VOLUMES OF CERTAIN LOCI OF POLYNOMIALS

AND THEIR APPLICATIONS

A Dissertation

by

SWAMINATHAN SETHURAMAN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2009

Major Subject: Mathematics

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ABSTRACT

Volumes of Certain Loci of Polynomials

and Their Applications. (May 2009)

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Chair of Advisory Committee: Dr. J. Maurice Rojas

To prove that a polynomial is nonnegative on $\mathbf{R}^n$, one can try to show that it is a sum of squares of polynomials (SOS). The latter problem is now known to be reducible to a semi-definite programming (SDP) computation that is much faster than classical algebraic methods, thus enabling new speed-ups in algebraic optimization. However, exactly how often nonnegative polynomials are in fact sums of squares of polynomials remains an open problem. Blekherman was recently able to show that for degree k polynomials in n variables with $k = 4$ fixed those that are SOS occupy a vanishingly small fraction of those that are nonnegative on $\mathbf{R}^n$, as $n \to \infty$. With an eye toward the case of small n, we refine Blekherman's bounds by incorporating the underlying Newton polytope, simultaneously sharpening some of his older bounds along the way. Our refined asymptotics show that certain Newton polytopes may lead to families of polynomials where efficient SDP can still be used for most inputs.

To my parents, sister and grandma

ACKNOWLEDGMENTS

I would like to acknowledge the constant support and direction of my advisor, Dr. Maurice Rojas, without which this dissertation would not have been possible. I also thank Dr. Paula Tretkoff and Dr. Stiller for useful comments and suggestions. Dr. Laszlo Kish, my M.S. thesis advisor has also been a source of help and support.

Above all I thank my parents and family for believing in me and providing all important moral support.

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CHAPTER I

INTRODUCTION

A. Motivation and previous work

In my dissertation I study certain quantitative versions of Hilbert's Nullstellensatz. We begin by recalling the Positivestellensatz of Stengle,

Theorem 1. *Stengle's Positivestellensatz [1] Let* $\mathbf{R}$ *be a real-closed field, and* $\mathbf{F}$ *a finite set of polynomials over* $\mathbf{R}$ *in* n *variables. Let* W *be the semialgebraic set*

$$W = \{x \in R^n \mid \forall f \in F \, f(x) \geq 0\},$$

and let C *be the cone generated by* $\mathbf{F}$ *(i.e., the subsemiring of* $\mathbf{R}[X_1, \ldots, X_n]$ *generated by* $\mathbf{F}$ *and arbitrary squares). Let* $p \in \mathbf{R}[X_1, \ldots, X_n]$ *be a polynomial. Then*

$$\forall x \in W \, : \, p(x) > 0 \textit{ if and only if } \exists f_1, f_2 \in \mathbf{C} \, : \, pf_1 = 1 + f_2$$

The Nullstellensatz gives an algebraic framework for the following decision problem: Given an $f \in \mathbf{R}\left[x_1, \ldots, x_n\right]$, decide if there exists an $x \in \mathbf{R}^n$ such that $f(x) = 0$. If we let n vary, this problem is known to be NP hard [2]. For $n = 1$, we can decide if f has real roots in time polynomial in the degree of f, $deg(f)$. This has been known since the beginning of the twentieth century [3], [4]. For sparse polynomials an upper bound polynomial in $\log deg(f)$ is unknown. For polynomials with 3 terms, we can find such a complexity bound due to a result of Rojas [5]. For a real polynomial to have roots it needs to be either always positive or always negative. So it makes sense to study positivity and in particular its relations to sums of squares. In particular we

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This dissertation follows the style of *IEEE Transactions on Automatic Control.*

have the following result:

$$\text{For, } n = 1, f(x) \geq 0 \Longleftrightarrow f = f_1^2 + \ldots + f_k^2 + c$$
$$\text{where } f_i \in \mathbf{R}[x] \text{ and } c \geq 0$$

Thus in the case $n = 1$, checking positivity is the same as checking sums of squares. Checking sums of squares can be done in polynomial time via semi definite programming, but its behavior for sparse polynomials is still not understood. For example, there are semi definite programming softwares which accomplish this (see [6], [7], [8]). Hilbert [9] also showed that for $n = 2$ and degree 4, positivity is the same as SOS (sums of squares). It was known that in all other cases positivity and SOS are not the same. Motzkin was the first to produce a counterexample. For example,

$$p(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$$

$p \geq 0$, for all $x, y, z \in \mathbf{R}$, yet p is not a sum of squares. Choi and Lam showed another counterexample in 1987 [10].

