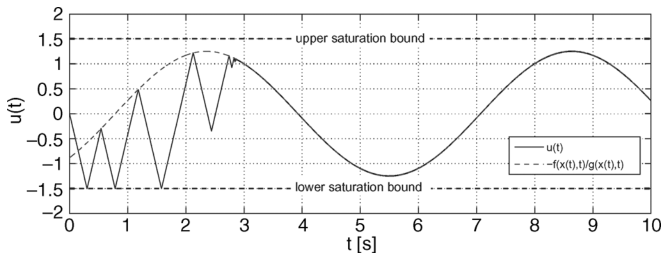

Fig. 8. Time evolution of the control signal for $\beta = \bar{\beta}$.

far as the convergence to zero of the sliding variable is concerned, preventing the enforcement of the desired second order sliding mode. In this technical note, a modified sub-optimal SOSM control algorithm is proposed. The modification is oriented to avoid the delay in the controller switching caused by actuator saturation. The proposed controller proves to guarantee the convergence of the sliding variable and of its first time derivative to zero in a finite time, in spite of the presence of uncertain terms affecting the system model and of the saturating actuators.

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A Parametrized Controller Reduction Technique via a New Frequency Weighted Model Reduction Formulation

Pantazis Houlis and Victor Sreeram

***Abstract*—In this technical note, the solution to the controller reduction problem via a double-sided frequency weighted model reduction technique is considered. A new method for finding low-order controllers based on new frequency weights derived using closed-loop system approximation criterion is proposed. The formulas of frequency weights are obtained in terms of the plant, the original controller and a matrix of free parameters. By varying the free parameters in the resulting two-sided frequency weighted model reduction problem, frequency weighted error can be significantly reduced to yield more accurate low-order controllers.**

***Index Terms*—Controller reduction, frequency weights, model reduction.**

I. INTRODUCTION

Controller reduction problems are usually solved via frequency weighted model reduction problem [1]–[6]. The frequency weighted model reduction problem can be classified into single-sided or double-sided frequency weighted problems. The single-sided frequency weighted model reduction problem is based on stability margin considerations. The reduced-order controller should satisfy the same conditions as are listed in the above references. The double-sided problem is based on closed-loop system approximation and attempts to minimize an index of the form

$$e = \|V_1(K - K_r)V_2\|_\infty$$
$$\text{where} \quad V_1 = (I + GK)^{-1}G \quad \text{and} \quad V_2 = (I + GK)^{-1}$$

and G, K, and K_r are the plant, the original controller and the reduced controller respectively.

There are a few methods for the solution of the frequency weighted model reduction problem [4], [5]. However, approximation errors obtained using these techniques are large. In general, the techniques are not as good as the techniques available for the unweighted case [7].

In this technical note, new techniques for obtaining lower approximation error using standard techniques by manipulating the frequency weights are proposed [4], [5], [8]–[12]. The techniques are based on deriving a new set of weights for double-sided frequency weighted model reduction. Out of the two new weights derived, one of the weights can be made to be a function of free parameters. By varying those free parameters in the resulting double-sided frequency weighted model reduction problem, the frequency weighted error can be significantly reduced, subsequently obtaining more accurate low-order controllers. Note that this technical note contains an improved and generalized version of the method presented in [13], [14].

Preliminary results on the Double Controller Techniques that were presented in [13], were only applicable to SISO (Single Input Single Output) systems. The method presented here is a generalization which

Manuscript received September 07, 2005; revised October 09, 2006. Current version published May 13, 2009. Recommended by Associate Editor C. Beck.

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Color versions of one or more of the figures in this technical note are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2008.2010993

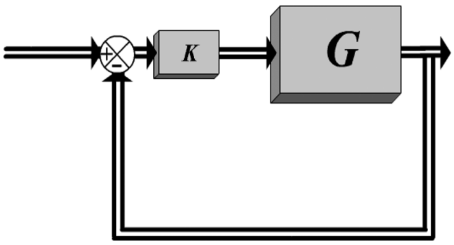

Fig. 1. Closed-loop system with plant G and controller K.

is applicable to MIMO (Multiple Input Multiple Output) systems. Furthermore, in this technical note we introduce an additional algorithm together with an efficient searching method.

II. Preliminaries

Consider the closed-loop system shown in Fig. 1, with the plant G and the controller K. The transfer function of the closed-loop system is given by

$$W = GK(I + GK)^{-1}. \quad (1)$$

In the closed-loop system configuration shown in Fig. 1, if the original controller K is replaced by a reduced-order controller, K_r, then the closed-loop system transfer function is given by

$$W_r = GK_r(I + GK_r)^{-1}. \quad (2)$$

Remark 1: In the controller reduction problem, the objective is to find a reduced-order controller, K_r such that the closed-loop systems are approximately equal. Because of the order simplification, it is not possible to have (in general) $W_r = W$. Therefore, a more realistic approach is to minimize the index $\|W - W_r\|_\infty$, so that the closed-loop systems W and W_r can become approximately equal, i.e., $W_r \approx W$.

