ON ORBITS OF OPERATORS ON HILBERT SPACE

A Dissertation

by

LIDIA SMITH

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2009

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ABSTRACT

On Orbits of Operators on Hilbert Space. (August 2009)

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In this dissertation we treat some problems about possible density of orbits for non-hypercyclic operators and we enlarge the class of known non-orbit-transitive operators. One of the questions related to hypercyclic operators that we answer is whether the density (in the set of positive real numbers) of the norms of the elements in the orbit for each nonzero vector in the Hilbert space is sufficient to imply that at least one vector has orbit dense in the Hilbert space. We show that the density of the norms is not a sufficient condition to imply hypercyclicity by constructing a weighted bilateral shift that, on one hand, satisfies the orbit-density property (in the sense defined above), but, on the other hand, fails to be hypercyclic. The second major topic that we study refers to classes of operators that are not hypertransitive (or orbit-transitive) and is related to the invariant subspace problem on Hilbert space. It was shown by Jung, Ko and Pearcy in 2005 that every compact perturbation of a normal operator is not hypertransitive. We extend this result, after introducing the related notion of weak hypertransitivity, by giving a sufficient condition for an operator to belong to the class of non-weakly-hypertransitive operators. Next, we study certain 2-normal operators and their compact perturbations. In particular, we consider operators with a slow growth rate for the essential norms of their powers. Using a new idea, of accumulation of growth for each given power on a set of different orthonormal vectors, we establish that the studied operators are not hypertransitive.

To my parents

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I would like to thank my advisor, Professor Carl M. Pearcy, for accepting to guide my work as a graduate student. I feel fortunate to have learned from such an accomplished mathematician. Seeing his great insight at work revitalized in me the enthusiasm for mathematics. His arduous work and continuous striving for better results set an inspiring model. In particular, I would like to thank my advisor for teaching me perseverance. The results of this dissertation would not exist without him saying "I'm not so sure about that" when during our meetings I would say "I don't think that this can work".

I would also like to thank my committee members for being supportive, and especially Dr. Ciprian Foias for expressing his interests in the problems that I was working on.

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CHAPTER I

INTRODUCTION

1.1. Terminology and notation

In what follows, $\mathbb{N}$ denotes, as usual, the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{D}$ is the open unit disc in $\mathbb{C}$. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and write $\mathcal{L}(\mathcal{H})$ for the algebra of all bounded linear operators $T : \mathcal{H} \to \mathcal{H}$. For a sequence $\{T_n\} \subset \mathcal{L}(\mathcal{H})$ converging in the strong operator topology to some $T_0 \in \mathcal{L}(\mathcal{H})$ we shall write $T_n \overset{SOT}{\to} T_0$, and similarly for the weak operator topology. Moreover, we will use the slightly unusual (but useful) notation $T_n \overset{SOT}{\to} +\infty$ to mean $|| \, T_n x|| \to +\infty$ for all nonzero x in $\mathcal{H}$, and we write, as is customary, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_e(T)$ for the spectrum, point spectrum, and essential (Calkin) spectrum of an operator T.

For any operator T in $\mathcal{L}(\mathcal{H})$ and any (nonzero) x in $\mathcal{H}$, we denote by $\mathcal{O}(x,T)$ *the orbit of x under T, i.e.,* $\mathcal{O}(x,T)$ is the set of vectors $\{T^n x : n \in \mathbb{N}_0\}$ in $\mathcal{H}$. An operator T in $\mathcal{L}(\mathcal{H})$ for which there exists a vector x with the property that the orbit $\mathcal{O}(x,T)$ is (norm) dense in $\mathcal{H}$ is called *a hypercyclic operator,* and such a vector x is called *hypercyclic* for T. If the same property holds when the norm topology is replaced by the weak topology T is called a *weakly hypercyclic operator* and the vector x is called *weakly hypercyclic* for T. One of the interesting open questions about operators in $\mathcal{L}(\mathcal{H})$ is the following orbit-transitive operator problem:

> Does there exist an operator T in $\mathcal{L}(\mathcal{H})$ such that for every (nonzero) x in $\mathcal{H}$, the orbit $\mathcal{O}(x,T)$ is norm (or weakly) dense in $\mathcal{H}$?

This dissertation follows the style of Proceedings of the American Mathematical Society.

When the norm topology is used we call such an operator T *orbit-transitive* (or *hypertransitive*) and in the case of the weak topology we call T *weakly hypertransitive.* A positive answer to the above question would solve the invariant subspace problem for Hilbert space in the negative.

1.2. History of the problem

The topics of hypercyclic vectors and operators have received much attention in the last twenty years. Rolewitz gave in 1969 the first example of an operator T in $\mathcal{L}(\mathcal{H})$ for which there do exist vectors x with the property that the orbit $\mathcal{O}(x,T)$ is dense in $\mathcal{H}$ [25]. Important early work on hypercyclic vectors and operators was done by Bernard Beauzamy , and much of his work is exposed in his book [4]. For instance, Beauzamy showed in [3] that there are operators T in $\mathcal{L}(\mathcal{H})$ for which there is a dense linear manifold $\mathfrak{M}_T$ in $\mathcal{H}$ with the property that every nonzero $x \in \mathfrak{M}_T$ is hypercyclic for T.

Beginning in the 1990's the theory of hypercyclic operators (on Hilbert space) and the structure of the set of all hypercyclic vectors for a given operator T in $\mathcal{L}(\mathcal{H})$ were studied extensively. See, for example, the excellent survey article [13] by Grosse-Erdmann. To mention but a few of the most striking of these "new" theorems, we note that S. Ansari showed in [1] that, for every $T\in \mathcal{L}(\mathcal{H})$, each of the operators in the set $\{T^n,\, n \in \mathbb{N}\}$ has exactly the same set of hypercyclic vectors. Also, Leon-Saavera and Muller proved in [17] that if $T\in \mathcal{L}(\mathcal{H})$ and $\theta \in \mathbb{R}$ then T and $e^{i\theta}T$ have exactly the same set of hypercyclic vectors. Moreover, Bourdon [5] showed that if T is any hypercyclic operator in $\mathcal{L}(\mathcal{H})$, then there is a dense linear manifold $\mathcal{D}_T \subset \mathcal{H}$ such that every vector in $\mathcal{D}_T\backslash(0)$ is hypercyclic for T (thus generalizing the example of Beauzamy mentioned above). Finally (to conclude this brief survey), Bourdon and

Feldman established in [6] that that if $T\in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$ have the property that $\{T^n x\}^-$ contains some (nonempty) open ball, then x is hypercyclic for T, and deduced as a corollary that if $T\in \mathcal{L}(\mathcal{H})$ has the property that there is a finite set of vectors $\{x_{1,}, \ldots, x_k\}$ in $\mathcal{H}$ such that $\bigcup_{j=1}^{k} \{T^n x_j : n \in \mathbb{N}_0\}$ is dense in $\mathcal{H}$ (such a T is called multi-hypercyclic), then T is a hypercyclic operator.

But, despite the existence of this mountain of work on hypercyclic vectors and operators, it is fair to say that results showing that there are no orbit-transitive operators in certain subclasses of $\mathcal{L}(\mathcal{H})$ have been scarce. (Of course, every new (or old) invariant subspace theorem for a class of operators in $\mathcal{L}(\mathcal{H})$ immediately gives a class of non-orbit-transitive operators, since obviously an operator T having a nontrivial invariant subspace cannot be orbit-transitive. However, this doesn't help much, since new invariant subspace theorems are themselves somewhat rare.)

In fact, one might say that the recent article [15], by Jung, Ko, and Pearcy, initiated the theory of non-orbit-transitive operators. In [15], for example, it was shown that if $T \in \mathcal{L}(\mathcal{H})$ and some T^n is essentially hyponormal, then no operator of the form $STS^{-1} + K$, where S is invertible and K is compact, is orbit-transitive. As an immediate corollary one gets that no operator of the form $SNS^{-1} + K$, where N is normal, is orbit-transitive. (Note that the invariant subspace problem for such operators T is still open.)

Some other known results that are pertinent to our problem are the following. Beauzamy [4, page 66] has shown that there exists a operator T on Hilbert space such that $||T^n||$ is asymptotic to $\sqrt{\log n}$ but there exists no vector x in $\mathcal{H}$ for which $||T^n x||$ tends to infinity. This was recently improved in [19] as follows. There exists an operator T in $\mathcal{L}(\mathcal{H})$ such that $||T^n||$ is asymptotic to $n^{1/2}$ but there exists no vector x whose orbit tends to $+\infty$. It is also shown in [19] that this is best possible (for the class $\mathcal{L}(\mathcal{H})$) by proving that every operator T in $\mathcal{L}(\mathcal{H})$ with growth rate of $\|T^n\|$

asymptotic to $n^{1/2+\varepsilon}$ (for some ε positive) has a vector x whose orbit does tend to $+\infty$. Chan and Sanders [9] recently exhibited a sequence of vectors $\{x_n\}$ in $\mathcal{H}$, with strictly increasing norms, that is weakly dense in $\mathcal{H}$.

1.3. N-normal operators

Recall that for any $n \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is called an n-normal operator if T is unitarily equivalent to an $n \times n$ operator matrix (N_{ij}) acting on $\mathcal{H}^{(n)}$ in the usual fashion, where the set $\{N_{ij}\}$ consists of mutually commuting normal operators in $\mathcal{L}(\mathcal{H})$. The theory of n-normal operators is quite well developed. In [14] it was proved that every n-normal operator has a nontrivial hyperinvariant subspace (n.h.s.).

A first step in the direction of showing that if $T \in \mathcal{L}(\mathcal{H})$ is n-normal and $K \in \mathbb{K}(\mathcal{H})$, then $T+K \in$(NHT) is the following, which depends on a (deep) result from [7] as well as the upper triangular form theorem for n-normal operators from [20]. We call a (commutative) collection $\{N_\alpha\}_{\alpha \in A} \subset \mathcal{L}(\mathcal{H})$ of normal operators a *simultaneously diagonalizable* family if there exists an orthonormal basis $\mathcal{E}$ for $\mathcal{H}$ such that the matrix for each N_α, $\alpha \in A$ with respect to $\mathcal{E}$ is diagonal.

Theorem 1.3.1. *For every n-normal operator $T \in \mathcal{L}(\mathcal{H})$ and compact operator K, there exist an n-normal operator $\widetilde{T} = \oplus_{k \in \mathbb{N}} T_k$, where each T_k is an $n \times n$ complex matrix (regarded as an operator on $\mathbb{C}^{(n)}$) in upper triangular form and a compact operator $\widetilde{K}$ such that $T+K$ is unitarily equivalent to $\widetilde{T}+\widetilde{K}$. Consequently, to show that $T+K$ belongs to* (NHT) *it suffices to show that $\widetilde{T}+\widetilde{K}$ does.*

Proof. One can write $T = (N_{ij}) \in \mathcal{L}(\mathcal{H}^{(n)})$, where the N_{ij} are mutually commuting normal operators and the matrix (N_{ij}) is in upper triangular form (cf. [10, Theorem 2]). Moreover via a deep result from [7] one knows that for $1 \le i, j \le n$, $N_{ij} = D_{ij} + K_{ij}$, where the D_{ij} are (mutually commuting) simultaneously diagonalizable normal operators in $\mathcal{L}(\mathcal{H})$ and the K_{ij} are all compact. Thus the $n\times n$ matrix $(K_{ij}) \in \mathcal{L}(\mathcal{H}^{(n)})$ is a compact operator and the $n\times n$ matrix (D_{ij}) is unitarily equivalent to a direct sum $\bigoplus_{j\in\mathbb{N}} \widetilde{T}_j$, where each $\widetilde{T}_j$ (in $\mathcal{L}(\mathbb{C}^{(n)})$) is an $n \times n$ complex matrix in upper triangular form. □

1.4. The first example of a hypercyclic operator

We present here Rolewitz's example of a hypercyclic operator on l_2 in the slightly modified version of Beauzamy. The basic idea from this proof is used in the example of a weakly hypercyclic vector constructed in [9] .

Example 1.4.1. Let T be a backward weighted shift on $l_2(\mathbb{Z})$, defined by

$$Te_k = w_k e_{k-1}, \quad k \in \mathbb{Z}.$$

If the weights w_k satisfy

$$\lim_{n\to\infty} \prod_0^n w_k = +\infty, \text{ and } w_k \geq 1 \text{ for } k > 0,$$

$$\lim_{n\to\infty} \prod_0^{-n} w_k = 0, \text{ and } 0 < w_k < 1 \text{ for } k < 0,$$

then the operator T has a hypercyclic point.

Proof. Note that the inverse S of T is given by

$$Se_k = \frac{1}{w_{k+1}} e_{k+1}.$$

It follows from the definition of the weights that

$$\lim_{n\to\infty} T^n e_k = 0, \text{ for all } k \in \mathbb{Z},$$

$$\lim_{n\to\infty} S^n e_k = 0, \text{ for all } k \in \mathbb{Z}.$$

Let $\{h_n\}$ be a dense sequence in $l_2(\mathbb{Z})$, each h_n having finite support. For $n \geq 0$ let k_n be an integer such that, if $k > k_n$ we have for $i = 1, \dots, n-1$,

$$||T^{k_n} h_i|| < 1/2^n,$$

$$||S^{k_n} h_i|| < 1/2^n.$$

Let $p_n = \sum_{i=1}^n k_i$ and $z = \sum_{k=1}^\infty S^{p_k} h_k$. Then

$$T^{p_n} z = T^{p_n - p_1} h_1 + \cdots + T^{p_n - p_i} h_i + \cdots + T^{p_n - p_{n-1}} h_{n-1} + h_n + \sum_{m=n+1}^\infty S^{p_m - p_n} h_m.$$

But, for $i = 1, \dots, n-1$:

$$||T^{p_n - p_i} h_k|| = ||T^{k_n + \cdots + k_{i+1}} h_i|| < 1/2^n,$$

and

$$||\sum_{m=n+1}^\infty S^{p_m - p_n} h_m|| \leq ||\sum_{m=n+1}^\infty S^{k_m + \cdots + k_{n+1}} h_m|| < 1/2^n.$$

Finally :

$$||T^{p_n} z - h_n|| < n/2^n,$$

and since the sequence $\{h_n\}_{n\geq 1}$ is norm-dense, so is $\{T^{p_n} z\}_{n\geq 1}$. □

1.5. Facts on weak topology

Lemma 1.5.1. *Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Consider the set $\{\sqrt{n} e_n\}_{n\in\mathbb{N}}$. Then $0 \in \overline{\{\sqrt{n} e_n\}}^{w}_{n\in\mathbb{N}}$.*

Proof. Let $\{y_1, \dots y_k\}$ be an arbitrary finite set in $\mathcal{H}$ and let $\varepsilon > 0$. It suffices to show that there exists $n \in \mathbb{N}$ such that $\sqrt{n}e_n \in \{x \in \mathcal{H} : |\langle x - 0, y_j\rangle| < \varepsilon, \quad j = 1, 2, \ldots, k\}$.

Suppose this is false. Then there exist $\varepsilon > 0$ and $y_1, \ldots, y_k$ such that for all $n \in \mathbb{N}$, $\sqrt{n}e_n \notin \{x \in \mathcal{H} : |\langle x, y_j\rangle| < \varepsilon, \quad j = 1, 2, \ldots, k\}$. That is, for each n, there exists y_{j_n}, $1 \leq j_n \leq k$ such that $|\langle \sqrt{n}e_n, y_{j_n}\rangle| \geq \varepsilon$. So $|\langle e_n, y_{j_n}\rangle| \geq \dfrac{\varepsilon}{\sqrt{n}}$.

Thus

$$\sum_{n\in\mathbb{N}} |\langle e_n, y_{j_n}\rangle|^2 \geq \sum_{n\in\mathbb{N}} \frac{\varepsilon^2}{n} = +\infty.$$

But for all $1 \leq j \leq k$, $||y_j||^2 = \sum_{m\in\mathbb{N}} |\langle e_m, y_j\rangle|^2$ and thus

$$\sum_{1\leq j\leq k} \sum_{m\in\mathbb{N}} |\langle e_m, y_j\rangle|^2 < +\infty.$$

This gives a contradiction. □

Lemma 1.5.2. *If $\{x_n\} \subset \mathcal{H}$ has disjoint support, i.e., $\langle x_n, x_m\rangle = 0$ when $m \neq n$, and $\sum \dfrac{1}{||x_n||^2} = \infty$, then $0 \in \overline{\{x_n\}}^{w}_{n\in\mathbb{N}}$.*

Proof. Let $x_n = ||x_n||u_n$ for every n, where u_n is a unitary vector. Then the set $\{u_n\}$ is orthonormal and can be completed to a basis for $\mathcal{H}$. The rest of the proof follows as in the lemma above with u_n replacing e_n and $||x_n||$ replacing $\sqrt{n}$. □

Lemma 1.5.3. *Given $\mathcal{S} \subset \mathcal{H}$, if $\bigvee \mathcal{S} \neq \mathcal{H}$, then $\mathcal{S}$ is not weakly dense.*

Proof. We have

$$\mathcal{S}^{\underline{wk}} \subset \left(\bigvee \mathcal{S}\right)^{\underline{wk}} = \bigvee \mathcal{S}.$$

□

Lemma 1.5.4. *Given $\{x_n\} \subset \mathcal{H}$, we have*

$$\{x_n\}^{\underline{wk}} = \mathcal{H} \iff \forall\{\theta_n\}, \{e^{i\theta_n} x_n\}^{\underline{wk}} = \mathcal{H}.$$

Proof. Assume $\underline{\{x_n\}^{wk}} = \mathcal{H}$. Let $\{y_1, \ldots, y_k\}$ be an arbitrary set in $\mathcal{H}$. Let $\varepsilon > 0$, and let x_0 be an arbitrary element of $\mathcal{H}$. A basic weak neighborhood of x_0 is of the form

$$\mathcal{U}_{x_0} = \{x \in \mathcal{H} : |x - x_0| < \varepsilon, j = 1, \ldots, k\}.$$

It is enough to show that there exists an element of $\{e^{i\theta_n} x_n\}$ contained in this neighborhood.

For an arbitrary θ, there exists $x' \in \{x_n\}$, say $x' = x_m$, such that

$$|x' - e^{-i\theta} x_0| < \varepsilon, j = 1, \ldots, k,$$

so

$$|x' e^{i\theta} - x_0| < \varepsilon, j = 1, \ldots, k.$$

If we let $\theta = \theta_m$ we have $e^{i\theta_m} x_m \in \mathcal{U}_{x_0}$. $\square$

Lemma 1.5.5. *Let $\{e_i\}_{i\in\mathbb{Z}}$ be the standard orthonormal basis for $l_2(\mathbb{Z})$. There exists a norm dense set $\{h_n : n \geq 1\}$ in $l_2(\mathbb{Z})$ such that for each $n \geq 1$*

$$||h_n||^2 \leq n, \text{ and } \langle h_n, e_i \rangle = 0 \text{ for } |i| > n.$$

Proof. We start by constructing a norm-dense sequence of vectors for each finite dimensional space $\mathcal{F}_n = \bigvee\{e_{-n}, \ldots, e_{-1}, e_0, \ldots e_n\}$, $n \in \mathbb{N}$ and denote it by $\{r_{n+j,n}\}_{j\in\mathbb{N}}$. Arrange these in a lower triangular matrix such that each column k contains the elements of the sequence r_k , with the first element on the row k, as follows

$$\begin{matrix} r_{1,0} & & & & \\ r_{2,0} & r_{2,1} & & & \\ r_{3,0} & r_{3,1} & r_{3,2} & & \\ r_{4,0} & r_{4,1} & r_{4,2} & r_{4,3} & \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix} .$$

Thus, for $j \in \mathbb{N}$, the elements of the jth column of this matrix form a dense set in $\mathcal{F}_j$. By writing the elements of this matrix in a single sequence $\{h'_n\}$: $r_{3,0}r_{1,0}, r_{2,0}, r_{2,1}, r_{3,0}, r_{3,1}r_{3,2}, \ldots$ (*i.e.*, row by row) we obtain that $\{h'_n : n \geq 1\}$ is dense in $l_2(\mathbb{Z})$ and $\langle h'_n, e_i\rangle = 0$ for $|i| > n$.

Finally, define

$$h_n = \begin{cases} h'_n & \text{if } ||h'_n|| \leq \sqrt{n} \\ e_1 & \text{if } ||h'_n|| > \sqrt{n} \end{cases} .$$

Then $||h_n|| \leq \sqrt{n}$ and $\langle h_n, e_i\rangle = 0$ for $|i| > n$. To show that $\{h_n\}$ is norm-dense in $l_2(\mathbb{Z})$, notice that since $\{h'_n : n \geq 1\}$ is dense in $l_2(\mathbb{Z})$, for $z \in \mathcal{H}$ and $\varepsilon > 0$ there exists a subsequence $\{h'_{n_j}\}$ converging to z and thus there exists $N_0 \in \mathbb{N}$ such that $||h'_{n_j} - z|| < \varepsilon$ for all $n_j \geq N_0$. Assume N_0 is large enough such that $||z|| + \varepsilon \leq \sqrt{N_0}$, then for $n_j \geq N_0$ we have $h_{n_j} = h'_{n_j}$ and thus $||h_{n_j} - z|| < \varepsilon$ for all $n_j \geq N_0$. This shows that $\{h_{n_j}\}$ converges to z. Thus $\{h_n\}$ is norm-dense in $l_2(\mathbb{Z})$.

An alternate way to build $\{h_n\}$ is the following : inductively choose h_k to be the earliest h'_n such that

$$h'_n \in \bigvee\{e_{-k}, \ldots, e_k\} \text{ and } ||h'_n|| \leq k.$$

Then $\{h_n\}$ goes over all elements in $\{h'_n\}$ and has the desired properties. $\square$

1.6. A weakly hypercyclic operator that is not norm hypercyclic

We include here the example of [9] together a proof that the operator given is weakly hypercyclic without being norm hypercyclic. We follow the proof given in [9] with the simplifications implied by considering the particular operator in the settings of Hilbert space.

The following lemma is one of the ingredients used in proving the existence of a weakly hypercyclic vector. The lemma is included here together with its proof as it appears in [9].

Lemma 1.6.1. *For every given real number $\lambda > 1$ there exists a bijective map $\nu : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that*

(i) *for each $r \geq 1$, the sequence $\{\nu(r,s)\}_{s=1}^{\infty}$ is strictly increasing*

(ii) *for each $r \geq 1$, we have $r \leq \nu(r,1)$;*

(iii) *there exists a sequence $\{a_r\}_{r=1}^{\infty}$ of positive integers such that if $\{c_r\}_{r=1}^{\infty}$ is a sequence of nonnegative real numbers such that $c_r^2 \leq r\lambda^{2r}$ for each $r \geq 1$, then the new sequence $\{d_n\}_{n=1}^{\infty}$ given by $d_n = d_{\nu(i,j)} = c_i$ satisfies the inequality*

$$\sum_{n=1}^{\nu(r,s)} d_n^2 \leq (a_r + s)\log(a_r + s), \ \textit{for each } r, s \geq 1.$$

Proof. Let $\{m_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of integers that satisfies

$$(1 + 2 + \cdots + i)\lambda^{2i} \leq \log m_i. \tag{1.1}$$

Let $\{\alpha_j\}_{j=1}^{\infty}$ be the sequence of integers given by

$$(\alpha_1, \alpha_2 \dots) = (\underbrace{G_1, \dots, G_1}_{m_2 \text{ copies}}, \underbrace{G_2, \dots, G_2}_{m_3 \text{ copies}}, \underbrace{G_3, \dots, G_3}_{m_4 \text{ copies}}, \dots),$$

where $G_i = (1, 2, \dots, i)$.

Let $\nu : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by $\nu(r,s) = n$ if $\alpha_n = r$ and $\alpha_j = r$ for exactly s positive integers $j \leq n$. Thus, $\nu(r,s)$ gives the position of the s-th appearance of r in the above sequence.

