decreasing in y.

Thus, there exists at most one $y \in (v_1^s, 1]$ such that $\lambda'(y)=0$. For the same reason as in the proof of Lemma 6, we only have one unique monotonic equilibrium.

■

Remark 8 When $n_1=n_2=1$, $xF(x)^{n_1}[\frac{n_1n_2}{(n_1-1)+\frac{F(x)}{xf(x)}}-(n_2-1)]$ can be simplified to $x^2f(x)$, which is a decreasing function of y; thus the second condition in the above lemma is redundant.

Lemma 12 *When $F(.)$ is strictly convex, there exists a unique type-symmetric neg-monotonic equilibrium when $c_1=c_m$ and at least two type-symmetric neg-monotonic equilibria when $c_m<c_1<c_2$. There is no type-symmetric neg-monotonic equilibrium when $c_1<c_m$.*

Proof. Consider $\phi(x)=y(x)F(y(x))^{n_2}F(x)^{n_1-1}+F(x)^{n_1-1}\int_{y(x)}^x F(v)^{n_2}dv$, where $y(x)$ is defined by $c_2=yF(y)^{n_2-1}F(x)^{n_1}$. We have

$$y'(x)=-\frac{n_1yf(x)F(y)}{F(x)[F(y)+(n_2-1)yf(y)]},$$

and

$$\begin{aligned}\phi'(x) &= F(x)^{n_1-2}\{(n_1-1)f(x)\int_y^x F(v)^{n_2}dv + F(x)^{n_1+1} \\ &\quad -yf(x)F(y)^{n_2}[\frac{n_1n_2}{(n_2-1)+\frac{F(y)}{yf(y)}} - (n_1-1)]\} \\ &= F(x)^{n_1-2}f(x)\{(n_1-1)\int_y^x F(v)^{n_2}dv + \frac{F(x)^{n_2+1}}{f(x)} \\ &\quad -yF(y)^{n_2}[\frac{n_1n_2}{(n_2-1)+\frac{F(y)}{yf(y)}} - (n_1-1)]\}.\end{aligned}$$

When $x = y = v_2^s$, we have

$$\phi'(v_2^s) = F(v_2^s)^{n_1-2}f(v_2^s)\{\frac{F(v_2^s)^{n_2+1}}{f(v_2^s)} - v_2^sF(v_2^s)^{n_2}[\frac{n_1n_2}{(n_2-1)+\frac{F(v_2^s)}{v_2^sf(v_2^s)}} - (n_1-1)]\}.$$

Since $v_2^sf(v_2^s) > F(v_2^s)$ by the strict convexity of $F(v)$, we have $\phi'(v_2^s) < 0$, which indicates that $\phi(x)$ is decreasing at $x = v_2^s$. Thus $\phi(x)$ has a minimum value $c_m < c_2$ in the interval $[v_2^s, 1]$ since $\phi(v_2^s) = c_2$. Let $\phi(x_m) = c_m$.

When $c_1 < c_m$, we have $\phi(x) > c_1$ for $x \in [v_2^s, 1]$. However, for us to have a neg-monotonic equilibrium, we need $\phi(x) \leq c_1$. Therefore we do not have type-symmetric neg-monotonic equilibria.

When $c_1 = c_m$, since $\phi(x_m) = c_m$, then (x, y) is the unique neg-monotonic equilibrium, where $x = x_m$ and y is determined by $c_2 = yF(y)^{n_2-1}F(x_m)^{n_1}$. Also note that when $c_1 = c_m$, we do have a neg-monotonic equilibrium in which bidder 1 never participates since $\phi(1) \geq c_1$.

When $c_m < c_1 < c_2$, we have at least two type-symmetric neg-monotonic equilibria. Indeed, since $\phi(x_m) = c_m < c_1$ and $\phi(v_2^s) = c_2 > c_1$, there is an x_1 such that $\phi(x_1) = c_1$. On the other hand, when $\phi(1) < c_1$, we have a neg-monotonic equilibrium in which bidder 1 never participates. When $\phi(1) \geq c_1$, we can find $x_2 \in (x_m, 1]$ such that $\phi(x_2) = c_1$ since $\phi(1) \geq c_1$ and $\phi(x_m) = c_m < c_1$. Thus we can find at least two

type-symmetric neg-monotonic equilibria. ■

Remark 9 The condition that $F(.)$ is concave in Lemma 9 can be weakened to $F(v) \geq vf(v)$ for all $v \in [c_1, 1]$ and the condition that $F(v)$ is strictly convex in Lemma 12 can be weakened to $F(v) < vf(v)$ for all $v \in [c_2, 1]$.

Proof of Proposition 5:

5.(1):

Taking derivatives with respect to c_1 on both sides of (2.15) and (2.17) and making simplifications, we have

$$\begin{aligned} 1 &= n_1 v_1^{*n_1-1} v_2^{*n_2} \frac{\partial v_1^*}{\partial c_1} + n_2 v_2^{*n_2-1} v_1^{*n_1} \frac{\partial v_2^*}{\partial c_1}, \\ 0 &= n_1 v_1^{*n_1} v_2^* \frac{\partial v_1^*}{\partial c_1} + [\frac{n_1(n_1-1)}{n_1+1} v_1^{*n_1+1} + \frac{n_1+n_2}{n_1+1} v_2^{*n_1+1}] \frac{\partial v_2^*}{\partial c_1}. \end{aligned}$$

Solving for $\frac{\partial v_1^*}{\partial c_1}$ gives us

$$\frac{\partial v_1^*}{\partial c_1} = \frac{n_1(n_2-1)v_1^{*n_1+1} + (n_1+n_2)v_2^{*n_1+1}}{n_1(n_1+n_2)(v_1^{*n_1-1} v_2^{*n_2}(v_2^{*n_1+1} - v_1^{*n_1+1}))} > 0$$

by $v_2^* > v_1^*$. Then we have

$$\frac{\partial v_2^*}{\partial c_1} = -\frac{n_1 v_1^{*n_1} v_2^*}{\frac{n_1(n_1-1)}{n_1+1} v_1^{*n_1+1} + \frac{n_1+n_2}{n_1+1} v_2^{*n_1+1}} \frac{\partial v_1^*}{\partial c_1} < 0.$$

Now taking derivatives with respect to c_2 on both sides of (2.15) and (2.17) and making simplifications, we have

$$\begin{aligned} 0 &= n_1 v_2^* \frac{\partial v_1^*}{\partial c_2} + n_2 v_1^* \frac{\partial v_2^*}{\partial c_2}, \\ 1 &= \frac{v_2^{*n_2-2}}{n_1+1} [n_1(n_1+1) v_1^{*n_1} v_2^* \frac{\partial v_1^*}{\partial c_2} + (n_1(n_2-1) v_1^{*n_1+1} + (n_1+n_2) v_2^{*n_1+1}) \frac{\partial v_2^*}{\partial c_2}]. \end{aligned}$$

Solving for $\frac{\partial v_2^*}{\partial c_2}$, we have

$$\frac{\partial v_2^*}{\partial c_2} = \frac{n_1+1}{(n_1+n_2)(v_2^{*n_1+1} - v_1^{*n_1+1})v_2^{*n_2-2}} > 0$$

by $v_2^* > v_1^*$. Thus we have

$$\frac{\partial v_1^*}{\partial c_2} = -\frac{n_2 v_1^*}{n_1 v_2^*}\frac{\partial v_2^*}{\partial c_2} < 0.$$

5.(2):

Note that (2.15) and (2.17) implicitly define v_1^* and v_2^* as functions of n_1 and n_2, denoted by $v_1^* = v_1^*(n_1, n_2)$ and $v_2^* = v_2^*(n_1, n_2)$. Let

$$\mathbf{V}_{\mathbf{n}}^* = \begin{pmatrix} \frac{\partial v_1^*}{\partial n_1} & \frac{\partial v_1^*}{\partial n_2} \\ \frac{\partial v_2^*}{\partial n_1} & \frac{\partial v_2^*}{\partial n_2} \end{pmatrix}.$$

Taking logs of both equations and defining

$$H(n_1, n_2; v_1^*, v_2^*) = \begin{cases} H_1(n_1, n_2; v_1^*, v_2^*) = n_1 \ln(v_1^*) + n_2 \ln(v_2^*) - \ln(c_1) \\ H_2(n_1, n_2; v_1^*, v_2^*) = \ln(\frac{v_2^{*n_1+n_2}}{n_1+1}) + \ln(1 + n_1(\frac{v_1^*}{v_2^*})^{n_1+1}) - \ln(c_2), \end{cases}$$

we have

$$\mathbf{H}_{\mathbf{v}^*} = \begin{pmatrix} \frac{\partial H_1}{\partial v_1^*} & \frac{\partial H_1}{\partial v_2^*} \\ \frac{\partial H_1}{\partial v_1^*} & \frac{\partial H_1}{\partial v_2^*} \end{pmatrix} = \begin{pmatrix} \frac{n_1}{v_1^*} & \frac{n_2}{v_2^*} \\ \frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} & \frac{n_1+n_2}{v_2^*} + \frac{n_1(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}(-\frac{v_1^*}{(v_2^*)^2})}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} \end{pmatrix}.$$

Since

$$\begin{aligned}
\det(H_{v^*}) &= \frac{n_1}{v_1^*}\left(\frac{n_1+n_2}{v_2^*} + \frac{n_1(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}(-\frac{v_1^*}{(v_2^*)^2})}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}\right) - \frac{n_2}{v_2^*}\frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} \\
&= \frac{n_1}{v_1^*}\frac{n_1+n_2}{v_2^*} - \frac{n_1(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}\frac{n_1}{(v_2^*)^2} - \frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}\frac{n_2}{v_2^2} \\
&= \frac{n_1}{v_1^*}\frac{n_1+n_2}{v_2^*} - \frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}\frac{n_2+n_1}{(v_2^*)^2} \\
&= \frac{n_1(n_1+n_2)}{v_2^*}\left(\frac{1}{v_1^*} - \frac{(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}\frac{1}{v_2^*}\right) \\
&= \frac{n_1(n_1+n_2)}{v_1^* v_2^*}\frac{1-(\frac{v_1^*}{v_2^*})^{n_1+1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}
\end{aligned}$$

and $v_1^* < v_2^*$, at the equilibrium (v_1^*, v_2^*), we have $\det(H_{v^*}) > 0$. Then, by the Implicit Function Theorem, we have

$$V_n^* = -H_{v^*}^{-1}H_n = \begin{pmatrix} \frac{\partial v_1^*}{\partial n_1} & \frac{\partial v_1^*}{\partial n_2} \\ \frac{\partial v_2^*}{\partial n_1} & \frac{\partial v_2^*}{\partial n_2} \end{pmatrix},$$

where

$$\mathbf{H_n} = \begin{pmatrix} \frac{\partial H_1}{\partial n_1} & \frac{\partial H_1}{\partial n_2} \\ \frac{\partial H_1}{\partial n_1} & \frac{\partial H_1}{\partial n_2} \end{pmatrix} = \begin{pmatrix} \ln(v_1^*) & \ln(v_2^*) \\ \ln(v_2^*) + \frac{(\frac{v_1^*}{v_2^*})^{n_1+1} + n_1(\frac{v_1^*}{v_2^*})^{n_1+1}(\ln(v_1^*)-\ln(v_2^*))}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} - \frac{1}{1+n_1} & \ln(v_2^*) \end{pmatrix}$$

and

$$\mathbf{H_{v^*}^{-1}} = \frac{1}{\det(H_(v^*))}\begin{pmatrix} \frac{n_1+n_2}{v_2^*} + \frac{n_1(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}(-\frac{v_1^*}{(v_2^*)^2})}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} & -\frac{n_2}{v_2^*} \\ -\frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}} & \frac{n_1}{v_1^*} \end{pmatrix}.$$

To determine the sign for each term in V_n^*, we ignore $\det(H_{v^*})$ which is positive.

The sign of $\frac{\partial v_1^*}{\partial n_1}$ is determined by the opposite sign of the following term:

$$(\frac{n_1+n_2}{v_2^*}+\frac{n_1(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}(-\frac{v_1^*}{(v_2^*)^2})}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}})\ln(v_1^*)-\frac{n_2}{v_2^*}\ln(v_2^*)$$

$$-\frac{n_2}{v_2^*}\frac{(\frac{v_1^*}{v_2^*})^{n_1+1}+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}(\ln(v_1^*)-\ln(v_2^*))}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}+\frac{1}{1+n_1}\frac{n_2}{v_2^*}$$

which can be simplified to

$$\frac{1}{v_2^*(1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1})}[(n_1+n_2-n_1(\frac{v_1^*}{v_2^*})^{n_1+1})\ln(v_1)-n_2\ ln(v_2)+\frac{n_2}{1+n_1}(1-(\frac{v_1^*}{v_2^*})^{n_1+1})].$$

Let

$$f(v_1^*)=n_2(1-(\frac{v_1^*}{v_2^*})^{1+n1})-(1+n_1)n_2\ln(v_2^*)+\ln(v_1^*)[(1+n_1)(n_1+n_2)-(1+n_1)n_1(\frac{v_1^*}{v_2^*})^{n_1+1}]$$

be a function of v_1^*, where $0\le v_1^*\le v_2^*<1$.

From $f(v_2^*)=0$ and

$$f'(v_1^*)=-n_2(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}+\frac{1}{v_1^*}(1+n_1)(n_1+n_2)-\frac{1}{v_2^*}(1+n_1)n_1(\frac{v_1^*}{v_2^*})^{n_1}$$

$$-(1+n_1)^2n_1(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}\ln(v_1^*)$$

$$=-(1+n_1)^2n_1(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}\ln(v_1^*)+(1+n_1)(n_1+n_2)(\frac{1}{v_1^*}-\frac{1}{v_2^*}(\frac{v_1^*}{v_2^*})^{n_1}),$$

we can see that both terms are positive when $0\le v_1^*\le v_2^*<1$. Thus, $f(v_1^*)$ is monotonic in the range $0\le v_1^*\le v_2^*<1$; hence $f(v_1^*)<0$. Then we have $-f(v_1^*)>0$, and thus $\frac{\partial v_1^*}{\partial n_1}>0$.

Now for $\frac{\partial v_2^*}{\partial n_1}$, we check the sign of

$$
\begin{aligned}
- & \frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}\ln(v_1^*) \\
+ & \frac{n_1}{v_1^*}[\ln(v_2^*)+\frac{(\frac{v_1^*}{v_2^*})^{n_1+1}+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}(\ln(v_1^*)-\ln(v_2^*))}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}-\frac{1}{1+n_1}],
\end{aligned}
$$

which can be simplified to

$$
-\frac{n_1}{v_1^*(1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1})}[(\frac{v_1^*}{v_2^*})^{n_1+1}\ln(v_1^*)-\ln(v_2^*)+\frac{1-(\frac{v_1^*}{v_2^*})^{n_1+1})}{1+n_1}].
$$

Let

$$
f(v_1^*)=(1-(\frac{v_1^*}{v_2^*})^{1+n_1})-(1+n_1)\ln(v_2^*)+(1+n_1)\ln(v_1^*)(\frac{v_1^*}{v_2^*})^{n_1+1}.
$$

We can check that

$$
f(v_2^*)=0,
$$

and

$$
\begin{aligned}
f'(v_1^*) &= -(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}+\frac{1}{v_2^*}(\frac{v_1^*}{v_2^*})^{n_1}(1+n_1)+\ln(v_1^*)(\frac{v_1^*}{v_2^*})^{n_1}(1+n_1)^2\frac{1}{v_2^*} \\
&= \ln(v_1^*)(\frac{v_1^*}{v_2^*})^{n_1}(1+n_1)^2\frac{1}{v_2^*}
\end{aligned}
$$

is negative and it is monotonic over the domain. Thus $\frac{\partial v_2^*}{\partial n_1}>0$.

Now for $\frac{\partial v_1^*}{\partial n_2}$, we check the sign of

$$
-[(\frac{n_1+n_2}{v_2^*}+\frac{n_1(n_1+1)(\frac{v_1^*}{v_2^*})^{n_1}(-\frac{v_1^*}{(v_2^*)^2})}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}-\frac{n_2}{v_2^*})\ln(v_2^*)].
$$

It can be simplified to

$$
-\frac{n_1}{v_2^*}\frac{1-(\frac{v_1^*}{v_2^*})^{n_1+1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}\ln(v_2^*)>0.
$$

Thus, $\frac{\partial v_1^*}{\partial n_2}>0$.

Now for $\frac{\partial v_2^*}{\partial n_2}$, we check the sign of

$$-[-\frac{n_1(1+n_1)(\frac{v_1^*}{v_2^*})^{n_1}\frac{1}{v_2^*}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}+\frac{n_1}{v_1^*}]\ln(v_2^*)$$

that can be simplified to

$$-\frac{n_1\ln(v_2^*)}{v_1^*}\frac{1-(\frac{v_1^*}{v_2^*})^{n_1+1}}{1+n_1(\frac{v_1^*}{v_2^*})^{n_1+1}}>0.$$

Thus, $\frac{\partial v_2^*}{\partial n_2}>0$.

Proof of Proposition 7:

7.(1) Suppose by contradiction that there does not exist any type of equilibrium. We then have no monotonic equilibrium. Thus, $\lambda(y)>c_2=v_2^sF_1(v_2^s)$ for all $y\in[v_1^s,1]$, and particularly, we have $\lambda(v_1^s)=v_1^sF_1(v_1^s)>v_2^sF_1(v_2^s)$. Then we have $\frac{v_1^s}{v_2^s}>\frac{F_1(v_2^s)}{F_1(v_1^s)}$. Since there is no neg-monotonic equilibrium either, we have $\phi(x)>c_1=v_1^sF_2(v_1^s)$ for all $x\in[v_2^s,1]$, and particularly, we have $\phi(v_2^s)=v_2^sF_2(v_2^s)>v_1^sF_2(v_2^s)$. Then we have $\frac{v_1^s}{v_2^s}<\frac{F_2(v_2^s)}{F_2(v_1^s)}$. Combining these two cases, we have

$$\frac{F_1(v_2^s)}{F_1(v_1^s)}<\frac{v_1^s}{v_2^s}<\frac{F_2(v_2^s)}{F_2(v_1^s)}.$$

Now we prove that these two inequalities cannot hold simultaneously. Indeed, if $v_1^s\le v_2^s$, then $1\le\frac{F_1(v_2^s)}{F_1(v_1^s)}<\frac{v_1^s}{v_2^s}\le 1$, which is impossible. On the other hand, if $v_1^s>v_2^s$, $1<\frac{v_1^s}{v_2^s}<\frac{F_2(v_2^s)}{F_2(v_1^s)}<1$, which is also impossible. Thus, there must exist an equilibrium for any $F_1(v)$ and $F_2(v)$ under consideration.