Assuming that the second order terms are negligible in $K - K_r$, we write the following [1]:

$$W - W_r = (I + GK)^{-1}G[K - K_r](I + GK)^{-1}. \quad (3)$$

Therefore, the controller reduction problem can be reduced to a double-sided frequency weighted model reduction problem, which aims to minimize an index of the form

$$e = \|V_1(K - K_r)V_2\|_\infty, \text{ where } V_1 = (I + GK)^{-1}G \text{ and } V_2 = (I + GK)^{-1}. \quad (4)$$

Since the method proposed in the technical note is applicable to both continuous and discrete systems, the notation used throughout the technical note for transfer functions (plants, controllers, and any combination of them) will represent both continuous and discrete cases. For example, a plant G will represent both $G(s)$ (continuous case) and $G(z)$ (discrete case), unless stated otherwise.

III. Main Results

Consider the closed-loop block diagram shown in Fig. 1. This system has a plant G and controller K and a closed-loop transfer function, W. We will first show that this system can be expressed in another closed-loop configuration (see Fig. 2). The new configuration uses the original plant G and two controllers KC and $K(I - C)$ instead of one, where

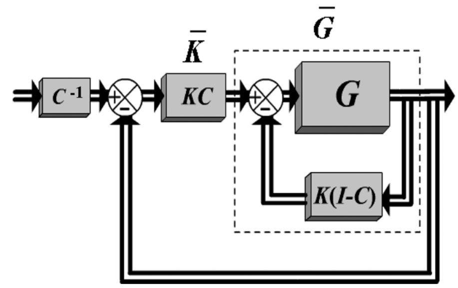

Fig. 2. Plant $\bar{G}$ is itself a closed-loop system with plant G and controller $K - \bar{K} = K(I - C)$.

$C = [c_{ij}]$ is a non-singular constant matrix. Note that both closed-loop configurations in Figs. 1 and have the same input and output and hence the same closed-loop system transfer function. Recall that for deriving frequency weights in standard techniques [1], [4], [8]–[12], we use the closed-loop system configuration shown in Fig. 1. In this technical note, for deriving the new set of frequency weights we use the closed-loop configuration in Fig. 2.

Definition 1: The configuration in Fig. 2 will be called the *Double Controller Form* of the closed-loop system W. Moreover, the use of the Double Controller Form to manipulate the frequency weights by changing the matrix C of free parameters, will be called the *Double Controller Technique*.

The advantage of using the Double Controller Form is that one of the weights will be a function of the matrix C. By varying the parameters of the matrix, we can significantly reduce the approximation error when using any frequency weighted model reduction technique.

A. Relationships Between Closed-Loop Configurations

In this subsection we derive the relationships between the closed-loop configurations shown in the block diagrams, Figs. 1, 2. In particular we will derive the relationships between the new plant $\bar{G}$ and the new controller $\bar{K}$ in terms of the old plant G, the old controller K and a matrix C of free parameters.

Let $W = GK(I + GK)^{-1}$, $\bar{W} = \bar{G}\bar{K}(I + \bar{G}\bar{K})^{-1}$ be closed-loop systems with plants and controllers G, K and $\bar{G}$, $\bar{K}$ respectively. Let us also define $H(C) = I + GK(I - C)$. By assuming that $(I + GK)^{-1}$ exists, it can be shown that $(I + GK(I - C))^{-1}$ also exists for given C, except for a finite number of values for s (continuous case) or z (discrete case). The proof (which is a simple extended version of the proof in [13], but for matrices) is omitted due to space restrictions. We will disregard those finite number of values, as we have infinite choices for s or z. For $C = I$ we have $H(I) = I$. We also define $H = I + GK$ and $\bar{H} = I + \bar{G}\bar{K}$.

Lemma 1: Assume we have the closed-loop systems $W = GKH^{-1}$ and $\bar{W} = \bar{G}\bar{K}\bar{H}^{-1}$ as defined above. If C is a non-singular real matrix such that $\bar{K} = KC$ and $\bar{G} = H^{-1}(C)G$, then we have

$$\bar{W} = WC. \quad (5)$$

Proof: First we have

$$\begin{aligned}\bar{H}^{-1} &= (I + \bar{G}\bar{K})^{-1} = \left(I + H^{-1}(C)GKC\right)^{-1} \\ &= (H(C) + GKC)^{-1} H(C) \\ &= (I + GK(I - C) + GKC)^{-1} H(C) \\ &= (I + GK)^{-1}H(C) = H^{-1}H(C). \quad (6)\end{aligned}$$