It follows immediately that ν is a bijective map and for each $r \geq 1$, the sequence $\{\nu(r,s)\}_{s=1}^{\infty}$ is increasing in s. Also note that $\nu(1,1) = 1$ and

$$\nu(r,1) = m_2 + 2m_3 + \cdots + (r-1)m_r + r. \tag{1.2}$$

Thus (i) and (ii) are satisfied.

To prove prove property (iii) note that $\{d_n\}_{n=1}^{\infty}$ is given by

$$(d_1, d_2 \ldots) = (\underbrace{F_1, \ldots, F_1}_{m_2 \text{ copies}}, \underbrace{F_2, \ldots, F_2}_{m_3 \text{ copies}}, \underbrace{F_3, \ldots, F_3}_{m_4 \text{ copies}}, \ldots),$$

where $F_i = (c_1, c_2, \ldots, c_i)$ and $\nu(r,s)$ gives the position of the s-th appearance of c_r in $\{d_n\}_{n=1}^{\infty}$.

Let $a_1 = m_1$ and $a_r = v(r,1)$ (that is the index of first appearance of c_r in $\{d_n\}_{n=1}^{\infty}$) for $r \geq 2$ and proceed by induction.

For $r = 1$ we have $\nu(r,1) = \nu(1,1) = 1$ and thus by (1.7)

$$\sum_{n=1}^{\nu(r,1)} d_n^2 = c_1^2 \leq \lambda^2 \leq \log m_1 \leq (a_1 + 1)\log(a_1 + 1).$$

For $r \geq 2$ by (1.2) we have

$$\sum_{n=1}^{\nu(r,1)} d_n^2 = m_2c_1^2 + \cdots + m_r(c_1^2 + \cdots + c_{r-1}^2) + (c_1^2 + \cdots + c_r^2) \tag{1.3}$$
$$\le (m_2 + m_3 + \cdots + m_r + 1)(c_1^2 + \cdots + c_r^2)$$
$$\le (\nu(r,1) + 1)(1 + 2 + \cdots + r)\lambda^{2r} \le (a_r + 1)\log m_r$$
$$\le (a_r + 1)\log(a_r + 1).$$

The induction assumption is that for some $s \ge 1$ the inequality $\sum_{n=1}^{\nu(r,s)} d_n^2 \le (a_r + s)\log(a_r + s)$ holds for each $r \ge 1$. We must show that the inequality holds for $s + 1$. Note that c_r makes its first appearance in the sequence $\{d_n\}_{n=1}^{\infty}$ as the last member in the first F_r , thus we need to separate the induction step in two cases.

Case 1. $s \le m_{r+1}$. In this case, the s-th appearance of c_r lies in an F_r. Hence,

$$\sum_{n=1}^{\nu(r,s+1)} d_n^2 = \sum_{n=1}^{\nu(r,s)} d_n^2 + (c_1^2 + \cdots + c_r^2) \tag{1.4}$$
$$\le (a_r + s)\log(a_r + s) + (1 + 2 + \cdots + r)\lambda^{2r}$$
$$\le (a_r + s)\log(a_r + s) + \log m_r$$
$$\le (a_r + s)\log(a_r + s) + \log a_r$$
$$\le (a_r + s + 1)\log(a_r + s + 1).$$

Case 2. $s > m_{r+1}$. In this case, the s-th appearance of c_r lies in some F_j for some $j \geq r+1$. Hence, $m_{r+1} + \cdots + m_j < s \leq m_{+1} + \cdots + m_j + m_{j+1}$ and

$$\begin{aligned}\sum_{n=1}^{\nu(r,s+1)} d_n^2 = \sum_{n=1}^{\nu(r,s)} d_n^2 + (c_{r+1}^2 + \cdots + c_j^2 + c_1^2 + \cdots + c_r^2) \qquad (1.5)\\ \leq (a_r + s)\log(a_r + s) + (1 + 2 + \cdots + j)\lambda^{2j}\\ \leq (a_r + s)\log(a_r + s) + \log m_j\\ \leq (a_r + s)\log(a_r + s) + \log s\\ \leq (a_r + s + 1)\log(a_r + s + 1).\end{aligned}$$

This concludes the proof of the lemma. □

Next we present the example from [9] of a weakly hypercyclic operator that is not hypercyclic.

Theorem 1.6.2. *The backward weighted shift on $l_2(\mathbb{Z})$, defined by*

$$Te_i = \begin{cases} e_{i-1}, & \textit{if } i \leq 0, \\ 2e_{i-1}, & \textit{if } i \geq 1. \end{cases}$$

is weakly hypercyclic and satisfies the inequality $||Tx|| \geq ||x||$ for all $x \in l_2(\mathbb{Z})$, hence, T is not hypercyclic.

Proof. The idea of proof is the one from [9], but we tried to simplify the details of it, since the proof given in [9] is for a more general setting.

Note that the inverse S of T is given by

$$Se_i = \begin{cases} e_{i+1}, & \text{if } i \leq -1, \\ 2^{-1}e_{i+1}, & \text{if } i \geq 0. \end{cases}$$

We start with $\{h_n : n \geq 1\}$ a dense sequence in $l_2(\mathbb{Z})$, as in the above lemma, each h_n having finite support,

$$||h_n||^2 \leq n, \text{ and } \langle h_n, e_i \rangle = 0 \text{ for } |i| > n.$$

We will construct a vector g such that, for given $\varepsilon > 0$, $r \geq 1$ and t nonzero vectors $x_1, \ldots, x_t$ in $l_2(\mathbb{Z})$, we will exhibit an N_r for which we have:

$$|\langle T^{N_r} g - h_r, x_j \rangle| \leq \varepsilon, \text{ for } j = 1, \ldots, t.$$

The vector g will be built based on the sequence $\{h_n\}$, as a convergent series. The series $T^{n_r} g$ will have h_r as one of its terms, the tail made of all the elements after h_r will converge to zero in norm as $n_r \to \infty$ and if we denote by v_n the partial sum consisting of the elements up to h_r we would like our construction to have the property that

$$||v_n||^2 \leq (c+n)\log(c+n), \text{ for all } n,$$

where c is a constant to be determined. Moreover, in the construction, the v_n's will have disjoint support. The series $\sum_{k=1}^{\infty} 1/||v_k||^2$ is divergent and using a similar argument with the one in Lemma 1.5.2 we obtain that $0 \in \overline{\{v_n\}}^{w}_{n \in \mathbb{N}}$.

Rather than directly using $\{h_n\}$ in the construction of the weakly hypercyclic vector g, we will use $\{f_n = T^n h_n\}$, since it is convenient to have all nonzero Fourier coefficients with index smaller or equal to zero; f_n has the property that

$$||f_n||^2 \leq n \cdot 2^{2n}, \text{ and } \langle f_n, e_i \rangle \neq 0 \text{ only for } i = -2n, \ldots, -1, 0. \tag{1.6}$$

Also, instead of using each vector f_n (or h_n) just once in a series that would define g as in Rolewitz example, each f_n will appear infinitely many times in the series that defines g.

Let $\lambda = ||T||$. Note that $||T|| = 2$, so $\lambda = 2$.

Let $\{m_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of integers that satisfies

$$(1+2+\cdots+i)\lambda^{2i} \leq \log m_i. \tag{1.7}$$

Let $\{g_j\}_{j=1}^{\infty}$ be the sequence given by

$$\begin{aligned}\{g_1, g_2 \dots\} &= \{\underbrace{F_1, \dots, F_1}_{m_2 \text{ copies}}, \underbrace{F_2, \dots, F_2}_{m_3 \text{ copies}}, \underbrace{F_3, \dots, F_3}_{m_4 \text{ copies}}, \dots\} \\ &= \{\underbrace{f_1, \dots, f_1}_{m_2 \text{ copies}}, \underbrace{f_1, f_2, \dots, f_1, f_2}_{m_3 \text{ copies}}, \underbrace{f_1, f_2, f_3, \dots, f_1, f_2, f_3}_{m_4 \text{ copies}}, \dots\},\end{aligned}$$

such that the group $F_i = f_1, \dots, f_i$ appears m_{i+1} consecutive times. If we set $n = \nu(i,j)$, where ν is the function from Lemma 1.6.1, it follows from that lemma that

$$\sum_{n=1}^{\nu(r,s)} ||g_n||^2 \leq (a_r + s)\log(a_r + s), \tag{1.8}$$

where $a_1 = m_1$ and $a_r = v(r, 1)$.

Now, going back to T: we have $w_i = 1$ for all $i \leq 0$, and $w_i = 2$ for $i \geq 1$. Note that $||T|| = 2$ and for fixed $n \geq 1$ and for every $m \geq 2n$, we have

$$||S^m e_i|| = \frac{1}{2^{m+i}}, \text{ for } i \leq 0, \tag{1.9}$$

and

$$||T^m e_i|| = 1, \text{ for } i \leq 0.$$

Let $\{k_n\}_{n=0}^{\infty}$ to be such that

$$k_0 = 0, \text{ and } \quad k_n = 4^n \text{ for } n \geq 1. \tag{1.10}$$

Then it satisfies

$$k_n \geq \max\{4k_{n-1}, 4n\} \text{ for } n \geq 1. \tag{1.11}$$

We have

$$||S^{k_n} g_n||^2 = || \sum_{i=-2n}^{0} \widehat{g_n}(i) S^{k_n} e_i ||^2 \leq n 2^{2n} \left(\frac{1}{2^{k_n - 2n}} \right)^2 = \left(\frac{\sqrt{n}}{2^{k_n - 3n}} \right)^2 \leq \left(\frac{1}{2^{k_{n-1}} 2^n} \right)^2,$$

using (1.9) , $||g_n||^2 \leq n2^{2n}$ by (1.6) and

$$k_n \geq k_{n-1} + 4n + \log_2 \sqrt{n}, \text{ from (1.10)}.$$

Thus $\sum_{n=1}^{\infty} ||S^{k_n} g_n|| < \infty$ and can define $g = \sum_{n=1}^{\infty} S^{k_n} g_n$ as a vector in $l_2(\mathbb{Z})$. The claim is that g is a weakly hypercyclic vector for T.

Define the vectors

$$\varphi_m = \sum_{n=1}^{m} T^{k_m - r} S^{k_n} g_n \text{ and } \psi_m = \sum_{n=m+1}^{\infty} T^{k_m - r} S^{k_n} g_n.$$

We'll show that for every given $r \geq 1$, $||\psi_{\nu(r,s)}|| \to 0$ as $s \to \infty$ and $0 \in \{\varphi_{\nu(r,s)}\}^{w}_{s \in \mathbb{N}}$.

$$||\psi_{\nu(r,s)}|| \leq \sum_{n=\nu(r,s)+1}^{\infty} ||T||^{k_{\nu(r,s)} - r} ||S^{k_n} g_n|| \leq \sum_{n=\nu(r,s)+1}^{\infty} 2^{k_{\nu(r,s)} - r} \frac{1}{2^{k_{n-1}} 2^n} \leq$$
$$\leq \sum_{n=\nu(r,s)+1}^{\infty} \frac{1}{2^n} \leq \frac{1}{2^{\nu(r,s)}} \to 0 \text{ as } s \to \infty.$$

Next we show that

$$||\varphi_{\nu(r,s)}||^2 \leq (a_r + s) \log(a_r + s). \tag{1.12}$$

First note that

$$\langle S^{k_n} g_n, e_i \rangle \neq 0 \text{ only if } k_n - 2n \leq i \leq k_n,$$

and if $n' > n \geq 1$ then $k_{n'} - 2n' \geq 4k_{n'-1} - 2n' > k_{n'-1} \geq k_n$, since $3 \cdot 4^{n'-1} >$

$2n'$ for $n' \geq 2$. It follows that for fixed $m = \nu(r,s)$, for any integer i, there exists at most one n such that $\langle T^{k_m - r} S^{k_n} g_n, e_i \rangle \neq 0$. Thus the terms that make up the sum that defines φ_m have disjoint support (and thus, are orthogonal), so

$$||\varphi_m|| = \sum_{n=1}^{m} ||T^{k_m - r} S^{k_n} g_n||^2.$$

But from $r \leq \nu(r,s) = m$ we have $k_m - r - k_n \geq 4k_{m-1} - r - k_n > (k_{m-1} - r) + (k_{m-1} - k_n) \geq 0$, so we can write

$$||T^{k_m - r} S^{k_n} g_n||^2 = || \sum_{i=-2n}^{0} \widehat{g_n}(i) T^{k_m - r - k_n} e_i ||^2 = ||g_n||^2$$

Next we show that for fixed $r \geq 1$ the elements of the sequence $\{\varphi_{\nu(r,s)}\}_{s \in \mathbb{N}}$ have disjoint support. Observe first that

$$\langle \varphi_{\nu(r,s)}, e_i \rangle \neq 0 \text{ only if } -k_{\nu(r,s)} + r + k_1 - 2 \leq i \leq -k_{\nu(r,s)} + r + k_{\nu(r,s)-1}.$$

But if $s' > s \geq 1$ then

$$k_{\nu(r,s')} \geq 4k_{\nu(r,s')-1} > k_{\nu(r,s')-1} + 2 + k_{\nu(r,s)},$$

and thus

$$-k_{\nu(r,s')} + r + k_{\nu(r,s')-1} < -k_{\nu(r,s)} + r - 2 < -k_{\nu(r,s)} + r + k_1 - 2.$$

This implies that for every given integer i, there is at most one integer s with $\langle \varphi_{\nu(r,s)}, e_i \rangle \neq 0$, so the elements of the sequence have disjoint support. This together with (1.12) gives by Lemma 1.2 that $0 \in \overline{\{\varphi_{\nu(r,s)}\}}^{w}_{s \in \mathbb{N}}$.

Now we show that g is weakly hypercyclic for T. Since $\{h_r\}_{r \geq 1}$ is norm dense, it suffices to show that $\{h_r : r \geq 1 \subset \overline{Orb(T,g)}^{w}$. Let $r \geq 1$, $\varepsilon > 0$, and $y_1, \ldots, y_t$ be t

nonzero vectors in $l_2(\mathbb{Z})$. Let $\gamma = \max\{||y_j|| : 1 \leq j \leq t\}$. There exists S such that

$$||\psi_{\nu(r,s)}|| \leq \frac{\varepsilon}{2\gamma} \text{ for all } s \geq S,$$

and there exists $s_0 > S$ such that

$$|\langle \varphi_{\nu(r,s_0)}, y_j \rangle| \leq \frac{\varepsilon}{2} \text{ for } 1 \leq j \leq t.$$

If we let $N = k_{\nu(r,s_0)} - r$ then

$$T^N g = \varphi_{\nu(r,s_0)} + h_r + \psi_{\nu(r,s)}.$$

Hence, for all $j = 1, \ldots, t$ we have

$$\begin{aligned} |\langle T^N g - h_r, y_j \rangle| &\leq \langle \varphi_{\nu(r,s_0)}, y_j \rangle| + ||\psi_{\nu(r,s)}|| \cdot ||y_j|| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2\gamma}\gamma = \varepsilon, \end{aligned}$$

which completes the proof. □

1.7. Overview of the results of this dissertation

My research, which might be regarded as a continuation of the investigation begun in [15], is concerned with enlarging the class of non-orbit-transitive operators. For this purpose, it is important to have more examples of operators in $\mathcal{L}(\mathcal{H})$ with "strange" orbits. (There are several such examples in [15]).

The first problem we have set ourselves was to produce an operator T in $\mathcal{L}(\mathcal{H})$ with $\sigma_p(T) \cup \sigma_p(T^*) = \emptyset$, such that for every $x \neq 0$ in $\mathcal{H}$,

$$\liminf_{n\in\mathbb{N}} \|T^n x\| = 0 \ , \quad \limsup_{n\in\mathbb{N}} \ || \ T^n x|| = \infty, \tag{1.13}$$

(Beauzamy sketched a construction in this direction in [4, Section 4 of Chapter III], but without details and, in any case, any such T satisfying his requirements has $\sigma_p(T^*) \neq \emptyset$). This is done in Chapter II, where we construct an operator T in $\mathcal{L}(\mathcal{H})$ such that for every nonzero vector x, the sequence $\{||T^n x||\}_{n\in\mathbb{N}}$ is dense in $\mathbb{R}_+$, but despite this, T is not hypercyclic (*i.e.*, no vector in $\mathcal{H}$ has a dense orbit).

In Chapter III we show that certain classes of operators consist entirely of non-weakly-hypertransitive operators, thereby generalizing the results of [15]. In particular, we show that if $T \in \mathcal{L}(\mathcal{H})$ and two of the three numbers representing the essential spectral radius, essential numerical radius, and essential norm of T, coincide, then for every invertible $S \in \mathcal{L}(\mathcal{H})$ and every compact K in $\mathcal{L}(\mathcal{H})$, $STS^{-1} + K$ fails to be weakly hypertransitive. As a corollary we have that no operator of the form $SNS^{-1} + K$, where S is invertible, N is normal, and K in compact, is weakly hypertransitive. Along the way we show that K. Ball's complex-plank theorem [2] is equivalent to a (slightly stronger) version of an old theorem of Beauzamy [4].

Recall that an operator $T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ is 2-normal if T is unitarily equivalent to a 2×2 matrix (N_{ij}), where the N_{ij} are mutually commuting normal operators, and it is known [14] that 2-normal operators have nontrivial hyperinvariant subspaces. In Chapter IV we show that no compact perturbation of certain 2-normal operators (which in general satisfy $\|T\|_e > r_e(T)$) can be orbit-transitive. This answers a question raised in [15]. Our main result herein is that if T belongs to a certain class of 2-normal operators in $\mathcal{L}(\mathcal{H}^{(2)})$ and there exist two constants $\delta, \rho > 0$ satisfying $\|T^k\|_e > \rho k^\delta$ for all $k \in \mathbb{N}$, then for every compact operator K, the operator $T + K$ is not orbit-transitive. This seems to be the first result that yields non-orbit-transitive operators in which such a modest growth rate on $\|T^k\|_e$ is sufficient to give an orbit $\{T^k x\}$ tending to infinity.

CHAPTER II

A NONHYPERCYCLIC OPERATOR*

In this chapter we construct an operator T on a (separable, complex) Hilbert space such that for every nonzero vector x, the sequence $\{||T^n x||\}_{n\in\mathbb{N}}$ is dense in $\mathbb{R}_+$, but despite this, T is not hypercyclic (*i.e.*, no vector in $\mathcal{H}$ has a dense orbit). In addition, this operator has the property that there are subsequences $\{r_n\}$ and $\{q_n\}$ of $\mathbb{N}$ such that $T^{r_n} \to 0$ and $T^{q_n} \to +\infty$ (properly defined) in the strong operator topology. Finally, neither T nor T^* has point spectrum. This partially answers a question in [15] and provides a counterexample to some reasonable conjectures.

2.1. Problem settings

In the last fifteen or so years, the interest in properties of orbits of an operator has increased dramatically. (It should be said at once that much of the early work on orbits of operators – on Hilbert spaces or otherwise – is due to Bernard Beauzamy; *cf.* [4, Chapter III]). In particular, recall that an operator T in $\mathcal{L}(\mathcal{H})$ is called *hypercyclic* if there is at least one vector x (and therefore a dense G_δ of such vectors) whose orbit $\{T^n x\}_{n\in\mathbb{N}_0}$ is dense in $\mathcal{H}$. There has been considerable progress in the direction of showing that the class of hypercyclic operators is much larger than was originally suspected. (See, for example, the excellent survey article [13]). The reader will recall that the question whether there exists an operator in $\mathcal{L}(\mathcal{H})$ such that every nonzero vector in $\mathcal{H}$ has dense orbit (such an operator is sometimes called *hypertransitive*) is still open. For some recent progress in that direction see [15]. This leads naturally to

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the question of finding necessary and sufficient conditions that an operator T in $\mathcal{L}(\mathcal{H})$ be hypercyclic. (A useful sufficient condition for hypercyclicity has been known for some time; *cf.* [13, Theorem 4].) In this connection, observe that if T is hypercyclic, then there exists a dense-G_δ set $\mathcal{D}$ of vectors x, such that $\{||T^n x||\}_{n\in\mathbb{N}_0}$ is dense in $\mathbb{R}_+$, and for all $y \in \mathcal{H}\backslash(0)$, $\{\langle T^n x, y\rangle\}$ is dense in $\mathbb{C}$. Therefore the following problem would seem to be of interest:

(P) If $T \in \mathcal{L}(\mathcal{H})$ and there is a dense set $\mathcal{D} \subset \mathcal{H}$ such that for $x \in \mathcal{D}$, $||T^n x||\}_{n\in\mathbb{N}_0}$ is dense in $\mathbb{R}_+$, must T be hypercyclic?

It is the purpose of this note to answer some questions, raised by Carl Pearcy, related to Problem P. In particular, the main result is the following theorem which shows that (even with a stronger hypothesis) Problem P has a negative answer:

Theorem 2.1.1. *There exists an operator T in $\mathcal{L}(\mathcal{H})$ with the following properties*:

(a) *for every $x \neq 0$ in $\mathcal{H}$ the sequence $\{||T^n x||\}_{n\in\mathbb{N}_0}$ is dense in $\mathbb{R}_+$*;

(b) *there exist subsequences $\{r_n\}$ and $\{q_n\}$ of $\mathbb{N}$ such that*

$$T^{r_n} \overset{SOT}{\rightarrow} 0, \qquad T^{q_n} \overset{SOT}{\rightarrow} +\infty;$$

(c) *there exists no vector x in $\mathcal{H}$ such that the orbit $\{T^n x\}_{n\in\mathbb{N}}$ is dense in $\mathcal{H}$* (i.e., *T is not hypercyclic*); *and*

(d) $\sigma_p(T) \;\cup\; \sigma_p(T^*) = \varnothing$.

Remark 2.1.2. Theorem 2.1.1 shows clearly that to make progress on the program proposed in [15] (*i.e.*, to show that no operator in $\mathcal{L}(\mathcal{H})$ is hypertransitive), one cannot hope to succeed by consideration only of the collection $\{\{||T^n x||\}\colon\ x \in \mathcal{H}\backslash(0)\}$ of sequences of norms. Thus the above theorem forecloses one approach to establishing that no operator in $\mathcal{L}(\mathcal{H})$ is hypertransitive.

The proof of Theorem 2.1.1 is set forth in next 3 sections below.

2.2. Some bilateral weighted shifts

We begin to develop the machinery needed to prove Theorem 1.1 by introducing some notation that will remain fixed throughout the paper. In particular, we first choose an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$. If $w = \{w_j\}_{j\in\mathbb{N}}$ is any bounded sequence of (strictly) positive real numbers, we denote by S_w the (forward, unilateral, weighted) shift in $\mathcal{L}(\mathcal{H})$ defined by

$$S_w e_j = w_j e_{j+1}, \qquad j \in \mathbb{N}. \tag{2.1}$$

Let $\{\tilde{e}_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}}$ is another copy of $\mathcal{H}$ and let $w = \{w_j\}_{j\in\mathbb{N}}$ and $\tilde{w} = \{\tilde{w}_j\}_{j\in\mathbb{N}}$ be bounded sequences of (strictly) positive real numbers. Let $S_{\tilde{w}}$ and S_w be the weighted unilateral forward shifts on $\tilde{\mathcal{H}}$ and $\mathcal{H}$ corresponding to the sequences $\tilde{w}$ and w, respectively, and let $B(= B_{\tilde{w},w})$ be the (injective) weighted bilateral shift on $\mathcal{L}(\tilde{\mathcal{H}} \oplus \mathcal{H})$, defined matricially by

$$B = \begin{bmatrix} S_{\tilde{w}}^* & 0 \\ e_1 \otimes \tilde{e}_1 & S_w \end{bmatrix}, \tag{2.2}$$

where, as usual $e_1 \otimes \tilde{e}_1 : \tilde{\mathcal{H}} \to H$ is the rank one operator satisfying $(e_1 \otimes \tilde{e}_1)(\tilde{x}) = \langle \tilde{x}, \tilde{e}_1 \rangle e_1$ for $\tilde{x}$ in $\tilde{\mathcal{H}}$. That B is, indeed, a weighted bilateral shift is clear from the equations

$$B\tilde{e}_j = \tilde{w}_{j-1}\tilde{e}_{j-1}, \; B\tilde{e}_1 = e_1, \; Be_j = w_j e_{j+1}, \quad j \in \mathbb{N}.$$

Moreover, it is easy to see that every injective bilateral weighted shift can be written in the form (2.2) .