7.(2) First note that $\lambda(v_1^s)=v_1^sF_1(v_1^s)<v_1^sF_2(v_2^s)=c_1<c_2$ by $F_1(v)<F_2(v)$ and $\lambda(y)$ is monotonically increasing by the concavity of $F_1(v)$ and $F_2(v)$. Thus, if $\lambda(y)<c_2$ for all $y\in(v_1^s,1]$, bidder 2 will never participate (i.e., $v_2^*>1$), and bidder 1 uses $v_1^*=c_1$ as the cutoff point. Otherwise bidder 2 will use $v_2^*>v_1^s$

which is determined by $\lambda(y) = c_2$. Thus, in both cases, (v_1^*, v_2^*) compose a monotonic equilibrium. Since $\lambda(y)$ is monotonically increasing, such a monotonic equilibrium must be unique.

Finally, we show there does not exist any neg-monotonic equilibrium. To do so, we only need to focus on $\phi(x)$ with $x > v_2^s$. Since $\phi(v_2^s) = v_2^s F_2(v_2^s) > v_2^s F_1(v_1^s) = c_1$ and $\phi(x)$ is monotonic increasing, $\phi(x) > c_1$ for all $x \in (v_2^s, 1]$. Thus we do not have a neg-monotonic equilibrium. Hence, there is a unique equilibrium that is monotonic.

7.(3.i) Suppose $v_1^s < v_2^s$. We have $\phi(v_2^s) = v_2^s F_2(v_2^s) > v_1^s F_2(v_1^s) = c_1$. By $\phi'(x) > 0$ we have $\phi(x) > c_1$ for all $x \in (v_2^s, 1]$. Thus no neg-monotonic exists. Also, we have $\lambda(v_1^s) = v_1^s F_1(v_1^s) < v_2^s F_1(v_2^s) = c_2$. Then by the monotonicity of $\lambda(y)$, there is a unique equilibrium that is monotonic.

7.(3.ii) Suppose $v_1^s > v_2^s$. We have $\lambda(v_1^s) = v_1^s F_1(v_1^s) > v_1^s F_2(v_1^s) > v_2^s F_2(v_2^s) = c_2$. By $\lambda'(y) > 0$ we have $\lambda(y) > c_2$ for all $y \in (v_1^s, 1]$. So no monotonic equilibrium exists. On the other hand, we have $\phi(v_2^s) = v_2^s F_2(v_2^s) < v_1^s F_2(v_1^s) = c_1$. By $\phi'(x) > 0$, if for all $x \in (v_2^s, 1]$ we have $\phi(x) < c_1$, bidder 1 never participates in the auction (i.e., $v_1^* > 1$). Thus, $v_1^* > 1$ and $v_2^* = c_2$ will be the unique neg-monotonic equilibrium. Otherwise $v_2^* > v_2^s$ is uniquely determined by $\phi(x) = c_1$. Then $v_1^* < v_2^s$ and $v_2^* > v_2^s$ is the unique neg-monotonic equilibrium. Thus we have a unique equilibrium that is neg-monotonic.

7.(3.iii) Now suppose $v_1^s = v_2^s = v^s$. We then have $c_1 = v^s F_2(v^s)$ and $c_2 = v^s F_1(v^s)$. Then $\lambda(v^s) = v^s F_1(v^s) = c_2$ and $\phi(v^s) = v^s F_2(v^s) = c_1$. Thus $v_1^* = v_2^* = v^s$ is the equilibrium that is a special neg-monotonic equilibrium. The uniqueness comes from the monotonicity of $\lambda(y)$ and $\phi(x)$.

CHAPTER III

SECOND PRICE AUCTIONS WITH TWO-DIMENSIONAL PRIVATE INFORMATION ON VALUES AND PARTICIPATION COSTS

This chapter studies equilibria of second price auctions when values and participation costs are both privation information and are drawn from general distribution functions. We consider the existence and uniqueness of equilibrium. It is shown that there always exists an equilibrium for this general economy, and further there exists a unique symmetric equilibrium when all bidders are ex ante homogeneous. Moreover, we identify a sufficient condition under which we have a unique equilibrium in a heterogeneous economy with two bidders. Our general framework covers many relevant models in the literature as special cases.

A. Introduction

The study of participation costs in auctions mainly focuses on second price auctions due to the simplicity of bidding behavior. In a standard second price auction, bidding one's true valuation is a weakly dominant strategy. There are also other equilibria in the standard second price auction as shown in Blume and Heidhues [1] for example, the bidder with the highest value bids his true value and all others bid zero. This is referred as the asymmetric bidding equilibrium in the standard second price auction. However, in second price auctions with participation costs, so long as a bidder finds participating optimal, he cannot do better than bidding his true value. Therefore in this paper we only consider equilibria in which each bidder uses a cutoff strategy; i.e., bids his true if one finds participating optimal, does not participate otherwise. All of our results about the uniqueness or multiplicity of equilibria, then, should be interpreted accordingly.

Laffont and Green [12] studied the second price auction with participation costs in a general framework where bidders' valuations and participation costs are both private information. However, their proof on the existence and uniqueness of equilibrium is incomplete. They wanted to show the existence and uniqueness of symmetric equilibrium via contract mapping theorem. However, the condition for that theorem to hold does not satisfied. Besides, they imposed a restrictive assumption of uniform distributions for both values and participation costs and only considered symmetric equilibrium. Recently, some work in the literature has been done on equilibria of the second price auction with participation costs in simplified versions where either only valuations or participation costs are private.

Campbell [5] and Tan and Yilankaya [40] studied equilibria and their properties in an economic environment when bidders' values are private information and participation costs are common knowledge and the same. They did find asymmetric equilibria when bidders are ex ante homogeneous. Uniqueness of the equilibrium cannot be guaranteed. Some other studies, including Samuelson [37], Stegaman [38], Levin and Smith [21], etc, also assumed that participation costs are the same across players. While the assumption of equal participation costs is stringent and unrealistic, Cao and Tian [3] investigated the equilibria when bidders may have differentiated participation costs. They introduced the notions of monotonic equilibrium and neg-monotonic equilibrium.

Kaplan and Sela [16] simplified the framework of Laffont and Green [12] in another way. They studied equilibria of second price auctions with participation costs when bidders' participation costs are private information and drawn from the same distribution function, while valuations are common knowledge.

Thus, up to now, the problem considered in Laffont and Green [12] has only been answered in some special settings: either participation costs are commonly known or

values are publicly known. However, in reality, it is possible that both the valuations and participation costs are private information. Some participation costs are observable to the seller such as the entry fee; some are unobservable to the seller such as the learning costs. A natural way to deal with this is to allow both valuations and participation costs of bidders are private information and their distribution functions are general and may be different. This paper aims to give an answer to the question raised in Laffont and Green [12] in a general framework.

This chapter studies equilibria of second price auctions with general distribution functions on valuations and participation costs. The special cases of this general specification includes that either the valuations or participation costs are common knowledge, as those have been investigated in previous literature.

Under a general two-dimensional distribution of the bidders' participation costs and valuations we prove that the equilibria always exist. When bidders have the same distributions, there exists a unique symmetric equilibrium. Moreover, we identify the conditions under which we have a unique equilibrium in a simple two bidder economy. Special cases in which multiple equilibria exist are also discussed. There may exist an equilibrium in which one bidder never participates or an equilibrium in which one bidder always participates.

As compared to the work by Laffont and Green [12], our general framework can not only establish the existence of equilibrium and uniqueness of symmetric equilibrium in the two-dimensional uniform setting, but can also do that in many other two-dimensional settings such as truncated normal distributions, exponential distributions etc. Not restricted to the symmetric equilibrium when all bidders are homogenous, our framework can deal with the asymmetric equilibria which have been seen in literature with one-dimensional private information, like those in Tan and Yilankaya [40].

The existence of asymmetric equilibria has important consequences for the strategic behavior of bidders and the efficiency of the auction mechanism. When an auction has a participation cost, a bidder would expect less bidders to submit their bids. When symmetric equilibrium is unique, every bidder has to follow the symmetric cutoff and has no other choices. However, when asymmetric equilibria exist, bidders may choose an equilibrium they are more desirable. In this case, some bidders may form a collusion to cooperate at the entrance stage by choosing a smaller cutoff point that may decrease the probability that other bidders enter the auction, and consequently, may reduce the competition in the bidding stage. An asymmetric equilibrium may become more desirable when an auction can run repeatedly. Also, an asymmetric equilibrium may be ex-post inefficient. The item being auctioned is not necessarily allocated to the bidder with the highest valuation.

The remainder of the chapter proceeds as follows. In Section B, we describe a general setting of economic environments. We establish the existence of equilibrium in Section C. The uniqueness of equilibrium is discussed in section D. In section E we give a brief discussion about the existence of multiple equilibria. Concluding remarks are provided in Section F. All the proofs are relegated to the Section G.

B. The Setup

We consider an independent value economic environment with one seller and n buyers. Let $N = \{1, 2, \ldots, n\}$. The seller is risk neutral and has an indivisible object to sell to one of the buyers. The seller values the object at zero. The auction format is the sealed-bid second price auction (see Vickrey [44]). In order to submit a bid, bidder i must pay a participation cost c_i. Buyer i's value for the object, v_i, and participation cost c_i are private and independently drawn from the distribution function $K_i(v_i, c_i)$

with the support $[0,1] \times [0,1]$. Let $k_i(v_i, c_i)$ denote the corresponding density function. In particular, when v_i and c_i are independent, we have $K_i(v_i, c_i) = F_i(v_i)G_i(c_i)$ and $k_i(v_i, c_i) = f_i(v_i)g_i(c_i)$, where $F_i(v_i)$ and $G_i(c_i)$ are the cumulative distribution functions of bidder i's valuation and participation cost, $f_i(v_i)$ and $g_i(c_i)$[1] are the corresponding density functions.

Each bidder knows his own value and participation cost before he makes his entrance decision and does not know others' decisions when one makes his own.[2] If bidder i decides to participate in the auction, he pays a non-refundable participation cost c_i and submits a bid. The bidder with the highest bid wins the object and pays the second highest bid. If there is only one person in the auction, he wins the object and pays 0. If there is a tie, the allocation is determined by a fair lottery. The bidder who wins the object pays his own bid.

In this second price auction mechanism with participation costs, the individually rational action set for any type of bidder is :$\{No\} \cup [0,1]$, where "$\{No\}$" denotes not participating in the auction. Bidder i incurs the participation cost if and only if his action is different from "$\{No\}$".

If a bidder finds participating in this second price auction optimal, he cannot do better than bidding his true valuation (i.e., bidding his true is a weakly dominant strategy). Therefore, we can restrict our attention to Bayesian-Nash equilibria in which each bidder uses a cutoff strategy; i.e., one bids his true valuation if his participation cost is less than some cutoff point and does not enter otherwise. An

[1]When v_i or c_i takes discrete values, their density functions $f_i(v)$ and $g_i(c_i)$ are reduced to the discrete probability distribution functions, which can be represented by the Dirac delta function. The density at the discrete point is infinity.

[2]We share the same assumptions as Lu [23] and Laffont and Green [12]. This differs from the other branch of literature on endogenous entry and entry cost, which assumes that bidders learn their valuations after incurring the entry costs, including Tan [39] and Ye [43] among others.

equilibrium strategy of each bidder i is then determined by the expected revenue of participating in the auction $c_i^*(v_i)$ when his value is v_i[3]. Let $b_i(v_i, c_i)$ denote bidder i's strategy. Then the bidding decision function can be characterized by

$$b_i(v_i, c_i) = \begin{cases} v_i & \text{if } 0 \leq c_i \leq c_i^*(v_i) \\ \text{No} & \text{otherwise.} \end{cases}$$

Remark 10 At an equilibrium, $c_i^*(v_i) > 0$ is a cost cutoff (critical) point such that individual i is indifferent from participating in the auction or not. Bidder i will participate in the auction whenever $0 < c_i \leq c_i^*(v_i)$. $c_i^*(v_i)$ can be interpreted as the maximal amount one would like to pay to participate in the auction. Note that at equilibrium, we have $c_i^*(v_i) \leq v_i$.

The description of the equilibria can be slightly different under different informational structures on $K_i(v_i, c_i)$:

(1) v_i is a private information and c_i is common knowledge to all bidders. In this case, $K_i(v_i, c_i) = F_i(v_i)$. Campbell (1998), Tan and Yilankaya [40] and Cao and Tian [3] studied this special case. The equilibrium is described by a valuation cutoff v_i^* for each bidder i. Bidder i submits a bid when $v_i \geq v_i^*$.

(2) c_i is a private information and v_i is common knowledge to all bidders. In this case, $K_i(v_i, c_i) = G_i(c_i)$. Kalpan and Sela [16] investigated this kind of economic environment. The equilibrium is described by a cost cutoff point cost c_i^* for each bidder i. Bidder i submits a bid when $c_i \leq c_i^*$.

[3]In equilibrium, $c_i^*(v_i)$ depends on the distributions of all bidders' valuations and participation costs.

C. The Existence of Equilibrium

Suppose, provisionally, there exists an equilibrium in which each bidder i uses $c_i^*(v_i)$ as his entrance decision making. Then for bidder i with value v_i, when his participation cost $c_i \leq c_i^*(v_i)$, the bidder will participate in the auction and submit his weakly dominant bid, or else he will stay out[4]. For bidder i, to submit a bid v_i, he should participate in the auction first; i.e., $c_i \leq c_i^*(v_i)$. So the density of submitting a bid v_i is

$$f_{c_i^*(v_i)}(v_i) = \int_0^{c_i^*(v_i)} k_i(v_i, c_i) dc_i.$$

Remark 11 When v_i and c_i are independent, bidder i with value v_i will submit the bid v_i with probability $G_i(c_i^*(v_i))$ and stay out with probability $1 - G_i(c_i^*(v_i))$.

$f_{c_i^*(v_i)}(0)$ refers the probability (density) that bidder i does not submit a bid. Let $F_{c_i^*(v_i)}(v_i)$ be the corresponding cumulative probability. Note that there is a mass at $v_i = 0$ for $F_{c_i^*(v_i)}(v_i)$.

For each bidder i, let the maximal bid of the other bidders be m_i. Note that, if $m_i > 0$, at least one of other bidders participates in the auction. If $m_i = 0$, no other bidders or at most some bidders with value zero participate in the auction.

The revenue of participating in the auction for bidder i with value v_i is given by $\int_0^{v_i}(v_i - m_i)d\prod_{j\neq i} F_{c_j^*}(m_i)$, and thus the zero expected net-payoff condition for bidder i to participate in the auction when his valuation is v_i requires that

$$c_i^*(v_i) = \int_0^{v_i}(v_i - m_i)d\prod_{j\neq i} F_{c_j^*}(m_i).$$

[4] $c_i^*(v_i)$ can be interpreted as the maximal amount that bidder i would like to pay to participate in the auction when his value is v_i.

If $m_i = 0$, none of the other bidders or at most some bidders with value zero participate in the auction. The probability of the first case is

$$\prod_{j \neq i} F_{c_j^*}(0) = \prod_{j \neq i} \int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau,$$

while the probability for the second case is neglected since the second case will not affect the expected revenue of participating in the auction for a bidder with value $v_i > 0$.

Otherwise, at least one other bidder submits a bid. Then

$$\prod_{j \neq i} F_{c_j^*}(m_i) = \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau].$$

Thus, the cutoff curve for individual i, $i \in 1, 2, ...n$, can be characterized by

$$c_i^*(v_i) = \int_0^{v_i} (v_i - m_i) d \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] + v_i \prod_{j \neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau].$$

Now, integrating the first part by parts, we have

$$\begin{aligned}
c_i^*(v_i) &= \int_0^{v_i} (v_i - m_i) d \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] \\
&+ v_i \prod_{j \neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau] \\
&= (v_i - m_i) \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] \mid_0^{v_i} \\
&+ v_i \prod_{j \neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau] \\
&\quad + \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i \\
&= -v_i \prod_{j \neq i} [1 - \int_0^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] + v_i \prod_{j \neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau] \\
&\quad + \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i
\end{aligned}$$

Since

$$\int_0^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau + \int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau = \int_0^1 \int_0^1 k_j(\tau, c_j) dc_j d\tau = 1,$$

we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i. \tag{3.1}$$

Remark 12 When v_i and c_i are independent, $K_i(v_i, c_i) = F_i(v_i) G_i(c_i)$ and $k_i(v_i, c_i) = f_i(v_i) g_i(c_i)$, we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 G_j(c_j(\tau)) f_j(\tau) d\tau] dm_i.$$

Take derivative of equation (3.1) with respect to v_i, we have

$$c_i^{*\prime}(v_i) = \prod_{j \neq i} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau]. \tag{3.2}$$

Notice that $c_i^*(0) = 0$, thus the above equation is a functional differential equation with the initial condition. Specially when v_i and c_i are independent,

$$c_i^{*\prime}(v_i) = \prod_{j \neq i} [1 - \int_{v_i}^{1} G_j(c_j^*(\tau)) f_j(\tau) d\tau].$$

Lemma 13 *$c_i^*(v_i)$ has the following properties:*

(i) $c_i^*(0) = 0$.

(ii) $0 \leq c_i^*(v_i) \leq v_i$.

(iii) $c_i^{*\prime}(1) = 1$.

(iv) $\frac{dc_i^*(v_i)}{dn} < 0$.

(v) $\frac{dc_i^*(v_i)}{dv_i} \geq 0$ *and* $\frac{d^2 c_i^*(v_i)}{dv_i^2} \geq 0$

(i) means that, when bidder i's value for the object is 0, the value of participating in the auction for bidder i is zero and thus the cutoff cost point for the bidder to enter the auction is zero. Then, as long as the bidder has participation cost bigger than zero, he will not participate in the auction.

(ii) means that a bidder will not be willing to pay more than his value to participate in the auction.

(iii) means that, when a bidder's value is 1, the marginal willingness to pay to enter the auction is 1. The intuition is that when his value for the object is 1, he will win the object almost surely. Then the marginal willingness to pay is equal to the marginal increase in the valuation.

(iv) states that the participation cutoff point is a nondecreasing function in the number of bidders. As the number of bidders increases, the probability to win the object will decrease, holding other things constant. More bidders will increase the competition among the bidders and thus reduce the expected revenue.