Note that for any closed-loop system W, the commutative property $GKH^{-1} = H^{-1}GK$ holds [2]. Then we use this property and the definitions of $\bar{G}$, $\bar{K}$ and $\bar{H}^{-1}$ to obtain $\bar{W}$

$$\begin{aligned}\bar{W} &= \bar{G}\bar{K}\bar{H}^{-1} = \bar{H}^{-1}\bar{G}\bar{K} = H^{-1}H(C)H^{-1}(C)GKC \\ &= H^{-1}GKC = GKH^{-1}C = WC \end{aligned} \tag{7}$$

■

Therefore, for C nonsingular, we could replace the closed-loop system W in Fig. 1 by the closed-loop system $\bar{W}C^{-1}$ in Fig. 2.

Remark 2: By using the definitions of $\bar{G}$ and $\bar{K}$ we have

$$\bar{G} = \left(I + G(K - \bar{K})\right)^{-1} G. \tag{8}$$

And as shown by the dashed lines in Fig. 2, $\bar{G}$ can itself be regarded as a closed-loop feedback system with plant G and controller $K - \bar{K} = K(I - C)$.

The real gain here is the revelation of a new matrix C of free parameters. By looking at the figures, it is clear that Fig. 2 is a generalization of Fig. 1, and by setting $C = I$ both block diagrams will become identical. The input and output always remain the same regardless of C, which means that we may manipulate the parameter C without affecting the system, but we may improve the non-linear procedure for calculating a reduced-order controller. In our technique, instead of directly calculating the reduced system from the original system, we use three steps:

1) We multiply the system W by the constant matrix C (linear procedure).
2) We reduce the system by using the standard controller reduction technique (non-linear procedure). For large C, there is more room for improvement between WC and W_rC, than between W and W_r.
3) Finally, after finding $W_r'C$, we multiply it by C^{-1} (linear procedure).

We will show that the constant matrix C plays a major role in decreasing the approximation error in controller reduction.

B. Derivation of New Weights

The main aim of controller reduction is to obtain a low-order controller by approximating the closed-loop behavior of the system. In this subsection, we derive the new set of frequency weights using the closed-loop configuration shown in Fig. 2. This is achieved by approximating the difference between the closed-loop systems $\bar{W}$ and $\bar{W}_r$, where $\bar{W}_r = W_rC$, and $\bar{W}_r$, W_r are the closed-loop systems with the lower order controllers $\bar{K}_r(C)$, K_r respectively. The reduced controller $\bar{K}_r(C)$ is obtained from a system $\bar{W}$ with plant $\bar{G} = (I + GK(I - C))^{-1}G$ and controller $\bar{K} = KC$. Therefore it is dependent on C. Since the procedure for obtaining a reduced controller is non-linear, for $C_1 \neq C_2$ we should also have $\bar{K}_r(C_1) \neq \bar{K}_r(C_2)$.

Let us now consider the closed-loop system $\bar{W} = WC$ with the assumptions used in Lemma 1.

We can express the difference $WC - W_rC$ by the difference $\bar{W} - \bar{W}_r$. More specifically

$$(W - W_r)C = \bar{W} - \bar{W}_r. \tag{9}$$

From (3) we have

$$\begin{aligned} W - W_r &= H^{-1}G(K - K_r)H^{-1} \text{ and} \\ \bar{W} - \bar{W}_r &= \bar{H}^{-1}\bar{G}\left(\bar{K} - \bar{K}_r(C)\right)\bar{H}^{-1}. \end{aligned}$$

Therefore, (9) may be rewritten as

$$H^{-1}G(K - K_r)H^{-1}C = \bar{H}^{-1}\bar{G}\left(\bar{K} - \bar{K}_r(C)\right)\bar{H}^{-1}. \tag{10}$$

Now define $K_r(C) = \bar{K}_r(C)C^{-1}$. For $C = I$, it is clear that $H(I) = I, \bar{G} = G$ and $\bar{K} = K$, which implies $K_r = K_r(I)$. We also define $I(C) = HCH^{-1}H(C)C^{-1}$.

Theorem 1: Using the assumptions in Lemma 1, we have

$$H^{-1}G\left(K - K_r(I)\right)H^{-1} = H^{-1}G\left(K - K_r(C)\right)H^{-1}I(C). \tag{11}$$

Proof: To prove the above Theorem, we must bring the RHS of (10) into a more explicit form.

