UPPER ESTIMATES FOR BANACH SPACES

A Dissertation

by

DANIEL BECKER FREEMAN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2009

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ABSTRACT

Upper Estimates for Banach Spaces. (August 2009)

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We study the relationship of dominance for sequences and trees in Banach spaces. In the context of sequences, we prove that domination of weakly null sequences is a uniform property. More precisely, if (v_i) is a normalized basic sequence and X is a Banach space such that every normalized weakly null sequence in X has a subsequence that is dominated by (v_i), then there exists a uniform constant $C \geq 1$ such that every normalized weakly null sequence in X has a subsequence that is C-dominated by (v_i). We prove as well that if $V = (v_i)_{i=1}^{\infty}$ satisfies some general conditions, then a Banach space X with separable dual has subsequential V upper tree estimates if and only if it embeds into a Banach space with a shrinking FDD which satisfies subsequential V upper block estimates. We apply this theorem to Tsirelson spaces to prove that for all countable ordinals α there exists a Banach space X with Szlenk index at most $\omega^{\alpha\omega+1}$ which is universal for all Banach spaces with Szlenk index at most $\omega^{\alpha\omega}$.

ACKNOWLEDGMENTS

I thank my advisor Thomas Schlumprecht for his professional guidance, collaboration, and time spent in both discussion with me and in reading and editing my writings. He has helped tremendously in my growth as a mathematician. I thank as well the other mathematicians with whom I was lucky to collaborate as a graduate student, Steve Dilworth, Edward Odell and Andras Zsák. I appreciate all the opportunities that I had to work with them. I thank my committee members William Johnson, David Larson, and Fred Dahm. I appreciate the time they took in hearing me talk and reading my dissertation. I thank as well my Mom and Dad, Blaise, Lydia, and Nikki.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

A. Background

A Banach space is a complete normed vector space. The generality of this definition leads to a huge variety in the class of Banach spaces. Because of this, many of the tools and properties used to study Banach spaces can only be applied in certain cases. Researchers often get around this problem by exploiting relations between the Banach space they are studying and ones which are well understood or have a strong property. For example, not every separable Banach space has a basis. However, every separable Banach space is isometric to a subspace of $C([0,1])$, which has a basis. This is one reason why Banach spaces are often studied in the context being subspaces of $C([0,1])$. With this idea in mind, understanding particular relationships between Banach spaces is not only of intrinsic interest, but can also be applied to study particular Banach spaces themselves. The general theme of this dissertation relates to studying the relationship of domination and upper estimates, together with some of their applications.

Definition A.1. If $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$ are sequences in (possibly different) Banach spaces and $C > 0$ is a constant, we say that (x_i) *C-dominates* (y_i) if

$$\| \sum a_i x_i \| \geq C \| \sum a_i y_i \| \quad \text{for all } (a_i) \in c_{00}.$$

We say that $(x_i)_{i=1}^\infty$ *dominates* $(y_i)_{i=1}^\infty$ if $(x_i)_{i=1}^\infty$ C-dominates $(y_i)_{i=1}^\infty$ for some $C > 0$.

Thus domination is a relationship between sequences in Banach spaces. A quick

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The journal model is *Studia Mathematica*.

example of the utility of this relationship arises when studying weak convergence. For example, it is easy to show that if (x_i) is weakly null and (x_i) dominates (y_i) then (y_i) is weakly null. Similarly, if (x_i) is weakly Cauchy and (x_i) dominates (y_i) then (y_i) is weakly Cauchy. For many of the studied properties, it is often the case that normalized weakly null sequences contain a subsequence having that property. Some important examples of this is the fact that every normalized weakly null sequence has a basic subsequence and the fact that every normalized weakly null sequence has a subsequence with an unconditional spreading model. In the subsequent chapter we will consider how domination relates to subsequences of normalized weakly null sequences. In particular, we will prove that domination is a uniform relationship.

The relationships of domination and upper estimates have become more pertinent in recent years due to their extension from sequences to trees and finite dimensional decompositions (FDD) [OS1]. One application of this is that the Szlenk index of a Banach space with separable dual can be measured using upper estimates of Tsirelson spaces [OSZ3]. We will first give a basic introduction to FDDs, and then discuss upper estimates for FDDs and show how they are naturally related to upper estimates for weakly null trees.

Definition A.2. A *finite dimensional decompostion* or *FDD* for an infinite dimensional Banach space X, is a sequence of nonzero finite dimensional subspaces $(E_n)_{n=1}^\infty$ of X such that for every $x \in X$ there exists a unique sequence of vectors (x_i) in X such that $x_i \in E_i$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^\infty x_i$ converges to x in norm. We denote the support of the vector $\sum_{i=1}^\infty x_i$ by $supp_E(x) = \{i \in \mathbb{N} | x_i \neq 0\}$.

An FDD (F_i) is called a *blocking* of (E_i) if $F_i = \sum_{j=N_i}^{N_{i+1}-1} E_j$ for some sequence $(N_i) \in [\mathbb{N}]^\omega$ with $N_1 = 1$. A sequence (x_i) in X is called a *block sequence* of (E_i) if there exists a blocking (F_i) of (E_i) such that $x_i \in F_i$ for all $i \in \mathbb{N}$. An FDD is called

shrinking if every block sequence is weakly null. An FDD is called *boundedly complete* if every block sequence is boundedly complete, or in other words if $(y_i)_{i=1}^{\infty}$ is a block sequence of (F_i) such that $\sup_{n \in \mathbb{N}} \|\sum_{i=1}^{n} y_i\| < \infty$ then the series $\sum_{i=1}^{\infty} y_i$ converges in norm. If $C \geq 1$ is a constant, then an FDD is called *C-unconditional* if every block sequence is C-unconditional, or equivalently if $\epsilon_i = \pm 1$ and $v_i \in E_i$ for all $i \in \mathbb{N}$ then $\|\sum \epsilon_i v_i\| \leq C\|\sum v_i\|$. An FDD is called *unconditional* if it is C-unconditional for some $C \geq 1$.

Finite dimensional decompositions can be thought of as a generalization of bases, and are useful in many of the same ways that bases are. One of these ways, which is particularly relevant to us, is in determining when a Banach space is reflexive or has separable dual. As is the case for bases, a Banach space with an FDD is reflexive if and only if the FDD is shrinking and boundedly complete. Furthermore, if a Banach space has a shrinking FDD then it has separable dual. Zippin proved that every Banach space with separable dual embeds into a Banach space with a shrinking basis. We will quantify this result by considering upper estimates of FDDs and upper tree estimates. In particular, we will show that every Banach space with separable dual satisfies some subsequential upper tree estimate and embeds into a Banach space with an FDD which satisfies the same subsequential upper estimate. This particular upper estimate will be determined by the Szlenk index of the space, which can be considered as measuring the size of a Banach space's dual. We define now subsequential upper estimates for FDDs.

Definition A.3. If $V = (v_i)_{i=1}^{\infty}$ is an unconditional sequence in some Banach space, $(E_i)_{i=1}^{\infty}$ is an FDD for some Banach space X, and $C > 0$ is a constant then we say that (E_i) satisfies *subsequential C-V-upper estimates* if every normalized block sequence (y_i) of (E_i) is C-dominated by $(v_{minsupp(y_i)})$. We say that (E_i) satisfies *subsequential*

V-upper estimates if (E_i) satisfies subsequential C-V-upper estimates for some $C > 0$.

Thus this relationship between the sequence (v_i) and the FDD (E_i) is an extension of domination to blocks of an FDD. The upper estimate here is referred to as subsequential as each block of (E_i) is dominated by a corresponding subsequence of V. We can now consider the analogue of our example of weak convergence. In this scenario, it is easy to show that if $V = (v_i)_{i=1}^{\infty}$ is weakly null and (E_i) is an FDD for a Banach space X which satisfies subsequential V-upper estimates then (E_i) is shrinking. Thus in particular X has separable dual. The class of Banach spaces of separable dual is an interesting and important class, and these upper estimates give one way of determining when a Banach space is an element of it.

When a Banach space X has an FDD $(E_i)_{i=1}^{\infty}$ with some particular property, there is often some related structure which is imposed on the subspaces of X. This subspace structure can often be intrinsically characterized as a *tree property*. Before defining the tree property which corresponds to subsequential upper estimates, we first need to define *weakly null even trees*. Each even tree will be a family in a Banach space indexed by

$$T_{\infty}^{\text{even}} = \{(n_1, n_2, ..., n_{2\ell}) : n_1 < n_2 < ... < n_{2\ell} \text{ are in } \mathbb{N} \text{ and } \ell \in \mathbb{N}\}.$$

Definition A.4. If X is a Banach space, an indexed family $(x_\alpha)_{\alpha \in T_{\infty}^{\text{even}}} \subset X$ is called an *even tree*. Sequences of the form $(x_{n_1,...,n_{2\ell-1},k})_{k=n_{2\ell-1}+1}^{\infty}$ are called *nodes*. Sequences of the form $(n_{2\ell-1}, x_{n_1,...,n_{2\ell}})_{\ell=1}^{\infty}$ are called *branches*. A normalized tree, i.e. one with $||x_\alpha|| = 1$ for all $\alpha \in T_{\infty}^{\text{even}}$, is called *weakly null* if every node is a weakly null sequence.

If Z is a Banach space with an FDD (E_n), and X is a closed subspace of Z then any weakly null even tree in X has a branch $(n_{2\ell-1}, x_{n_1,...,n_{2\ell}})_{\ell=1}^{\infty}$ such that $(x_{n_1,...,n_{2\ell}})_{\ell=1}^{\infty}$ is equivalent to a block sequence $(y_\ell)_{\ell=1}^{\infty}$ with respect to (E_n) so that

$minsupp(y_\ell) = n_{2\ell-1}$. Thus if (E_n) satisfies subsequential V-upper block estimates, then every weakly null even tree in X has a branch $(n_{2\ell-1}, x_{n_1,\dots,n_{2\ell}})_{\ell=1}^\infty$ such that $(x_{n_1,\dots,n_{2\ell}})_{\ell=1}^\infty$ is dominated by $(v_{n_{2\ell-1}})_{\ell=1}^\infty$. This example also illustrates why we need even trees, as opposed to trees which are of the form $(x_{(n_1,\dots,n_k)})_{(n_1,\dots,n_k)\in[\mathbb{N}]^{<\omega}}$. To consider V-upper tree estimates, we need a branch to contain both a sequence $(x_{n_1,\dots,n_{2\ell}})_{\ell=1}^\infty$ as well as the subsequence of (v_i) which dominates it. If (v_i) were subsymmetric, then the trees we would consider would not need to be even. We make these ideas into a coordinate free condition with the following definition.

Definition A.5. Let X be a Banach space, $V = (v_i)$ be a normalized 1-unconditional basis, and $1 \leq C < \infty$. We say that X *satisfies subsequential C-V-upper tree estimates* if every weakly null even tree $(x_\alpha)_{\alpha \in T_\infty^{\mathrm{even}}}$ in X has a branch $(n_{2\ell-1}, x_{n_1,\dots,n_{2\ell}})_{\ell=1}^\infty$ such that $(x_{n_1,\dots,n_{2\ell}})_{\ell=1}^\infty$ is C-dominated by $(v_{n_{2\ell-1}})_{\ell=1}^\infty$.

We say that X *satisfies subsequential V-upper tree estimates* if it satisfies subsequential C-V-upper tree estimates for some $1 \leq C < \infty$.

If X is a subspace of a dual space, we say that X *satisfies subsequential C-V-lower w^* tree estimates* if every w^* null even tree $(x_\alpha)_{\alpha \in T_\infty^{\mathrm{even}}}$ in X has a branch $(n_{2\ell-1}, x_{n_1,\dots,n_{2\ell}})_{\ell=1}^\infty$ such that $(x_{n_1,\dots,n_{2\ell}})_{\ell=1}^\infty$ C-dominates $(v_{n_{2\ell-1}})_{\ell=1}^\infty$.

B. Weakly Null Trees vs Weakly Null Sequences

One reason that weakly null trees were introduced was to give an intrinsic characterization of when a Banach space embeds into a Banach space with an FDD having some desired property. Before then, it was hoped that it would be possible to describe this characterization in terms of subsequences of normalized weakly null sequences. However, examples showed that this was not true. For instance, for each $1 < p < \infty$, Odell and Schlumprecht created a reflexive Banach space X with the property that

there exists a constant $C > 1$ such that every normalized weakly null sequence in X has a subsequence C-equivalent to the usual basis for ℓ_p, but there exists a normalized weakly tree in X such that no branch is equivalent to the usual basis for ℓ_p. Thus in particular, the space X does not embed into a space of the form $(\sum_{i=1}^{\infty} E_i)_{\ell_p}$ where E_i is finite dimensional for all $1 \leq i < \infty$. Unfortunately, the space they constructed is technical and somewhat difficult to understand. We show here that actually this property can be witnessed by a simple and classical construction.

Theorem B.1. *If $1 < p < \infty$, then the Banach space $X = (\sum_{n=1}^{\infty}(\sum_{k=1}^{n} \ell_p)_{\ell_1})_{\ell_p}$ has the property that for all $\epsilon > 0$ every weakly null sequence in S_X has a subsequence $(2 + \epsilon)$-equivalent to the usual basis for ℓ_p, but there exists a weakly null tree in S_X such that no branch is dominated by the usual basis for ℓ_p.*

Proof. We denote the usual basis for $X = (\sum_{n=1}^{\infty}(\sum_{k=1}^{n} \ell_p)_{\ell_1})_{\ell_p}$ by $(e_{n,k,i})$ with biorthogonal functionals $(e_{n,k,i}^*)$. In other words, if $(a_{i,k,n}) \in c_{00}(\mathbb{N}^3)$ then

$$\Big\| \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{i=1}^{\infty} a_{n,k,i} e_{n,k,i} \Big\|_X = \Big(\sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} \Big(\sum_{i=1}^{\infty} |a_{i,k,n}|^p \Big)^{1/p} \Big)^p \Big)^{1/p}.$$

For each $k \leq n$, we denote $X_n^k := \overline{span}_{i \in \mathbb{N}}(e_{n,k,i})$ and for each $n \in \mathbb{N}$ we denote $X_n := \overline{span}_{k \leq n, i \in \mathbb{N}}(e_{n,k,i}) = (\sum_{k=1}^{n} X_n^k)_{\ell_1}$. We will use $P_{X_n^k}$ and P_{X_n} to denote the natural projections of X onto X_n^k and X_n respectively.

We claim that X contains a normalized weakly null tree such that no branch is equivalent to the usual basis for ℓ_p. Indeed, the basis $(e_{n,k,i})$ can be given a tree structure that witnesses this. To show this we lexicographically order the set $\{(n,k) \in \mathbb{N}^2 | n \in \mathbb{N}, k \leq n\}$ into $(n_i, k_i)_{i=1}^{\infty}$. We define the normalized weakly null tree $(x_{\ell_1,\dots,\ell_j})_{(\ell_1,\dots,\ell_j) \in [\mathbb{N}]^{<\omega}}$ by $x_{\ell_1,\dots,\ell_j} = e_{n_j,k_j,\ell_j}$ for each $(\ell_1,\dots,\ell_j) \in [\mathbb{N}]^{<\omega}$. This tree does not have a branch equivalent to the unit vector basis for ℓ_p as for each $n \in \mathbb{N}$ every branch has a finite subsequence equivalent to the unit vector basis of ℓ_1^n.

Thus we just need to show for each $\epsilon > 0$ that every normalized weakly null sequence in X has a subsequence $(2 + \epsilon)$-equivalent to the usual basis for ℓ_p. Let $(x_j)_{i=1}^{\infty} \subset S_X$ be weakly null and let $\epsilon > 0$. We will first show that (x_j) has a subsequence which dominates the usual basis for ℓ_p. For $x \in X$, we will denote the support of x by $supp(x) := \{(n,k,i)|e_{n,k,i}^*(x) \neq 0\}$. By passing to a subsequence of (x_j) and perturbing, we may assume that $supp(x_j) \cap supp(x_i) = \emptyset$ if $i \neq j$. We let $(b_j) \in c_{00}$ and calculate the following.

$$\left(\sum_{j=1}^{\infty} |b_j|^p\right)^{1/p} = \left(\sum_{j=1}^{\infty} |b_j|^p \|x_j\|_X^p\right)^{1/p}$$

$$= \left(\sum_{j=1}^{\infty} |b_j|^p \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} \|P_{X_n^k} x_j\|_X\right)^p\right)^{1/p}$$

$$= \left(\sum_{n=1}^{\infty}\left(\left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{n} \|P_{X_n^k} b_j x_j\|_X\right)^p\right)^{1/p}\right)^p\right)^{1/p}$$

$$= \left(\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} a_j \sum_{k=1}^{n} \|P_{X_n^k} b_j x_j\|_X\right)^p\right)^{1/p} \quad \text{for some } (a_j) \in S_{\ell_p^*} = S_{\ell_{p'}}$$

$$= \left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} \sum_{j=1}^{\infty} a_j \|P_{X_n^k} b_j x_j\|_X\right)^p\right)^{1/p}$$

$$\leq \left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\sum_{j=1}^{\infty} \|P_{X_n^k} b_j x_j\|_X^p\right)^{1/p}\right)^p\right)^{1/p} \quad \text{by Hölder's inequality}$$

$$= \|\sum_{j=1}^{\infty} b_j x_j\|_X \quad \text{because } supp(x_j) \cap supp(x_i) = \emptyset \text{ if } i \neq j.$$

Hence, (x_j) 1-dominates the usual basis for ℓ_p. Since the above argument included passing to a perturbation of a subsequence of (x_j), we deduce that every normalized weakly null sequence in X has a subsequence which $(1+\epsilon)$-dominates the usual basis for ℓ_p.

We will now show that (x_j) has a subsequence which is dominated by the usual

basis for ℓ_p. After passing to a perturbation of a subsequence of (x_j), we may assume that there exists $(N_n) \in [\mathbb{N}]^\omega$ and $(a_{n,k}) \in (\sum_{n=1}^{\infty} \ell_1^n)_{\ell_p}$ such that the following conditions are satisfied.

(i) $supp(x_i) \cap supp(x_j) = \emptyset$ if $i \neq j$,

(ii) $x_j \in (\sum_{n=1}^{N_j} X_n)_{\ell_p}$ for all $j \in \mathbb{N}$,

(iii) $\|P_{X_n^k} x_j\|_X = a_{n,k}$ for all $k \leq n$ and all $j \in \mathbb{N}$ such that $N_{j-1} \geq n$.

We now fix $(b_j)_{j=1}^{\infty} \in c_{00}$. We first fix $n \in \mathbb{N}$ and estimate $\|P_{X_n}(\sum_{j=1}^{\infty} b_j x_j)\|$. Let $m \in \mathbb{N}$ such that $N_{m-1} < n \leq N_m$.

$$\|P_{X_n}(\sum_{j=1}^{\infty} b_j x_j)\| \leq \|P_{X_n} b_m x_m\| + \sum_{k=1}^{n}(\sum_{j=m+1}^{\infty} |a_{n,k} b_j|^p)^{1/p} \text{ by (ii) and (iii)}$$

$$= |b_m|\|P_{X_n} x_m\| + \sum_{k=1}^{n} |a_{n,k}|(\sum_{j=m+1}^{\infty} |b_j|^p)^{1/p}$$

$$= |b_m|\|P_{X_n} x_m\| + \|P_{X_n} x_M\|(\sum_{j=m+1}^{\infty} |b_j|^p)^{1/p} \quad \text{for all } M \geq m+1$$

$$= |b_m|\|P_{X_n} x_m\| + (\sum_{j=m+1}^{\infty} |b_j|^p \|P_{X_n} x_j\|^p)^{1/p}.$$

We thus have that $\|P_{X_n}(\sum_{j=1}^{\infty} b_j x_j)\| \leq |b_m|\|P_{X_n} x_m\| + (\sum_{j=m+1}^{\infty} |b_j|^p \|P_{X_n} x_j\|^p)^{1/p}$. We are now prepared to estimate $\|\sum_{j=1}^{\infty} b_j x_j\|$.

$$\|\sum_{j=1}^{\infty} b_j x_j\|^p = \sum_{m=1}^{\infty} \sum_{n=N_{m-1}+1}^{N_m} \|P_{X_n}(\sum_{j=1}^{\infty} x_j)\|^p$$

$$\leq \sum_{m=1}^{\infty} \sum_{n=N_{m-1}+1}^{N_m} (|b_m|\|P_{X_n} x_m\| + (\sum_{j=m+1}^{\infty} |b_j|^p \|P_{X_n} x_j\|^p)^{1/p})^p$$

$$\leq \sum_{m=1}^{\infty} \sum_{n=N_{m-1}+1}^{N_m} 2^p |b_m|^p \|P_{X_n} x_m\|^p + 2^p \sum_{j=m+1}^{\infty} |b_j|^p \|P_{X_n} x_j\|^p$$

$$= \sum_{m=1}^{\infty} \sum_{n=N_{m-1}+1}^{N_m} 2^p \sum_{j=1}^{\infty} |b_j|^p \|P_{X_n} x_j\|^p \quad \text{by (ii)}$$

$$= 2^p \sum_{j=1}^{\infty} |b_j|^p \sum_{n=1}^{\infty} \|P_{X_n} x_j\|^p$$

$$= 2^p \sum_{j=1}^{\infty} |b_j|^p \quad \text{as} \quad \sum_{n=1}^{\infty} \|P_{X_n} x_j\|^p = \|x_j\|^p = 1$$

Thus we have that (x_j) is 2-dominated by the usual basis for ℓ_p. We thus conclude that every normalized weakly null sequence in X has a subsequence which is $(2 + \epsilon)$-equivalent to the usual basis for ℓ_p. $\qquad \square$

CHAPTER II

WEAKLY NULL SEQUENCES WITH UPPER ESTIMATES

A. Introduction

The work in this chapter was published in the paper *Weakly null sequences with upper estimates,* Studia Mathematica **184** (2008), no. 1, 79–102 [F]. We thank the editors of Studia Mathematica for allowing the paper to be included here. In some circumstances, local estimates give rise to uniform global estimates. An elementary example of this is that every continuous function on a compact metric space is uniformly continuous. Uniform estimates are especially pertinent in functional analysis, as one of the cornerstones to the subject is the Uniform Boundedness Principle. Because uniform estimates are always desirable, it is important to determine when they occur. In this paper, we are concerned with uniform upper estimates of weakly null sequences in a Banach space. Before stating precisely what we mean by this, we give some historical context.