Clearly, with B as in (2.2),

$$B^n = \begin{bmatrix} S_{\tilde{w}}^{*n} & 0 \\ F_n & S_w^n \end{bmatrix}, \quad n \in \mathbb{N},$$

where $F_n : \tilde{\mathcal{H}} \to \mathcal{H}$ is given by

$$F_n = (F_n(\tilde{w}, w)) = \sum_{k=0}^{n-1} w_1 \dots w_{n-k-1} e_{n-k} \otimes \tilde{w}_1 \dots \tilde{w}_k \tilde{e}_{k+1}, \qquad n \in \mathbb{N},$$

and we have

$$F_n \tilde{e}_j = \begin{cases} 0, & j > n, \\ (\tilde{w}_1 \dots \tilde{w}_{j-1})(w_1 \dots w_{n-j}) e_{n-j+1}, & 1 \le j \le n. \end{cases} \tag{2.3}$$

For every $\tilde{x} \oplus y \in \tilde{\mathcal{H}} \oplus \mathcal{H}$ we have

$$B^n \begin{bmatrix} \tilde{x} \\ y \end{bmatrix} = \begin{bmatrix} S_{\tilde{w}}^{*n} \tilde{x} \\ F_n \tilde{x} + S_w^n y \end{bmatrix}, \quad n \in \mathbb{N},$$

and since $\operatorname{ran}(F_n) \subset \bigvee\{e_1, \dots, e_n\}$ and the range of S_w^n is orthogonal to $e_1, \dots, e_n$, we obtain

$$||B^n(\tilde{x} \oplus y)||^2 = ||S_{\tilde{w}}^{*n} \tilde{x}||^2 + ||F_n \tilde{x}||^2 + ||S_w^n y||^2, \; n \in \mathbb{N}, \; \tilde{x} \oplus y \in \tilde{\mathcal{H}} \oplus \mathcal{H}. \tag{2.4}$$

Lemma 2.2.1. *Suppose the (bounded) weight sequences $w = \{w_j\}_{j \in \mathbb{N}}$ and $\tilde{w} = \{\tilde{w}_j\}_{j \in \mathbb{N}}$ of positive numbers are bounded below (by some $\varepsilon > 0$), and w has the property that there exists a subsequence $\{q_n\}$ of $\mathbb{N}$ such that $S_w^{q_n} \overset{SOT}{\to} +\infty$. Then $B := B_{\tilde{w}, w}$ satisfies $B^{q_n} \overset{SOT}{\to} +\infty$.*

Proof. Observe first that since w, $\tilde{w}$ are bounded below, $B^{-1} \in \mathcal{L}(\tilde{\mathcal{H}} \oplus \mathcal{H})$. For each nonzero vector $\tilde{x} \oplus y \in \tilde{\mathcal{H}} \oplus \mathcal{H}$, and every $k_0 \in \mathbb{N}$, we have $||B^{q_n}(\tilde{x} \oplus y)|| =$

$||B^{-k_0}B^{q_n}B^{k_0}(\tilde{x}\oplus y)|| \geq ||B||^{-k_0}||B^{q_n}(S_{\tilde{w}}^{*k_0}\tilde{x}, F_{k_0}\tilde{x}+S_w^{k_0}y)|| \geq ||B||^{-k_0}||S^{q_n}(F_{k_0}\tilde{x}+S_w^{k_0}y)||$ by (2.4), and since obviously $F_{k_0}\tilde{x}+S_w^{k_0}y$ is nonzero for $k_0\in\mathbb{N}$ sufficiently large, the result is immediate. □

The following elementary lemma, which needs no proof, will be useful below.

Lemma 2.2.2. *Let $\{T_n\}$ be a sequence in $\mathcal{L}(\widetilde{\mathcal{K}},\mathcal{K})$ where $\widetilde{\mathcal{K}}$ and $\mathcal{K}$ are Hilbert spaces. Suppose $\{f_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for $\widetilde{\mathcal{K}}$, and there exist subsequences $\{T_{s_n}\}$ and $\{T_{t_n}\}$ of $\{T_n\}$ satisfying*

(i) *for all $n\in\mathbb{N}$, and $j,k\in\mathbb{N}$ with $j\neq k$, $\langle T_nf_j,\ T_nf_k\rangle=0$,*

(ii) *for all $j\in\mathbb{N}$, $\lim_{n\to\infty}||T_{s_n}f_j||=0$,*

(iii) *there exists a positive constant M such that for all $j,n\in\mathbb{N}$, $||T_{s_n}f_j||<M$,*

(iv) *for all $j\in\mathbb{N}$, $\lim_{n\to\infty}||T_{t_n}f_j||=+\infty$.*

Then $T_{s_n}\overset{SOT}{\to}0$ and $T_{t_n}\overset{SOT}{\to}+\infty$.

The next lemma, which is also elementary, shows that all operators T in $\mathcal{L}(\mathcal{K})$ satisfying (b) of Theorem 2.1.1 have empty point spectrum.

Lemma 2.2.3. *If $T\in\mathcal{L}(\mathcal{K})$ and there exist subsequences $\{p_n\}$ and $\{q_n\}$ of $\mathbb{N}$ with $T^{p_n}\overset{SOT}{\to}0$ and $T^{q_n}\overset{SOT}{\to}+\infty$, then $\sigma_p(T)=\varnothing$ and $\sigma_p(T^*)\subset\mathbb{D}$.*

Proof. Suppose $\lambda\in\sigma_p(T)$ and $x\in\mathcal{K}\backslash(0)$ is such that $Tx=\lambda x$. Then $T^{p_n}x=\lambda^{p_n}x$, $T^{q_n}x=\lambda^{q_n}x$, and from the hypothesis we get that $\lambda^{p_n}\to 0$, and $\lambda^{q_n}\to+\infty$, which is impossible. Moreover, we have $T^{p_n}\overset{WOT}{\to}0$ and $(T^*)^{p_n}\overset{WOT}{\to}0$. Thus, if $T^*x=\xi x$ with $x\in\mathcal{K}\backslash(0)$, we get $\langle(T^*)^{p_n}x,x\rangle=\xi^{p_n}||x||^2\to 0$, so $\xi\in\mathbb{D}$. □

Can the conclusions of Lemma 2.2.3 be strengthened to give also that $\sigma_p(T^*)=\varnothing$? The answer is negative as Example 2.3.4 (below) shows. On the other hand, if the hypotheses are strengthened somewhat, the answer is positive, as we now demonstrate.

Lemma 2.2.4. *Let $B = B_{\tilde{w},w}$ be an injective bilateral weighted shift in $\mathcal{L}(\tilde{\mathcal{H}} \oplus \mathcal{H})$ with the properties*

(i) *there exist subsequences $\{p_n\}$ and $\{q_n\}$ of $\mathbb{N}$ with $B^{p_n} \overset{SOT}{\to} 0$ and $B^{q_n} \overset{SOT}{\to} +\infty$, and*

(ii) *$\tilde{w}$ has the property that the sequence of products $\{\tilde{w}_1 \dots \tilde{w}_n\}_{n\in\mathbb{N}}$ does not converge to zero.*

Then $\sigma_p(B) = \varnothing$ and $\sigma_p(B^*) = \varnothing$.

Proof. By the previous lemma we have that $\sigma_p(B) = \varnothing$ and $\sigma_p(B^*) \subset \mathbb{D}$. Moreover, since B^* is injective with dense range, $0 \notin \sigma_p(B^*)$. If some $0 \neq \lambda \in \mathbb{D}$ were an eigenvalue of B^*, a trivial computation shows that all of the Fourier coefficients of any corresponding eigenvector $\tilde{x} \oplus y$ would be nonzero. If we write $\tilde{x} = \sum\limits_{j\in\mathbb{N}} \tilde{\alpha}_j \tilde{e}_j$, then, in particular, $\tilde{\alpha}_1$ is nonzero. We have $(B^*)^n(\tilde{x} \oplus y) = \lambda^n(\tilde{x} \oplus y)$, $n \in \mathbb{N}$, and evaluating $\langle (B^*)^n(\tilde{x} \oplus y), \tilde{e}_{n+1}\rangle$ gives

$$\tilde{\alpha}_1(\tilde{w}_1 \dots \tilde{w}_n)\tilde{e}_{n+1} = \tilde{\alpha}_{n+1}\lambda^n \tilde{e}_{n+1},\ n \in \mathbb{N}.$$

It follows that $\tilde{w}_1 \dots \tilde{w}_n = (\tilde{\alpha}_{n+1}/\tilde{\alpha}_1)\lambda^n \to 0$ as $n \to \infty$, contradicting hypothesis (ii). □

2.3. Some particular weight sequences

In order to prove Theorem 2.1.1, we will use the notation and terminology of Section 2, and construct a particular operator $T = B_{\tilde{w},w}$ having the desired properties (a)-(d). (Note that Lemma 2.2.4 shows that if $B_{\tilde{w},w}$ satisfies (b) of Theorem 2.1.1 and (ii) of Lemma 2.2.4, then $B_{\tilde{w},w}$ also satisfies (d).) Thus we must construct particular weight sequences w and $\widetilde{w}$. To this end, we first construct the sequence w and we need some purely arithmetical properties of sequences of products of positive real numbers.

Let $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ be two strictly increasing subsequences of $\mathbb{N}$ defined recursively by

$$\begin{cases} p_1 = 1, \quad p_{n+1} = 2n(p_n+1), \quad n \in \mathbb{N}, \\ q_n = 2np_n, \quad n \in \mathbb{N}, \end{cases} \tag{2.5}$$

(and hence $p_n < q_n < p_{n+1}$), and let $w = \{w_j\}_{j\in\mathbb{N}}$ be defined by:

$$\begin{cases} w_1 = 1, \\ w_j = 2^{1/p_n}, \quad p_n < \ j \le q_n, \\ w_j = 1/2, \quad q_n < j \le p_{n+1}. \end{cases} \tag{2.6}$$

Lemma 2.3.1. *The sequence w just defined has the following properties*:

(i) *the product $w_1 \dots w_j$ reaches a local minimum (as a function of j) when $j = p_n$ and a local maximum when $j = q_n$; moreover $w_1 \dots w_{p_n} = (1/2)^{n-1}$ and $w_1 \dots w_{q_n} = 2^n$ for all $n \in \mathbb{N}$;*

(ii) *for all $l, n \in \mathbb{N}$, $w_{l+1} \dots w_{l+p_n} \le 2$;*

(iii) *for all $k, l, n \in \mathbb{N}$ satisfying $k < n$ and $p_k < l \le p_{k+1}$, we have*

$$w_{l+1} \dots w_{l+p_n} \le 4(1/2)^{n-k},$$

and consequently for each l, the product of p_n consecutive weights with the first index equal to $l+1$ tends to zero as n tends to infinity;

(iv) *for all $k, n \in \mathbb{N}$, $w_k \cdots w_{q_n+k-1} > 2^{n-2k}$.*

Proof. (i) Note that $w_{p_n+1} \cdots w_{q_n} = 2^{2n-1}$ and $w_{q_n+1} \cdots w_{p_{n+1}} = (1/2)^{2n}$. Thus $w_{p_n+1} \cdots w_{p_{n+1}} = 1/2$. It follows that for $n > 1$, $w_1 \cdots w_{p_n} = w_2 \cdots w_{p_n} = (1/2)^{n-1}$ and $w_1 \cdots w_{q_n} = 2^n$. The product $w_1 \cdots w_j$ is increasing for j increasing between p_n and q_n, since weights larger than 1 are being inserted into the product, and is decreasing for j between q_n and p_{n+1}, since the weights inserted are equal to $1/2$.

(ii) For $p_n < j \leq q_n$ we have $w_j = 2^{1/p_n}$ and for $q_n \leq j, \quad w_j \leq 2^{1/p_n}$. Thus for $l > p_n$ we have $w_{l+1} \dots w_{l+p_n} \leq 2^{1/p_n} \dots 2^{1/p_n} = 2$. For $l \leq p_n$, we have $p_n + l \leq 2p_n < q_n$, so

$$w_{l+1} \dots w_{l+p_n} = \frac{(w_1 \dots w_{p_n})(w_{p_n+1} \dots w_{l+p_n})}{w_1 \dots w_l} = \frac{(1/2)^{n-1}(2^{1/p_n})^l}{w_1 \dots w_l} \leq 2,$$

since $w_1 \dots w_l$ is bounded below by $(1/2)^{n-1}$, by (i). Thus (ii) is satisfied.

(iii) Let $k \in \mathbb{N}$, with $k < n$, be such that $p_k < l \leq p_{k+1}$. Then we have $l \leq p_{k+1} \leq p_n$ and $p_n + l \leq 2p_n < q_n$, and thus

$$w_{l+1} \dots w_{l+p_n} = \frac{(w_{p_k+1} \dots w_{p_n})(w_{p_n+1} \dots w_{l+p_n})}{w_{p_k+1} \dots w_l} \leq 4 \left(\frac{1}{2}\right)^{n-k},$$

since $w_{p_k+1} \dots w_{p_n} = (1/2)^{n-k}, \quad w_{p_n+1} \dots w_{l+p_n} = (2^{1/p_n})^l \leq 2$, and $w_{p_k+1} \dots w_l \geq 1/2$. This last inequality follows since the product $w_{p_k+1} \cdots w_l$ increases with l for $p_k < l \leq q_k$ (since for these indices $w_l > 1$), and decreases for $q_k < l \leq p_{k+1}$ (since for these indices $w_l = 1/2$), reaching a a minimum value of $w_{p_k+1} \cdots w_{p_{k+1}} = 1/2$.

(iv) $w_k \cdots w_{q_n+k-1} = (w_1 \cdots w_{q_n})(w_{q_n+1} \cdots w_{q_n+k-1})/(w_1 \cdots w_{k-1}) > 2^{n-2k}$, since $w_{q_n+1} \cdots w_{q_n+k-1} \geq (1/2)^k$, and we have that the maximum value attained by the product of all weights up to index $k \ (< q_k)$ is smaller than 2^k. □

Remark 2.3.2. The construction just given of the weight sequence w is a modification and an extension of a construction of Beauzamy of a certain unilateral weighted shift (*cf.* [4, page 69]).

Proposition 2.3.3. *With the subsequences* $\{p_n\}$ *and* $\{q_n\}$ *of* $\mathbb{N}$ *as defined in* (2.5) *and the weight sequence* w *as defined in* (2.6)*, the unilateral weighted shift* S_w *satisfies* $S_w^{p_n} \overset{SOT}{\to} 0$ *and* $S_w^{q_n} \overset{SOT}{\to} +\infty$.

Proof. By property (ii) of Lemma 2.3.1 we get $||S_w^{p_n} e_j|| \leq 2$ for all n and j, and by (iii) of the same lemma we have $||S_w^{p_n} e_j|| \to 0$ as $n \to \infty$. Thus the operator S_w

(with $s_n = p_n$ and $M = 2$) satisfies the properties (ii) and (iii) of Lemma 2.2.2. For a fixed basis vector e_k, we have that $||S_w^{q_n} e_k|| > 2^{n-2k}$ by (iv) of Lemma 2.3.1. Thus the operator S_w with the sequence q_k satisfies (iv) of Lemma 2.2.2 (with $t_n = q_n$) and the conclusion follows from Lemma 2.2.2. $\square$

Lemma 2.2.3 gives rise to the question whether under its hypotheses the stronger conclusion that $\sigma_p(T^*) = \varnothing$ is true. The following example shows that this is not the case.

Example 2.3.4. Let w be the sequence of weights given by (2.6) and let the subsequences $\{p_n\}$ and $\{q_n\}$ of $\mathbb{N}$ be given by (2.5). By Proposition 2.3.3 we have that $S_w^{p_n} \overset{SOT}{\to} 0$ and $S_w^{q_n} \overset{SOT}{\to} +\infty$. Now define $\widehat{w} = \{\tilde{w}_j\}_{j\in\mathbb{N}}$ by $\tilde{w}_j = 1/2$, $j \in \mathbb{N}$. Then with $B = B_{\widehat{w},w}$ we have immediately $B^{q_n} \overset{SOT}{\to} +\infty$, by Lemma 2.2.1. Since $||S_{\widehat{w}}^*|| = 1/2 < 1$, we have that $S_{\widehat{w}}^{*p_n} \overset{SOT}{\to} 0$. Also, for $j, n \in \mathbb{N}$ we have by (5), $||F_{p_n}\tilde{e_j}|| = (\tilde{w}_1 \dots \tilde{w}_{j-1})(w_1 \dots w_{p_n-j}) = (1/2)^{j-1}(w_1 \dots w_{p_n-j}) \leq w_1 \dots w_{p_n-1} = (1/2)^{n-2}$. Thus by Lemma 2.2.2, $F_{p_n}(\widehat{w}, w) \overset{SOT}{\to} 0$. It follows by (6) that we also have $B^{p_n} \overset{SOT}{\to} 0$, so B satisfies the hypotheses of Lemma 2.2.3. But using a well known result (*cf.* [26, Theorem 9, page 71]) we have that $\sigma_p(B^*)$ includes the annulus $\{\frac{1}{2} < |z| < 1\}$. $\square$

Next we shall construct a sequence $\widetilde{w}$ such that, with w defined by (2.6), the operator $T := B_{\widetilde{w},w}$ satisfies Theorem 2.1.1. We define recursively a sequence $\{s_n\}$ in terms of $\{p_n\}_{n\in\mathbb{N}}$, the sequence defined by (2.5). Let

$$s_1 = 1, \qquad s_{n+1} = s_n \cdot (p_{s_n+1} - 2s_n) + 2s_n, \; n \in \mathbb{N}, \tag{2.7}$$

and denote by $\{r_n\}_{n\in\mathbb{N}}$ the subsequence of $\{p_n\}_{n\in\mathbb{N}}$ defined by

$$r_n = p_{s_n+1}, \qquad n \in \mathbb{N}. \tag{2.8}$$

We will show that

$$s_n < 2s_n < r_n \le s_{n+1}, \qquad n \in \mathbb{N}, \tag{2.9}$$

and thereafter define the weight sequence $\widetilde{w} = \{\tilde{w}_j\}_{j\in\mathbb{N}}$ by

$$\begin{cases} \tilde{w}_1 = 1, \\ \tilde{w}_j = 1/2, \qquad s_n < j \le 2s_n, \quad n \in \mathbb{N}, \\ \tilde{w}_j = 2^{1/(r_n - 2s_n)}, \qquad 2s_n < j \le s_{n+1}, \quad n \in \mathbb{N}. \end{cases} \tag{2.10}$$

We notice that $p_{n+1} \ge 4n$ for all $n \in \mathbb{N}$, since, from the definition of p_n we have $p_1 = 1$ and $p_{n+1} \ge 2np_n$, and thus $p_{n+1} \ge 2^n n!$. It follows that $p_{s_n+1} \ge 4s_n$, so $r_n \ge 4s_n$, and this gives $2s_n < r_n$ and also

$$r_n - 2s_n \ge (1/2)r_n, \quad n \in \mathbb{N}. \tag{2.11}$$

Since $r_1 = s_2 = 4$, and $s_n > 2$ for $n \ge 2$, we have $r_n \le s_{n+1}$ for $n \in \mathbb{N}$, so (2.9) is established.

Lemma 2.3.5. *The sequence $\widetilde{w}$ defined by* (2.10) *has the properties*:

(i) *the product $\tilde{w}_1 \dots \tilde{w}_m$ of the first m weights from $\widetilde{w}$ attains a local maximum equal to* 1 (*as a function of m*) *when $m = s_n$, and a local minimum equal to* $(1/2)^{s_n}$ *when $m = 2s_n$*;

(ii) *for $j, n \in \mathbb{N}$, we have $\tilde{w}_{j+1} \dots \tilde{w}_{j+r_n} \le 8$* ;

(iii) *for $j, n \in \mathbb{N}$, with $1 \le j \le r_n$ we have $(\tilde{w}_1 \dots \tilde{w}_{j-1})(w_1 \dots w_{r_n - j}) \le 2$; moreover, if $1 \le j < s_n - n$ then $(\tilde{w}_1 \dots \tilde{w}_{j-1})(w_1 \dots w_{r_n - j}) \le (1/2)^{n-1}$*; and

(iv) $(S^*_{\widetilde{w}})^{r_n} \overset{SOT}{\to} 0$ *and* $F_{r_n}(\widetilde{w}, w) \overset{SOT}{\to} 0$.

Proof. (i) For n fixed and $s_n < j \le 2s_n$, $\tilde{w}_j = 1/2$, and thus $\tilde{w}_{s_n+1} \dots \tilde{w}_{2s_n} = (1/2)^{s_n}$ and the product $\tilde{w}_1 \dots \tilde{w}_j$ is decreasing. On the other hand, for $2s_n < j \le s_{n+1}$, $\tilde{w}_j = 2^{1/(r_n - 2s_n)}$, so $\tilde{w}_{2s_n+1} \dots \tilde{w}_{s_{n+1}} = (2)^{s_n}$ since $s_{n+1} - 2s_n = s_n \cdot (r_n - 2s_n)$, and the product $\tilde{w}_1 \dots \tilde{w}_j$ and is increasing for such j. Thus $\tilde{w}_{s_n+1} \dots \tilde{w}_{s_{n+1}} = 1$, and it follows that

$$\tilde{w}_1 \dots \tilde{w}_{s_n} = 1 \ , \quad n \in \mathbb{N}, \tag{2.12}$$

and

$$\tilde{w}_1 \dots \tilde{w}_{2s_n} = (1/2)^{s_n}, \ n \in \mathbb{N}. \tag{2.13}$$

We also note that, via (2.12), we have

$$\tilde{w}_{2s_n+1} \dots \tilde{w}_{r_n} = 2, \quad n \in \mathbb{N}. \tag{2.14}$$

Turning to (ii), from (2.13) and (2.14) we have that $\tilde{w}_1 \dots \tilde{w}_{r_n} = (1/2)^{s_n - 1}, \quad n \in \mathbb{N}$. For $j \geq r_n$, we get that $\tilde{w}_j$ is either $1/2$ or smaller than $\tilde{w}_{r_n} = 2^{1/(r_n - 2s_n)}$, since by (2.9) and (2.11), $r_{n+1} - 2s_{n+1} \geq (1/2)r_{n+1} > r_n \geq r_n - 2s_n$. Thus $\tilde{w}_j \leq \tilde{w}_{r_n} = 2^{1/(r_n - 2s_n)} \leq 2^{2/r_n}$, and it follows that

$$\tilde{w}_{j+1} \dots \tilde{w}_{j+r_n} \leq 4, \quad j \geq r_n.$$

For $1 \leq j < r_n$ we write $\tilde{w}_{j+1} \dots \tilde{w}_{j+r_n} = (\tilde{w}_1 \dots \tilde{w}_{r_n})(\tilde{w}_{r_n+1} \dots \tilde{w}_{j+r_n})/(\tilde{w}_1 \dots \tilde{w}_j)$, and we have the following estimates for the three products in parentheses: $\tilde{w}_1 \dots \tilde{w}_{r_n} = (1/2)^{s_n - 1}$, as already noted. Next, $\tilde{w}_{r_n+1} \dots \tilde{w}_{j+r_n} \leq 4$, since the weights involved are fewer than r_n and smaller than $2^{2/r_n}$. Finally, $\tilde{w}_1 \dots \tilde{w}_j \geq (1/2)^{s_n}$, since $(1/2)^{s_n}$ is the smallest value that can be attained by the product of the first j weights in $\tilde{w}$ for $j < r_n$. Thus

$$\tilde{w}_{j+1} \dots \tilde{w}_{j+r_n} \leq 8, \qquad 1 \leq j < r_n.$$

and consequently (ii) is true.

