COMMUTATORS ON BANACH SPACES

A Dissertation

by

DETELIN TODOROV DOSEV

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2009

Major Subject: Mathematics

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ABSTRACT

Commutators on Banach Spaces. (August 2009)

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Chair of Advisory Committee: Dr. William B. Johnson

A natural problem that arises in the study of derivations on a Banach algebra is to classify the commutators in the algebra. The problem as stated is too broad and we will only consider the algebra of operators acting on a given Banach space X. In particular, we will focus our attention to the spaces ℓ_1 and ℓ_∞.

The main results are that the commutators on ℓ_1 are the operators not of the form $\lambda I + K$ with $\lambda \neq 0$ and K compact and the operators on ℓ_∞ which are commutators are those not of the form $\lambda I + S$ with $\lambda \neq 0$ and S strictly singular. We generalize Apostol's technique (1972, Rev. Roum. Math. Appl. 17, 1513 - 1534) to obtain these results and use this generalization to obtain partial results about the commutators on spaces $\mathcal{X}$ which can be represented as $\mathcal{X} \simeq \left(\bigoplus_{i=0}^{\infty} \mathcal{X}\right)_p$ for some $1 \leq p \leq \infty$ or $p = 0$. In particular, it is shown that every non - E operator on L_1 is a commutator. A characterization of the commutators on $\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n}$ is also given.

To my wife Asya and my children Ivan and Alex

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CHAPTER I

INTRODUCTION

The commutator of two elements A and B in a Banach algebra, or more generally, in any algebra, is given by

$$[A, B] = AB - BA.$$

To classify the commutators of operators acting on an infinite dimensional Banach space, it is natural first to understand the structure of commutators of operators acting on a finite dimensional space. In the latter case, every operator can be represented as a matrix with respect to a basis in the space and it is not hard to see (the proof of which we will present in Chapter II) that the operators on a finite dimensional Banach space which are commutators are the ones which have trace equal to zero. Commutators also appear in the study of derivations on a Banach algebra where it is essential to classify the commutators in the algebra. Wintner([31]) first proved that the identity in a unital Banach algebra is not a commutator. Here we sketch the Wienladt's proof of this result.

Assume that $AB - BA = I$ in some Banach algebra $\mathcal{C}$. Multiplying the last equation on the left by A and then on the right again by A and adding up the two resulting equations we obtain $A^2B - BA^2 = 2A$. Now it is not hard to see how we can proceed by induction in order to obtain

$$A^nB - BA^n = nA^{n-1}$$

for all $n \in \mathbb{N}$. From this equation it follows that $n\|A^{n-1}\| \leq \|A^{n-1}\|(\|AB\| + \|BA\|)$ and $A^n \neq 0$ for all n. Hence $\|AB\| + \|BA\| \geq n$ for every $n \in \mathbb{N}$ which is a

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contradiction. With no additional effort one can also show that no element of the form $\lambda I+K$, where K belongs to a norm closed ideal $\mathcal{I}(\mathcal{C})$ of the Banach algebra $\mathcal{C}$ and $\lambda \neq 0$, is a commutator. Indeed, just observe that the quotient algebra $\mathcal{C}/\mathcal{I}(\mathcal{C})$ also satisfies the conditions of Wintner's theorem. In particular, the above observation applies to the Banach algebra $\mathcal{L}(\mathcal{X})$ of all bounded linear operators on the Banach space $\mathcal{X}$. In our study of commutators, by *ideal* we will always mean a closed, proper, non-zero ideal.

Eighteen years after Winter's result, Brown and Pearcy ([5]) made a breakthrough in the study of commutators by proving that the only operators on ℓ_2 that are not commutators are the ones of the form $\lambda I + K$, where K is compact and $\lambda \neq 0$. Their result suggests what the classification of the commutators on the other classical sequence spaces might be, and, in 1972, Apostol ([3]) proved that every non-commutator on the space ℓ_p for $1 < p < \infty$ is of the form $\lambda I+K$, where K is compact and $\lambda \neq 0$. One year later he proved that the same classification holds in the case of $\mathcal{X} = c_0$ ([4]). Apostol proved some partial results on ℓ_1, but only 30 year later we was able to show that the same classification is valid in the case $\mathcal{X} = \ell_1$ ([6]). This result will be presented in Chapter IV.

Note that if $\mathcal{X} = \ell_p$ $(1 \leq p < \infty)$ or $\mathcal{X} = c_0$, the ideal of compact operators $K(\mathcal{X})$ is the largest proper ideal in $\mathcal{L}(\mathcal{X})$ ([9]; see also [30, Theorem 6.2]). The classification of the commutators on ℓ_p for $1 \leq p < \infty$ and partial results on other spaces suggest the following

Conjecture. Let $\mathcal{X}$ be a Banach space such that $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$, $1 \leq p \leq \infty$ or $p = 0$ (we say that such a space admits Pełczyński decomposition). Assume that $\mathcal{L}(\mathcal{X})$ has a largest ideal, $\mathcal{M}$. Then every non-commutator on $\mathcal{X}$ has the form $\lambda I + K$, where $K \in \mathcal{M}$ and $\lambda \neq 0$.

In [3] Apostol obtained a partial result regarding the commutators on ℓ_∞. He proved that if $T \in \mathcal{L}(\ell_\infty)$ and there exists a sequence of projections $(P_n)_{n=1}^\infty$ on ℓ_∞ such that $P_n(\ell_\infty) \simeq \ell_\infty$ (we say that two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic, and denote it by $\mathcal{X} \simeq \mathcal{Y}$, if there exists an onto, bounded and invertible linear operator $T : \mathcal{X} \to \mathcal{Y}$) for $n = 1, 2, \ldots$ and $\|P_n T\| \to 0$ as $n \to \infty$, then T is a commutator. This condition is clearly satisfied if T is a compact operator, but, as we show in Chapter V, it is also satisfied if T is strictly singular. An operator T from $\mathcal{X}$ to $\mathcal{Y}$ is called strictly singular if whenever the restriction of T to a subspace X of $\mathcal{X}$ has a continuous inverse, X is finite dimensional. In Chapter V we will obtain a complete classification of the commutators on ℓ_∞, that will show that the conjecture is valid in this case as well.

In order to give a positive answer to the conjecture one has to prove

- Every operator $T \in \mathcal{M}$ is a commutator;
- If $T \in \mathcal{L}(\mathcal{X})$ is not of the form $\lambda I + K$, where $K \in \mathcal{M}$ and $\lambda \neq 0$, then T is a commutator.

The proofs we provide for these two steps in the cases $\mathcal{X} = \ell_1$ and $\mathcal{X} = \ell_\infty$ are quite different but use a common idea which we develop in Chapter III. Namely, we use the idea of decompositions of Banach spaces to develop a technique that will allow us to reduce the question of whether an operator T on a Banach space $\mathcal{X}$ is a commutator to a question that is related to the structure of the Banach space $\mathcal{X}$ itself.

In Chapter VI we provide applications of our technique to a class of Banach spaces and provide particular examples.

In the last chapter we give a brief summary of our results and raise some open problems we are currently working on.

CHAPTER II

PRELIMINARY RESULTS

In this chapter we state some of the well known facts that we will use in the sequel and we also give a proof or sketch of a proof of some of the results. A Banach space $\mathcal{X}$ is a complete normed vector space. We are mainly interested in Banach spaces built over a complex vector space. A linear functional on a Banach space $\mathcal{X}$ is a linear map from $\mathcal{X}$ to $\mathbb{C}$. By a *subspace* we always mean a linear subspace. The dual of $\mathcal{X}$ is denoted by $\mathcal{X}^*$ and it is defined to be the set of all continuous linear functionals on $\mathcal{X}$, endowed with the norm $\|f\|^* = \sup_{\|x\|=1} |f(x)|$. It is not hard to see that $(\mathcal{X}^*, \|\cdot\|^*)$ is also a Banach space. By $\mathcal{L}(\mathcal{X})$ we denote the space of all linear operators $T : \mathcal{X} \to \mathcal{X}$ and it is a well known fact that $\mathcal{L}(\mathcal{X})$ is a Banach space with respect to the norm defined by $\|T\| = \sup_{\|x\|=1} \|Tx\|$. The set of finite rank operators on $\mathcal{X}$, i.e the operators T for which $\dim T(\mathcal{X}) < \infty$, is denoted by $\mathcal{F}(\mathcal{X})$.

A. Commutators on finite dimensional Banach spaces

An n-dimensional Banach space X can be viewed as $\mathbb{C}^n$ with some norm, so a linear operator A can be viewed as a $n \times n$ matrix $(a_{i,j})_{i,j=1}^n$ with complex entries. We denote by M_n the set of all such $n \times n$ matrices. The *trace* of a matrix $A \in M_n$ is defined by

$$tr(A) = \sum_{i=1}^n a_{ii}$$

and recall that $tr(AB) = tr(BA)$ for any two $A, B \in M_n$. The last fact is easy to see by just observing that

$$tr(AB) = \sum_{i=1}^n (\sum_{j=1}^n a_{ij} b_{ji}) = \sum_{j=1}^n (\sum_{i=1}^n a_{ji} b_{ij}) = tr(BA). \tag{2.1}$$

From (2.1) it is clear that if a matrix $C \in M_n$ is a commutator, say $C = AB - BA$ for some $A, B \in M_n$, then $tr(C) = 0$, so in the finite dimensional setting, the interesting question is whether all matrices with trace 0 are commutators. This question was answered by Shoda ([27]), who showed that this is in fact the case. The proof presented here (cf. [14]) gives more information about the structure of the $n \times n$ matrices.

Theorem II.1. *Let $Z \in M_n$ be such that $tr(Z) = 0$. Then Z is a commutator.*

Proof. We will divide the proof into three steps.

Step I.

First we show that if $Z \in M_n$ is a commutator, then $Z' \in M_{n+1}$, defined by $Z' = \left(\begin{smallmatrix} 0 & \mathbf{r} \\ \mathbf{c} & Z \end{smallmatrix}\right)$ for some row $\mathbf{r}$ and some column $\mathbf{c}$, is also a commutator. To show this let $Z = AB - BA$ and assume in addition that A is invertible (this can always be achieved by replacing A with $A + \alpha I$ for some α). Define

$$A' = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix} \quad , \quad B' = \begin{pmatrix} 0 & -\mathbf{r}A^{-1} \\ A^{-1}\mathbf{c} & B \end{pmatrix}.$$

Now a simple computation shows that $A'B' - B'A' = Z'$.

Step II.

Now we show that if a matrix Z is not a scalar multiple of the identity, then there exists an invertible matrix S such that the element in the $(1, 1)$ position of the matrix $S^{-1}ZS$ is equal to 0. To do this first assume that there exists a row $\mathbf{r}$ and a column $\mathbf{c}$ such that $\mathbf{r}Z\mathbf{c} = 0$ and $\mathbf{r}\mathbf{c} = 1$. If this is the case, then choose S to be a matrix $[\mathbf{c}, \mathbf{c}_2, \mathbf{c}_3, \ldots, \mathbf{c}_n]$ such that the columns $\{\mathbf{c}_i\}_{i=2}^n$ span the hyperplane orthogonal to $\mathbf{r}$. This will ensure that the first row of the matrix S^{-1} is actually equal to $\mathbf{r}$. Then a simple computation shows that the element in $(1, 1)$ position of the matrix $S^{-1}ZS$ is equal to $\mathbf{r}Z\mathbf{c}$ and hence we are done.

Now assume that no such row $\mathbf{r}$ and column $\mathbf{c}$ exist. This is possible only if $\mathbf{r}Z = \lambda\mathbf{r}$ for some λ (depending of $\mathbf{r}$), because otherwise we can choose $\mathbf{c}$ to be orthogonal to $\mathbf{r}Z$ and then multiply $\mathbf{c}$ by a constant to ensure $\mathbf{rc} = 1$. Since $\mathbf{r}$ was an arbitrary row, this implies that $\mathbf{r}Z = \lambda(\mathbf{r})\mathbf{r}$ for every row $\mathbf{r}$ such that $\mathbf{r}Z \neq \mathbf{0}$. The last conclusion automatically implies that $S^{-1}ZS$ is a diagonal matrix for every choice of S. Now we observe that $\lambda(\mathbf{r})$ does not depend on $\mathbf{r}$. Indeed, from $\mathbf{r}Z = \lambda(\mathbf{r})\mathbf{r}$ for every row $\mathbf{r}$, we obtain that $\mathbf{r}_1 Z - \mathbf{r}_2 Z = \lambda(\mathbf{r}_1)\mathbf{r}_1 - \lambda(\mathbf{r}_2)\mathbf{r}_2$ for every two rows in any invertible matrix which is only possible when $\lambda(\mathbf{r}_1) = \lambda(\mathbf{r}_2)$ (otherwise $\mathbf{r}_1$ and $\mathbf{r}_2$ have to be linearly dependent, which is impossible). But this is a contradiction to the statement that Z is not a multiple of the identity, hence the claim in the beginning is proved.

Step III.

Finally, we are ready to finish the proof via induction on the size of the matrix. If $Z \in M_1$ then $tr(Z) = 0 \Leftrightarrow Z \equiv \mathbf{0}$ and there is nothing to prove. For $n \geq 2$, $tr(Z) = 0$ implies that Z is not a non-zero multiple of the identity and hence we can apply the argument in **Step II** . There exists an invertible matrix S such that $S^{-1}ZS = \left(\begin{smallmatrix} 0 & \mathbf{r} \\ \mathbf{c} & Z' \end{smallmatrix}\right)$, where $Z' \in M_{n-1}$ and clearly, $tr(Z') = tr(Z) = 0$. According to the induction step Z' is a commutator and now using **Step I** we conclude that Z is a commutator. $\square$

B. Subspaces of Banach spaces

When studying Banach spaces, it is often important to find a subspace of a given Banach space that has certain properties. A subspace Y of a Banach space $\mathcal{X}$ is called a complemented subspace if there exists an idempotent operator $P : \mathcal{X} \to \mathcal{X}$ such that $P(\mathcal{X}) = Y$ and $P^2 = P$ (such operators we call projections). In finite dimensional settings, all subspaces of a given finite dimensional Banach spaces $\mathcal{X}$ are complemented, but this is not always true if $\mathcal{X}$ is infinite dimensional. In some cases we have that a subspace Y of $\mathcal{X}$ is automatically complemented, an example of which we state below.

Lemma II.2. *[22, Lemma 1] Assume that $\mathcal{X} = c_0$ or $\mathcal{X} = \ell_p, 1 \leq p < \infty$. Let (z_n) be a sequence in $\mathcal{X}$ such that there exists an increasing sequence of indices $0 = p_0 < p_1 < \ldots$ such that the expansion of z_m in the unit vector basis of $\mathcal{X}$ is of the form*

$$z_m = \sum_{i=p_{m-1}+1}^{p_m} t_i^m e_i \qquad (m = 1, 2, \ldots)$$

(such a sequence is called a block basis sequence). Then the subspace $[z_n]$ (the closed linear span of the vectors (z_n)) is isometrically isomorphic to $\mathcal{X}$ and is complemented in $\mathcal{X}$ by a norm one projection.

The following theorem is well known and it will be used several times throughout this dissertation.

Theorem II.3. *Let $\mathcal{X}$ is a Banach space such that $\mathcal{X} = c_0$ or $\mathcal{X} = \ell_p, 1 \leq p \leq \infty$ or $\mathcal{X} = L_p, 1 \leq p \leq \infty$. If Y is an infinite dimensional subspace of $\mathcal{X}$ such that $Y \simeq \mathcal{X}$, then Y contains a subspace Z such $Z \simeq \mathcal{X}$ and Z is complemented in X.*

Proof. For $\mathcal{X} = \ell_p, 1 \leq p \leq \infty$, this result is due to Pełczyński ([22]), except for the case $\mathcal{X} = \ell_\infty$, which is due to Lindenstrauss ([18]). If $\mathcal{X} = c_0$ the theorem is due to Sobczyk ([28]). Note also that in the cases $\mathcal{X} = c_0, \ell_\infty, \ell_2$ we can take $Z = Y$ ([22, Theorem 3]). Also, if $\mathcal{X} = \ell_p$, $1 \leq p < \infty$, then Y could be any infinite dimensional subspace of $\mathcal{X}$ and the conclusion of the theorem remains valid. If $\mathcal{X} = L_1$ the result is due to Enflo and Starbird ([8]). In this case, for any $\varepsilon > 0$ we can take Z to be $(1+\varepsilon)$ isomorphic to $\mathcal{X}$. Finally, if $\mathcal{X} = L_p$, $1 < p \neq 2 < \infty$, the theorem was proved by Johnson et al. ([13, Theorem 9.1]). □

C. Ideals in Banach spaces

1. Ideals in c_0, ℓ_p, and L_p

In this section we define the notion of *ideal* in $\mathcal{L}(\mathcal{X})$ for an infinite dimensional Banach space $\mathcal{X}$ and show some basic facts about the ideals in spaces of operators. Ideals will appear quite often in our study, and as will become apparent in the later chapters, they will play an important role in the study of the commutators on Banach spaces.