(v) states that the marginal willingness to pay is positive and increasing. The intuition is that when a bidder's value increases, the probability of winning the auction increases. The willingness to pay increases and so is the marginal willingness to pay.

Definition 4 Given the economic environment and the properties described above, a cutoff curve equilibrium is a n-dimensional plane compromised by $(c_1^*(v_1), c_2^*(v_2), ...c_n^*(v_n))$ that is a solution of the following equation system:

$$(P1)\begin{cases} c_1^*(v_1) = \int_0^{v_1} \prod_{j\neq 1}[1 - \int_{m_1}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_1 \\ c_2^*(v_2) = \int_0^{v_2} \prod_{j\neq 2}[1 - \int_{m_2}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_2 \\ \qquad\qquad\qquad\qquad \vdots \\ c_n^*(v_n) = \int_0^{v_n} \prod_{j\neq n}[1 - \int_{m_n}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_n, \end{cases}$$

or equivalently the following differential equation system problem with initial conditions:

$$(P2)\begin{cases} c_1^{*\prime}(v_1) = \prod_{j\neq 1}[1 - \int_{v_1}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau] \text{ with } c_1^*(0) = 0 \\ c_2^{*\prime}(v_2) = \prod_{j\neq 2}[1 - \int_{v_2}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau] \text{ with } c_2^*(0) = 0 \\ \qquad\qquad\qquad\qquad \vdots \\ c_n^{*\prime}(v_n) = \prod_{j\neq n}[1 - \int_{v_n}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau] \text{ with } c_n^*(0) = 0. \end{cases}$$

We then have the following result on the existence of equilibrium

$$(c_1^*(v_1), \ldots, c_i^*(v_i), \ldots, c_n^*(v_n))$$

for $v_i \in [0, 1]$, $i \in \{1, 2, ...n\}$.

Proposition 8 (The Existence Theorem) *For the economic environment under consideration, the integral equation system (P1) or the differential equation system (P2) with initial conditions $c_i(0) = 0$ for all i has at least one solution*

$$(c_1^*(v_1), c_2^*(v_2), ...c_n^*(v_n)),$$

i.e., there is always an equilibrium in which every bidder i uses his cutoff curve $c_i^*(v_i)$.

The differential equation system above is a partial functional differential equation system, but not a partial differential equation system. The derivatives of $c_i^*(v_i)$ at v_i depends not only on v_i itself, but also on the future path of $c_i^*(v_j)$ with $j \neq i$ and $v_j \geq v_i$. Beyond that, we have multiple variables in the functional differential equation system which increases the difficulty to show the existence of equilibrium. However, we can transfer the original differential equation system to the following differential equation system

$$(P3)\begin{cases} c_1^*(v) = \int_0^v \prod_{j\neq 1}[1 - \int_{m_1}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_1 \\ c_2^*(v) = \int_0^v \prod_{j\neq 2}[1 - \int_{m_2}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_2 \\ \qquad\qquad\qquad \vdots \\ c_n^*(v) = \int_0^v \prod_{j\neq n}[1 - \int_{m_n}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_n. \end{cases}$$

Lemma 14 Problem (P1) and problem (P3) are equivalently solvable in the sense that

(1), if $(c_1^*(v_1), c_2^*(v_2), ..., c_n^*(v_n))$ is a solution to problem (P1), then $(c_1^*(v), c_2^*(v), ..., c_n^*(v))$ is a solution to problem (P3).

(2), if $(c_1^*(v), c_2^*(v), ..., c_n^*(v))$ to problem (P3), then $(c_1^*(v_1), c_2^*(v_2), ..., c_n^*(v_n))$ is a solution to problem (P1).

Thus we have reduced the multiple variables functional differential equation system to a single variable functional equation system.

Remark 13 When v_i and c_i are independent, the equilibrium is a n-dimensional plane composed by $(c_1^*(v), c_2^*(v), ...c_n^*(v))$ that is a solution of the following equation

system:

$$(P4)\begin{cases} c_1^*(v) = \int_0^v \prod_{j\neq 1}[1-\int_{m_1}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_1 \\ c_2^*(v) = \int_0^v \prod_{j\neq 2}[1-\int_{m_2}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_2 \\ \qquad\qquad\vdots \\ c_n^*(v) = \int_0^v \prod_{j\neq n}[1-\int_{m_n}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_n, \end{cases}$$

or equivalently the following differential equation system problem with initial conditions:

$$(P5)\begin{cases} c_1^{*\prime}(v) = \prod_{j\neq 1}[1-\int_v^1 G_j(c_j^*(\tau))f_j(\tau)d\tau] \text{ with } c_1^*(0)=0, \\ c_2^{*\prime}(v) = \prod_{j\neq 2}[1-\int_v^1 G_j(c_j^*(\tau))f_j(\tau)d\tau] \text{ with } c_2^*(0)=0, \\ \qquad\qquad\vdots \\ c_n^{*\prime}(v) = \prod_{j\neq n}[1-\int_v^1 G_j(c_j^*(\tau))f_j(\tau)d\tau] \text{ with } c_n^*(0)=0. \end{cases}$$

This general model with two-dimensional private values and participation costs with general distribution functions is very general and contains many existing results as special cases. In the following, for simplicity, we assume v_i and c_i are independent to illustrate the generality of our setting.

Case 1. Suppose there is a subset, denoted by A, of bidders whose valuations are common knowledge. Then for all $i \in \bar{A} = N \setminus A$, we have

$$c_i^*(v) = \int_0^v \prod_{j\in\bar{A}\setminus\{i\}}[1-\int_{m_i}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau] \prod_{j\in A\setminus\{i\},v_j>v}[1-G_j(c_j^*(v_j))]$$
$$\times \prod_{j\in\bar{A}\setminus\{i\},v_j<v}[1-\int_{m_i}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_i.$$

For all $i \in A$,

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j\in\bar{A}\setminus\{i\}}[1-\int_{m_i}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau] \prod_{j\in A\setminus\{i\},v_j>v_i}[1-G_j(c_j^*(v_j))]$$
$$\times \prod_{j\in A\setminus\{i\},v_j<v_i}[1-\int_{m_i}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_i.$$

In this case, one needs to distinguish the difference between $v_i > v_j$ and $v_j > v_i$, since under these two situations the expected revenue has different expressions.

Example 2 Suppose $n = 2$ and $v_1 < v_2$ is common knowledge, we have two bidders. Then for the bidder with value v_1,

$$\begin{aligned} c_2^*(v_2) &= \int_0^{v_2} [1 - \int_{m_2}^1 G_1(c_1^*(\tau)) f_1(\tau) d\tau] dm_2 \\ &= \int_0^{v_1} [1 - \int_{m_2}^1 G_1(c_1^*(\tau)) f_1(\tau) d\tau] dm_2 \\ &\quad + \int_{v_1}^{v_2} [1 - \int_{m_2}^1 G_1(c_1^*(\tau)) f_1(\tau) d\tau] dm_2 \\ &= v_1(1 - G_1(c_1^*(v_1))) + (v_2 - v_1), \end{aligned}$$

and

$$\begin{aligned} c_1^*(v_1) &= \int_0^{v_1} [1 - \int_{m_1}^1 G_2(c_2^*(\tau)) f_2(\tau) d\tau] dm_1 \\ &= \int_0^{v_1} [1 - G_2(c_2^*(v_2)] dm_1 = v_1(1 - G_2(c_2^*(v_2))). \end{aligned}$$

which can be reduced to the formula obtained in Kaplan and Sela [16] when the cost distribution functions are the same.

Case 2. On the contrary, suppose there is a subset, denoted by B, of bidders whose participation costs are common knowledge, as discussed in Tan and Yilankaya

[40] and Cao and Tian [3]. Let $\bar{A} = N \setminus A$. Then, for all $i \in N$, we have

$$\begin{aligned}
c_i^*(v_i) &= \int_0^{v_i} \prod_{j \in B \setminus \{i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] \prod_{j \in \bar{B} \setminus \{i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] dm_i \\
&= \int_0^{v_i} \prod_{j \in B \setminus \{i\}} [1 - \int_{m_j^*}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] \prod_{j \in \bar{B} \setminus \{i\}} [1 - \int_{m_j}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] dm_i \\
&= \int_0^{v_i} \prod_{j \in B \setminus \{i\}, m_j > v} [1 - \int_{m_j}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] \\
&\quad \times \prod_{j \in B \setminus \{i\}, m_j < v} [1 - \int_{m_j}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] \\
&\quad \times \prod_{j \in \bar{B} \setminus \{i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] dm_i,
\end{aligned}$$

where m_j^* is determined by $c_j^*(m_j) = c_j$ for $j \in B$. It may be remarked that $c_i^*(v_i)$ may have different functional forms when v_i is in the different regions of $v_i > m_j^*$ and $v_i \leq m_j^*$.

Example 3 Consider an economic environment with two bidders whose values are drawn from the same continuous distribution function $F(v)$. Bidders' participation costs are common knowledge and the same, $c_1 = c_2 = c$. This is an economy studied in Tan and Yilankaya [40] for $n = 2$. Let $c_1^*(m_1^*) = c_2^*(m_2^*) = c$.

Then for bidder 1, we have

$$c_1^*(v_i) = \int_0^{v_i} [1 - \int_{m_1}^1 G(c_2^*(\tau)) f(\tau) dm_1] d\tau.$$

As such, we have

$$c_1^*(v_1) = \int_0^{v_1} [1 - \int_{m_2^*}^1 G(c_2^*(\tau)) f(\tau) dm_1] d\tau = F(m_2^*) v_1$$

when $v_1 < m_2^*$, and

$$c_1^*(v_1) = \int_0^{m_2^*} [1 - \int_{m_2^*}^1 G(c_2^*(\tau)) f(\tau) dm_1] d\tau + \int_{m_2^*}^{v_1} [1 - \int_{m_2^*}^1 G(c_2^*(\tau)) f(\tau) dm_1] d\tau$$

$$= F(m_2^*)m_2^* + \int_{m_2^*}^{v_1} F(m_1)dm_1$$

when $v_1 \geq m_2^*$.

Similarly, for bidder 2, we have

$$c_2^*(v_2) = \int_0^{v_2} [1 - \int_{m_2}^1 G(c_1^*(\tau))f(\tau)d\tau]dm_2.$$

Then, we have $c_2^*(v_2) = F(m_1^*)v_2$ when $v_2 < m_1^*$, and $c_2^*(v_2) = F(m_1^*)m_1^* + \int_{m_1^*}^{v_2} F(m_2)dm_2$ when $v_2 \geq m_1^*$.

We can use these equations to find the cutoff points. It is clear that there is a symmetric equilibrium in which both bidders use the same cutoff point $m_1^* = m_2^* = m^*$, which satisfies the equation

$$m^* F(m^*) = c.$$

Indeed, by the monotonicity of $m^* F(m^*)$, the symmetric equilibrium exists and is unique.

Now if we provisionally suppose that $m_1^* < m_2^*$, then we should have

$$c_1^*(m_1^*) = m_1^* F(m_2^*) = c,$$

and

$$c_2^*(m_2^*) = m_1^* F(m_1^*) + \int_{m_1^*}^{m_2^*} F(m_2)dm_2 = c.$$

Tan and Yilankaya [40] showed that when $F(v)$ is strictly convex, there exists $m_1^* < m_2^*$ satisfying the above two equations.

We can use Figure 4 to illustrate the equilibria in Example 3. There are three curves in Figure 4. The middle curve indicates both bidders use the same cutoff point $c^*(v)$, and then they have the same cutoff point m^*. The curve most above is bidder 1's reaction curve $c_1^*(v_1)$. There is a kink at $v_1 = m_2^*$. Before this point, the curve is

a straight line passing through the original point with slope $F(m_2^*)$. After m_2^*, it is a smooth curve with the slope changing along the curve, which is $F(v)$. We can see as $v \to 1$, the slope goes to 1, which is consistent with properties of the cutoff curves discribed in Lemma 13. The lowest curve is bidder 2's reaction curve $c_2^*(v_2)$. The equilibrium is the intersection of the horizontal line c and each bidder's cutoff curve.

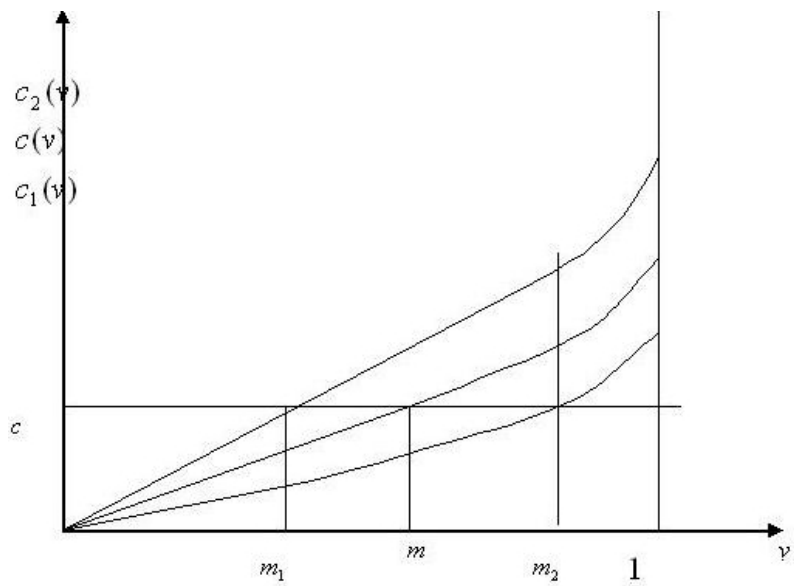

Fig. 4. Symmetric & Asymmetric Equilibrium with Two Bidders

Case 3. When all participation costs are zero, $G_i(c_i^*(\tau)) = 1$ for all τ and all i. Then

$$c_i^*(v) = \int_0^v \prod_{j \neq i} [1 - \int_{m_i}^1 f_j(\tau) d\tau] dm_i = \int_0^v \prod_{j \neq i} F_j(m_i) dm_i > 0,$$

and thus, a bidder with positive value for the object will always participate in the auction and submit a bid. Under this circumstance the entrance equilibrium curve is unique.

Case 4. When all participation costs are 1, $G_j(c_j^*(\tau)) = 0$ for all $c_j^*(\tau) < 1$, and thus $c_i^{*\prime}(v) = 1$. Considering the initial condition, we have $c_i^*(v_i) = v_i$, i.e., a bidder with value v_i would like to pay at most v_i to enter the auction. Now since the designed participation cost is 1 for all bidders, then there will be no one participating in the auction.

D. Uniqueness of Equilibrium

To investigate the uniqueness of the equilibrium $c^*(v)$, we can focus on uniqueness of the solution of (P3) by Lemma 14. We first consider the case that all bidders are ex ante homogeneous in the sense that they have the same joint distribuiton functions of valuations and participation costs and focus on the symmetric equilibrium in which all bidders use the same cutoff curve, and then study the uniqueness of equilibrium for a more general case. Then (P3) can be simply written as

$$c^*(v) = \int_0^v [1 - \int_m^1 \int_0^{c^*(\tau)} k_j(\tau, c) dc d\tau]^{n-1} dm, \tag{3.3}$$

and correspondingly we have

$$c^{*\prime}(v) = [1 - \int_v^1 \int_0^{c^*(\tau)} k(\tau, c) dc d\tau]^{n-1}, c^*(0) = 0. \tag{3.4}$$

We first give the uniqueness of the symmetric equilibrium when all bidders are ex ante homogeneous.

Proposition 9 (Uniqueness of Symmetric Equilibrium) *For the economic environment under consideration in this section, suppose that all bidders have the same distribution function $K(v, c)$. There is a unique solution $c^*(v)$ to integral equation (3.3) or differential equation (3.4) with initial condition. Consequently, there exists a unique symmetric equilibrium at which each bidder uses the same cutoff curve for*

his entrance decision making.

Remark 14 Uniqueness of the symmetric equilibrium has been established in some special cases.

1) In Campbell [5] and Tan and Yilankaya [40], when bidders have the same participation cost and continuously differentiable valuation distribution function, there is a unique symmetric equilibrium in which each bidder uses a same cutoff point v^s for their entrance decision making.
2) In Kaplan and Sela [16], when all bidders have the same valuations for the object and continuously differentiable participation cost distribution functions, there is a unique symmetric cutoff point c^*.
3) More earlier, Laffont and Green [12] investigated the existence of equilibria when both valuation and participation costs are uniform distributed. They got the uniqueness of the symmetric equilibrium under the simple two-dimensional economic environment. However their proof is incomplete.

Remark 15 Note that the above proposition only shows that the uniqueness of symmetric equilibrium when bidders are ex ante homogeneous. It does not exclude the possibility of the asymmetric equilibrium. As those in Tan and Yilankaya [5], Kalpan and Sela [16], there are some examples where ex-ante homogeneous bidders may use different cutoffs which means the equilibria are not unique.

Now under the assumption of independence of v_i and c_i, We consider the uniqueness of the functional differential equation system (P3). For simplicity we consider a simple economy with only two bidders. The corresponding functional differential equation system can be written as:

$$(P6)\begin{cases} c_1^{*\prime}(v) = [1 - \int_v^1 G_2(c_2^*(\tau))f_2(\tau)d\tau], c_1^*(0) = 0, \\ c_2^{*\prime}(v) = [1 - \int_v^1 G_1(c_1^*(\tau))f_1(\tau)d\tau], c_2^*(0) = 0. \end{cases}$$

Proposition 10 (Uniqueness of Equilibrium) In the two bidder economy with $G_i(c)$ is continuously differentiable on $[0,1]$ and $\delta_i = \max_c g_i(c)$, there is a unique equilibrium when $\delta_i \int_0^1 (1 - F_i(s))ds < 1$.

When $G_i(c_i)$ is uniform on $[0,1]$, $\delta_i = 1$ and $\int_0^1 (1-F_i(s))ds < 1$, we have a unique equilibrium. Specially when bidders are ex ante homogenous, the unique equilibrium is symmetric. To see this, consider the following examples.

Example 4 This example follows from Example 2. Assume that $G_i(c_i)$ is uniform on $[0,1]$. Then we have

$$\begin{cases} c_1^*(v_1) = v_1(1 - c_2^*(v_2)), \\ c_2^*(v_2) = v_2 - v_1 c_1^*(v_1). \end{cases}$$

There is a unique equilibrium given by $c_1^*(v_1) = \frac{v_1(1-v_2)}{1-v_1^2}$ and $c_2^*(v_2) = \frac{v_2(1-v_2)}{1-v_1^2}$. Further we can check that when $v_1 = v_2 = v$, the unique equilibrium is symmetric with $c_1^*(v) = c_2^*(v) = \frac{v}{1+v}$.