(iii) With the notation from (2.3) we show first that for $1 \leq j \leq r_n$ we have $||F_{r_n}\tilde{e}_j|| = (\tilde{w}_1 \dots \tilde{w}_{j-1})(w_1 \dots w_{r_n-j}) \leq 2$. By Lemma 2.3.1(i), for $1 \leq j \leq r_n$, we get that $w_1 \dots w_{r_n-j} \leq 2^{s_n}$, with equality for $j = r_n - q_{s_n} = 2s_n$. It is easier to estimate $||F_{r_n}\tilde{e}_j||$ in the case $2s_n < j \leq r_n$: for such j we have $||F_{r_n}\tilde{e}_j|| = (\tilde{w}_1 \dots \tilde{w}_{2s_n})(\tilde{w}_{2s_n+1} \dots \tilde{w}_{j-1})(w_1 \dots w_{r_n-j}) \leq (\tilde{w}_{2s_n+1} \dots \tilde{w}_{j-1}) < 2$, by using (2.13) and (2.14).

For the case $1 \leq j \leq 2s_n$ we notice that $||F_{r_n}\tilde{e}_{2s_n}|| = 2$ and we can obtain $||F_{r_n}\tilde{e}_j||$ from $||F_{r_n}\tilde{e}_{2s_n}||$ by replacing those weights $\tilde{w}_j, \dots, \tilde{w}_{2s_n-1}$, of $\widetilde{w}$ that are either 1/2 or larger than 1, with the weights $w_{r_n-2s_n+1}, \dots, w_{r_n-j}$ which are all equal to 1/2 (since these weights have indices between q_{s_n} and p_{s_n+1}). This gives $||F_{r_n}\tilde{e}_j|| \leq ||F_{r_n}\tilde{e}_{2s_n}||$.

Note that if we have $1 \leq j < s_n - n$, then the above estimates hold; moreover, we have at least n weights $\tilde{w}_j$ from $\widetilde{w}$ that are larger than 1 (the ones with j between $s_n - n$ and s_n, since $s_n - n > 2s_{n-1}$) that are replaced with weights with value 1/2 from w, and thus $||F_{r_n}\tilde{e}_j|| \leq (1/2)^{n-1}$.

Finally to prove (iv), from (ii) we have $||(S^*_{\widetilde{w}})^{r_n}\tilde{e}_j|| \leq 8$ for all $j,\ n \in \mathbb{N}$, and since $(S^*_{\widetilde{w}})$ is a backward shift, we have $\lim_n ||(S^*_{\widetilde{w}})^{r_n}\tilde{e}_j|| = 0$ for all $j \in \mathbb{N}$. Thus Lemma 2.2.2 gives $(S^*_{\widetilde{w}})^{r_n} \overset{SOT}{\rightarrow} 0$. Using that $\lim_{n\to\infty}(s_n - n) = +\infty$, the same lemma gives $F_{r_n}(\widetilde{w}, w) \overset{SOT}{\rightarrow} 0$, since from (iii) we have $||F_{r_n}\tilde{e}_j|| \leq 2$ for all $j,\ n \in \mathbb{N}$ and $\lim_n ||F_{r_n}\tilde{e}_j|| = 0$ for all $j \in \mathbb{N}$. □

2.4. Completion of the proof

For the sequences of weights w and $\widetilde{w}$ defined by (2.6) and (2.10) and the subsequences $\{q_n\}$ and $\{r_n\}$ of $\mathbb{N}$ defined by (2.5) and (2.8), respectively, we have from Proposition 2.3.3 that $S_w^{r_n} \overset{SOT}{\rightarrow} 0$ and $S_w^{q_n} \overset{SOT}{\rightarrow} +\infty$. By Lemma 2.2.1 it follows immediately that the operator $B(= B_{\widetilde{w},w})$ satisfies $B^{q_n} \overset{SOT}{\rightarrow} +\infty$. Lemma 2.3.5(iv) gives that

$(S^*_{\widetilde{w}})^{r_n} \overset{SOT}{\to} 0$ and $F_{r_n}(\widetilde{w}, w) \overset{SOT}{\to} 0$; thus by (2.4) we have that $B^{r_n} \overset{SOT}{\to} 0$. It follows that B satisfies condition (b) of Theorem 2.1.1. Moreover (2.12) gives that the hypothesis (ii) of Lemma 2.2.4 is satisfied and therefore, as noted earlier, B has the desired property (d) of Theorem 2.1.1.

The fact that B is not hypercyclic follows from [12]. Indeed, according to [12, Theorem 3.2], one knows that a necessary and sufficient condition for B to be hypercyclic is that there exists a strictly increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ such that $w_1 \dots w_{n_k} \to 0$ and $\widetilde{w}_1 \dots \widetilde{w}_{n_k} \to +\infty$. But the weight sequence $\widetilde{w}$ has the property that $\widetilde{w}_1 \dots \widetilde{w}_n \leq 1$ for all $n \in \mathbb{N}$ (Lemma 2.3.5(i)), so the second condition cannot be satisfied, and thus B has the desired property (c) of Theorem 2.1.1. $\square$

Finally, we establish below that B satisfies condition (a) of Theorem 2.1.1.

Definition 2.4.1. A sequence $\{\rho_n\}_{n\in S}$ is ε-*dense* in an interval $[0, M]$ if every subinterval of $[0, M]$ of length ε contains at least one ρ_n.

Obviously if an infinite sequence has the property that for all $\varepsilon, M > 0$, it is ε-dense in the interval $[0, M]$, then the sequence is dense in $\mathbb{R}_+$.

The following elementary lemma needs no proof.

Lemma 2.4.2. *Suppose that the sequence $\{\rho_j\}_{j=0}^{\infty} \subset \mathbb{R}_+$ has the properties that for given $\varepsilon, M > 0$ there exists an index $t \in \mathbb{N}$ such that*

(i) $0 < \rho_0 < \varepsilon$;

(ii) *for* $0 \leq j < t$, $\rho_{j+1} - \rho_j < \varepsilon$;

(iii) $\rho_t \geq M$;

Then the finite sequence $\{\rho_j\}_{j=0}^{t}$ is ε-dense in $[0, M]$, and thus the infinite sequence $\{\rho_j\}_{j=0}^{\infty}$ is ε-dense in $[0, M]$.

The following proposition completes the proof of Theorem 2.1.1.

Proposition 2.4.3. *Let $T := B_{\widetilde{w},w}$ with w and $\widetilde{w}$ given by (2.6) and (2.10), and let $\tilde{x} \oplus y$ be a nonzero vector in $\tilde{\mathcal{H}} \oplus \mathcal{H}$. Let $\varepsilon > 0$ and $M > 0$ be arbitrary. Let p_n, q_n, s_n, r_n be as defined by (2.5), (2.7) and (2.8) and set $t_n := q_{s_n+1} - r_n$. For $n \in \mathbb{N}$, define $\rho_j(n) := ||T^{r_n+j}\tilde{x} \oplus y||$, where $0 \leq j \leq t_n$. Then there exists $n_0 \in \mathbb{N}$ such that $\{\rho_j(n)\}_{j=0}^{t_n}$ is $\varepsilon-$dense in $[0, M]$ for all $n \geq n_0$. Consequently, the sequence $\{||T^n(\tilde{x} \oplus y)||\}_{n\in\mathbb{N}_0}$ is dense in $[0, +\infty)$.*

Proof. We have $\rho_0(n) = ||T^{r_n}(\tilde{x} \oplus y)||$ and $\rho_{t_n}(n) = ||T^{q_{s_n+1}}(\tilde{x} \oplus y)||$. From condition (b) of the Theorem 2.1.1 (proved above) it follows that there exists $n_1 = n_1(\varepsilon, M)$ such that for $n \geq n_1$, $0 < \rho_0(n) < \varepsilon$ and $\rho_{t_n}(n) \geq M$.

We will show that there exists $n_2 = n_2(\varepsilon)$ such that for $n \geq n_2$ $j = 0, \ldots, t_n - 1$, $\rho^2_{j+1}(n) - \rho^2_j(n) < \varepsilon^2$. We have that, for $n \in \mathbb{N}$,

$$\rho^2_{j+1}(n) - \rho^2_j(n) = ||T^{r_n+j+1}(\tilde{x} \oplus y)||^2 - ||T^{r_n+j}(\tilde{x} \oplus y)||^2 =$$

$$||(S^*_{\tilde{w}})^{r_n+j+1}\tilde{x}||^2 - ||(S^*_{\tilde{w}})^{r_n+j}\tilde{x}||^2 + ||F_{r_n+j+1}\tilde{x}||^2 - ||F_{r_n+j}\tilde{x}||^2 + ||S^{r_n+j+1}_w y||^2 - ||S^{r_n+j}_w y||^2$$

Let $\tilde{x} = \sum_{k=1}^{\infty} \widetilde{\alpha}_k \widetilde{e}_k$ and $y = \sum_{k=1}^{\infty} \alpha_k e_k$. We show first that there exists $n_{2,0} = n_{2,0}(\varepsilon)$ such that for $n \geq n_{2,0}$, $j = 0, \ldots, t_n - 1$, $||S^{r_n+j+1}_w y||^2 - ||S^{r_n+j}_w y||^2 < \varepsilon^2/3$. We have

$$||S^{p_n+j+1}_w y||^2 - ||S^{p_n+j}_w y||^2 = \sum_{k=1}^{\infty} |\alpha_k|^2 \left(\prod_{i=k}^{k+p_n+j-1} w_i^2 \right) (w^2_{k+p_n+j} - 1).$$

From (2.6), since $p_n \leq k + p_n + j$, we have that $w_{k+p_n+j} \leq 2^{1/p_n}$, and thus

$$w^2_{k+p_n+j} - 1 \leq 2^{2/p_n} - 2^0 \leq \frac{4 \ln 2}{p_n} \leq \frac{4 \ln 2}{2^{n-1}(n-1)!}, \quad n, k, j \in \mathbb{N},$$

The second inequality follows by the Mean Value Theorem applied to the function $f(x) = 2^x$ on the interval $[0,2/p_n]$, and the last inequality follows from $p_n \geq 2^{n-1}(n-1)!$ which is immediate from (2.5). It follows also from (2.6) that the product $\prod_{i=k}^{k+p_n+j-1} w_i$ is bounded above by $\prod_{i=p_n+1}^{q_n} w_i = 2^{2n+1}$, since it contains $p_n + j$ weights, with $p_n + j \leq q_n$. Thus

$$\left(\prod_{i=k}^{k+p_n+j-1} w_i^2\right)(w_{k+p_n+j}^2 - 1) \leq (2^{2n+1})^2 \frac{4\ln 2}{2^{n-1}(n-1)!}, \; n, k \in \mathbb{N}, \; 0 \leq j < t_n.$$

The right-hand side of this inequality tends to zero as $n \to +\infty$, and since $\sum_{k=1}^{\infty} |\alpha_k|^2$ is bounded we can choose n_2 large enough such that for $n \geq n_{2,0}$ $\forall j = 0, \ldots, t_n - 1$, $||S_w^{r_n+j+1} y||^2 - ||S_w^{r_n+j} y||^2 < \varepsilon^2/3$.

Next we evaluate the remaining terms.

$$\begin{aligned}
||(S_{\tilde{w}}^*)^{r_n+j+1}\tilde{x}||^2 - ||(S_{\tilde{w}}^*)^{r_n+j}\tilde{x}||^2 + ||F_{r_n+j+1}\tilde{x}||^2 - ||F_{r_n+j}\tilde{x}||^2 = \\
= \sum_{k=r_n+j+2}^{\infty} |\widetilde{\alpha}_k|^2 \left(\prod_{i=k-(r_n+j)}^{k-1} \widetilde{w}_i^2\right)(\widetilde{w}_{k-1-(r_n+j)}^2 - 1) + \\
+ \sum_{k=1}^{r_n+j} |\widetilde{\alpha}_k|^2 \left(\prod_{i=1}^{k-1} \widetilde{w}_i^2\right)\left(\prod_{i=1}^{r_n+j+1-k} w_i^2\right)(w_{r_n+j+2-k}^2 - 1).
\end{aligned}$$

Let $m \in \mathbb{N}$ be such that $s_m < k - 1 - (r_n + j) \leq s_{m+1}$. Then we have that $\prod_{i=1}^{k-1} \widetilde{w}_i^2 \leq 1$ and $\prod_{i=1}^{k-1-(r_n+j)} \widetilde{w}_i^2 \geq (1/2)^{s_m}$, by Lemma 2.3.5(i), so $\prod_{i=k-(r_n+j)}^{k-1} \widetilde{w}_i^2 \leq 2^{s_m}$, and

$$\widetilde{w}_{k-1-(r_n+j)}^2 - 1 \leq 2^{2/(r_m - 2s_n)} - 1 \leq 2^{4/r_m} - 2^0 \leq \frac{8\ln 2}{r_m} \leq \frac{8\ln 2}{2^{s_m-1}(s_m-1)!},$$

and thus

$$\left(\prod_{i=k-(r_n+j)}^{k-1} \widetilde{w}_i^2\right)(\widetilde{w}_{k-1-(r_n+j)}^2 - 1) \leq \frac{16\ln 2}{(s_m-1)!}.$$

It follows that there exists a constant $C > 0$ such that

$$\left(\prod_{i=k-(r_n+j)}^{k-1} \widetilde{w}_i^2\right)(\widetilde{w}_{k-1-(r_n+j)}^2 - 1) \leq C, \quad n \in \mathbb{N}, \; 0 \leq j < t_n, \; k \geq r_n + j + 2.$$

Since $\lim_n \sum_{k=r_n+j+2}^{\infty} |\widetilde{\alpha}_k|^2 = 0$ we can choose $n_{2,1}$ large enough such that

$$\sum_{k=r_n+j+2}^{\infty} |\widetilde{\alpha}_k|^2 \left(\prod_{i=k-(r_n+j)}^{k-1} \widetilde{w}_i^2 \right) (\widetilde{w}_{k-1-(r_n+j)}^2 - 1) < \frac{\varepsilon^2}{3}, \quad n \geq n_{2,1},\ 0 \leq j < t_n.$$

For the other sum we have that $\prod_{i=1}^{k-1} \widetilde{w}_i^2 \leq 1$ and if $m \in \mathbb{N}$ is such that $p_m < r_n + j + 2 - k \leq p_{m+1}$, we have $\left(\prod_{i=1}^{r_n+j+1-k} w_i^2 \right) \leq 2^{2m}$ and

$$(w_{r_n+j+2-k}^2 - 1) \leq (2^{2/p_m} - 1) \leq \frac{4 \ln 2}{p_m} \leq \frac{4 \ln 2}{2^{m-1}(m-1)!}.$$

Now we write

$$\begin{aligned} &\sum_{k=1}^{r_n+j} |\widetilde{\alpha}_k|^2 \left(\prod_{i=1}^{r_n+j+1-k} w_i^2 \right) (w_{r_n+j+2-k}^2 - 1) = \\ &= \sum_{k=1}^{r_n/2-1} |\widetilde{\alpha}_k|^2 \left(\prod_{i=1}^{r_n+j+1-k} w_i^2 \right) (w_{r_n+j+2-k}^2 - 1) + \\ &\quad + \sum_{k=r_n/2}^{r_n+j} |\widetilde{\alpha}_k|^2 \left(\prod_{i=1}^{r_n+j+1-k} w_i^2 \right) (w_{r_n+j+2-k}^2 - 1) \end{aligned}$$

If $k < r_n/2$ we have that $r_n + j + 1 - k > r_n/2$ and thus $m \geq s_n$, so for n large enough we can make $\left(\prod_{i=1}^{r_n+j+1-k} w_i^2 \right) (w_{r_n+j+2-k}^2 - 1)$ arbitrarily small, thus the first sum tends to zero with n. If $k \geq r_n/2$ then $\sum_{k=r_n/2}^{r_n+j} |\widetilde{\alpha}_k|^2$ can be made arbitrarily small with n, and the factor $\left(\prod_{i=1}^{r_n+j+1-k} w_i^2 \right) (w_{r_n+j+2-k}^2 - 1)$ is bounded. Thus we can choose $n_{2,2}$ such that

$$\sum_{k=1}^{r_n+j} |\widetilde{\alpha}_k|^2 \left(\prod_{i=1}^{k-1} \widetilde{w}_i^2 \right) \left(\prod_{i=1}^{r_n+j+1-k} w_i^2 \right) (w_{r_n+j+2-k}^2 - 1) \leq \frac{\varepsilon^2}{3} \quad n \geq n_{2,2},\ 0 \leq j < t_n.$$

Let n_2 be the largest of $n_{2,0}$, $n_{2,1}$ and $n_{2,2}$. We have that for $n \geq n_2$, $j = 0, \ldots, t_n - 1$, $\rho_{j+1}^2(n) - \rho_j^2(n) < \varepsilon^2$. From this since the terms involved are positive it follows that for $n \geq n_2$, $j = 0, \ldots, t_n - 1$, $\rho_{j+1}(n) - \rho_j(n) < \varepsilon$.

Choose n_0 to be the largest of n_1 and n_2; then for the given ε and M we found n_0

such that the hypotheses of Lemma 2.4.2 are satisfied. This shows that the sequence $\{\rho_j(n)\}_{j=0}^{t_n}$ is ε-dense in $[0, M]$ for all $n \geq n_0$.

Since every operator that is similar to T has these properties, we see that, in fact, $\mathcal{L}(\mathcal{H})$ contains many operators with these same properties. □

We conclude this section with a result to be used in forthcoming work, containing more information about the bilateral weighted shift $B = B_{\widetilde{w},w}$ just constructed.

Proposition 2.4.4. *For $B = B_{\widetilde{w},w}$ we have $\sigma_e(B) = \sigma(B) = \{z : \frac{1}{2} \leq |z| \leq 1\}$, $||B|| = \sqrt[4]{2}$, $||B||_e = 1$ and $||B^{-1}||_e = ||B^{-1}|| = 2$, where $||\cdot||_e$ denotes the essential(Calkin) norm.*

Proof. Let $r(A)$ and $r_e(A)$ denote the spectral radius and essential spectral radius of an operator $A \in \mathcal{L}(\mathcal{H})$. Since the norm of a weighed shift equals the modulus of the largest weight, we have $||B|| = \sqrt[4]{2}$ and $||B^{-1}|| = 2$. Moreover, since $(1/2)^{s_n} \leq ||B^{r_n}|| \leq 8$, for all n, and $(s_n/r_n) \to 0$, it follows that $r(B) = 1$. Also, since B^{-1} has arbitrarily long sequences of consecutive weights equal to 2 we get $||(B^{-1})^n|| = 2^n$. Thus, $r(B^{-1}) = 2$. According to [26, Theorem 5, page 67] we have that

$$\sigma(B) = \{z : r(B^{-1})^{-1} \leq |z| \leq r(B)\} = \{z : \frac{1}{2} \leq |z| \leq 1\}.$$

Since one knows that for every $A \in \mathcal{L}(\mathcal{H})$, every non-isolated point of the boundary of $\sigma(A)$ is contained in $\sigma_e(A)$, one also obtains easily, using the fact that neither B nor B^* has an eigenvalue, that $\sigma_e(B) = \sigma(B)$. It follows that $r_e(B) = 1$ and $r_e(B^{-1}) = 2$.

Clearly, $r_e(B) \leq ||B||_e \leq ||B||$. If we replace all the weights of B that are roots of 2 by weights equal to 1, we obtain a compact perturbation of B that has norm 1, and thus $||B||_e = 1$. Finally, we have that $r_e(B^{-1}) = ||B^{-1}||_e = ||B^{-1}|| = 2$. □

2.5. Remarks and questions

Remark 2.5.1. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *supercyclic* if there exists a vector $x \in \mathcal{H}$ such that $\{\lambda T^n x : n \in \mathbb{N}, \lambda \in \mathbb{C}\}$ is dense in $\mathcal{H}$. The operator $B_{\widetilde{w},w}$ constructed in this chapter is not even supercyclic. Indeed, according to [12], if $B_{\widetilde{w},w}$ were supercyclic, there would exist a strictly increasing sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ such that

$$(w_1 \dots w_{n_k}) \; / \; (\widetilde{w}_1 \dots \widetilde{w}_{n_k}) \to 0.$$

Some calculations based on the fact that

$$(w_1 \dots w_{s_n}) \, / \, (\widetilde{w}_1 \dots \widetilde{w}_{s_n}) = w_1 \dots w_{s_n} \to 1/2,$$

show that the supercyclicity condition is not satisfied.

Remark 2.5.2. Note that the operator $B_{\widetilde{w},w}$ constructed above has a dense set of noncyclic vectors (*i.e.*, vectors that lie in some proper invariant subspace of $B_{\widetilde{w},w}$).

Problem 2.5.3. Theorem 2.1.1 establishes the existence of an operator in $\mathcal{L}(\mathcal{H})$ such that every (nonzero) orbit has certain property – namely, density in $\mathbb{R}_+$ of the sequence of norms. Moreover, in [15] an example was given of an operator T in $\mathcal{L}(\mathcal{H})$ with $||T||_e = 1$ such that the orbit of every nonzero vector x satisfies $\{||T^n x||\} \to +\infty$. What other properties that are common to every (nonzero) orbit can an operator in $\mathcal{L}(\mathcal{H})$ have? For example, does there exist an operator $T \in \mathcal{L}(\mathcal{H})$ such that for all nonzero vectors $x, y \in \mathcal{H}$, $\{\langle T^n x, y\rangle\}$ is dense in $\mathbb{C}$? (Of course, such a T would be transitive.)

CHAPTER III

WEAK HYPERCYCLICITY ON HILBERT SPACE

3.1. Definitions and known results

Recently, Chan and Sanders [9] discussed the concept of *a weakly hypercyclic operator*, defined as a T in $\mathcal{L}(\mathcal{H})$ with the property that there exists a vector x in $\mathcal{H}$ such that $\mathcal{O}(x,T)$ is weakly dense in $\mathcal{H}$. They obtained there several interesting results, including the following: a) there exist weakly hypercyclic operators in $\mathcal{L}(\mathcal{H})$ that are not hypercyclic, and b) there exists a sequence $\{x_n\} \subset \mathcal{H}$ that is weakly dense in $\mathcal{H}$ and satisfies $||x_{n+1}|| > ||x_n||$ for all $n \in \mathbb{N}$. An important fact in these considerations was a result by Dilworth and Troitsky [11] to the effect that if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in a complex Banach space $\mathcal{X}$ whose norms satisfy, for some $c > 1$,

$$||x_n|| \geq c^n, \quad n \in \mathbb{N}, \tag{3.1}$$

then $0 \notin \overline{\{x_n\}}^w$ (so, in particular, $\{x_n\}$ is not weakly dense in $\mathcal{X}$). This led the present authors to ask whether in $\mathcal{H}$, where more structure is present, a weaker growth rate than that in (5.11) would ensure the weak non-density of a sequence in $\mathcal{H}$. After we obtained some partial results in this direction we found that this question was already answered by the following complex-plank theorem of Keith Ball [2].

Theorem 3.1.1 (Ball). *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of unit vectors in $\mathcal{H}$ and let $\{a_n\}$ be a sequence of positive numbers such that $\sum a_n^2 = 1$. Then there exists a unit vector $y \in \mathcal{H}$ such that $|\langle x_n, y\rangle| \geq a_n$ for each $n \in \mathbb{N}$.*

The following corollaries are due to Kadets [16] and Shkarin [27], respectively.

Corollary 3.1.2 (Kadets). *For a sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers, the following are equivalent:*

a) there exists at least one sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ satisfying $\|x_n\| = a_n$ for all n and having 0 as a weak cluster point;

b) the series $\sum_n 1/a_n^2$ diverges.

Corollary 3.1.3 (Shkarin). *If $x \in \mathcal{H}$ and $T \in \mathcal{L}(\mathcal{H})$ are such that the orbit $\mathcal{O}(x, T)$ satisfies*

$$\sum_{n\in\mathbb{N}} 1/||T^n x||^2 < +\infty, \tag{3.2}$$

then x is not weakly hypercyclic for T. In fact, the orbit $\mathcal{O}(x, T)$ is weakly closed.

We will also need the following nice result of Beauzamy [4, Ch. 3].