Definition II.4. *An operator ideal $\mathcal{I}(X)$ in $\mathcal{L}(\mathcal{X})$ is a subset of $\mathcal{L}(\mathcal{X})$ such that*

1. $\mathcal{I}(X) \subset \mathcal{L}(\mathcal{X})$
2. *If $T \in \mathcal{F}(\mathcal{X})$ then $T \in \mathcal{I}(X)$*
3. *If $T_1, T_2 \in \mathcal{I}(X)$ then $T_1 + T_2 \in \mathcal{I}(X)$*
4. *If $R, S \in \mathcal{L}(\mathcal{X})$ are arbitrary operators and $T \in \mathcal{I}(X)$ then $RTS \in \mathcal{I}(X)$*

The ideal $\mathcal{I}(X)$ is called closed *if it is closed in the usual operator norm. $\mathcal{I}(X)$ is called proper if $\mathcal{I}(X) \neq \mathcal{L}(\mathcal{X})$.*

It is an easy observation that $\mathcal{F}(\mathcal{X})$ is the smallest operator ideal, but it is not closed if $\mathcal{X}$ is infinite dimensional. A less trivial (but also quite easy) fact is that the ideal of the compact operators $\mathcal{K}(\mathcal{X})$ (operators that map the closed unit ball of $\mathcal{X}$ into a set that is relatively compact in the norm topology) is the smallest closed operator ideal if $\mathcal{X}$ has the approximation property (a Banach spaces where every compact operator is a limit, in the operator norm, of finite rank operators).

Proposition II.5. *[[7, Lemma 3] , [9]]Let $\mathcal{X}$ be an infinite dimensional Banach space having the approximation property. Then $\mathcal{K}(\mathcal{X})$ is the smallest closed operator ideal in $\mathcal{L}(\mathcal{X})$. If $\mathcal{X} = c_0$ or $\mathcal{X} = \ell_p, 1 \leq p < \infty$, then $\mathcal{K}(X)$ is the largest proper operator ideal in $\mathcal{L}(\mathcal{X})$.*

Proof. This proposition is well known and here we present only a sketch of the proof. The set of all compact operators is clearly an ideal, so we only have to check that it is closed. This is done by a showing that $\overline{TB_{\mathcal{X}}}$ is a totally bounded set for every T which is a limit of compact operators. The fact that $\mathcal{K}(\mathcal{X})$ is the smallest ideal follows from the definition of the approximation property.

To see that $\mathcal{K}(\mathcal{X})$ is the largest proper ideal if $\mathcal{X} = c_0$ or $\mathcal{X} = \ell_p, 1 \leq p < \infty$, one can show that if an operator is not compact then the identity factors through it and hence no non-compact operator can belong to a proper ideal. In order to see this factorization result one can show that T is not a compact operator if and only if it is an isomorphism on some complemented subspace of $\mathcal{X}$ which is also isomorphic to $\mathcal{X}$ (this equivalence we will use in the future as well). Now it is easy to finish the proof. □

The complete ideal structure in L_p, $1 \leq p < \infty$, is not known, but we know what the largest ideals in these spaces are. In [8] Enflo and Starbird introduced the ideal of non-E operators on L_1 and proved that $T \in \mathcal{L}(L_1)$ is an E-operator if and only if it is an isomorphism on a subspace of L_1 which is isomorphic to L_1. The original definition we will omit since we do not need it. Using the properties of the subspaces of L_1 which are isomorphic to L_1 one can show that the ideal of non-E operators is the largest ideal in $\mathcal{L}(L_1)$. In [13] the ideal of non - A operators was introduced. It was also shown there that a non-A operator is an operator on L_p which does not preserve an isomorphic copy of L_p. As in the case of L_1, one can show that non-A operators are the largest ideal in $\mathcal{L}(L_p)$, $1 < p < \infty$. One can immediately observe that the ideals of the non-E and non-A operators have a quite similar definition which can be generalized, but we will leave this till Chapter VI.

We will also be interested in the ideals in $\mathcal{L}(\ell_\infty)$. The complete classification of those ideals is not known, but fortunately we know that there is a largest ideal in $\mathcal{L}(\ell_\infty)$. Denote by $\mathcal{S}(\mathcal{X})$ the ideal of strictly singular operators (operators that are not isomorphisms on any infinite dimensional subspace of $\mathcal{X}$). It is a known fact, the proof of which we present below, that $\mathcal{S}(\ell_\infty)$ is the largest ideal in $\mathcal{L}(\ell_\infty)$.

Lemma II.6. *The ideal of strictly singular operators is the largest ideal in $\mathcal{L}(\ell_\infty)$.*

Proof. Assume that T is not a strictly singular operator. Our goal is to prove that any ideal that contains T must coincide with $\mathcal{L}(\ell_\infty)$. Note first that on ℓ_∞ the ideals of the weakly compact and the strictly singular operators coincide ([30, Theorem 1.2]). Then we use the fact that any non-weakly compact operator on ℓ_∞ is an isomorphism on some subspace Y of ℓ_∞ isomorphic to ℓ_∞ ([24, Corollary 1.4]). The subspaces Y and TY will be automatically complemented in ℓ_∞ because ℓ_∞ is an injective space. This yields that I_{ℓ_∞} factors through T and hence any ideal containing T coincides with $\mathcal{L}(\ell_\infty)$. □

As we saw in the proof of the preceding lemma, an operator T is strictly singular if and only if it is an isomorphism on a subspace isomorphic to ℓ_∞. This property we have already observed when considering the largest ideals in ℓ_p and L_p, $1 \leq p < \infty$, and the following corollary summarizes these observations on the spaces considered so far.

Corollary II.7. *If $\mathcal{X}$ is a Banach space such that $\mathcal{X} \simeq c_0$ or $\mathcal{X} \simeq \ell_p, 1 \leq p \leq \infty$ or $\mathcal{X} \simeq L_p, 1 \leq p \leq \infty$, then $\mathcal{X}$ has a largest ideal $\mathcal{M}$. Moreover, $T \notin \mathcal{M}$ if and only if T is an isomorphism on a complemented subspace of $\mathcal{X}$ which is also isomorphic to $\mathcal{X}$.*

2. Ideals in other Banach spaces

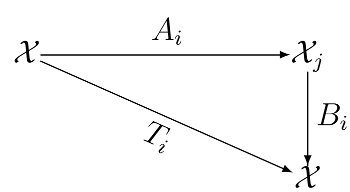
There are Banach spaces $\mathcal{X}$ for which there is no largest ideal in $\mathcal{L}(\mathcal{X})$, but of course there are maximal ideals in $\mathcal{L}(\mathcal{X})$. In the space $\ell_p \oplus \ell_q$, $1 \leq p < q < \infty$, there are exactly two maximal ideals ([23]), namely, the closure of the ideal of the operators that factor through ℓ_p, which we denote by α_p, and the closure of the ideal of the operators that factor through ℓ_q, which we denote by α_q. Instead of considering a sum of just two different ℓ_p spaces, we consider a finite sum of arbitrary length of different ℓ_p spaces. As we show in the lemma below, the structure of the maximal ideals in this sum is similar to the structure of the maximal ideals in the case of just two summands.

Lemma II.8. *Let $p_1, p_2, \ldots, p_n$ are n distinct numbers and denote $\mathcal{X} = \ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n}$. Then there are exactly n maximal ideals in $\mathcal{L}(\mathcal{X})$, namely, the closure of the operators that factor through $\oplus_{i \neq j} \ell_{p_i}$ for $j = 1, 2, \ldots, n$.*

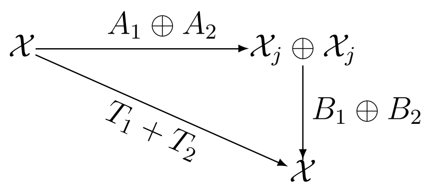
Proof. Denote $\mathcal{X}_j = \oplus_{i \neq j} \ell_{p_i}$. It is not hard to see that the set

$$M_j = \overline{\{T \in \mathcal{L}(\mathcal{X}) \,:\, T \text{ factors through } \mathcal{X}_j\}}$$

is an ideal. It is clearly closed under multiplication by an arbitrary operator from left and right and we have to show that the sum of two operators in M_j belongs to M_j. Let $T_1, T_2 \in M_j$ and let A_i, B_i, $i = 1, 2$, be such that the following diagram commutes



for $i = 1, 2$. We immediately obtain that the following diagram commutes



where the operators $A_1 \oplus A_2$ and $B_1 \oplus B_2$ are defined by $(A_1 \oplus A_2)(x) = (A_1(x), A_2(x))$, $(B_1 \oplus B_2)(x, y) = B_1(x) + B_2(y)$. Now using the fact that $\mathcal{X}_j \oplus \mathcal{X}_j \simeq \mathcal{X}_j$, $j = 1, 2, \ldots, n$, we conclude that $T_1 + T_2 \in M_j$ which shows that M_j is closed under addition. One can also show that an equivalent definition of the ideal M_j is:

$$M_j = \{T \in \mathcal{L}(\mathcal{X}) \,:\, P_j T P_j \in \mathcal{K}(\ell_{p_j})\}, \tag{2.2}$$

where P_i is the natural projection from $\mathcal{X}$ onto ℓ_{p_i}. This can be shown with an argument similar to the argument where we proved that M_j is closed under addition. First observe that the operator $T - \sum_{i=1}^{n} P_j T P_i$ factors through $\mathcal{X}_j$ simply because its range lies into $\mathcal{X}_j$. Then we have to observe that if $i \neq j$ then $P_j T P_i$ always factors through $\mathcal{X}_j$ (in a obvious way $P_j T P_i = (P_j T) \circ P_i$), thus $T - P_j T P_j \in M_j$. If we assume that $T \in M_j$, then we have that $P_j T P_j \in M_j$ which is only possible if $P_j T P_j$ is a compact operator, because for $p \neq q$ the spaces ℓ_p and ℓ_q are *totally incomparable* (do not have isomorphic infinite dimensional subspaces).

The ideal M_j is maximal, because if we assume that there exists a proper ideal M, such that $M_j \subsetneq M$, then there exists $S \in M$ such that $P_j S P_j$ is not a compact operator. This implies that $I_{\ell_{p_j}}$ - the identity on ℓ_{p_j} factors through $P_j S P_j$ hence there exists operators $A', B' \in \mathcal{L}(\ell_{p_j})$, such that $A' P_j S P_j B' = I_{\ell_{p_j}}$. Let J_j be the natural injection from ℓ_{p_j} into $\mathcal{X}$ and consider operators $A, B \in \mathcal{L}(\mathcal{X})$ defined by $A := J_j A' P_j + I - P_j$ and $B := J_j B' P_j + I - P_j$, and, also let $T = I - P_j + P_j S P_j$. We have that $T \in M$ because $I - P_j \in I_j$ and $P_j S P_j \in M$. Now from the equality

$I_{\mathcal{X}} = ATB$, which follows from the definition of A and B, we have that $I \in M$ which is a contradicts the fact that M is a proper ideal.

Assume now that M is a maximal ideal not in $\{M_j\}_{j=1}^n$. Then $M \backslash M_j \neq \emptyset$ for $j = 1, 2 \ldots, n$. This implies that for every $j = 1, 2, \ldots, n$ there exists an operator $S_j \in M$ such that $P_j S_j P_j$ is not a compact operator as an operator acting on ℓ_{p_j}. As before, for every $j = 1, 2, \ldots, n$ we find operators $A'_j, B'_j \in \mathcal{L}(\ell_{p_j})$, such that $A'_j P_j S P_j B'_j = I_{\ell_{p_j}}$. Finally, define an operator $T \in \mathcal{L}(\mathcal{X})$ by

$$T = \sum_{j=1}^{n} J_j A'_j P_j S_j P_j B'_j P_j.$$

Since $S_j \in M$ for every $j = 1, 2, \ldots, n$ we have that $T \in M$. On the other side

$$T = \sum_{j=1}^{n} J_j A'_j P_j S_j P_j B'_j P_j = \sum_{j=1}^{n} J_j I_{\ell_{p_j}} P_j = \sum_{j=1}^{n} P_j = I_{\mathcal{X}}$$

which is a contradiction with the assumption that M is a proper ideal.

$\square$

Let us also mention that there are more Banach spaces for which the maximal ideals in the Banach algebra of the operators on that space are known. For example $X_0 = (\sum \ell_2^n)_{c_0}$ and $X_1 = (\sum \ell_2^n)_{\ell_1}$ are such spaces. In [16] it was shown that $\mathcal{L}(X_0)$ has exactly two non-trivial ideals - the compact operators and the closure of the set of operators factoring through c_0. Because $X_1 = (X_0)^*$, one can expect that we have a similar classification of the norm closed ideals in $\mathcal{L}(X_1)$. Indeed, in [17] it was shown that $\mathcal{L}(X_1)$ also has exactly two non-trivial ideals - the compact operators and the closure of the set of operators factoring through ℓ_1. Note also that $X_0 \simeq \left(\sum X_0\right)_0$ and $X_1 \simeq \left(\sum X_1\right)_1$, which (as it will become apparent later) is important in our study.

CHAPTER III

COMMUTATORS ON $\left(\sum Y\right)_p$

A. Notation and basic results

For a Banach space $\mathcal{X}$ denote by the $\mathcal{L}(\mathcal{X})$, $\mathcal{K}(\mathcal{X})$, $\mathcal{C}(\mathcal{X})$ and $S_{\mathcal{X}}$ the space of all bounded linear operators, the ideal of the compact operators, the set of all finite co-dimensional subspaces of $\mathcal{X}$ and the unit sphere of $\mathcal{X}$. A map from a Banach space $\mathcal{X}$ to a Banach space $\mathcal{Y}$ is said to be strictly singular if whenever the restriction of T to a subspace M of X has a continuous inverse, M is finite dimensional. In the case of $\mathcal{X} \equiv \mathcal{Y}$, the set of strictly singular operators forms an ideal which we will denote by $\mathcal{S}(\mathcal{X})$. Recall that for $\mathcal{X} = \ell_p$, $1 \leq p < \infty$, $\mathcal{S}(\mathcal{X}) = \mathcal{K}(X)$ ([9]) and on ℓ_∞ the ideals of strictly singular and weakly compact operators coincide ([1, Theorem 5.5.1]). A Banach space $\mathcal{X}$ is called *prime* if each infinite-dimensional complemented subspace of $\mathcal{X}$ is isomorphic to $\mathcal{X}$. For any two subspaces (possibly not closed) $\mathcal{X}$ and $\mathcal{Y}$ of a Banach space $\mathcal{Z}$ let

$$d(\mathcal{X}, \mathcal{Y}) = \inf\{\|x - y\| : x \in S_{\mathcal{X}},\ y \in \mathcal{Y}\}.$$

A well known consequence of the open mapping theorem is that for any two closed subspaces $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{Z}$, if $\mathcal{X} \cap \mathcal{Y} = \{0\}$ then $\mathcal{X} + \mathcal{Y}$ is a closed subspace of $\mathcal{Z}$ if and only if $d(\mathcal{X}, \mathcal{Y}) > 0$. Note also that $d(\mathcal{X}, \mathcal{Y}) = 0$ if and only if $d(\mathcal{Y}, \mathcal{X}) = 0$. First we prove a proposition that will later allow us to consider translations of an operator T by a multiple of the identity instead of the operator T itself.

Proposition III.1. *Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}(\mathcal{X})$ be such that there exists a subspace $Y \subset \mathcal{X}$ for which T is an isomorphism on Y and $d(Y, TY) > 0$. Then for every $\lambda \in \mathbb{C}$, $(T - \lambda I)_{|Y}$ is an isomorphism and $d(Y, (T - \lambda I)Y) > 0$.*

Proof. First, note that the two hypotheses on Y (that T is an isomorphism on Y and $d(Y,TY)>0$) are together equivalent to the existence of a constant $c>0$ such that for all $y\in S_Y$, $d(Ty,Y)>c$. To see this, let us first assume that the hypotheses of the theorem are satisfied. Then there exists a constant C such that $\|Ty\|\geq C$ for every $y\in S_Y$. For an arbitrary $y\in S_Y$, let $z_y=\frac{Ty}{\|Ty\|}$ and then we clearly have

$$d(Ty,Y)=\|Ty\|d(z_y,Y)\geq Cd(TY,Y)=:c>0.$$

To show the other direction note that for $y\in S_Y$, $0<c<d(Ty,Y)=\|Ty\|d(z_y,Y)\leq\|T\|d(z_y,Y)$. Taking the infimum over all $z_y\in S_Y$ in the last inequality, we obtain that $d(Y,TY)>0$. On the other hand, for all $y\in S_Y$ we have

$$0<c<d(Ty,Y)\leq\|Ty-\frac{c}{2}y\|\leq\|Ty\|+\frac{c}{2},$$

hence $\|Ty\|\geq\frac{c}{2}$, which in turn implies that T is an isomorphism on Y.

Now it is easy to finish the proof. The condition $d(Ty,Y)>c$ for all $y\in S_Y$ is clearly satisfied if we substitute T with $T-\lambda I$ since for a fixed $y\in S_Y$,

$$d((T-\lambda I)y,Y)=\inf_{z\in Y}\|(T-\lambda I)y-z\|=\inf_{z\in Y}\|Ty-z\|=d(Ty,Y),$$

hence $(T-\lambda I)_{|Y}$ is an isomorphism and $d(Y,(T-\lambda I)Y)>0$. □

Note the following two simple facts:

- If $T\colon\mathcal{X}\to\mathcal{X}$ is a commutator on $\mathcal{X}$ and $S\colon\mathcal{X}\to\mathcal{Y}$ is an onto isomorphism, then STS^{-1} is a commutator on $\mathcal{Y}$.