For each $1 < p < \infty$, Johnson and Odell [JO] have constructed a Banach space X such that every normalized weakly null sequence in X has a subsequence equivalent to the standard basis for ℓ_p, and yet there is no fixed $C \geq 1$ such that every normalized weakly null sequence in X has a subsequence $C-$equivalent to the standard basis for ℓ_p. A basic sequence (x_i) is equivalent to the unit vector basis for ℓ_p if it has both a lower and an upper ℓ_p estimate. That is there exist constants $C, K \geq 1$ such that:

$$\frac{1}{K}(\sum |a_i|^p)^{1/p} \leq ||\sum a_i x_i|| \leq C(\sum |a_i|^p)^{1/p} \quad \forall (a_i) \in c_{00}.$$

The examples of Johnson and Odell show that the upper constant C and the lower constant K cannot always both be chosen uniformly. It is somewhat surprising then

that Knaust and Odell proved [KO2] that the upper estimate can always be chosen uniformly. Specifically, they proved that for every Banach space X if each normalized weakly null sequence in X has a subsequence with an upper ℓ_p estimate, then there exists a constant $C \geq 1$ such that each normalized weakly null sequence in X has a subsequence with a C-upper ℓ_p estimate. They also proved earlier the corresponding theorem for upper c_0 estimates [KO1]. The standard bases for ℓ_p, $1 < p < \infty$ and c_0 enjoy many strong properties which Knaust and Odell employ in their papers. It is natural to ask what are some necessary and sufficient properties for a basic sequence to have in order to guarantee the uniform upper estimate. In this paper we show that actually all semi-normalized basic sequences give uniform upper estimates. We make the following definition to formalize this.

Definition A.1. Let $V = (v_n)_{n=1}^{\infty}$ be a semi-normalized basic sequence. A Banach space X has property (S_V) if every normalized weakly null sequence (x_n) in X has a subsequence (y_n) such that for some constant $C < \infty$

$$\left\| \sum_{n=1}^{\infty} \alpha_n y_n \right\| \leq C \quad \text{for all } (\alpha_n) \in c_{00} \text{ with } \left\| \sum_{n=1}^{\infty} \alpha_n v_n \right\| \leq 1. \tag{2.1}$$

X has property (U_V) if C may be chosen uniformly. We say that (y_n) has a C-upper V-estimate (or that V C-dominates (y_n)) if (1) holds for C, and that (y_n) has an upper V-estimate (or that V dominates (y_n)) if (1) holds for some C.

Using these definitions, we can formulate the main theorem of our paper as:

Theorem A.2. *A Banach space has property (S_V) if and only if it has property (U_V).*

(S_V) and (U_V) are isomorphic properties of V, so it is sufficient to prove Theorem A.2 for only normalized bimonotone basic sequences. This is because every semi-normalized basic sequence is equivalent to a normalized bimonotone basic sequence.

Indeed, if $0 < A \leq ||v_i|| \leq B$ for all $i \in \mathbb{N}$, then we can define a new norm $||| \cdot |||$ on $[v_i]$ by $|||x||| = \frac{1}{B} \sup_{n<m} ||P_{[n,m]}x|| \vee \sup_{i \in \mathbb{N}} |v_i^*(x)|$ for all $x \in [v_i]$ where $P_{[n,m]}$ denotes the projection of $[v_i]$ onto the the span of $\{v_n, ..., v_m\}$. The norm $||| \cdot |||$ is equivalent to $|| \cdot ||$ on $[v_i]$ and (v_i) is normalized and bimonotone in the new norm.

In section 2 we present the necessary definitions and reformulate our main results. We break up the main proof into two parts which we give in sections 3 and 4. In section 5 we give some illustrative examples which show in particular that our result is a genuine extension of [KO2] and not just a corollary.

For a Banach space X we use the notation B_X to mean the closed unit ball of X and S_X to mean the unit sphere of X. If $F \subset X$ we denote $[F]$ to be the closed linear span of F in X. If N is a sequence in $\mathbb{N}$, we denote $[N]^\omega$ to be the set of all infinite subsequences of N.

B.   Main Results

Here we introduce the main definitions and theorems of the paper. Many of our theorems and lemmas are direct generalizations of corresponding results in [KO2]. We specify when we are able to follow the same outline as a proof in [KO2], and also when we are able to follow a proof exactly.

Definition B.1. Let X be a Banach space and $V = (v_n)_{n=1}^\infty$ be a normalized bimonotone basic sequence. With the exception of (ii), the following definitions are adapted from [KO2].

(i) A sequence (x_n) in X is called a *uV-sequence* if $||x_n|| \leq 1$ for all $n \in \mathbb{N}$, (x_n) converges weakly to 0, and

$$\sup_{\left\|\sum_{n=1}^\infty \alpha_n v_n\right\| \leq 1} \left\|\sum_{n=1}^\infty \alpha_n x_n\right\| < \infty.$$

(x_n) is called a *C-uV-sequence* if

$$\sup_{\left\|\sum_{n=1}^{\infty} \alpha_n v_n\right\| \leq 1} \left\|\sum_{n=1}^{\infty} \alpha_n x_n\right\| < C.$$

(ii) A sequence (x_n) in X is called a *hereditary uV-sequence*, if every subsequence of (x_n) is a uV-sequence, and is called a *hereditary C-uV-sequence* if every subsequence of (x_n) is a C-uV-sequence.

(iii) A sequence (x_n) in X is called an *M-bad-uV sequence* for a constant $M < \infty$, if every subsequence of (x_n) is a uV-sequence, and no subsequence of (x_n) is an M-uV-sequence.

(iv) An array $(x_i^n)_{i,n=1}^{\infty}$ in X is called a *bad uV-array*, if each sequence $(x_i^n)_{i=1}^{\infty}$ is an M_n-bad uV-sequence for some constants M_n with $M_n \to \infty$.

(v) $(y_i^k)_{i,k=1}^{\infty}$ is called a *subarray* of $(x_i^n)_{i,n=1}^{\infty}$, if there is a subsequence (n_k) of $\mathbb{N}$ such that every sequence $(y_i^k)_{i=1}^{\infty}$ is a subsequence of $(x_i^{n_k})_{i=1}^{\infty}$.

(vi) A bad uV-array $(x_i^n)_{i,n=1}^{\infty}$ is said to satisfy the *V-array procedure*, if there exists a subarray (y_i^n) of (x_i^n) and there exists $(a_n) \subseteq \mathbb{R}^+$ with $a_n \leq 2^{-n}$, for all $n \in \mathbb{N}$, such that the weakly null sequence (y_i) with $y_i := \sum_{n=1}^{\infty} a_n y_i^n$ has no uV-subsequence.

(vii) X satisfies the *V-array procedure* if every bad uV-array in X satisfies the V-array procedure. X satisfies the *V-array procedure for normalized bad uV-arrays* if every normalized bad uV-array in X satisfies the V-array procedure.

Note: A subarray of a bad uV-array is a bad uV-array. Also, a bad uV-array satisfies the V-array procedure if and only if it has a subarray which satisfies the V-array procedure.

Our Theorem A.2 is now an easy corollary of the theorem below.

Theorem B.2. *Every Banach space satisfies the V-array procedure for normalized bad uV-arrays.*

Theorem B.2 implies Theorem A.2 because if a Banach space X has property S_V and not U_V then there exists a normalized bad uV-array, and the V-array procedure gives a weakly null sequence in B_X which has no uV-subsequence. The sequence must be semi-normalized, so we could pass to a basic subsequence on which the norm of each element is essentially constant then renormalize. This would give a weakly null sequence with no uV-subsequence; contradicting X being U_V.

The proof for Theorem B.2 will be given first for the following special case.

Proposition B.3. *Let K be a countable compact metric space. Then $C(K)$ satisfies the V-array procedure.*

The case of a general Banach space reduces to this special case by the following proposition.

Proposition B.4. *Let $(x_i^n)_{i,n=1}^{\infty}$ be a normalized bad uV-array in a Banach space X. Then there exists a subarray (y_i^n) of (x_i^n) and a countable w^*-compact subset K of B_{Y^*}, where $Y := [y_i^n]_{i,n=1}^{\infty}$, such that $(y_i^n|_K)$ is a bad uV-array in $C(K)$.*

Theorem B.2 is an easy consequence of Proposition B.3 and B.4. Note that Proposition B.4 is only proved for normalized bad uV-arrays. This makes the proof a little less technical.

Before we prove anything about sub-arrays though, we need to first consider just a single weakly null sequence. One of the many nice properties enjoyed by the standard basis for ℓ_p which we denote by (e_i) is that (e_i) is 1-spreading. This is the property that every subsequence of (e_i) is 1-equivalent to (e_i). Spreading is of particular importance because it implies the following two properties which are implicitly used in [KO2]:

(i) If (e_i) C-dominates a sequence (x_i) then (e_i) C-dominates every subsequence of (x_i).

(ii) If a sequence (x_i) C-dominates (e_i) then (x_i) C-dominates every subsequence of (e_i).

Throughout the paper, we will be passing to subsequences and subarrays, so properties (i) and (ii) would be very useful for us. In our paper we have to get by without property (ii). On the other hand, for a given sequence that does not have property (i), we may use the following two results, which are both easy consequences of Ramsey's theorem (c.f. [O]), and will be needed in subsequent sections.

Lemma B.5. *Let $V = (v_i)_{i=1}^{\infty}$ be a normalized bimonotone basic sequence. If $(x_i)_{i=1}^{\infty}$ is a sequence in the unit ball of some Banach space X, such that every subsequence of $(x_i)_{i=1}^{\infty}$ has a further subsequence which is dominated by V then there exists a constant $1 \leq C < \infty$ and a subsequence $(y_i)_{i=1}^{\infty}$ of $(x_i)_{i=1}^{\infty}$ so that every subsequence of $(y_i)_{i=1}^{\infty}$ is C-dominated by V.*

Proof. Let $A_n = \{(m_k)_{k=1}^{\infty} \in [\mathbb{N}]^{\omega} \mid (x_{m_k}) \text{ is } 2^n \text{ dominated by V}\}$.

A_n is Ramsey, thus for all $n \in \mathbb{N}$ there exists a sequence $(m_i^n)_{i=1}^{\infty} = M_n \in [M_{n-1}]^{\omega}$ such that $[M_n]^{\omega} \subseteq A_n$ or $[M_n]^{\omega} \subseteq A_n^c$. We claim that $[M_n]^{\omega} \subseteq A_n$ for some $n \in \mathbb{N}$, in which case we could choose $(y_i)_{i=1}^{\infty} = \left(x_{m_i^n}\right)_{i=1}^{\infty}$. Every subsequence of $(y_i)_{i=1}^{\infty}$ is then 2^n-dominated by V.

If our claim where false, we let $(y_n)_{n=1}^{\infty} = \left(x_{m_n^n}\right)_{n=1}^{\infty}$ and $(y_{k_n})_{n=1}^{\infty}$ be a subsequence of $(y_n)_{n=1}^{\infty}$ for which there exists $C < \infty$ such that $(y_{k_n})_{n=1}^{\infty}$ is C-dominated by V. Let $N \in \mathbb{N}$ such that $2^N - 2N > C$ and set

$$
\ell_i = \begin{cases} m_i^N & \text{if } i \leq N, \\ m_{k_i}^{k_i} & \text{if } i > N. \end{cases}
$$

Then $(\ell_i)_{i=1}^\infty \in [M_N]^\omega \subset A_N^c$ which implies that some $(a_i)_{i=1}^L \subset [-1, 1]$ exists such that $\left\|\sum_{i=1}^L a_i v_i\right\| \leq 1$ and $\left\|\sum_{i=1}^L a_i x_{\ell_i}\right\| > 2^N$. This yields

$$2^N < \left\|\sum_{i=1}^L a_i x_{l_i}\right\| \leq \sum_{i=1}^N |a_i| + \left\|\sum_{i=N+1}^L a_i x_{m_{k_i}^{k_i}}\right\|$$

$$\leq N + \left\|\sum_{i=N+1}^L a_i y_{k_i}\right\|$$

$$\leq 2N - \left\|\sum_{i=1}^N a_i y_{k_i}\right\| + \left\|\sum_{i=N+1}^L a_i y_{k_i}\right\|$$

$$\leq 2N + \left\|\sum_{i=1}^L a_i y_{k_i}\right\|$$

which implies

$$C < 2^N - 2N < \left\|\sum_{i=1}^L a_i y_{k_i}\right\|.$$

Thus $(y_{k_n})_{n=1}^\infty$ being C-dominated by V is contradicted. $\qquad\square$

The following lemma is used for a given (x_i) to find a subsequence (y_i) and a constant $C \geq 1$ such that (v_i) C-dominates every subsequence of (y_i) and that C is approximately minimal for every subsequence of (y_i).

Lemma B.6. *Let $V = (v_n)_{n=1}^\infty$ be a normalized bimonotone basic sequence, $(x_n)_{n=1}^\infty$ be a sequence in the unit ball of some Banach space X, and $a_n \nearrow \infty$ with $a_1 = 0$. If every subsequence of $(x_n)_{n=1}^\infty$ has a further subsequence which is dominated by V then there exists a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ and an $N \in \mathbb{N}$ such that every subsequence of $(y_n)_{n=1}^\infty$ is a_{N+1}-dominated by V but not a_N-dominated by V.*

Proof. By the previous lemma, we may assume by passing to a subsequence that there exists $C < \infty$ such that every subsequence of $(x_n)_{n=1}^\infty$ is C-dominated by V. Let

$M \in \mathbb{N}$ such that $a_M < C \le a_{M+1}$. For $1 \le n \le M$ let

$$A_n = \left\{ (m_k) \in [\mathbb{N}]^{\omega} \,\Big|\, \begin{array}{l} (x_{m_k})_{k=1}^{\infty} \text{ is } a_{n+1}\text{-dominated by V} \\ \text{and is not } a_n\text{-dominated by V.} \end{array} \right\}$$

A_n is Ramsey, and $\{A_n\}_{n=1}^{M}$ forms a finite partition of $[\mathbb{N}]^{\omega}$ which implies that there exists $N \le M$ and $(m_k) \in [\mathbb{N}]^{\omega}$ such that $[(m_k)_{k=1}^{\infty}]^{\omega} \subset A_N$. Every subsequence of $(y_n) := (x_{m_n})$ is a_{N+1}-dominated by V and not a_N-dominated by V. $\qquad \square$

C.   Proof of Proposition B.3

Proposition B.3 will be shown to follow easily from a characterization of countable compact metric spaces along with transfinite induction using the following result.

Lemma C.1. *Let (X_n) be a sequence of Banach spaces each satisfying the V-array procedure. Then $(\sum_{n=1}^{\infty} X_n)_{c_0}$ satisfies the V-array procedure.*

To prove Lemma C.1 we will need the following lemma which is stated in [KO2] for ℓ_p as Lemma 3.6. The proof for general V closely follows its proof.

Lemma C.2. *Let (X_n) be a sequence of Banach spaces each satisfying the V-array procedure and let (x_i^n) be a bad uV-array in some Banach space X. Suppose that for all $m \in \mathbb{N}$ there is a bounded linear operator $T_m : X \to X_m$ with $\|T_m\| \le 1$ such that $(T_m x_i^m)_{i=1}^{\infty}$ is an m-bad uV-sequence in X_m. Then (x_i^n) satisfies the V-array procedure.*

Proof. We first consider Case 1: There exists $m \in \mathbb{N}$ and a subarray (y_i^n) of (x_i^n) such that $(T_m y_i^n)_{i,n=1}^{\infty}$ is a bad uV-array in X_m. $(T_m y_i^n)_{i,n=1}^{\infty}$ satisfies the V-array procedure because X_m does. Therefore, there exists a subarray $(T_m z_i^n)_{i,n=1}^{\infty}$ of $(T_m y_i^n)_{i,n=1}^{\infty}$ and $(a_n) \subset \mathbb{R}^+$ with $a_n \le 2^{-n}$ such that $(\sum_{n=1}^{\infty} a_n T_m z_i^n)_{i=1}^{\infty}$ has no uV-subsequence.

$\left(\sum_{n=1}^{\infty} a_n z_i^n\right)_{i=1}^{\infty}$ has no uV-subsequence because $\|T_m\| \leq 1$. Therefore $(y_i^n)_{i,n=1}^{\infty}$ and hence $(x_i^n)_{i,n=1}^{\infty}$ satisfies the V-array procedure.

Case 2: If Case 1 is not satisfied then for all $m \in \mathbb{N}$ and every subarray (y_i^n) of (x_i^n), we have that $(T_m y_i^n)$ is not a bad uV-array in X_m. We may assume by passing to a subarray and using Lemma B.5 that there exists $(N_n)_{n=1}^{\infty} \in [\mathbb{N}]^{\omega}$ such that

$$(x_i^n)_{i=1}^{\infty} \text{ is a hereditary } N_n - uV - \text{sequence} \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

By induction we choose for each $m \in \mathbb{N}_0$ a subarray $(z_{m,i}^n)_{i,n=1}^{\infty}$ of $(x_i^n)_{i,n=1}^{\infty}$ and an $M_m \in \mathbb{N}$ so that

$$(z_{m,i}^n)_{i,n=1}^{\infty} \text{ is a sub-array of } (z_{m-1,i}^n)_{i,n=1}^{\infty} \quad \text{if } m \geq 1, \tag{2.3}$$

$$z_{m,i}^n = z_{m-1,i}^n \quad \text{if } N_n \leq m \text{ and } i \in \mathbb{N}, \tag{2.4}$$

$$(T_m(z_{m,i}^n))_{i=1}^{\infty} \text{ is a hereditary } M_m\text{-u}V\text{-sequence } \forall n \in \mathbb{N} \quad \text{if } m \geq 1. \tag{2.5}$$

For $m = 0$ let $\left(z_{0,i}^n\right)_{i,n=1}^{\infty} = (x_i^n)_{i,n=1}^{\infty}$. Now let $m \geq 1$. For each $n \in \mathbb{N}$ such that $N_n \leq m$ let $\left(z_{m,i}^n\right)_{i=1}^{\infty} = \left(z_{m-1,i}^n\right)_{i=1}^{\infty}$ and $K_n = m$. For each $n \in \mathbb{N}$ such that $N_n > m$, using Lemma B.6, we let $\left(z_{m,i}^n\right)_{i=1}^{\infty}$ be a subsequence of $\left(z_{m-1,i}^n\right)_{i=1}^{\infty}$ for which there exists $K_n \in \mathbb{N}_{\not\sim}$ such that $\left(T_m z_{m,i}^n\right)_{i=1}^{\infty}$ is a K_n-bad-uV sequence and is also a hereditary $(K_n + 1)$-uV-sequence. $(K_n)_{n=1}^{\infty}$ is bounded because otherwise we are in Case 1. Let $M_m = \max_{n \in \mathbb{N}} K_n + 1$. This completes the induction.