Theorem 3.1.4 (Beauzamy). *Let $\{T_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of operators in $\mathcal{L}(\mathcal{H})$, let ε be any positive number, and let $\{a_n\}_{n\in\mathbb{N}}$ be any sequence of positive numbers in (l_2). Then there exists a nonzero vector $y \in \mathcal{H}$ such that $\|y\| \leq (1 + \varepsilon)\left(\sum a_n^2\right)^{1/2}$ and for each $n \in \mathbb{N}$, $||T_n y|| \geq (1 - \varepsilon)a_n \|T_n\|_e$.*

In [15], Jung, Ko, and Pearcy initiated the study of non-hypertransitive operators (sometimes called non-orbit-transitive [23]), with the stated goal of eventually showing that $\mathcal{L}(\mathcal{H})$ contains no hypertransitive operator. (By definition, an operator $T \in \mathcal{L}(\mathcal{H})$ is *hypertransitive* if every nonzero vector in $\mathcal{H}$ is hypercyclic for T.) They showed, for example, that no operator of the form $H + K$, where H is essentially hyponormal and K is compact, can be hypertransitive, but recall that it is still unknown whether these operators have nontrivial invariant subspaces. (On the other hand, C. Read showed in [24] that on the Banach space (l_1) there do exist hypertransitive operators.) This led the present authors to the following.

Definition 3.1.5. An operator T in $\mathcal{L}(\mathcal{H})$ is called *weakly hypertransitive* (or *weakly* *orbit-transitive*) if every nonzero vector x in $\mathcal{H}$ is weakly hypercyclic for T.

Of course, it remains an open problem whether weakly hypertransitive or hypertransitive operators exist in $\mathcal{L}(\mathcal{H})$.

In this note we first establish, in Section 3.2, the perhaps surprising result that Theorem 3.1.1 is equivalent to a slightly stronger version of Theorem 3.1.4 (namely, Theorem 3.2.1). We then obtain (in Section 3.3) a theorem that complements Corollary 3.1.2 above. Finally, we show that all of the results of [15] can be improved to obtain that various classes of operators in $\mathcal{L}(\mathcal{H})$ are not weakly hypertransitive. This is done in Section 3.5 below.

3.2. An equivalence

In this section, we show that Theorem 3.1.1 is equivalent to the following slightly stronger version of Theorem 3.1.4.

Theorem 3.2.1. *Let $\{T_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of operators in $\mathcal{L}(\mathcal{H})$, let $\varepsilon > 0$, and suppose $\{a_n\}_{n\in\mathbb{N}}$ is an (l_2)-sequence of nonnegative numbers such that $\sum a_n^2 = 1$. Then there exists a unit vector $y \in \mathcal{H}$ such that*

$$\|T_n y\| > (1-\varepsilon)a_n\|T_n\|, \quad n \in \mathbb{N}.$$

Proof of Theorem 3.2.1 (using Theorem 3.1.1). We construct a sequence $\{w_n\}$ of unit vectors as follows. There exists a unit vector $w_n \in \mathcal{H}$ such that

$$\|T_n w_n\| = (1-\varepsilon_n)\|T_n\|, \quad n \in \mathbb{N},$$

where $0 \le \varepsilon_n < \varepsilon/2$. Applying Theorem 1.1 to the sequence $x_n = T_n w_n/(1-\varepsilon_n)\|T_n\|$ of unit vectors and the sequence $\{a_n\}$, we obtain that there exists a unit vector $y \in \mathcal{H}$

such that

$$\left|\left\langle \frac{T_n w_n}{(1-\varepsilon_n)\|T_n\|}, y \right\rangle\right| \geq a_n, \quad n \in \mathbb{N},$$

and thus

$$\|T_n^* y\| \geq |\langle w_n, T_n^* y\rangle| > (1-\varepsilon)\|T_n\| a_n, \quad n \in \mathbb{N}.$$

Since $\|T_n^*\| = \|T_n\|$, upon interchanging T_n and T_n^* we get the desired result. $\square$

Proof of Theorem 3.1.1 (using Theorem 3.2.1). Let a sequence $\varepsilon_k \searrow 0$ be given. We define a sequence of operators $\{T_n\}$ as follows. For $w \in \mathcal{H}$ set $T_n(w) = \langle w, x_n\rangle x_n$. Since $\|x_n\| = 1$ for all n, $\|T_n\| = 1$ too. By Theorem 3.2.1, for each $k \in \mathbb{N}$ there exists a unit vector y_k in $\mathcal{H}$ such that

$$\|T_n y_k\| = |\langle y_k, x_n\rangle| > (1-\varepsilon_k)a_n, \quad k, n \in \mathbb{N}. \tag{3.3}$$

Now choose a subsequence $\{y_{k_m}\}$ of $\{y_k\}$ converging weakly, say to (the nonzero vector) y_0, satisfying $\|y_0\| \leq 1$. It follows now from (3.3) that, with $y := y_0/\|y_0\|$, we have $|\langle x_n, y\rangle| \geq a_n$ *for each* $n \in \mathbb{N}$. $\square$

3.3. On Kadets' result

The following result was motivated by a construction of Chan and Sanders [9], and complements Corollary 3.1.2 above. The weak closure of a set S in $\mathcal{H}$ will be written as $\overline{S}^w$.

Theorem 3.3.1. *Let* $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *satisfy* $0 \in \overline{\{x_n\}}^w$, *and* $\lim(\ln\|x_n\|/\ln n) = \alpha > 0$. *Then there exists a sequence* $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *such that* $\overline{\{w_n\}}^w = \mathcal{H}$ *and* $\lim(\ln||w_n||/\ln n) = \alpha$ *also.*

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$ and let $S \in \mathcal{L}(\mathcal{H})$ be the forward unilateral shift defined by $Se_n = e_{n+1}$ for $n \in \mathbb{N}$. Moreover, for $k \in \mathbb{N}$ let P_k be the orthogonal projection of $\mathcal{H}$ onto $\bigvee\{e_1, \ldots, e_k\}$. Let $\{x_n\}$ be as in the hypothesis and let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence that is strongly dense in $\mathcal{H}$ and has the additional property that for each $n \in \mathbb{N}$ and all $k = 1, \ldots, n$, $P_n z_k = z_k$ and $\|z_k\| \leq k$. (One way to accomplish this is to first choose, for each n, a finite $1/(n+1)$ net in the ball of radius n centered at origin in $P_n\mathcal{H}$, then note that the union over n is a dense set $\mathcal{D}$ in $\mathcal{H}$, and finally order $\mathcal{D}$ by first counting those points in the ε-net with $\varepsilon = 1/2$, followed by those in the ε-net with $\varepsilon = 1/3$, etc.). Next, for each pair $(n, k) \in \mathbb{N} \times \mathbb{N}$ with $n \leq 1 + \ln k$, we define

$$y_{n,k} = z_n + S^n x_k, \tag{3.4}$$

and note that since 0 is a weak cluster point of $\{x_k\}$, we get, by holding n fixed and letting k run,

$$z_n \in \overline{\{y_{n,k} : k \in \mathbb{N}, n \leq 1 + \ln k\}}^{w};$$

hence

$$\overline{\{y_{n,k} : k \in \mathbb{N}, n \leq 1 + \ln k\}}^{w} = \mathcal{H}. \tag{3.5}$$

Next we order the elements $\{y_{n,k} : k \in \mathbb{N}, n \leq 1 + \ln k\}$ in a sequence $\{w_j\}$ by associating to every $j \in \mathbb{N}$ a pair (n_j, k_j) of natural numbers such that

$$w_j := y_{n_j,k_j} = z_{n_j} + S^{n_j} x_{k_j}, \tag{3.6}$$

where

$$\begin{array}{lll} n_1 := 1, & k_1 := 1 & \\ n_{j+1} := n_j + 1, & k_{j+1} := k_j, & \text{if } n_j + 1 \leq 1 + \ln k_j, \\ n_{j+1} := 1, & k_{j+1} := k_j + 1, & \text{if } n_j + 1 > 1 + \ln k_j. \end{array}$$

Observe that as j runs through $\mathbb{N}$, w_j runs through all $y_{n,k}$, and thus from (3.5) we deduce that $\overline{\{w_j : j \in \mathbb{N}\}}^{w} = \mathcal{H}$. Also note that since z_{n_j} is orthogonal to $S^{n_j} x_{k_j}$, we obtain from (3.6) that, for a given $j \in \mathbb{N}$,

$$||w_j|| = (||z_{n_j}||^2 + ||x_{k_j}||^2)^{1/2} \geq (||z_{n_j}|| + ||x_{k_j}||)/\sqrt{2}.$$

On the other hand, since $\|z_{n_j}\| \leq n_j$ and $n_j \leq \ln k_j$ we have

$$\|w_j\| \leq \|z_{n_j}\| + \|x_{k_j}\| \leq \ln k_j + \|x_{k_j}\|.$$

Since $\lim(\ln \|x_n\|/\ln n) = \alpha > 0$, we have that, for j large enough, $\ln k_j \leq \|x_{k_j}\|$, so $\|w_j\| \leq 2\|x_{k_j}\|$. The nondecreasing sequence $\{k_j\}_j$ will take a given value k exactly $(1 + [\ln k])$ times (where, as usual, $[\beta]$ denotes the greatest integer in β), so we have that

$$k_j + [\ln 2] + \cdots + [\ln(k_j - 1)] < j \leq k_j + [\ln 2] + \cdots + [\ln k_j]\,, \quad j \in \mathbb{N}.$$

Thus

$$\frac{\ln ||w_j||}{\ln j} \geq \frac{\ln(||z_{n_j}|| + ||x_{k_j}||) - (1/2)\ln 2}{\ln j} \geq \frac{\ln ||x_{k_j}|| - 1/2}{\ln(k_j + k_j \ln k_j)},$$

and it follows that

$$\lim_{j\to\infty} \frac{\ln ||w_j||}{\ln j} \geq \lim_{j\to\infty} \frac{\ln ||x_{k_j}||}{\ln k_j + \ln(1 + \ln k_j)} = \alpha.$$

For the other inequality we note that

$$\frac{\ln ||w_j||}{\ln j} \leq \frac{\ln(||x_{k_j}||) + \ln 2}{\ln j} \leq \frac{\ln ||x_{k_j}|| + \ln 2}{\ln k_j},$$

and thus

$$\lim_{j\to\infty} \frac{\ln ||w_j||}{\ln j} = \alpha.$$

$\square$

3.4. Some useful tools

In this section we list the results needed as tools to show that the results from [15] asserting that certain classes of operators consist entirely of non-orbit-transitive operators can be improved to say that these classes are also subsets of the smaller class of non-weakly-orbit-transitive operators. The first proposition is elementary and needs no proof.

Proposition 3.4.1. *If $T \in \mathcal{L}(\mathcal{H})$ and there exist nonzero vectors x and y such that $\{\langle T^n x, y\rangle\}_n$ is not dense in $\mathbb{C}$, then x is not a weakly hypercyclic vector for T. Thus, if one of the equalities*

$$\sigma(T) = \sigma_e(T) = \sigma_{le}(T) = \sigma_{re}(T)$$

fails to hold, then T or T^* has point spectrum and T has a nontrivial invariant subspace, so T is not weakly orbit-transitive. Moreover, T is weakly orbit-transitive if and only if (some or) every operator similar to T is weakly orbit-transitive.

The following result is a modest improvement of inequalities due to S. Brown, V. Lomonosov, and A. Simonic, developed over time (cf. [8], [28], and [18]). We write $w_e(A)$ for the essential numerical radius of an operator A.

Theorem 3.4.2. *Let $\mathcal{A}$ denote any unital, proper subalgebra of $\mathcal{L}(\mathcal{H})$ that is closed in the weak operator topology. Then there exist nonzero vectors x, y in $\mathcal{H}$ such that the linear functional on $\mathcal{A}$ defined by $\varphi_{x,y}(A) = \langle Ax, y\rangle$ is a positive functional on $\mathcal{A}$ (i.e., $\varphi(1_{\mathcal{H}}) = \|\varphi\|$) and satisfies*

$$|\langle Ax, y\rangle| \leq w_e(A)\langle x, y\rangle, \quad A \in \mathcal{A}.$$

As noted in [27], the following extension of Ansari's theorem from [1] to weakly hypercyclic operators also holds.

Theorem 3.4.3 (Ansari-Shkarin). *For every $T \in \mathcal{L}(\mathcal{H})$ and every $n \in \mathbb{N}$, the operators T and T^n have exactly the same set of weakly hypercyclic vectors.*

3.5. Consequences

Our first improvement of the results of [15] is this.

Theorem 3.5.1. *If $T \in \mathcal{L}(\mathcal{H})$ and T is weakly orbit-transitive, then $r_e(T) = r(T) = 1$.*

Proof. By Proposition 3.4.1 we may suppose that $\sigma(T) = \sigma_e(T)$. It is shown in [11] that every weakly hypercyclic operator in $\mathcal{L}(\mathcal{H})$ satisfies $\sigma(T) \cap \mathbb{T} \neq \varnothing$, and thus it suffices to show that $r_e(T) > 1$ is impossible. But if $r_e(T) = c > 1$, then $\|T^n\|_e \geq r_e(T^n) > c^n$ for $n \in \mathbb{N}$, and by Theorem 3.1.4 there exists a vector y in $\mathcal{H}$ such that $\|T^n y\| > c^n/n$ for all n, and hence by Corollary 3.1.2, $0 \notin \mathcal{O}(y,T)^{\underline{w}}$, a contradiction. □

This next result may be thought of as the principal result of this section.

Theorem 3.5.2. *Suppose that $T \in \mathcal{L}(\mathcal{H})$ and that there exists $n \in \mathbb{N}$ such that two of the numbers $r_e(T^n), w_e(T^n), \|T^n\|_e$ coincide. Then for every invertible $S \in \mathcal{L}(\mathcal{H})$ and every compact K in $\mathcal{L}(\mathcal{H})$, $STS^{-1} + K$ fails to be weakly orbit-transitive.*

Proof. Using Proposition 3.4.1, Theorem 3.4.3 and the fact that $r_e(T)$, $w_e(T)$ and $\|T\|_e$ remain the same if T is replaced by a compact perturbation of T, one easily sees (by changing notation if necessary) that it suffices to show that if two of $r_e(T), w_e(T), \|T\|_e$ coincide, then T is not weakly orbit-transitive. But, as is well-known, the coincidence of two of the above three numbers implies that there exists an operator similar to T such that all three coincide, cf. [30]. Thus we may suppose that $r_e(T) = w_e(T) = \|T\|_e = 1$ (via Theorem 3.5.1). But then according to Theorem 3.4.2, there exist nonzero vectors x, y in $\mathcal{H}$ such that

$$|\langle T^n x, y\rangle| \le w_e(T^n)\langle x, y\rangle \le \|T^n\|_e\langle x, y\rangle = \langle x, y\rangle, \quad n \in \mathbb{N},$$

so obviously $\mathcal{O}(x, T)$ is not weakly dense in $\mathcal{H}$. □

This result has some immediate corollaries.

Corollary 3.5.3. *If T is essentially hyponormal or a Toeplitz operator, S is invertible and K is compact, then $STS^{-1} + K$ is not weakly orbit-transitive.*

Proof. One knows that for such an operator T, $\|T\|_e = r_e(T)$. □

Corollary 3.5.4. *No operator of the form $SNS^{-1} + K$, where S is invertible, N is normal, and K in compact, is weakly orbit-transitive.*

We close with some problems that arise from the above considerations.

Problem 3.5.5. If $T \in \mathcal{L}(\mathcal{H})$ is invertible and weakly orbit-transitive, must T^{-1} also be weakly orbit-transitive? (One knows that an operator can be weakly hypercyclic without its inverse being weakly hypercyclic [9, Cor. 3.6].)

Problem 3.5.6. If $T \in \mathcal{L}(\mathcal{H})$ is weakly orbit-transitive, must $\sigma(T) \subset \mathbb{T}$?

Problem 3.5.7. If $T \in \mathcal{L}(\mathcal{H})$ and $\{\langle T^n x, y\rangle\}$ is dense in $\mathbb{C}$ for every pair x, y of nonzero vectors in $\mathcal{H}$, must T be weakly orbit-transitive? (One knows form [29] that there exist nonhypercyclic operators T in $\mathcal{L}(\mathcal{H})$ such that for every $x \neq 0$ in $\mathcal{H}$, $\{\|T^n x\|\}$ is dense in $\mathbb{R}^+$.)

Problem 3.5.8. Recall that an operator $T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ is 2-normal if T is unitarily equivalent to a 2×2 matrix (N_{ij}), where the N_{ij} are mutually commuting normal operators, and it is known [14] that 2-normal operators have nontrivial hyperinvariant subspaces. Can a sum $T + K$, where T is 2-normal and K is compact, be weakly orbit-transitive?

Problem 3.5.9. The following question, which logically falls between the Orbit-transitive Operator Problem and the Invariant Subspace Problem, seems not to have received any attention: Is it true that for every T in $\mathcal{L}(\mathcal{H})$ there exist nonzero vectors x and y such that the sequence $\{\langle T^n x, y\rangle\}_{n\in\mathbb{N}}$ is bounded?

CHAPTER IV

MORE CLASSES OF NON-ORBIT-TRANSITIVE OPERATORS

In [15] the authors initiated a program whose (announced) goal is to eventually show that no operator in $\mathcal{L}(\mathcal{H})$ is orbit transitive. In [15] it is shown, for example, that if $T \in \mathcal{L}(\mathcal{H})$ and the essential (Calkin) norm of T is equal to its essential spectral radius, then no compact perturbation of T is orbit-transitive, and in Chapter II this result was extended to say that no element of this same class of operators is weakly orbit-transitive. Here we show that no compact perturbation of certain 2-normal operators (which in general satisfy $\|T\|_e > r_e(T)$) can be orbit-transitive. This answers a question raised in [15]. Our main result herein is that if T belongs to a certain class of 2-normal operators in $\mathcal{L}(\mathcal{H}^{(2)})$ and there exist two constants $\delta, \rho > 0$ satisfying $\|T^k\|_e > \rho k^\delta$ for all $k \in \mathbb{N}$, then for every compact operator K, the operator $T + K$ is not orbit-transitive. This seems to be the first result that yields non-orbit-transitive operators in which such a modest growth rate on $\|T^k\|_e$ is sufficient to give an orbit $\{T^k x\}$ tending to infinity.

4.1. Definitions

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote the algebra of all bounded linear operators on $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$. If $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, the countable (finite or infinite) set $\{T^n x\}_{n=0}^{\infty}$ is called the *orbit* of x *under* T, and is denoted by $\mathcal{O}(x, T)$. If $\mathcal{O}(x, T)$ is dense in $\mathcal{H}$, then x is called a *hypercyclic vector* for T, and T is said to be a *hypercyclic operator*. The question of which operators in $\mathcal{L}(\mathcal{H})$ are hypercyclic and properties of the set of hypercyclic vectors of a hypercyclic operator have been much studied during the past twenty years. An operator T in $\mathcal{L}(\mathcal{H})$ is called *transitive* if T has no invariant subspace (closed linear manifold) other than $\{0\}$ and $\mathcal{H}$, and is called *orbit-transitive* (or *hypertransitive* as in [15]) if every nonzero vector in $\mathcal{H}$ is hypercyclic for T. Presently one does not know whether there exist transitive or orbit-transitive operators in $\mathcal{L}(\mathcal{H})$. (It is obvious that every orbit-transitive operator is transitive, and Read [24] has constructed an operator on the Banach space (l_1) that is orbit-transitive.) Denote the set of all nontransitive operators in $\mathcal{L}(\mathcal{H})$ by (NT) and the set of all non-orbit-transitive operators in $\mathcal{L}(\mathcal{H})$ by (NOT). The *invariant subspace problem* is the open question whether (NT)$=\mathcal{L}(\mathcal{H})$, and the *orbit-transitive operator problem* is the question whether (NOT)$=\mathcal{L}(\mathcal{H})$. (The orbit-transitive operator problem is sometimes referred to as the hypertransitive-operator problem [15] or the nontrivial-invariant-closed-set problem. At present, neither of the terms "hypertransitive" nor "orbit-transitive" has been in use long enough to be considered standard, but note that if an operator T in $\mathcal{L}(\mathcal{H})$ is called "supertransitive" if every nonzero vector y in $\mathcal{H}$ is supercyclic for T, i.e., $\{\rho T^n y : n \in \mathbb{N}, \rho > 0\}^- = \mathcal{H}$, then "hypertransitive" would seem to be a reasonable alternative to "orbit-transitive".)

This chapter continues the study of classes of non-orbit-transitive operators, with the purpose (as mentioned explicitly in [15]) of eventually showing that (NOT)$=$ $\mathcal{L}(\mathcal{H})$, and thus to give convincing evidence that operators on Hilbert space are very different creatures from operators on more general complex Banach spaces. More exactly, in this article we make progress on the problem of showing that if T is n-normal and K is compact, then $T+K\in$(NOT). In particular, we produce the only known subset of (NOT) invariant under compact perturbations, consisting of operators which satisfy the very modest growth condition $\|T^k\|_e \geq \rho k^\delta$ for some $\rho, \delta > 0$ and all $k \in \mathbb{N}$.

All the notation and terminology to follow is consistent with that of [15] and [?], but for the readers convenience, we briefly review the main points. The sets of positive and nonnegative integers will be denoted by $\mathbb{N}$ and $\mathbb{N}_0$, and the complex plane by $\mathbb{C}$. The ideal of compact operators in $\mathcal{L}(\mathcal{H})$ will be written as $\mathbb{K}(\mathcal{H})$, or more simply as $\mathbb{K}$, and the quotient (Calkin) map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbb{K}$ by π. For T in $\mathcal{L}(\mathcal{H})$ we write $\sigma(T)$ and $\sigma_p(T)$ for the spectrum and point spectrum of T, respectively, and $\sigma_e(T) := \sigma(\pi(T))$, $\sigma_{lre}(T) := \sigma_{lr}(\pi(T))$ (the intersection of the left and right spectra of $\pi(T)$). We also write $r(T)$ and $r_e(T)$ for the spectral radii of T and $\pi(T)$, as well as $\|T\|_e := \|\pi(T)\|$.

Finally, for any positive integer n we write $\mathcal{H}^{(n)}$ for the direct sum of n copies of $\mathcal{H}$.

4.2. Some new ideas

Our first new result is a supplement to the following theorem of Ansari.

Theorem 4.2.1 (Ansari [1]). *For every $T \in \mathcal{L}(\mathcal{H})$ and for every $n \in \mathbb{N}$, T and T^n have exactly the same set of hypercyclic operators.*

Our supplement is as follows:

Proposition 4.2.2. *Suppose $T \in \mathcal{L}(\mathcal{H})$, $x \in \mathcal{H}$, and there exists $n_0 \in \mathbb{N}$ such that $\| \left(T^{n_0}\right)^k x\| \xrightarrow{k} +\infty$. Then$\|T^k x\| \xrightarrow{k} +\infty$ too. Consequently,*

$$\{y \in \mathcal{H} : \|T^k y\| \xrightarrow{k} +\infty\} = \{y \in \mathcal{H} : \exists n_0 \in \mathbb{N} \textit{ with } \| \left(T^{n_0}\right)^k y\| \xrightarrow{k} +\infty\}.$$

Proof. Assume that $\|T^k x\| \not\to +\infty$. Then there exist $M \geq 0$ and a subsequence $\{k_j\} \subset \mathbb{N}$ such that

$$\|T^{k_j} x\| \to M. \tag{4.1}$$

If, for $k_j > n_0$, we write $k_j = n_0 q_j - r_j$, where $q_j, r_j \in \mathbb{N}_0$, $0 \leq r_j < n_0$, we have

$$\|(T^{n_0})^{q_j} x\| = \|T^{r_j + k_j} x\| \leq \|T\|^{r_j} \cdot \|T^{k_j} x\|. \tag{4.2}$$

But under the assumption (4.1), (4.2) contradicts the fact that $\| (T^{n_0})^k x\| \xrightarrow{k} +\infty$.

$\square$

The essence of the main new technique of this note is contained in the following proposition.