- Let $T\colon\mathcal{X}\to\mathcal{X}$ be such that there exists $X_1\subset\mathcal{X}$ for which $T_{|X_1}$ is an isomorphism and $d(X_1,TX_1)>0$. If $S\colon\mathcal{X}\to\mathcal{Y}$ is an onto isomorphism, then there exists $Y_1\subset\mathcal{Y}$, $Y_1\simeq X_1$, such that $STS^{-1}{}_{|Y_1}$ is an isomorphism and $d(Y_1,STS^{-1}Y_1)>0$ (in fact $Y_1=SX_1$). Note also that if X_1 is complemented in $\mathcal{X}$, then Y_1 is complemented in $\mathcal{Y}$.

Using the two facts above, sometimes we will replace an operator T by an operator T_1 which is similar to T and possibly acts on another Banach space.

If $\{Y_i\}_{i=0}^{\infty}$ is a sequence of arbitrary Banach spaces, by $\left(\sum_{i=0}^{\infty} Y_i\right)_p$ we denote the space of all sequences $\{y_i\}_{i=0}^{\infty}$ where $y_i \in Y_i$, $i = 0, 1, \ldots$, such that $(\|y_i\|_{Y_i}) \in \ell_p$ with the norm $\|(y_i)\| = \|\|y_i\|_{Y_i}\|_p$ (if $Y_i \equiv Y$ for every $i = 0, 1, \ldots$ we will use the notation $\left(\sum Y\right)_p$). We will only consider the case where all the spaces Y_i, $i = 0, 1 \ldots$, are uniformly isomorphic to a Banach space Y, that is, there exists a constant $\lambda > 0$ and sequence of isomorphisms $\{T_i \colon Y_i \to Y\}_{i=0}^{\infty}$ such that $\|T^{-1}\| = 1$ and $\|T\| \leq \lambda$. In this case we define an isomorphism $U \colon \left(\sum_{i=0}^{\infty} Y_i\right)_p \to \left(\sum Y\right)_p$ via (T_i) by

$$U(y_0, y_1, \ldots) = (T_0(y_0), T_1(y_1), \ldots), \tag{3.1}$$

and it is easy to see that $\|U\| \leq \lambda$ and $\|U^{-1}\| = 1$. Sometimes we will identify the space $\left(\sum_{i=0}^{\infty} Y_i\right)_p$ with $(\sum Y)_p$ via the isomorphism U when there is no ambiguity how the properties of an operator T on $\left(\sum_{i=0}^{\infty} Y_i\right)_p$ translate to the properties of the operator UTU^{-1} on $\left(\sum Y\right)_p$.

For $y = (y_i) \in (\sum Y)_p$, $y_i \in Y$, define the following two operators :

$$R(y) = (0, y_0, y_1, \ldots) \qquad , \qquad L(y) = (y_1, y_2, \ldots).$$

The operators L and R are, respectively, the left and the right shift on the space $(\sum Y)_p$. Denote by P_i, $i = 0, 1, \ldots$, the natural, norm one projection from $\left(\sum Y\right)_p$ onto the i-th component of $\left(\sum Y\right)_p$, which we denote by Y^i. Our first proposition shows some basic properties of the left and the right shift as well as the fact that all the powers of L and R are uniformly bounded, which will play an important role in the sequel.

Proposition III.2. *Consider the Banach space* $\left(\sum Y\right)_p$*. We have the following identities*

$$\|L^n\| = 1 \ , \ \|R^n\| = 1 \ \textit{for every}\ n = 1, 2, \dots \tag{3.2}$$

$$LR = I \ , \ RL = I - P_0 \ , \ RP_i = P_{i+1}R \ , \ P_iL = LP_{i+1} \ \textit{for}\ i \geq 0. \tag{3.3}$$

$$\lim_{n\to\infty} \|L^n(x)\| = 0 \ \textit{for all}\ 1 \leq p < \infty \ \textit{and}\ p = 0. \tag{3.4}$$

Proof. The relations in (3.2) and (3.3) follow immediately from the definitions of the left and right shift. The relation in (3.4) follows from the definition of the norm on $\left(\sum Y\right)_p$ in the cases $1 \leq p < \infty$ and $p = 0$. □

Note that we can define a left and right shift on $\left(\sum_{i=0}^{\infty} Y_i\right)_p$ by $\tilde{L} = U^{-1}LU$ and $\tilde{R} = U^{-1}RU$, and, using the above proposition, we immediately have $\|\tilde{R}^n\| \leq \lambda$ and $\|\tilde{L}^n\| \leq \lambda$. If there is no ambiguity, will denote the left and the right shift on $\left(\sum_{i=0}^{\infty} Y_i\right)_p$ simply by L and R.

Following the ideas in [3], for $1 \leq p < \infty$ and $p = 0$ define the set

$$\mathcal{A} = \{T \in \left(\sum Y\right)_p : \sum_{n=0}^{\infty} R^nTL^n \ \text{ is strongly convergent}\}, \tag{3.5}$$

and for $T \in \mathcal{A}$ define

$$T_{\mathcal{A}} = \sum_{n=0}^{\infty} R^nTL^n.$$

Note that an operator T is a commutator if and only if T is in the range of D_S for some S, where D_S is the inner derivation determined by S, defined by $D_S(T) = ST - TS$. Our next lemma shows that each operator $T \in \mathcal{A}$ is a commutator and also gives an explicit expression for T as the commutator of two operators.

Lemma III.3. *([6, Lemma 3]) Let* $T \in \mathcal{A}$ *for some decomposition* $\mathcal{D} = \{X_i\}$ *of* $\mathcal{X}$*. Then we have*

$$T = D_L(RT_{\mathcal{A}}) = -D_R(T_{\mathcal{A}}L), \tag{3.6}$$

hence T is a commutator.

Proof. We will show one of the equalities via direct computation. The proof of the other is similar.

$$\begin{aligned} D_L(RT_{\mathcal{A}}) &= LRT_{\mathcal{A}} - RT_{\mathcal{A}}L = T_{\mathcal{A}} - R(\sum_{n=0}^{\infty} R^n T L^n)L \\ &= T_{\mathcal{A}} - \sum_{n=1}^{\infty} R^n T L^n = T. \end{aligned}$$

In the computation above we used the convention $L^0 = R^0 = I$. □

B. Main result

The ideas in this section are similar to the ideas in [6], but here we present them from a different point of view, in a more general setting and we also include the case $p = \infty$. The following lemma is a generalization of [3, Lemma 2.8] in the case $p = \infty$ and [6, Corollary 7] in the case $1 \leq p < \infty$ and $p = 0$. The proof presented here follows the ideas of the proofs in [3] and [6].

Lemma III.4. *Let $T \in \mathcal{L}\big(\big(\sum Y\big)_p\big)$. Then the operators P_0T and TP_0 are commutators.*

Proof. The proof shows that P_0T is in the range of D_L and TP_0 is in the range of D_R. We will consider two cases depending on p.

Case I : $p = \infty$

In this case we first observe that the series

$$S_0 = \sum_{n=0}^{\infty} R^n P_0 T L^n$$

is pointwise convergent coordinatewise. Indeed, let $x \in \big(\sum Y\big)_\infty$ and define $y_n =$

$R^nP_0TL^nx$ for $n = 0, 1, \ldots$. Note that from the definition we immediately have $y_n \in Y^n$ so the sum $\sum_{n=0}^{\infty} y_n$ converges in the product topology on $\left(\sum Y\right)_\infty$ to a point in $\left(\sum Y\right)_\infty$ since $\|y_n\| \leq \|R^n\|\|P_0\|\|T\|\|L^n\|\|x\| \leq \|T\|\|x\|$.

Secondly, we observe that S_0 and L commute. Because L and R are continuous operators on $\left(\sum Y\right)_\infty$ with the product topology and $LR = I$, we have

$$\begin{aligned} S_0Lx &= \sum_{n=0}^{\infty} R^nP_0TL^{n+1}x = L\left(\sum_{n=1}^{\infty} R^nP_0TL^nx\right) = L\left(\sum_{n=0}^{\infty} R^nP_0TL^nx\right) - LP_0Tx \\ &= LS_0x - 0 \end{aligned} \tag{3.7}$$

since $LP_0 = 0$. That is, $D_LS_0 = 0$, as desired.

On the other hand, again using $LP_0 = 0$,

$$\begin{aligned} (I - RL)S_0x &= \sum_{n=0}^{\infty}(I - RL)R^nP_0TL^nx = (I - RL)P_0Tx + \underbrace{\sum_{n=1}^{\infty}(I - RL)R^nP_0TL^nx}_{0} \\ &= (I - RL)P_0Tx = P_0Tx. \end{aligned} \tag{3.8}$$

Therefore

$$D_L(RS_0) = (D_LR)S_0 + R(D_LS_0) = (I - RL)S_0 + 0 = P_0S_0 = P_0T. \tag{3.9}$$

The proof of the statement that TP_0 is a commutator involves a similar modification of the proof of [3, Lemma 2.8]. Again, consider the series

$$S = \sum_{n=0}^{\infty} R^nP_0TP_0L^n.$$

This is pointwise convergent coordinatewise and $SL = LS$ (from the above reasoning applied to the operator TP_0), and

$$D_R(-SL) = -D_R(LS) = -RLS + LSR = -(I - P_0)S + LSR$$
$$= -S + P_0S + SLR = -S + P_0S + S - P_0S = P_0TP_0.$$

Now it is easy to see that

$$D_R(LTP_0 - SL) = RLTP_0 - \underbrace{LTP_0R}_{0} + P_0TP_0 = (I - P_0)TP_0 + P_0TP_0 = TP_0.$$

Case II : $1 \leq p < \infty$ or $p = 0$

In this case the proof is similar to the proof of [6, Lemma 6 and Corollary 7] and we include it for completeness. Let us consider the case $p \geq 1$ first. For any $y \in \left(\sum Y\right)_p$ we have

$$\begin{aligned} \|\sum_{n=m}^{m+r} R^n P_i T P_j L^n y\|^p &= \|\sum_{n=m}^{m+r} R^n P_i T P_j L^n P_{j+n} y\|^p \leq \|P_i T P_j\|^p \sum_{n=m}^{m+r} \|P_{j+n} y\|^p \\ &\leq \|P_i T P_j\|^p \sum_{n=m}^{\infty} \|P_{j+n} y\|^p. \end{aligned}$$

Since $\sum_{n=m}^{\infty} \|P_{j+n} y\|^p \to 0$ as $m \to \infty$ we have that $\sum_{n=0}^{\infty} R^n P_i T P_j L^n$ is strongly convergent and $P_i T P_j \in \mathcal{A}$.

For $p = 0$ a similar calculation shows

$$\begin{aligned} \|\sum_{n=m}^{m+r} R^n P_i T P_j L^n y\| &= \|\sum_{n=m}^{m+r} R^n P_i T P_j L^n P_{j+n} y\| = \max_{m \leq n \leq m+r} \|R^n P_i T P_j L^n P_{j+n} y\| \\ &\leq \|P_i T P_j\| \max_{m \leq n \leq m+r} \|P_{j+n} y\| \end{aligned}$$

and since $\max_{m \leq n \leq m+r} \|P_{j+n} y\| \to 0$ as $m \to \infty$ we apply the same argument as in the case $p \geq 1$ to obtain $P_i T P_j \in \mathcal{A}$.

Using $P_i T P_j \in \mathcal{A}$ for $i = j = 0$ and (3.6) we have $P_0 T P_0 = D_L(R(P_0 T P_0)_{\mathcal{A}}) =$

$-D_R((P_0TP_0)_{\mathcal{A}}L)$. Again, as in [6, Corollary 7], via direct computation we obtain

$$TP_0 = D_R(LTP_0 - (P_0TP_0)_{\mathcal{A}}L) \tag{3.10}$$

$$P_0T = D_L(-P_0TR + R(P_0TP_0)_{\mathcal{A}}). \tag{3.11}$$

□

Now we switch our attention to Banach spaces which in addition satisfy $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$ for some $1 \le p \le \infty$ or $p = 0$. Note that the Banach space $\left(\sum Y\right)_p$ satisfies this condition regardless of the space Y, hence we will be able to use the results we proved so far in this chapter.

Definition III.5. *Let $\mathcal{X}$ be a Banach space such that $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$, $1 \le p \le \infty$ or $p = 0$. We say that $\mathcal{D} = \{X_i\}_{i=0}^{\infty}$ is a decomposition of $\mathcal{X}$ if it forms an ℓ_p or c_0 decomposition of $\mathcal{X}$ into subspaces which are uniformly isomorphic to $\mathcal{X}$; that is, if the following three conditions are satisfied:*

1. *There are uniformly bounded projections P_i on $\mathcal{X}$ with $P_i\mathcal{X} = X_i$ and $P_iP_j = 0$ for $i \neq j$*
2. *There exists a collection of isomorphisms $\psi_i : X_i \to \mathcal{X}$, $i \in \mathbb{N}$, such that $\|\psi_i^{-1}\| = 1$ and $\lambda = \sup_{i\in\mathbb{N}} \|\psi_i\| < \infty$*
3. *The formula $Sx = (\psi_i P_i x)$ defines a surjective isomorphism from $\mathcal{X}$ onto $\left(\sum \mathcal{X}\right)_p$*

Remark III.6. *We should point out that immediately from the definition of decomposition we have*

- *For every $x \in \mathcal{X}$, $\sum_{n=m}^{\infty} \|P_n x\|^p \to 0$ as $m \to \infty$. This is easy to see just* *noticing that if* $\|x\| = 1$ *then*

$$\sum_{n=0}^{\infty} \|P_n x\|^p \leq \sum_{n=0}^{\infty} \|\psi_n P_n x\|^p = \|Sx\|^p \leq \|S\|^p < \infty$$

- *If* $\widetilde{P}_n = \sum_{i=0}^{n} P_i$ *then there exists a constant* C *depending only on the decomposition* $\mathcal{D}$ *such that* $\|\widetilde{P}_n\| \leq C$ *for all* $n = 0, 1, \ldots$. *To see this, note that for every* $x \in \mathcal{X}$, $\|x\| = 1$ *we have*

$$\begin{aligned}\|\widetilde{P}_n(x)\|^p &\leq \|S^{-1}\|^p \|S\widetilde{P}_n(x)\|^p = \|S^{-1}\|^p \sum_{i=0}^{n} \|\psi_i P_i x\|^p \\ &\leq \|S^{-1}\|^p \sum_{i=0}^{\infty} \|\psi_i P_i x\|^p = \|S^{-1}\|^p \|S(x)\|^p \leq (\|S^{-1}\| \|S\|)^p\end{aligned}$$

If $\mathcal{D} = \{X_i\}_{i=0}^{\infty}$ is a decomposition of $\mathcal{X}$ we have $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p \simeq (\sum_{i=0}^{\infty} X_i)_p$, where the second isomorphic relation is via the isomorphism U defined in (3.1). Using this simple observation we will often identify $\mathcal{X}$ with $(\sum_{i=0}^{\infty} X_i)_p$. Our next theorem is similar to [6, Theorem 16] and [3, Theorem 4.6], but we state it and prove it in a more general setting and also include the case $p = \infty$.

Theorem III.7. *Let* $\mathcal{X}$ *be a Banach space such that* $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$, $1 \leq p \leq \infty$ *or* $p = 0$. *Let* $T \in \mathcal{L}(\mathcal{X})$ *be such that there exists a subspace* $X \subset \mathcal{X}$ *such that* $X \simeq \mathcal{X}$, $T_{|X}$ *is an isomorphism,* $X + T(X)$ *is complemented in* $\mathcal{X}$ *and* $d(X, T(X)) > 0$. *Then there exists a decomposition* $\mathcal{D}$ *of* $\mathcal{X}$ *such that* T *is similar to a matrix operator of the form*

$$\begin{pmatrix} * & L \\ * & * \end{pmatrix}$$

on $\mathcal{X} \oplus \mathcal{X}$, *where* L *is the left shift associated with* $\mathcal{D}$.

Proof. Clearly $\mathcal{X} = X \oplus T(X) \oplus Z$ where Z is complemented in $\mathcal{X}$. Note that without loss of generality we can assume that Z is isomorphic to $\mathcal{X}$. Indeed, if this is not the case, let $X = X_1 \oplus X_2$, $X \simeq X_1 \simeq X_2$ and X_1, X_2 complemented in X (hence also complemented in $\mathcal{X}$). Then $d(X_1, T(X_1)) > 0$ and $\mathcal{X} = X_1 \oplus T(X_1) \oplus Z_1$ where Z_1 is a complemented subspace of $\mathcal{X}$, which contains the subspace $X_2 \subset \mathcal{X}$, such that X_2 is isomorphic to $\mathcal{X}$ and complemented in Z. Applying the Pełczýnski decomposition technique ([22, Proposition 4]), we conclude that Z_1 is isomorphic to X. This observation plays an important role and will allow us to construct the decompositions we need during the rest of the proof.