For all $n, i \in \mathbb{N}$ we have by (2.4) that $\left(z_{m,i}^n\right)_{m=1}^{\infty}$ is eventually constant. Let $(z_i^n)_{i,n=1}^{\infty} = \lim_{m \to \infty} \left(z_{m,i}^n\right)_{i,n=1}^{\infty}$. We have that $(z_i^n)_{i,n=1}^{\infty}$ is a subarray of $(x_i^n)_{i,n=1}^{\infty}$, and by (2.5), $(z_i^n)_{i,n=1}^{\infty}$ satisfies:

$$(T_m(z_i^n))_{i=1}^{\infty} \text{ is a hereditary } M_m\text{-u}V\text{-sequence for all } m, n \in \mathbb{N}. \tag{2.6}$$

We will now inductively choose $(m_n) \in [\mathbb{N}]^{\omega}$ and $(a_n) \subset \mathbb{R}^+$ so that for all $n \in \mathbb{N}$

we have:

$$(T_{m_n} z_i^{m_n})_{i=1}^{\infty} \text{ is an } m_n\text{-bad uV sequence in } X_{m_n}, \tag{2.7}$$

$$a_n m_n > n, \tag{2.8}$$

$$\sum_{j=1}^{n-1} a_j N_{m_j} < \frac{a_n m_n}{4}, \text{ and} \tag{2.9}$$

$$0 < a_n < \min_{1 \le k < n} \{2^{-n}, 2^{-n} \frac{a_k m_k}{4 M_{m_k}}\}. \tag{2.10}$$

Property (2.7) has been assumed in the statement of the Lemma. For n=1 let $a_1 = \frac{1}{2}$ and $m_1 \in \mathbb{N}$ such that $a_1 m_1 > 1$, so (2.8) is satisfied. (2.9) and (2.10) are vacuously true for n=1, so all conditions are satisfied for $n = 1$.

Let $n > 1$ and assume $(a_j)_{j=1}^{n-1}$ and $(m_j)_{j=1}^{n-1}$ have been chosen to satisfy (2.8), (2.9) and (2.10). Choose $a_n > 0$ small enough such that $a_n < \min_{1 \le k < n} \left\{ 2^{-n}, 2^{-n} \frac{a_k m_k}{4 M_k} \right\}$, thus satisfying (2.10). Choose $m_n > 0$ large enough to satisfy (2.8) and (2.9). This completes the induction.

By (2.10), we have for all $n \in \mathbb{N}$ that

$$\sum_{j=n+1}^{\infty} a_j M_{m_n} < \frac{a_n m_n}{4}. \tag{2.11}$$

We have by (2.10) that $a_j < 2^{-j}$ for all $j \in \mathbb{N}$, so $y_k := \sum_{j=1}^{\infty} a_j z_k^{m_j}$ is a valid choice for the V-array procedure. Let $C > 0$ and (y_{k_i}) be a subsequence of (y_k). We need to show that (y_{k_i}) is not a C-uV-sequence. Using (2.8), choose $n \in \mathbb{N}$ so that $a_n m_n > 2C$. Using (2.7) choose $\ell \in \mathbb{N}$ and $(\beta_i)_{i=1}^{\ell} \in B_{[v_i]_{i=1}^{\ell}}$ such that

$$\left\| \sum_{i=1}^{\ell} \beta_i T_{m_n} \left(z_{k_i}^{m_n} \right) \right\| > m_n. \tag{2.12}$$

We now have the following

$$\left\|\sum_{i=1}^{\ell}\beta_i y_{k_i}\right\| = \left\|\sum_{i=1}^{\ell}\sum_{j=1}^{\infty}\beta_i a_j z_{k_i}^{m_j}\right\|$$

$$\geq \left\|\sum_{i=1}^{\ell}\sum_{j=n}^{\infty}T_{m_n}\left(\beta_i a_j z_{k_i}^{m_j}\right)\right\| - \left\|\sum_{i=1}^{\ell}\sum_{j=1}^{n-1}\beta_i a_j z_{k_i}^{m_j}\right\| \quad \text{since } \|T_{m_n}\| \leq 1$$

$$\geq a_n\left\|\sum_{i=1}^{\ell}\beta_i T_{m_n} z_{k_i}^{m_n}\right\| - \sum_{j=n+1}^{\infty}a_j\left\|\sum_{i=1}^{\ell}\beta_i T_{m_n}z_{k_i}^{m_j}\right\| - \sum_{j=1}^{n-1}a_j\left\|\sum_{i=1}^{\ell}\beta_i z_{k_i}^{m_j}\right\|$$

$$> a_n m_n - \sum_{j=n+1}^{\infty}a_j M_{m_n} - \sum_{j=1}^{n-1}a_j N_{m_j} \quad \text{by } (2.12), (2.6), \text{and } (2.2)$$

$$\geq a_n m_n - a_n m_n/4 - a_n m_n/4 \quad \text{by } (2.9) \text{ and } (2.11)$$

$$= a_n m_n/2 > C.$$

Therefore, (y_{k_i}) is not a C-uV-sequence. $(y_i)_{i=1}^{\infty} = \left(\sum_{j=1}^{\infty}a_j z_i^{m_j}\right)_{i=1}^{\infty}$ has no uV-subsequence, so (x_i^n) satisfies the V-array procedure which proves the lemma. $\qquad\square$

Now we are prepared to give a proof of Lemma C.1. We follow the outline of the proof of Lemma 3.5 in [KO2].

Proof of Lemma C.1. let (x_i^n) be a bad uV-array in $X = \left(\sum X_n\right)_{c_0}$ and $R_m : X \to X_m$ be the natural projections.

Claim: For all $M < \infty$ there exists $n, m \in \mathbb{N}$ and a subsequence $(y_i)_{i=1}^{\infty}$ of $(x_i^n)_{i=1}^{\infty}$ such that $(R_m y_i)_{i=1}^{\infty}$ is an M-bad uV-sequence.

Assuming the claim, we can find $(N_n)_{n=1}^{\infty} \in [\mathbb{N}]^{\omega}$, $(m(n))_{n=1}^{\infty} \subset \mathbb{N}$, and subsequences $(y_i^n)_{i=1}^{\infty}$ of $\left(x_i^{N_n}\right)_{i=1}^{\infty}$ such that $\left(R_{m(n)}y_i^n\right)_{i=1}^{\infty}$ is an n-bad uV sequence for all $n \in \mathbb{N}$. By passing to a subsequence, we may assume either that $m(n) = m$ is constant, or that $(m(n))_{n=1}^{\infty} \in [\mathbb{N}]^{\omega}$. If $m(n) = m$, then $R_m(y_i^n)_{n,i=1}^{\infty}$ is a bad uV-array in X_m. $R_m(y_i^n)_{n,i=1}^{\infty}$ satisfies the V-array procedure, and thus $(y_i^n)_{n,i=1}^{\infty}$ satisfies the

V-array procedure. If $(m(n))_{n=1}^\infty \in [\mathbb{N}]^\omega$ let $T_n := R_{m(n)}|_{[y_i^r]_{i,r=1}^\infty}$ and apply Lemma C.2 to the array $(y_i^n)_{i,n=1}^\infty$ to finish the proof.

To prove the claim, we assume it is false. There exists $M < \infty$ such that for all $m, n \in \mathbb{N}$ every subsequence of $(x_i^n)_{i=1}^\infty$ contains a further subsequence $(y_i)_{i=1}^\infty$ such that $(R_m y_i)_{i=1}^\infty$ is an M-uV-sequence.

By Ramsey's theorem, for each $n \in \mathbb{N}$ and $m \in \mathbb{N}$ every subsequence of $(x_i^n)_{i=1}^\infty$ contains a further subsequence $(y_i)_{i=1}^\infty$ such that $(R_m y_i)_{i=1}^\infty$ is a hereditary M-uV-sequence. Fix $n \in \mathbb{N}$ such that $(x_i^n)_{i=1}^\infty$ is an (M+3)-bad uV-sequence. We now construct a nested collection of subsequences $\{(y_{k,i})_{i=1}^\infty\}_{k=0}^\infty$ of $(x_i^n)_{i=1}^\infty$ (where $(y_{0,i})_{i=1}^\infty = (x_i^n)_{i=1}^\infty$) as well as $(m_i) \in [\mathbb{N}]^\omega$ so that for all $k \in \mathbb{N}$ we have

$$\sup_{m > m_k} \|R_m y_{k-1,k}\| \leq 2^{-k}, \tag{2.13}$$

$$(y_{k,i})_{i=1}^\infty \text{ is a subsequence of } (y_{k-1,i})_{i=1}^\infty, \tag{2.14}$$

$$(R_m y_{k,i})_{i=1}^\infty \text{ is a hereditary M-uV-sequence } \forall m \leq m_k. \tag{2.15}$$

For k=1 we choose $m_1 \in \mathbb{N}$ such that $\sup_{m > m_1} \|R_m y_{0,1}\| \leq 2^{-1}$. Pass to a subsequence $(y_{1,i})_{i=1}^\infty$ of $(y_{0,i})_{i=1}^\infty$ such that $(R_m y_{1,i})_{i=1}^\infty$ is a hereditary M-uV-sequence for all $m \leq m_1$.

For $k > 1$ given $m_{k-1} \in \mathbb{N}$ and a sequence $(y_{k-1,i})_{i=1}^\infty$. Choose $m_k > m_{k-1}$ so that $\sup_{m > m_k} \|R_m y_{k-1,k}\| \leq 2^{-k}$, thus satisfying (2.13). Let $(y_{k,i})_{i=1}^\infty$ be a subsequence of $(y_{k-1,i})_{i=1}^\infty$ so that $(R_m y_{k,i})_{i=1}^\infty$ is a hereditary M-uV-sequence for all $m \leq m_k$, thus satisfying (2.14) and (2.15). This completes the induction.

We define $y_k = y_{k-1,k}$ for all $k \in \mathbb{N}$. By (2.14), we have that $(y_{k,i})_{i=1}^k \cup (y_i)_{i=k+1}^\infty$

is a subsequence of $(y_{k,i})_{i=1}^{\infty}$. Therefore, (2.15) gives that

$$(v_i)_{i=k+1}^{\infty} \text{ M-dominates } (R_m y_{q_i})_{i=k+1}^{\infty} \forall m \le m_k, \ (q_i) \in [\mathbb{N}]^{\omega}, \text{ and } k \in \mathbb{N}. \qquad (2.16)$$

We have that $(x_i^n)_{i=1}^{\infty}$ is a $(M+3)$-bad uV sequence, so there exists $(\alpha_i) \in B_{[V]}$ such that

$$\left\| \sum_{i=1}^{\infty} \alpha_i y_i \right\| > M+3. \qquad (2.17)$$

For all $k \in \mathbb{N}$ and $m \in (m_{i-1}, m_i]$ (with $m_0 = 0$) we have that

$$\left\| \sum_{i=1}^{\infty} R_m (\alpha_i y_i) \right\| \le \sum_{i=1}^{k-1} |\alpha_i| \, \|R_m y_i\| + \|R_m (\alpha_k y_k)\| + \left\| \sum_{i=k+1}^{\infty} R_m (\alpha_i y_i) \right\|$$

$$\le \sum_{i=1}^{k-1} 2^{-i} + 1 + \left\| \sum_{i=k+1}^{\infty} \alpha_i R_m(y_i) \right\| \qquad \text{by (2.13)}$$

$$\le 1 + 1 + M \qquad \text{by (2.16)}$$

which implies

$$\left\| \sum_{i=1}^{\infty} \alpha_i y_i \right\| = \sup_{m \in \mathbb{N}} \| \sum_{i=1}^{\infty} R_m (\alpha_i y_i) \| \le M+2.$$

This contradicts (2.17), so the claim and hence the lemma is proved. $\qquad \square$

The proof for proposition B.3 now follows in exactly the same way as in [KO2].

Proof of Proposition B.3. If K is a countable compact metric space then there is a countable limit ordinal α such that $C(K)$ is isomorphic to $C(\alpha)$ (see [BP]). Thus if the V-array procedure fails for $C(K)$, then there is a first limit ordinal α such that the V-array procedure fails for $C(\alpha)$. If α is the first infinite ordinal then $C(\alpha)$ is isomorphic to c_0 and satisfies the V-array procedure. Otherwise, we can find a

sequence $\beta_n < \alpha$ of limit ordinals such that $C(\alpha)$ is isomorphic to $\left(\sum C\left(\beta_n\right)\right)_{c_0}$. Thus $C(\alpha)$ satisfies the V-array procedure by Lemma C.1. □

D.   Proof of Proposition B.4

The proof of Theorem B.2 will be complete once we have proven proposition B.4. To make notation easier, we now consider the triangulated version $(x_i^n)_{1 \leq n \leq i < \infty}$ of the square array $(x_i^n)_{i,n=1}^{\infty}$. The benefit of using a triangular array is that a natural sequential order can be put on a triangular array. As the following proposition shows, we can then pass to a basic sequence in that order.

Lemma D.1. *For all $\epsilon > 0$, a triangular bad uV-array $(x_i^n)_{n \leq i}$ admits a triangular subarray $(y_i^n)_{n \leq i}$ which is basic in its lexicographical order (where i is the first letter and n is the second letter), and its basis constant is not greater than $1 + \epsilon$. In other words $y_1^1, y_2^1, y_2^2, y_3^1, y_3^2, y_3^3, y_4^1...$ is a basic sequence.*

Proof. The proof is an easy adaptation of the proof that a weakly null sequence has a basic subsequence. □

The following lemma shows that we need to prove Proposition B.4 only for triangular arrays.

Lemma D.2. *A square array satisfies the V-array procedure if and only if it's triangulated version does.*

Proof. If $(y_i^n)_{i,n=1}^{\infty}$ is a subarray of $(x_i^n)_{i,n=1}^{\infty}$ then $(y_i^n)_{1 \leq n \leq i < \infty}$ is a triangular subarray of $(x_i^n)_{1 \leq n \leq i < \infty}$. Also, if $(y_i^n)_{1 \leq n \leq i < \infty}$ is a triangular subarray of $(x_i^n)_{1 \leq n \leq i < \infty}$ then $(y_i^n)_{1 \leq n \leq i < \infty}$ may be extended to a subarray of $(x_i^n)_{i,n=1}^{\infty}$ by letting $(y_i^n)_{i<n} = (x_i^{m_n})_{i<n}$, where $(m_n) \in [\mathbb{N}]^{\omega}$ is such that $(y_i^n)_{i=1}^{\infty} \subset (x_i^{m_n})_{i=1}^{\infty}$ for all $n \in \mathbb{N}$.

We now show that applying the V-array procedure to $(y_i^n)_{i,n=1}^\infty$ and $(y_i^n)_{1\le n\le i<\infty}$ yield sequences which either both satisfy the V-array procedure or both fail the V-array procedure. For all $n \in \mathbb{N}$ let $0 \le |\alpha_n| \le 2^{-n}$, $z_i = \sum_{n=1}^i \alpha_n y_i^n$, and $y_i = \sum_{n=1}^\infty \alpha_n y_i^n$. For all $m \in \mathbb{N}$ if $(\beta_i)_{i=1}^\infty \in B_{[V]}$ then

$$\left\| \sum_{i=1}^m \beta_i z_i - \sum_{i=1}^m \beta_i y_i \right\| = \left\| \sum_{i=1}^m \beta_i \sum_{n=i+1}^\infty \alpha_n y_i^n \right\| \le \sum_{i=1}^m |\beta_i| \sum_{n=i+1}^\infty |\alpha_n| \le \sum_{i=1}^m 2^{-i} < 1.$$

Thus $\sup_{m\in\mathbb{N}} \|\sum_{i=1}^m \beta_i z_i\| = \infty$ if and only if $\sup_{m\in\mathbb{N}} \|\sum_{i=1}^m \beta_i y_i\| = \infty$, which implies the claim.

$\square$

We now assume that the given bad uV-array (x_i^n) is labeled triangularly and that it is a bimonotone basic sequence in its lexicographical order. This assumption is valid because the properties "being a bad uV-array" and "satisfying the V-array procedure" are invariant under isomorphisms. We also assume that (x_i^n) is normalized.

The following theorem is our main tool used to construct the subarray (y_i^n) of (x_i^n) and the countable w^*-compact set $K \subset B_{[y_i^n]}$ for Proposition B.4.

Theorem D.3. *Assume that $(x_i^n)_{1\le n\le i}$ is a normalized triangular array in X, such that for every $n \in \mathbb{N}$ the sequence $(x_i^n)_{i=1}^\infty$ is weakly converging to 0. Let $V = (v_i)$ be a normalized basic sequence and let $(C_n) \subset [0,\infty)$ and $\epsilon > 0$.*

Then (x_i^n) has a triangular sub-array (y_i^n) with the following property:

For all $m, s \in \mathbb{N}$ and all $m \le m_1 < m_2 \ldots < m_s$ all $(\alpha_j)_{j=1}^s \in B_V$ with $\|\sum_{j=1}^s \alpha_j y_{m_j}^m\| \ge C_n$ there is a $g \in (2+\epsilon)B_{X^*}$ and $(\beta_j)_{j=1}^s \in B_V$, so that

$$\sum_{j=1}^{s} \beta_j g(y_{m_j}^m) \geq C_n, \tag{2.18}$$

$$g(y_j^{m'}) = 0 \text{ whenever } m' \leq j \text{ and } j \notin \{m_1, m_2, \ldots m_s\}. \tag{2.19}$$

If we also assume that $(x_i^n)_{1 \leq n \leq i}$ is a bimonotone basic sequence in its lexicographical order then there exists $(j_i) \in [\mathbb{N}]^\omega$ so that we may choose the sub-array (y_i^n) by setting $y_i^n = x_{j_i}^n$ for all $n \leq i$. In this case we have the above conclusion for some $g \in (1 + \epsilon) B_{Y^*}$.

Proof. After passing to a sub-array using Lemma D.1 we can assume that (x_i^n) is a basic sequence in its lexicographical order and that it's basis constant does not exceed the value $1 + \epsilon$. We first renorm $Z = [x_i^n]$ by a norm $||| \cdot |||$ in the standard way so that $\|z\| \leq |||z||| \leq (2 + 2\epsilon)\|z\|$ and so that (x_i^n) is bimonotone in Z. We therefore can assume that (x_i^n) is a bimonotone basis and need to show the claim of Theorem D.3 for $(1 + \epsilon) B_{X^*}$ instead of $(2 + \epsilon) B_{X^*}$.

Let $(\epsilon_k) \subset (0, 1)$ with $\sum_{k=1}^{\infty} k \epsilon_k < \epsilon/4$. By induction on $k \in \mathbb{N}_0$ we choose $i_k \in \mathbb{N}$ and a sequence $L_k \in [\mathbb{N}]^\omega$, and define $y_j^m = x_{i_j}^m$ for $m \leq k$ and $m \leq j \leq k$ so that the following conditions are satisfied.

a) $i_k = \min L_{k-1} < \min L_k$ and $L_k \subset L_{k-1}$, if $k \geq 1$ ($L_0 = \mathbb{N}$).

b) For all $s, t \in \mathbb{N}_0$, all $1 \leq m \leq k$, all $m \leq m_1 < m_2 < \ldots m_s \leq k$ and $\ell_0 < \ell_1 < \ldots \ell_t$ in L_k, if $\exists f \in B_{X^*}$ with

$$\sum_{j=1}^{s} \alpha_j f(y_{m_j}^m) + \sum_{j=1}^{t} \alpha_{j+s} f(x_{\ell_j}^m) \geq C_m \text{ for some } (\alpha_j)_{j=1}^{s+t} \in B_{[V]} \tag{2.20}$$

$$\text{then } \exists g \in B_{X^*} \text{ such that} \tag{2.21}$$

(a) $\sum_{j=1}^{s} \beta_j g(y_{m_j}^m) + \sum_{j=1}^{t} \beta_{j+s} g(x_{\ell_j}^m) \geq C_m$ for some $(\beta_j)_{j=1}^{s+t} \in B_{[V]}$

(b) $|g(y_j^{m'})| < \epsilon_j$ if $m' \leq k$ and $j \in \{m', \ldots k\} \setminus \{m_1, \ldots m_s\}$, and

(c) $|g(x_{\ell_0}^{m'})| < \epsilon_{k+1}$ if $m' \leq k+1$.

(in the case that $s = 0$ condition (b) is defined to be vacuous, also note that in (c) we allow $m' = k+1$).

We first note for $(i_j) \in [\mathbb{N}]^\omega$ that $(x_{i_j}^n)_{n \leq j}$ is a subsequence of $(x_j^n)_{n \leq j}$ in their lexicographic orders. Thus $(x_{i_j}^n)_{n \leq j}$ is a bimonotone basic sequence in its lexicographic order.