Example 5 Now we assume $G_i(c)$ and $F_i(v)$ are both uniform on $[0,1]$. At equilibrium we have

$$c_1^{*\prime}(v) = 1 - \int_v^1 c_2^*(\tau)d\tau,$$

$$c_2^{*\prime}(v) = 1 - \int_v^1 c_1^*(\tau)d\tau.$$

Then $c_1^{*\prime\prime}(v) = c_2^*(v)$ and $c_2^{*\prime\prime}(v) = c_1^*(v)$. Thus we have $c_1^{*(4)}(v) = c_1^*(v)$ and $c_2^{*(4)}(v) = c_2^*(v)$ with $c_1^*(0) = 0$, $c_1^{*\prime}(1) = 1$, $c_2^*(0) = 0$ and $c_2^{*\prime}(1) = 1$. One can check that the only equilibrium is $c_1^*(v) = c_2^*(v) = ae^v - ae^{-v}$, where $a = \frac{e}{e^2+1}$.

E. Discussions

There are in general multiple equilibria in the setting under consideration. Examples can be found in Campbell [5], Tan and Yilankaya [40], Cao and Tian [3] and Kaplan and Sela [16] where either participation costs or valuations are common knowledge. In this section we provide evidence for the multiplicity of equilibria even when both the participation costs and valuations are private information.

Suppose the support of v_i and c_i to be $[0,1] \times [\epsilon, \delta]$, where $[\epsilon, \delta]$ is a subset of $[0,1]$ and $\epsilon > 0$. To investigate the existence of equilibrium, we construct a new density function $\widetilde{k_i}(v_i, c_i)$ with support $[0,1] \times [0,1]$ which has the same density as $k_i(v_i, c_i)$ on the interval $[0,1] \times [\epsilon, \delta]$ and 0 otherwise and $\widetilde{K_i}(v_i, c_i)$ is the corresponding cumulative density function. The same as in Section C, the equilibrium cutoff curve for individual i, $i \in 1, 2, ...n$, is given by

$$c_i^*(v_i) = \int_0^{v_i} (v_i - m_i) d \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} \widetilde{k_j}(\tau, c_j) dc_j d\tau] + v_i \prod_{j \neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 \widetilde{k_j}(\tau, c_j) dc_j d\tau].$$

After integration by parts we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} \widetilde{k_j}(\tau, c_j) dc_j d\tau] dm_i.$$

Then

$$c_i^{*\prime}(v_i) = \prod_{j \neq i} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} \widetilde{k_j}(\tau, c_j) dc_j d\tau]. \tag{3.5}$$

By the fixed point theorem, an equilibrium exists. However the uniqueness of the equilibrium cannot be guaranteed. Specially when bidders are ex ante homogenous, asymmetric equilibrium may exist.

One special type of asymmetric equilibrium is that some bidders may never participate in the auction. This can happen when the support of participation costs, c, has non-zero lower bound. Such an equilibrium can be called a corner equilibrium.

One implication of such equilibrium is that in this economic environment, some of the bidders can form a collusion to enter the auction regressively so that they can prevent some others enter the auction and thus can reduce the competition among those who participate in the auction which in turn will increase the benefits from participating.

The expected revenue of participating in the auction is a non-decreasing function of one's true value. Thus the sufficient and necessary condition for a bidder to never participate is when his value is 1, participating in the auction still gives him an expected revenue that is less than the minimum participation cost, ϵ, giving the strategies of other bidders. Formally, suppose in equilibrium, a subset $A = \{1, 2, \ldots, k\} \subset \{1, 2, 3 \ldots, n\}$ of bidders choose to participate in the auction when their valuations are big enough and bidders in $B = \{k+1, \ldots, n\}$ choose never participating in the auction. Then for all $i \in A$ we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i, j \in A} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} \widetilde{k}_j(\tau, c_j) dc_j d\tau] dm_i.$$

For bidders in B never participate, it is required that for all $j \in B$,

$$c_j^*(1) = \int_0^1 \prod_{i \in A} [1 - \int_{m_j}^1 \int_0^{c_i^*(\tau)} \widetilde{k}_i(\tau, c_j) dc_j d\tau] dm_j < \epsilon,$$

which raises a requirement for the lower and upper bound of the participation costs and the distributions of valuations and participation costs. To see this, we assume that there are only two bidders and v_i and c_i are independent. The distribution functions are $F(v_i)$ and $G(c_i)$ separately.

Suppose bidder 2 never participates, then bidder 1 enters if and only if $v_1 \geq c_1$ and thus we have $c_1^*(v_1) = v_1$. Given this, the expected revenue of bidder 2 when he participates in the auction is

$$F(\epsilon) + \int_\epsilon^\delta [(1 - v_2)G(v_2) + (1 - G(v_2))] dF(v_2) + \int_\delta^1 (1 - v_2) dF(v_2)$$

when $v_2 = 1$. We have three terms in the above equation. When bidder 1's value is less than ϵ he will not enter the auction and bidder 2 will get revenue 1, the probability is $F(\epsilon)$; the second term is the revenue when bidder 1's value is between ϵ and δ. For any $v_2 \in (\epsilon, \delta)$, bidder 2's revenue is $1 - v_2$ when bidder 1 participates, and is 1 when bidder 1 does not participate, the probabilities are $G(v_2)$ and $1 - G(v_2)$ separately. The third term is the revenue when bidder 1's value is greater than δ and in this case bidder 1 participates for sure.

In order to have a corner equilibrium, we need

$$F(\epsilon) + \int_{\epsilon}^{\delta} [(1 - v_2)G(v_2) + (1 - G(v_2))]dF(v_2) + \int_{\delta}^{1} (1 - v_2)dF(v_2) < \epsilon. \tag{3.6}$$

It can be seen that in the two homogenous bidders economy, when $F(\cdot)$ is concave, there is no corner equilibrium. To see this, note that when $F(\cdot)$ is concave, we have $F(v_i) \geq v_i$, equation (4.1) can not hold; i.e, corner equilibrium does not exist.

Remark 16 if $\epsilon = \delta$; i.e., c_i is common knowledge to all bidders, (4.1) can be simplified to $F(\epsilon) + \int_{\epsilon}^{1}(1 - v_2)dF(v_2) < \epsilon$; i.e., $\epsilon F(\epsilon) + \int_{\epsilon}^{1} F(v_2)dv_2 < \epsilon$.

Example 6 Assume v_i and c_i to be joint uniform distributed (then they are independent) and there are only two bidders. Suppose bidder 2 never participates. We have $c_1^*(v_1) = v_1$.

Then we have

$$c_2^{*\prime}(v_2) = 1 - \int_{v_2}^{1} G(c_1(\tau))d\tau = 1 - \int_{v_2}^{1} min\{1, max\{\frac{\tau - \epsilon}{\delta - \epsilon}, 0\}\}d\tau,$$

which results

$$c_2^{*\prime}(v_2) = \begin{cases} 1 - \int_{\epsilon}^{\delta} \frac{\tau - \epsilon}{\delta - \epsilon} d\tau - \int_{\delta}^{1} d\tau = \frac{\epsilon + \delta}{2} & \text{if } v_2 < \epsilon \\ \delta - \frac{\delta^2 - 2\epsilon\delta - v_2^2 + 2\epsilon v_2}{2(\delta - \epsilon)} = \frac{\delta^2 + v_2^2 - 2\epsilon v_2}{2(\delta - \epsilon)} & \text{if } \epsilon \leq v_2 < \delta \\ v_2 \quad \text{if } v_2 \geq \delta \end{cases}.$$

Given the above and the initial condition $c_2^*(0) = 0$, we have

$$c_2^*(v_2) = \begin{cases} \frac{\epsilon+\delta}{2} v_2 & \text{if } v_2 < \epsilon \\ \frac{v_2^3 - 3\epsilon v_2^2 - \epsilon^3 + 3\delta^2 v_2}{6(\delta-\epsilon)} & \text{if } \epsilon \leq v_2 < \delta \\ \frac{\delta^2+\delta\epsilon+\epsilon^2+3v_2^2}{6} & \text{if } v_2 \geq \delta \end{cases} \cdot$$

For bidder 2 never participates, we need $c_2^*(1) = \frac{\delta^2+\delta\epsilon+\epsilon^2+3}{6} \leq \epsilon$, which is equivalent to $\epsilon^2 + (\delta - 6)\epsilon + \delta^2 + 3 \leq 0$. So when

$$\frac{(6-\delta) - \sqrt{-3(\delta^2 + 4\delta - 8)}}{2} \leq \epsilon \leq \frac{(6-\delta) + \sqrt{-3(\delta^2 + 4\delta - 8)}}{2},$$

the required condition is satisfied. For this to be true, we need $\delta^2 - 2\delta + 1 < 0$ which cannot be true.

However when $F(\cdot)$ is strictly convex, given proper ϵ and δ, there may be an equilibrium in which one bidder never participates while the other enters the auction whenever his valuation is greater than his participation cost. As an illustration, we assume $F(v_i) = v_i^2$ and $G(c_i)$ is uniformly distributed on $[\epsilon, \delta]$. (4.1) becomes

$$\frac{\delta^3 + \delta\epsilon^2 + \delta^2\epsilon + \epsilon^3 + 2}{6} < \epsilon.$$

One can check that when $\epsilon = 0.5$ and $\delta = 0.744$, there exists a corner equilibrium. It can be concluded that if there is a corner equilibrium in the homogenous two bidder economy, there exists a corner equilibrium in which $n - 1$ bidders never participate in the homogenous n bidder economy.

When the lower bound of valuation is positive and when bidders are ex ante homogenous, there may exists an asymmetric equilibrium in which one bidder always participates. To see this, suppose the c_i is distributed on $[c_l, c_h]$ with distribution $G(c_i)$ and v_i is distributed on $[v_l, v_h]$ with distribution $F(v_i)$. Assume $v_h > v_l > c_h > c_l$. Suppose we have an equilibrium in which bidder 1 always enter and bidder 2 never

participates. Then bidder 1 always participates is a best response. For bidder 2's strategy to be a best response, we need

$$\int_{v_l}^{v_h} (v_h - v_1) dF(v_1) - c_l < 0,$$

the maximum expected revenue is less than the lowest participation cost. Integration by parts we have

$$\int_{v_l}^{v_h} F(v_1) dv_1 < c_l.$$

One sufficiently condition for this to be true is $v_h - v_l < c_l$.

F. Conclusion

This paper investigates equilibria of second price auctions with general distribution functions of private values and participation costs. We show that there always exists an equilibrium cutoff curve for each bidder. Moreover, when all bidders are ex ante homogeneous, there is a unique symmetric equilibrium. In a simple two bidder economy, a sufficient condition for the uniqueness of the equilibrium is identified. This general two-dimensional framework covers many models as special cases.

We find evidence that multiple equilibria exist. Specifically, when bidders are ex ante homogeneous, besides the symmetric equilibrium, there may be an equilibrium at which one bidder always participates or never participates. Future research may be focused on identifying sufficient conditions to guarantee the uniqueness of equilibrium but not only the uniqueness of the symmetric equilibrium. Uniqueness of the equilibrium has important policy implications. The seller can modify the economic environment such that the economy has a unique equilibrium and thus have more predictable power for the final outcomes. Welfare analysis with participation costs is another interesting topic to be tackled.

G. Proofs of the Main Results

Proof of Lemma 13:

Proof: (i) Letting $v_i = 0$ in the expression of $c_i^*(v_i)$, we have the result.

(ii) Since

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i \leqslant \int_0^{v_i} dv_i = v_i$$

by the nonnegativity of $\int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j$ and

$$\int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau \leq \int_{m_i}^1 \int_0^1 k_j(\tau, c_j) dc_j d\tau \leq \int_0^1 \int_0^1 k_j(\tau, c_j) dc_j d\tau = 1,$$

we have $0 \leq c_i^*(v_i) \leq v_i$.

(iii) Letting $v_i = 1$ in (3.5), we have the result.

(iv) Since n is the number of bidders, as n increases, say, from n to $n+1$, the product term inside the integral will be increased by one more term. Also, note that $0 < 1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau < 1$. So given more bidders, $c_i^*(v_i)$ will decrease.

(v)

$$\frac{dc_i^*(v_i)}{dv_i} = \prod_{j \neq i} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] \geq 0$$

by noting that

$$\int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau \leq \int_0^1 \int_0^1 k_j(\tau, c_j) dc_j d = 1.$$

We then have

$$\frac{d^2 c_i^*(v_i)}{dv_i^2} = \sum_{k \neq i} \prod_{j \neq i, j \neq k} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] \int_0^{c_k^*(v_i)} k_j(\tau, c_j) dc_j d\tau \geq 0.$$

Proof of Proposition 8:

Proof: For $i = 1, 2, ...n$, let

$$\phi_i(c^*(v_i)) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i$$

where $c^*(v) = (c_1^*(v_1), c_2^*(v_2), \ldots, c_n^*(v_n))$. Then $\phi_i(c^*)$ is a continuous function and $0 \leq \phi_i(c^*) \leq v_i$ by Lemma 13.(ii). Thus, $\phi_1(.), \phi_2(.), \cdots, \phi_n(.)$ is a continuous mapping from the non-empty compact and convex domain $[0, v_1] \times [0, v_2] \times \cdots \times [0, v_n]$ to itself, and therefore, by Brower's Fixed Point Theorem, there exists $c^*(v) = (c_1^*(v_1), c_2^*(v_2), \ldots, c_n^*(v_n))$ such that $c_i^*(v_i) = \phi_i(c^*(v))$, and consequently, it is a solution to (P2) or (P3) with initial condition.

Proof of Lemma 14:

Proof: Suppose $(c_1^*(v_1), c_2^*(v_2), ..., c_n^*(v_n))$ is a solution to problem (P1), then we have for any $i \in \{1, 2, ...n\}$,

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i,$$

then by changing the variable v_i to v we have

$$c_i^*(v) = \int_0^{v} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i$$

for all $i \in \{1, 2, ...n\}$. So $(c_1^*(v), c_2^*(v), ..., c_n^*(v))$ is a solution to (P3). On the contrary, if $(c_1^*(v), c_2^*(v), ..., c_n^*(v))$ is a solution to (P3), then we have for any $i \in \{1, 2, ...n\}$,

$$c_i^*(v) = \int_0^{v} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i.$$

Then by changing the variable v to v_i in the ith equation we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i.$$

Thus $(c_1^*(v_1), c_2^*(v_2), ..., c_n^*(v_n))$ is a solution to (P1).

Proof of Proposition 9:

Proof: The existence of the symmetric equilibria can be established by the Brower' Fixed point Theorem. Here we only need to prove the uniqueness of the symmetric equilibrium. Suppose not, by way of contradiction, we have two different symmetric equilibria $x(v)$ and $y(v)$ to the economic environment we consider. Then we have

$$x'(v) = [1 - \int_v^1 \int_0^{x(\tau)} k(\tau, c) dc d\tau]^{n-1}$$

$$y'(v) = [1 - \int_v^1 \int_0^{y(\tau)} k(\tau, c) dc d\tau]^{n-1}.$$

Suppose $x(1) > y(1)$, then by the continuity of $x(v)$ and $y(v)$ we can find a v^* such that $x(v^*) = y(v^*) = c(v^*)$ and $x(v) > y(v)$ for all $v \in (v^*, 1]$ by noting that $x(0) = y(0)$.

Case 1: if $k(v, c) > 0$ with positive probabiltiy measure on $(v^*, 1) \times (c(v^*), 1)$, then for $\tau \in (v^*, 1]$ we have

$$\int_0^{x(\tau)} k(\tau, c) dc > \int_0^{y(\tau)} k(\tau, c) dc$$

for $\tau \in (v^*, 1)$. Then we have $x'(v^*) < y'(v^*)$ which is a contradiction to $x(v) > y(v)$ for $v > v^*$. So we have $x(1) = y(1)$. By the same logic above we can prove that $x(v) = y(v)$ for all $v \in [0, 1]$ and thus the symmetric equilibrium is unique.

Case 2: if $k(v, c) > 0$ with zero probabilty measure on $(v^*, 1) \times (c(v^*), 1)$, then we have $x'(v) = y'(v)$ for all $v \in (v^*, 1]$. By $x(v^*) = y(v^*)$ we have $x(v) = y(v)$ for all

$v > v^*$, which is a contradiction to $x(v) > y(v)$. Thus there is a unique symmetric equilibrium.

Then in both cases we proved that there is a unique symmetric equilibrium.

Proof of Proposition 10:

Proof: Define a mapping

$$(Pc)(v) = \int_0^v ds - \int_0^v \int_s^1 \begin{pmatrix} 0 & f_1(\tau) \\ f_2(\tau) & 0 \end{pmatrix} \begin{pmatrix} G_1(c_1(\tau)) \\ G_2(c_2(\tau)) \end{pmatrix} d\tau ds,$$

where $c = (c_1, c_2)'$.

Take any $x(v) = (x_1(v), x_2(v))'$ and $y(v) = (y_1(v), y_2(v))'$ with $x(v)$, $y(v) \in \varphi$ where φ is the space of monotonic increasing continuous functions defined on $[0,1] \to [0,1]$. Then we have

$$\begin{aligned}
|(Px)(v) \;-\; (Py)(v)| & \\
&\leq \int_0^v \int_s^1 \begin{pmatrix} 0 & g_1(\widehat{x}_1(\tau))f_1(\tau) \\ g_2(\widehat{x}_2(\tau))f_2(\tau) & 0 \end{pmatrix} \left| \begin{pmatrix} x_1(\tau) - y_1(\tau) \\ x_2(\tau) - y_2(\tau) \end{pmatrix} \right| d\tau ds \\
&= \int_0^v \int_s^1 \begin{pmatrix} 0 & g_1(\widehat{x}_1(\tau))f_1(\tau) \\ g_2(\widehat{x}_2(\tau))f_2(\tau) & 0 \end{pmatrix} d\tau ds \sup_{0<v\leq 1} |x(v) - y(v)| \\
&\leq \int_0^1 \int_s^1 \begin{pmatrix} 0 & g_1(\widehat{x}_1(\tau))f_1(\tau) \\ g_2(\widehat{x}_2(\tau))f_2(\tau) & 0 \end{pmatrix} d\tau ds \sup_{0<v\leq 1} |x(v) - y(v)| \\
&\leq \int_0^1 \begin{pmatrix} 0 & \delta_1(1 - F_1(s)) \\ \delta_2(1 - F_2(s)) & 0 \end{pmatrix} ds \sup_{0<v\leq 1} |x(v) - y(v)|, \qquad (3.7)
\end{aligned}$$

where the first equality comes from mean value theorem, and $\widehat{x}_i(\tau)$ is some number between $x_i(\tau)$ and $y_i(\tau)$, δ_i is the maximum of $g_i(c)$, $i = 1, 2$. Thus when $\delta_i \int_0^1 (1 -$

$F_i(s))ds < 1$, the above mapping is a contraction, there exists a unique equilibrium.