Denote by $I - P$ the projection onto $T(X)$. Consider two decompositions $\mathcal{D}_1 = \{X_i\}_{i=0}^{\infty}$, $\mathcal{D}_2 = \{Y_i\}_{i=0}^{\infty}$ of $\mathcal{X}$ such that $T(X) = Y_0 = X_1 \oplus X_2 \oplus \ldots$, $X_0 = Y_1 \oplus Y_2 \oplus \ldots$ and $Y_1 = X$. Define a map S

$$S\varphi = L_{\mathcal{D}_1}\varphi \oplus L_{\mathcal{D}_2}\varphi, \qquad \varphi \in \mathcal{X}$$

from $\mathcal{X}$ to $\mathcal{X} \oplus \mathcal{X}$. The map S is invertible ($S^{-1}(a,b) = R_{\mathcal{D}_1}a + R_{\mathcal{D}_2}b$). Just using the definition of S and the formula for S^{-1} it is easy to see that

$$\begin{aligned} STS^{-1}(a,b) &= ST(R_{\mathcal{D}_1}a + R_{\mathcal{D}_2}b) = S(TR_{\mathcal{D}_1}a + TR_{\mathcal{D}_2}b) \\ &= (L_{\mathcal{D}_1}TR_{\mathcal{D}_1}a + L_{\mathcal{D}_1}TR_{\mathcal{D}_2}b) \oplus (L_{\mathcal{D}_2}TR_{\mathcal{D}_1}a + L_{\mathcal{D}_2}TR_{\mathcal{D}_2}b), \end{aligned}$$

hence

$$STS^{-1} = \begin{pmatrix} * & L_{\mathcal{D}_1}TR_{\mathcal{D}_2} \\ * & * \end{pmatrix}.$$

Let

$$A = P_{Y_0}TR_{\mathcal{D}_2} = (I-P)TR_{\mathcal{D}_2} \tag{3.12}$$

and note that $A_{|P_{Y_0}\mathcal{X}} \equiv A_{|(I-P)\mathcal{X}} : (I-P)\mathcal{X} \to (I-P)\mathcal{X}$ is onto and invertible since $R_{\mathcal{D}_2}$ is an isomorphism on $P_{Y_0}\mathcal{X}$ and $R_{\mathcal{D}_2}(P_{Y_0}\mathcal{X}) = Y_1 = X$. Here we used the fact that $P_{Y_0}T$ is an isomorphism on X ($PX = X$). Denote by T_0 the inverse of $A_{|P_{Y_0}\mathcal{X}}$ (note that T_0 is an automorphism on $(I-P)\mathcal{X}$) and consider

$$G = I + T_0(I-P) - T_0A.$$

We will show that $G^{-1} = A + P$. In fact, from the definitions of A and T_0 it is clear that

$$AT_0(I-P) = T_0A(I-P) = I - P\,,\; PT_0 = PA = 0\,,\; (I-P)A = A \tag{3.13}$$

and since A maps onto $(I-P)\mathcal{X}$ and $AT_{0|(I-P)\mathcal{X}} = I_{|(I-P)\mathcal{X}}$ we also have

$$A - AT_0A = 0. \tag{3.14}$$

Now using (3.13) and (3.14) it is easy to see that

$$\begin{aligned}
(A+P)G &= (A+P)(I + T_0(I-P) - T_0A)\\
&= A + AT_0(I-P) - AT_0A + P = I - P + P = I\\
G(A+P) &= (I + T_0(I-P) - T_0A)(A+P)\\
&= A + P + T_0(I-P)A + T_0(I-P)P - T_0AA - T_0AP\\
&= A + P + T_0A - T_0AA - T_0AP\\
&= P + (I - T_0A)A + T_0A(I-P)\\
&= P + (I - T_0A)(I-P)A + (I-P)\\
&= I + ((I-P) - T_0A(I-P))A\\
&= I + (I - P - (I-P))A = I.
\end{aligned}$$

Using a similarity we obtain

$$\begin{pmatrix} I & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} * & L_{\mathcal{D}_1} T R_{\mathcal{D}_2} \\ * & * \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix} = \begin{pmatrix} * & L_{\mathcal{D}_1} T R_{\mathcal{D}_2} G \\ * & * \end{pmatrix}.$$

It is clear that we will be done if we show that $L_{\mathcal{D}_1} = L_{\mathcal{D}_1} T R_{\mathcal{D}_2} G$. In order to do this consider the equation $(A + P)G = I \Leftrightarrow AG + PG = I$. Multiplying both sides of the last equation on the left by $L_{\mathcal{D}_1}$ gives us $L_{\mathcal{D}_1} AG + L_{\mathcal{D}_1} PG = L_{\mathcal{D}_1}$. Using $L_{\mathcal{D}_1} P \equiv L_{\mathcal{D}_1} P_{X_0} = 0$ we obtain $L_{\mathcal{D}_1} AG = L_{\mathcal{D}_1}$. Finally, substituting A from (3.12) in the last equation yields

$$L_{\mathcal{D}_1} = L_{\mathcal{D}_1} AG = L_{\mathcal{D}_1} P_{Y_0} T R_{\mathcal{D}_2} G = L_{\mathcal{D}_1} (I - P_{X_0}) T R_{\mathcal{D}_2} G = L_{\mathcal{D}_1} T R_{\mathcal{D}_2} G$$

which finishes the proof. □

The following theorem was proved in [3] for $X = \ell_p$, $1 < p < \infty$, but inessential modifications give the result in these general settings.

Theorem III.8. *Let $\mathcal{X}$ be a Banach space such that $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$. Let $\mathcal{D}$ be a decomposition of $\mathcal{X}$ and let L be the left shift associated with it. Then the matrix operator*

$$\begin{pmatrix} T_1 & L \\ T_2 & T_3 \end{pmatrix}$$

acting on $\mathcal{X} \oplus \mathcal{X}$ is a commutator.

Proof. Let $\mathcal{D} = \{X_i\}_{i=0}^{\infty}$ be the given decomposition. Consider a decomposition $\mathcal{D}_1 = \{Y_i\}$ such that $Y_0 = \bigoplus_{i=1}^{\infty} X_i$ and $X_0 = \bigoplus_{i=1}^{\infty} Y_i$. Now there exists an operator G such that $D_{L_{\mathcal{D}}} G = R_{\mathcal{D}_1} L_{\mathcal{D}_1} (T_1 + T_3)$. This can be done using Lemma III.4, since

$R_{\mathcal{D}_1} L_{\mathcal{D}_1} = I - P_{Y_0} = P_{X_0}$. By making the similarity

$$\widetilde{T} := \begin{pmatrix} I & 0 \\ G & I \end{pmatrix} \begin{pmatrix} T_1 & L \\ T_2 & T_3 \end{pmatrix} \begin{pmatrix} I & 0 \\ -G & I \end{pmatrix} = \begin{pmatrix} T_1 - LG & L \\ * & T_3 + GL \end{pmatrix}$$

we have $T_1 + T_3 - LG + GL = T_1 + T_3 - D_L G = T_1 + T_3 - R_{\mathcal{D}_1} L_{\mathcal{D}_1}(T_1 + T_3) = P_{Y_0}(T_1 + T_3)$. Using Corollary III.4 again we deduce that $T_1 + T_3 - LG + GL$ is a commutator. Thus by replacing T by $\widetilde{T}$ we can assume that $T_1 + T_3$ is a commutator, say $T_1 + T_3 = AB - BA$ and $\|A\| < \dfrac{1}{2}$ (this can be done by scaling). Denote by M_T left multiplication by the operator T. Then $\|M_R D_A\| < 1$ where R is the right shift associated with $\mathcal{D}$. The operator $T_0 = (M_I - M_R D_A)^{-1} M_R (T_3 B - T_2)$ is well defined and it is easy to see that

$$\begin{pmatrix} A & 0 \\ T_3 & A - L \end{pmatrix} \begin{pmatrix} B & I \\ T_0 & 0 \end{pmatrix} - \begin{pmatrix} B & I \\ T_0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ T_3 & A - L \end{pmatrix} = \begin{pmatrix} T_1 & L \\ T_2 & T_3 \end{pmatrix}.$$

This finishes the proof. □

Finally, we are in position to state the main theorem for this chapter. The proof follows immediately from Theorem III.7 and Theorem III.8 and is omitted.

Theorem III.9. *Let $\mathcal{X}$ be a Banach space such that $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$, $1 \le p \le \infty$ or $p = 0$. Let $T \in \mathcal{L}(\mathcal{X})$ be such that there exists a subspace $X \subset \mathcal{X}$ such that $X \simeq \mathcal{X}$, $T_{|X}$ is an isomorphism, $X + T(X)$ is complemented in $\mathcal{X}$ and $d(X, T(X)) > 0$. Then T is a commutator.*

CHAPTER IV

COMMUTATORS ON ℓ_p AND $L_p\,(1 \le p < \infty)^*$

In this chapter we focus our attention to the classical Banach spaces ℓ_p and $L_p\,(1 \le p < \infty)$. The commutators on ℓ_p, $1 < p < \infty$, have already been classified by Apostol in [3], but the method we use gives this classification as well so we include these cases in the statement of the results for completeness.

A. Notation and basic results

We begin this chapter with a lemma which gives us a description of the set $\mathcal{A}$ introduced in (3.5).

Lemma IV.1. *[6, Lemma 4] For a decomposition $\mathcal{D} = \{X_i\}_{i=0}^{\infty}$ of $\mathcal{X}$ we have the following relations*

$$\mathcal{A} = D_R(\mathcal{L}(\mathcal{X})RL) = D_L(RL\mathcal{L}(\mathcal{X})).$$

Proof. We will show the first of the relations. The proof of the second relation, as one may expect, is similar.

If $T \in \mathcal{A}$, then $T_{\mathcal{A}}L = T_{\mathcal{A}}LRL = (T_{\mathcal{A}}L)RL \in \mathcal{L}(\mathcal{X})RL$. Then using $T = -D_R(T_{\mathcal{A}}L)$ from (3.6) we have $T \in D_R(\mathcal{L}(\mathcal{X})RL)$. To prove the other direction, assume that $T \in \mathcal{L}(\mathcal{X})RL$ and let $T = SRL$ for some operator S (hence $TR = SR$). Then

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$$\begin{aligned}\sum_{n=0}^{m} R^n(D_R T)L^n &= \sum_{n=0}^{m} R^n(RT-TR)L^n = \sum_{n=0}^{m} R^{n+1}TL^n - \sum_{n=0}^{m} R^n TRL^n \\ &= \sum_{n=0}^{m} R^{n+1}SRLL^n - \sum_{n=0}^{m} R^n SRLRL^n \\ &= \sum_{n=0}^{m} R^{n+1}SRL^{n+1} - \sum_{n=0}^{m} R^n SRL^n \\ &= R^{m+1}SRL^{m+1} - SR = R^{m+1}TL^m - TR.\end{aligned}$$

Since $\lim_{m\to\infty} \|L^m(x)\| = 0$ for any $x \in \mathcal{X}$ from (3.4) and $\|R^m\| \leq 1$ for every $m > 0$, we have $\lim_{m\to\infty} \sum_{n=0}^{m} R^n(D_R T)L^n = -TR$. From the last equation we conclude that $D_R T \in \mathcal{A}$ and $(D_R T)_{\mathcal{D}} = -TR$. Moreover, from $TR = SR$ we have $(D_R T)_{\mathcal{D}} = -SR$ and multiplying both sides by L we obtain $(D_R T)_{\mathcal{D}} L = -T$. □

We proved in Lemma III.3 that for a given decomposition $\mathcal{D}$ all operators $T \in \mathcal{A}$ are commutators, but in general the condition in (3.5) is hard to check for a given operator T. We want to have a condition on T which is easy to check and which ensures the containment $T \in \mathcal{A}$. To be more precise, given an operator T, we want to have a condition on T which will allow us to build a decomposition $\mathcal{D}$ for which $T \in \mathcal{A}$. Our next lemma gives us such a condition (as will become clear later) and is our main tool for constructing decompositions in the sequel.

Lemma IV.2. *[6, Lemma 5] Let $T \in \mathcal{L}(\mathcal{X})$ and $\mathcal{D} = \{X_i\}_{i=0}^{\infty}$ be a decomposition of $\mathcal{X}$. Fix $\varepsilon > 0$ and denote $\widetilde{P}_n = \sum_{i=0}^{n} P_i$, where P_i is the projection onto X_i. Let us also assume that*

$$\lim_{n\to\infty} \|(I - \widetilde{P}_n)T\| = \lim_{n\to\infty} \|T(I - \widetilde{P}_n)\| = 0. \tag{4.1}$$

Then there exists an increasing sequence of integers $\{m_j\}_{i=0}^{\infty}$ such that

$$\sum_{j=0}^{\infty}\|(I-\widetilde{P}_{m_j})T\|+\sum_{j=0}^{\infty}\|T(I-\widetilde{P}_{m_j})\|+\sum_{i,j=0}^{\infty}\|(I-\widetilde{P}_{m_i})T(I-\widetilde{P}_{m_j})\|<\varepsilon.$$

Proof. Note first that $\|I-\widetilde{P}_i\|\leq\|I\|+\|\widetilde{P}_i\|\leq\|\widetilde{P}_i\|+1=C_1$ for every $i\in\mathbb{N}$. This estimate follows directly from the remark after Definition III.5. Let $\{n_j\}_{j=0}^{\infty}$ be an increasing sequence of integers such that

$$\sum_{j=0}^{\infty}\|T(I-\widetilde{P}_{n_j})\|<\frac{\varepsilon}{3C_1}\ ,\ \sum_{j=0}^{\infty}\|(I-\widetilde{P}_{n_j})T\|<\frac{\varepsilon}{3C_1}.$$

Now we can use the inequality

$$\sum_{j=0}^{\infty}\|(I-\widetilde{P}_i)T(I-\widetilde{P}_{n_j})\|\leq\sum_{j=0}^{m}\|(I-\widetilde{P}_i)T(I-\widetilde{P}_{n_j})\|+C_1\sum_{j=m+1}^{\infty}\|T(I-\widetilde{P}_{n_j})\|$$

to deduce that

$$\lim_{i\to\infty}\sum_{j=0}^{\infty}\|(I-\widetilde{P}_i)T(I-\widetilde{P}_{n_j})\|=0.$$

Using the last equation we can find an increasing sequence of integers $\{m_j\}_{j=0}^{\infty}$, $m_j\geq n_j$ such that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|(I-\widetilde{P}_{m_i})T(I-\widetilde{P}_{n_j})\|<\frac{\varepsilon}{3C_1}.$$

Now it is easy to deduce that the sequence $\{m_j\}_{j=0}^{\infty}$ satisfies the condition of the lemma. In fact

$$\begin{aligned}\|T(I-\widetilde{P}_{m_j})\| &= \|T(I-\widetilde{P}_{n_j})(I-\widetilde{P}_{m_j})\|\leq C_1\|T(I-\widetilde{P}_{n_j})\|\\ \|(I-\widetilde{P}_{m_j})T\| &= \|(I-\widetilde{P}_{m_j})(I-\widetilde{P}_{n_j})T\|\leq C_1\|(I-\widetilde{P}_{n_j})T\|\\ \|(I-\widetilde{P}_{m_i})T(I-\widetilde{P}_{m_j})\| &= \|(I-\widetilde{P}_{m_i})T(I-\widetilde{P}_{n_j})(I-\widetilde{P}_{m_j})\|\\ &\leq C_1\|(I-\widetilde{P}_{m_i})T(I-\widetilde{P}_{n_j})\|.\end{aligned}$$

This finishes the proof. □

Lemma IV.3. *[6, Lemma 6] Let $\mathcal{D} = \{X_i\}_{i=0}^{\infty}$ be a decomposition of $\mathcal{X}$. Then for any $T \in \mathcal{L}(\mathcal{X})$ we have*

$$P_iTP_j \in \mathcal{A}\,,\ \|(P_iTP_j)_{\mathcal{D}}\| \leq C\|P_iTP_j\|$$

where C depends on $\mathcal{D}$ only.

Proof. The proof of the lemma is basically included in the proof of Lemma III.4, **Case II**, and is omitted. □

The following theorem shows the importance of the decompositions in determining whether an operator is a commutator.

Theorem IV.4. *[6, Theorem 8] Under the hypotheses of Lemma IV.2, there is a decomposition $\mathcal{D}$ of $\mathcal{X}$ for which*

$$T \in \mathcal{A},\ \ \|T_{\mathcal{D}}\| \leq C\|T\| + \varepsilon$$

where C depends on $\mathcal{D}$ only. In particular, using Lemma III.3 we conclude that T is a commutator.