For $k = 0$, if $f \in B_{X^*}$ satisfies (2.20) then $g = P^*_{[x_{\ell_1}^n, \infty)} f$ satisfies (2.21) by our assumed assumed bimonotonicity.

Assume $k \geq 1$ and we have chosen $i_1 < i_2 < \ldots < i_{k-1}$. We let $i_k = \min L_{k-1}$.

Fix an infinite $M \subset L_{k-1} \setminus \{i_k\}$, a positive integer $m \leq k$, an integer $0 \leq s \leq k - m + 1$, and positive integers $m \leq m_1 < m_2 < \ldots m_s \leq k$ and define

$$A = A(m, s, (m_j)_{j=1}^s) = \bigcap_{t \in \mathbb{N}_0} A_t, \text{ where}$$

$$A_t = \left\{ (\ell_j)_{j=0}^\infty \in [M]^\omega : \begin{array}{c} \text{If } (m_j)_{j=1}^s \text{ and } (\ell_j)_{j=0}^t \text{ satisfy } (2.20) \\ \text{then they also satisfy } (2.21) \end{array} \right\}.$$

For $t \in \mathbb{N}$ the set A_t is closed as a subset of $2^\mathbb{N}$ in the product topology, thus A is closed and, thus, Ramsey. We will show that there is an infinite $L \subset M$ so that $[L]^\omega \subset A$. Once we verified that claim we can finish our induction step by applying that argument successively to all choices of $m \leq k$, $0 \leq s \leq k$ and $m \leq m_1 < m_2 < \ldots m_s \leq k$, as there are only finitely many.

Assume our claim is wrong and, using Ramsey's Theorem, we could find an $L = (\ell_j)_{j=1}^\infty$ so that $[L]^\omega \cap A = \emptyset$.

Let $n \in \mathbb{N}$ be fixed, and let $p \in \{1, 2 \ldots, n\}$. Then $L^{(p)} = \{\ell_p, \ell_{n+1}, \ldots\}$ is not in A and we can choose $t_n \in \mathbb{N}_0$, $(\alpha_j^n)_{j=1}^{t_n+s}$ and $f_n \in B_{X^*}$ so that (2.20) is satisfied

(for $(\ell_{n+1}, \ldots \ell_{\ell+t})$ replacing $(\ell_1, \ldots \ell_t)$ and ℓ_p replacing ℓ_0) but for no $g \in B_{X^*}$ and $(\beta_j)_{j=1}^{s+t_n} \in B_{[V]}$ condition (2.21) holds. By choosing t_n to be minimal so that (2.20) is satisfied, we can have t_n, $(\alpha_j^n)_{j=1}^{t_n+s}$ and f_n be independent of p.

We now show that there is a $g_n \in B_X$ satisfying (a) and (b) of (2.21).

Let $k' = \max\{m - 1 \le i \le k : i \notin \{m_1, m_2, \ldots m_s\}\}$. If $k' \le m$ then $\{m_1, \ldots, m_s\} = \{k' + 1, k' + 2, \ldots, k\}$ and by our assumed bimonotonicity $g_n :=$ $P^*_{[y_{k'+1}^m, \infty)} f_n \in B_X^*$ satisfies (a) and (b) of (2.20). If $k' > m$ let $0 \le s' \le s$, such that $m_1 < m_2 < \ldots < m_{s'} < k'$, and apply the $k' - 1$ step of the induction hypothesis to f_n, $(\alpha_j^n)_{j=1}^{t_n+s}$, $m \le m_1 < \ldots < m_{s'}$ (replacing $m \le m_1 < \ldots < m_s$), and $k' < k' + 1 < \ldots < m_s < \ell_{n+1} < \ldots < \ell_{t_n}$ (replacing $\ell_p < \ell_{n+1} < \ldots < \ell_{t_n}$) to obtain a functional $g_n \in B_{X^*}$ which satisfies (a) and (b) of (2.21).

Since g_n cannot satisfy all three conditions of (2.21) (for any choice of $1 \le p \le n$), we deduce that $|g_n(x_{\ell_p}^{m_p})| \ge \epsilon_{k+1}$ for some choice of $m_p \in \{1, 2, \ldots k+1\}$.

Let g be a w^* cluster point of $(g_n)_{n \in \mathbb{N}}$. As the set $\{1, 2, \ldots k+1\}$ is finite, we have for all $p \in \mathbb{N}_0$ that $|g(x_{\ell_p}^{m_p})| \ge \epsilon_{k+1}$ for some $m_p \in \{1, 2, \ldots k+1\}$. Which implies there exists $1 \le m \le k+1$ such that $|g(x_{\ell_p}^m)| \ge \epsilon$ for infinitely many $p \in \mathbb{N}$. This is a contradiction with the sequence $(x_{\ell_i}^m)_{i=1}^\infty$ being weakly null. Our claim is verified, and we are able to fulfill the induction hypothesis.

The conclusion of our theorem now follows by the following perturbation argument. If we have $n \le i_1 < i_2 \ldots < i_q$ and $(\alpha_j)_{j=1}^q \in B_V$ with $\| \sum_{j=1}^q \alpha_j y_{i_j}^n \| \ge C_n$, then there exists $f \in B_{X^*}$ so that $\sum_{j=1}^q \alpha_j f(y_{i_j}^n) \ge C_n$. Our construction gives an $h \in B_{X^*}$ with $\sum_{j=1}^q \alpha_j h(y_{i_j}^n) \ge C_n$ and $|h(y_j^m)| < \epsilon_j$ if $m \le q$ and $j \in \{m', \ldots k\} \setminus \{i_1, \ldots i_q\}$. Because (y_i^n) is bimonotone, we may assume that $h(y_i^n) = 0$ for all $i \ge n$ with $i > i_q$. We perturb h by small multiples of the biorthogonal functionals of (y_i^n) to achieve $g \in X^*$ with $g(y_i^n) = h(y_i^n)$ for $i \in \{i_1, \ldots, i_q\}$ and $g(y_i^n) = 0$ for $i \notin \{i_1, \ldots, i_q\}$. Thus g satisfies (2.18) and (2.19). All that remains is to check that $g \in (1+\epsilon)B_{X^*}$.

Because (y_i^n) is normalized and bimonotone, we can estimate $\|g\|$ as follows:

$$\|g\| \leq \|h\| + \|g - h\| \leq 1 + \sum_{j=1}^{i_q-1} j\epsilon_j < 1 + \frac{\epsilon}{4}.$$

□

We are now prepared to give the proof of Proposition B.4. We follow the same outline as the proof given in [KO2] for Proposition 3.4.

Proof of Proposition B.4. Let (x_i^n) be a normalized bad uV-array in X and let M_n, for $n \in \mathbb{N}$, be chosen so that the sequence $(x_i^n)_{i=n}^\infty$ is an M_n-bad uV-sequence and $\lim_{n\to\infty} M_n = \infty$. By Lemma D.2 we just need to consider the triangular array $(x_i^n)_{n\leq i}$. By passing to a subarray using Lemma D.1 and then renorming, we may assume that $(x_i^n)_{n\leq i}$ is a normalized bimonotone basic sequence in its lexicographical order.

We apply Theorem D.3 for $\epsilon = 1$ and $(C_n) = (M_n)$ to obtain a subarray $(y_i^n)_{n\leq i}$ that satisfies the properties (2.18) and (2.19). Moreover (y_i^n) in its lexicographical order is a subsequence of (x_i^n) in its lexicographical order, and thus is bimonotone. Furthermore, $(y_i^n)_{i=n}^\infty$ is a subsequence of $(x_i^n)_{i=n}^\infty$ for all $n \in \mathbb{N}$. We denote $Y = [y_i^n]_{n\leq i}$.

Let $F(n)$ be a finite $\frac{1}{2n2^n}$-net in $[-2, 2]$ which contains the points 0,-2, and 2. Whenever we have a functional $g \in 2B_{X^*}$ which satisfies conditions (2.18) and (2.19) we may perturb g by small multiples of the biorthogonal functions of $(y_i^n)_{n\leq i}$ to obtain $f \in 3B_{X^*}$ which satisfies (2.18), (2.19), and the following new condition

$$f(y_i^n) \in F(n) \qquad \text{for all } n \leq i. \tag{2.22}$$

We now start the construction of K. Let $Y = [y_i^n]_{n\leq i}$ and $m \in N$. We define the

following,

$$L_m = \left\{ (k_1, ..., k_q) \mid \begin{array}{l} m \le k_1 < k_2 < ... < k_q, \\ || \sum_{i=1}^{q-1} \alpha_i y_{k_i}^m || \le M_m \quad \text{for all } (\alpha_i) \in B_V \\ || \sum_{i=1}^{q} \alpha_i y_{k_i}^m || > M_m \quad \text{for some } (\alpha_i) \in B_V \end{array} \right\}$$

It is important to note that if $(k_i) \in [\mathbb{N}]^\omega$ and $k_1 \ge m$ then there is a unique $q \in N$ such that $(k_1, ..., k_q) \in L_m$.

Whenever $\vec{k} = (k_1, ..., k_q) \in L_m$, our application of Theorem D.3 and then perturbation gives a functional $f \in 3B_{Y^*}$ which satisfies the properties (2.18),(2.19), and (2.22). In particular we have that $\sum_{i=1}^{q} f(\alpha_i y_{k_i}^m) > M_m$ for some $(\alpha_i) \in B_V$. We denote $f/3$ by $f_{\vec{k}}$ and let for any $n \in \mathbb{N}$,

$$K_n = \{ Q_m^* f_{\vec{k}} \mid m \in \mathbb{N} \ \vec{k} \in L_n \}.$$

Here Q_m denotes the natural norm 1 projection from Y onto $[(y_i^n)]_{1 \le n \le i \le m}$. Finally, we define

$$K = \bigcup_{n=1}^{\infty} K_n \cup \{0\}.$$

We first show that $(y_i^n|_K)_{n \le i}$ is a bad uV-array as an array in $C_b(K)$. Fix an $n_0 \in \mathbb{N}$. $(y_i^{n_0})_{i=n_0}^\infty$ is an M_{n_0}-bad uV-sequence. Consequently, given a subsequence $(y_{k_i}^{n_0})_{i=1}^\infty$ of $(y_i^{n_0})_{i=n_0}^\infty$ we have that $\vec{k} := (k_1, ..., k_q) \in L_{n_0}$ for some $q \in \mathbb{N}$. By (2.22), $f_{\vec{k}} = Q_{q+1}^* f_{\vec{k}}$ and thus $f_{\vec{k}} \in K_{n_0} \subset K$. $\sum_{i=1}^{q} f_{\vec{k}}(\alpha_i y_{k_i}^{n_0}) > \frac{M_{n_0}}{3}$ for some $(\alpha_i) \in B_V$, and so we obtain that $(y_i^{n_0}|_K)_{i=n_0}^\infty$ is an $(M_{n_0}/3)$-bad sequence in $C_b(K)$, thus proving that $(y_i^n|_K)_{n \le i}$ is a bad uV-array.

K is obviously a countable subset of B_{Y^*}. Since Y is separable, K is w^*-metrizable. Thus we need to show that K is a w^*-closed subset of B_{Y^*} in order to finish the proof.

Let $(g_j) \subset K$ and assume that (g_j) converges w^* to some $g \in B_{Y^*}$. We have to

show that $g \in K$. Every g_j is of the form $Q^*_{m_j} f_{\vec{k}_j}$ for some $m_j \in \mathbb{N}$, $\vec{k}_j \in L_{n_j}$, and some $n_j \in \mathbb{N}$.

By passing to a subsequence of (g_j), we may assume that either $n_j \to \infty$ as $j \to \infty$ or that there is an $n \in N$ such that $n_j = n$ for all $j \in \mathbb{N}$. We will start with the first alternative. Let i_j be the first element of $\vec{k}_j$. Since $i_j \geq n_j$, we have that $i_j \to \infty$. We also have that $f_{\vec{k}_j}(y^n_i) = 0$ for all $n \leq i < i_j$. Thus $f_{\vec{k}_j} \to 0$ in the w^* topology as $j \to \infty$, so $g = 0 \in K$.

From now on we assume that there is an $n \in \mathbb{N}$ such that $\vec{k}_j \in L_n$ for all $j \in \mathbb{N}$. L_n is relatively sequentially compact as a subspace of $\{0,1\}^{\mathbb{N}}$ endowed with the product topology. Thus we may assume by passing to a subsequence of (g_j) that $\vec{k}_j \to \vec{k}$ for some $\vec{k} \in \overline{L_n}$, the closure of L_n in $\{0,1\}^{\mathbb{N}}$.

We now show that $\vec{k}$ is finite. Suppose to the contrary that $\vec{k} = (k_i)^\infty_{i=1}$. We have that $\vec{k} \in \overline{L_n}$, so for all $r \in \mathbb{N}$ there exists $N_r \in \mathbb{N}$ such that $\vec{k}_j = (k_1, ..., k_r, \ell_1, ..., \ell_s)$ for some $\ell_1, ..., \ell_s$ for all $j \geq N_r$. Because $\vec{k}_j \in L_n$ we have that $k_1 \geq n$, which implies that there exists $q \in \mathbb{N}$ such that $(k_1, ...k_q) \in L_n$. By uniqueness, L_n does not contain any sequence extending $(k_1, ..., k_q)$. Therefore, $\vec{k}_{N_{q+1}} = (k_1, ..., k_{q+1}, \ell_1, ..., \ell_s) \notin L_n$, a contradiction.

Since B_{Y^*} is w^*-sequentially compact, we may assume that $f_{\vec{k}_j}$ converges w^* to some $f \in B_{Y^*}$. We claim that $f \in K$. To prove this we first show that $Q^*_m f \in K$ for all $m \in \mathbb{N}$. By (2.19) and (2.22) the set $\{Q^*_m f_{\vec{k}_j}(y^n_i) \mid j \in \mathbb{N} \ 1 \leq n \leq i\}$ has only finitely many elements. Since $Q^*_m f_{\vec{k}_j} \to Q^*_m f$ as $j \to \infty$ we obtain that $Q^*_m f_{\vec{k}_j} = Q^*_m f$ for $j \in \mathbb{N}$ large enough. In particular $Q^*_m f \in K$. Next let $q = \max \vec{k}$. Since $\vec{k}_j \to \vec{k}$ and $\vec{k}$ is finite, we have $Q^*_q f = f$ and thus $f \in K$.

Now we show that $g \in K$. By passing again to a subsequence of (g_j) we can assume that either $m_j \geq \max \vec{k}$ for all $j \in \mathbb{N}$ or that there exists $m < \max \vec{k}$ such that $m_j = m$ for all $j \in \mathbb{N}$. If the first case occurs, then $g_j = Q^*_{m_j} f_{\vec{k}_j}$ converges w^* to

f, and hence $g = f \in K$. If the second case occurs then $g_j = Q_m^* f_{\vec{k}_j}$ converges w^* to $Q_m^* f$, and hence $g = Q_m^* f \in K$. □

E.   Examples

In previous sections, we introduced for any semi-normalized basic sequence (v_i) the property $U_{(v_i)}$, and then proved that if a Banach space X is $U_{(v_i)}$ then there exists a constant $C \geq 1$ such that X is $C - U_{(v_i)}$. As Knaust and Odell proved that result for the cases in which (v_i) is the standard basis for c_0 or ℓ_p with $1 \leq p < \infty$, we need to show that our result is not a corollary of theirs. For example, if (v_i) is a basis for $\ell_p \bigoplus \ell_q$ with $1 < q < p < \infty$ which consists of the union of the standard bases for ℓ_p and ℓ_q then a Banach space is $U_{(v_i)}$ or $C - U_{(v_i)}$ if and only if X is U_{ℓ_p} or $C - U_{\ell_p}$ respectively. Thus the result for this particular (v_i) follows from [KO2]. We make this idea more formal by defining the following equivalence relation:

Definition E.1. If (v_i) and (w_i) are normalized basic sequences then we write $(v_i) \sim_U (w_i)$ (or $(v_i) \sim_{CU} (w_i)$) if each reflexive Banach space is $U_{(v_i)}$ (or $C - U_{(v_i)}$) if and only if it is $U_{(w_i)}$ (or $C - U_{(w_i)}$).

We define the equivalence relation strictly in terms of reflexive spaces to avoid the unpleasant case of ℓ_1. Because ℓ_1 does not contain any normalized weakly null sequence, ℓ_1 is trivially $U_{(v_i)}$ for every (v_i). This is counter to the spirit of what it means for a space to be $U_{(v_i)}$. By considering reflexive spaces, we avoid ℓ_1, and we also make the propositions included in this section formally stronger. Reflexive spaces are also especially nice when considering properties of weakly null sequences because the unit ball of a reflexive spaces is weakly sequentially compact. That is every sequence in the unit ball of a reflexive space has a weakly convergent subsequence.

In order to show that our result is not a corollary of the theorem of Knaust and

Odell, we give an example of a basic sequence (v_i) such that $(v_i) \not\sim_U (e_i)$ where (e_i) is the standard basis for c_0 or ℓ_p with $1 \leq p < \infty$. To this end we consider a basis (v_i) for a reflexive Banach space X with the property that ℓ_p is not $U_{(v_i)}$ for any $1 < p < \infty$, but that X is $U_{(v_i)}$ and not U_{c_0}. We will be interested in particular with the dual of the following space.

Definition E.2. Tsirelson's space, T, is the completion of c_{00} under the norm satisfying the implicit relation:

$$||x|| = ||x||_\infty \vee \sup_{n \in \mathbb{N}, (E_i)_1^n \subset [\mathbb{N}]^\omega, n \leq E_1 < ... < E_n} \frac{1}{2} \sum_{i=1}^n ||E_i(x)||.$$

(t_i) is the unit vector basis of T and (t_i^*) are the biorthogonal functionals to (t_i).

Tsirelson constructed the dual of T as the first example of a Banach space which does not contain c_0 or ℓ_p for any $1 \leq p < \infty$ [T]. Though we are more interested in T^* and (t_i^*), we use the implicit definition of T (which was formulated by Figiel and Johnson in [FJ]) as it is nice to work with. The properties of (t_i^*) that will be most useful for us are that (t_i^*) dominates all of its normalized block bases, and has spreading model equivalent to the standard basis for c_0. Though we will not be using this directly, (t_i^*) also has the interesting property of being block stable. Casazza, Johnson, and Tzafriri showed in [CJT] that (t_i^*) has the property that if (x_i) is a normalized block bases of (t_i^*) then (x_i) is equivalent to $(t_{n_i}^*)$ where $n_i \in \text{supp } x_i^*$ for all $i \in \mathbb{N}$. As we have defined T, but wish to know about sequences in T^*, we need the following proposition which relates sequences in a space to sequences in its dual.

Proposition E.3. *If (v_i) and (x_i) are normalized basic sequences, then*

(i) *(v_i) dominates (x_i) if and only if (v_i^*) is dominated by (x_i^*),*

(ii) *If (v_i) is unconditional, then (v_i) dominates all of its normalized block bases if and only if (v_i^*) is dominated by all of its normalized block bases.*

Proof. Without loss of generality we may assume that (v_i) and (x_i) are bimonotone. We assume that (v_i) C-dominates (x_i) and let $(a_i) \in c_{00}$. Because (v_i) is bimonotone, there exists $(b_i) \in c_{00}$ such that $\sum a_i v_i^*(\sum b_i v_i) = || \sum a_i v_i^*||$ and $|| \sum b_i v_i|| = 1$. We have that

$$\left\| \sum a_i v_i^* \right\| = \sum a_i b_i = \sum a_i x_i^* (\sum b_i x_i) \leq C \left\| \sum a_i x_i^* \right\|.$$

Thus (v_i^*) is $C-$dominated by (x_i^*). The converse is true by duality in the sense that we replace the roles of (v_i) and (x_i) by (x_i^*) and (v_i^*) respectively. We have that (x_i^{**}) is equivalent to (x_i) and (v_i^{**}) is equivalent to (v_i) and thus the converse follows and hence (i) is proven.