Recall that $\bar{K} = KC$ and $K_r(C) = \bar{K}_r(C)C^{-1}$. By using (6) it can be shown that

$$\bar{H}^{-1}\bar{G}\left(\bar{K} - \bar{K}_r(C)\right)\bar{H}^{-1} = H^{-1}G(K - K_r(C))CH^{-1}H(C). \tag{12}$$

Then we have

$$\begin{aligned} &H^{-1}G\left(K - K_r(C)\right)CH^{-1}H(C) \\ &\quad = H^{-1}G\left(K - K_r(C)\right)H^{-1}\left(HCH^{-1}H(C)\right) \\ &\quad = H^{-1}G\left(K - K_r(C)\right)H^{-1}\left(HCH^{-1}H(C)C^{-1}\right)C \\ &\quad = H^{-1}G\left(K - K_r(C)\right)H^{-1}I(C)C. \end{aligned}$$

Thus, by substituting the last part of the above Equation into (10), and then canceling out the constant matrix C, we obtain the required result. ■

Remark 3: The new weights follow immediately from the RHS of (11)

$$V_1 = H^{-1}G = (I + GK)^{-1}G \text{ and } V_2(C) = H^{-1}I(C). \tag{13}$$

For $C = I$, we have $V_2(C) = V_2$ and we get the standard weights defined in (4).

Remark 4: Comparing the two set of weights in (4) and (13), observe that the output weights, V_1 are exactly the same while the input weights are different. The new input weight $V_2(C)$, is now a function of a matrix C of free parameters, whereas the input weight V_2 in the standard techniques is a fixed transfer function [1], [4]. The proposed method is based on varying this free parameter to achieve lower approximation errors and hence better low-order controllers.

Lemma 2: If in Theorem 1 we have $C = cI$, where c is a real number, then C commutes with GK, that is $CGK = GKC$, and (11) will be simplified into the form

$$H^{-1}G\left(K - K_r(I)\right)H^{-1} = H^{-1}G\left(K - K_r(C)\right)H^{-1}H(C). \tag{14}$$

Proof: Since $C = cI$, it commutes with all the matrix terms GK, H, H^{-1}, $H(C)$ and $H^{-1}(C)$. Therefore, regarding $I(C)$, the term C will be canceled out by the term C^{-1}, and consequently, H will be canceled out by H^{-1}. Therefore, $I(C)$ will be simplified to $H(C)$. ■

Remark 5: According to the above Lemma, if C commutes with GK, the new weights will now be

$$V_1 = H^{-1}G = (I+GK)^{-1}G \tag{15}$$
$$V_2(C) = H^{-1}H(C) = (I+GK)^{-1}\left[I+GK(I-C)\right]. \tag{16}$$

In this technical note, we consider the simplest of all commutative matrices, that is $C = cI$. The case of non-commutative matrices C is under investigation (i.e., using $I(C)$ instead of $H(C)$, which restricts C to only one parameter). Specifically, we seek to generalize they theory along the following lines:

1) Let $C = diag[c_1, c_2, \ldots, c_n]$, which requires more complex calculations, but would provide improved error reduction. The entries, c_i, can be viewed as weighting parameters in the optimization of the error.
2) Let C be a full matrix.
3) For a known plant G and controller K, perform a search over the general form matrices C that commute with GK.

By defining $C = cI$ where c is a constant, the matrix clearly commutes with GK, and we can safely use the weights defined in (15) and (16), that is, we may use $H(C)$ instead of $I(C)$. The matrix $I(C)$ (which is a part of the right weight $V_2(C)$ as shown in (13)) is too complex to use in this technical note as it will involve *any* matrix C, and a much more expanded error analysis.

Furthermore, we will have $H(C) = I + GK(1-c)$. Since C now depends on the scalar parameter c, in the rest of this technical note the notations $H(c)$, $K_r(c)$ and $\bar{K}_r(c)$ will be used instead of $H(C)$, $K_r(C)$ and $\bar{K}_r(C)$.

C. Error Analysis

Given G, K and c, it is standard procedure to derive a lower order controller $K_r = K_r(c)$ by minimizing $\|H^{-1}G(K - K_r)H^{-1}H(c)\|_\infty$ using any of the standard double-sided frequency weighted model reduction techniques [3], [4]. Let us define $E(c_1, c_2)$ and $e(c_1, c_2)$ as

$$E(c_1,c_2) = H^{-1}G\left(K - K_r(c_1)\right)H^{-1}H(c_2) \tag{17}$$
$$e(c_1,c_2) = \|E(c_1,c_2)\|_\infty. \tag{18}$$

In other words, the term $e(c_1, c_2)$ symbolizes the approximation error that uses $K_r(c_1)$ as the reduced controller, and $H^{-1}G$ and $H^{-1}H(c_2)$ as left and right weights. Note that, the controller $K_r(c_1)$ is defined as being found by minimizing the error $e(c_1, c_1)$, not $e(c_1, c_2)$. Moreover, we have $H(1) = I$, and $e(1, 1)$ corresponds to the approximation error obtained using standard weights (4) in double-sided frequency weighted model reduction techniques [3], [4].