An Alternative Proof for The Existence of Equilibria:

We give the proof of the existence of equilibrium based on (P3), the transferred single variable functional differential equation system.

Proposition 11 (The Existence Theorem) *For the general economic environment under consideration in the paper, the integral equation system (P3) has at least one solution* $(c_1^*(v), c_2^*(v), ...c_n^*(v))$*; i.e., there is always an equilibrium in which every bidder* i *uses his own cutoff curve* $c_i^*(v)$.

To prove the above proposition we introduce the following lemma:

Lemma 15 (Schauder-Tychonov Fixed-point Theorem Cf. Smart (1980, p.15)) *Let* M *be a compact convex nonempty subset of a locally convex topological space and* $P : M \to M$ *be continuous. Then* P *has a fixed point.*

Proof of Proposition 11:

Proof: Let $h_i = \prod_{j \neq i}[1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c)dcd\tau]$ and $H = (h_1, h_2, \cdots, h_n)'$. Define

$$M = \{c \in \varphi \mid |c| \leq n, |c(v_1) - c(v_2)| < n|v_1 - v_2|\},$$

where φ is the space of continuous of function ϕ defined on $[0, 1] \to R^n$ with the supremum norm. Then by Ascoli's theorem M is compact and M is certainly convex. Define an operator $P : M \to M$ by

$$(Pc)(v) = \int_0^v H(s, c(.))ds$$

To see $P : M \to M$, note that

$$|(P\psi)(v_1) - (P\psi)(v_2)| \leq |\int_{v_2}^{v_1} H(s, c(.))ds| \leq n|v_1 - v_2|,$$

and also it can be easily check that

$$|(P\psi)(v) - 0| \leq n.$$

To see P is continuous, let $\phi \in M$ and let $\mu > 0$ be given. We must find $\eta > 0$ such that $\|\phi - \psi\| < \eta$ implies $\|(P\phi)(t) - (P\psi)(t)\| \leq \mu$. Now

$$|(P\phi)(t) - (P\psi)(t)| = |\int_0^t [H(s, \phi(s)) - H(s, \psi(s))]ds|$$

and H is uniformly continuous so for the $\mu > 0$ there is an $\eta > 0$ such that $|\phi(s) - \psi(s)| < \eta$ implies $|H(s, \phi(s)) - H(s, \psi(s))| \leq \mu$ and thus $|(P\phi)(t) - (P\psi)(t)| \leq \mu$ by noting that $0 < t \leq 1$ as required. Then by Lemma 15, there exists a fixed point; i.e., a solution for the functional differential equation system, also the solution is continuously differentiable.

CHAPTER IV

FIRST PRICE AUCTIONS WITH PARTICIPATION COSTS

This chapter characterizes equilibria of first price auctions with participation costs in the independent private values environment. We focus on the cutoff strategies in which each bidder participates and submits a bid if his value is greater than or equal to a critical value. It is shown that, when bidders are homogenous, there always exists a unique symmetric equilibrium, and further, there is no other equilibrium when valuation distribution functions are concave. However, when distribution functions are elastic at the symmetric equilibrium, there exists an asymmetric equilibrium. We find similar results when bidders are heterogenous.

A. Introduction

The studies of participation costs in auctions so far have mainly focused on the second price auction due to its simplicity of bidding behavior in the interim information acquisition setting.[1] In second price auctions (Vickrey [44]), bidders cannot do better than bidding their valuations when they find participating optimal. Much of the existing literature investigates equilibria of second price auctions with participation costs. Laffont and Green [12] study the second price auction with participation costs in a general framework where bidders' valuations and participation costs are both private information and establish the existence of symmetric equilibrium with uniform distribution. Gal et al. [11] study equilibria in a two dimensional framework with more general distributions, focusing on symmetric equilibrium only. Campbell [5]

[1]There is also some work in the ex ante information acquisition setting, in which bidders make the entrance decisions before they know their valuations (cf. McAfee and McMillan [30, 31], Engelbrecht-Wiggans [9], Levin and Smith [21, 22] and Chakraborty and Kosmopolov [7]).

and Tan and Yilankaya [40] study equilibria and their properties of second price auctions in an economic environment with equal participation costs when bidders' values are private information. Cao and Tian [3] investigate equilibria in second price auctions when bidders may have differentiated participation costs. They introduced the notions of monotonic equilibrium and neg-monotonic equilibrium. Kaplan and Sela [16] consider a private entry model in second price auctions in which they assume all bidders' valuations are common knowledge while participation costs are private information.

Studies of first price auctions in the presence of participation costs, however, have received little attention, although they are used more often in practice[2], like the auctions for tendering, particularly for government contracts and auctions for mining leases. The difficulty partly lies in the fact that in first price auctions, bidding strategies are not so explicit, as compared with the strategies in second price auctions. Bidders in first price auctions no longer bid their true valuations. The degree of shading relies heavily on who others enter the auction and the information inferred from the entrance behavior of those bidders. The effect of the information inferred on the bidding strategy of first price auctions is greater than that on second price auctions. Moreover, when bidders use different thresholds to enter an auction, the valuation distributions updated from their entrance behavior are different so that there may be no explicit bidding function and some bidders may use mixed strategies. As such, it is more technically difficult to solve the cutoff strategy since it is determined by the expected revenue of participating in the auction at the thresholds, which in turn depends on the more complicated bidding functions of bidders who submit bids.

[2]Samuelson [37] studies the entrance equilibrium of first price competitive procurement auctions and related welfare problem, focusing on the symmetric cutoff threshold.

There are some studies on equilibrium behavior in economic environments with different valuation distributions which can be used to study the equilibria of first price auctions with participation costs. Kaplan and Zamir [17, 18] discuss the properties of bidding functions when valuations are uniformly distributed with different supports. Martinez-Pardina [25] study the first price auction in which bidders' valuations are common knowledge. They show that in equilibrium bidders whose valuations are common knowledge randomize their bids.

In this chapter, we investigate Bayesian-Nash equilibria of sealed-bid first price auctions in the independent private values environment with participation costs. We assume bidders know their valuations and participation costs before they make their decisions. Participation costs are assumed to be the same across all the bidders.

When bidders are homogenous, there is a unique symmetric equilibrium. We show that there is no other equilibrium when valuation distribution functions are concave. However, when valuation distribution functions are elastic at the symmetric equilibrium, there always exists an asymmetric equilibrium. It may be remarked that, when a distribution function is strictly convex, it is elastic everywhere, specifically at the symmetric equilibrium, and therefore there exists an asymmetric equilibrium. Moreover, when bidders are in two different groups, the cutoffs used by one group can always be different from those used by the other group.

The existence of asymmetric equilibria has important consequences for the strategic behavior of bidders and the efficiency of the auction mechanism. When an auction has a participation cost, a bidder would expect less bidders to submit their bids. When symmetric equilibrium is unique, every bidder has to follow the symmetric cutoff and has no other choices. However, when asymmetric equilibria exist, bidders may choose an equilibrium that is more desirable. In this case, some bidders may form a collusion to cooperate at the entrance stage by choosing a smaller cutoff point that may

decrease the probability that other bidders enter the auction, and consequently, may reduce the competition in the bidding stage. An asymmetric equilibrium may become more desirable when an auction can run repeatedly. Also, an asymmetric equilibrium may be ex-post inefficient. The item being auctioned is not necessarily allocated to the bidder with the highest valuation.

We also consider the existence of equilibria in an economy with heterogenous bidders in the sense that the distribution functions are different. Specifically, we consider the case where one distribution (called a weak bidder) is first order dominated by another (called a strong bidder). We concentrate on equilibria that the bidders in the same group use the same threshold. We show that there is always an equilibrium in which the strong bidders are more likely to enter the auction by using a smaller cutoff point for valuations. When the distribution functions are concave, the equilibrium is unique. However, when the distribution functions for the weak bidders are strictly convex, and the participation costs are sufficiently large, there exists an equilibrium in which weak bidders are more likely to enter the auction.

The remainder of the chapter is structured as follows. Section B presents a general setting of economic environment. Section C studies the existence and uniqueness of equilibria for homogenous bidders. Section D studies equilibria for heterogenous bidders. Concluding remarks are provided in Section E. All the proofs are presented in Section F.

B. Economic Environment

We consider an independent private values economic environment with one seller and $n \geq 2$ risk-neutral buyers (bidders). The seller is also risk-neutral and has an indivisible object to sell to one of the buyers. The seller values the object as 0. Each

buyer i's valuation for the object is v_i $(i = 1, 2, ..., n)$, which is private information to the other bidders. It is assumed that v_i is independently distributed with a cumulative distribution function $F_i(\cdot)$ that has continuously differentiable density $f_i(\cdot) > 0$ everywhere with support $[0, 1]$.

The auction format is the sealed-bid first price auction. The bidder with the highest bid wins the auction and pays the price equal to his bid. His payoff is equal to the difference between his valuation and the price. The other bidders have zero payoff from submitting a bid. If the highest bid is submitted by more than one bidder, there is a tie which is broken by a fair lottery.

There is a participation cost, common to all bidders, denoted by $c \in (0, 1)$. Bidders must incur c in order to submit bids. It is assumed that each bidder knows his own valuation and who will participate, but does not know the others' valuations so that we are in the interim information setting. Specifically, the timing of the game is as follows:

- Nature draws a valuation v_i for each bidder i and tells the bidder only what his own valuation is.

- Bidder i decides whether or not to submit a bid. If he chooses to submit a bid, he pays the participation cost c which is not refundable, otherwise the game ends for him.

- All the bidders who pay the participation costs observe who others also participate in the auction and submit bids. The item is awarded to the bidder who submits the highest bid and pays his own bid. If more than one bidder submit the highest bid, the allocation is determined by a fair lottery.

The individual action set for any bidder can be characterized as $No \cup [0, 1]$, where "No" denotes not submitting a bid. Bidder i incurs the participation cost c if

and only if his action is different from "*No*." While it is always a weakly dominant strategy to bid one's true valuation in second price auctions, this is not true for first price auctions. In first price auctions, a bidder may submit a bid that may not be his true valuation. Nevertheless, given the strategies of all other bidders, a bidder's expected revenue from participating in the auction is a non-decreasing function of his valuation. Thus bidders use the cutoff strategy[3], i.e., a bidder submits a bid if and only if his valuation is greater than or equal to a cutoff point and does not enter otherwise[4].

An equilibrium strategy whether to participate is then given by a profile of the bidders' cutoff points, which are a vector of the minimum valuations for each bidder i to cover the cost. Let $v^* = (v_1^*, \ldots, v_n^*)$ denote the profile of bidders' cutoff points and $S_i(v^*)$ denote the set of bidders who also participate in the auction beside bidder i. The bidding decision function $b_i(\cdot)$ of each bidder is characterized by

$$b_i(v_i, v^*, S_i(v^*)) = \begin{cases} \lambda_i(v_i, v^*, S_i(v^*)) & \text{if } 1 \geq v_i \geq v_i^* \\ No & \text{if } v_i < v_i^*, \end{cases}$$

where $\lambda_i(v_i, v^*, S_i(v^*))$ is a contingent bidding function when bidder i participates in the auction. Note that, if bidder i enters the auction while all the others do not enter, bidder i will bid zero. If some other bidders also participate in the auction, the bid depends on the cutoff points and the valuation distributions of all others. For notational simplicity, we use $b_i(v_i, v^*)$ to denote $b_i(v_i, v^*, S_i(v^*))$ and $\lambda_i(v_i, v^*)$ to

[3]In Lu and Sun [24], they show that for any auction mechanism with participation costs, the participating and nonparticipating types of any bidder are divided by a nondecreasing and equicontinuous shutdown curve. Thus in our framework, when participation cost is given, the participating and nonparticipating types of any bidder can be divided by a cutoff value and the threshold form is the only form of equilibria.

[4]In Milgrom and Weber [33], the term of "screening level" is used instead of using "cutoff point."

denote $\lambda_i(v_i, v^*, S_i(v^*))$ in the remainder of the chapter.

For the game described above, each bidder's action is to choose a cutoff and decide how to bid when he participates. Thus, a (Bayesian-Nash) equilibrium of the game is composed of bidders' cutoff strategies, together with participants' bidding strategies.

Formally, we have the following definitions:

Definition 5 An equilibrium

$$(v^*, \mathbf{b}(v_i, v^*)) = ((v_1^*, b_1(v_i, v^*)), \ldots, (v_n^*, b_n(v_i, v^*))) \in \mathbb{R}_+^{2n}$$

is a profile of cutoff points together with optimal bidding functions such that each bidder i's action is optimal, given others' strategies.

Note that, once the cutoff points are given, for those bidders who participate in the auction, the game is reduced to the standard first price auction and the optimal bidding functions are uniquely determined (see Maskin and Riley [29]). As such, it is sufficient for us to focus on cutoff points $v^* = (v_1^*, \ldots, v_n^*) \in \mathbb{R}_+^n$ to describe the equilibrium. All of our results about uniqueness or multiplicity of equilibria, should be interpreted in terms of cutoffs, accordingly.

As usual, when bidders' distribution functions are the same; i.e., $F_1(\cdot) = F_2(\cdot) = \ldots = F_n(\cdot) = F(\cdot)$, we introduce the definition of symmetric and asymmetric equilibria focusing on the cutoffs:

Definition 6 For the economic environment with the same distribution functions, an equilibrium $v^* = (v_1^*, \ldots, v_n^*) \in \mathbb{R}_+^n$ is a symmetric (resp. asymmetric) equilibrium if the bidders have the same cutoff points; i.e., $v_1^* = v_2^* = \ldots = v_n^*$, (resp. different cutoff points). Denote the symmetric equilibrium by $v_s = (v^s, \ldots, v^s)$.

Remark 17 It may be worth to mention the following remarks on the cutoff points:

(1) $v_i^* > 1$ means that bidder i will never participate in the auction, no matter what his valuation is. This is the case where the bidder's revenue from participating in the auction is less than c even when $v_i = 1$.

(2) When $v_i^* < v_i \leq 1$, bidder i will enter the auction and submit a bid $\lambda_i(v_i, v^*)$. When $v_i = v_i^*$, bidder i is indifferent between participating in the auction and holding out. For discussion convenience, we assume he enters the auction. When $v_i < v_i^*$, bidder i does not participate in the auction.

(3) $v_i^* \geq c$.

(4) As shown in Cao and Tian [3], $v_i^* \leq 1$ for at least one bidder i.

Note that, once a bidder enters the auction, he can observe who has also entered the auction and thus can update his belief about others' valuation distributions. If we observe that bidder i participates in the auction, it can be inferred that bidder i's value is bigger than or equal to v_i^*. Then, by Bayes's rule, bidder i's value is distributed on $[v_i^*, 1]$ with

$$Pr(\xi \leq v | v \geq v_i^*) = \frac{Pr(v_i^* \leq \xi < v)}{Pr(\xi \geq v_i^*)} = \frac{F_i(v) - F_i(v_i^*)}{1 - F_i(v_i^*)}.$$

The corresponding density function is given by $\frac{f_i(v)}{1-F_i(v_i^*)}$.

C. Homogenous Bidders

In this section we analyze the case in which bidders' valuations are drawn from the same distribution function; i.e., $F_i(.) = F(.)$ for all i. We first study the symmetric equilibrium. For any two bidders who use the same cutoff point v^s, the supports of their updated valuation distributions have the same lower bound when they both

participate in the auction. Then the minimal bids they submit should be equal. Thus when $v_i = v^s$, bidder i can only win the item when all others do not participate. In equilibrium we have

$$c = v^s F(v^s)^{n-1}.$$

Since $\rho(v) = vF(v)^{n-1} - c$ is an increasing function of v with $\rho(0) < 0$ and $\rho(1) > 0$, there exists a unique symmetric equilibrium. To illustrate how the bidders submit bids when they face different number of other bidders who enter the auction, consider the following example:

Example 7 Suppose $F(v)$ is uniform on $[0, 1]$. Then by $v^s F(v^s)^{n-1} = c$ we have $v^s = \sqrt[n]{c}$. Then when $v_i \geq \sqrt[n]{c}$, bidding function for i is $\lambda_i(v_i, v^*) = v_i - \frac{v_i - \sqrt[n]{c}}{1+S_i(v^*)}$ if $S_i(v^*) \in \{1, 2, \cdots, n-1\}$ and zero if $S_i(v^*) = 0$. Otherwise, bidder i will not participate in the auction. Hence, the unique symmetric equilibrium is $(\sqrt[n]{c}, \sqrt[n]{c} \cdots, \sqrt[n]{c})$ and the bidding function is given by

$$b_i(v_i, v^*) = \begin{cases} \lambda_i(v_i, v^*) & 1 \geq v_i \geq \sqrt[n]{c} \\ No & v_i < \sqrt[n]{c}, \end{cases}$$

where

$$\lambda_i(v_i, v^*) = \begin{cases} 0 & \text{if } S_i(v^*) = 0 \\ v_i - \frac{v_i - \sqrt[n]{c}}{1+S_i(v^*)} & \text{if } S_i(v^*) \in \{1, 2, \cdots, n-1\}. \end{cases}$$

Now we consider the existence of asymmetric equilibria. Suppose there are only two different cutoff points used by the bidders. Bidders $i = 1, \ldots, m$ use v_1^* and bidders $j = m+1, \ldots, n$ use v_2^* as the cutoff point. Without loss of generality, we assume $v_1^* < v_2^*$. By Remark 1, we must have $v_1^* \leq 1$. Thus we divide the bidders into two types or groups. Bidders in type 1 use v_1^* and bidders in type 2 use v_2^* as

their cutoffs separately.

When bidder i in group 1 participates in the auction, his updated valuation is distributed on $[v_1^*, 1]$ with cumulative distribution function $G_i(v) = \frac{F(v)-F(v_1^*)}{1-F(v_1^*)}$, and when bidder j in group 2 participates in the auction, her updated valuation is distributed on $[v_2^*, 1]$ with cumulative distribution function $G_2(v) = \frac{F(v)-F(v_2^*)}{1-F(v_2^*)}$. The two distributions have the same upper bounds and different lower bounds. Thus if both types of bidders participate in the auction, the bidders are involved in an asymmetric first price auction in the sense that they have valuation distributions with different supports. To get the expected revenue at the cutoffs, we need to know how the bidders bid when there are both types of bidders participating in the auction.