Nevertheless, these results dont shed any light on the volume of positive polynomials that are not SOS. This question is very important to assess the effectiveness of algorithms [6], [7] which employ SOS methods to approximate maximum value of a multivariate polynomial. Blekherman [11] was the first to study positive polynomials and sums of squares as convex bodies. He showed that for homogeneous polynomials of n variables and of degree $2k$, the volume of positive polynomials goes down as $O(n^{-1/2})$, whereas that of SOS polynomials goes down as $O(n^{-k/2})$. This means that as k increases, there are significantly more positive polynomials than sums of squares.

In this dissertation, I extend Blekherman's dissertation to multihomogeneous polynomials. There are two main reasons for doing this:

1. There are important problems in control systems theory where SOS methods are used to determine the postivity of multihomogeneous polynomials [12].

2. Addition of more symmetries occurs in the multihomogeneous case and it is unknown whether that would have any impact on the ratio of SOS versus positive polynomials.

We now briefly describe the mathematical setup and our results.

B. Organization of the dissertation

In Chapter II, we introduce the notation and background in a general setting for our results. After the definition of the objects of interest namely the three cones of multihomogeneous polynomials - nonnegative polynomials, sums of squares(SOS) polynomials and sums of powers of linear forms we move onto describe a natural inner product in the space of multihomogeneous polynomials. Choosing the correct inner product is of paramount importance in this work, because a suitable choice can greatly simplify the calculations of the bounds we seek. Chapter II also provides a basic introduction to the convexity results used in this dissertation. For a greater understanding of these concepts we provide sufficient references. Chapter II ends with a discussion of Barvinok's results which form an important tool for most of the results we obtain and also an introduction to some of the exotic metrics we use in this dissertation.

In Chapter III, we examine the case of bihomogeneous polynomials in detail. We do this for two reasons. The first one being the difficulty posed by the compact general notation of Chapter II towards the understanding of methods employed. Secondly, we use the example to help calculate the improvements in our bounds to the ones obtained by Blekherman. Armed with the insight obtained from Chapter II, we proceed to the general multihomogeneous case in Chapter IV.

Chapter V talks about some applications in motion planning of robot systems arising from our joint work with Mayank Lal [13]. This is one of the first instances of the application of discriminant variety to motion planning and it provides an insight as to how recent developments in real algebraic geometry [14] offer fresh approaches in practical applications of great current interest.

C. Summary of results

Multihomogeneous polynomials are a natural extension of $P_{n,k}$ to the setting where the variable are divided into l blocks, with n_l variables in each block. Furthermore each block is homogeneous of degree k_l. More specifically if we set $N = (n_1, \ldots, n_l)$ and $K = (k_1, \ldots, k_l)$, then,

$$P_{N,2K} = \otimes_{i=1}^{l} P_{n_i,2k_i}$$

It is clear that $P_{N,2K}$ is a vector space. Chapter II describes in greater detail the properties of this vector space, namely its dimension and also describes a basis for this vector space. Inside $P_{N,2K}$, are three cones of interest, the nonnegative polynomials, $Pos_{N,2K}$, the sums of squares $Sq_{N,2K}$ and the sums of powers of linear forms $Lf_{N,2K}$. For precise definitions we refer the reader to the first section in Chapter II.

One of the goals of this dissertation is to provide a comparison of the volumes of Pos, Sq and Lf. We use a measure called relative volume which takes into account the effect of high dimensions [15]. We need to overcome a main hurdle in attempting to use methods of convex geometry towards this problem. The theorems in convexity are geared towards convex bodies with origins in their interior. But unfortunately Pos, Sq and Lf do not have this property. Also they are not compact. The way around is described in Blekherman [11]. We extend this to our general setting. The solution is to take sections of these cones with the linear space of multihomogeneous forms which integrate to zero over the products of spheres. The next step is to translate these sections so that the origin lies in their interior. We name these translated sections $\widetilde{Pos}_{N,2K}, \widetilde{Sq}_{N,2K}$ and $\widetilde{Lf}_{N,2K}$.