Proof. Using the sequence $\{m_j\}$ from Lemma IV.2, define a decomposition $\{\widetilde{X}_i\}$, where $\widetilde{X}_0 = \bigoplus_{k=0}^{m_0} X_k$, and $\widetilde{X}_i = \bigoplus_{k=m_{i-1}+1}^{m_i} X_k$ for $i > 0$. Note that the new decomposition also satisfies the conditions in Definition III.5 for being a decomposition. Condition (1) follows from the remark after the definition and (3) is clearly satisfied since we are taking only finite direct sums. Hence we will only check condition (2). Let $\mathcal{X} \overset{C}{\simeq} \left(\bigoplus_{i=0}^{\infty} \mathcal{X}\right)_p$ and let $X_i \overset{\lambda}{\simeq} \mathcal{X}$ for $i = 0, 1, \ldots$ (for two Banach spaces X and Y we say that $X \overset{\lambda}{\simeq} Y$ if there exist an onto isomorphism $T\colon X \to Y$ such that $\|T\|\|T^{-1}\| \leq \lambda$). Then for $1 \leq r < s$ we have $\bigoplus_{k=r}^{s} X_k \overset{\lambda}{\simeq} \left(\bigoplus_{k=r}^{s} \mathcal{X}\right)_p \overset{C}{\simeq} \Big(\bigoplus_{k=r}^{s} \left(\bigoplus_{i=0}^{\infty} \mathcal{X}\right)_p\Big)_p \equiv$

$(\bigoplus_{i=0}^{\infty}\mathcal{X})_p \overset{C}{\simeq} \mathcal{X}$, so all the terms of the decomposition after the first one are $C^2\lambda$ isomorphic to $\mathcal{X}$. The first term in the new decomposition is also isomorphic to $\mathcal{X}$ thus we showed (2) for the new decomposition. For simplicity of notation, denote the new decomposition by $\{X_i\}_{i=0}^{\infty}$ and the projections onto X_i by P_i. In the new notation the conclusion from Lemma IV.2 can be written as

$$\sum_{j=0}^{\infty}\|(I-\widetilde{P}_j)T\| + \sum_{j=0}^{\infty}\|T(I-\widetilde{P}_j)\| + \sum_{i,j=0}^{\infty}\|(I-\widetilde{P}_i)T(I-\widetilde{P}_j)\| < \varepsilon.$$

Now using $P_i(I-\widetilde{P}_{i-1}) = (I-\widetilde{P}_{i-1})P_i = P_i$ we have

$$\begin{aligned}
\sum_{i,j=0}^{\infty}\|P_iTP_j\| &\le \|P_0TP_0\| + C_1\sum_{i=1}^{\infty}\|P_iT\| + C_1\sum_{j=1}^{\infty}\|TP_j\| + \sum_{i,j=1}^{\infty}\|P_iTP_j\| \\
&\le \|P_0TP_0\| + C_1\sum_{i=1}^{\infty}\|P_i(I-\widetilde{P}_{i-1})T\| \\
&+ C_1\sum_{j=1}^{\infty}\|T(I-\widetilde{P}_{j-1})P_j\| \\
&+ \sum_{i,j=1}^{\infty}\|P_i(I-\widetilde{P}_{i-1})T(I-\widetilde{P}_{j-1})P_j\| \\
&\le \|P_0TP_0\| + C_1^2\sum_{i=1}^{\infty}\|(I-\widetilde{P}_{i-1})T\| + C_1^2\sum_{j=1}^{\infty}\|T(I-\widetilde{P}_{j-1})\| \\
&+ C_1^2\sum_{i,j=1}^{\infty}\|(I-\widetilde{P}_{i-1})T(I-\widetilde{P}_{j-1})\| \\
&\le \|P_0TP_0\| + C_1^2\varepsilon.
\end{aligned}$$

Since the series $\sum_{i=0}^{\infty}P_i$ is strongly convergent to I, we have $T = \sum_{i,j=0}^{\infty}P_iTP_j$ in the norm topology of $\mathcal{L}(\mathcal{X})$. Using Lemma IV.3 and the estimate from above, the operator

$$S = \sum_{i,j=0}^{\infty}(P_iTP_j)_{\mathcal{D}}$$

is well defined and using Lemma III.3 for each term in the sum of the definition of S we have that $T = D_R(-SL) \in \mathcal{A}$. Now $D_R(T_{\mathcal{A}}L - SL) = 0$ and by the proof of Lemma IV.1 we have

$$0 = -(D_R(T_{\mathcal{A}}L - SL))_{\mathcal{D}}L = (T_{\mathcal{A}} - S)L.$$

From the equation above we conclude that $T_{\mathcal{A}} = S$ and $\|T_{\mathcal{A}}\| \leq C\|T\| + \varepsilon$. □

B. Compactness and commutators

In order to prove the conjecture about the structure of the commutators on a given space we have to show that all the elements in the largest proper ideal are commutators. We prove a lemma that takes care of this in the case $\mathcal{X} = \ell_1$ and also shows that the ideal of compact operators consists of commutators only, provided the space $\mathcal{X}$ has some additional structure. Before that we show a lemma about the operators T on $\mathcal{X}$ which do not preserve a copy of $\mathcal{X}$ in the cases of $\mathcal{X} = \ell_1$ and $\mathcal{X} = L_1$, which will be used later and it is interesting on its own.

Lemma IV.5. *[6, Lemma 9] Let $\mathcal{X} = L_1$ or $\mathcal{X} = \ell_1$ and suppose that $T \in \mathcal{L}(\mathcal{X})$ does not preserve a copy of $\mathcal{X}$. Then, for every $\delta > 0$ and for every $\tilde{X} \subset \mathcal{X}$, $\tilde{X} \equiv \mathcal{X}$, there exists $Y \subset \tilde{X}$, such that Y is $(1+\delta)$ isomorphic to $\mathcal{X}$, $(1+\delta)$ complemented in $\mathcal{X}$, and $\|T_{|Y}\| < \delta$.*

Proof. Consider the case $\mathcal{X} = L_1$ first. By assumption T does not preserve a copy of L_1 which implies that T is not an E-operator (actually this can be taken as an equivalent definition for an operator not to be an E-operator [8, Theorem 4.1]) and hence it is not sign-preserving either ([25, Theorem 1.5]). Now [25, Lemma 3.1] gives us a subspace $Z \subset \tilde{X}$ such that $Z \simeq \tilde{X}$ and $\|T_{|Z}\| < \delta$. Using Theorem II.3 we find $Y \subset Z$, which is $(1+\delta)$ isomorphic to $\tilde{X} \equiv L_1$, $(1+\delta)$ complemented in $\tilde{X}$ and Y

clearly satisfies $\|T_{|Y}\| < \delta$. If Q is the norm one projection onto $\tilde{X}$, and $R : \tilde{X} \to Y$ is a projection of norm less than $1+\delta$, then $P := RQ$ is a projection from L_1 onto Y and $\|P\| < 1+\delta$.

For the case $\mathcal{X} = \ell_1$ we use the fact that if $\tilde{X}$ is isometric to ℓ_1, then $\tilde{X} = \overline{\text{span}}\{\psi_i : i = 0, 1, \ldots\}$ for some vectors $\{\psi_i\}_{i=1}^{\infty}$ of norm one, such that

$$\psi_j = \sum_{i \in \sigma_j} \lambda_i e_i, \qquad \text{with } \sigma_j \cap \sigma_k = \emptyset \text{ for } j \neq k$$

where $\{e_i\}_{i=1}^{\infty}$ is the standard unit vector basis of ℓ_1. This follows trivially from the observation that Ue_i and Ue_j must have disjoint supports if $U : \tilde{X} \to \ell_1$ is an into isometry (cf. [21, Proposition 2.f.14]). Note also that since every infinite dimensional subspace of ℓ_1 contains an isomorphic copy of ℓ_1 (see proof of Theorem II.3), then the operator T is automatically strictly singular and hence compact ([9]). Then, $\{T\psi_i\}_{i=0}^{\infty}$ is relatively compact in ℓ_1 and hence there exist $y \in \ell_1$ and a subsequence $\{\psi_{i_j}\}$ such that $T\psi_{i_j} \to y$. Without loss of generality we may assume that $T\psi_i \to y$. Finally, define $\varphi_i = \dfrac{\psi_{2i} - \psi_{2i+1}}{2}$ for $i = 0, 1, \ldots$. Clearly $\{\varphi_i\}_{i=0}^{\infty}$ is a normalized block basis of $\tilde{X}$ such that $\|T\varphi_i\|_1 \to 0$. Assume without loss of generality that $\|T\varphi_i\|_1 < \varepsilon$ (this can be easily achieved by passing to a subsequence). Then for $Y = \overline{\text{span}}\{\varphi_i : i = 0, 1, \ldots\}$ we have $\|T_{|Y}\| < \varepsilon$. Note also that $Y \subset \tilde{X}$ is 1-complemented in $\tilde{X}$ as it is the closed linear span of a normalized block basis and clearly is isometric to $\tilde{X}$ ([22, Lemma 1]). Finally, let $R : \tilde{X} \to Y$ be the norm one projection onto Y and $Q : \ell_1 \to \tilde{X}$ be the norm one projection onto $\tilde{X}$. Then clearly $P := RQ$ is a norm one projection onto Y. $\square$

Lemma IV.6. *[6, Lemma 10] Let $\mathcal{X}$ be a Banach space for which $\mathcal{X} \simeq \left(\bigoplus_{i=0}^{\infty} \mathcal{X}\right)_p$ for some $1 \leq p < \infty$ or $p = 0$. In the case $p = 1$ we will assume that $\mathcal{X} = L_1$ or $\mathcal{X} = \ell_1$. Let $T \in \mathcal{L}(\mathcal{X})$ be a compact operator and $\varepsilon > 0$. Then there exists a decomposition $\mathcal{D}$*

of $\mathcal{X}$ such that $T \in \mathcal{A}$ and $\|T_{\mathcal{A}}\| \leq C\|T\| + \varepsilon$ for some constant C depending on $\mathcal{D}$ only. Consequently, T is a commutator and $T = -D_R(T_{\mathcal{A}}L)$.

Proof. The result is known in the case of $\mathcal{X} = L_p$ and $\mathcal{X} = \ell_p$ for $1 < p < \infty$ (cf. [26] and [3]), and for $\mathcal{X} = c_0$ and $\mathcal{X} = C(K)$ ([4]). The proof presented here in these cases follows Apostol's ideas from [3] and our generalized context gives a shorter proof in the case of L_p for $1 < p < \infty$. Partial results were known in the case $\mathcal{X} = \ell_1$ ([3, Theorem 2.6]).

Case I. $p > 1$ or $p = 0$. In this case we proceed as in Theorem 2.4 in [3], but instead of considering a particular type of decomposition as in [3], we consider an arbitrary decomposition $\mathcal{D}$ of $\mathcal{X}$ and denote $\widetilde{P}_n = \sum_{i=0}^{n} P_i$. Now we have

$$\lim_{n\to\infty} \|(I - \widetilde{P}_n)T\| = \lim_{n\to\infty} \|T(I - \widetilde{P}_n)\| = 0.$$

Choose $\varphi_i, \psi_i \in \mathcal{X}$ such that

$$\|(I - \widetilde{P}_n)T\varphi_n\| > \|(I - \widetilde{P}_n)T\| - \frac{1}{n+1},\ \|\varphi_n\| = 1$$

$$\|T(I - \widetilde{P}_n)\psi_n\| > \|T(I - \widetilde{P}_n)\| - \frac{1}{n+1},\ \|\psi_n\| = 1,\ (I - \widetilde{P}_n)\psi_n = \psi_n.$$

Since the set $\{T\varphi_i\}_{i=0}^{\infty}$ is relatively compact in $\mathcal{X}$ and the sequence $\{(I - \widetilde{P}_i)\}_{i=0}^{\infty}$ converges strongly to 0 we have $\lim_{n\to\infty} \|(I - \widetilde{P}_n)T\| = 0$. On the other hand, the sequence $\{\psi_i\}_{i=0}^{\infty}$ is weakly convergent to 0. Using the fact that T is compact, it follows that the sequence $\{T\psi_i\}_{i=0}^{\infty}$ converges to 0 in norm and hence $\lim_{n\to\infty} \|T(I - \widetilde{P}_n)\| = 0$. Now Theorem IV.4 gives the result.

Case II. $p = 1$. Fix $\varepsilon > 0$ and let $\mathcal{D} = \{X_i\}_{i=0}^{\infty}$ be the fixed decomposition of $\mathcal{X}$ defined by $X_i = L_1[\frac{1}{2^{i+1}}, \frac{1}{2^i})$ in the case of $\mathcal{X} = L_1$ and by $X_i = P_{N_i}\ell_1$ (where $\mathbb{N} = \cup_{i=0}^{\infty} N_i$ such that $\operatorname{card} N_i = \operatorname{card} \mathbb{N}$ for all $i \in \mathbb{N}$ and $N_j \cap N_j = \emptyset$ for $i \neq j$) in the case of $\mathcal{X} = \ell_1$. Using Lemma IV.5 for each X_i with $\delta = \frac{\varepsilon}{2^i}$ will give us $1 + \varepsilon$

complemented subspaces $\{Y_i\}$ of $\mathcal{X}$ which are isomorphic to $\mathcal{X}$ and $\|T_{|Y_i}\| < \frac{\varepsilon}{2^i}$. Set $Y_0 = (I - \sum_{i=1}^{\infty} P_i)\mathcal{X}$. Note that $\mathcal{D} = \{Y_i\}$ is a decomposition for $\mathcal{X}$ since all the spaces are complemented and isomorphic to $\mathcal{X}$. This is clear for Y_i for $i = 1, 2, \ldots$ and it also holds for Y_0, since $X_0 \subset Y_0$ is complemented in $\mathcal{X}$, isomorphic to $\mathcal{X}$, and using [8, Corollary 5.3] in the case $\mathcal{X} = L_1$, and [22, Proposition 4] in the case $\mathcal{X} = \ell_1$, it follows that Y_0 is isomorphic to $\mathcal{X}$ as well. Now, if $\widetilde{P}_n = \sum_{i=0}^{n} P_i$, then we clearly have $\lim_{n\to\infty} \|T(I - \widetilde{P}_n)\| = 0$. Since T is a compact operator, we have $\lim_{n\to\infty} \|(I - \widetilde{P}_n)T\| = 0$ as well (the argument provided in **Case I** above works in this case as well), so using Theorem IV.4 we conclude that T is a commutator. □

Remark IV.7. *Using the previous lemma we immediately conclude that [26, Theorem 4.3] holds for $p = 1$. Namely, a multiplication operators M_ϕ on L_1 is a commutator if and only if the spectrum of M_ϕ contains more than one limit point or contains zero as the unique limit point.*

Corollary IV.8. *[6, Corollary 12]Let $\mathcal{X}$ be a Banach space for which $\mathcal{X} \simeq \left(\bigoplus_{i=0}^{\infty} \mathcal{X}\right)_p$ for some $1 \le p < \infty$ or $p = 0$. In the case $p = 1$ we will assume that $\mathcal{X} = L_1$ or $\mathcal{X} = \ell_1$. Let $T \in \mathcal{L}(\mathcal{X})$ and suppose that P is a projection on $\mathcal{X}$ such that $P\mathcal{X} \simeq \mathcal{X} \simeq (I - P)\mathcal{X}$ and that either TP or PT is a compact operator. Then T is a commutator.*

Proof. First we treat the case when TP is compact operator. Let $\mathcal{D} = \{X_i\}_{i=0}^{\infty}$ be a decomposition for which $TP \in \mathcal{A}$ and $\|(TP)_{\mathcal{D}}\|_{\mathcal{X}} \le C\|TP\|_{\mathcal{X}} + \frac{\varepsilon}{2}$ for a fixed $\varepsilon > 0$ (by Lemma IV.6). We also want $\mathcal{D}$ to be such that $(I - P)\mathcal{X} = X_0$ hence we may assume $(I - P) = P_0$, where P_0 is the projection onto X_0. This can obviously be done for $1 < p < \infty$ (since the decomposition used in the proof was arbitrary). In the case of L_1 we consider the operator $\tilde{T} = GTG^{-1}$ where $G : P\mathcal{X} \oplus (I - P)\mathcal{X} \to (I - P_0)\mathcal{X} \oplus X_0$

is an isomorphism such that $GP\mathcal{X} = (I - P_0)\mathcal{X}$, $G(I - P)\mathcal{X} = X_0$. In this case $\tilde{T}GPG^{-1}$ is compact and clearly we can choose the decomposition as in Lemma IV.6 and apply the same argument. Now without loss of generality we can assume that $\tilde{T} = T$. In the case of ℓ_1 we can make a similarity as in the previous case and reduce to the case where TP_M is a compact operator for some infinite set $M \subset \mathbb{N}$. Define

$$S = LT(I - P) - (P_0T(I - P)P_0)_{\mathcal{D}}L - (TP)_{\mathcal{D}}L.$$

Use equation (3.6) applied to TP and $P_0T(I - P)P_0$ (recall that $P_0T(I - P)P_0 \in \mathcal{A}$ by Lemma IV.3) to get

$$\begin{aligned} -D_R((TP)_{\mathcal{D}}L) &= TP && (4.2)\\ -D_R((P_0T(I-P)P_0)_{\mathcal{D}}L) &= P_0T(I-P)P_0 = P_0T(I-P). && (4.3) \end{aligned}$$

Now

$$D_R(LT(I - P)) = RLT(I - P) - LT(I - P)R = (I - P_0)T(I - P) \qquad (4.4)$$

since $(I - P)R = 0$. Combining (4.2), (4.3) and (4.4) we conclude that $D_RS = T$. If PT is compact we consider $S = -(I - P)TR + R(P_0(I - P)TP_0)_{\mathcal{D}} + R(PT)_{\mathcal{D}}$ and a similar calculation shows that $T = D_L(S)$. □

C. General operators on ℓ_p and L_p

We already saw in the previous section that the compact operators on ℓ_1 are commutators and in order to prove the conjecture in the case of $\mathcal{X} = \ell_1$ we have to show that all operators not of the form $\lambda I + K$, where K is compact and $\lambda \neq 0$, are commutators. To do that we are going to show that if T is not of the form $\lambda I + K$, then there exist complemented subspaces X and Y of $\mathcal{X}$ which are isomorphic to $\mathcal{X}$, such that $d(X,Y) > 0$ and $T_{|X} : X \to Y$ is an onto isomorphism. As we already saw (Theorem III.9), this property of T is enough to show that T is a commutator on any space $\mathcal{X}$ for which $\mathcal{X} \simeq \left(\bigoplus_{i=0}^{\infty} \mathcal{X}\right)_p$.