Remark 6: The controller $K_r(c_1)$ depends on c_1 and $H(c_1)$, but the way $K_r(c_1)$ is influenced by $H(c_1)$ (in making the approximation error smaller) is more complex, and it mostly depends on the mechanism of the frequency weighted balanced truncation method that is used at the time (for example Enns's [4] or Wang *et al.*'s [5] method).

Lemma 3:

$$E(c,c) = E(1,1). \tag{19}$$
$$E(c,1) = E(1,1)H^{-1}(c). \tag{20}$$

Proof: Equation (19) is directly derived from (14) by using the definitions of $E(c, c)$ and $E(1, 1)$.

Regarding (20), we use the formula in (17) and the definitions of $E(c, 1)$, $E(c, c)$ to get $E(c, 1)H(c) = E(c, c)$. Then, by using (19) we obtain $E(c, 1)H(c) = E(1, 1)$, which finally gives $E(c, 1) = E(1, 1)H^{-1}(c)$. ■

We will also have

$$E(c,1)H(c) = E(1,1)$$
$$E(c,1)\left[(1-c)I + cH^{-1}\right] = E(1,1)H^{-1}. \tag{21}$$

Lemma 4: Let $h(c) = \|H(c)\|_\infty$ and $\hat{h}(c) = \|H^{-1}(c)\|_\infty$. Then we have

$$e(c,c) = e(1,1). \tag{22}$$
$$e(1,1)h^{-1}(c) \le e(c,1) \le e(1,1)\hat{h}(c). \tag{23}$$

Proof: Equation (22) is a direct consequence of (19).

Regarding Inequality (23), we will first prove the left part of the above expression and then the right part. From (20) we have

$$\|E(c,1)\|_\infty = \left\|E(1,1)H^{-1}(c)\right\|_\infty$$
$$\|E(c,1)\|_\infty \le \|E(1,1)\|_\infty \left\|H^{-1}(c)\right\|_\infty.$$
$$e(c,1) \le e(1,1)\hat{h}(c)$$

Similarly, we have

$$e(c,1) \ge e(1,1)h^{-1}(c)$$

■

D. The Behavior of $H(c)$

It is clear from the above that the term $H(c)$ and its two forms of infinity norm $h(c)$ and $\hat{h}^{-1}(c)$ are of great importance, and so it is essential to understand their properties.

Remark 7: Recall (21). We can see very clearly that by increasing the value of the parameter c, the term $[I(1-c) + cH^{-1}]$ will become large. And since the RHS of this Equation is a constant with respect to c, the size of the term $E(c, 1)$ will decrease, and the derived controller $K_r(c)$ will give a smaller approximation error.

By looking at (21), the approximation error $e(c, 1)$ becomes smaller as c increases. Because of the difference between the high order and low-order transfer functions, the approximation error $e(c, 1)$ can never become zero, but we will still achieve a significant error reduction as large values of c will force the approximation error to converge to a better minimum. And as c goes to plus or minus infinity, this minimum becomes a constant value, which may not be the optimal for all c, but it is always less than the original error when $c = 1$. By using some rough assumptions, we may approximate a value for c, that will give us an approximation error which will be almost equal to the constant number that $e(c, 1)$ is converging when c tends to $\pm\infty$. This means, that if we choose greater values for c, there will be no significant difference.

Let us choose some number γ, such that we want $e(c, 1) = \gamma e(1, 1)$, where $0 < \gamma < 1$. From (21) we have

$$\left\|E(c,1)\left[(1-c)I + cH^{-1}\right]\right\|_\infty = \left\|E(1,1)H^{-1}\right\|_\infty$$
$$\|E(c,1)\|_\infty \left\|(1-c)I + cH^{-1}\right\|_\infty \ge \left\|E(1,1)H^{-1}\right\|_\infty. \tag{24}$$

If we choose a large c, then it can be assumed without loss of generality that $[I(1-c) + cH^{-1}] = [I(-c) + cH^{-1}] = [c(H^{-1} - I)]$.

Then (24) becomes

$$\begin{aligned}&\|E(c,1)\|_\infty \left\|c(H^{-1}-I)\right\|_\infty \geq \left\|E(1,1)H^{-1}\right\|_\infty\\&|c|\,\|E(c,1)\|_\infty \left\|(H^{-1}-I)\right\|_\infty \geq \left\|E(1,1)H^{-1}\right\|_\infty\\&|c| \geq \left\|E(1,1)H^{-1}\right\|_\infty \left[\|E(c,1)\|_\infty \left\|(H^{-1}-I)\right\|_\infty\right]^{-1}. \qquad (25)\end{aligned}$$