After possibly renorming, We may assume that (v_i) is 1-unconditional. For the first direction, we assume that (v_i) C-dominates all of its normalized block bases. Let $a_i \in c_{00}$ and (w_i^*) be a normalized block basis of (v_i^*). As (v_i) is bimonotone, there exists normalized block basis (w_i) of (v_i) such that $w_i^*(w_j) = \delta_{ij}$. Let $x \in S_{[v_i]}$ such that $\sum a_i v_i^*(x) = || \sum a_i v_i^*||$. We have now,

$$\left\| \sum a_i v_i^* \right\| = \sum a_i v_i^*(x) = \sum a_i w_i^* \sum v_j^*(x) w_j \leq \left\| \sum a_i w_i^* \right\| \left\| \sum v_j^*(x) w_j \right\|$$
$$\leq C \left\| \sum a_i w_i^* \right\| \left\| \sum v_j^*(x) v_j \right\| = C \left\| \sum a_i w_i^* \right\|.$$

Thus (v_i^*) is C-dominated by (w_i^*), and we have proven the first direction. For the converse, we now assume that (v_i^*) is C-dominated by all of its normalized block bases. Let $(a_i) \in c_{00}$ and (w_i) be a normalized block basis of (v_i). There exists $f \in B_{[v_i]^*}$ such that $f(\sum a_i w_i) = || \sum a_i w_i||$. Choose $(k_n) \in [\mathbb{N}]^\omega$ such that $\mathrm{supp}(w_n) \subset [k_n, k_{n+1})$ for all $n \in \mathbb{N}$. There is a normalized block basis (f_i) of (v_i^*) and $(b_i) \in c_{00}$ such that $f = \sum b_i f_i$ and $\mathrm{supp}(f_n) \subset [k_n, k_{n+1})$ for all $n \in \mathbb{N}$. As (v_i) is 1-unconditional, we may assume that $a_i, b_i, f_i(w_i) \geq 0$. This gives that $\sum a_i b_i f_i(w_i) \leq \sum a_i b_i$, as $f_i(w_i) \leq 1$.

We now have that,

$$\left\|\sum a_i w_i\right\| = (\sum b_i f_i)(\sum a_i w_i) \leq (\sum b_i v_i^*)(\sum a_i v_i) \leq C \left\|\sum a_i v_i\right\|.$$

Hence, (v_i) C-dominates (w_i) and (ii) is proven. $\qquad\square$

We will use Proposition E.3 together with some basic properties of (t_i) to prove the following proposition.

Proposition E.4. $(t_i^*) \nsim_U (e_i)$ *where* (e_i) *is the standard basis for* c_0 *or* ℓ_p *for* $1 \leq p < \infty$.

Proof. It easily follows from the definition that (t_i) is an unconditional normalized basic sequence and that (t_i) is dominated by each of its normalized block bases. Also, the spreading model for (t_i) is isomorphic to the standard ℓ_1 basis. By proposition E.3, (t_i^*) is an unconditional basic sequence that dominates all of its block bases and has its spreading model isomorphic to the standard basis for c_0. T^* is reflexive because (t_i^*) is unconditional and T^* does not contain an isomorphic copy of c_0 or ℓ_1. As (t_i^*) has the standard basis for c_0 as its spreading model, we have that ℓ_p is not $U_{(t_i^*)}$ for all $1 < p < \infty$. Therefore $(t_i^*) \nsim_U \ell_p$ for all $1 \leq p < \infty$. As (t_i^*) dominates all of its normalized block bases and every normalized weakly null sequence in T^* has a subsequence equivalent to a normalized block basis of (t_i^*), we have that T^* is $U_{(t_i^*)}$. T^* does not contain c_0 isomorphically thus T^* is not U_{c_0}. Therefore, $(t_i^*) \nsim_U c_0$. $\quad\square$

We have shown that $(t_i^*) \nsim (e_i)$ where (e_i) is the usual basis for c_0 or ℓ_p for $1 \leq p < \infty$, but we can actually show something much stronger than this. One of the main properties of ℓ_p used in [KO2] is that ℓ_p is subsymmetric. If for each basic sequence (v_i) there existed a constant $C \geq 1$ and a subsymmetric basic sequence (w_i) such that $(v_i) \sim_{CU} (w_i)$ then actually the first half of [KO2] would apply to all basic

sequences without changing anything. The following example shows in particular that this is not true for even the weaker condition of spreading (the property that all subsequences are equivalent).

Proposition E.5. *If (v_i) is a normalized spreading basic sequence, then $(v_i) \not\sim_U (t_i^*)$.*

In general, it can be fairly difficult to check if a Banach space is $U_{(v_i)}$, as every normalized weakly null sequence in the space needs to be checked. In contrast to this, it is very easy to check if T^* is $U_{(v_i)}$. This is because (t_i) is dominated by all of its block bases, and thus by Proposition E.3 T^* is $U_{(v_i)}$ if and only if (v_i) dominates a subsequence of (t_i^*). In proving Proposition E.5 we will carry this idea further by considering a class of spaces, each of which have a subsymmetric basis (e_i) such that (e_i) is dominated by all of its normalized block bases. The additional condition of subsymmetric gives that $[e_i^*]$ is $U_{(v_i)}$ if and only if (v_i) dominates (e_i^*). Hence, we need to check only one sequence instead of all weakly null sequences in $[e_i^*]$.

We consider generalizations of the spaces introduced by Schlumprecht [S] as the first known arbitrarily distortable Banach spaces. We put less restriction on the function f given in the following proposition, but we also infer less about the corresponding Banach space. The techniques used in [S] are used to prove the following proposition.

Proposition E.6. *Let $f : \mathbb{N} \to [1,\infty)$ increase to ∞, $f(1) = 1 < f(2)$, and $\lim_{n\to\infty} n/f(n) = \infty$. If X is defined as the closure of c_{00} under the norm $|| \cdot ||$ which satisfies the implicit relation:*

$$||x|| = ||x||_\infty \vee \sup_{m \geq 2, E_1 < ... < E_m} \frac{1}{f(m)} \sum_{j=1}^m ||E_j(x)|| \quad \text{for all } x \in c_{00},$$

then X is reflexive.

Proof. Let (e_n) denote the standard basis for c_{00}. It is straightforward to show that

the norm $||\cdot||$ as given in the statement of the theorem exists, as well as that (e_n) is a normalized, 1-subsymmetric and 1-unconditional basis for X. Furthermore, (e_n) is 1-dominated by all of its normalized block bases. We will prove that X is reflexive by showing that (e_n) is boundedly complete and shrinking.

We first prove that (e_n) is boundedly complete. As (e_n) is unconditional, if (e_n) is not boundedly complete then it has some normalized block basis which is equivalent to the standard c_0 basis. However, (e_n) is 1-dominated by all its normalized block bases, so (e_n) is also equivalent to the standard c_0 basis. Hence $\sup_{N \in \mathbb{N}} || \sum_{n=1}^{N} e_n || < \infty$. This contradicts that $|| \sum_{n=1}^{N} e_n || \geq N/f(N) \to \infty$. Thus (e_n) is boundedly complete.

We now assume that (e_n) is not shrinking. As (e_n) is unconditional, it has a normalized block basis (x_n) which is equivalent to the standard basis for ℓ_1. We will use James' Blocking Lemma [Ja] to show that this leads to a contradiction. In one of its more basic forms, James' blocking lemma states that if (x_n) is equivalent to the standard basis for ℓ_1 and $\epsilon > 0$ then (x_n) has a normalized block basis which is $(1 + \epsilon)$-equivalent to the standard basis for ℓ_1. Let $0 < \epsilon < \frac{1}{2}(f(2) - 1)$. By passing to a normalized block basis using James' blocking lemma, we may assume that (x_n) is $(1+\epsilon)$-equivalent to the standard basis for ℓ_1, and thus any normalized block basis of (x_n) will also be $(1 + \epsilon)$-equivalent to the standard basis for ℓ_1. Let $\epsilon_n > 0$ such that $\sum_{n=1}^{\infty} \epsilon_n < \epsilon$.

We denote $||\cdot||_m$ to be the norm on X which satisfies:

$$||x||_m = \sup_{E_1 < ... < E_m} \frac{1}{f(m)} \sum_{j=1}^{m} ||E_j(x)|| \quad \text{for all } x \in c_{00}.$$

We will construct by induction on $n \in \mathbb{N}$ a normalized block basis (y_i) of (x_i) such that for all $m \in \mathbb{N}$ we have:

$$\text{If } ||y_j||_m > \epsilon_j \text{ for some } 1 \leq j < n, \text{ then } ||y_n||_m < \frac{1 + \epsilon_n}{f(m)}. \tag{2.23}$$

For $n = 1$ we let $y_1 = x_1$, and note that (2.23) is vacuously satisfied.

We now assume that we are given $n \geq 1$ and finite block sequence $(y_i)_{i=1}^n$ of (x_i) which satisfies (2.23). We have $\lim_{m \to \infty} ||y_i||_m \leq \lim_{m \to \infty} \frac{\#\text{supp}(y_i)}{f(m)} = 0$ (where $\text{supp}(y_i)$ denotes the support of y_i). Thus, there exists $N > \text{supp}(y_n)$ such that $||y_i||_m < \epsilon_i$ for all $1 \leq i \leq n$ and all $m \geq N$. Using James' blocking lemma, we block $(x_i)_{i=N}^\infty$ into $(z_i)_{i=1}^\infty$ such that $(z_i)_{i=1}^\infty$ is $(1 + \epsilon_{n+1}/3)-$equivalent to the standard ℓ_1 basis. Let $M \geq 6N/\epsilon_{n+1}$ and define $y_{n+1} = \frac{1}{||\sum_{i=1}^M z_i||} \sum_{i=1}^M z_i$. Let $m \in \mathbb{N}$ such that $||y_j||_m > \epsilon_j$ for some $1 \leq j \leq n$. By our choice of $N \in \mathbb{N}$, we have that $m < N$. There exists disjoint intervals $E_1 < \ldots < E_m$ in $\mathbb{N}$ and integers $1 = k_0 \leq k_1 \leq \ldots \leq k_m$ such that:

$$f(m)||y_{n+1}||_m = \frac{1}{||\sum_1^M z_i||} \sum_{i=1}^m \left\| E_i \sum_{j=k_{i-1}}^{k_i} z_j \right\|$$

$$\leq \frac{1 + \epsilon_{n+1}/3}{M} \sum_{i=1}^m \left(||E_i z_{k_{i-1}}|| + ||\sum_{j=k_{i-1}+1}^{k_i-1} z_j|| + ||E_i z_{k_i}|| \right)$$

$$\leq \frac{1 + \epsilon_{n+1}/3}{M} (M + 2m) < (1 + \epsilon_{n+1}/3) (1 + 2N/M)$$

$$\leq (1 + \epsilon_{n+1}/3) (1 + \epsilon_{n+1}/3) < 1 + \epsilon_{n+1}.$$

Hence, the induction hypothesis is satisfied.

We now show that property (2.23) leads to a contradiction with (y_i) being $(1 + \epsilon)-$equivalent to the standard ℓ_1 basis. Let $n \in \mathbb{N}$. We have for some $m \geq 2$ that $||\sum_{i=1}^n \frac{y_i}{n}|| = ||\sum_{i=1}^n \frac{y_i}{n}||_m$. By (2.23) there exists $1 \leq j \leq n+1$ such that $||y_i||_m < \epsilon_i$

for all $1 \le i < j$ and $f(m)||y_i||_m < 1 + \epsilon_i$ for all $j < i \le n$. We have that:

$$||\sum_{i=1}^{n} \frac{y_i}{n}|| = ||\sum_{i=1}^{n} \frac{y_i}{n}||_m \le \frac{1}{n}\sum_{i=1}^{j-1}||y_i||_m + \frac{1}{n}||y_j||_m + \frac{1}{n}\sum_{i=j+1}^{n}||y_i||_m$$

$$< \frac{1}{n}\sum_{i=1}^{j-1}\epsilon_i + \frac{1}{n} + \frac{1}{nf(m)}\sum_{i=j+1}^{n}1 + \epsilon_i$$

$$< \frac{\epsilon}{n} + \frac{1}{n} + \frac{1}{f(2)} + \frac{\epsilon}{nf(2)}$$

$$< \frac{\epsilon}{n} + \frac{1}{n} + \frac{1}{1+2\epsilon} + \frac{\epsilon}{n(1+2\epsilon))}.$$

Thus $\inf_{n\in\mathbb{N}}||\sum_{i=1}^{n}\frac{y_i}{n}|| < \frac{1}{1+2\epsilon}$. This contradicts that (y_i) is $(1 + \epsilon)$ equivalent to the standard ℓ_1 basis. Hence (e_i) is shrinking, and X is reflexive. $\square$

Using the reflexive spaces presented in Proposition E.6, we can prove the following lemma. Proposition E.5 will then follow easily.

Lemma E.7. *If (v_i) is a 1-suppression unconditional normalized basic sequence such that (v_{k_i}) dominates (v_i) for all $(k_i) \in [\mathbb{N}]^\omega$ and (v_i) is not equivalent to the unit vector basis for c_0 then there exists a reflexive Banach space X which is $U_{(v_i)}$ and not $U_{(t_i^*)}$.*

Proof. There exists $K \ge 1$ such that (v_{k_i}) K-dominates (v_i) for all $(k_i) \in [\mathbb{N}]^\omega$. We define $\langle \cdot \rangle$ to be the norm on (v_i) determined by:

$$\left\langle \sum_{i\in\mathbb{N}} a_i v_i^* \right\rangle = \sup_{(k_i)\in[\mathbb{N}]^\omega} \left\|\sum_{i\in\mathbb{N}} a_i v_{k_i}^*\right\| \qquad \text{for all } (a_i) \in c_{00}.$$

Where (v_i^*) is the sequence of biorthogonal functionals to (v_i). The norm $\langle \cdot \rangle$ is K-equivalent to the original norm $||\cdot||$. Furthermore, under the new norm (v_{k_i}) 1-dominates (v_i) for all $(k_i) \in [\mathbb{N}]^\omega$. Thus after possibly renorming, we may assume that $K=1$.

Let $\epsilon > 0$ and $\epsilon_i \searrow 0$ such that $\prod \frac{1}{1-\epsilon_i} < 1+\epsilon$. We have that (v_i) is unconditional and is not equivalent to the unit vector basis of c_0, so there exists $(N_k) \in [\mathbb{N}]^\omega$ such that for all $k \in \mathbb{N}$ we have $N_k \geq k^2$ and

$$\left\| \sum_{i=1}^{N_k} v_i \right\| > \frac{k+1}{\epsilon_{k+1}}. \tag{2.24}$$

We define the function $f : \mathbb{N} \to [1, \infty)$ by:

$$f(n) = \begin{cases} 1 & \text{if } n = 1, \\ \frac{1}{1-\epsilon_1} & \text{if } 1 < n \leq N_1, \\ k+1 & \text{if } N_k < n \leq N_{k+1} \text{ for } k \in \mathbb{N}. \end{cases}$$

We denote $||| \cdot |||$ to be the norm on c_{00} determined by the following implicit relation:

$$|||x||| = ||x||_\infty \vee \sup_{m \geq 2, E_1 < \ldots < E_m} \frac{1}{f(m)} \sum_{j=1}^m |||E_j(x)||| \quad \text{for all } x \in c_{00}.$$

The completion of c_{00} under the norm $||| \cdot |||$ is denoted by X, and its standard basis is denoted by (e_i). We have that $N_k > k^2$ which implies that $\lim_{k \to \infty} k/f(k) = \infty$ and hence X is reflexive by proposition E.5.

We now show by induction on $k \in \mathbb{N}$ that if $(a_i)_{i=1}^{N_k} \in c_{00}$ then

$$\left(\prod_{i=1}^k \frac{1}{1-\epsilon_i} \right) ||| \sum_{i=1}^{N_k} a_i e_i ||| \geq || \sum_{i=1}^{N_k} a_i v_i^* ||. \tag{2.25}$$

For k=1, we have that $\frac{1}{1-\epsilon_1} ||| \sum_{i=1}^{N_1} a_i e_i ||| \geq \sum_{i=1}^{N_1} |a_i| \geq || \sum_{i=1}^{N_1} a_i v_i^* ||$. Thus (2.25) is satisfied. Now we assume that $k \in \mathbb{N}$ and that (2.25) holds for k.

Let $(a_i)_{i=1}^{N_{k+1}} \subset \mathbb{R}$ such that $|| \sum_{i=1}^{N_{k+1}} a_i v_i^* || = 1$. There exists $(\beta_i)_{1=1}^{N_{k+1}} \subset \mathbb{R}$ such that $\sum \beta_i a_i = || \sum \beta_i v_i || = 1$. Let $I = \{ j \in \mathbb{N} : \quad |\beta_j| < \frac{\epsilon_{k+1}}{k+1} \}$. If $\sum_{i \in I} |a_i| \geq k+1$ then $||| \sum_{i=1}^{N_{k+1}} a_i e_i ||| \geq \frac{1}{k+1} \sum_{i \in I} |a_i| \geq 1 = || \sum a_i v_i^* ||$ and we are done. Therefore we assume that $\sum_{i \in I} |a_i| < k+1$, and thus $\sum_{i \in I} \beta_i a_i \leq \sum_{i \in I} \frac{\epsilon_{k+1}}{k+1} |a_i| < \epsilon_{k+1}$. We let

$\{j_i\}_{i=1}^{\sharp I^C} = I^C$, and claim that $\sharp I^C \leq N_k$. Indeed, if we assume to the contrary that $\sharp I^C > N_k$, then

$$1 \geq ||\sum_{i=1}^{\sharp I^C} \beta_{j_i} v_{j_i}|| \geq ||\sum_{i=1}^{\sharp I^C} \beta_{j_i} v_i|| \geq \frac{\epsilon_{k+1}}{k+1} ||\sum_{i=1}^{N_k} v_i|| > \frac{\epsilon_{k+1}}{k+1} \frac{k+1}{\epsilon_{k+1}} = 1.$$

The first inequality is due to (v_i) being 1-suppression unconditional, and the second inequality is due to (v_i) being 1-dominated by (v_{j_i}). Thus we have a contradiction and our claim that $\sharp I^C \leq N_k$ is proven. We now have that

$$1 = \sum \beta_i a_i = \sum_I \beta_i a_i + \sum_{I^C} \beta_i a_i$$

$$< \epsilon_{k+1} + ||\sum_{i=1}^{\sharp I^C} a_{j_i} v_{j_i}^*||$$

$$\leq \epsilon_{k+1} + ||\sum_{i=1}^{\sharp I^C} a_{j_i} v_i^*||$$

$$\leq \epsilon_{k+1} + (\prod_{i=1}^{k} \frac{1}{1-\epsilon_i}) |||\sum_{i=1}^{\sharp I^C} a_{j_i} e_i||| \quad \text{by induction hypothesis}$$

$$\leq \epsilon_{k+1} + (\prod_{i=1}^{k} \frac{1}{1-\epsilon_i}) |||\sum_{i=1}^{N_{k+1}} a_i e_i||| \quad \text{by 1-subsymetric.}$$

The last inequality gives that $1 \leq (\prod_{i=1}^{k+1} \frac{1}{1-\epsilon_i}) |||\sum_{i=1}^{N_{k+1}} a_i e_i|||$. Thus the induction hypothesis is satisfied.

We have that (e_i) dominates (v_i^*), and hence (v_i) dominates (e_i^*). (e_i^*) is subsymmetric and dominates all its block bases, so $[e_i^*]$ is $U_{(v_i)}$. (e_i^*) is weakly null and is not equivalent to the unit vector basis of c_0, so $[e_i^*]$ is not $U_{(t_i^*)}$. $\square$

The proof of Proposition E.5 now follows easily.