Definition IV.9. *The left essential spectrum of $T \in \mathcal{L}(\mathcal{X})$ is the set ([2] Def 1.1)*

$$\sigma_{l.e.}(T) = \{\lambda \in \mathbb{C} : \inf_{x \in S_Y} \|(\lambda - T)x\| = 0 \text{ for all } Y \subset \mathcal{X} \text{ s.t. } codim(Y) < \infty\}.$$

Apostol [2, Theorem 1.4] proved that for any $T \in \mathcal{L}(\mathcal{X})$, $\sigma_{l.e.}(T)$ is a closed non-void set.

The following lemma is a characterization of the operators not of the form $\lambda I + K$ on the classical Banach sequence spaces. The proof presented here follows Apostol's ideas [3, Lemma 4.1], but it is presented in a more general way.

Lemma IV.10. *Let $\mathcal{X}$ be a Banach space isomorphic to ℓ_p for $1 \le p < \infty$ or c_0 and let $T \in \mathcal{L}(\mathcal{X})$. Then the following are equivalent*

(1) *$T - \lambda I$ is not a compact operator for any $\lambda \in \mathbb{C}$.*

(2) *There exists an infinite dimensional complemented subspace $Y \subset \mathcal{X}$ such that $Y \simeq \mathcal{X}$, $T_{|Y}$ is an isomorphism and $d(Y, T(Y)) > 0$.*

(3) *There exists an infinite dimensional complemented subspace $Y \subset \mathcal{X}$ such that $Y \simeq \mathcal{X}$, $T_{|Y}$ is an isomorphism, $d(Y, T(Y)) > 0$, and $Y + TY$ is closed complemented subspace of $\mathcal{X}$ (and hence isomorphic to $\mathcal{X}$).*

Proof. (**(2)** $\Longrightarrow$ **(1)**)

Assume that $T = \lambda I + K$ for some $\lambda \in \mathbb{C}$ and some $K \in \mathcal{K}(\mathcal{X})$. Clearly $\lambda \neq 0$ since $T_{|Y}$ is an isomorphism. Now there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset S_Y$ such that

$\|Kx_n\| \to 0$ as $n \to \infty$. Let $y_n = T\left(\frac{x_n}{\lambda}\right)$ and note that

$$\|x_n - y_n\| = \left\|x_n - (\lambda I + K)\left(\frac{x_n}{\lambda}\right)\right\| = \left\|x_n - x_n - K\left(\frac{x_n}{\lambda}\right)\right\| = \frac{\|Kx_n\|}{\lambda} \to 0$$

as $n \to \infty$ which contradicts the assumption $d(Y, T(Y)) > 0$. Thus $T - \lambda I$ is not a compact operator for any $\lambda \in \mathbb{C}$.

(**(1)** $\implies$ **(2)**)

The proof in this directions follows the ideas of the proof of Lemma 4.1 from [3]. Let $\lambda \in \sigma_{l.e.}(T)$. Then $T_1 = T - \lambda I$ is not a compact operator and $0 \in \sigma_{l.e}(T_1)$. Using just the definition of the left essential spectrum, we find a normalized block basis sequence $\{x_i\}_{i=1}^{\infty}$ of the standard unit vector basis of $\mathcal{X}$ such that $\|T_1 x_n\| < \frac{1}{2^n}$ for $n = 1, 2, \ldots$. Thus if we denote $Z = \overline{\text{span}}\{x_i : i = 1, 2, \ldots\}$ we have $Z \simeq \mathcal{X}$ and $T_{1|Z}$ is a compact operator. Let $I - P$ be a bounded projection from $\mathcal{X}$ onto Z ([22, Lemma 1]) so that $T_1(I - P)$ is compact. Now consider the operator $T_2 = (I - P)T_1 P$. We have two possibilities:

Case I. Assume that $T_2 = (I - P)T_1 P$ is not a compact operator. Then there exists an infinite dimensional subspace $Y_1 \subset P\mathcal{X}$ on which T_2 is an isomorphism and hence, using [22, Lemma 2] if necessary, we find a complemented subspace $Y \subset P\mathcal{X}$, such that T_2 is an isomorphism on Y. By the construction of the operator T_2 we immediately have $d(Y, (I - P)T_1 P(Y)) > 0$ and hence $d(Y, T_1(Y)) > 0$. Note that since $\mathcal{X}$ is prime and Y is complemented in $\mathcal{X}$, $Y \simeq \mathcal{X}$ is automatic. Now we are in position to use Proposition III.1 to conclude that $d(Y, T(Y)) > 0$.

Case II. Now we can assume that the operator $(I - P)T_1 P$ is compact. Since $T_1(I - P)$ is compact and using

$$T_1 = T_1(I - P) + (I - P)T_1 P + PT_1 P$$

we conclude that the operator PT_1P is not compact. Using $\mathcal{X} \equiv P\mathcal{X} \oplus (I-P)\mathcal{X}$, we identify $P\mathcal{X} \oplus (I-P)\mathcal{X}$ with $\mathcal{X} \oplus \mathcal{X}$ via an isomorphism U, such that U maps $P\mathcal{X}$ onto the first copy of $\mathcal{X}$ in the sum $\mathcal{X} \oplus \mathcal{X}$. Without loss of generality we assume that $T_1 = \left(\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{smallmatrix}\right)$ is acting on $\mathcal{X} \oplus \mathcal{X}$. Denote by $P = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$ the projection from $\mathcal{X} \oplus \mathcal{X}$ onto the first copy of $\mathcal{X}$. In the new settings, we have that T_{11} is not compact and T_{21}, T_{22} and T_{12} are compact operators. Define the operator S on $\mathcal{X} \oplus \mathcal{X}$ in the following way:

$$\sqrt{2}S = \begin{pmatrix} I & I \\ I & -I \end{pmatrix}.$$

Clearly $S^2 = I$ hence $S = S^{-1}$. Now consider the operator $2(I-P)S^{-1}T_1SP$. A simple calculation shows that

$$2(I-P)S^{-1}T_1SP = \begin{pmatrix} 0 & 0 \\ T_{11} + T_{12} - T_{21} - T_{22} & 0 \end{pmatrix}$$

hence $(I-P)S^{-1}T_1SP$ is not compact. Now we can continue as in the previous case to conclude that there exists a complemented subspace $Y \subset \mathcal{X}$ in the first copy of $\mathcal{X} \oplus \mathcal{X}$ for which $d(Y, S^{-1}T_1S(Y)) > 0$ and hence $d(SY, T_1(SY)) > 0$. Again using Proposition III.1, we conclude that $d(SY, T(SY)) > 0$.

(**(3)** $\Longrightarrow$ **(2)**)

This implication is obvious and follows immediately.

(**(2)** $\Longrightarrow$ **3)**)

The only thing we have to prove is that we can choose Y in such a way that $Y + TY$ is a complemented subspace of $\mathcal{X}$. Note that $Y + TY$ is a priori isomorphic to $\mathcal{X}$ because $d(Y, T(Y)) > 0$. To do this, we are going to use the so called “gliding hump” argument. The idea is to build a sequence $\{z_i\}_{i=0}^{\infty}$ such that $z_{2i} \in Y$ and $z_{2i+1} = Tz_{2i}$ for $i = 0, 1, \ldots$, which is almost a block basis of the unit vector basis in $\mathcal{X}$. Let

$\{e_i\}_{i=1}^{\infty}$ is the unit vector basis for $\mathcal{X}$. Choose a subspace Y as in **(2)** and without loss of generality we can assume that $Y = \overline{\text{span}}\{e_{2k} : k = 1, 2, \ldots\}$. This can be easily achieved by considering a operator T_1 which is similar to T. Note that here we use the fact that Y is a complemented subspace of $\mathcal{X}$. With P_M we denote the projection onto the coordinates of a vector with indices in M.

Fix an $\varepsilon > 0$ and let $z_0 = e_2$. Define $n_0 = 1$ and let $z_1 = Tz_0$. There exists n_1 such that $\max(\|P_{(n_1,\infty)}z_0\|, \|P_{(n_1,\infty)}z_1\| < \varepsilon$ and denote $C_0 = [1, n_1]$. Now consider $Y_1 = TY \cap P_{(n_1,\infty)}\mathcal{X}$. Since $P_{(n_1,\infty)}\mathcal{X}$ is finite co-dimensional, we have that Y_1 is a non-empty subspace of $\mathcal{X}$ which is finite co-dimensional in TY. Similarly, $T_{|Y}^{-1}(Y_1) \cap P_{(n_1,\infty)}\mathcal{X}$ is finite co-dimensional subspace of Y and let $z_2 \in T_{|Y}^{-1}(Y_1) \cap P_{(n_1,\infty)}\mathcal{X}$ be a finite linear combination of vectors in $\{e_{2k}\}_{k>n_1}$. This can always be done due to the fact that $T_{|Y}^{-1}(Y_1) \cap P_{(n_1,\infty)}\mathcal{X}$ is finite co-dimensional subspace of Y. Denote $z_3 = Tz_2$, and, as before, let n_2 be such that $\text{supp}(z_2) \subset [n_1 + 1, n_2]$, $\max(\|P_{(n_2,\infty)}z_2\|, \|P_{(n_2,\infty)}z_3\| < \frac{\varepsilon}{2}$ and denote $C_1 = [n_1 + 1, n_2]$.

Continuing this process, we obtain a sequence of vectors $\{z_i\}_{i=0}^{\infty}$ and a sequence of numbers $\{n_i\}_{i=1}^{\infty}$ such that

- $z_{2i} \in S_Y$ and $z_{2i+1} = Tz_{2i} \in TY$ for $i = 0, 1, \ldots$
- There exist a sequence of disjoint sets $C_i = [n_i + 1, n_{i+1}]$, such that $\max(\|P_{C_i^c}z_{2i}\|, \|P_{C_i^c}z_{2i+1}\| < \frac{\varepsilon}{2^i}$ for $i = 0, 1, \ldots$

Denote $Z = \overline{\text{span}}\{z_{2k} : k = 0, 1, \ldots\}$ and note that $\text{supp}(z_{2i}) \subset C_i$ and the support of z_{2i+1} is "almost" in C_i for $i = 0, 1, \ldots$. If we denote $W_i = \text{span}\{z_{2i}, P_{C_i}z_{2i+1}\}$ and $Z_i = \text{span}\{z_{2i}, z_{2i+1}\}$, $i = 0, 1, \ldots$, then each W_i is the range of a projection from $\mathcal{X}$ of a norm at most $\sqrt{2}$ ([12]). Since the W_i's have disjoint supports, we automatically have that $W_0 + W_1 + \cdots$ is complemented in $\mathcal{X}$ by a projection of norm at most $\sqrt{2}$. From this fact and the way we constructed the sequence $\{Z_i\}_{i=0}^{\infty}$, we deduce that $Z_0 + Z_1 + \cdots$ is also complemented in $\mathcal{X}$. Finally, from the assumption that $Y + TY$ is a closed subspace of X (which is equivalent to $d(Y, T(y)) > 0$) we have that $Z_0 + Z_1 + \cdots = \overline{\text{span}}\{z_k : k = 0, 1, \ldots\} = Z + T(Z)$ and hence $Z + T(Z)$ is complemented in $\mathcal{X}$. Note that Z and $T(Z)$ are also complemented subspaces of $\mathcal{X}$ as a subspaces spanned by a sequences which are "close" to a block basis sequences. This finishes the proof of the last implication and the theorem. □

Remark IV.11. *We should note that the third condition in the preceding lemma is what was used for proving the complete classification of the commutators on ℓ_p, $1 < p < \infty$, and c_0 in [3] and [4] and what we will use in the case of ℓ_1. The last mentioned condition will also play an important role in the proof of the complete classification of the commutators on ℓ_∞, but we should point out that once we have an infinite dimensional subspace $Y \subset \ell_\infty$ such that $Y \simeq \ell_\infty$, $T_{|Y}$ is an isomorphism and $d(Y, T(Y)) > 0$, then Y and $Y + T(Y)$ will be automatically complemented in ℓ_∞.*

Finally, we are in position to apply Theorem III.9 in order to obtain a complete classification of the commutators on ℓ_1.

Theorem IV.12. *Let $\mathcal{X} = \ell_1$. An operator $T \in \mathcal{L}(\mathcal{X})$ is a commutator if and only if $T - \lambda I$ is not compact for any $\lambda \neq 0$.*

Proof. We have two cases depending on λ:

Case I. If T is compact operator ($\lambda = 0$), the statement of the theorem follows from Lemma IV.6.

Case II. If $T - \lambda I$ is not compact for any $\lambda \in \mathbb{C}$, then we apply Lemma IV.10 first which allows us to apply Theorem III.9 in order to conclude that T is a commutator.

□

CHAPTER V

COMMUTATORS ON ℓ_∞

In order to show that the conjecture we stated in Chapter I holds for the space ℓ_∞ we first have to show that all strictly singular operators (which as we saw in Lemma II.6 is the largest ideal in $\mathcal{L}(\ell_\infty)$) are commutators. Then, as in the case of ℓ_1, we have to show that if an operator T is such that $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any $\lambda \in \mathbb{C}$ then T is a commutator.

A. Strictly singular operators

In the case of the spaces ℓ_p, $1 \leq p < \infty$ we proved that if T is a compact operator on ℓ_p then we can find a subspace Y isomorphic to ℓ_p, such that $\|T_{|Y}\|$ is arbitrary small. Intuitively, this should be true for the compact operators on ℓ_∞ as well, but we need a similar statement to be true for the strictly singular operators on ℓ_∞ in order to use some of the results we have previously proved. In fact, we show that if $T \in \mathcal{S}(\ell_\infty)$ then there exists a $Y \subset \ell_\infty$, $Y \simeq \ell_\infty$, such that $T_{|Y} = 0$ which will allow us to conclude that T is a commutator.

Theorem V.1. *Let $T \in \mathcal{L}(\ell_\infty)$ be a strictly singular operator. Then T is a commutator.*

Proof. Since T is a strictly singular operator, T is weakly compact ([24, Corollary 1.4]). Thus it follows that $T\ell_\infty$ is separable (since any weakly compact subset of the dual to any separable space is metrizable) and let $Y = \overline{T\ell_\infty}$. The space ℓ_∞/Y must be non-reflexive since assuming otherwise gives us that Y has a subspace isomorphic to ℓ_∞ ([20, Theorem 4]). Now consider the quotient map $Q : \ell_\infty \to \ell_\infty/Y$. Q is not weakly compact and hence (using again [24, Corollary 1.4]) there exists $X \simeq$

$\ell_\infty, X \subset \ell_\infty$ such that $Q_{|X}$ is an isomorphism. Let P' be a projection onto QX and set $P = (Q_{|X})^{-1}P'Q$. P is a projection in ℓ_∞, $PY = \{0\}$ and by the construction, $P\ell_\infty$ is isomorphic to ℓ_∞. Thus it follows that $PT = 0$ and we obtain that T is similar to an operator T' for which there exists an infinite $M \subset \mathbb{N}$ such that $P_M T' = 0$. Now we are in position to apply Lemma III.4. In order to do so, choose a sequence of disjoint infinite sets $\{M_i\}_{i=0}^\infty$ such that $M = \cup_{i=0}^\infty M_i$ and let $Y_0 = (I - P_{M\setminus M_0})\ell_\infty$ and $Y_i = P_{M_i}\ell_\infty$ for $i = 1, 2, \ldots$. Clearly $\ell_\infty = \big(\oplus_{i=0}^\infty Y_i\big)_\infty$ and Y_i is isometric to ℓ_∞ for all $i = 0, 1, \ldots$ (in the construction above we had to ensure that Y_0 is isometric to ℓ_∞, because it may happen that $\mathbb{N}\setminus M$ is a finite set). Now it is clear how to finish. If P_i is the natural projection onto Y_i for $i = 1, 2, \ldots$, then $T' \equiv P_0 T'$ because from $P_M T' = 0$ it follows that $T \equiv (I - P_M)T' = (I - (P_{M\setminus M_0} + P_{M_0}))T' = (I - P_{M\setminus M_0})T' - P_{M_0}T' = (I - P_{M\setminus M_0})T' = P_0 T'$. Finally, Lemma III.4 gives us that T' (and hence T) is a commutator. $\square$

B. General operators

In this section we follow the ideas in Chapter IV in order to obtain a complete classification of the commutators on ℓ_∞. In order to do this, we want to show that if $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for every $\lambda \in \mathbb{C}$ then we can find a subspace Y which is isomorphic to ℓ_∞ and satisfies $d(Y, TY) > 0$. This task we accomplish by first showing that a similar statement is true when considering the operator T restricted to the subspaces of ℓ_∞ which are isomorphic to c_0, and then we extend the result using known techniques of Rosenthal ([24]).