By assuming that the original transfer function $W = GKH^{-1}$ has no poles on the $j\omega$-axis, we can easily show that the terms $I - GKH^{-1}, H^{-1}$, $H^{-1} - I$, and $E(1,1)H^{-1}$ have no poles on the $j\omega$-axis. Thus, we may safely use their corresponding infinity norms to calculate c. And if we substitute $e(c,1)$ by $\gamma e(1,1)$ we finally have

$$|c| \geq \left\|E(1,1)H^{-1}\right\|_\infty \left[\gamma e(1,1) \left\|(H^{-1}-I)\right\|_\infty\right]^{-1}. \qquad (26)$$

For $\gamma = 1/3$, we may choose a big enough c that gives the required approximation error. As we have stated, in practice, we may not expect to derive an approximation error $e(c,1)$ as small as γ times (in this case 1/3) of the original error $e(1,1)$, but we will derive a big improvement of this error, equal to the value of $e(c,1)$ for c converging to $\pm\infty$. A satisfactory assumption would always require a number γ which is "small enough".

E. Double Controller Technique

The proposed double controller technique solves the frequency weighted model reduction problem $e(c_1,c_2) = \|V_1(K - K_r(c_1))V_2H(c_2)\|_\infty$ to find a new improved controller $K_r(c_1)$, by replacing the old weights with new ones (V_1 and V_2 are the standard weights (4)). To solve the frequency weighted model reduction problem, we can use any of the standard techniques, e.g. Enns' technique [4] and Wang *et al.*'s technique [5]. Since the weight $V_2H(c_1)$ is a function of c_1, we get different low-order controllers $K_r(c_1)$ for different c_1. Note that, $c_1 = 1$ corresponds to the lower order controller $K_r(1)$ obtained by using the standard weights (4). A logical approximation for $e(c,1)$ which lies between $e(1,1)h^{-1}(c)$ and $e(1,1)\hat{h}(c)$, would be

$$e(c,1) \approx \hat{e}(c,1) = e(1,1)\frac{h^{-1}(c)+\hat{h}(c)}{2} \qquad (27)$$

that is, by taking the mean of the lower and upper bounds of $e(c,1)$, which is very easy to calculate. This sum is able to reveal important changes of the behavior of $e(c,1)$. The local minimum points of the smooth curve represented by $\hat{e}(c,1)$ are candidate values for which c gives optimal controllers. Note that, this method depends on the quality of the approximation given in (27). Thus, we may not get the best controller (found after an inefficient exhaustive search), but we will get a controller whose approximation error is close to the optimum and has better approximation error to the one using the original method with the standard weights.

Summarizing, there are two ways to improve the approximation error.

1) Calculate $\hat{e}(c,1)$, and use the values $c_{\min}$ that correspond to its local minimum points to construct an optimal controller.
2) Construct an optimal controller by using a large value c (as shown in (26)).

Based on the approximation error used, we propose an algorithm (of one parameter) for finding the optimum controller $K_r(c)$ by using weights $V_1, V_2H(c)$.

F. The Algorithm

1) Given plant G and controller K, define the weights $V_1 = (I + GK)^{-1}G$ and $V_2(c) = (I+GK)^{-1}(I+GK(1-c))$.
2) Calculate $\hat{e}(c,1)$ and find the values $c_{\min}$ that correspond to its local minimum points. Alternatively, a large value c may be used instead.

$$A_G = \begin{pmatrix} -1 & 0 & 4 & 5 & -3 & -2\\ -2 & 4 & -7 & -2 & 0 & 3\\ -6 & 9 & -5 & 0 & 2 & -1\\ -8 & 4 & 7 & -1 & -3 & 0\\ 2 & 5 & 8 & -9 & 1 & -4\\ 3 & -5 & 8 & 0 & 2 & -6 \end{pmatrix},\ B_G = \begin{pmatrix} -3 & -4\\ 2 & 0\\ -5 & -7\\ 4 & -6\\ -3 & 9\\ 1 & -2 \end{pmatrix}$$

$$C_G = \begin{pmatrix} 1 & -1 & 2 & -4 & 0 & -3\\ -3 & 0 & 5 & -1 & 1 & 1 \end{pmatrix},\ D_G = \begin{pmatrix} 1 & -2\\ 0 & 4 \end{pmatrix}$$

$$A_K = \begin{pmatrix} -2.8043 & 14.7367 & 4.6658 & 8.1596 & 0.0848 & 2.5290\\ 4.6609 & 3.2756 & -3.5754 & -2.8941 & 0.2393 & 8.2920\\ -15.3127 & 23.5592 & -7.1229 & 2.7599 & 5.9775 & -2.0285\\ -22.0691 & 16.4758 & 12.5523 & -16.3602 & 4.4300 & -3.3168\\ 30.6789 & -3.9026 & -1.3868 & 26.2357 & -8.8267 & 10.4860\\ -5.7429 & 0.0577 & 10.8216 & -11.2275 & 1.5074 & -10.7244 \end{pmatrix}$$