Assume that a bidder with zero probability of winning bids his true value when he participates[5]. Then, by Maskin and Riley [29], there is a unique optimal bidding strategy, which is characterized in the following lemma:

Lemma 16 *Suppose k_1 bidders in type 1 whose values are distributed on the interval $[v_1^*, 1]$ with cumulative distribution function $G_1(v) = \frac{F(v)-F(v_1^*)}{1-F(v_1^*)}$ and k_2 bidders in type 2 whose values are distributed on the interval $[v_2^*, 1]$ with cumulative distribution function $G_2(v) = \frac{F(v)-F(v_2^*)}{1-F(v_2^*)}$ participate in the auction, where $v_1^* < v_2^*$. Let $\underline{b} = \max \arg\max_b (F(b) - F(v_1^*))^{k_1} (F(b) - F(v_2^*))^{k_2-1} (v_2^* - b)$. The optimal inverse bidding functions $v_1(b)$ and $v_2(b)$ are uniquely determined by*

(1) $v_1(b) = b$ for $v_1^* \leq b \leq \underline{b}$;

[5]Without this assumption a bidder with value v_i, who in optimum has zero probability of winning, can sometimes bid more than his value. However, this bidding strategy can be eliminated by a trembling-hand argument. Once a bidder bids above his value, he may have a positive probability to win the object which gives him a negative revenue. For a bidder, bidding below his value when he has zero probability of winning can also be supported in an optimal bidding strategy. However the allocation is the same as the optimal bidding strategy where he bids his value. For simplicity, we eliminate it.

(2) for $\underline{b} < b \leq \bar{b}$, the inverse bidding functions are determined by the following differential equation system:

$$\begin{cases} \frac{k_1 f(v_1(b))v_1'(b)}{F(v_1(b))-F(v_1^*)} + \frac{(k_2-1)f(v_2(b))v_2'(b)}{F(v_2(b))-F(v_2^*)} = \frac{1}{v_2(b)-b} \\ \frac{(k_1-1)f(v_1(b))v_1'(b)}{F(v_1(b))-F(v_1^*)} + \frac{k_2 f(v_2(b))v_2'(b)}{F(v_2(b))-F(v_2^*)} = \frac{1}{v_1(b)-b}. \end{cases}$$

with boundary conditions $v_2(\underline{b}) = v_2^*$, $v_1(\underline{b}) = \underline{b}$ and $v_1(\bar{b}) = v_2(\bar{b}) = 1$.

By Lemma 16, bidders in type 2 have an advantage in distribution so that they can benefit from the auction. Indeed, for a bidder in type 2 with any value on her support, she has a positive probability to win the auction. However, bidders in type 1, when $v_1 \in [v_1^*, \underline{b})$, have no chance to win the auction when any bidder in type 2 also submits a bid. From the above lemma, when there are two bidders using the same cutoff participating in the auction, the bidder with the value at the cutoff has zero expected revenue from the auction.

Remark 18 When there are k bidders in type 1 and one bidder in type 2 participating in the auction, the lower bound of the bid submitted by bidders in type 2 is $\max \arg\max_b (F(b) - F(v_1^*))^k (v_2^* - b)$.

Bidder $i \in \{1, \ldots, m\}$ with $v_i = v_1^*$ can only win the object when none of the others enters the auction. He bids zero when he is the only participant. Indeed, if another bidder $i' \in \{1, 2, \ldots, i-1, i+1, \ldots, m\}$ participates, we have $v_{i'} \geq v_i = v_1^*$, so $\lambda_{i'}(v_{i'}, v^*) \geq \lambda_i(v_i, v^*)$. Then bidder i gains zero revenue from the participation. When any bidder $j = m+1, \ldots, n$ also enters the auction, we have $v_j \geq v_2^* > v_1^* = v_1$, bidder i will lose the auction for sure.

Thus, at equilibrium we then have

$$c = v_1^* F(v_1^*)^{m-1} F(v_2^*)^{n-m}.$$

For bidder $j \in \{m+1, ..., n\}$ with $v_j = v_2^*$, she can bid zero and has revenue v_2^* when none of others enters the auction. If other bidders in type 2 enter the auction, she will lose the bid. If $k \leq m$ bidders in type 1 enter the auction, the optimal bid $\underline{b}_k$ for bid j is decided by

$$\underline{b}_k = \max \arg\max_b (F(b) - F(v_1^*))^k (v_2^* - b).$$

The first order condition for $\underline{b}_k$ gives

$$\underline{b}_k + \frac{F(\underline{b}_k) - F(v_1^*)}{kf(\underline{b}_k)} = v_2^*.$$

$\underline{b}_k$ is chosen with probability $C_m^k F(v_1^*)^{m-k}(1 - F(v_1^*))^k$. C_m^k is the combination number for choosing k candidates from the n items that are available and $C_m^k = \frac{m!}{k!(m-k)!}$. Thus in equilibrium we have

$$c \geq v_2^* F(v_1^*)^m F(v_2^*)^{n-m-1} + F(v_2^*)^{n-m-1} \sum_{k=1}^{m} C_m^k F(v_1^*)^{m-k} (F(\underline{b}_k) - F(v_1^*))^k (v_2^* - \underline{b}_k),$$

where the first part is the expected revenue when none of others enters the auction, which happens with probability $F(v_1^*)^m F(v_2^*)^{n-m-1}$; the second part is the expected revenue when no bidders in type 2 enters the auction and there are exactly $k \leq m$ bidders in the auction, which happens with probability $F(v_2^*)^{n-m-1} F(v_1^*)^{m-k}$. The inequality holds whenever bidders in type 2 never participate in the auction, i.e., $v_2^* > 1$.

Summarizing our discussion, we have the following proposition.

Proposition 12 *In an economic environment with n homogeneous bidders,*

(1) there is a unique symmetric equilibrium in which all bidders use the same cutoff point v^s that is determined by $v^s F(v^s)^{n-1} = c$;

(2) if $F(\cdot)$ is elastic at v^s, i.e., $F(v^s) < v^s f(v^s)$, then, for any $m \in$

$\{1, 2, ..., n-1\}$, *there exists an asymmetric equilibrium where* m *bidders use the cutoff point* v_1^* *and the others use the other cutoff point* v_2^* *that satisfy*

$$\begin{aligned} c &= v_1^* F(v_1^*)^{m-1} F(v_2^*)^{n-m}, \\ c &\geq v_2^* F(v_1^*)^m F(v_2^*)^{n-m-1} \\ &+ F(v_2^*)^{n-m-1} \sum_{k=1}^{m} C_m^k F(v_1^*)^{m-k} (F(\underline{b}_k) - F(v_1^*))^k (v_2^* - \underline{b}_k), \end{aligned}$$

with equality whenever $v_2^* \leq 1$ *and* $v_1^* < v^s < v_2^*$, *where*

$$\underline{b}_k = \max \arg \max_b (F(b) - F(v_1^*))^k (v_2^* - b);$$

(3) if $F(\cdot)$ *is concave, there exists no asymmetric equilibrium.*

In words, when $F(\cdot)$ is elastic at the symmetric equilibrium, if the bidders are randomly divided into two groups, there is an equilibrium where all bidders within one group use the same cutoff that is different from the cutoff used by bidders in the other group. One implication of this result is that some bidders can coordinate by choosing a smaller cutoff threshold so that they can reduce the probability that the others enter the auction which, in turn, can reduce the competition among the bidders who participate in the auction. However, when $F(\cdot)$ is concave, there is no such equilibrium.

Remark 19 When $F(.)$ is strictly convex, it is elastic at any point on its support, specifically at v^s, and therefore there exists an asymmetric equilibrium. For instance, when $c = 0.1$, $F(v) = v^2$ and $n = 2$, there is a symmetric equilibrium (0.466, 0.466) and an asymmetric equilibrium $(0.141, 0.842)$.

The intuition for the existence of asymmetric equilibria when $F(.)$ is strictly convex is the following. When bidders in type 1 use a smaller cutoff v_1^* to enter the

auction, the expected payoff for any bidder j in type 2 with a lower value to enter the auction is smaller even when he wins the auction. This is true because bidder j's expected payment to the seller, which is equal to the expected value of the highest valuation of bidders in type 1, is higher when $F(.)$ is strictly convex. In this case, bidders in type 1 would stay out of the auction by using a larger cutoff point, and thus we have an asymmetric equilibrium. When $F(.)$ is concave, the above argument cannot be applied. Now the expected payment of bidder j to the seller when he wins is smaller since other bidders tend to have smaller valuations. Bidder j with lower value may also benefit from participating in the auction which can prevent bidders in type 1 from entering the auction with a smaller cutoff value.

Remark 20 Further remarks can be given as follows:

(1) When the lower bound of support is positive, $F(.)$ may be elastic at v^s even if $F(\cdot)$ is concave on its support. Thus we may have an asymmetric equilibrium.

(2) Since $v_i^* \geq c$, the condition that $F(\cdot)$ is concave can be weakened to $F(v) \geq vf(v)$ for all $v \in [c, 1]$.

(3) As $c \to 0$, one can check that both symmetric equilibrium and asymmetric equilibrium (if it exists) go to zero. Thus when $c = 0$, all bidders participate in the auction.

(4) Similar to Cao and Tian [3, 4], one can study equilibrium properties when bidders may have differentiated participation costs or when both values and participation costs are private information.

It may be remarked that asymmetric equilibria inevitably lead to inefficient allocation in first price auctions with participation costs. Indeed, like second price auctions with participation costs, they are not efficient because the object may be

not allocated to the bidder with the highest valuation when bidders use different cutoff points. However, unlike second price auctions with participation costs, they are not even weakly efficient (Miralles [34]). The bidder who wins the object may not have the highest valuation among those who participate. To see this, suppose $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ are the equilibrium bidding functions for any two bidders who use different cutoff points. Suppose $\lambda_1(v) < \lambda_2(v)$. By the continuity of the functions, $\lambda_1(v+\epsilon) < \lambda_2(v-\epsilon)$ when ϵ is sufficiently small. Thus bidder 2 will win the object even though he has a lower valuation. In conclusion, we have the following proposition:

Proposition 13 *The first price auctions with participation costs are not efficient and are not even weakly efficient at asymmetric equilibrium.*

In reality, we can expect that when the auction can run repeatedly, bidders may use asymmetric equilibria at earlier periods while using the symmetric equilibrium at later periods. We now investigate the welfare effect of participation costs on sellers when it is just one shot game, focusing on the symmetric equilibrium.[6] When bidders use the same threshold and participate in the auction, the optimal bidding function is unique, which is symmetric and monotonic increasing, given by

$$\lambda(v_i, v_s) = v_i - \frac{\int_{v^s}^{v_i} (F(y) - F(v^s))^{k-1} dy}{(F(v_i) - F(v^s))^{k-1}}$$

when there are $k \geq 2$ participants. Thus the seller's expected revenue from the auction is

$$R = \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} \int_{v^s}^{1} (v_i - \frac{\int_{v^s}^{v_i} (F(y) - F(v^s))^{k-1} dy}{(F(v_i) - F(v^s))^{k-1}}) k (F(v_i) - F(v^s))^{k-1} f(v_i) dv_i,$$

[6]The welfare analysis for the case of asymmetric equilibrium is much more complicated. Letting the bidders know the number of other bidders who submit bids may have different welfare implications for the sellers. We leave the welfare analysis at the asymmetric equilibrium for future research.

and consequently, we have

$$R = n(n-1)\int_{v^s}^{1}(1-F(x))xf(x)F(x)^{n-2}dx \tag{4.1}$$

with integration by parts and changing the order of the integration in the double integrals[7].

There are several effects of increasing the magnitude of participation costs. First, as c increases, the probability to have k participants decreases. Secondly, participants bid more regressively. The reason is that to win the auction, a player has to bid the expected value of the highest among his opponents with values between v^s and 1. Lower participation reduces the expected revenue while more regressive bidding increases it. One might conjecture that there exists an optimal participation cost that will maximize the seller's expected revenue. However from the equation above, this is not true, which leads to the following result:

Proposition 14 *At the symmetric equilibrium, the seller's expected revenue decreases as the participation cost c increases.*

One implication of the above proposition is that in reality, the seller may give the potential bidders some subsidy to encourage them to participate in the auction to increase the expected revenue.

Remark 21 When participation costs are part of the seller's revenue, like the entry fee, the above conclusion no longer holds. In this case, the seller's expected revenue is

$$n(n-1)\int_{v^s}^{1}(1-F(x))xf(x)F(x)^{n-2}dx + nc(1-F(v^s)),$$

[7]See details in the Section F of this chapter.

which is equivalent to

$$n(n-1)\int_{v^s}^{1}(1-F(x))xf(x)F(x)^{n-2}dx + nv^sF(v^s)^{n-1}(1-F(v^s)).$$

First order condition $v^sf(v^s) = 1 - F(v^s)$ determines the optimal entry fee from the perspective of seller.

Remark 22 Menezes and Monteiro [32] consider first price auctions with participation costs. However, they adopt a different specification on information structure. A bidder does not know who others are in the auction when he is to submit a bid. Besides, they only focus on the symmetric equilibrium in which all bidders use the same cutoff point (which is equal to v^s) and submit bids via the same bidding function. They mainly focus on comparing the revenue from first price auctions and second price auctions and investigate the effect of the number of potential bidders on seller's revenue. Within their framework, when a bidder decides to participate in the auction, he will bid as if all others are in the auction since he cannot observe any other's entrance behavior and the bidding function is given by

$$\lambda^*(v_i, v_s) = \frac{\int_{v^s}^{v_i}(n-1)yF(y)^{n-2}f(y)dy}{F(v)^{n-1}}$$

when $v \geq v^s$, and consequently the expected revenue is given by

$$\widetilde{R} = \int_{v^s}^{1}\lambda^*(v_i, v_s)nF^{n-1}(x)f(x)dx,$$

which can be shown to be equivalent to (4.1). Thus at symmetric equilibrium, letting the bidders observe or not observe who else participates will give the seller the same expected revenue.

D. Heterogenous Bidders

Now consider the case where we have n_1 strong bidders with value distribution $F_1(\cdot)$ and n_2 weak bidders with value distribution $F_2(\cdot)$. The total number of bidders is $n = n_1 + n_2$. We concentrate on type-symmetric equilibrium in which all strong (resp., weak) bidders use the same cutoff point.

We first assume, provisionally, that the cutoff points v_1^* and v_2^* satisfy $v_1^* < v_2^*$. Then for a strong bidder i with $v_i = v_1^*$, he can only get the object when all the others do not participate in the auction. (If any strong bidder i' enters the auction, he must have a value greater than v_1^* and thus bids higher than bidder i; or if any weak bidder j enters, then it must be the case that $v_j \geq v_2^* > v_1^*$. As seen in the previous section, bidder i will lose the item for sure.) Thus, at equilibrium we have

$$c = v_1^* F_1(v_1^*)^{n_1 - 1} F_2(v_2^*)^{n_2}.$$

For a weak bidder j with $v_j = v_2^*$, we have the following three cases:

Case 1: All the other bidders do not enter the auction. Then bidder j bids zero and gains a surplus of v_2^*. The probability of this event is $F_1(v_1^*)^{n_1} F_2(v_2^*)^{n_2 - 1}$. In this case the expected revenue for bidder j is $v_2^* F_1(v_1^*)^{n_1} F_2(v_2^*)^{n_2 - 1}$.

Case 2: At least another weak bidder enters. Then bidder j will lose the auction, deriving zero revenue from participating.

Case 3: None of the other weak bidders enters and there are exactly $k \in \{1, 2, ..., n_1\}$ strong bidders participating in the auction. In this case bidder j with value v_2^* will submit a bid

$$\underline{b}_k = \max \arg\max_b [(F_1(b) - F_1(v_1^*))^k (v_2^* - b)].$$

The first order condition for $\underline{b}_k$ gives

$$\underline{b}_k + \frac{F_1(\underline{b}_k) - F_1(v_1^*)}{k f_1(\underline{b}_k)} = v_2^*.$$

The probability of this event is $C_{n_1}^k F_1(v_1^*)^{n_1-k}(1 - F_1(v_1^*))^k$. The expected revenue in this case is $C_{n_1}^k F_1(v_1^*)^{n_1-k} F_2(v_2^*)^{n_2-1}(F_1(\underline{b}_k) - F_1(v_1^*))^k(v_2^* - \underline{b}_k)$.

Then at equilibrium we have

$$c \geq v_2^* F_1(v_1^*)^{n_1} F_2(v_2^*)^{n_2-1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(v_1^*)^{n_1-k} F_2(v_2^*)^{n_2-1}(F_1(\underline{b}_k) - F_1(v_1^*))^k(v_2^* - \underline{b}_k).$$

Proposition 15 *When $F_1(v) < F_2(v)$ for all $v \in (0,1)$, there always exists a type-symmetric equilibrium in which $v_1^* < v_2^*$. Further, the type-symmetric equilibrium $v_1^* < v_2^*$ is unique when both distributions are concave.*

Similarly for the case where $v_1^* \geq v_2^*$, at equilibrium we have

$$c = v_2^* F_2(v_2^*)^{n_2-1} F_1(v_1^*)^{n_1},$$

and

$$c \geq v_1^* F_2(v_2^*)^{n_2} F_1(v_1^*)^{n_1-1} + \sum_{k=1}^{n_2} C_{n_2}^k F_2(v_2^*)^{n_2-k} F_1(v_1^*)^{n_1-1}(F_2(\underline{b}_k) - F_2(v_2^*))^k(v_1^* - \underline{b}_k),$$

where the first part on the right side of the inequality is the expected revenue when none of the others (no matter whether they are strong or weak bidders) participates in the auction. The second part is the expected revenue when at least one weak bidder participates and no other strong bidders participates.

Proposition 16 *In the heterogenous economy involving any number of bidders,*

(1) if $F_2(\cdot)$ is concave, there is no type-symmetric equilibrium with $v_2^* \leq v_1^*$;

(2) if $F_2(\cdot)$ is strictly convex, there exists $c^* < 1$ such that there exists a type-symmetric equilibrium with $v_2^* \leq v_1^*$ for all $c > c^*$.

This result indicates that, when the participation cost is sufficiently large, strong bidders may choose a higher cutoff point. The intuition behind this is that, when c is sufficiently large and the weak bidder is more likely to have higher valuation, the expected revenue of the strong bidder from entering the auction is low. Strong bidders' advantage in valuations is attenuated by the weak bidders' value distribution and a higher participation cost.