Lemma V.2. *Let $T \in \mathcal{L}(\ell_\infty)$ and denote by I the identity operator on ℓ_∞. Then the following are equivalent*

(a) For each subspace $X \subset \ell_\infty, X \simeq c_0$, there exists a constant λ_X and a compact

operator $K_X : X \to \ell_\infty$ *depending on* X *such that* $T_{|X} = \lambda_X I_{|X} + K_X$.

(b) There exists a constant λ *such that* $T = \lambda I + S$, *where* $S \in \mathcal{S}(\ell_\infty)$.

Proof. Clearly (b) implies (a), since every strictly singular operator from c_0 to any Banach space is compact ([1, Theorem 2.4.10]). For proving the other direction we first show that for every two subspaces X, Y such that $X \simeq Y \simeq c_0$ we have $\lambda_X = \lambda_Y$. We have several cases.

Case I. $X \cap Y = \{0\}$, $d(X, Y) > 0$.

Let $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ be bases for X and Y, respectively, which are equivalent to the usual unit vector basis of c_0. Consider the sequence $\{z_i\}_{i=1}^\infty$ such that $z_{2i} = x_i$, $z_{2i-1} = y_i$ for $i = 1, 2, \ldots$. If we denote $Z = \overline{\text{span}}\{z_i \, : \, i = 1, 2, \ldots\}$, then clearly $Z \simeq c_0$, and, using the assumption of the lemma, we have that $T_{|Z} = \lambda_Z I_{|Z} + K_Z$. Now using $X \subset Z$ we have that $\lambda_X I_{|X} + K_X = (\lambda_Z I_{|Z} + K_Z)_{|X}$, hence

$$(\lambda_X - \lambda_Z) I_{|X} = (K_Z)_{|X} - K_X.$$

The last equation is only possible if $\lambda_X = \lambda_Z$ since the identity is never a compact operator on a infinite dimensional subspace. Similarly $\lambda_Y = \lambda_Z$ and hence $\lambda_X = \lambda_Y$.

Case II. $X \cap Y = \{0\}$, $d(X, Y) = 0$.

Again let $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ be bases of X and Y, respectively, which are equivalent to the usual unit vector basis of c_0 and assume also that $\lambda_X \neq \lambda_Y$. There exists a normalized block basis $\{u_i\}_{i=1}^\infty$ of $\{x_i\}_{i=1}^\infty$ and a normalized block basis $\{v_i\}_{i=1}^\infty$ of $\{y_i\}_{i=1}^\infty$ such that $\|u_i - v_i\| < \dfrac{1}{i}$. Then $\|u_i - v_i\| \to 0 \Rightarrow \|Tu_i - Tv_i\| \to 0 \Rightarrow \|\lambda_X u_i + K_X u_i - \lambda_Y v_i - K_Y v_i\| \to 0$. Since $u_i \to 0$ weakly (as a bounded block basis of the standard unit vector basis of c_0) we have $\|K_X u_i\| \to 0$ and using $\|u_i - v_i\| \to 0$ we conclude that

$$\|(\lambda_X - \lambda_Y) v_i - K_Y v_i\| \to 0.$$

Then there exists $N \in \mathbb{N}$ such that $\|K_Y v_i\| > \frac{|\lambda_X - \lambda_Y|}{2}\|v_i\|$ for $i > N$, which is impossible because K_Y is a compact operator. Thus, in this case we also have $\lambda_X = \lambda_Y$.

Case III. $X \cap Y = Z \neq \{0\},\ \dim(Z) = \infty.$

In this case we have $(\lambda_X I_{|X} + K_X)_{|Z} = (\lambda_Y I_{|Y} + K_Y)_{|Z}$. Since Z is infinite dimensional, we can argue as in the first case to conclude that $\lambda_X = \lambda_Y$.

Case IV. $X \cap Y = Z \neq \{0\},\ \dim(Z) < \infty.$

Let $X = Z \bigoplus X_1$ and $Y = Z \bigoplus Y_1$. Then $X_1 \cap Y_1 = \{0\}$, $X_1 \simeq Y_1 \simeq c_0$ and we can reduce to one of the previous cases.

Let us denote $S = T - \lambda I$ where $\lambda = \lambda_X$ for arbitrary $X \subset \ell_\infty$, $X \simeq c_0$. If S is not a strictly singular operator, then there is a subspace $Z \subset \ell_\infty$, $Z \simeq \ell_\infty$ such that $S_{|Z}$ is an isomorphism ([24, Corollary 1.4]), hence we can find $Z_1 \subset Z \subset \ell_\infty$, $Z_1 \simeq c_0$, such that $S_{|Z_1}$ is an isomorphism. This contradicts the assumption that $S_{|Z_1}$ is a compact operator. □

The following corollary is an immediate consequence of Lemma V.2.

Corollary V.3. *Suppose $T \in \mathcal{L}(\ell_\infty)$ is such that $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any $\lambda \in \mathbb{C}$. Then there exist a subspace $X \subset \ell_\infty$, $X \simeq c_0$ such that $(T - \lambda I)_{|X}$ is not a compact operator for any $\lambda \in \mathbb{C}$.*

Theorem V.4. *Let $T \in \mathcal{L}(\ell_\infty)$ be such that $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any λ. Then there exists a subspace $X \subset \ell_\infty$ such that $X \simeq c_0$, $T_{|X}$ is an isomorphism and $d(X, T(X)) > 0$.*

Proof. By Corollary V.3 we have a subspace $X \subset \ell_\infty$, $X \simeq c_0$ such that $(T - \lambda I)_{|X}$ is not a compact operator for any λ. Let $Z = \overline{X \oplus T(X)}$ and let P be a projection from Z onto X (such exists since Z is separable and $X \simeq c_0$). We have two cases:

Case I. The operator $T_1 = (I-P)TP$ is not compact. Since T_1 is a non-compact operator from $X \simeq c_0$ into a Banach space we have that T_1 is an isomorphism on some subspace $Y \subset X$, $Y \simeq c_0$ ([1, Theorem 2.4.10]). Clearly, from the form of the operator T_1 we have $d(Y, T_1(Y)) = d(Y, (I-P)TP(Y)) > 0$ and hence $d(Y, T(Y) > 0$.

Case II. If $(I-P)TP$ is compact and $\lambda \in \mathbb{C}$, then $(I-P)TP + PTP - \lambda I_{|Z} = TP - \lambda I_{|Z}$ is not compact and hence $PTP - \lambda I_{|Z}$ is not compact. Now for $T_2 := PTP\colon X \to X$ we apply Lemma IV.10 to conclude that there exists a subspace $Y \subseteq X$, $Y \simeq c_0$ such that $d(Y, PT(Y)) = d(Y, PTP(Y)) > 0$ and hence $d(Y, T(Y)) > 0$. □

The following theorem is an analog of Lemma IV.10 for the space ℓ_∞.

Theorem V.5. *Let $T \in \mathcal{L}(\ell_\infty)$ be such that $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any $\lambda \in \mathbb{C}$. Then there exists a subspace $X \subset \ell_\infty$ such that $X \simeq \ell_\infty$, $T_{|X}$ is an isomorphism and $d(X, T(X)) > 0$.*

Proof. From Theorem V.4 we have a subspace $Y \subset \ell_\infty$, $Y \simeq c_0$ such that $T_{|Y}$ is an isomorphism and $d(Y, T(Y)) > 0$. Let $N_k = \{3i + k\colon i = 0, 1, \ldots\}$ for $k = 1, 2, 3$. There exists an isomorphism $\overline{S}\colon Y \oplus TY \to c_0(N_1) \oplus c_0(N_2)$ such that $\overline{S}(Y) = c_0(N_1)$ and $\overline{S}(TY) = c_0(N_2)$. Note that the space $Y \oplus TY$ is indeed a closed subspace of ℓ_∞ due to the fact that $d(Y, T(Y)) > 0$. Now we use [20, Theorem 3] to extend $\overline{S}$ to an automorphism S on ℓ_∞. Let $T_1 = STS^{-1}$ and consider the operator $(P_{N_2}T_1)_{|\ell_\infty(N_1)}\colon \ell_\infty(N_1) \to \ell_\infty(N_2)$, where P_{N_2} is the natural projection onto $\ell_\infty(N_2)$. Since $T_1(c_0(N_1)) = c_0(N_2)$, by [24, Proposition 1.2] there exists an infinite set $M \subset N_1$ such that $(P_{N_2}T_1)_{|\ell_\infty(M)}$ is an isomorphism. This immediately yields

$$d(\ell_\infty(M), P_{N_2}T_1(\ell_\infty(M))) > 0$$

and hence

$$d(\ell_\infty(M), T_1(\ell_\infty(M))) > 0. \tag{5.1}$$

Finally, recall that $T_1 = STS^{-1}$, thus

$$d(\ell_\infty(M), STS^{-1}(\ell_\infty(M))) > 0$$

and hence $d(S^{-1}(\ell_\infty(M)), TS^{-1}(\ell_\infty(M))) > 0$. □

Finally, we can prove our main result.

Theorem V.6. *An operator $T \in \mathcal{L}(\ell_\infty)$ is a commutator if and only if $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any $\lambda \neq 0$.*

Proof. Note first that if T is a commutator, from the remarks we made in the introduction it follows that $T - \lambda I$ cannot be strictly singular for any $\lambda \neq 0$. For proving the other direction we have to consider two cases:

Case I. If $T \in \mathcal{S}(\ell_\infty)$ ($\lambda = 0$), the statement of the theorem follows from Theorem V.1.

Case II. If $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any $\lambda \in \mathbb{C}$, then we apply Theorem V.5 to get $X \subset \ell_\infty$ such that $X \simeq \ell_\infty$, $T_{|X}$ an isomorphism and $d(X, TX) > 0$. The subspace $X + TX$ is isomorphic to ℓ_∞ and thus is complemented in ℓ_∞. Theorem III.9 now yields that T is a commutator. □

CHAPTER VI

COMMUTATORS ON OTHER BANACH SPACES

In this chapter we apply some of the techniques we have developed in the previous chapters in order to get a better insight or in some cases a complete characterization of the commutators on other classical Banach spaces.

A. Commutators on $\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n}$

Lemma VI.1. *[6, Lemma 19] Let X and Y be Banach spaces and* $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *an operator from $X \oplus Y$ into $X \oplus Y$. If A and D are commutators on the corresponding spaces then T is a commutator on $X \oplus Y$.*

Proof. Let $A = [A_1, A_2]$ and $D = [D_1, D_2]$. Assume without loss of generality that $\max(\|A_2\|, \|D_2\|) < \frac{1}{4}$. We need to find operators E_1 and E_2 such that

$$T = \begin{pmatrix} A_1 & E_1 \\ E_2 & D_1 \end{pmatrix} \begin{pmatrix} A_2 + I & 0 \\ 0 & D_2 \end{pmatrix} - \begin{pmatrix} A_2 + I & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} A_1 & E_1 \\ E_2 & D_1 \end{pmatrix},$$

or equivalently, we have to solve the equations

$$B = E_1 D_2 - (A_2 + I) E_1 \tag{6.1}$$

$$C = E_2 (A_2 + I) - D_2 E_2 \tag{6.2}$$

for E_1 and E_2. Let $G : \mathcal{L}(X, Y) \to \mathcal{L}(X, Y)$ be defined by $G(S) = -SA_2 + D_2 S$. It is clear that $\|G\| < 1$ by our choice of A_2 and D_2, hence $I - G$ is invertible. Now it is enough to observe that (6.2) is equivalent to $C = (I - G)(E_2)$ which gives us $E_2 = (I - G)^{-1} C$. Analogously, we define $F : \mathcal{L}(Y, X) \to \mathcal{L}(Y, X)$ by $F(S) = -A_2 S + S D_2$ and then (6.1) becomes equivalent to $-B = (I - F)(E_1)$.

Applying the same argument as before, we get that $I - F$ is invertible and hence $E_1 = (I-F)^{-1}(-B)$. □

Theorem VI.2. *[6, Theorem 20] Let $\mathcal{X} = \ell_p \oplus \ell_q$ where $1 \le q < p < \infty$ and $T \in \mathcal{L}(\mathcal{X})$. Let P_{ℓ_p} and P_{ℓ_q} be the natural projections from $\mathcal{X}$ onto ℓ_p and ℓ_q respectively. Then T is a commutator if and only if $P_{\ell_p}TP_{\ell_p}$ and $P_{\ell_q}TP_{\ell_q}$ are commutators as operators acting on ℓ_p and ℓ_q, respectively.*

Proof. Throughout the proof we will work with the matrix representation of T as an operator acting on $\mathcal{X}$. Let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A : \ell_p \to \ell_p, D : \ell_q \to \ell_q, B : \ell_q \to \ell_p, C : \ell_p \to \ell_q$. The well known fact that the operator C is compact ([21, Proposition 2.c.3]) plays an important role in the proof. If T is a commutator, then $T = [T_1, T_2]$ for some $T_1, T_2 \in \mathcal{L}(\mathcal{X})$. Write $T_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ for $i = 1, 2$. A simple computation shows that

$$T = \begin{pmatrix} [A_1, A_2] + B_1C_2 - B_2C_1 & A_1B_2 + B_1D_2 - A_2B_1 - B_2D_1 \\ C_1A_2 + D_1C_2 - C_2A_1 - D_2C_1 & [D_1, D_2] + C_1B_2 - C_2B_1 \end{pmatrix}.$$

From the classification of the commutators on ℓ_p for $1 \le p < \infty$ and the fact that the C_i's are compact we immediately deduce that the diagonal entries in the last representation of T are commutators. For the other direction we apply Lemma VI.1 which concludes the proof. □

The classification given in the theorem can be immediately generalized to a space which is finite sum of ℓ_p spaces, namely, we have the following

Corollary VI.3. *[6, Corollary 21] Let $\mathcal{X} = \ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n}$ where $1 \le p_n < p_{n-1} < \ldots < p_1 < \infty$ and $T \in \mathcal{L}(\mathcal{X})$. Let $P_{\ell_{p_i}}$ be the natural projections from $\mathcal{X}$ onto*

ℓ_{p_i} *for* $i = 1, 2, \ldots, n$. *Then* T *is a commutator if and only if for each* $1 \leq i \leq n$, $P_{\ell_{p_i}} T P_{\ell_{p_i}}$ *is a commutator as an operator acting on* ℓ_{p_i}.

Proof. We proceed by induction on n and clearly Theorem VI.2 gives us the result for $n = 2$. If the statement is true for some n, then to show it for $n+1$, denote $Y = \ell_{p_2} \oplus \ell_{p_3} \oplus \cdots \oplus \ell_{p_n}$. Now $\mathcal{X} = \ell_{p_1} \oplus Y$ and using the same argument as in Theorem VI.2 we see that if T is a commutator, then both $P_{\ell_{p_1}} T P_{\ell_{p_1}}$ and $P_Y T P_Y$ are commutators on ℓ_{p_1} and Y respectively. Here we use the induction step to show that compact perturbation of a commutator on Y is still a commutator. The other direction is exactly as in Theorem VI.2. It is worthwhile noticing that for this direction we do not need any assumption on the spaces in the sum. □

Using the description of the maximal ideals in $\mathcal{L}(\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n})$ we now give an alternate classification of the commutators on $\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n}$, which we will use later.

Theorem VI.4. *Let* $\mathcal{X} = \ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n}$ *where* $1 \leq p_n < p_{n-1} < \ldots < p_1 < \infty$. *An operator* $T \in \mathcal{L}(\mathcal{X})$ *is a commutator if and only if* $T - \lambda I$ *is not in any of the* n *maximal ideals in* $\mathcal{L}(\mathcal{X})$ *for any* $\lambda \neq 0$.

Proof. The proof of the theorem is immediate from Corollary VI.3 and the alternate description of the maximal ideals in $\mathcal{L}(\mathcal{X})$ we gave in Lemma II.8. Note that we also use the classification of the commutators on ℓ_p, $1 \leq p < \infty$, we obtained in Theorem IV.12. □

It is not hard to see that the Theorem VI.4 remains valid if we substitute one of the spaces ℓ_{p_i} with c_0. The proof does not change at all because the classification of the commutators on c_0 and ℓ_p is the same and all operators from c_0 to ℓ_p are compact ([21, Proposition 2.c.3]), which was essential in the proof.