Proof of Proposition E.5. Let (v_i) be a normalized C-spreading basic sequence. Because (v_i) is spreading, Rosenthal's ℓ_1 theorem gives that (v_i) must be either equivalent to the standard basis for ℓ_1 or be weakly Cauchy. In the first case, it is obvious that $(v_i) \nsim_U (t_i^*)$ as every Banach space is U_{ℓ_1}. Thus we assume that (v_i) is weakly Cauchy. The difference sequence defined by $(w_i) = (v_{2i-1} - v_{2i})$ is weakly null. (w_i) is weakly null and spreading, and is thus unconditional. We have for all $(a_i) \in c_{00}$ that

$$\left\| \sum a_i w_i \right\| \leq \left\| \sum a_i v_{2i-1} \right\| + \left\| \sum a_i v_{2i} \right\| \leq 2C \left\| \sum a_i v_i \right\|$$

Thus, (v_i) dominates (w_i). If (w_i) is not equivalent to the standard basis for c_0 then by Lemma E.7, there exists a Banach space which is $U_{(w_i)}$ and hence $U_{(v_i)}$, but is not $U_{(t_i^*)}$. If (w_i) is equivalent to the standard basis for c_0 then

$$\sup_n \left\| \sum_{i=1}^n (-1)^{n-1} v_i \right\| = \sup_n \left\| \sum_{i=1}^n w_i \right\| < \infty.$$

However, $\sup_n || \sum_{i=1}^n (-1)^n t_{k_i}^* || = \infty$ for all $(k_i) \in [\mathbb{N}]^\omega$. Thus T^* is not $U_{(v_i)}$, and $(v_i) \nsim_U (t_i^*)$. $\qquad\square$

We also considered the question: "Does there exist a basic sequence (v_i) such that $(v_i) \nsim_U (w_i)$ for any unconditional (w_i)?". This is a much harder question, which is currently open. Neither the summing basis for c_0, nor the standard basis for James' space give a solution, as these are covered by the following proposition:

Proposition E.8. *If (v_i) is a basic sequence such that $\sup_{n \in \mathbb{N}} || \sum_{i=1}^n \epsilon_i v_i || < D$ for some $(\epsilon_i) \in \{-1, 1\}^\mathbb{N}$ and constant $D < \infty$ then $(v_i) \sim_U c_0$.*

Proof. Let X be a C-U_V Banach space, and let $(x_i) \in S_X$ be weakly null. By Ramsey's theorem, we may assume by passing to a subsequence that (v_i) C-dominates every subsequence of (x_i). By a theorem of John Elton [E], there exists $K < \infty$ and a subsequence (y_i) of (x_i) such that if $(a_i)_1^\infty \in [-1, 1]^\mathbb{N}$ and $I \subset \{i : \ |a_i| = 1\}$ is finite

then $||\sum_I a_i y_i|| \le K \sup_{n\in\mathbb{N}} ||\sum_{i=1}^n \epsilon_i y_i||$. Thus we have for all $A \in [\mathbb{N}]^{<\omega}$ that

$$||\sum_{i\in A} \epsilon_i y_i|| \le K \sup_{n\in\mathbb{N}} ||\sum_{i=1}^n \epsilon_i y_i|| \le KC \sup_{n\in\mathbb{N}} ||\sum_{i=1}^n \epsilon_i v_i|| < KCD.$$

As this is true for all $A \in [N]^{<\omega}$, (y_i) is equivalent to the unit vector basis of c_0. Every normalized weakly null sequence in X has a subsequence equivalent to c_0, so X is U_{c_0}. $\qquad\square$

CHAPTER III

SUBSEQUENTIAL UPPER ESTIMATES

A. Introduction

This chapter contains the author's joint work with Edward Odell, Thomas Schlump-recht, and Andras Zsák [FOSZ]. The added structure and rich theory of coordinate systems can be of significant help when studying Banach spaces. Because of this, it is often the case that Banach spaces are studied in the context of being a subspace or quotient of some space with a coordinate system. Two early results in this area are that every separable Banach space is the quotient of ℓ_1 and also every separable Banach space may be embedded as a subspace of $C[0,1]$. Both ℓ_1 and $C[0,1]$ have bases, and so we have in particular that every separable Banach space is a quotient of a Banach space with a basis and may also be embedded as a subspace of a Banach space with a basis. However, it is often that one has a Banach space with a particular property, and one wishes that the coordinate system has some associated property.The first important step in this direction was made by Davis, Figiel, Johnson and Pełczyński [DFJP] who proved that if a reflexive space embeds into a space with a shrinking basis then it embeds into a space with a shrinking and boundedly complete basis. Later, Zippin [Z] proved the following two major results: every separable reflexive Banach space may be embedded as a subspace of a space with shrinking and boundedly complete basis and every Banach space with separable dual may be embedded in a Banach space with a shrinking basis. Further results in this area give intrinsic characterizations on when a space may be embedded as a subspace of a reflexive space with unconditional basis [JZh], or reflexive space with an asymptotic ℓ_p FDD [OSZ]. These are only a portion of the recent results in this area. These new

characterizations are all based on the relatively recent tools of weakly null trees. One important result in particular for us is a characterization of subspaces of reflexive spaces with an FDD satisfying subsequential V upper block estimates and subsequential U lower block estimates where V is an unconditional, block stable, and right dominant basic sequence and U is an uncondition, block stable, and left dominant basic sequence [OSZ2]. This characterization when applied to Tzirelson's spaces was shown to have strong applications to the Szlenk index of reflexive spaces [OSZ3]. Our main result adds to this theory with the following theorem which extends the results in [OSZ2] and [OSZ3] to spaces with separable dual. The notions and concepts used, will be introduced in the next section.

Theorem A.1. *Let X^* be separable and $V = (v_i)$ be an unconditional, block stable, and right dominant basic sequence. Then the following are equivalent.*

1) *X has subsequential V upper tree estimates.*

2) *X is a quotient of a space Z with Z^* separable and Z has subsequential V upper tree estimates.*

3) *X is a quotient of a space Z with a shrinking FDD satisfying subsequential V upper block estimates.*

4) *There exists $w^* - w^*$ continuous embedding of X^* into Z^*, a space with boundedly complete FDD (F_i^*) (so $Z = \oplus F_i$ defines Z^*) satisfying subsequential V^* lower block estimates.*

5) *X is isomorphic to a subspace of a space Z with a shrinking FDD satisfying subsequential V upper block estimates.*

Using our characterization, we are able to achieve the following universality result:

Theorem A.2. *Let $V = (v_i)$ be an unconditional, shrinking, block stable, and right dominant basic sequence. There is a Banach space Z with an FDD (F_i) satisfying subsequential V upper block estimates such that if a Banach space X has subsequential V upper tree estimates, then X embeds into Z.*

We will apply Theorems 1 and 2 for the case that V is the canonical basic sequence for Tsirelson's space of order α, which will allow us to prove some new results for the Szlenk index. As shown in [OSZ3], the Szlenk index is closely related to a space having subsequential $T_{\alpha,c}$ upper tree estimates for some $0 < c < 1$. In particular, for each $\alpha < \omega_1$ a Banach space X with separable dual has Szlenk index at most $\omega^{\alpha\omega}$ if and only if X satisfies subsequential $T_{\alpha,c}$ upper tree estimates for some $c \in (0,1)$. Our characterization allows us to add a further equivalence, giving the following theorem.

Theorem A.3. *Let $\alpha < \omega_1$. For a space X with separable dual, the following are equivalent:*

  (i) X has Szlenk index at most $\omega^{\alpha\omega}$.

 (ii) X satisfies subsequential $T_{\alpha,c}$ upper tree estimates for some $c \in (0,1)$.

(iii) X embeds into a space Z with an FDD (E_i) which satisfies subsequential $T_{\alpha,c}$ upper block estimates in Z for some $c \in (0,1)$.

We are able to combine the previous two theorems using ideas in [OSZ3] to prove the following universality result.

Theorem A.4. *For each $\alpha < \omega_1$ there exists a Banach space Z with a shrinking FDD and Szlenk index at most $\omega^{\alpha\omega+1}$ such that Z is universal for the collection of spaces with separable dual and Szlenk index at most $\omega^{\alpha\omega}$.*

In particular, the universal space Z will be of the form $(\sum_{n \in \mathbb{N}} X_n)_{\ell_2}$, where X_n has an FDD satisfying subsequential $T_{\alpha, \frac{n}{n+1}}$ upper block estimates and X_n is universal for all Banach spaces with separable dual which satisfy subsequential $T_{\alpha, \frac{n}{n+1}}$ upper tree estimates.

Theorem A.4 represents a quantitative version of a result first shown by Dodos and Ferenczi [DF], which states that for every $\alpha < \omega_1$ there is a Banach space with separable dual which is universal for all separable Banach spaces whose Szlenk index does not exceed α. While the proofs in [DF] use methods of descriptive set theory developed by Bossard, our proofs will rely on the concepts like *infinite asymptotic games*, trees and branches as introduced in [OS1] and [OS2].

B.   Definitions and Lemmas

Our main result characterizes subspaces and quotients of spaces having an FDD with subsequential V upper block estimate, where V is an unconditional, right dominant, and block stable basic sequence. The case when $V = T_{\alpha,c}$ is Tsirelson's space is intimately related to the Szlenk index (see Proposition B.5) , and has become an important property in the fertile area between descriptive set theory and the classification of Banach spaces [OSZ3]. For basic notions like (shrinking and boundedly complete) FDDs and their projection constants and blockings we refer to [OSZ3]. If Z is a Banach spaces with an FDD $E = (E_i)$, we denote by $c_{00}(\oplus E_i)$ the dense linear subspace of Z spanned by (E_i) and its closure by $[E_i] = [E_i]_Z$. We denote the closure of $c_{00}(\oplus E_i^*)$ inside Z^* by $Z^{(*)}$. If (E_i), is shrinking it follows that $Z^{(*)} = Z^*$ and if (E_i) is boundedly complete, then $Z^{(*)}$ is the predual of Z. If $A \subset \mathbb{N}$, is finite, or cofinite, we denote the natural projection onto the closed span of $(E_i : i \in A)$ by P_A^E,

i.e.

$$P_A^E : Z \to Z, \quad P\Big(\sum_{i=1}^{\infty} x_i\Big) = \sum_{i \in A} x_i, \text{ whenever } x_i \in E_i, \text{ for } i \in \mathbb{N} \text{ so that } \sum_{i=1}^{\infty} x_i \in Z.$$

For $\alpha < \omega_1$ and $c \in (0,1)$, the definition of the Tsirelson space $T_{(\alpha,c)}$ and the relevant properties of $T_{(\alpha,c)}$ for us can also be found in [OSZ2]. Let us also recall the following notion from [OSZ2].

Definition B.1. Let Z be a Banach space with an FDD (E_n), let $V = (v_i)$ be a normalized 1-unconditional basis, and let $1 \leq C < \infty$. We say that (E_n) *satisfies subsequential C-V-upper block estimates* if every normalized block sequence (z_i) of (E_n) in Z is C-dominated by (v_{m_i}), where $m_i = \min \operatorname{supp}_E(z_i)$ for all $i \in \mathbb{N}$. We say that (E_n) *satisfies subsequential C-V-lower block estimates* if every normalized block sequence (z_i) of (E_n) in Z C-dominates (v_{m_i}), where $m_i = \min \operatorname{supp}_E(z_i)$ for all $i \in \mathbb{N}$. We say that (E_n) *satisfies subsequential V-upper (or lower) block estimates* if it satisfies subsequential C-V-upper (or lower) block estimates for some $1 \leq C < \infty$.

Subsequential V-upper block estimates and subsequential V-lower block estimates are dual properties, as shown in the following proposition from [OSZ2].

Proposition B.2. [OSZ2, Proposition 3] *Assume that Z has an FDD (E_i), and let $V = (v_i)$ be a 1-unconditional normalized basic sequence with biorthogonal functionals $V^* = (v_i^*)$. The following statements are equivalent:*

(a) (E_i) satisfies subsequential V-upper block estimates in Z.

(b) (E_i^*) satisfies subsequential V^*-lower block estimates in $Z^{(*)}$.

Moreover, if (E_i) is bimonotone in Z, then the equivalence holds true if one replaces,

for some $C \geq 1$, V-upper estimates by C-V-upper estimates in (a) and V^*-lower block estimates by C-V^*-lower block estimates in (b).

It is important to note that if a Banach space Z has an FDD (E_n) which satisfies subsequential V-upper block estimates where $V = (v_i)$ is weakly null, then (E_n) is shrinking. Indeed, any normalized block sequence of (E_n) is dominated by a weakly null sequence, and is thus weakly null. Thus if V is weakly null, a necessary condition for a Banach space X to be isomorphic to a quotient or subspace of a Banach space with an FDD satisfying subsequential V-upper block estimates is that X have separable dual. This is important as spaces with separable dual may be analyzed using *weakly null trees* . In this paper we will need in particular *weakly null even trees* (see [OSZ3]).

In order to index weakly null even trees, we denote

$$T^{\text{even}}_\infty = \{(n_1, n_2, ..., n_{2\ell}) : \ n_1 < n_2 < ... < n_{2\ell} \text{ are in } \mathbb{N} \text{ and } \ell \in \mathbb{N}\}.$$

Definition B.3. If X is a Banach space, an indexed family $(x_\alpha)_{\alpha \in T^{\text{even}}_\infty} \subset X$ is called an *even tree*. Sequences of the form $(x_{n_1,...,n_{2\ell-1},k})^\infty_{k=n_{2\ell-1}+1}$ are called nodes. Sequences of the form $(n_{2\ell-1}, x_{n_1,...,n_{2\ell}})^\infty_{\ell=1}$ are called branches. A normalized tree is called *weakly null* if every node is a weakly null sequence. In case that X has an FDD (E_i), we say that a normalized tree is a *block tree* (with respect to (E_i)) if every node is a block sequence with respect to (E_i).

Note that if (E_i) is shrinking, every weakly null tree can be *refined* (passing to subsequences of the nodes) to a tree which is a perturbation of a block tree.

If Z is a Banach space with an FDD (E_n), and X is a closed subspace of Z then any weakly null even tree has a branch equivalent to a block basis of (E_n). Thus if (E_n) satisfies subsequential V-upper block estimates then every weakly null even

tree in X has a branch dominated by a subsequence of V. We can carry this even further if V is *right dominant*: we say that (v_i) is *C-right-dominant* (respectively, *C-left-dominant*) if for all sequences $m_1 < m_2 < \ldots$ and $n_1 < n_2 < \ldots$ of positive integers with $m_i \leq n_i$ for all $i \in \mathbb{N}$ we have that (v_{m_i}) is C-dominated by (respectively, C-dominates) (v_{n_i}). We say that (v_i) is *right-dominant* or *left-dominant* if for some $C \geq 1$ it is C-right-dominant or C-left-dominant, respectively.

If (v_i) is right dominant and X is a subspace of a space with an FDD satisfying subsequential V-upper block estimates, then every weakly null even tree $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ has a branch $(n_{2\ell-1}, x_{n_1, \ldots, n_{2\ell}})_{\ell=1}^\infty$ such that $(x_{n_1, \ldots, n_{2\ell}})_{\ell=1}^\infty$ is dominated by $(v_{n_{2\ell-1}})_{\ell=1}^\infty$. We make this into a coordinate free condition with the following definition.

Definition B.4. Let X be a Banach space, $V = (v_i)$ be a normalized 1-unconditional basis, and $1 \leq C < \infty$. We say that X *satisfies subsequential C-V-upper tree estimates* if every weakly null even tree $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in X has a branch $(n_{2\ell-1}, x_{n_1, \ldots, n_{2\ell}})_{\ell=1}^\infty$ such that $(x_{n_1, \ldots, n_{2\ell}})_{\ell=1}^\infty$ is C-dominated by $(v_{n_{2\ell-1}})_{\ell=1}^\infty$.

We say that X *satisfies subsequential V-upper tree estimates* if it satisfies subsequential C-V-upper tree estimates for some $1 \leq C < \infty$.

If X is a subspace of a dual space, we say that X *satisfies subsequential C-V-lower w^* tree estimates* if every w^* null even tree $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in X has a branch $(n_{2\ell-1}, x_{n_1, \ldots, n_{2\ell}})_{\ell=1}^\infty$ such that $(x_{n_1, \ldots, n_{2\ell}})_{\ell=1}^\infty$ C-dominates $(v_{n_{2\ell-1}})_{\ell=1}^\infty$.

A comprehensive survey on the Szlenk index can be found in [L]. We will need the following description of the Szlenk index using Tsirelson spaces. It can be deduced from results in [OSZ3]: the implication (i)$\Rightarrow$(ii) follows from [OSZ3, Corollary 19] in the same way as the implication (i)$\Rightarrow$(ii) of [OSZ3, Theorem 21] (the assumed reflexivity is irrelevant for that part). The implication (ii)$\Rightarrow$(i) follows from the computation of the Szlenk index of $T_{\alpha,c}$ in [OSZ3, Proposition 16] as well as the

description of the Szlenk index provided in [AJO](see also [OSZ3, Theorem 12]).

We denote the Szlenk index of a separable Banach space by Sz(X).

Proposition B.5. *For a Banach space X with separable dual and an ordinal $\alpha < \omega_1$ the following conditions are equivalent:*

(i) $\mathrm{Sz}(X) \leq \omega^{\alpha\omega}$.

(ii) There is a $0 < c < 1$ so that X satisfies subsequential $(t_i^{(\alpha,c)})$ upper tree estimates, where $(t_i^{(\alpha,c)})$ denotes the canonical basis of $T_{\alpha,c}$.

We have a property of trees and a property of FDDs, and our goal is to show how they are related. Zippin's theorem allows us to embed a Banach space with separable dual into a space with shrinking FDD. Our next step will be to then pass information about trees in the space to information about $\bar{\delta}$-skipped blocks of the FDD, which we define here.

Definition B.6. Let $\mathbf{E} = (E_i)$ be an FDD for a Banach space Y and let $\bar{\delta} = (\delta_i)$ with $\delta_i \downarrow 0$. A sequence $(y_i) \subset S_Y$ is called a $\bar{\delta} - skipped\ block$ w.r.t. (E_i) if there exists integers $1 = k_0 < k_1 < ...$ so that for all $i \in \mathbb{N}$,

$$\|P^E_{(k_{i-1},k_i)}y_i - y_i\| < \delta_i.$$

The following proposition is an adaptation of Proposition 5 in [OSZ2] for the case (E_i) is shrinking, but not necessarily boundedly complete and for the case where X is a w^*-closed subspace of a dual space. We will need to first recall some notation introduced in [OSZ2].

Given a Banach space X, we let $(\mathbb{N} \times S_X)^\omega$ denote the set of all sequences (k_i, x_i), where $k_1 < k_2 < \dots$ are positive integers, and (x_i) is a sequence in S_X. We equip the set $(\mathbb{N} \times S_X)^\omega$ with the product topology of the discrete topologies of $\mathbb{N}$ and S_X.

Given $A \subset (\mathbb{N} \times S_X)^\omega$ and $\varepsilon > 0$, we let

$$A_\varepsilon = \left\{ (\ell_i, y_i)_{i \in \mathbb{N}} \in (\mathbb{N} \times S_X)^\omega : \exists\, (k_i, x_i)_{i \in \mathbb{N}} \in A \quad k_i \leq \ell_i\,, \ \|x_i - y_i\| < \varepsilon \cdot 2^{-i} \ \forall\, i \in \mathbb{N} \right\},$$

and we let $\overline{A}$ be the closure of A in $(\mathbb{N} \times S_X)^\omega$.

Given $A \subset (\mathbb{N} \times S_X)^\omega$, we say that an even tree $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in X *has a branch in* A if there exist $n_1 < n_2 < \dots$ in $\mathbb{N}$ such that $\big((n_{2i-1}, x_{(n_1, n_2, \dots, n_{2i})}) : i \in \mathbb{N}\big) \in A$.

Let Z be a Banach space with an FDD (E_i) and assume that Z contains X. For each $m \in \mathbb{N}$ we set $Z_m = \overline{\bigoplus_{i > m} E_i}$. Given $\varepsilon > 0$, we consider the following game between players S (subspace chooser) and P (point chooser). The game has an infinite sequence of moves; on the nth move $(n \in \mathbb{N})$ S picks $k_n, m_n \in \mathbb{N}$ and P responds by picking $x_n \in S_X$ with $d(x_n, Z_{m_n}) < \varepsilon' \cdot 2^{-n}$, where $\varepsilon' = \min\{\varepsilon, 1\}$. S wins the game if the sequence (k_i, x_i) the players generate ends up in $\overline{A_{5\varepsilon}}$, otherwise P is declared the winner. We will refer to this as the (A, ε)-game. More about this game and its connection to trees can be found in [OSZ2].

Proposition B.7. *Let X be an infinite-dimensional closed subspace of a space Z with an FDD (E_i). Let $A \subset (\mathbb{N} \times S_X)^\omega$. If (E_i) is shrinking, or if Z is a dual space with $X \subset Z$ w^* closed then the following are equivalent.*

(a) *For all $\varepsilon > 0$ there exists $(K_i) \subset \mathbb{N}$ with $K_1 < K_2 < ...$, $\overline{\delta} = (\delta_i) \subset (0, 1)$ with $\delta_i \searrow 0$, and a blocking $F = (F_i)$ of (E_i) such that if $(x_i) \subset S_X$ is a $\overline{\delta}$-skipped block sequence of (F_n) in Z with $\|x_i - P^F_{(r_{i-1}, r_i)} x_i\| < \delta_i$ for all $i \in \mathbb{N}$, where $1 \leq r_0 < r_1 < r_2 < ...$, then $(K_{r_{i-1}}, x_i) \in \overline{A_\varepsilon}$.*

(b) *For all $\varepsilon > 0$ S has a winning strategy for the (A, ε)-game.*

If (E_i) is shrinking, then (a) and (b) are equivalent to

(c) *for all $\varepsilon > 0$ every normalized, weakly null even tree in X has a branch in $\overline{A_\varepsilon}$.*

If Z is a dual space and $X \subset Z$ is w^* closed, then (a) and (b) are equivalent to

(d) for all $\varepsilon > 0$ every normalized, w^* null even tree in X has a branch in $\overline{A_\varepsilon}$.