We are now in a position to state our results.

Theorem 2. *Lower bound on the volume of non negative multihomogeneous polynomials*

The following is a lower bound on the volume of non negative multihomogeneous polynomials:

$$\left(\frac{\mathtt{Vol}\widetilde{Pos}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} \geq \frac{\beta}{\sqrt{\max_{i\in 1,\dots,l}\{n_i \ln(2k_i+1)\}}}$$

where $\beta = \frac{1}{9e^2}$

This clearly reduces to Barvinok's result [16] when $l = 1$. We also have the following upper bound on the volume of non negative multihomogeneous polynomials.

Theorem 3. *Upper bound on the volume of non negative multihomogeneous polynomials*

The following is an upper bound on the volume of non negative multihomogeneous polynomials:

$$\left(\frac{\widetilde{Pos}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} \leq 4\prod_{i=1}^{l}\left(\frac{2k_i^2}{4k_i^2+n_i-2}\right)^{1/2}$$

Theorem 4. *Upper bound on the volume of sums of squares(SOS) of multihomogeneous polynomials*

The following is an upper bound on the volume of SOS multihomogeneous polynomials:

$$\left(\frac{\widetilde{Sq}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} \leq \sqrt{24}\prod_{i=1}^{l}\left(\frac{4^{2k_i}(2k_i)!n_i^{-k_i/2}}{k_i!}\right)^{1/2}$$

Theorem 5. *Lower bound on the volume of SOS multihomogeneous polynomials*

The following is a lower bound on the volume of SOS multihomogeneous polynomials:

$$\left(\frac{\widetilde{Sq}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} \geq \frac{1}{\sqrt{24}}\prod_{i=1}^{l}\left(\frac{(k_i)!^2}{4^{2k_i}(2k_i)!(n_i/2+k_i)^{k_i}}\right)^{1/2}$$

Theorem 6. *Lower bound on the volume of sums of powers of linear forms of multihomogeneous polynomials*

The following is a lower bound on the volume of sums of powers of linear forms of multihomogeneous polynomials:

$$\left(\frac{\widetilde{Lf}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} \geq \prod_{i=1}^{l}\left(\frac{(k_i)!\sqrt{4k_i^2+n_i-2}}{4k_i\sqrt{2}(n_i/2+2k_i)^{k_i}}\right)^{1/2}$$

In the next section we shall discuss the improvements obtained by our results in contrast with Blekherman's results. Although we provide these numerical examples here, the focus of the dissertation is in the methods involved. Hence our numerical exploration will be brief and confined to this introductory chapter.

D. Discussion of results

All these bounds reduce to Blekherman's results in the case $l = 1$. Table I and Table II compare the improvements obtained by our bounds versus that of Blekherman's, keeping the number of variables and the number of blocks fixed. The greatest improvement is obtained when the blocks are even sized, that is when $k_1 = k_2$.

Our next two tables(Table III and Table IV) compare the case when the degrees are kept constant at $k_1 = k_2 = 3$, but the number of variables is allowed to vary. Again the results are similar to the one obtained above.

Table I. Lower bound on non negative polynomials with $n_1 = 3, n_2 = 3$

	$k_1 = 1$	$k_1 = 2$	$k_1 = 3$	k_1=4
k_2 =1	0.0046	0.0031	0.0026	0.0023
k_2 =2	0.0031	0.0031	0.0026	0.0023
k_2 =3	0.0026	0.0026	0.0026	0.0023
k_2 =4	0.0023	0.0023	0.0023	0.0023

Table II. Blekherman's lower bound on non negative polynomials with $n_1 = 3, n_2 = 3$

	$k_1 = 1$	$k_1 = 2$	$k_1 = 3$	k_1=4
k_2 =1	0.0016	0.0013	0.0011	0.0010
k_2 =2	0.0013	0.0011	0.0010	0.0010
k_2 =3	0.0011	0.0010	0.0010	0.0009
k_2 =4	0.0010	0.0010	0.0009	0.0009

Finally we shall investigate the effect of increasing the number of blocks l. We find that keeping the number of variables and the degrees fixed, increasing the blocks provides a greater improvement in our bounds. For the sake of definiteness, we fix $n = 10$ and $2k = 20$. Blekherman's bound for the non negatives is 0.0005. We compare this value with our results for different number of blocks in Table V.