Proposition 4.2.3. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of operators in $\mathcal{L}(\mathcal{H})$ with the property that there exists a sequence of infinite dimensional subspaces $\{\mathfrak{M}_n\}_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, the operator T_n is bounded below on the subspaces $\mathfrak{M}_n, \ldots, \mathfrak{M}_{2n-1}$ by some $M(n) > 0$. Moreover, let $\{\alpha_n\}_{n\in\mathbb{N}}$ be any sequence in $l_2(\mathbb{N})$. Then for every $x_0 \in \mathcal{H}$ there exists $y \in \mathcal{H}$ such that*

$$\|y - x_0\|^2 \leq \sum_{i=1}^{\infty} |\alpha_i|^2, \quad \|T_n y\|^2 \geq \sum_{i=n}^{2n-1} |\alpha_i|^2 M^2(n), \quad n \in \mathbb{N}. \tag{4.3}$$

Proof. From each subspace $\mathfrak{M}_n$ we will choose by induction a unit vector f_n such that the sequence $\{f_i\}_{i=1}^{\infty}$ will satisfy a set of orthogonality conditions, and we will define

$$y = x_0 + \sum_{i=1}^{\infty} \alpha_i f_i.$$

To simplify notation, let $f_0 := x_0$, $\alpha_0 := 1$, $T_0 := I$, and

$$y_n = \sum_{i=0}^{n} \alpha_i f_i, \quad n \in \mathbb{N}.$$

We will choose the vector $f_n \in \mathfrak{M}_n$, with $\|f_n\| = 1$, such that

$$T_k f_n \perp T_k f_j, \quad 0 \leq j \leq n-1, \quad 0 \leq k \leq n. \tag{4.4}$$

After defining f_j we will have, by hypothesis, for every $n \in \mathbb{N}$,

$$\|T_n f_j\| \geq M(n), \quad n \leq j \leq 2n-1, \tag{4.5}$$

and for $n \leq i, j, i \neq j$ we will have $T_n f_i \perp T_n y_{n-1}$ and $T_n f_j \perp T_n f_i$, so

$$\|T_n y\|^2 = \left\| T_n \left(y_{n-1} + \sum_{i=n}^{\infty} \alpha_i f_i \right) \right\|^2 = \|T_n y_{n-1}\|^2 + \left\| T_n \left(\sum_{i=n}^{\infty} \alpha_i f_i \right) \right\|^2 = \quad (4.6)$$
$$= \|T_n y_{n-1}\|^2 + \sum_{i=n}^{\infty} |\alpha_i|^2 \|T_n f_i\|^2.$$

Now we give more details on the recursive construction of the sequence $\{f_n\}$. If $\mathfrak{M}$ is a subspace of $\mathcal{H}$, we will write $\mathcal{P}_{\mathfrak{M}}$ for the (orthogonal) projection of $\mathcal{H}$ onto $\mathfrak{M}$.

Choose $f_1 \in \mathfrak{M}_1$ such that $\|f_1\| = 1$, $f_1 \perp f_0$ and $T_1 f_1 \perp T_1 f_0$. To accomplish this, define $\mathcal{S}_1 = \bigvee\{f_0, T_1^* T_1 f_0\}$ and choose $f_1 \in \mathfrak{M}_1 \ominus \mathcal{P}_{\mathfrak{M}_1}(\mathcal{S}_1)$.

In general, in order to have all the conditions in (4.4) satisfied, after $f_0, \ldots, f_{n-1}$ have been defined, set $\mathcal{S}_n = \bigvee_{0 \leq j \leq n-1,\ 0 \leq k \leq n} \{T_k^* T_k f_j\}$, which is a finite dimensional vector space, and choose f_n to be a unit vector in $\mathfrak{M}_n \ominus \mathcal{P}_{\mathfrak{M}_n}(\mathcal{S}_n)$. This defines the sequence $\{f_n\}_{n \in \mathbb{N}_0}$ with the desired properties.

Using (4.5) and (4.6) we have

$$\|T_n y\|^2 \geq \sum_{i=n}^{\infty} |\alpha_i|^2 \|T_n f_i\|^2 \geq \sum_{i=n}^{2n-1} |\alpha_i|^2 M^2(n), \quad n \in \mathbb{N}.$$

□

Remark 4.2.4. Notice first that the hypothesis of Proposition 4.2.3 could be modified so as to be valid only for n *sufficiently large* without changing the conclusion. Moreover, the hypothesis is implied by the statement that, given the sequence of operators $\{T_n\}$, there exist a sequence of infinite dimensional subspaces $\{\mathfrak{M}_n\}_n$ and a sequence of positive real numbers $\{N(n)\}_n$, such that for every n sufficiently large, all operators from the set $T_{[(n+1)/2]}, \ldots, T_n$ are bounded below by $N(n)$ on $\mathfrak{M}_n$. This is based on the fact that, if for every $j \in \mathbb{N}$ the operators $T_{[(j+1)/2]}, \ldots, T_j$ are bounded below on the subspace $\mathfrak{M}_\mathrm{j}$ by a constant $N(j)$, then for every $n \in \mathbb{N}$ the operator T_n is bounded below on the subspaces $\mathfrak{M}_n, \ldots \mathfrak{M}_{2n-1}$ by $M(n)$, if we let $M(n)$ to be the minimum of the set $\{N(n), \ldots, N(2n-1)\}$. We have also that the condition

$$\forall n, j \in \mathbb{N} \text{ with } n \leq j \leq 2n-1 : \quad \|T_n f_j\| \geq M(n),$$

implies the following:

$$\forall j, n \in \mathbb{N} \text{ with } \left(j+1\right)/2 \leq n \leq j : \quad \|T_n f_j\| \geq N(j),$$

with the notation $N(j)$ for the minimum of the set $\{M([(j+1)/2]), \ldots, M(j)\}$.

The following is one of our two main results.

Theorem 4.2.5. *Let $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ and $\{K_n\} \subset \mathbb{K}$ be such that there exist positive numbers ρ, δ with the property that for every $n \in \mathbb{N}$ sufficiently large, there exists an infinite dimensional subspace $\mathfrak{M}_n$ on which $T_{[(n+1)/2]}, \ldots, T_n$ are bounded below by ρn^{δ}, and define*

$$A_n = T_n + K_n, \quad n \in \mathbb{N}.$$

Then the set of vectors $y \in \mathcal{H}$ such that $\|A_n y\| \to +\infty$ is dense in $\mathcal{H}$.

Proof. Fix ε such that $0 < \varepsilon < 1$ and choose N_0 sufficiently large that for $n \geq N_0$, there exists an infinite dimensional subspace $\mathfrak{M}_n$ on which $T_{[(n+1)/2]}, \ldots, T_n$ are bounded below by ρn^{δ}. Note that the compact operators, $K_{[n/2]}, \ldots, K_n$, when restricted to $\mathfrak{M}_n$, remain compact operators, and thus for $j = [n/2], \ldots, n$, there exists a finite dimensional subspace $\mathfrak{F}_j \subset \mathfrak{M}_n$ such that $\|K_j| (\mathfrak{M}_n \ominus \mathfrak{F}_j)\| < \varepsilon\rho n^{\delta}$. Thus for $j = [(n+1)/2], \ldots, n$, $\left\| K_j \middle| \left(\mathfrak{M}_n \ominus \bigvee_{j=[(n+1)/2]}^{n} \mathfrak{F}_j\right) \right\| < \varepsilon\rho n^{\delta}$. Thus by defining $\mathfrak{N}_n = \mathfrak{M}_n \ominus \left(\bigvee_{j=[(n+1)/2]}^{n} \mathfrak{F}_j\right)$, we see that each of the operators $A_{[(n+1)/2]}, \ldots, A_n$ is bounded below on $\mathfrak{N}_n$ by $N(n) := (1-\varepsilon)\rho n^{\delta}$.

Next, note that the sequence of operators $\{A_n\}$, together with the sequence of subspaces $\{\mathfrak{N}_n\}$ and the sequence of lower bounds $\{N(n)\}$, satisfies the hypothesis of

Proposition 4.2.3 (and Remark 4.2.4). Given an arbitrary vector $x_0 \in \mathcal{H}$, let y be as in the conclusion of the Proposition 4.2.3. Then we have

$$\|y - x_0\|^2 \leq \sum_{i=1}^{\infty} |\alpha_i|^2, \quad ||A_n y||^2 \geq \sum_{i=n}^{2n-1} |\alpha_i|^2 N^2(n), \ n \in \mathbb{N}.$$

Take $\alpha_n = \varepsilon / n^{(1+\delta)/2}$. Then

$$\|A_n y||^2 \geq (1-\varepsilon)^2 \rho^2 \sum_{i=n}^{2n-1} \left(\frac{\varepsilon^2}{i^{1+\delta}} \cdot n^{2\delta} \right) \geq (1-\varepsilon)^2 \frac{\rho^2}{2^{1+\delta}} \cdot \sum_{i=n}^{2n-1} \frac{\varepsilon^2}{n^{1-\delta}} = (1-\varepsilon)^2 \frac{\rho^2 \varepsilon^2}{2^{1+\delta}} \cdot n^{\delta},$$

and

$$\|y - x_0\|^2 \leq \varepsilon^2 \sum_{i=1}^{\infty} \left(\frac{1}{i^{1+\delta}} \right),$$

from which is immediate that $\|A_n y\| \to \infty$ and that the set of vectors y with this property is dense in $\mathcal{H}$. □

The obvious application to powers of a single operator is this:

Corollary 4.2.6. *Assume $T \in \mathcal{L}(\mathcal{H})$ has the property that there exist positive numbers ρ, δ such that for every $n \in \mathbb{N}$ sufficiently large, there exists an infinite dimensional subspace $\mathfrak{M}_n$ on which the powers $T^{[(n+1)/2]}, \ldots, T^n$ are bounded below by ρn^{δ}. If $K \in \mathbb{K}$ and $A := T + K$, then there exists a dense set of vectors y in $\mathcal{H}$ such that $\|A^n y\| \to +\infty$.*

Proof. There exists a sequence of compact operators $\{K_n\}$ such that

$$A^n = T^n + K_n.$$

□

Theorem 4.2.5 has a generalization that should be quite useful in enlarging the class (NOT):

Theorem 4.2.7. *Suppose that* $\{B_n\} \in \mathcal{L}(\mathcal{H}^{(2)})$ *has the property that*

$$B_n = \begin{pmatrix} T_n & * \\ 0 & * \end{pmatrix},$$

where the asterisks denote arbitrary entries, and $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ *is such that there exist positive numbers* ρ, δ *with the property that for every* $n \in \mathbb{N}$ *sufficiently large, there exists an infinite dimensional subspace* $\mathfrak{M}_n$ *on which* $T_{[n/2]}, \ldots, T_n$ *are bounded below by* ρn^δ*. Let* $\{K_n\} \subset \mathbb{K}\left(\mathcal{H}^{(2)}\right)$ *and define* $A_n = B_n + K_n$*. Then the set of vectors* $y \in \mathcal{H}$ *satisfying* $\|A_n y\| \to +\infty$ *is dense in* $\mathcal{H} \oplus (0)$*.*

Proof. Write

$$K_n = \begin{pmatrix} K_{n1} & K_{n2} \\ K_{n3} & K_{n4} \end{pmatrix},$$

and

$$A_n = \begin{pmatrix} T_n + K_{n1} & * \\ K_{n3} & * \end{pmatrix}.$$

Apply Theorem 4.2.5 to the sequence of operators $\{A_{n1} := T_n + K_{n1}\}$ to build a vector $y_1 \in \mathcal{H}$ such that $\|A_{n1} y_1\| \to +\infty$. Then for the vector

$$y = \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \in \mathcal{H}^{(2)},$$

we have

$$A_n y = \begin{pmatrix} T_n + K_{n1} & * \\ K_{n3} & * \end{pmatrix} \begin{pmatrix} y_1 \\ 0 \end{pmatrix} = \begin{pmatrix} A_{n1} y_1 \\ K_{n3} y_1 \end{pmatrix},$$

and hence $\|A_n y\| \to +\infty$. □

We turn now to the application of the results of Section 4.2 to the class of n-normal operators.

4.3. N-normal operators

Recall that for any $n \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is called an n-normal operator if T is unitarily equivalent to an $n \times n$ operator matrix (N_{ij}) acting on $\mathcal{H}^{(n)}$ in the usual fashion, where the set $\{N_{ij}\}$ consists of mutually commuting normal operators in $\mathcal{L}(\mathcal{H})$. The theory of n-normal operators is quite well developed and in [14] it was proved that every n-normal operator has a nontrivial hyperinvariant subspace (n.h.s.).

We will show below that compact perturbations of certain classes of 2-normal operators are subsets of (NOT) by virtue of having an orbit that tends to infinity.

We begin with the following well-known fact from [10].

Proposition 4.3.1. *Let T be any n-normal operator in $\mathcal{L}(\mathcal{H})$. Then T is unitarily equivalent to an n-normal operator (N_{ij}), acting on $\mathcal{H}^{(n)}$ in the usual matricial fashion, which satisfies*

i) the N_{ij} are mutually commuting normal operators in $\mathcal{L}(\mathcal{H})$,

ii) $N_{ij} = 0$ whenever $i > j$, i.e., the matrix (N_{ij}) is in upper triangular form.

Notation 4.3.2. We shall say that an n-normal operator is *in standard form* if it is an $n \times n$ matrix (N_{ij}) acting as usual on $\mathcal{H}^{(n)}$ and satisfying i) and ii) of Proposition 4.3.1. (Of course, except in rare cases the standard form of an n-normal operator is not unique.)

The next lemma is elementary.

Lemma 4.3.3. *Let $T = (N_{ij}) \in \mathcal{L}(\mathcal{H}^{(2)})$ be a 2-normal operator in standard form, and suppose that the polar decompositions $N_{11} = V_1 P_1$ and $N_{22} = V_2 P_2$ (with $P_1, P_2 \geq 0$) satisfy $V_1 = V_2$. Then there exists a unitary operator $U \in \mathcal{L}(\mathcal{H})$ such that $N_{ii} = UP_i$, $i = 1, 2$, and U commutes with all the N_{ij}. Moreover, for every $K \in \mathbb{K}$, there exists* *a sequence* $\{J_k\}_{k\in\mathbb{N}} \subset \mathbb{K}$ *such that*

$$\|(T+K)^k x\| = \|(S^k + J_k)x\|, \quad k \in \mathbb{N}, x \in \mathcal{H}, \tag{4.7}$$

where S *is the 2-normal operator in* $\mathcal{L}(\mathcal{H}^{(2)})$ *in standard form given by the matrix*

$$S = \begin{pmatrix} P_1 & U^{-1}N_{12} \\ 0 & P_2 \end{pmatrix}. \tag{4.8}$$

Proof. The first statement of the lemma is an easy consequence of the spectral theorem for normal operators. Next, write $Diag(U)$ for the 2×2 diagonal matrix with each diagonal entry equal to U. Note that S and $Diag(U)$ are mutually commuting 2-normal operators in standard form, and $T = Diag(U)S$. Furthermore, if we write $(T+K)^k = T^k + K_k$ where $K_k \in \mathbb{K}$, we obtain

$$(T+K)^k = T^k + K_k = (Diag(U)S)^k + K_k = Diag(U^k)(S^k + J_k), \quad k \in \mathbb{N}, \tag{4.9}$$

where $J_k := Diag(U^{-k})K_k \in \mathbb{K}$, and since for each $k \in \mathbb{N}$, $Diag(U^k)$ is a unitary operator, (4.7) is immediate from (4.9). □

We turn now to some preliminary lemmas.

Lemma 4.3.4. *Let* $1 \geq a \geq b \geq 0$ *and let* k *be a given positive integer. Then, for every* $m \in \mathbb{N} \cap [k/2, k]$, *we have*

$$a^m - b^m \geq \frac{1}{2}(a^k - b^k), \tag{4.10}$$

and thus

$$\sum_{i=0}^{m-1} a^{m-1-i} b^i \geq \frac{1}{2}\left(\sum_{i=0}^{k-1} a^{k-1-i} b^i\right). \tag{4.11}$$

Proof. The inequality (4.10) is equivalent to

$$b^k - 2b^m \geq a^k - 2a^m \quad \text{for } \frac{k}{2} \leq m \leq k.$$

Consider the function $f(x) = x^k - 2x^m$. Then $f'(x) = kx^{k-1} - 2mx^{m-1} = x^{m-1}(kx^{k-m} - 2m)$. But since $2m \geq k$ we have $f'(x) \leq x^{m-1}(kx^{k-m} - k) = x^{m-1}k(x^{k-m} - 1)$, and for $x \in [0, 1]$ we have $f'(x) \leq 0$. Thus f is decreasing on the interval $[0, 1]$ and thus the inequality (4.10) follows.

The equation (4.11) is immediate in the case $a = b$, and it follows from (4.10) by dividing by $(a - b)$ when $a > b$. □

Lemma 4.3.5. *Suppose*

$$T = \begin{pmatrix} P_1 & N \\ 0 & P_2 \end{pmatrix}$$

is a 2-normal operator in standard form, where $P_i \geq 0$, and let $K \in \mathbb{K}$ and $A = T + K$. Then

$$T^k = \begin{pmatrix} P_1^k & N\sum_{i=0}^{k-1}\left(P_1^{k-1-i}P_2^i\right) \\ 0 & P_2^k \end{pmatrix}, \quad k \in \mathbb{N}, \tag{4.12}$$

and $A^k = T^k + K_k$, where $K_k \in \mathbb{K}$. Moreover, if $\max\{\|P_1\|_e, \|P_2\|_e\} > 1$, then there exists a vector $x \in \mathcal{H}^{(2)}$ such that $\|A^k x\| \to +\infty$.

Proof. Equation (4.12) results from an easy calculation, and since $r_e(A) = \max\{\|P_1\|_e, \|P_2\|_e\}$, if this maximum is greater than 1, then the existence of such a vector x is immediate from Corollary 1.6 of [15] . □

This next result is our second main theorem.

Theorem 4.3.6. *Suppose*

$$T = \begin{pmatrix} N_1 & M \\ 0 & N_2 \end{pmatrix}$$

is a 2-normal operator in standard form, with $N_1 = VP_1$ and $N_2 = VP_2$ (polar decompositions), $K \in \mathbb{K}$, and there exist $\rho, \delta > 0$ such that

$$\|T^k\|_e > \rho k^\delta, \quad k \in \mathbb{N}. \tag{4.13}$$

Then there exists $x \in \mathcal{H}^{(2)}$ such that $\|(T+K)^n x\| \to +\infty$, and consequently $T+K \in (NOT)$.

Proof. By Lemmas 4.3.3 and 4.3.5, with

$$S = \begin{pmatrix} P_1 & N \\ 0 & P_2 \end{pmatrix} \tag{4.14}$$

a normal operator in standard form, it suffices to show that if $\{J_k\}$ is any sequence of compact operators, then there exists $x \in \mathcal{H}^{(2)}$ such that

$$\|(S^k + J_k)x\| \to +\infty.$$

Moreover, we may suppose that $\{\|P_1\|_e, \|P_2\|_e\} \leq 1$, and that (via (4.9))

$$\|S^k\|_e = \|T^k\|_e \geq \rho k^\delta, \quad k \in \mathbb{N}. \tag{4.15}$$

Applying (4.15), (4.14) and Lemma 4.3.5, we obtain immediately that

$$\left\| N \sum_{i=0}^{k-1} P_1^{k-1-i} P_2^i \right\|_e > \rho k^\delta - 1 > \hat{\rho} k^\delta, \tag{4.16}$$

where $\hat{\rho} > 0$ is defined appropriately for k sufficiently large.

Since P_1, P_2 and N are mutually commuting normal operators, it follows from [7] that there exist compact operators J_1, J_2, J_3, an orthonormal basis $\mathcal{E} = \{e_n\}_{n \in \mathbb{N}}$ for $\mathcal{H}$, and sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in (l_∞), with $0 \leq \alpha_n, \beta_n \leq 1$, such that

$$P_1 = Diag(\alpha_n) + J_1, \quad P_2 = Diag(\beta_n) + J_2, \quad N = Diag(\gamma_n) + J_3,$$

and thus (see 4.16) the $(1,2)$ entry of S^k is

$$Diag\left(\gamma_n \sum_{i=0}^{k-1} \left(\alpha_n^{k-1-i} \beta_n^i\right)\right) + J_4^{(k)}, \quad k \in \mathbb{N},$$

where $J_4^{(k)}$ is compact. Moreover, since

$$\left\| N \sum_{i=0}^{k-1} P_1^{k-1-i} P_2^i \right\|_e = r_e \left(Diag \left(\gamma_n \sum_{i=0}^{k-1} \left(\alpha_n^{k-1-i} \beta_n^i \right) \right) \right)$$

is the largest (in modulus) limit point of the sequence

$$\left\{ \gamma_n \sum_{i=0}^{k-1} \alpha_n^{k-1-i} \beta_n^i \right\},$$

for each fixed $k \in \mathbb{N}$ there exists a subsequence

$$\left\{ |\gamma_{n_q}| \sum_{i=0}^{k-1} \alpha_{n_q}^{k-1-i} \beta_{n_q}^i \right\}_q ,$$

where $\{n_q\}$ depends on k, that converges to $\left\| N \sum_{i=0}^{k-1} P_1^{k-1-i} P_2^i \right\|_e$. Thus for n_q sufficiently large (say $q \geq q_0$), we have, with

$$\mathfrak{M}_k = \bigvee \{e_{n_q}\}_{q \geq q_0},$$

that $N \sum_{i=0}^{k-1} P_1^{k-1-i} P_2^i$ is bounded below on $\mathfrak{M}_k$ by $\hat{\rho} k^\delta$ (see (4.16)), and thus S^k has the same lower bound on $(0) \oplus \mathfrak{M}_k$.

By Lemma 4.3.4, for $m = [(k+1)/2], \ldots, k-1$, we have

$$|\gamma_n| \sum_{i=0}^{m-1} \alpha_n^{m-1-i} \beta_n^i \geq \frac{1}{2} \left(|\gamma_n| \sum_{i=0}^{k-1} \alpha_n^{k-1-i} \beta_n^i \right),$$

and using this fact for the subsequence of indices $\{n_q\}$ we get that $S^{[(k+1)/2]}, \ldots, S^k$ have $\frac{1}{2}\hat{\rho} k^\delta$ as a lower bound on $(0) \oplus \mathfrak{M}_k$. Now the conclusion follows from Theorem 4.2.5 . $\square$

This result combined with Theorem 4.2.7 yields the following:

Corollary 4.3.7. *If R is an operator that has an invariant subspace on which it is a 2-normal operator satisfying the hypothesis of Theorem 4.3.6, and $A = R + K$, where*

K is compact, then A has an orbit tending to infinity.

Remark 4.3.8. It follows as in Theorem 4.2.5 that the set of vectors x as in the conclusion of Theorem 4.3.6 is dense in $\mathcal{H}$.

We continue by recalling the definition of the operator A from Example 4.5 of [15]. (The question whether A belongs to (NOT) was left unresolved in [15].)

Example 4.3.9. Let H be a Hermitian 2-normal operator in $\mathcal{L}(\mathcal{H}^{(2)})$ represented as an operator matrix

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \tag{4.17}$$

where H_1 and H_2 are commuting Hermitian operators in $\mathcal{L}(\mathcal{H})$ satisfying $\sigma(H_i) = \sigma_e(H_i) = [1/2, 1]$, $i = 1, 2$, and on the space $\bigoplus_{n=2}^{\infty} \mathcal{H}^{(2)}$ consider the 2-normal operator

$$T = \bigoplus_{n=2}^{\infty} \begin{pmatrix} (1-1/n)1_{\mathcal{H}} & (1/\sqrt{n})1_{\mathcal{H}} \\ 0 & (1-1/n)1_{\mathcal{H}} \end{pmatrix}.$$

Then, as was noted in [15], for all $k \in \mathbb{N}$, $\|T^k\|_e$ satisfies

$$\frac{k}{\sqrt{2k-1}} \left(\frac{2k-2}{2k-1} \right)^{k-1} \leq \|T^k\|_e \leq 1 + \frac{k}{\sqrt{2k-1}} \left(\frac{2k-2}{2k-1} \right)^{k-1}. \tag{4.18}$$

Let K be an arbitrary compact operator on $\bigoplus_{n=1}^{\infty} \mathcal{H}^{(2)}$ and set

$$A = (H \oplus T) + K. \tag{4.19}$$

(The presence here of the direct summand H is simply to prevent A from having disconnected spectrum, and thus to have a nontrivial hyperinvariant subspace.) Then $\sigma_e(A) = \sigma_e(H \oplus T) = [1/2, 1]$, but A is not essentially power bounded. In fact, from (4.18) we see that, asymptotically, $\|A^k\|_e = \|T^k\|_e \sim \sqrt{k}$; and this growth is too slow for [15, Th. 1.5] to be applicable. However, that the operator A in (4.19) belongs to

(NOT) is now immediate from Theorem 4.3.6.