Proof. The proofs of the implications $(b) \Rightarrow (a) \Rightarrow (d) \Rightarrow (b)$ shown in the reflexive case in Proposition 5 in [OSZ2] still hold in the nonreflexive case when Z is a dual space and $X \subset Z$ is w^* closed. The proof still works, as in this case B_X is w^* compact and the proof in [OSZ2] only relies on reflexivity when it uses that the unit ball of a reflexive space is weakly compact.

For the case in which (E_i) is shrinking, the proofs of the implications $(b) \Rightarrow (a) \Rightarrow (c)$ shown for the reflexive case still work. The proof for the implication $(c) \Rightarrow (b)$ requires some adaptation which we provide here.

Let $Z_m = \bigoplus_{i>m} E_i$. Assume that S does not have a winning strategy for the (A, ε') game for some $1 > \varepsilon' > 0$. We let $\varepsilon = \frac{1}{5(1+C)}\varepsilon'$ where $C = \sup_{n \in \mathbb{N}} ||P^E_{[1,n]}||$. As this game is determined [M] , there exists a winning strategy ϕ for the point chooser. The function ϕ takes values in S_X: if $(k_i), (m_i) \in [\mathbb{N}]^\omega$ are the choices by player S and if $z_n = \phi(k_1, m_1, ..., k_n, m_n)$ for all $n \in \mathbb{N}$, then $d(z_i, Z_{m_i}) < \varepsilon' 2^{-i}$ for all $i \in \mathbb{N}$ and $(k_i, z_i) \notin \overline{A_{5\varepsilon'}}$. We will define a new winning strategy for P based on ϕ. For each $n \in \mathbb{N}$ and sequence $((k_1, m_1, ..., k_n, k_n + i) : i \in \mathbb{N}) \subset T^{even}_\infty$ we may pass to a subsequence $((k_1, m_1, ..., k_n, \ell_i) : i \in \mathbb{N})$ such that for $x_i = \phi(k_1, m_1, ..., k_n, \ell_i)$ we have that (x_i) is weakly Cauchy and

$$||P^E_{[1,n]}(x_i - x_j)|| < \varepsilon 2^{-j} \quad \forall n \in \mathbb{N} \text{ and } i > j \geq n. \tag{3.1}$$

We define the new winning strategy ψ for P by $\psi(k_1, m_1, ..., k_n, i) = \phi(k_1, m_1, ..., k_n, \ell_i)$. We use ψ to create a normalized even tree $(x_\alpha)_{\alpha \in T^{even}_\infty}$ by setting

$$x_{(k_1, m_1, ..., k_\ell, m_\ell)} = \psi(k_1, m_1, ..., k_\ell, m_\ell).$$

We have that $d(x_i, Z_{\ell_i}) < \varepsilon' 2^{-n}$, hence $||P^E_{[1,i]} x_i|| < C\varepsilon 2^{-n}$, where $C = \sup_{n \in \mathbb{N}} ||P^E_{[1,n]}||$. The nodes $(x_{(k_1,m_1,\ldots,k_n,i)})_{i>k_n}$ are weakly Cauchy, but may not be weakly null. Let $(k_1, m_1, \ldots, k_n, N) \in T^{even}_\infty$ and define $x_i = x_{(k_1,m_1,\ldots,k_n,i)}$ for $i \geq N$. The sequence $(P^E_{(i,\infty)} x_i)_{i \geq N}$ is weakly null so there exists $(a_i^{(N)})_{i=N}^{K(N)} = (a_i)_{i=N}^K \subset [0,1]$ such that $\sum_{i=N}^K a_i = 1$ and $|| \sum_{i=N}^K a_i P^E_{(i,\infty)} x_i || < \varepsilon 2^{-n}$. We now set

$$y_{(k_1,m_1,\ldots,k_n,N)} = \frac{x_N - \sum a_i x_i}{||x_N - \sum a_i x_i||}.$$

We have that

$$\left\| \sum_{i=N}^K a_i x_i \right\| \leq \sum_{i=N}^K a_i ||P^E_{[1,i]} x_i|| + \left\| \sum_{i=N}^K a_i P^E_{(i,\infty)} x_i \right\| < \sum_{i=N}^K a_i C\varepsilon 2^{-n} + \varepsilon 2^{-n} = (1+C)\varepsilon 2^{-n}.$$

Thus for all $(k_i), (m_i) \in [\mathbb{N}]^\omega$ we get that

$$\|x_{(k_1,m_1,\ldots,k_n,N)} - y_{(k_1,m_1,\ldots,k_n,N)}\| = \left\| \frac{(||x_N - \sum a_i x_i|| - 1)x_N + \sum a_i x_i}{||x_N - \sum a_i x_i||} \right\|$$
$$< \frac{2(1+C)\varepsilon 2^{-n}}{1 - (1+C)\varepsilon 2^{-n}} < 5(1+C)\varepsilon 2^{-n} = \varepsilon' 2^{-n}.$$

No branch of $(x_\alpha)_{\alpha \in T^{even}_\infty}$ is contained in $\overline{A_{5\varepsilon'}}$, thus no branch of $(y_\alpha)_{\alpha \in T^{even}_\infty}$ is contained in $\overline{A_{\varepsilon'}}$. Thus if we show that the nodes of $(y_\alpha)_{\alpha \in T^{even}_\infty}$ are weakly null, then we have that (a) does not hold. We consider the node $(y_{(k_1,m_1,\ldots,k_n,N)})_{N>k_n}$ and deduce from the choice of $(y_{(k_1,m_1,\ldots k_n,N)}$ and $(a_i^{(N)})$ and from (3.1) that

$$||P_{[1,N]} y_{(k_1,m_1,\ldots,k_n,N)}|| < 2 \left\| P_{[1,N]} (x_N - \sum a_i^{(N)} x_i) \right\| \leq 2 \sum a_i ||P_{[1,N]}(x_N - x_i)|| < 2\varepsilon 2^{-N}.$$

Thus $||P_{[1,N]} y_{(k_1,m_1,\ldots,k_n,N)}|| \to 0$ and hence $(y_{(k_1,m_1,\ldots,k_n,N)})$ is weakly null as (E_i) is shrinking.

$\square$

Remark. In the proof of Theorem A.1, we will apply Proposition B.7 for the case that $A = \{(n_i, x_i)_{i=1}^\infty \,|\, (v_{n_i})$ dominates $(x_i)\}$ where (v_i) is a 1-unconditional, basic

sequence. When applying the proposition, we are required to first block the FDD. As we will be applying multiple theorems that require us to block an FDD, we want our results about $\bar{\delta}$-skipped blocks to be preserved under blockings. However, if we have a blocking (H_i) of (E_i), a $\bar{\delta}$- skipped block of (H_i) may not be a $\bar{\delta}$-skipped block of (E_i) as skipped blocks are defined to skip exactly one coordinate. Fortunately if (E_i) is bimonotone, it will be true that a $\bar{\delta}$- skipped block of (H_i) will be a $\bar{\delta}$-skipped block of (E_i). For the case that (E_i) is not bimonotone, $\bar{\delta}$-skipped blocks of (H_i) will be $2K\bar{\delta}$-skipped blocks of (E_i) where K is the projection constant of (E_i). This will allow us to apply theorems about blockings in succession.

We will be concerned with a space X which satisfies subsequential V-upper tree estimates. However the nature of our proofs require us to work with X^* as well. This is because some of the blocking techniques which we use depend on the FDD being boundedly complete.

Lemma B.8. *Let X be a Banach space, $V = (v_i)$ be a normalized 1-unconditional basis. If X satisfies subsequential V-upper tree estimates, then X^* satisfies subsequential V^*-lower w^* tree estimates.*

Proof. X has separable dual, so by [DFJP, Corollary 8] there exists a space Z with a shrinking and bimonotone FDD (F_i) for which there is a quotient map $Q : Z \to X$. After renorming X we may assume that it has the quotient norm $||x|| = \inf_{Qy=x} ||y||$ for all $x \in X$. This gives that Q^* is an isometric embedding of X^* into Z^*. Furthermore, (F_i^*) is a boundedly complete FDD for Z^* as (F_i) is shrinking.

Let $(x_\alpha)_{\alpha \in T_\infty^{\mathrm{even}}} \subset S_{X^*}$ be a w^*-null tree. We will consider nodes of (x_α) individually, thus we fix an $s = (n_1, ..., n_{2k-1}) \in [\mathbb{N}]^{<\omega}$. We have that the sequence $(x_{(s,i)})_{i=n_{2k-1}+1}^\infty$ is w^* null, and thus $(Q^*x_{(s,i)})_{i=n_{2k-1}+1}^\infty$ is w^* null in Z^* as Q^* is w^* to w^* continuous. Hence after passing to a subsequence we may assume that

$||P_{[1,i)}^{F^*}Q^*x_{(s,i)}|| < 2^{-i}$. As (F_i) is bimonotone, there exists $y_{(s,i)} \in S_Z$ such that $||P_{[1,i)}^{F}y_{(s,i)}|| = 0$ and $Q^*x_{(s,i)}(y_{(s,i)}) > 1 - 2^{-i}$. The sequence $(y_{(s,i)})_{i=n_{2k-1}+1}^{\infty}$ is coordinate wise null and hence weakly null as (F_i) is shrinking. Thus $(Qy_{(s,i)})_{i=n_{2k-1}+1}^{\infty}$ is weakly null. By passing again to a subsequence, we may assume the conditions that $|x_{(s,i)}(Qy_{(n_1,...,n_{2\ell})})| < 2^{-i}$ and $|x_{(n_1,...,n_{2\ell})}(Qy_{(s,i)})| < 2^{-i}$ for all $1 \le \ell < k$ and $i \in \mathbb{N}$, $i > n_{2k-1}$. The first condition may be assumed because $(x_{(s,i)})_{i=1}^{\infty}$ is w^*-null, and the second condition may be assumed because $(Qy_{(s,i)})_{i=1}^{\infty}$ is w-null.

Let $(n_{2k-1}, Qy_{(n_1,...,n_{2k})})_{k=1}^{\infty}$ be a branch of the weak null tree $(Qy_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ such that $(v_{n_{2k-1}})_{k=1}^{\infty}$ C-dominates $(Qy_{(n_1,...,n_{2k})})_{k=1}^{\infty}$ for some $C > 0$. Let $(a_i) \in c_{00}$ such that $||\sum a_i v_i^*|| = 1$. There exists $(b_i) \in c_{00}$ such that $||\sum b_i v_i|| = 1$ and $\sum a_i b_i = 1$ as (v_i) is bimonotone. Furthermore, $\text{sign}(a_i) = \text{sign}(b_i)$ as (v_i) is 1-unconditional. We have that,

$$
\begin{aligned}
1 &= ||\sum_{i=1}^{\infty} a_i v_i^*|| \\
&= \sum_{i=1}^{\infty} a_i b_i \\
&\le \sum_{i=1}^{\infty} a_i b_i \frac{2^{2i}}{2^{2i} - 1} x_{(n_1,...,n_{2i})}(Qy_{(n_1,...,n_{2i})}) \\
&\le \frac{4}{3}(\sum_{k=1}^{\infty} a_k x_{(n_1,...,n_{2k})})(\sum_{\ell=1}^{\infty} b_\ell Qy_{(n_1,...,n_{2\ell})}) + \frac{4}{3}\sum_{k=1}^{\infty}\sum_{\ell\neq k}|x_{(n_1,...,n_{2k})}^*(Qy_{(n_1,...,n_{2\ell})})| \\
&\le \frac{4}{3}(\sum_{k=1}^{\infty} a_k x_{(n_1,...,n_{2k})})(\sum_{\ell} b_\ell Qy_{(n_1,...,n_{2\ell})}) + \frac{2}{3} \\
&< C\frac{4}{3}||\sum_{k=1}^{\infty} a_k x_{(n_1,...,n_{2k})}|| + \frac{2}{3}.
\end{aligned}
$$

Hence $(x_{(n_1,...,n_{2k})})_{k=1}^{\infty}$ $4C$-dominates $(v_{n_{2k-1}}^*)_{k=1}^{\infty}$. Thus X^* satisfies subsequential V^*-lower w^* tree estimates. $\qquad\square$

Proposition B.7 allows us to pass from information about trees to information about $\bar{\delta}$-skipped blocks of an FDD (E_n). To go from information about $\bar{\delta}$-skipped blocks to blocks in general, we will renorm the FDD (E_n) to form a new space.

Let Z be a space with an FDD $E = (E_n)$ and let $V = (v_i)$ be a normalized 1-unconditional basic sequence. The space $Z^V = Z^V(E)$ is defined to be the completion of $c_{00}(\bigoplus E_n)$ with respect to the following norm $\|\cdot\|_{Z^V}$.

$$\|z\|_{Z^V} = \max_{k \in \mathbb{N}, \, 1 \leq n_0 < n_1 < \ldots < n_k} \left\| \sum_{j=1}^{k} \left\| P^E_{[n_{j-1}, n_j)}(z) \right\|_Z \cdot v_{n_{j-1}} \right\|_V \quad \text{for } z \in c_{00}(E_i).$$

We note that if $\|\cdot\|$ and $\|\cdot\|'$ are equivalent norms on Z then the corresponding norms $\|\cdot\|_{Z^V}$ and $\|\cdot\|'_{Z^V}$ are equivalent on $c_{00}(\bigoplus E_n)$. This allows us, when examining the space Z^V, to assume that (E_n) is bimonotone in Z. The following proposition from [OSZ2] is what makes the space Z_V essential for us. Recall that a basic sequence is called *C-block stable* for some $C \geq 1$ if any two normalized block bases (x_i) and (y_i) with

$$\max \big(\mathrm{supp}(x_i) \cup \mathrm{supp}(y_i) \big) < \min \big(\mathrm{supp}(x_{i+1}) \cup \mathrm{supp}(y_{i+1}) \big) \qquad \text{for all } i \in \mathbb{N}$$

are C-equivalent. We say that (v_i) is *block-stable* if it is C-block-stable for some constant C.

The following Proposition recalls some properties of Z^V which where shown in [OSZ2].

Proposition B.9. [OSZ2, Corollary 7, Lemma 8 and 10] *Let $V = (v_i)$ be a normalized, 1-unconditional, and C-block-stable basic sequence. If Z is a Banach space with an FDD (E_i), then (E_i) satisfies 2C-V-lower block estimates in $Z^V(E)$.*

If the basis (v_i) is boundedly complete then (E_i) is a boundedly complete basis for $Z^V(E)$.

If the basis (v_i) is shrinking and if (E_i) is shrinking in Z, then (E_i) is a shrinking FDD for $Z^V(E)$.

In proving our main theorem we will show that if X satisfies subsequential V-upper tree estimates then it is isomorphic to a subspace of some $Z^V(E)$ and is isomorphic to quotient of some $Z^V(F)$.

C.   Proofs of Main Results

Proof of Theorem A.1. 1) $\Rightarrow$ 4) (v_i) is D-right-dominant for some $D \geq 1$, from which we can easily deduce that (v_i^*) is D-left-dominant. By [DFJP, Corollary 8] there exists a space Z with a shrinking and bimonotone FDD (E_i) for which there is a quotient map $Q : Z \to X$. The map $Q^* : X^* \to Z^*$ is an into isomorphism. By Lemma B.8 we have that X^* satisfies subsequential C-V^* lower w^* tree estimates for some $C \geq 1$. As $Q^*X^* \subset Z^*$ is w^* closed, we may apply Proposition B.7 with $A = \{(n_i, x_i)_{i=1}^\infty \,|\, (x_i) \text{ C-dominates } (v_{n_i})\}$ and $\varepsilon > 0$ such that $\bar{A}_\varepsilon \subset \{(n_i, x_i)_{i=1}^\infty \,|\, (x_i) \text{ 2C-dominates } (v_{n_i})\}$. This gives sequences $(K_i) \in [\mathbb{N}]^\omega$ and $\bar{\delta} = (\delta_i) \subset (0,1)$ and a blocking (F_i) of (E_i^*) such that if $(x_i) \subset S_{X^*}$ and $||x_i - P_{(r_{i-1}, r_i)}^F(x_i)|| < \delta_i$ for some $(r_i) \in [\mathbb{N}]^\omega$ then $(K_{r_{i-1}}, x_i) \in \bar{A}_\varepsilon$. Hence, the sequence (x_i) 2C-dominates $(v_{K_{r_i-1}})$.

We choose a blocking $G = (G_i)$ of (F_i) defined by $G_i = \sum_{i=m_{j-1}+1}^{m_j} F_j$ for some $(m_i) \in [\mathbb{N}]^\omega$ such that there exists $(e_n) \subset S_{X^*}$ with $||e_n - P_n^G(e_n)|| < \delta_n$ for all $n \in \mathbb{N}$.

In order to continue we need the following result from [OS1] which is based on an argument due to W. B. Johnson [J]. Corrollary 4.4 was stated in [OS1] for a reflexive spaces, but the proof shows that it is enough to assume that X is a w^*-closed subspace of a dual space with boundedly complete FDD. Also note that conditions (d) and (e) which where not stated follow easily from the proof.

Proposition C.1. [OS1, Lemma 4.3 and Corollary 4.4] *Let Y be a Banach space which is a subspace of a Banach space Z with a boundedly complete FDD $A = (A_i)$ having projection constant K. Let $\bar{\eta} = (\eta_i) \subset (0,1)$ with $\eta_i \downarrow 0$. Then there exists $(N_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ such that the following holds. Given $(k_i)_{i=0}^\infty \in [\mathbb{N}]^\omega$ and $x \in S_X$, there exists $x_i \in X$ and $t_i \in (N_{k_{i-1}-1}, N_{k_{i-1}})$ for all $i \in \mathbb{N}$ with $N_0 = 0$ and $t_0 = 0$ such that*

(a) $x = \sum_{i=1}^\infty x_i$, and for all $i \in \mathbb{N}$ we have

(b) either $||x_i|| < \eta_i$ or $||x_i - P^A_{(t_{i-1},t_i)}x_i|| < \eta_i||x_i||$,

(c) $||x_i - P^A_{(t_{i-1},t_i)}x|| < \eta_i$,

(d) $||x_i|| < K + 1$,

(e) $||P^A_{t_i}x|| < \eta_i$.

We apply Proposition C.1 to the FDD $A = G$ and $\bar{\eta} = \bar{\delta}$ which gives a sequence $(N_i) \in [\mathbb{N}]^\omega$. We set $H_j = \bigoplus_{i=N_{j-1}+1}^{N_j} G_i$ for each $j \in \mathbb{N}$. To make notation easier we let $V_M^* = (v_{M_i}^*)$ be the subsequence of (v_i^*) defined by $M_i = N_{K_{m_i}}$.

Fix $x \in S_{Q^*X^*}$ and a sequence $(n_i)_{i=0}^\infty \in [\mathbb{N}]^\omega$, the proof in [OSZ2, Theorem 12 (a)] shows

$$\left\| \sum_{i=1}^\infty \|P^H_{[n_{i-1},n_i)}(x)\|_{Z^*} \cdot v_{M_{n_{i-1}}}^* \right\|_{V^*} \leq 4D^2C(1 + 2\Delta + 2) + 2 + 3\Delta.$$

where $\Delta = \sum_{i=1}^\infty \delta_i$. Thus the norms $|| \cdot ||_Z$ and $|| \cdot ||_{(Z^*)V_M^*}$ are equivalent on Q^*X^*. As the norm on each H_j is unchanged, a coordinate wise null sequence in $Q^*X^* \subset Z^*$

will still be coordinate wise null in $(Z^*)^{V_M^*}$. Hence the map $Q^* : X^* \to (Z^*)^{V_M^*}$ is still w^* to w^* continuous.

We have that $(Z^*)^{V_M^*}$ has a boundedly complete FDD (H_j) which satisfies subsequential V_M^* lower block estimates by Proposition B.9. We can now fill in the FDD. We let $B_{M_j} = H_j$ for all $j \in \mathbb{N}$ and we let $B_j = \mathbb{R}$ for each $j \notin (M_i)$. For $x = (x_j) \in c_{00}(B_j)$ we define

$$\|x\| = \Big\| \sum_{j \in \mathbb{N}} x_{M_j} \Big\|_{(Z^*)^{V_M^*}} + \sum_{j \notin M} |x_j|.$$

We let Y be the completion of $c_{00}(\oplus B_j)$ under this norm. (B_j) satisfies subsequential V^* block lower block estimates. Y is clearly isometrically isomorphic to $(Z^*)^{V_M^*} \oplus \ell_1$ or $(Z^*)^{V_M^*} \oplus \ell_1^n$ for some $n \in \mathbb{N}_0$. Thus the natural embedding of $(Z^*)^{V_M^*}$ into Y is w^* to w^* continuous. Hence there is a w^* to w^* continuous embedding of X^* into Y. Finally, from the fact that (H_j) satisfies subsequential V_M^* lower block estimates in $(Z^*)^{V_M^*}$ it is not hard to deduce that (B_j) satisfies subsequential V^* lower block estimates in Y.