$$C_K = \begin{pmatrix} -0.2480 & -0.1713 & -0.0880 & 0.1534 & 0.5016 & -0.0730\\ 2.8810 & -0.3658 & 1.3007 & 0.3945 & 1.2244 & 2.5690 \end{pmatrix}$$

$$B_K = \begin{pmatrix} -0.1581 & -0.0793\\ -0.9237 & -0.5718\\ 0.7984 & 0.6627\\ 0.1145 & 0.1496\\ -0.6743 & -0.2376\\ 0.0196 & -0.7598 \end{pmatrix},\ D_K = \begin{pmatrix} 0.0554 & 0.1334\\ -0.3195 & 0.0333 \end{pmatrix}.$$

TABLE I
APPROXIMATION ERROR COMPARISON USING ENNS METHOD FOR FIRST UP TO FIFTH ORDER CONTROLLERS (MIMO EXAMPLE)

	Enns					Wang				
Order	$e(1,1)$	$e(c_{\min},1)$	$e(\infty,1)$	c_{opt}	$\varepsilon(c_{\text{opt}})$	$e(1,1)$	$e(c_{\min},1)$	$e(\infty,1)$	c_{opt}	$\varepsilon(c_{\text{opt}})$
1	3.6608	2.7510	3.6909	-0.6122	2.7510	4.6013	3.1764	3.8629	-1.0204	3.1764
2	3.1186	2.2697	2.2993	1.0204	2.2697	2.5725	2.3780	2.7582	-0.6122	2.3780
3	5.3129	2.8736	1.5485	1.0204	1.5485	1.9823	1.0293	7.1506	-0.2041	1.0293
4	0.9745	0.7643	1.3900	2.2449	0.7643	1.0475	0.8282	1.0028	0.2041	0.8282
5	0.2271	0.0846	0.3172	0.2041	0.0846	0.3209	0.2431	0.3477	-0.2041	0.2431

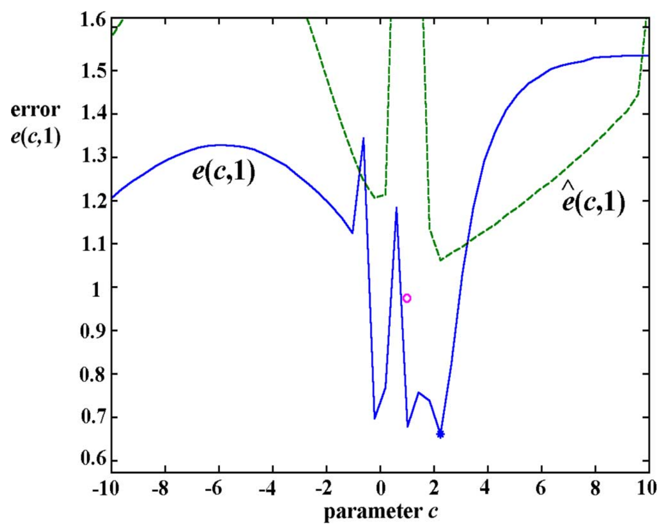

Fig. 3. Error functions $e(c,1)$ and $\hat{e}(c,1)$ for the MIMO example when applying controller reduction of order 4.

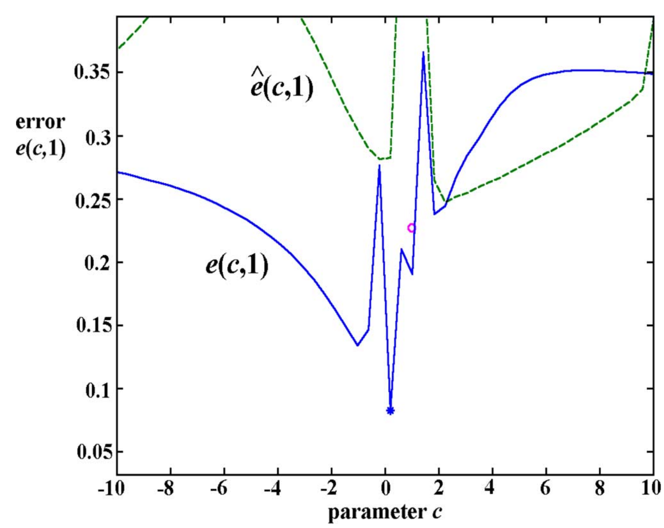

Fig. 4. Error functions $e(c,1)$ and $\hat{e}(c,1)$ for the MIMO example when applying controller reduction of order 5.

3) Solve the frequency weighted model reduction problem $\|V_1(K - K_r(c))V_2(c)\|$ for those c to compute $K_r(c)$ by using standard techniques (Enns [4] and Wang *et al.* [5]).