E. Conclusion

This chapter investigates (Bayesian-Nash) equilibria of sealed-bid first price auctions with participation costs. We focus on equilibria in cutoff strategies. Once a bidder participates in the auction, the bidding strategy depends on the valuation distributions and cutoff points of other bidders.

When bidders are ex-ante homogeneous with the same valuation distribution, there always exists a unique symmetric equilibrium in which all bidders use the same cutoff to enter the auction and there may also exist an asymmetric equilibrium. In particular, there is no asymmetric equilibrium when $F(\cdot)$ is concave, and there exists an asymmetric equilibrium when $F(\cdot)$ is elastic at the symmetric equilibrium. When bidders can be ranked by their valuation distributions, we find that bidders with higher probability to have higher valuations are more likely to enter the auction. However the opposite can be obtained when the participation cost is sufficiently large and weak bidders's valuation distributions are strictly convex.

In the presence of participation costs, not all bidders will participate in the auction and the seller's expected revenue decreases as the participation costs increase.

Then, it may be profitable for the sellers to subsidize the buyers to encourage their participating in the auction. How to implement this should be a potentially interesting question which will be left for future research.

F. Proofs of the Main Results

Proof of Lemma 16:

Proof. Denote the inverses of the bidding function as $v_1(b)$ with support $[\underline{b}_1, \overline{b}_1]$ and $v_2(b)$ with support $[\underline{b}_2, \overline{b}_2]$. Let $(\underline{b}, \overline{b}]$ be the range in which a bidder has a positive probability to win the object if he participants in the auction. First from Maskin and Reiley [29], the upper endpoint of the support of the distributions of the valuations is the the same for all bidders and thus the upper endpoints in the supports of all buyers' equilibrium bid distributions are the same. Thus $\overline{b}_1 = \overline{b}_2 = \overline{b}$ and $v_1(\overline{b}) = v_2(\overline{b}) = 1$.

Also from Maskin and Reiley [29] we have $\underline{b}_1 < \underline{b}_2 = \underline{b}$ which indicates that the minimum bid of a bidder in type 1 is always less than that of bidders in type 2 since bidders in type 2 have an advantage in valuation distribution.

Below $\underline{b}$, type 1 bidder has no chance to win the auction and bids his true value, so $v_1(\underline{b}) = \underline{b}$. For bidders in type 2, when $v_2 = v_2(\underline{b}) = v_2^*$, bidding $\underline{b}$ is his best strategy. Again, from Maskin and Reiley (2003), $\underline{b} = \max \arg\max_b (F(b) - F(v_1^*))^{k_1} (F(b) - F(v_2^*))^{k_2 - 1} (v_2^* - b)$.

In the interval $[\underline{b}, \overline{b}]$, a bidder in type i bids b which is determined by the following maximization problem:

$$\max_b \left(\frac{F(v_j(b)) - F(v_j^*)}{1 - F(v_j^*)}\right)^{k_j} \left(\frac{F(v_i(b)) - F(v_i^*)}{1 - F(v_i^*)}\right)^{k_i - 1} (v_i - b), j \neq i.$$

First order conditions give us

$$\begin{cases} \frac{k_1 f(v_1(b))v_1'(b)}{F(v_1(b))-F(v_1^*)} + \frac{(k_2-1)f(v_2(b))v_2'(b)}{F(v_2(b))-F(v_2^*)} = \frac{1}{v_2(b)-b} \\ \frac{(k_1-1)f(v_1(b))v_1'(b)}{F(v_1(b))-F(v_1^*)} + \frac{k_2 f(v_2(b))v_2'(b)}{F(v_2(b))-F(v_2^*)} = \frac{1}{v_1(b)-b}. \end{cases}$$

The boundary conditions for the differential equation system are $v_2(\underline{b}) = v_2^*$, $v_1(\underline{b}) = \underline{b}$ and $v_1(\bar{b}) = v_2(\bar{b}) = 1$. ■

Proof of Proposition 12:

Proof. (1) The existence and uniqueness of symmetric equilibrium is obvious. The proof is omitted here.

(2) Suppose $F(\cdot)$ is elastic at v^s so that $F(v^s) < v^s f(v^s)$. Consider the following two equations:

$$\begin{aligned} c &= xF(x)^{m-1}F(y)^{n-m} \\ c &\geq yF(x)^m F(y)^{n-m-1} + F(y)^{n-m-1}\sum_{k=1}^{m} C_m^k F(x)^{m-k}(F(\underline{b}_k) - F(x))^k (y - \underline{b}_k). \end{aligned}$$

where $\underline{b}_k$ satisfies $\underline{b}_k + \frac{F(\underline{b}_k)-F(x)}{kf(\underline{b}_k)} = y$, x corresponds to the cutoff point used by bidders in the first group, and y corresponds to the cutoff point used by bidders in the second group. Let v^s satisfy $c = v^s F(v^s)^{m-1}F(v^s)^{n-m}$. Define $x = \phi(y)$ implicitly from $c = xF(x)^{m-1}F(y)^{n-m}$. Notice that $\phi(y)$ is continuously differentiable and $\phi(v^s) = v^s$. Since $x \leq y$ we have $x = \phi(y)$ with $y \geq v^s$. Then we have

$$\phi'(y) = -\frac{(n-m)f(y)xF(x)}{(F(x) + (m-1)xf(x))F(y)},$$

and thus

$$\phi'(v^s) = -\frac{(n-m)v^s f(v^s)}{F(v^s) + (m-1)v^s f(v^s)}.$$

Define

$$h(y) = F(y)^{n-m-1}[yF(\phi(y))^m + \sum_{k=1}^{s} C_m^k F(\phi(y))^{m-k}(F(\underline{b}_k(y)) - F(\phi(y)))^k (y - \underline{b}_k(y))] - c.$$

with $y \geq v^s$. Notice that $h(y)$ is continuously differentiable and $\underline{b}_k(y) = v^s$ when $y = v^s$. So $h(v^s) = 0$. In order to have an asymmetric equilibrium, we only need to show that *either* there exists a $y^* \in (v^s, 1]$ such that $h(y^*) = 0$ (in which case we have $v_2^* = y^*$ and $v_1^* = h(v_2^*) < v^s$ as our asymmetric cutoff equilibrium.) *or* $h(1) < 0$ (in which case $v_2^* > 1$ and $v_1^* = c$.). So if $h(1) < 0$, then it is done.

Suppose $h(1) > 0$. Since $h(.)$ is continuous with $h(v^s) = 0$ and $h(1) > 0$, when $h(y)$ is decreasing at v^s, then there exists a $y^* \in (v^s, 1]$ such that $h(y^*) = 0$. This is true when $F(\cdot)$ is elastic at v^s. Indeed,

$$h'(y) = I(y) + F(y)^{n-m-1}[II(y) + \sum_{k=1}^{m} C_s^k (III(y) + IV(y))],$$

where

$$\begin{aligned}
I(y) &= (n-m-1)F(y)^{n-m-2} f(y)[yF(\phi(y))^m \\
&+ \sum_{k=1}^{m} F(\phi(y))^{m-k}(F(\underline{b}_k(y)) - F(\phi(y)))^k (y - \underline{b}_k(y))], \\
II(y) &= F(\phi(y))^m + y.mF(\phi(y))^{m-1} f(\phi(y))\phi'(y), \\
III(y) &= (m-k)F(\phi(y))^{m-k-1} f(\phi(y))\phi'(y)(F(\underline{b}_k(y)) - F(\phi(y)))^k (y - \underline{b}_k(y)), \\
IV(y) &= F(\phi(y))^{m-k}[k[F(\underline{b}_k(y)) - F(\phi(y))]^{k-1}(f(\underline{b}_k(y))\underline{b}_k'(y) - f(\phi(y))\phi'(x))(y - \underline{b}_k(y)) \\
&+ (F(\underline{b}_k(y)) - F(\phi(y)))^k (1 - \underline{b}_k'(y))].
\end{aligned}$$

When $x = y = v^s$, we have $\underline{b}_k(v^s) = v^s$. Then,

$$\begin{aligned} I(v^s) &= (n-m-1)F(v^s)^{n-m-2}f(v^s)v^sF(v^s)^m = (n-m-1)F(v^s)^{n-2}v^sf(v^s), \\ II(v^s) &= F(v^s)^m + v^s.mF(v^s)^{m-1}f(v^s)\phi'(v^s), \\ III(v^s) &= IV(v^s) = 0 \end{aligned}$$

and thus

$$h'(v^s) = F(v^s)^{n-2}[(n-m-1)v^sf(v^s) + mv^sf(v^s)\phi'(v^s) + F(v^s)].$$

Thus, $h'(v^s) < 0$ if and only if

$$|\phi'(v^s)| = \frac{(n-m)v^sf(v^s)}{F(v^s) + (m-1)v^sf(v^s)} > \frac{(n-m-1)v^sf(v^s) + F(v^s)}{mv^sf(v^s)},$$

which is true when $F(\cdot)$ is elastic at v^s. Indeed, when $F(\cdot)$ is elastic at v^s we have $v^sf(v^s) > F(v^s)$. So $F(v^s) + (m-1)v^sf(v^s) < mv^sf(v^s)$ and at the same time $(n-m)v^sf(v^s) > (n-m-1)v^sf(v^s) + F(v^s)$. Then if $h(1) > 0$, we have an asymmetric equilibrium in which $v_1^* < v^s < v_2^* \leq 1$, otherwise there is an asymmetric equilibrium in which bidders in group 2 never participate in the auction.

(3) When $F(\cdot)$ is concave, we prove the nonexistence of asymmetric equilibrium by way of contradiction. Suppose there is an asymmetric equilibrium with $v_1^* < v_2^*$. Then

$$\begin{aligned} c &= v_1^*F(v_1^*)^{m-1}F(v_2^*)^{n-m}, \\ c &\geq v_2^*F(v_1^*)^mF(v_2^*)^{n-m-1} + F(v_2^*)^{n-m-1}\sum_{k=1}^{m} C_s^kF(v_1^*)^{m-k}(F(\underline{b}_k) - F(v_1^*))^k(v_2^* - \underline{b}_k). \end{aligned}$$

One necessary condition for the system of these equations above to hold is

$$v_1^*F(v_1^*)^{m-1}F(v_2^*)^{n-m} \geq v_2^*F(v_1^*)^mF(v_2^*)^{n-m-1},$$

i.e., $\frac{F(v_2^*)}{v_2^*} \geq \frac{F(v_1^*)}{v_1^*}$, which cannot be true when $F(\cdot)$ is concave and $v_2^* > v_1^*$. Following

the same procedures above, we can prove there is no asymmetric equilibrium in which $v_1^* > v_2^*$. ■

Proof of Equation (4.1):

Proof. Rewrite

$$R = \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} \int_{v^s}^{1} (v_i - \frac{\int_{v^s}^{v_i} (F(y) - F(v^s))^{k-1} dy}{(F(v_i) - F(v^s))^{k-1}}) k(F(v_i) - F(v^s))^{k-1} f(v_i) dv_i$$

as

$$\begin{aligned} R &= \int_{v^s}^{1} \{v_i \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} k(F(v_i) - F(v^s))^{k-1} \\ &- \int_{v^s}^{v_i} \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} k(F(y) - F(v^s))^{k-1} dy\} dF(v_i). \end{aligned}$$

Integrating by parts for $\int_{v^s}^{v_i} \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} k(F(y) - F(v^s))^{k-1} dy$ and making simplifications, we have

$$\begin{aligned} R &= \int_{v^s}^{1} \int_{v^s}^{v_i} \sum_{k=2}^{n} C_n^k F(v_s)^{n-k} k(k-1)(F(y) - F(v^s))^{k-2} y dy f(v_i) dv_i \\ &= \int_{v^s}^{1} \int_{v^s}^{v_i} \sum_{k=2}^{n} C_{n-2}^{k-2} n(n-1) F(v_s)^{n-k} (F(y) - F(v^s))^{k-2} y dy f(v_i) dv_i \\ &= n(n-1) \int_{v^s}^{1} \int_{v^s}^{v_i} F(y)^{n-2} y dy f(v_i) dv_i \\ &= n(n-1) \int_{v^s}^{1} (1 - F(x)) x f(x) F(x)^{n-2} dx, \end{aligned}$$

where the second line comes from the fact that $C_n^k k(k-1) = n(n-1) C_{n-2}^{k-2}$ and the last line comes from changing the order of integration in the double integral. ■

Proof of Proposition 15:

Now consider the following two equations:

$$\begin{aligned} c &= xF_1(x)^{n_1-1}F_2(y)^{n_2} \\ c &\geq yF_1(x)^{n_1}F_2(y)^{n_2-1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}F_2(y)^{n_2-1}(F_1(\underline{b}_k) - F_1(x))^k(y - \underline{b}_k), \end{aligned}$$

with $c \leq x \leq y < 1$, where x corresponds to the cutoff point used by the strong bidders and y corresponds to the cutoff point used by the weak bidders. Let v_1^s satisfy $v_1^s F_1(v_1^s)^{n_1-1}F_2(v_1^s)^{n_2} = c$. Note that $\theta(v_1^s) = v_1^s F_1(v_1^s)^{n_1-1}F_2(v_1^s)^{n_2}$ is an increasing function of v_1^s with $\theta(1) = 1 > c$, so we have $v_1^s < 1$. For $y \geq v_1^s$, define $x = \phi(y)$ from $c = xF_1(x)^{n_1-1}F_2(y)^{n_2}$. Then x is a decreasing function of y and $\phi(v_1^s) = v_1^s$. Now let

$$\begin{aligned} h(y) &= yF_1(\phi(y))^{n_1}F_2(y)^{n_2-1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(\phi(y))^{n_1-k}F_2(y)^{n_2-1}(F_1(\underline{b}_k(y)) - F_1(\phi(y)))^k \\ &\quad (y - \underline{b}_k(y)) - c. \end{aligned}$$

Then $h(y)$ is a continuous function of $y \geq v_1^s$. The remainder of the proof is based on the following two lemmas:

Lemma 17 There always exists a type-symmetric equilibrium with $v_1^* < v_2^*$.

Proof. Note that we have $x \leq \underline{b}_k \leq y$. When $y = v_1^s$, we have $\underline{b}_k = v_1^s$. Then

$$h(v_1^s) = v_1^s F_1(v_1^s)^{n_1}F_2(v_2^s)^{n_2-1} - c < v_1^s F_1(v_1^s)^{n_1-1}F_2(v_1^s)^{n_2} - c = 0$$

since $F_1(v_1^s) < F_2(v_2^s)$ by assumption. We also have

$$h(1) = F_1(\phi(1))^{n_1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(\phi(1))^{n_1-k}(F_1(\underline{b}_k(1)) - F_1(\phi(1)))^k(1 - \underline{b}_k(1)) - c.$$

Now if $h(1) \geq 0$, then by the mean value theorem, there exists a $y = v_2^* \in (v_1^s, 1]$ such that $h(v_2^*) = 0$ so that there is an equilibrium in which $v_1^* = \phi(v_2^*) < v_1^s < v_2^* \leq 1$.

Otherwise if $h(1) < 0$, then there is an equilibrium in which $v_1^* = \phi(1) < 1$ and $v_2^* > 1$; i.e., weak bidders never participate in the auction. ■

Lemma 18 When $F_1(\cdot)$ and $F_2(\cdot)$ are both concave and $F_1(v) < F_2(v)$ for all $v \in (0,1)$, there exists a unique type-symmetric equilibrium with $v_1^* < v_2^*$.

Proof. Suppose $y \leq 1$. Substituting $c = xF_1(x)^{n_1-1}F_2(y)^{n_2}$ into

$$c = yF_1(x)^{n_1}F_2(y)^{n_2-1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}F_2(y)^{n_2-1}(F_1(\underline{b}_k) - F_1(x))^k(y - \underline{b}_k)$$

and making simplifications, we have

$$yF_1(x)^{n_1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k(y - \underline{b}_k) - xF_1(x)^{n_1-1}F_2(y) = 0. \quad (4.2)$$

We claim that the above equation implicitly defines x as a strictly increasing function of y. Consequently, it *either* has a unique intersection with $x = \phi(y)$ (which is strictly decreasing), *or* it does not intersect with $x = \phi(y)$, in which case the unique equilibrium is given by $x = \phi(1)$ and $y > 1$ (weak bidders never participate).