A natural question to ask, in view of Theorem 4.3.6, is whether the hypothesis (4.13) is really necessary in order to conclude that T has some orbit converging to infinity. Example 4.4.1 shows that (4.13) cannot be omitted in general, but this next proposition shows that there are some classes of n-normal operators in which (4.13) is not needed.

Proposition 4.3.10. *Suppose $K \in \mathbb{K}(\mathcal{H}^{(2)})$ and*

$$T = \begin{pmatrix} N_1 & N_3 \\ 0 & N_2 \end{pmatrix} \in \mathcal{L}(\mathcal{H}^{(2)})$$

is a 2-normal operator in standard form such that $\max\{r_e(N_1), r_e(N_2)\} < 1$. Then $T + K$ is essentially power bounded, and thus (by [15, Theorem 1.2]) belongs to (NOT). Moreover, if, on the other hand, $\sigma_e(N_1) = \sigma_e(N_2) = \{1\}$ and N_3 is essentially invertible (i.e., $\pi(N_3)$ is invertible), then either $T + K$ is essentially power bounded or $\|(T+K)^k\|_e \sim k$ asymptotically, so again (by [15, Theorems 1.2 and 1.5]), $T + K \in$ (NOT).

Proof. Since $r_e(T + K) = r_e(T) = \max\{r_e(N_1), r_e(N_2)\}$, if $r_e(T + K) < 1$, then one knows from the general theory of Banach algebras that $\|(T + K)^k\|_e \xrightarrow{k} 0$, so $T + K$ is essentially power bounded. Now suppose that $\sigma_e(N_1) = \sigma_e(N_2) = \{1\}$ and $\pi(N_3)$ is invertible. Then one knows from [21] that there exist compact operators $K_i, i = 1, 2, 3$, in $\mathcal{L}(\mathcal{H})$ such that $N_1 = 1_{\mathcal{H}} + K_1$, $N_2 = 1_{\mathcal{H}} + K_2$, and $N_3 = J + K_3$, where J is invertible. Thus

$$T = \begin{pmatrix} 1_{\mathcal{H}} & J \\ 0 & 1_{\mathcal{H}} \end{pmatrix} + K,$$

where $K \in \mathcal{L}(\mathcal{H}^{(2)})$ is compact. An easy matricial calculation (see [15, Prop. 3.2]) gives that $\|(T+K)^k\|_e = \|T^k\|_e \sim k$, and that $T+K$ has (a dense set of) orbits tending to infinity is now immediate from [15, Theorem 1.5] . $\square$

4.4. Oscillating behavior

Here we construct a 2-normal operator $B = (B_{ij}) \in \mathcal{L}(\mathcal{H}^{(2)})$ in standard form, with $B_{11} = B_{22} \geq 0$, but B is not essentially power bounded, and neither does it satisfy the hypothesis of Theorem 4.3.6. The operator B will have the property that there exists a subsequence of powers $\{k_n\}$ and $\rho, \delta > 0$ such that $\|B^{k_n}\|_e > \rho k_n^\delta$ for all k_n, but at the same time, there exists a subsequence of powers $\{j_n\}$ such that $\{\|B^{j_n}\|\}$ is bounded.

Example 4.4.1. Let

$$B = \bigoplus_{n=2}^{\infty} T_{m_n},$$

where T_n is given by

$$T_n := \begin{pmatrix} (1-1/n)1_{\mathcal{H}} & (1/\sqrt{n})1_{\mathcal{H}} \\ 0 & (1-1/n)1_{\mathcal{H}} \end{pmatrix}, \quad n \in \mathbb{N},$$

and $\{m_n\}$ is defined recursively as $m_2 = 2$, $m_{n+1} = m_n^4$, which gives $m_n = 2^{(4^{n-2})}$. The sequences of powers $\{k_n\}$ and $\{j_n\}$ are given by $k_n := m_n$ and $j_n := m_n^2$.

Note that

$$T_n^k = \begin{pmatrix} (1-1/n)^k 1_{\mathcal{H}} & (k/\sqrt{n})(1-1/n)^k 1_{\mathcal{H}} \\ 0 & (1-1/n)^k 1_{\mathcal{H}} \end{pmatrix},$$

Consider the magnitude of the $(1,2)$ entry of T_n^k as a function of n, with k as parameter, denoted as

$$g(n) = \frac{k}{\sqrt{n}}\left(1 - \frac{1}{n}\right)^{k-1}.$$

Note that the $(1,2)$ entry of T_n^k is the one that determines the growth of the essential norm, since the diagonal entries T_n^k have norms bounded by 1. The first derivative of g is

$$g'(n) = \frac{1}{n}\frac{k}{\sqrt{n}}\left(1 - \frac{1}{n}\right)^{k-2}\left(-\frac{1}{2}\frac{n-1}{n} + \frac{k-1}{n}\right).$$

with the only zero at $n = 2k-1$. By computing the second derivative we note that $n = 2k-1$ is a maximum point for g.

We are now ready to estimate the growth of the sequences $\{\|B^{k_n}\|_e\}$ and $\{\|B^{j_n}\|\}$. As noted above the growth of $T_{m_n}^k$ is determined by the expression

$$\frac{k}{\sqrt{m_n}}\left(1 - \frac{1}{m_n}\right)^{k-1}.$$

For the first sequence, we have that $\|B^{k_n}\|_e = \|B^{m_n}\|_e \geq \|T_{m_n}^{m_n}\|_e$ and since the sequence $\left\{\left(1 - \frac{1}{m_n}\right)^{m_n}\right\}_n$ is decreasing to $1/e$ we have

$$\|T_{m_n}^{m_n}\|_e \geq \sqrt{m_n}\left(1 - \frac{1}{m_n}\right)^{m_n-1} > \sqrt{m_n}\cdot\frac{1}{e}$$

It follows that here we can take $\delta = 1/2$ and $\rho = 1/3$ to get

$$\|B^{k_n}\|_e > \rho k_n^{\delta}.$$

Next, to show that $\{\|B^{j_n}\|\}$ is a bounded sequence, it is sufficient to show that there exists a constant M such that for all n and p we have

$$\|T_{m_p}^{j_n}\| \leq M.$$

By the above remarks, for the fixed power $k := j_n = m_n^2$, (and thus for a fixed n), we have that the entry of T_q^k that determines the growth of the $\|T_q^k\|$ has a maximum at $q_0 = 2k - 1 (= 2m_n^2 - 1)$, increasing for q up to that value and decreasing after the maximum is attained. Notice that the sequence $\{m_r\}_r$ does not assume the value q_0; in particular $m_n < q_0 < m_{n+1}$. Thus, to show $\|T_{m_p}^{j_n}\|$ is bounded for all p and n, it is sufficient to show that, for fixed n, we have that $\|T_{m_n}^{j_n}\|$ and $\|T_{m_{n+1}}^{j_n}\|$ have upper bounds independent of n.

The norm of the $(1, 2)$ entry of $T_{m_{n+1}}^{j_n}$ is

$$\left(1 - \frac{1}{m_n^4}\right)^{m_n^2 - 1} \leq 1,$$

and the norm of the $(1, 2)$ entry of $T_{m_n}^{j_n}$ is

$$\frac{m_n^2}{\sqrt{m_n}} \left(1 - \frac{1}{m_n}\right)^{m_n^2 - 1} = 2 \cdot m_n^{3/2} \left(1 - \frac{1}{m_n}\right)^{m_n^2} \leq 2 \cdot m_n^{3/2} \left(\frac{1}{2}\right)^{m_n},$$

since the sequence $\left\{\left(1 - \dfrac{1}{m_n}\right)^{m_n}\right\}_n$ begins with $1/2$ and is decreasing; thus $\|T_{m_n}^{j_n}\| \xrightarrow{n} 0$.

CHAPTER V

ADDITIONAL FACTS

We include here some facts related to the material in Chapter IV. First we present an alternative proof that the operator A defined in equation (4.19) is not orbit-transitive. The proof relies on induction to construct a vector y that is not hypercyclic for the operator A. The second section shows that the operator T defined by equation (5.1) is not weakly hypertransitive. We were not able to extend the proof to show that any compact perturbation of T is not weakly hypertransitive. The third section includes results related to the growth rate of the powers of a 2-normal operator, and the last section contains a slight generalization of a proposition from Chapter IV that has as corollary a well known theorem.

5.1. An alternative proof for the non-hypertransitivity of the operator A

Consider the 2-normal operator T as in [15, Example 4.5]; that is, on the space $\oplus_{n=1}^{\infty}\mathcal{H}_n^{(2)}$ define

$$T = \oplus_{n=2}^{\infty} \begin{pmatrix} (1-1/n)1_{\mathcal{H}} & (1/\sqrt{n})1_{\mathcal{H}} \\ 0 & (1-1/n)1_{\mathcal{H}} \end{pmatrix}. \tag{5.1}$$

We want to show that, if K is an arbitrary compact operator and H is defined as in [15, Example 4.5], than we can construct a vector y_0 of the type (5.8) such that the operator $A = K + (H \oplus T)$ has the property that $\{||A^n y_0||\}_n$ tends to infinity. Such a vector y_0 would not be hypercyclic and it would follow that the operator A is not hypertransitive.

Let T_n denote the matrix

$$T_n = \begin{pmatrix} (1-1/n)1_{\mathcal{H}} & (1/\sqrt{n})1_{\mathcal{H}} \\ 0 & (1-1/n)1_{\mathcal{H}} \end{pmatrix}$$

and let $S_n := \begin{pmatrix} (1-1/n) & (1/\sqrt{n}) \\ 0 & (1-1/n) \end{pmatrix}$ be the corresponding scalar matrix. We have that

$$S_n^k = \begin{pmatrix} \left(\frac{n-1}{n}\right)^k & \frac{k}{\sqrt{n}}\left(\frac{n-1}{n}\right)^{k-1} \\ 0 & \left(\frac{n-1}{n}\right)^k \end{pmatrix}.$$

Let $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

$$S_n^k v = \begin{pmatrix} \frac{k}{\sqrt{n}}\left(\frac{n-1}{n}\right)^{k-1} \\ \left(\frac{n-1}{n}\right)^k \end{pmatrix}$$

If we take $n = 2k-1$ we get that $||T_{2k-1}^k|| \geq ||S_{2k-1}^k v|| \geq \frac{k}{\sqrt{2k-1}}\left(\frac{2k-2}{2k-1}\right)^{k-1}$, so the growth of $||T^k||$ is at least of order $\sqrt{k}$.

Choose $\alpha_n = 1/n^{(1+\delta)/2}$, with $0 < \delta < 1$ (in particular we can take $\delta = 1/2$); then $\{\alpha_n\}_{n=1}^{\infty} \in l_2$. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal sequence in $\oplus_{k=1}^{\infty}\mathcal{H}_k^2$ such that

$$e_n = \begin{pmatrix} 0 \\ f_n \end{pmatrix}, \text{where } f_n \in \oplus_{k=1}^{\infty}\mathcal{H}_k \text{ has a nonzero component only for } k = n, \quad (5.2)$$

Define $y \in \oplus_{n=1}^{\infty} \mathcal{H}_n^{(2)}$ by

$$y = \sum_{n=2}^{\infty} \alpha_n e_n, \tag{5.3}$$

Then

$$T^k y = \sum_{n=2}^{\infty} \alpha_n T_n^k e_n = \sum_{n=2}^{\infty} \alpha_n T_n^k \begin{pmatrix} 0 \\ f_n \end{pmatrix} = \sum_{n=2}^{\infty} \alpha_n \begin{pmatrix} \frac{k}{\sqrt{n}} \left(\frac{n-1}{n}\right)^{k-1} f_n \\ \left(\frac{n-1}{n}\right)^{k} f_n \end{pmatrix}. \tag{5.4}$$

We want to show that, if K is an arbitrary compact operator and H is defined as in [15, Example 4.5], than we can construct a vector y_0 of the type (5.8) such that the operator $A = K + (H \oplus T)$ has the property that $\{||A^n y_0||\}_n$ tends to infinity. Such a vector y_0 would not be hypercyclic and it would follow that the operator A is not hypertransitive.

In order to show this, we show first that for an arbitrary orthonormal sequence $\{f_n\}$ as above (f_n is a unitary vector from $\oplus_{k=1}^{\infty} \mathcal{H}_k$ with a nonzero component only for $k = n$) we have that

$$\{||T^n y||\}_n \to \infty.$$

Let

$$c_k(n) := \frac{k}{\sqrt{n}} \left(\frac{n-1}{n}\right)^{k-1},$$

and consider the terms in (5.11) from $n = k$ to $n = 2k$. We have

$$c_k(k) := \frac{k}{\sqrt{k}} \left(\frac{k-1}{k}\right)^{k-1}, \quad c_k(2k) := \frac{k}{\sqrt{2k}} \left(\frac{2k-1}{2k}\right)^{k-1},$$

Since

$$\lim_k \left(\frac{k-1}{k}\right)^{k-1} = e^{-1} \text{ and } \lim_k \left(\frac{2k-1}{2k}\right)^{k-1} = e^{-1/2},$$

there exist a constant $c_1 > 0$ and an index $n_1 \in \mathbb{N}$ such that for all $k \geq n_1$ and for all

n such that $k \leq n \leq 2k$ we have

$$c_k(n) \geq c_1 \cdot \sqrt{k}.$$

Thus for $n = k, .., 2k$ we have $||T^k(e_n)|| \geq c_1 \cdot \sqrt{k}$ and

$$||T^k y|| \geq \sum_{n=k}^{2k} |\alpha_n|^2 ||T_n^k e_n||^2 \geq c_1 \cdot \sum_{n=k}^{2k} \left(\frac{1}{n^{1+\delta}} \cdot k \right) \geq c_1 \cdot \frac{1}{2^{1+\delta}} \cdot \sum_{n=k}^{2k} \frac{1}{k^\delta} = c_2 \cdot k^{1-\delta}.$$

Thus, for $\delta = 1/2$ there exists a constant $c_2 > 0$ such that for $k \geq n_1$,

$$||T^k y|| \geq c_2 \cdot \sqrt{k}.$$

For each $n \in \mathbb{N}$ we choose an orthonormal sequence $\{e_k^{(n)}\}_{k=1}^\infty \in \oplus_{n=1}^\infty \mathcal{H}_n^{(2)}$ that it lives on $\mathcal{H}_n^{(2)}$ (all other components are zero) and define

$$y_j = \sum_{n=2}^{j} \alpha_n e_{k_n}^{(n)},$$

and $y = \lim_{j \to \infty} y_j$. We will choose by induction the subsequence $\{e_{k_n}^{(n)}\}_{n=1}^\infty$ as in [15, Example 4.5]. Let $\varepsilon > 0$.

It would be enough to get

$$||A^n y||^2 \geq (1-\varepsilon)^2 \sum_{i=n}^{2n} |\alpha_i|^2 ||T^n e_{k_i}^{(i)}||^2, \text{ for all } n. \tag{5.5}$$

That would work if we can get the following (our induction hypothesis) to be true:

$$||A^n \sum_{i=2}^{j} \alpha_i e_{k_i}^{(i)}||^2 \geq (1-\varepsilon)^2 \sum_{i=n}^{j} |\alpha_i|^2 ||T^n e_{k_i}^{(i)}||^2, \text{ for } n = 1, \ldots, j. \tag{5.6}$$

Since for $\{e_k^{(2)}\}$ the operator T produces vectors of equal norm, while the compact K_2 takes them to a sequence of vectors decreasing to zero we can choose $e_{k_2}^{(2)}$ far enough in the sequence such that

$$||A\alpha_2 e_{k_2}^{(2)}|| \geq (1-\varepsilon)\alpha_2||Te_{k_2}^{(2)}||$$

$$|\langle T\alpha_2 e_{k_2}^{(2)}, K_2\alpha_2 e_{k_2}^{(2)}\rangle| \leq |\alpha_2|^2\varepsilon.$$

Now suppose $\{e_{k_i}^{(i)}\}$, for $i = 2, \ldots, j$, have been chosen from $\{e_k^{(n)}\}_{k=1}^{\infty}$ such that

$$||A^n \sum_{i=2}^{j} \alpha_i e_{k_i}^{(i)}||^2 \geq (1-\varepsilon)^2 \sum_{i=n}^{j} |\alpha_i|^2||T^n e_{k_i}^{(i)}||^2, \text{ for } n = 1, \ldots, j. \tag{5.7}$$

Since the sequence $\{e_k^{(n)}\}_{k=1}^{\infty}$ converges weakly to zero, the same is true for every sequence $\{A^m e_k^{(n)}\}_{k=1}^{\infty}$ for fixed $m, n \in \mathbb{N}$. Thus for the fixed vector $\sum_{i=2}^{j} \alpha_i e_{k_i}^{(i)}$ we have

$$\left\langle A^n \alpha_{j+1} e_k^{(j+1)}, A^n \left(\sum_{i=2}^{j} \alpha_i e_{k_i}^{(i)}\right)\right\rangle \to 0, \text{ with } k$$

It follows, using the inductive hypothesis (5.7), that we can pick $e_{k_{j+1}}^{(j+1)}$ far enough in the $\{e_k^{(j+1)}\}$ sequence such that

$$\left\|A^n \left(\alpha_{j+1} e_{k_{j+1}}^{(j+1)} + \sum_{i=2}^{j} \alpha_i e_{k_i}^{(i)}\right)\right\|^2 \geq (1-\varepsilon)^2 \sum_{i=n}^{j+1} |\alpha_i|^2||T^n e_{k_i}^{(i)}||^2, \text{ for } n = 1, \ldots, j.$$

and

$$\left\|A^{j+1} \left(\sum_{i=2}^{j+1} \alpha_i e_{k_i}^{(i)}\right)\right\|^2 \geq (1-\varepsilon)^2|\alpha_{j+1}|^2||T^{j+1} e_{k_{j+1}}^{(j+1)}||^2.$$

5.2. Weak non-orbit-transitivity of T

We will show that we can construct two vectors y and $\tilde{y}$, such that the operator T defined by (5.1) has the property that $\{|\langle T^n y, \tilde{y}\rangle|\}_n$ tends to infinity. Then the vector

y would not be weakly hypercyclic and it would follow that the operator T is not weakly hypertransitive.

Let $\{e_n\}_{n=1}^{\infty}$ and $\{\widetilde{e}_n\}_{n=1}^{\infty}$ be two orthonormal sequences in $\oplus_{k=1}^{\infty}\mathcal{H}_k^2$ such that

$$e_n = \begin{pmatrix} 0 \\ f_n \end{pmatrix}, \quad \widetilde{e}_n = \begin{pmatrix} f_n \\ 0 \end{pmatrix}.$$

Define

$$y = \sum_{n=2}^{\infty} \alpha_n e_n = \sum_{n=2}^{\infty} \alpha_n \begin{pmatrix} 0 \\ f_n \end{pmatrix}, \tag{5.8}$$

and

$$\tilde{y} = \sum_{n=2}^{\infty} \alpha_n \begin{pmatrix} f_n \\ 0 \end{pmatrix}. \tag{5.9}$$

Then

$$T^k y = \sum_{n=2}^{\infty} \alpha_n T_n^k e_n = \sum_{n=2}^{\infty} \alpha_n T_n^k \begin{pmatrix} 0 \\ f_n \end{pmatrix} = \sum_{n=2}^{\infty} \alpha_n \begin{pmatrix} \frac{k}{\sqrt{n}} \left(\frac{n-1}{n}\right)^{k-1} f_n \\ \left(\frac{n-1}{n}\right)^{k} f_n \end{pmatrix}.$$

and

$$\langle T^k y, \tilde{y}\rangle = \sum_{n=2}^{\infty} \alpha_n^2 \frac{k}{\sqrt{n}} \left(\frac{n-1}{n}\right)^{k-1}.$$

We will show that

$$\{\langle T^n y, \tilde{y}\rangle\}_n \to \infty.$$

For $n = k, .., 2k$, there exists $\rho > 0$ such that

$$\frac{k}{\sqrt{n}} \left(\frac{n-1}{n}\right)^{k-1} \geq \rho \cdot \sqrt{k}$$

and thus

$$\langle T^n y, \tilde{y}\rangle = \sum_{n=1}^{\infty} \alpha_n^2 \frac{k}{\sqrt{n}} \left(\frac{n-1}{n}\right)^{k-1} \geq \sum_{n=k}^{2k} \alpha_n^2 \cdot \rho \cdot \sqrt{k} \geq \rho \cdot \sum_{n=k}^{2k} \left(\frac{1}{n^{1+\delta}} \cdot \sqrt{k}\right)$$
$$\geq \rho \cdot \frac{1}{2^{1+\delta}} \cdot \sum_{n=k}^{2k} \left(\frac{1}{k^{1+\delta}} \cdot \sqrt{k}\right) = \rho \cdot \frac{1}{2^{1+\delta}} \cdot k \left(\frac{1}{k^{1+\delta}} \cdot \sqrt{k}\right) = \rho \cdot \frac{1}{2^{1+\delta}} \cdot k^{1/2-\delta}.$$

So, for $\delta < 1/2$ we have that

$$\langle T^n y, \tilde{y}\rangle \xrightarrow{n} +\infty,$$

and this finishes the argument that the operator T is not weakly-orbit transitive.

5.3. Results related to the growth rate of a 2-normal operator

Let $\{T_n\}_n$ be a sequence of operators and $\{e_j\}_j$ be a sequence of vectors such that, for every j, when the operators in the set $\{T_{[(j+1)/2]}, \dots, T_j\}$ are applied to e_j give vectors with a common lower bound $M(j)$. It follows that when applying the operator T_n to the vectors $\{e_n, \dots, e_{2n-1}\}$ we obtain vectors with norm bounded below by $\min\{M(n), \dots, M(2n-1)\}$.

Note that if we restrict the condition above to hold for just a ratio of the operators in the set $\{T_{[(j+1)/2]}, \dots, T_j\}$ when applied to the vector e_j, this does not necessarily imply that every operator T_n will have a similar bound as above on a subset of the set of vectors $\{e_n, \dots, e_{2n-1}\}$.

We are showing below that for a sequence of operators $\{T_n\}$ that are represented as 2×2 matrices and satisfy certain hypothesis, we can construct a sequence of vectors $\{e_j\}$ such that, on each vector e_j, we have that $1/3$ of the operators in the set $\{T_{[(j+1)/2]}, \dots, T_j\}$ have a common lower bound of the same order as $\|T_j e_j\|$. We are interested in showing that the dual statement holds, that is, for every n, the fixed operator T_n when applied to 1/3 of the vectors in the set $\{e_n, \ldots, e_{2n-1}\}$ produces vectors that have a common lower bound of the same order as $\|T_n e_n\|$. This would enable us to extend the statement the Theorem 4.3.6 to all 2-normal operators that satisfy the given growth condition, without the restriction that the diagonal entries of the 2-normal operator are either equal or Hermitian.

We stat with two lemmas that are easy exercises in linear algebra.