4) $\Rightarrow$ 3) This is clear because if (F_i^*) is a boundedly complete FDD of Z^* then (F_i) is a shrinking FDD of Z and a $w^* - w^*$ continuous embedding $T : X^* \to Z^*$ must be the dual of some quotient map $Q : Z \to X$. Also, (F_i^*) having subsequential V^*-lower block estimates is equivalent to (F_i) having subsequential V-upper block estimates due to Proposition B.2.

3) $\Rightarrow$ 1) Let (F_i) be a bimonotone shrinking FDD which satisfies subsequential V upper block estimates, and $Q : Z \to X$ be a quotient map. There exists $C > 0$ such that $B_X \subset Q(CB_{[F_i]})$. We will need the following lemma.

Lemma C.2. *Let X and Z be Banach spaces, $F = (F_i)$ be a bimonotone FDD for Z, and $Q : Z \to X$ be a quotient map. If $(x_i) \subset S_X$ is weakly null and $Q(CB_Z) \supseteq B_X$*

for some $C > 0$ then for all $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ and $z \in 2CB_Z$ such that $P_{[1,n]}z = 0$ and $||Qz - x_N|| < \varepsilon$.

Proof. Let $z_i \in CB_X$ such that $Qz_i = x_i$. After passing to a subsequence (z_{k_i}) and perturbing we may assume instead that $P_{[1,n]}^F z_{k_i} = z_0$ for some $z_0 \in CB_X$, and that $||Qz_{k_i} - x_{k_i}|| < \varepsilon/3$. As (x_{k_i}) is weakly null, 0 must be in the closure of the convex hull of (x_{k_i}). Hence there is some finite sequence $(a_i)_{i=2}^m \subset [0,1]$ such that $||\sum_{i=2}^m a_i x_{k_i}|| < \varepsilon/3$ and $\sum_{i=2}^m a_i = 1$. Let $z = z_{k_1} - \sum_{i=2}^m a_i z_{k_i}$. Hence, $z \in 2CB_Z$ and $P_{[1,n]}^F z = 0$. We have that

$$||Qz - x_{k_1}|| = ||Qz_{k_1} - x_{k_1} + \sum_{i=2}^m Qa_i z_{k_i}|| \leq ||Qz_{k_1} - x_{k_1}|| + \sum_{i=2}^m a_i ||Qz_{k_i} - x_{k_i}|| + ||\sum_{i=2}^m a_i x_{k_i}|| < \varepsilon.$$

$\square$

Continuation of proof of Theorem A.1. Let $\varepsilon_i = 2^{-i}$ and $(x_t)_{t \in T_\infty^{even}} \subset S_X$ be a weakly null even tree. By Lemma C.2 we may pass to a subtree of (x_t) so that there exists a block tree $(z_t)_{t \in [\mathbb{N}]^{<\omega}} \subset 2CB_{[F_i]}$ such that $||Q(z_t) - x_t|| < \eta 2^{-\max(t)}$ for all $t \in T_\infty^{even}$, where $\eta \in (0,1)$ is chosen small enough so that the following holds: if $(k_{2i-1}, z_{(k_1,...,k_{2i})})_{i=1}^\infty$ is a branch of (z_t) then $(Qz_{(k_1,...,k_{2i})})_{i=1}^\infty$ is 2-equivalent to $(x_{(k_1,...,k_i)})_{i=1}^\infty$. Choose a branch $(k_{2i-1}, z_{(k_1,...,k_{2i})})_{i=1}^\infty$ such that $\max \operatorname{supp}(z_{(k_1,...,k_{2i})}) < k_{2i+1} < \min \operatorname{supp}(z_{(k_1,...,k_{2i+2})})$ for all $i \in \mathbb{N}$. Thus $(z_{(k_1,...,k_{2i})})$ is dominated by $(v_{k_{2i-1}})$ as (v_i) is block stable. Hence $(x_{(k_1,...,k_i)})$ is dominated by $(v_{k_{2i-1}})$. Thus X satisfies subsequential V upper tree estimates.

2) $\Rightarrow$ 1) We assume that X is a quotient of a space Z with separable dual such that Z satisfies subsequential V upper tree estimates. By the implication 1) $\Rightarrow$ 3) applied to Z, Z is the quotient of a space Y with an FDD satisfying subsequential V upper block estimates. X is then also a quotient of Y, so by the implication 3) $\Rightarrow$ 1) we have that X satisfies subsequential V upper tree estimates.

1) $\Rightarrow$ 5) From the already shown equivalences of the conditions (1) and (3) we know that there exists a Banach space W with a shrinking FDD (E_j) having subsequential V upper block estimates and a quotient map $Q : W \twoheadrightarrow X$. By Zippins theorem, we may assume that there also exists a Banach space Z with shrinking FDD (F_j) and an isometric embedding $i : X \hookrightarrow Z$. Thus we have a quotient map $i^* : [F_j^*] = Z^* \twoheadrightarrow X^*$ and an embedding $Q^* : X^* \hookrightarrow [E_j^*] = W^*$. We have that (E_j^*) satisfies subsequential V^* lower block estimates. We shall put a new norm $\|\cdot\|$ on $c_{00}(\oplus F_j^*)$ so that if we denote the completion of $c_{00}(\oplus F_j^*)$ with respect to that norm by $\tilde{Z}$ the following holds: the map $i^* : c_{00}(\oplus F_j^*) \to X^*$ extends to a continuous quotient map from $\tilde{Z}$ to X^*, $\tilde{Z}$ is still a dual space, the extension is still w^* continuous, and (F_j^*) satisfies subsequential V^* lower block estimates in $\tilde{Z}$.

After passing to blockings of (E_j) and (F_j) we may assume using Proposition B.7 and Johnson and Zippin's blocking lemma [JZ] that there exists $\delta_j \searrow 0$ and $(m_j) \in [\mathbb{N}]^\omega$ such that every $\bar{\delta}$-skipped block in $i(X)$ with respect to (F_j) has subsequential (v_{m_j}) upper block estimates and for all $k < \ell$ that

$$\|P_{[1,k)}^F Qz\| < \delta_k \text{ and } \|P_{[\ell,\infty)}^F Qz\| < \delta_\ell \text{ for all } z \in S_{\bigoplus_{j\in(k,\ell)} F_j}.$$

Let $k < \ell$, $z^* \in \bigoplus_{j\in(k,\ell)} F_j^*$, and $w \in c_{00}(\oplus E_j)$. We have that $Q^* i^* z^*(w) = z^*|_X(Qw) = z^*(Qw)$. Thus,

$$|P_{[1,k)}^{E^*} Q^* i^* z^*(w)| = |z^*(QP_{[1,k)}^E w)| = |P_{(k,\ell)}^{F^*} z^*(QP_{[1,k)}^E w)|$$
$$= |z^*(P_{(k,\ell)}^F QP_{[1,k)}^E w)| \le \|P_{(k,\infty)}^F QP_{[1,k)}^E w\| < \delta_k.$$

Hence we have that $\|P_{[1,k)}^{E^*} Q^* i^* z^*\| < \delta_k$. Similarly, we have that $\|P_{[\ell,\infty)}^{E^*} Q^* i^* z^*\| < \delta_\ell$. This verifies the conclusion of Proposition 18 in [OSZ2] for $Q^* i^* : [F_j^*] \to [E_j^*]$. Proposition 18 was only stated reflexive spaces in [OSZ2], and that is exactly what

limited the proof of in [OSZ2, Theorem 12 part b)] to reflexive spaces. Thus, we may follow the proof of Theorem 12 part b) to finish our proof of $(5) \Rightarrow (1)$.

Finally, since the missing implications $(1) \Rightarrow (5)$ and $(3) \Rightarrow (2)$ are trivial, we finished the proof of the theorem. $\qquad \square$

The proof of the following result is an adaptation of the proof Theorem 21 in [OSZ2] to the non reflexive case.

Corollary C.3. *Let $V = (v_i)$ be an unconditional, shrinking, block stable, and right dominant normalized basic sequence. There is a Banach space Y with a shrinking FDD (E_i) satisfying subsequential V upper block estimates such that if a Banach space X has subsequential V upper tree estimates, then X embeds into Y.*

Proof. By Schechtman's result [Sc] there exists a space W with a bimonotone FDD $E = (E_i)$ with the property that any space X with bimonotone FDD $F = (F_i)$ naturally almost isometrically embeds into W, i.e. for any $\varepsilon > 0$ there is a $(1 + \varepsilon)$-embedding $T : X \to W$ and a subsequence (k_i) of N, such that $T(F_i) = E_{k_i}$.

Since V^* is boundedly complete it follows from Proposition B.7 that the sequence (E_i^*) is a boundedly complete FDD of the space $(W^{(*)})^{V^*}$. It follows that (E_i) is a shrinking FDD of the space $Y = \left((W^{(*)})^{V^*}\right)^{(*)}$ and that $Y^* = (W^{(*)})^{V^*}$. We denote by $\| \cdot \|_W$, $\| \cdot \|_{W^{(*)}}$, $\| \cdot \|_Y$, $\| \cdot \|_{Y^*}$ the norms in W, $W^{(*)}$, Y and Y^* respectively.

By Proposition B.9 (E_i^*) satisfies subsequential V^* lower block estimates in $(W^{(*)})^{V^*}$, and, thus, by Proposition B.2 (E_i^*) satisfies subsequential V upper block estimates in Y (recall that $Y^{(*)} = Y^* = (W^{(*)})^{V^*}$).

We now have to show that a space X which satisfies subsequential V upper tree estimates embeds in Y. By Theorem A.1 we can assume that X has a bimonotone FDD (F_i) satisfying subsequential V upper block estimates. By our choice of W we can assume that X is the complemented subspace of W generated by a subsequence

(E_{k_i}) of (E_i). We need to show that on X the norms $\|\cdot\|_W$ and $\|\cdot\|_Y$ are equivalent.

Let $C \geq 1$ be chosen so that (v_i) is C-block stable and C-right dominant (thus (v_i^*) is C-block stable and C-left dominant) and such that $(E_{k_i}^*)$ satisfies subsequential V^* C-lower block estimates in $X^{(*)}$. Let $w^* \in c_{00}(\oplus F_i^*) = c_{00}(\oplus E_{k_i}^*)$. Clearly, we have $\|w^*\|_{W^{(*)}} \leq \|w^*\|_{Y^*}$. Choose $1 \leq m_0 < m_1 < \dots$ such that

$$\|w^*\|_{Y^*} = \left\| \sum_{i=1}^{\infty} \|P_{[m_{i-1},m_i)}^{E^*}(w^*)\|_{W^{(*)}} v_{m_{i-1}}^* \right\|_{V^*}.$$

W.l.o.g we can assume that $m_0 = 1$ and that $P_{[m_{i-1},m_i)}^{E^*}(w^*) \neq 0$, for $i \in \mathbb{N}$. Since $w^* \in c_{00}(\oplus E_{k_i}^*)$, we can choose $j_1 < j_2 < \dots$ such that $k_{j_i} = \min \mathrm{supp} P_{[m_{i-1},m_i)}^{E^*}$ and deduce

$$\begin{aligned}
\|w^*\|_{Y^*} &= \left\| \sum_{i=1}^{\infty} \|P_{[m_{i-1},m_i)}^{E^*}(w^*)\|_{W^{(*)}} v_{m_{i-1}}^* \right\|_{V^*} \\
&\leq C \left\| \sum_{i=1}^{\infty} \|P_{[m_{i-1},m_i)}^{E^*}(w^*)\|_{W^{(*)}} v_{k_{j_i}}^* \right\|_{V^*} \\
&\leq C^2 \left\| \sum_{i=1}^{\infty} \|P_{[j_{i-1},j_i)}^{F^*}(w^*)\|_{W^{(*)}} v_{j_i}^* \right\|_{V^*} \leq C^3 \|w^*\|_{W^{(*)}}.
\end{aligned}$$

This proves that $\|\cdot\|_{W^{(*)}}$ and $\|\cdot\|_{Y^{(*)}}$ are equivalent on $c_{00}(\oplus E_{k_i})$. Since X is 1-complemented in W, and X^* is 1-complemented in $W^{(*)}$ and since $\sum_i P_{k_i}^{E^*}$ is still a norm 1-projection from Y^* onto $\overline{c_{00}(\oplus(E_{k_i})}^{Y^*}$ it follows for any $w \in c_{00}(\oplus E_{k_i})$ that $\frac{1}{C^3}\|w\|_W \leq \|W\|_Y \leq C^3 \|w\|_W$, which finishes the proof of our claim. $\qquad \square$

Combining Proposition B.5 with Theorem A.1 we obtain the following

Corollary C.4. *Let $\alpha < \omega_1$. For a space X with separable dual, the following are equivalent:*

(i) X has Szlenk index at most $\omega^{\alpha\omega}$,

(ii) X satisfies subsequential $T_{\alpha,c}$ upper block estimates for some $c \in (0,1)$,

(iii) X embeds into a space Z with an FDD (E_i) which satisfies subsequential $T_{\alpha,c}$ upper block estimates in Z for some $c \in (0,1)$.

Corollary C.5. *For each $\alpha < \omega_1$ there exists a Banach space Z_α with a shrinking FDD and Szlenk index at most $\omega^{\alpha\omega+1}$ such that Z_α is universal for the collection of spaces with separable dual and Szlenk index at most $\omega^{\alpha\omega}$.*

Proof. By Corollary C.3 for all $n \in \mathbb{N}$ there exists a Banach space X_n with an FDD satisfying subsequential $T_{\alpha,\frac{n}{n+1}}$ upper estimates which is universal for all Banach spaces with separable dual which satisfy subsequential $T_{\alpha,\frac{n}{n+1}}$ upper tree estimates. Let $Z_\alpha = (\bigoplus X_n)_{\ell_2}$. We have that Z_α is universal for the collection of spaces with separable dual and Szlenk index at most $\omega^{\alpha\omega}$ by Corollary C.4. The Szlenk index of Z_α is at most $\omega^{\alpha\omega+1}$ as proven in [OSZ3]. □

D. Applications to $\mathcal{L}_\infty$ Banach Spaces

A Banach space X is $\mathcal{L}_\infty$ if there exists a constant C such that every finite dimensional subspace of X is contained in some finite dimensional superspace which is C-isomorphic to ℓ_∞^n for some $n \in \mathbb{N}$. This local structure has a strong impact on the infinite dimensional structure of the Banach space, as every infinite dimensional $\mathcal{L}_\infty$ Banach space is non-reflexive. Furthermore, every $\mathcal{L}_\infty$ Banach space with separable dual has its dual isomorphic to ℓ_1. Due to these properties, it was surprising that in 1980 Bourgain and Delbaen constructed a class of infinite dimensional $\mathcal{L}_\infty$ Banach space with separable dual which are reflexive saturated [BD]. In 2000 Haydon showed that each of the spaces in that class are in fact ℓ_p saturated for some $1 < p < \infty$[H]. Recently, Theorem A.3 was applied to show that every Banach space X with separable dual embeds into a $\mathcal{L}_\infty$ space with a shrinking basis [FOS]. With

additional assumptions made on X, additional properties can be guaranteed for the $\mathcal{L}_\infty$ super-space. In particular, if ℓ_1 does not embed into X^*, then the $\mathcal{L}_\infty$ space can be constructed to not contain c_0, and if X is reflexive then the $\mathcal{L}_\infty$ space can be constructed to be reflexive saturated. These constructions solve a problem posed by Alspach [A], who asked essentially if there exists $\mathcal{L}_\infty$ Banach spaces of arbitrarily large Szlenk index which do not contain c_0.

Upper estimates are applied in [FOS] by connecting them to c-decompositions of linear functionals. These c-decompositions are used both in the construction of the $\mathcal{L}_\infty$ Banach spaces as well as the proof that they have separable dual. To further illustrate the utility of upper estimates, we present here without proof the following definition and lemma from [FOS].

Definition D.1. Let X be Banach space with an FDD $E = (E_i)$ and let $0 < c < 1$. If $x \in X \cap c_{00}(\oplus E_i)$ we call a finite block sequence $(x_1, x_2, \ldots, x_\ell)$ a c-*decomposition of* x if

$$x = \sum_{i=1}^{\ell} x_i, \text{ and for every } i = 1, \ldots \ell, \text{ either } \#supp_E(x_i) = 1 \text{ or } \|x_i\| \leq c. \quad (3.2)$$

It is easy to see that every $x \in X \cap c_{00}(\oplus E_i)$ has a c-decomposition. The *optimal c-decomposition* $(x_1, \ldots x_\ell)$ of $x \in X \cap c_{00}(\oplus E_i)$ is defined as follows. We put $n_1 := minsupp_E(x)$, and assuming we defined $n_1 < n_2 < \ldots n_j$, for some $j \in \mathbb{N}$

$$n_{j+1} = \begin{cases} n_j + 1 & \text{if } \|P_{n_j}(x)\| > c, \\ \min\{n : \|P_{[n_j,n]}(x)\| > c\} & \text{if } \|P_{n_j}(x)\| \leq c \text{ and if that minimum exists,} \\ 1 + maxsupp(x) & \text{otherwise.} \end{cases}$$

There will be a smallest $\ell \in \mathbb{N}$ so that $n_\ell = maxsupp(x)$. We put $x_j = P_{[n_j,n_{j+1})}(x)$ for $j = 1, \ldots, \ell$. It follows that $(x_j : j = 1, 2 \ldots, \ell)$ is a c-decomposition and that,

moreover, for all $j = 1, 2 \ldots \lfloor \ell/2 \rfloor$ we have $\|x_{2j-1} + x_{2j}\| > c$.

Let $\mathcal{A} \subset [\mathbb{N}]^{<\omega}$ be compact, we call a c-decomposition $(x_1, x_2, \ldots, x_\ell)$ of x $\mathcal{A}$-*admissible* if

$$\left\{ minsupp_E(x_i) : i = 1, 2 \ldots \ell \right\} \in \mathcal{A}.$$

We say (E_i) is $(c, \mathcal{A})$-*admissible in* X if every $x \in S_X \cap c_{00}(\oplus E_i)$ has an $\mathcal{A}$-admissible c-decomposition.

Lemma D.2. *Let X be a Banach space with a bimonotone FDD $E = (E_i)$. Assume that $\mathcal{A} \subset [\mathbb{N}]^{<\omega}$ is compact, hereditary, and spreading, and that $0 < c < \frac{1}{2}$ and $C \geq 1$.*

a) *Assume that $D \subset B_{X^{(*)}} \cap c_{00}(E_i^*)$ is d-norming X for some $c < d \leq 1$ and that every element of D has an $\mathcal{A}$-admissible c-decomposition. Then (E_i) satisfies subsequential $\frac{1}{c}$-$T_{c/d,\mathcal{B}}$ - upper estimates in X, where*

$$\mathcal{B} = \mathcal{B}_\mathcal{A} = \left\{ \{n\} \cup B_1 \cup B_2 : n \in \mathbb{N}, B_1, B_2 \in \mathcal{A} \right\}.$$

b) *Assume that (E_i) satisfies subsequential $C - T_{\mathcal{A},c}$ upper estimates, then there exists a compact, hereditary, and spreading family $\mathcal{G}$, depending on c, C and $\mathcal{A}$, such that (E_i^*) is $(c, \mathcal{G})$ admissible in $Z^{(*)}$.*

CHAPTER IV

CONCLUSION

We have proven that domination of weakly null sequences is a uniform property. That is if (v_i) is a normalized basic sequence and X is a Banach space such that every normalized weakly null sequence in X has a subsequence that is dominated by (v_i), then there exists a uniform constant $C \geq 1$ such that every normalized weakly null sequence in X has a subsequence that is C-dominated by (v_i). This is an extension of theorems of Knaust and Odell who proved the case that (v_i) is the unit vector basis for c_0 or ℓ_p with $1 < p < \infty$.

In the context of trees and finite dimensional decompositions, we have proven that if $V = (v_i)_{i=1}^{\infty}$ satisfies some general conditions, then a Banach space X with separable dual has subsequential V upper tree estimates if and only if it embeds into a Banach space with a shrinking FDD which satisfies subsequential V upper block estimates. Every Banach space with separable dual satisfies subsequential $T_{\alpha,c}$ upper tree estimates for some countable ordinal α and constant $0 < c < 1$, and thus our theorem has application to every Banach space with separable dual. This connection has allowed us to prove new theorems related to the Szlenk index and new theorems related to embedding Banach spaces into $\mathcal{L}_{\infty}$ spaces with separable dual.

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