B. Commutators and property **P**

In this section we try to generalize some of the concepts we have encountered during our work. First consider the set

$$\mathcal{M}_{\mathcal{X}} = \{T \in \mathcal{L}(\mathcal{X}) \,:\, I_{\mathcal{X}} \text{ does not factor through } T\}.$$

This set comes naturally from our investigation of the commutators on ℓ_p for $1 \le p \le \infty$. We know (Theorem IV.12, [3, Theorem 4.8], [4, Theorem 2.6]) that the non-commutators on ℓ_p, $1 \le p < \infty$ and c_0 have the form $\lambda I + K$ where $K \in \mathcal{M}_{\mathcal{X}}$ and $\lambda \neq 0$, where $\mathcal{M}_{\mathcal{X}} = \mathcal{K}(\ell_p)$ is actually the largest ideal in $\mathcal{L}(\ell_p)$ ([9]), and, in this dissertation we showed (Theorem V.6) that the non-commutators on ℓ_∞ have the form $\lambda I + S$ where $S \in \mathcal{M}_{\mathcal{X}}$ and $\lambda \neq 0$, where $\mathcal{M}_{\mathcal{X}} = \mathcal{S}(\ell_\infty)$. Thus, it is natural to ask the question for which Banach spaces $\mathcal{X}$ is the set $\mathcal{M}_{\mathcal{X}}$ the largest ideal in $\mathcal{L}(\mathcal{X})$? Let us also mention that in addition to the already mentioned spaces, if $\mathcal{X} = L_p(0,1)$, $1 \le p < \infty$, then $\mathcal{M}_{\mathcal{X}}$ is again the largest proper ideal in $\mathcal{L}(\mathcal{X})$ (cf. [8] for the case $p = 1$ and [13, Proposition 9.11] for $p > 1$).

First note that the set $\mathcal{M}_{\mathcal{X}}$ is closed under left and right multiplication with operators from $\mathcal{L}(\mathcal{X})$, so the question whether $\mathcal{M}_{\mathcal{X}}$ is an ideal is equivalent to the question whether $\mathcal{M}_{\mathcal{X}}$ is closed under addition. Note also that if $\mathcal{M}_{\mathcal{X}}$ is an ideal then it is automatically the largest ideal in $\mathcal{L}(\mathcal{X})$ and hence closed, so the question we will consider is under what conditions we have

$$\mathcal{M}_{\mathcal{X}} + \mathcal{M}_{\mathcal{X}} \subseteq \mathcal{M}_{\mathcal{X}}. \tag{6.3}$$

The following proposition gives a sufficient condition for (6.3) to hold.

Proposition VI.5. *Let $\mathcal{X}$ be a Banach space such that for every $T \in \mathcal{L}(\mathcal{X})$ we have $T \notin \mathcal{M}_{\mathcal{X}}$ or $I - T \notin \mathcal{M}_{\mathcal{X}}$. Then $\mathcal{M}_{\mathcal{X}}$ is the largest (hence closed) ideal in $\mathcal{L}(\mathcal{X})$.*

Proof. Let $S, T \in \mathcal{M}_{\mathcal{X}}$ and assume that $S + T \notin \mathcal{M}_{\mathcal{X}}$. By our assumption, there exist two operators $U \colon \mathcal{X} \to \mathcal{X}$ and $V \colon \mathcal{X} \to \mathcal{X}$ which make the following diagram commute:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\;S+T\;} & \mathcal{X} \\ {\scriptstyle U}\uparrow & & \downarrow{\scriptstyle V} \\ \mathcal{X} & \xrightarrow{\;\;I\;\;} & \mathcal{X} \end{array}$$

Denote $W = (S + T)U(\mathcal{X})$ and let $P \colon \mathcal{X} \to W$ be a projection onto W (we can take $P = (S + T)UV$). Clearly $VP(S + T)U = I$. Now $S, T \in \mathcal{M}_{\mathcal{X}}$ implies $VPSU, VPST \in \mathcal{M}_{\mathcal{X}}$ which is a contradiction since $VPSU + VPTU = I$. □

The conditions of the previous proposition are clearly satisfied for all the spaces ℓ_p and L_p, $1 \leq p \leq \infty$, but we already saw in the introduction that $\mathcal{M}_{\mathcal{X}}$ is in fact the largest ideal in these spaces. Let us just mention that the conditions of the proposition above are satisfied for $\mathcal{X} = C([0, 1])$ ([19, Proposition 2.1]) hence $\mathcal{M}_{\mathcal{X}}$ is the largest ideal in $\mathcal{L}(C([0, 1]))$ as well.

We should point out that there are Banach spaces for which $\mathcal{M}_{\mathcal{X}}$ is not an ideal in $\mathcal{L}(\mathcal{X})$. We already saw (Lemma II.8) that $\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n}$ has exactly n maximal ideals and for this particular space we provided a necessary and sufficient condition for an operator to be a commutator (Theorem VI.2). Namely, we showed that an operator T is a commutator if and only of $P_{\ell_p} T P_{\ell_p}$ and $P_{\ell_q} T P_{\ell_q}$ are commutators as operators acting on ℓ_p and ℓ_q respectively, where P_{ℓ_p} and P_{ℓ_q} are the natural projections from $\ell_p \oplus \ell_q$ onto ℓ_p and ℓ_q, respectively. The alternative classification of the commutators on $\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n}$ we gave in Theorem VI.4 can be generalized, but first we need a definition and a Lemma that follows easily from [6, Corollary 21].

Property P. We say that a Banach space $\mathcal{X}$ has property **P** if $T \in \mathcal{L}(\mathcal{X})$ is not a commutator if and only if $T = \lambda I + S$, where $\lambda \neq 0$ and S belongs to some proper ideal of $\mathcal{L}(\mathcal{X})$.

All Banach spaces we have considered so far have property **P** and our goal now is to show that property **P** is closed under taking finite sums under certain conditions imposed on the spaces in the sum.

Lemma VI.6. *Let $\{X_i\}_{i=1}^n$ be a finite sequence of Banach spaces that have property **P**. Assume also that all operators $A\colon X_i \to X_i$ that factor through X_j are in the intersection of all maximal ideals in $\mathcal{L}(X_i)$ for each $i, j = 1, 2, \ldots, n$, $i \neq j$. Let $\mathcal{X} = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ and let P_i be the natural projections from $\mathcal{X}$ onto X_i for $i = 1, 2, \ldots, n$. Then $T \in \mathcal{L}(\mathcal{X})$ is a commutator if and only if for each $1 \leq i \leq n$, P_iTP_i is a commutator as an operator acting on X_i.*

Proof. The proof is by induction and it mimics the proof of Corollary VI.3. First consider the case $n = 2$. Let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A : X_1 \to X_1, D : X_2 \to X_2, B : X_2 \to X_1, C : X_1 \to X_2$. If T is a commutator, then $T = [T_1, T_2]$ for some $T_1, T_2 \in \mathcal{L}(\mathcal{X})$. Write $T_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ for $i = 1, 2$. A simple computation shows that

$$T = \begin{pmatrix} [A_1, A_2] + B_1C_2 - B_2C_1 & A_1B_2 + B_1D_2 - A_2B_1 - B_2D_1 \\ C_1A_2 + D_1C_2 - C_2A_1 - D_2C_1 & [D_1, D_2] + C_1B_2 - C_2B_1 \end{pmatrix}.$$

From the assumption that X_1 and X_2 have property **P** and the fact that the B_1C_2, B_2C_1 lie in the intersection of all maximal ideals in $\mathcal{L}(X_1)$, and C_1B_2, C_2B_1 lie in the intersection of all maximal ideals in $\mathcal{L}(X_2)$, we immediately deduce that the diagonal entries in the last representation of T are commutators. In the preceding argument we used the fact that a perturbation of a commutator on a Banach space X having property **P** by an operator that lies in the intersection of all maximal ideals in $\mathcal{L}(X)$ is still a commutator. To show this fact assume that $A \in \mathcal{L}(X)$ is a commutator, $B \in \mathcal{L}(X)$ lies in the intersection of all maximal ideals in $\mathcal{L}(X)$ and $A + B = \lambda I + S$ where S is an element of some ideal M in $\mathcal{L}(X)$. Now using the simple observation that every ideal is contained in some maximal ideal, we conclude that $S - B$ is contained in a maximal ideal, say $\tilde{M}$ containing M hence $A - \lambda I \in \tilde{M}$, which is a contradiction with the assumption that X has property **P**.

For the other direction we apply Lemma VI.1 which concludes the proof in the case $n = 2$. The general case follows from the same considerations as in the case $n = 2$ in a obvious way. □

Our last corollary shows that property **P** is preserved under taking finite sums of Banach spaces having property **P** and some additional assumptions as in Lemma VI.6.

Corollary VI.7. *Let $\{X_i\}_{i=1}^n$ be a finite sequence of Banach spaces that have property **P**. Assume also that all operators $A\colon X_i \to X_i$ that factor through X_j are in the intersection of all maximal ideals in $\mathcal{L}(X_i)$ for each $i, j = 1, 2, \dots, n$, $i \neq j$. Then $\mathcal{X} = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ has property **P**.*

Proof. Assume that $T \in \mathcal{L}(\mathcal{X})$ is not a commutator. Using Lemma VI.6, this can happen if and only if $P_i T P_i$ is not commutator on X_i for some $i \in \{1, 2, \dots, n\}$ and without loss of generality assume that $i = 1$. Since $P_1 T P_1$ is not a commutator and X_1 has property **P** then $P_1 T P_1 = \lambda I_{X_1} + S$ where S belongs to some maximal ideal J of $\mathcal{L}(X_1)$. Consider

$$M = \{T \in \mathcal{L}(\mathcal{X}) \,:\, P_1 T P_1 \in J\}. \tag{6.4}$$

Clearly, if $T \in M$ and $A \in \mathcal{L}(\mathcal{X})$, then $AT, TA \in M$ because of the assumption on the operators from X_1 to X_1 that factor through X_j. It is also obvious that M is closed under addition, hence M is an ideal. Now it is easy to see that $T - \lambda I \in M$ which shows that all non-commutators have the form $\lambda I + S$, where $\lambda \neq 0$ and S belongs to some proper ideal of $\mathcal{L}(\mathcal{X})$.

The other direction follows from our comment in the beginning of the introduction that no operator of the form $\lambda I + S$ can be a commutator for any $\lambda \neq 0$ and any operator S which lies in a proper ideal of $\mathcal{L}(\mathcal{X})$. $\square$

CHAPTER VII

SUMMARY

In our final chapter we give a brief summary of the result we proved in the dissertation.

A. Commutators on Banach spaces

1. Commutators on $\left(\sum Y\right)_p$

First we considered Banach spaces $\mathcal{X}$ of the form $\left(\sum Y\right)_p$ for an arbitrary Banach space Y and $1 \leq p \leq \infty$. For those type of spaces we were able to show that the following theorem holds.

Theorem. Let $T \in \mathcal{L}\big(\left(\sum Y\right)_p\big)$. Then the operators $P_0 T$ and TP_0 are commutators, where P_0 is the natural projection onto the first copy of Y in the sum $\left(\sum Y\right)_p$.

If we impose the additional condition that $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$, then the above theorem says that on those type of spaces, the operators with large kernels or large complements of the range are commutators. We also proved a theorem which played a significant role in our further results.

Theorem. Let $\mathcal{X}$ be a Banach space such that $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$, $1 \leq p \leq \infty$ or $p = 0$. Let $T \in \mathcal{L}(\mathcal{X})$ be such that there exists a subspace $X \subset \mathcal{X}$ such that $X \simeq \mathcal{X}$, $T_{|X}$ is an isomorphism, $X + T(X)$ is complemented in $\mathcal{X}$ and $d(X, T(X)) > 0$. Then T is a commutator.

2. Commutators on ℓ_p, $1 \leq p < \infty$, and c_0

The classification of the commutators on ℓ_p for $1 < p < \infty$ and c_0 was already known and our contribution is the case ℓ_1. Since our methods give the classification for all ℓ_p, $1 \leq p < \infty$, and c_0 we state the theorem in its full generality.

Theorem. Let $\mathcal{X} = \ell_p$, $1 \leq p < \infty$ or $\mathcal{X} = c_0$. An operator $T \in \mathcal{L}(\mathcal{X})$ is a commutator if and only if $T - \lambda I$ is not compact for any $\lambda \neq 0$.

3. Commutators on ℓ_∞

Despite the fact that the space ℓ_∞ satisfies the condition $\ell_\infty \simeq \left(\sum \ell_\infty\right)_\infty$, we can not apply exactly the same methods we used for classifying the commutators on ℓ_1, though the general idea is the same. The lack of an unconditional basis for ℓ_∞ and the fact that the largest ideal in $\mathcal{L}(\ell_\infty)$ is the ideal of strictly singular operators (not the compact operators which is the case in $\mathcal{L}(\ell_1)$) were some of the obstacles we had to overcome. We were able to obtain a complete classification of the commutators on ℓ_∞, which is what we conjectured in the introduction of the dissertation.

Theorem. An operator $T \in \mathcal{L}(\ell_\infty)$ is a commutator if and only if $T - \lambda I \notin \mathcal{S}(\ell_\infty)$ for any $\lambda \neq 0$.

4. Commutators on other spaces

Using the techniques developed in Chapter III and the classification of the commutators on ℓ_p, $1 \leq p \leq \infty$ and c_0 given in Chapter IV we obtained a classification of the commutators on $\ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n}$.

Theorem. Let $\mathcal{X} = \ell_{p_1} \oplus \ell_{p_2} \oplus \cdots \oplus \ell_{p_n}$ where $1 \leq p_n < p_{n-1} < \ldots < p_1 < \infty$. An operator $T \in \mathcal{L}(\mathcal{X})$ is a commutator if and only if $T - \lambda I$ is not in any of the n maximal ideals in $\mathcal{L}(\mathcal{X})$ for any $\lambda \neq 0$.

We say that a Banach space $\mathcal{X}$ has property **P** if $T \in \mathcal{L}(\mathcal{X})$ is not a commutator if and only if $T = \lambda I + S$, where $\lambda \neq 0$ and S belongs to some proper ideal of $\mathcal{L}(\mathcal{X})$. The theorem above can be generalized using the notion of **Property P**.

Corollary. Let $\{X_i\}_{i=1}^n$ be a finite sequence of Banach spaces that have property **P**. Assume also that all operators $A\colon X_i \to X_i$ that factor through X_j are in the intersection of all maximal ideals in $\mathcal{L}(X_i)$ for each $i, j = 1, 2, \ldots, n,\ i \neq j$. Then $\mathcal{X} = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ has property **P**.

B. Open problems

We end this dissertation with some comments and questions that arise from our work.

1. Commutators on L_p, $1 \leq p < \infty$

Let $\mathcal{X} = L_p$ for some p, $1 \leq p < \infty$. We know that $\mathcal{X}$ has largest ideal (the ideal of non-E operators if $\mathcal{X} = L_1$ and the ideal of non-A operators if $\mathcal{X} = L_p$, $1 < p < \infty$) and the space $\mathcal{X}$ satisfies the condition $\mathcal{X} \simeq \left(\sum \mathcal{X}\right)_p$. Thus, we can try to apply the techniques developed in Chapter III in order to obtain a complete classification of the commutators on $\mathcal{X}$. We already saw (in the proof of Lemma IV.5) that non-E operators are commutators, but we have to deal with non-A operators, as well as a general operator in $\mathcal{L}(\mathcal{X})$.

Question VII.1. *Does every non-commutator on L_p, $1 \leq p < \infty$, have the form*

$\lambda I + K$, where $\lambda \neq 0$ and K belongs to the largest ideal in $\mathcal{L}(L_p)$?

If the answer to this question is positive, the spaces L_p, $1 \leq p < \infty$, will be one more example of class of spaces for which conjecture we stated in Chapter I is valid.

2. Distance between finite dimensional Banach spaces

In 1948 F. John showed in [11] that the Banach-Mazur distance between any n-dimensional Banach space and ℓ_2^n does not exceed $\sqrt{n}$. One can consider a similar question, but instead of considering ℓ_2^n we consider ℓ_∞^n (or by duality ℓ_1^n). The best upper bound is due to Giannopoulos ([10]) who proved that $d(X, \ell_\infty^n) \leq Cn^{5/6}$ and in [29] Szarek also proved that $\max\{d(X, \ell_\infty^n) : \dim X = n\} \geq cn^{1/2} \log n$. Since the exponent $5/6$ does not seem natural, we can ask the following

Question VII.2. *Find the essential upper bound of the expression $d(X, \ell_\infty^n)$, where X is an n-dimensional Banach space.*

3. Best constant in Grotendieck's inequality

The famous Grothendieck inequality

$$\max_{S_i, T_j \in S_{B(H)}} \left| \sum_{i,j=1}^{\infty} a_{i,j} (S_i, T_j) \right| \leq C \max_{s_i, t_j \in [-1,1]} \left| \sum_{i,j=1}^{\infty} a_{i,j} s_i t_j \right|$$

has a tremendous impact in many areas of mathematics. There are very tight bounds for the constant C in this inequality due to Krivine ([15]) who showed that $1.67696\ldots \leq C \leq 1.7822139781$, but so far the best constant C is unknown. If we fix a number $n > 2$, it is also not known what is the best constant C_n in the Grothendick inequality if we impose the restriction $1 \leq i, j \leq n$. Clearly $C_1 = 1$ and it is not hard to prove that $C_2 = \sqrt{2}$.

Question VII.3. *What is the best constant C in the Grothendieck inequality? What about the best constant C_n in the same inequality?*

An answer to the second question will also answer the first one because $C = \lim_{n\to\infty} C_n$.

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