Lemma 5.3.1. *Suppose* $A \in \mathbb{C}^{2,2}$, $A = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$, *and* $|\alpha| \leq 1$, $|\beta| \leq 1$, $\alpha \neq \beta$. *Then for all* $k \in \mathbb{N}$,

$$\|A^k\| \leq \left(1 + \frac{\|A\|}{|\alpha - \beta|}\right)^2.$$

Proof. Use the similarity $S = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, with $x = -\gamma/(\alpha - \beta)$ to diagonalize A. $\square$

Lemma 5.3.2. *Suppose* $A \in \mathcal{L}(\mathbb{C}^2)$, $A = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$, *and* $|\alpha|, |\beta| \leq 1$. *Then for all* $k \in \mathbb{N}$,

$$\|A^k\,(0,1)^t\,\|^2 \geq \|A^k\|^2 - 1 > \|A^k\| - 1.$$

Proof. We write

$$A^k = \begin{pmatrix} \alpha^k & \mu_k \\ 0 & \beta^k \end{pmatrix} = \begin{pmatrix} \alpha^k & \gamma(\alpha^{k-1} + \alpha^{k-2}\beta + \cdots + \beta^{k-1}) \\ 0 & \beta^k \end{pmatrix}, \quad k \in \mathbb{N}.$$

Thus

$$\|A^k(0,1)^t\|^2 = |\mu_k|^2 + |\beta|^{2k},$$

and if $y = (\gamma_1, \gamma_2)^t$ is any unit vector in $\mathbb{C}^2$, then

$$\|A^k y\|^2 = |\gamma_1\alpha^k + \gamma_2\mu_k|^2 + |\gamma_2|^2|\beta|^{2k} \leq (|\gamma_1|^2 + |\gamma_2|^2)(|\alpha|^{2k} + |\mu_k|^2) + |\gamma_2|^2|\beta|^{2k} \leq$$
$$\leq |\mu_k|^2 + |\beta|^{2k} + 1 = \|A^k(0,1)^t\|^2 + 1,$$

from which the result follows. □

Lemma 5.3.3. *Let $\{e_n^{(1)}, e_n^{(2)}\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$, and let $T \in \mathcal{L}(\mathcal{H})$ be a 2-normal operator of the form $T = \oplus_n T_n$, where each $T_n \in \mathcal{L}(\mathbb{C}^2)$ has associated matrix, relative to the orthonormal basis $\{e_n^{(1)}, e_n^{(2)}\}$,*

$$\begin{pmatrix} \alpha_n & \gamma_n \\ 0 & \beta_n \end{pmatrix}.$$

Let

$$\Lambda := \{L \in \mathbb{C}^{2,2} = \mathcal{L}(\mathbb{C}^{(2)}) : \exists \text{ a subsequence } \{T_{n_k}\} \text{ of } \{T_n\} \text{ satisfying } \|T_{n_k} - L\| \to 0\}.$$

Then

$$\|T\|_e = \max\{\|L\| : L \in \Lambda\}. \tag{5.10}$$

(It turns out that the set Λ coincides with the essential 2×2 matricial spectrum of T, cf. [22], [21, Ch. 3].)

Proof. Since Λ is compact and $L \to \|L\|$ is continuous, the maximum in (5.10) exists. Let $L_M \in \Lambda$ satisfy $\|L_M\| = \max\{\|L\| : L \in \Lambda\}$ and have matrix

$$L_M = \begin{pmatrix} \alpha_M & \gamma_M \\ 0 & \beta_M \end{pmatrix}.$$

We know that

$$\|T\|_e^2 = \|T^*T\|_e = r_e(T*T) = r_e\left(\bigoplus_n \begin{pmatrix} |\alpha_n|^2 & \bar{\alpha}_n\gamma_n \\ \alpha_n\bar{\gamma}_n & |\beta_n|^2+|\gamma_n|^2 \end{pmatrix}\right),$$

and thus that $\|T\|_e$ is the largest limit point of the union of sets of eigenvalues of $T_n^*T_n$, i.e., the largest limit point of the set $\{\lambda : \exists n \in \mathbb{N} \text{ such that } \lambda^2 - (|\alpha_n|^2 + |\beta_n|^2 + |\gamma_n|^2)\lambda + |\alpha_n|^2|\beta_n|^2 = 0\}$. Moreover, since $\|L_M\|^2 = r(L_M^*L_M) = \max\{\lambda : \lambda^2-(|\alpha_M|^2+|\beta_M|^2+|\gamma_M|^2)\lambda+|\alpha_M|^2|\beta_M|^2 = 0\}$ and $\alpha_{n_k} \to \alpha_M$, $\beta_{n_k} \to \beta_M$, $\gamma_{n_k} \to \gamma_M$ and the eigenvalues of a matrix are continuous functions of the matrix entries, the result follows. □

Lemma 5.3.4. *Let $1 \geq a > b \geq 0$ and let k be a given positive integer. Then for every $m \in \mathbb{N} \cap [k/2, k]$ we have*

$$a^m - b^m \geq \frac{1}{2}(a^k - b^k). \tag{5.11}$$

Proof. The inequality (5.11) is equivalent to

$$b^k - 2b^m \geq a^k - 2a^m \quad \text{for } \frac{k}{2} \leq m \leq k.$$

Consider the function $f(x) = x^k-2x^m$. Then $f'(x) = kx^{k-1}-2mx^{m-1} = x^{m-1}(kx^{k-m}-2m)$. But since $2m \geq k$ we have $f'(x) \leq x^{m-1}(kx^{k-m} - k) = x^{m-1}k(x^{k-m} - 1)$, and for $x \in [0, 1]$ we have $f'(x) \leq 0$. Thus f is decreasing on the interval $[0, 1]$ and thus the estimate (5.11) follows. □

Lemma 5.3.5. *Let $\alpha, \beta \in \mathbb{C}$ with $1 \geq |\alpha| \geq |\beta|$. Given $k \in \mathbb{N}$, for at least $1/3$ of the natural numbers m in the interval $[k/2, k]$ we have*

$$|\alpha^m - \beta^m| \geq \frac{1}{8}|\alpha^k - \beta^k|. \tag{5.12}$$

Proof. By writing $|\alpha^{k/2} - \beta^{k/2}| = |\alpha^k - \beta^k|/|\alpha^{k/2} + \beta^{k/2}|$ it is immediate that for

$m = k/2$,

$$|\alpha^{k/2} - \beta^{k/2}| \geq |\alpha^k - \beta^k|/2. \tag{5.13}$$

Let $\alpha = ae^{i\theta_1}$ and $\beta = be^{i\theta_2}$, where $1 \geq a \geq b \geq 0$. Then

$$|\alpha^m - \beta^m| = |a^m e^{im\theta_1} - b^m e^{im\theta_2}| = |a^m - b^m e^{im(\theta_2 - \theta_1)}|.$$

Denote $\theta_2 - \theta_1$ by θ and let $\delta_{k/2}$, δ_m and δ_k be the values of $(k/2)\theta$, $m\theta$ and $k\theta$ modulo 2π. We have that $\delta_{k/2}$, δ_m and δ_k are in the interval $[0, 2\pi)$ and we ca assume that $\theta \in [0, \pi]$ since the case $\theta \in [-\pi, 0]$ is symmetric to the positive case. Then we can write

$$|\alpha^m - \beta^m|^2 = |a^m - b^m e^{im\theta}|^2 = (a^m - b^m \cos\delta_m)^2 + (b^m \sin\delta_m)^2 =$$

$$= a^{2m} + b^{2m} - 2a^m b^m \cos\delta_m.$$

We show first that the estimate (5.12) holds for all m for which one of the following is true:

$$\delta_m \in [\frac{\pi}{3}, 5\frac{\pi}{3}] \text{ or } 0 \leq \delta_k/2 \leq \delta_m \leq \delta_k \leq \frac{\pi}{2}.$$

To complete the proof we will show that for at least 1/3 of the natural numbers in the interval $[k/2, k]$ one of these 2 cases happens.

(1) If $\delta_m \in [\pi/3, 5\pi/3]$ then $\cos\delta_m \leq 1/2$ and

$$|\alpha^m - \beta^m|^2 \geq a^{2m} + b^{2m} - a^m b^m \geq \frac{1}{2}\left(a^{2m} + b^{2m}\right) \geq$$
$$\geq \frac{1}{4}(a^m + b^m)^2 \geq \frac{1}{4}(a^k + b^k)^2 \geq \frac{1}{4}|\alpha^k - \beta^k|^2,$$

so (5.12) holds in this case.

(2) If δ_m satisfies $0 \leq \delta_k/2 \leq \delta_m \leq \delta_k \leq \pi/2$, first we compare $|a^m - b^m e^{ik\theta}|$

with $|a^k - b^k e^{ik\theta}|$, then we compare $|a^m - b^m e^{im\theta}|$ with $|a^m - b^m e^{ik\theta}|$ to obtain the estimates:

$$|a^m - b^m e^{im\theta}| \geq \frac{1}{4}|a^m - b^m e^{ik\theta}| \geq \frac{1}{8}|a^k - b^k e^{ik\theta}|.$$

First we show that, if $\delta_m = \delta_k = \delta$, then a similar estimate as in the real case holds:

$$(a^m - b^m \cos\delta)^2 + (b^m \sin\delta)^2 \geq \frac{1}{4}\left((a^k - b^k \cos\delta)^2 + (b^k \sin\delta)^2\right), \tag{5.14}$$

which is equivalent to

$$(a^m - b^m)^2 + 2a^m b^m(1 - \cos\delta) \geq \frac{1}{4}\left((a^k - b^k)^2 + 2a^k b^k(1 - \cos\delta)\right),$$

but from the real case we know that

$$(a^m - b^m)^2 \geq \frac{1}{4}(a^k - b^k)^2.$$

In order to prove

$$|a^m - b^m e^{im\theta}| \geq \frac{1}{4}|a^m - b^m e^{ik\theta}|,$$

we prove first a general estimate to be used in this case. From the fact

$$\frac{3}{4} \geq (\cos\frac{\delta}{2} - \frac{1}{4}\cos\delta), \text{ for all } \delta,$$

which is equivalent to

$$0 \leq (\cos\frac{\delta}{2} - 1)^2,$$

we have, for $a \geq b \geq 0$ and for any δ, that

$$\left(b\sin\frac{\delta}{2}\right)^2 + \left(a - b\cos\frac{\delta}{2}\right)^2 \geq \frac{1}{16}\left((b\sin\delta)^2 + (a - b\cos\delta)^2\right) \tag{5.15}$$

since

$$\left(b\sin\frac{\delta}{2}\right)^2 \geq \left(\frac{1}{2}b\sin\delta\right)^2 \text{ and } a - b\cos\frac{\delta}{2} \geq \frac{1}{4}(a - b\cos\delta).$$

If $0 \leq \delta_k/2 \leq \delta_m \leq \delta_k \leq \pi/2$, since $\sin x$ is increasing on $[0, \pi/2]$ and $\cos x$ is decreasing on the same interval, we also have

$$(b \sin \delta_m)^2 + (a - b \cos \delta_m)^2 \geq \left(b \sin \frac{\delta_k}{2}\right)^2 + \left(a - b \cos \frac{\delta_k}{2}\right)^2 \tag{5.16}$$

It follows from the equations (5.15), (5.16) by replacing a and b by a^m and b^m respectively,

$$(b^m \sin \delta_m)^2 + (a^m - b^m \cos \delta_m)^2 \geq \frac{1}{16} \left((b^m \sin \delta_k)^2 + (a^m - b^m \cos \delta_k)^2\right).$$

This finishes the proof of case (2).

Next we show that for at least 1/3 of the natural numbers in the interval $[k/2, k]$ one of the above two cases happens. When drawn in the unit disc, the unit vectors corresponding to the angles δ_m are equally spaced; the angle between two consecutive vectors has measure θ.

If $k\theta \leq \pi/2$ then for all $m \in [k/2, k]$, δ_m will satisfy $0 \leq \delta_{k/2} \leq \delta_m \leq \delta_k \leq \pi/2$, which is case (2).

If $k\theta \geq 2\pi$ then $k\theta - (k/2)\theta \geq \pi$, so at least 1/3 of the δ_m's, for $m \in [k/2, k]$, will be in the interval $[\pi/3, 5\pi/3]$.

If $\pi/2 < k\theta \leq 2\pi$ then at least 1/2 of the δ_m's satisfy $\delta_m \in [\pi/3, 5\pi/3]$. $\square$

Lemma 5.3.6. *If*

$$T_n = \begin{pmatrix} \alpha_n & \gamma_n \\ 0 & \beta_n \end{pmatrix},$$

with $1 \geq |\alpha_n| \geq |\beta_n|$ *and for some fixed* $k \in \mathbb{N}$ *we have that* $\|T_n^k e_n^{(2)}\| > M$ *then for at least* $1/3$ *of the natural numbers* m *in the interval* $[k/2, k]$ *we have* $\|T_n^m e_n^{(2)}\| \geq (1/8)M$.

Proof. If $\alpha_n \neq \beta_n$

$$T_n^k = \begin{pmatrix} \alpha_n^k & \gamma_n(\alpha_n^k - \beta_n^k)(\alpha_n - \beta_n)^{-1} \\ 0 & \beta_n^k \end{pmatrix},$$

and if $\alpha_n = \beta_n$ then

$$T_n^k = \begin{pmatrix} \alpha_n^k & \gamma_n k \alpha_n^{k-1} \\ 0 & \alpha_n^k \end{pmatrix}.$$

Thus

$$\|T_n^k e_n^{(2)}\|^2 = |\gamma_n(\alpha_n^k - \beta_n^k)(\alpha_n - \beta_n)^{-1}|^2 + |\beta_n^k|$$

or

$$\|T_n^k e_n^{(2)}\|^2 = |\gamma_n k \alpha_n^{k-1}|^2 + |\beta_n^k|.$$

If $\alpha_n = \beta_n$ then, since $0 \leq |\alpha_n| \leq 1$, the function $f(x) = |\alpha_n|^x$ is decreasing, so $|\alpha_n|^{m-1} \geq |\alpha_n|^{k-1}$, thus the desired estimate is immediate.

For the case $\alpha_n \neq \beta_n$ we use Lemma 5.3.4 and the fact that the function $g(x) = |\beta_n|^x$ is decreasing. $\square$

5.4. A generalization

The proof of Proposition 4.2.3 can be adapted to yield the following generalization of the statement.

Proposition 5.4.1. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of operators in $\mathcal{L}(\mathcal{H})$ such that there exists a positive integer valued function k and a sequence of infinite dimensional subspaces $\{\mathfrak{M}_n\}_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, the operator T_n is bounded below on the subspaces $\mathfrak{M}_n, \ldots, \mathfrak{M}_{n+k(n)-1}$ by a constant $M(n) > 0$. Next, let $\{\alpha_n\}$ be any sequence in $l_2(\mathbb{N})$. Then for every $x_0 \in \mathcal{H}$ there exists $y \in \mathcal{H}$ such that*

$$\|y - x_0\|^2 \leq \sum_{i=1}^{\infty} |\alpha_i|^2, \quad ||T_n y||^2 \geq \sum_{i=n}^{n+k(n)-1} |\alpha_i|^2 M(n), \quad n \in \mathbb{N}.$$

Let s be an integer valued function defined such that $s(j)$ is given by the smallest solution for n in terms of j to the inequality:

$$j \leq n + k(n) - 1.$$

For example, if $k(n) = n$ then $s(n) = [(n+1)/2]$.

Remark 5.4.2. Notice that the hypothesis is equivalent to the statement that given the sequence of operators $\{T_n\}$ there exist a sequence of infinite dimensional subspace $\{\mathfrak{M}_n\}_n$ and a sequence of positive real numbers $\{N(n)\}_{n\in\mathbb{N}}$, such that for every $n \in \mathbb{N}$, every operator T_j from the set $T_{s(n)}, \ldots, T_n$ is bounded below by $N(j)$ on $\mathfrak{M}_n$.

When taking $k(n) = 1$ in Proposition 5.4.1, we obtain the following result of Beauzamy as a corollary:

Theorem (Beauzamy). *Let $\{T_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of operators in $\mathcal{L}(\mathcal{H})$, let ε be any positive number, and let $\{\alpha_n\}_{n\in\mathbb{N}}$ be any sequence of positive numbers in (l_2). Then for every $x_0 \in \mathcal{H}$ there exists $y \in \mathcal{H}$ such that for each $n \in \mathbb{N}$,*

$$\|y - x_0\|^2 \leq \sum_{i=1}^{\infty} |\alpha_i|^2, \quad ||T_n y|| \geq (1-\varepsilon)\alpha_n \|T_n\|_e.$$

Proof. We need to show that for each $n \in \mathbb{N}$ there exists an infinite dimensional subspace $\mathfrak{M}_n$ such that

$$T_n|_{\mathfrak{M}_n} \geq (1-\varepsilon)\|T_n\|_e.$$

This follows easily from the fact $\|T_n\|_e = \|T_n^* T_n\|_e^{1/2} = r_e(T_n^* T_n)^{1/2}$ and that $r_e(T_n^* T_n)^{1/2} \in \sigma_{le}(T_n^* T_n)$ and thus there exists an infinite orthonormal set $\{e_k\}_{k\in\mathbb{N}}$ such that

$$\|T_n^* T_n e_k - r_e(T_n^* T_n) e_k\| \to 0.$$

□

CHAPTER VI

SUMMARY

6.1. Review of the results

In the first chapter we gave some of the definitions related to hypercyclicity and hypertransitivity and discussed two examples known in the literature as the first hypercyclic operator and the first weakly hypercyclic operator that is not hypercyclic.

In Chapter II, we construct an operator T in $\mathcal{L}(\mathcal{H})$ with "strange" orbits. This operator has the property that for every nonzero vector x, the sequence $\{||T^n x||\}_{n\in\mathbb{N}}$ is dense in $\mathbb{R}_+$, but despite this, T is not hypercyclic (*i.e.*, no vector in $\mathcal{H}$ has a dense orbit). The example shows that to make progress on the program proposed in [15] (*i.e.,* to show that no operator in $\mathcal{L}(\mathcal{H})$ is hypertransitive), one cannot hope to succeed by consideration only of the collection $\{\{||T^n x||\}\colon x \in \mathcal{H}\backslash(0)\}$ of sequences of norms. The construction is based on an example of Beauzamy, but it is extended for a weighted bilateral shift. T has the property that there are subsequences $\{r_n\}$ and $\{q_n\}$ of $\mathbb{N}$ such that $\|T^{r_n}x\| \to 0$ and $\|T^{q_n}x\| \to +\infty$ for all nonzero x, and neither T nor T^* has point spectrum.

In Chapter III we show that certain classes of operators consist entirely of non-weakly-hypertransitive operators. In particular, we show that if $T \in \mathcal{L}(\mathcal{H})$ and two of the three numbers representing the essential spectral radius, essential numerical radius, and essential norm of T, coincide, then for every invertible $S \in \mathcal{L}(\mathcal{H})$ and every compact K in $\mathcal{L}(\mathcal{H})$, $STS^{-1} + K$ fails to be weakly hypertransitive. As a corollary we have that no compact perturbation of a normal operator is weakly hypertransitive. Along the way we show that K. Ball's complex-plank theorem [2] is equivalent to a (slightly stronger) version of an old theorem of Beauzamy [4].

In Chapter IV we show that no compact perturbation of certain 2-normal operators (which in general satisfy $\|T\|_e > r_e(T)$) can be orbit-transitive. Our main result herein is that if T belongs to a certain class of 2-normal operators in $\mathcal{L}(\mathcal{H}^{(2)})$ and there exist two constants $\delta, \rho > 0$ satisfying $\|T^k\|_e > \rho k^\delta$ for all $k \in \mathbb{N}$, then for every compact operator K, the operator $T + K$ is not orbit-transitive. This seems to be the first result that yields non-orbit-transitive operators in which such a modest growth rate on $\|T^k\|_e$ is sufficient to give an orbit $\{T^k x\}$ tending to infinity. One of the sufficient conditions that gives $T \in$ (NOT),which we used in our proof, is that there exists a vector y such that $\|T^n y\| \to \infty$. One of the new ideas is to use accumulation of growth on different orthonormal vectors to compensate for a slow growth rate of the essential norms of the powers of the studied operators.

If we denote by (T) the class of transitive operators in $\mathcal{L}(\mathcal{H})$, that is, operators with no nontrivial invariant subspace, and by (HT) the class of hypertransitive operators, which is just the complement of (NOT) in $\mathcal{L}(\mathcal{H})$, we have that

$$\text{(HT)} \subset \text{(T)}.$$

Thus finding an operator in the class (HT) would give automatically an example of a transitive operator and thus the invariant subspace problem would be solved in the negative.

Our work is in the direction of showing that there are not hypertransitive operators in $\mathcal{L}(\mathcal{H})$. As future work, we hope to enlarge the class of non-hypertransitive operators to encompass the whole $\mathcal{L}(\mathcal{H})$.

6.2. Open problems

We include here a set of open problems related to the topics of hypercyclic and non-orbit-transitive operators that seem to be of interest. Some of these are given in previous chapters, but we include them here for completeness.

Problem 6.2.1. Theorem 2.1.1 establishes the existence of an operator in $\mathcal{L}(\mathcal{H})$ such that every (nonzero) orbit has certain property – namely, density in $\mathbb{R}_+$ of the sequence of norms. Moreover, in [15] an example was given of an operator T in $\mathcal{L}(\mathcal{H})$ with $||T||_e = 1$ such that the orbit of every nonzero vector x satisfies $\{||T^n x||\} \to +\infty$. What other properties that are common to every (nonzero) orbit can an operator in $\mathcal{L}(\mathcal{H})$ have? For example, does there exist an operator $T \in \mathcal{L}(\mathcal{H})$ such that for all nonzero vectors $x, y \in \mathcal{H}$, $\{\langle T^n x, y\rangle\}$ is dense in $\mathbb{C}$? (Of course, such a T would be transitive.)

Problem 6.2.2. If $T \in \mathcal{L}(\mathcal{H})$ is invertible and weakly orbit-transitive, must T^{-1} also be weakly orbit-transitive? (One knows that an operator can be weakly hypercyclic without its inverse being weakly hypercyclic [9, Cor. 3.6].)

Problem 6.2.3. Does every orbit-transitive operator T in $\mathcal{L}(\mathcal{H})$ satisfy $\sigma(T) \subset \{z \in \mathbb{C} : |z| = 1\}$? (One knows that an orbit-transitive operator must satisfy $\sigma(T) \subset \{z \in \mathbb{C} : |z| \leq 1\}$.)

Problem 6.2.4. If $T \in \mathcal{L}(\mathcal{H})$ is weakly orbit-transitive, must $\sigma(T) \subset \mathbb{T}$?

Problem 6.2.5. If $T \in \mathcal{L}(\mathcal{H})$ and $\{\langle T^n x, y\rangle\}$ is dense in $\mathbb{C}$ for every pair x, y of nonzero vectors in $\mathcal{H}$, must T be weakly orbit-transitive? (One knows form [29] that there exist nonhypercyclic operators T in $\mathcal{L}(\mathcal{H})$ such that for every $x \neq 0$ in $\mathcal{H}$, $\{\|T^n x\|\}$ is dense in $\mathbb{R}^+$.)

Problem 6.2.6. Recall that an operator $T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ is 2-normal if T is unitarily equivalent to a 2×2 matrix (N_{ij}), where the N_{ij} are mutually commuting normal operators, and it is known [14] that 2-normal operators have nontrivial hyperinvariant subspaces. Can a sum $T + K$, where T is 2-normal and K is compact, be weakly orbit-transitive?

Problem 6.2.7. The following question, which logically falls between the Orbit-transitive Operator Problem and the Invariant Subspace Problem, seems not to have received any attention: Is it true that for every T in $\mathcal{L}(\mathcal{H})$ there exist nonzero vectors x and y such that the sequence $\{\langle T^n x, y\rangle\}_{n\in\mathbb{N}}$ is bounded?

Problem 6.2.8. If the operator H is as in (4.17) and B is the operator from the Example 4.4.1, how does one show that a compact perturbation of $H \oplus B$ belongs to (NOT)?

Problem 6.2.9. Does every compact perturbation of an arbitrary n-normal operator belong to (NOT)?

Problem 6.2.10. The following question, which logically falls between the orbit-transitive operator problem and the invariant subspace problem, seems not to have received any attention: is it true that for every T in $\mathcal{L}(\mathcal{H})$ there exist nonzero vectors x and y such that the sequence $\{\langle T^n x, y\rangle\}_{n\in\mathbb{N}}$ is bounded?

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VITA

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