To see this, taking derivatives with respect to y (notice that $\underline{b}_k$ is also a function of y) on both sides of the above equation, we have

$$\begin{aligned}
0 &= F_1(x)^{n_1} + n_1 yF_1(x)^{n_1-1}f_1(x)\frac{dx}{dy} - xF_1(x)^{n_1-1}f_2(y) - F_2(y)(F_1(x)^{n_1-1} \\
&+ (n_1-1)xf_1(x)F_1(x)^{n_1-2})\frac{dx}{dy} \\
&+ \sum_{k=1}^{n_1} C_{n_1}^k\{(n_1-k)F_1(x)^{n_1-k-1}f_1(x)(F_1(\underline{b}_k) - F_1(x))^k(y - \underline{b}_k)\frac{dx}{dy} \\
&+ F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^{k-1}[(F_1(\underline{b}_k) - F_1(x))(1 - \underline{b}_k') \\
&+ k(y - \underline{b}_k)(f(\underline{b}_k)\underline{b}_k' - f_1(x)\frac{dx}{dy})]\},
\end{aligned}$$

where

$$\begin{aligned}(F_1(\underline{b}_k) - F_1(x))(1 - \underline{b}'_k) &+ k(y - \underline{b}_k)(f(\underline{b}_k)\underline{b}'_k - f_1(x)\frac{dx}{dy}) \\ &= F_1(\underline{b}_k) - F_1(x)) - k(y - \underline{b}_k)f_1(x)\frac{dx}{dy}\end{aligned}$$

by noting that $F_1(\underline{b}_k(y)) - F_1(x) = kf_1(\underline{b}_k(y))(y - \underline{b}_k(y))$. Thus we have

$$\begin{aligned}0 &= F_1(x)^{n_1} + n_1 y F_1(x)^{n_1-1} f_1(x)\frac{dx}{dy} - xF_1(x)^{n_1-1} f_2(y) - F_2(y)(F_1(x)^{n_1-1} \\ &+ (n_1 - 1)xf_1(x)F_1(x)^{n_1-2})\frac{dx}{dy} \\ &+ \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k-1} f_1(x)(F_1(\underline{b}_k) - F_1(x))^{k-1} \\ &\quad (y - \underline{b}_k)\frac{dx}{dy}\{n_1(F_1(\underline{b}_k) - F_1(x)) - kF_1(\underline{b}_k)\} \\ &+ \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k.\end{aligned}$$

Then

$$\frac{dx}{dy} = \frac{F_1(x)^{n_1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k - xF_1(x)^{n_1-1} f_2(y)}{-n_1 y F_1(x)^{n_1-1} f_1(x) - II + F_2(y)(F_1(x)^{n_1-1} + (n_1 - 1)xf_1(x)F_1(x)^{n_1-2})}$$

with

$$\begin{aligned}II &= \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k-1} f_1(x)(F_1(\underline{b}_k) - F_1(x))^{k-1}(y - \underline{b}_k)\{n_1(F_1(\underline{b}_k) - F_1(x)) - kF_1(\underline{b}_k)\} \\ &= I - \alpha,\end{aligned}$$

where

$$\begin{aligned}I &= n_1 \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k-1} f_1(x)(F_1(\underline{b}_k) - F_1(x))^k (y - \underline{b}_k), \\ \alpha &= \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k-1} f_1(x)(F_1(\underline{b}_k) - F_1(x))^{k-1}(y - \underline{b}_k)kF_1(\underline{b}_k) \geq 0.\end{aligned}$$

Now we prove the denominator and numerator are strictly positive separately. First we prove the numerator is positive. From equation (4.2), we have

$$yF_1(x)^{n_1} - xF_1(x)^{n_1-1}F_2(y) = -\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k(y - \underline{b}_k).$$

When $F_2(\cdot)$ is concave, we have

$$\begin{aligned} & F_1(x)^{n_1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k - xF_1(x)^{n_1-1}f_2(y) \\ \geq\ & F_1(x)^{n_1} + \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k - xF_1(x)^{n_1-1}\frac{F_2(y)}{y}. \end{aligned}$$

Then,

$$\begin{aligned} yF_1(x)^{n_1} \quad &+ \quad y\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k - xF_1(x)^{n_1-1}F_2(y) \\ &= \quad -\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k(y - \underline{b}_k) \\ &+ \quad y\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k \\ &= \quad \sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k \underline{b}_k > 0. \end{aligned}$$

So the numerator is positive.

We now prove the denominator is also positive. Again from (4.2) we have

$$\begin{aligned} -I \quad &- \quad n_1yF_1(x)^{n_1-1}f_1(x) = -n_1f_1(x)/F_1(x)\sum_{k=1}^{n_1} C_{n_1}^k F_1(x)^{n_1-k}(F_1(\underline{b}_k) - F_1(x))^k(y - \underline{b}_k) \\ &= \quad -n_1f_1(x)/F_1(x)(xF_1(x)^{n_1-1}F_2(y) - yF_1(x)^{n_1}) - n_1yF_1(x)^{n_1-1}f_1(x) \\ &= \quad n_1f_1(x)(-xF_1(x)^{n_1-2}F_2(y) + yF_1(x)^{n_1-1} - yF_1(x)^{n_1-1}) \\ &= \quad -n_1f_1(x)xF_1(x)^{n_1-2}F_2(y). \end{aligned}$$

Then we have

$$-n_1 f_1(x) x F_1(x)^{n_1-2} F_2(y) + F_2(y)(F_1(x)^{n_1-1} + (n_1-1) x f_1(x) F_1(x)^{n_1-2})$$
$$= F_2(y)(F_1(x)^{n_1-1} - x f_1(x) F_1(x)^{n_1-2}) > 0$$

since $F_1(x) > x f_1(x)$ by the concavity of $F_1(.)$. Thus we have $\frac{dx}{dy} > 0$. The uniqueness of the equilibrium is established. ■

Proof of Proposition 16:

Proof. We first prove that when $F_2(\cdot)$ is concave, there is no type symmetric equilibrium with $v_1^* \geq v_2^*$. Suppose not. Then a necessary condition is

$$v_2^* F_2(v_2^*)^{n_2-1} F_1(v_1^*)^{n_1} \geq v_1^* F_2(v_2^*)^{n_2} F_1(v_1^*)^{n_1-1},$$

or

$$\frac{F_1(v_1^*)}{v_1^*} \geq \frac{F_2(v_2^*)}{v_2^*}.$$

Note that when $F_2(\cdot)$ is concave and $v_1^* \geq v_2^*$, we have $\frac{F_2(v_2^*)}{v_2^*} \geq \frac{F_2(v_1^*)}{v_1^*}$, and thus $\frac{F_1(v_1^*)}{v_1^*} \geq \frac{F_2(v_1^*)}{v_1^*}$ which cannot be true since $F_2(v_1^*) > F_1(v_1^*)$ by assumption.

We now show that when $F_2(\cdot)$ is strictly convex, there exists an equilibrium in which $v_1^* \geq v_2^*$ when c is sufficiently large.

Let v_2^s satisfy

$$c = v_2^s F_2(v_2^s)^{n_2-1} F_1(v_2^s)^{n_1}$$

and let v_1^s satisfy

$$c = v_1^s F_2(v_1^s)^{n_2-1}.$$

For $y \in [v_1^s, v_2^s]$, define $x = \phi(y)$ from $c = y F_2(y)^{n_2-1} F_1(x)^{n_1}$. Then x is a decreasing

function of y satisfying $\phi(v_2^s) = v_2^s$ and $\phi(v_1^s) = 1$. Now define

$$\begin{aligned} h(y) &= \phi(y)F_2(y)^{n_2}F_1(\phi(y))^{n_1-1} + \sum_{k=1}^{n_2} C_{n_2}^k F_2(y)^{n_2-k}F_1(\phi(y))^{n_1-1}(F_2(\underline{b}_k(y)) - F_2(y))^k \\ &\quad (\phi(y) - \underline{b}_k(y)) - c. \end{aligned}$$

There is the required equilibrium if $\exists y \in [v_1^s, v_2^s]$ with $h(y) = 0$. Note that

$$h(v_2^s) = v_2^s F_2(v_2^s)^{n_2}F_1(v_2^s)^{n_1-1} - c > v_2^s F_2(v_2^s)^{n_2-1}F_1(v_2^s)^{n_1} - c = 0$$

since $F_2(v_2^s) > F_1(v_1^s)$ by assumption. Since $h(y)$ is continuous, we only need

$$h(v_1^s) = F_2(v_1^s)^{n_2} + \sum_{k=1}^{n_2} C_{n_2}^k F_2(v_1^s)^{n_2-k}(F_2(\underline{b}_k(v_1^s)) - F_2(v_1^s))^k(1 - \underline{b}_k(v_1^s)) - c < 0.$$

From the definition we know v_1^s is a monotonically increasing function of c, denoted by $v_1^s(c)$. It is obvious that $v_1^s(1) = 1$ and $v_1^{s\prime}(c) = \frac{F_2(v_1^s)v_1^s}{c(F_2(v_1^s)+(n_2-1)f_2(v_1^s))}$. So we have $v_1^{s\prime}(1) = \frac{1}{1+(n_2-1)f_2(1)}$. It suffices to show

$$\begin{aligned} \widehat{h}(c) &= F_2(v_1^s(c))^{n_2} + \sum_{k=1}^{n_2} C_{n_2}^k F_2(v_1^s(c))^{n_2-k}(F_2(\underline{b}_k(v_1^s(c))) - F_2(v_1^s(c)))^k \\ &\quad (1 - \underline{b}_k(v_1^s(c))) - c < 0 \end{aligned}$$

for some c. Note that we have $\widehat{h}(1) = 0$ and

$$\begin{aligned} \widehat{h}'(c) &= n_2 F_2(v_1^s(c))^{n_1-1} f_2(v_1^s(c))v_1^{s\prime}(c) \\ &+ \sum_{k=1}^{n_2} C_{n_2}^k [(n_2 - k)F_2(v_1^s(c))^{n_2-k-1} f_2(v_1^s(c))v_1^{s\prime}(c) \\ &\quad (F_2(\underline{b}_k(v_1^s(c))) - F_2(v_1^s(c)))^k(1 - \underline{b}_k(v_1^s(c))) \\ &+ F_2(v_1^s(c))^{n_2-k}(k(F_2(\underline{b}_k(v_1^s(c))) - F_2(v_1^s(c)))^{k-1} \\ &\quad (1 - \underline{b}_k(v_1^s(c)))(f_2(\underline{b}_k(v_1^s(c)))\underline{b}_k{}'(v_1^s(c)) \\ &- f_2(v_1^s(c))))v_1^{s\prime}(c) - (F_2(\underline{b}_k(v_1^s)) - F_2(v_1^s))^k \underline{b}_k{}'(v_1^s(c))] - 1 \end{aligned}$$

As $c \to 1$, we have $\underline{b}_k(v_1^s(1)) \to 1$, and thus

$$\widehat{h}'(1) = n_2 f_2(1) v_1^{s\prime}(1) - 1 = \frac{f_2(1) - 1}{1 + (n_2 - 1) f_2(1)} > 0$$

when $F_2(\cdot)$ is strictly convex. Hence, $\exists c^* < 1$ s.t. $\widehat{h}(c) < 0$ whenever $c > c^*$. ■

CHAPTER V

CONCLUSION

This dissertation investigates the entry equilibria of auctions when bidders have participation costs.

We began in Chapter II from second price auctions with differentiated participation costs in which we assume bidders only have private information about their values while they have public information about all bidders' participation costs. In this setting We identify two types of equilibria: monotonic equilibria in which a higher participation cost results in a higher cutoff point for submitting a bid, and neg-monotonic equilibria in which a higher participation cost results in a lower cutoff point. We show that there always exists a monotonic equilibrium, and further, that the equilibrium is unique for concave distribution functions and strictly convex distribution functions with some additional conditions. There exists a neg-monotonic equilibrium when the distribution function is strictly convex and the difference of the participation costs is sufficiently small.

Then in Chapter III we consider an economic environment in which bidders hold private information both in their values and participation costs. We consider the existence and uniqueness of equilibrium. It is shown that there always exists an equilibrium for this general economy, and further there exists a unique symmetric equilibrium when all bidders are ex ante homogeneous. Moreover, we identify a sufficient condition under which we have a unique equilibrium in a heterogeneous economy with two bidders. Our general framework covers many relevant models in the literature as special cases.

While bidding the true value is a dominant strategy when one decides to participate in a second price auction, the same cannot be said to the first price auctions.

In Chapter IV, we consider the entry equilibria of first price auctions with participation costs. We focus on the cutoff strategies in which each bidder participates and submits a bid if his value is greater than or equal to a critical value. It is shown that, when bidders are homogenous, there always exists a unique symmetric equilibrium, and further, there is no other equilibrium when valuation distribution functions are concave. However, when distribution functions are elastic at the symmetric equilibrium, there exists an asymmetric equilibrium. We find similar results when bidders are heterogenous.

There are several directions of which the current research can be extended. First in this research we only consider the pure strategy equilibria. Alternatively one can consider the case of mixed strategy. Secondly, this research mainly focus on the positive analysis. One can go further by investigating the normative analysis. As we have found, there are multiple equilibria in some cases. Then one direct question is which equilibrium is the best for the seller or for the society. How can we induce the buyers to choose that equilibrium. Last but not the least, one can also consider the question of designing the optimal auction that maximize the seller's expected revenue or the total surplus of the trade.

REFERENCES

[1] A. Blume, P. Heidhues, All Equilibria of the Vickrey Auction, Journal of Economic Theory 114 (2004) 170-177.

[2] T.A. Burton, Stability and Periodic Solutions of Ordinary and Functional Differential Equations, Academic Press, Orlando, 1985.

[3] X. Cao, G. Tian, Second Price Auctions with Differentiated Participation Costs, Working Paper, 2008.

[4] X. Cao, G. Tian, Second Price Auctions with Two-Dimensional Provate Information on Values and Participation Costs, Working Paper, 2008.

[5] C.M. Campbell, Coordination Auctions with Entry, Journal of Economic Theory 82 (1998) 425-450.

[6] G. Celik, O. Yilankaya, Optimal Auctions with Simultaneous and Costly Participation, Working Paper, 2008.

[7] I. Chakraborty, G. Kosmopoulou, Auctions whth Endogenous Entry, Economics Letters 72 (2001) 195-200.

[8] O. Compte, J. Jehiel, Auctions and Information Acquisition: Sealed Bid or Dynamic Formats? RAND Journal of Economics 38 (2007) 355-372.

[9] R. Engelbrecht-Wiggans, The Effect of Entry and Information Costs on Oral versus Sealed-bid Auctions, Economics Letters 70 (2001) 195-202.

[10] D. Fudenberg, J. Tirole, Game Theory, The MIT Press, Cambridge, 1991.

[11] S. Gal, M. Landsberger, A. Nemirovski, Participation in Auctions, Games and Economic Behavior 60 (2007) 75-103.

[12] J. Green, J.J. Laffont, Participation Constraints in the Vickrey Auction, Economics Letters 16 (1984) 31-36.

[13] R.M. Harstad, Alternative Common Value Auctions Procedure: Revenue Comparisions with Free Entry, Journal of Political Economy 98 (1990) 421-429.

[14] R.M. Harstad, J.H. Kagel, D. Levin, Equilibrium Bid Functions for Auctions with an Uncertain Number of Bidders, Economics Letters 33 (1990) 35-40.

[15] G. A. Jehle, P. J. Reny, Advanced Microeconomic Theory (the second edition), Addison Wesley, New York, 2001.

[16] T.R. Kaplan, A. Sela, Second Price Auctions with Private Entry Costs, Working Paper, 2006.

[17] T.R. Kaplan, S. Zamir, The Strategic Use of Seller Information in Private-Value Auctions, Working Paper, 2000.

[18] T.R. Kaplan, S. Zamir, Asymmetric First-Price Auctions with Uniform Distribution: Analytic Solution to the General Case, Working Paper, 2007.

[19] V. Krishna, Auction Theory, Academic Press, San Diego, 2002.

[20] B. Lebrun, Uniqueness of the Equilibrium in First-price Auctions, Games and Economic Behavior 55 (2006) 131-151.

[21] D. Levin, J.L. Smith, Equilibrium in Auctions with Entry, American Economic Review 84 (1994) 585-599.

[22] D. Levin, J.L. Smith, Ranking Auctions with Risk Averse Bidders, Journal of Economic Theory 68 (1996) 549-561.

[23] J. Lu, Endogenous Entry and Auctions Design with Private Participation Costs, Working Paper, 2006.

[24] J. Lu, Y. Sun, Efficient Auctions with Private Participation Costs, Working Paper, 2007.

[25] I. Martinez-Partina, First-price Auctions where one of the Bidders' Valuation is Common Knowledge, Rev. Econ. Design 10 (2006) 31-51.

[26] E.S. Maskin, J.G. Riley, Optimal Auctions with Risk Averse Buyers, Econometrica 52 (1984) 1473-1518.

[27] E.S. Maskin, J.G. Riley, Asymmetric Auctions, Rev. Econ. Stud. 67 (2000) 413-438.

[28] E.S. Maskin, J.G. Riley, Equilibrium in Sealed High-bid Auctions, Rev. Econ. Stud 67, (2000) 439-454.

[29] E.S. Maskin, J.G. Riley, Uniqueness of Equilibrium in Sealed High-bid Auctions, Games and Economic Behavior 45 (2003) 395-409.

[30] R.P. McAfee, J. McMillan, Auctions with Entry, Economics Letters 23 (1987) 343-347.

[31] R.P. McAfee, J. McMillan, Auctions with a Stochastic Number of Bidders, Journal of Economic Theory 43 (1987) 1-19.

[32] F.M. Menezes, P.K. Monterio, Auctions with Endogeneous Participation, Review of Economic Design 5 (2000) 71-89.

[33] P.R. Milgrom, J.W. Weber, A Theory of Auctions and Competitive Bidding, Econometrica 50 (1982) 1089-1122.

[34] A. Miralles, Intuitive and Noncompetitive Equilibria in Weakly Efficient Auctions with Entry Costs, Mathematical Social Sciences 56 (2008) 448-445.

[35] R.B. Myerson, Optimal Auction Design, Mathematics of Operations Research 6 (1981) 58-73.

[36] P. Persico, Information Acquisition in Auctions, Econometrica 68 (2000) 135-148.

[37] W.F. Samuelson, Competitive Bidding with Entry Costs, Economics Letters 17 (1985) 53-57.

[38] M. Stegeman, Participation Costs and Efficient Auctions, Journal of Economic Theory 71 (1996) 228-259.

[39] G. Tan, Entry and R&D in Procurement Contracting, Journal of Economic Theory 58 (1992) 41-60.

[40] G. Tan, O. Yilankaya, Equilibria in Second Price Auctions with Participation Costs, Journal of Economic Theory 130 (2006) 205-219.

[41] G. Tan, O. Yilankaya, Ratifiability of Efficient Collusive Mechanisms in Second-price Auctions with Participation Costs, Games and Economic Behavior 59 (2007) 383-396.

[42] G. Tian, Micro Economic Theory (Lecture Notes), Department of Economics, Texas A&M University, 2006.

[43] L. Ye, Optimal Auctions with Endogenous Entry, Contributions to Theoretical Economics 4 (2004) article 8.

[44] W. Vickrey, Counterspection, Auctions, and Competitive Sealed Tenders, Journal of Finance 16 (1961) 8-37.

[45] E. Wolfstetter, Topics in Microeconomics: Industrial Organization, Auctions, and Incentives, Cambridge University Press, New York, 1999.

VITA

Xiaoyong Cao
Phone: (979) 571-0392 (Cell)
Email: yongcx2000@hotmail.com
Homepage: http://econweb.tamu.edu/xcao

School of International Trade and Economics
UIBE
10 East Huixin Street, Chaoyang District
Beijing, 100029, China

EDUCATION

Ph.D in Economics, **Texas A&M University** August 2009

B.A. in Economics, **Huazhong University of Science & Technology**, China June 2003

B.S. in Mathematics, **Huazhong University of Science & Technology**, China June 2003

FIELDS OF SPECIALIZATION

Primary: Auction Theory, Microeconomics, Industrial Organization.

Secondary: Economic Theory.

FELLOWSHIPS & AWARDS

Bradley Fellowship, by Private Enterprise Research Center, Texas A&M University, 2007-2008

Texas A&M University Graduate Assistantship, 2004-2007, 2008-2009

Referee

Referee for *Economic Inquiry*.

COMPUTER SKILLS

Familiar with programming and application of skills in Gauss, Matlab, Eviews.

The typist for this dissertation was Xiaoyong Cao.