IV. MIMO EXAMPLE

We now consider an example from the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ Optimization Toolbox in SLICOT [15], defined with a plant G and a designed controller K (of sixth order, four inputs, and four outputs), whose state space matrices $(A_G, B_G, C_G, D_G$, and A_K, B_K, C_K, D_K, respectively, are given in the equation, shown at the bottom of the previous page.

Figs. 3 and 4 reveal that in the fourth order and fifth order reduced controllers cases (for Enns' Method), $\hat{e}(c,1)$'s local minimums directly point out the possible c's which give optimal results. Sometimes though, it is $e(\infty,1)$ which gives a better approximation error. This usually happens in the cases where the order reduction is done by an odd number, which could result in replacing a pair of complex poles by a real one, and yield results which are not as accurate as the ones when the order reduction is done by an even number [16]. However, there are also cases where $e(\infty,1)$ becomes too large. In general, we notice again a very big improvement of error reduction by using the double controller technique.

All the results for controllers with a reduced order from 1 to 5, can be seen in Table I. We may comment that in most cases, c_{opt} is directly found by the $c_{\min}$'s for each different order.

V. CONCLUSION

Formulas for new set of weights required for solving controller reduction problem via double-sided frequency weighted model reduction techniques are derived. It is shown that one of the frequency weights in a double-sided frequency weighted model reduction problem can be expressed as a function of a free matrix parameter C. It is shown that by varying this matrix parameter, the approximation error in double-sided frequency weighted model reduction problem can be greatly reduced, yielding more accurate controllers.

Remark 8: Note that, it is the standard double-sided frequency weighted model reduction techniques that dictate the stability status of the reduced-order controller, and not the double controller technique. The introduction of the matrix parameter c is only an intermediate step to temporarily modify the high order controller, but it plays no role in deciding the stability of the reduced-order controller. Moreover, it should be reminded to the reader that for any matrix parameter C, the system input and output are always equal to the ones for $C = I$. Therefore, the technique does not interfere with the stability of the system. Finally, for future purposes, the use of a constant matrix C with more than one parameters (instead of one) has clearly the potential to give even better approximation errors, as well as constructing a much more accurate graph for $\hat{e}(c,1)$.

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To Zero or to Hold Control Inputs With Lossy Links?

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***Abstract*—This technical note studies the linear quadratic (LQ) performance of networked control systems where control packets are subject to loss. In particular we explore the two simplest compensation strategies commonly found in the literature: the zero-input strategy, in which the input to the plant is set to zero if a packet is dropped, and the hold-input strategy, in which the previous control input is used if packet is lost. We derive expressions for computing the optimal static gain for both strategies and we compare their performance on some numerical examples. Interestingly, none of the two can be claimed superior to the other, even for simple scalar systems, since there are scenarios where one strategy performs better then the other and scenarios where the converse occurs.**

***Index Terms*—LQ control, networked systems, packet loss, stability.**

I. INTRODUCTION

Today's technological advances in wireless communications and in the fabrication of inexpensive embedded electronic devices, are creating a new paradigm where a large number of systems are interconnected, thus providing an unprecedented opportunity for totally new distributed control applications, commonly referred as networked control systems [1]. One of the most common problems in networked control systems, especially in wireless sensor networks, is packet drop, i.e. packets can be lost due to communication noise, interference, or congestion. If the controller is not co-located with the sensors and the actuators and it is placed in a remote location, then both sensor measurement packets and control packets can be lost.

A large number of works in the literature have analyzed estimation and filter design under lossy communication between the sensors and the controller [2]–[9]. However, there are also several works that studied the close loop performance when control packets can be dropped [4], [10]–[15]. In general, in most of the literature two different strategies are considered for dealing with packet drops. In the first one, which we refer as *zero-input*, the actuator input to the plant is set to zero when the control packet from the controller to the actuator is lost [12]–[15], while in the second, which we refer as *hold-input*, the latest control input stored in the actuator buffer is used when a packet is lost [4], [10], [11]. These are not the only strategies that can be adopted. In fact, if smart actuators are available, i.e. if actuators are provided with computational resources, then the whole controller [14] or a compensation filter [16] can be placed on the actuator. Another strategy is to use a model predictive controller which sends not only the current input but also a finite window of future control inputs into a single packet so that if a packet is lost the actuator can pop up from its buffer the corresponding predicted input from the latest received packet [17], [18]. Nonetheless, even this strategy requires more computational resources and communication bandwidth than the zero-input or hold-input strategies.

Manuscript received November 09, 2007; revised May 28, 2008. Current version published May 13, 2009. This work was supported in part by Italian CaRiPaRo Foundation under project "Wise-Wai" and by EU STREP FP7:ICT-2007.3.7 project "FeedNetBack." Recommended by Associate Editor K. Johansson.

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Digital Object Identifier 10.1109/TAC.2008.2010999