E. An algebraic geometry method for motion coordination of mobile agents

In this section, we sketch a novel method for motion coordination of mobile agents. Autonomous mobile agents have a number of applications these days with a lot of research being done in building them with better capabilities. Multiple robots are

Table III. Lower bound on non negative polynomials with $k_1 = 3, k_2 = 3$

	$n_1 = 2$	$n_1 = 3$	$n_1 = 4$	n_1=5
n_2 =2	0.0039	0.0026	0.0019	0.0015
n_2 =3	0.0026	0.0026	0.0019	0.0015
n_2 =4	0.0019	0.0019	0.0019	0.0015
n_2 =5	0.0015	0.0015	0.0015	0.0015

Table IV. Blekherman's lower bound on non negative polynomials with $k_1 = 3, k_2 = 3$

	$n_1 = 2$	$n_1 = 3$	$n_1 = 4$	n_1=5
n_2 =2	0.0015	0.0012	0.0010	0.0008
n_2 =3	0.0012	0.0010	0.0008	0.0007
n_2 =4	0.0010	0.0008	0.0007	0.0007
n_2 =5	0.0008	0.0007	0.0007	0.0006

more useful than single robots and are capable of doing many tasks which cannot be done by single robots. Applications include deployment of a group of mobile agents with sensors mounted on them in an area affected by earthquake, flood etc. so that data regarding the damage can be assessed and relief provided accordingly. Most of the research that has been done in the area of motion planning uses the composite configuration approach or the decoupled planning approach. The method we describe differs from these methods in that the planning is being done in polynomial space.

The basic idea is that given n agents moving in a 2-D space, we represent them as point objects. Let (x_{1i}, y_{1i}), .., (x_{ni}, y_{ni}) be the initial configuration of the agents and (x_{1f}, y_{1f}), .., (x_{nf}, y_{nf}) be the desired final configuration. We create two polyno-

Table V. Lower bound on non negative polynomials with different block sizes

	Lower bound
N =(2, 2, 2, 2, 2) and K = (2,2,2,2,2)	0.0047
N =(3, 3, 2, 2) and K = (3,3,2,2)	0.0026
N =(3, 4, 3) and K = (3,4,3)	0.0017
N =(5, 5) and K = (5,5)	0.0013

mials P_i and P_f from the initial and final configurations respectively, by mapping the configuration in $\mathbf{R}^2$ to $\mathbf{C}^2$ and using the n points in $\mathbf{C}^2$ as roots of the corresponding polynomials. Then we deform the intial polynomial P_i to the final polynomial P_f by means of a straight line path connecting each coefficient. Now the set of polynomials of degree n having multiple roots is called the discriminant variety, Σ_n. There is a result [17] which states that the complement of the discriminant variety in $\mathbf{C}^n$ is connected. There is also a simple parametrization of Σ_n due to [14]. This enables us to give a quick method to verify that our deformation indeed does not pass through the discriminant variety. It is important that the path does not pass through Σ_n to ensure that the agents do not collide at any point in time. We shall now briefly describe the algorithm.

1. Algorithm description

If $(x_{1i}, y_{1i}), (x_{2i}, y_{2i}), \ldots, (x_{ni}, y_{ni})$ are the coordinates of the agents in the initial configuration and $(x_{1f}, y_{1f}), (x_{2f}, y_{2f}), \ldots, (x_{nf}, y_{nf})$ are the coordinates of the agents in the final configuration, then we define the initial and final polynomials as follows:

$$P_i(x) := (x - \imath(x_{1i} + y_{1i})) \ldots \imath(x - (x_{ni} + y_{ni}))$$

$$P_g(x) := (x - \imath(x_{1f} + y_{1f})) \ldots \imath(x - (x_{nf} + y_{nf}))$$

Let, Σ_n be the discriminant variety of polynomials P_n of degree n. Then we know [17] that the complement of the discriminant variety, that is $P_n\ \Sigma_n$, is connected. We consider the following "straight line" path in the parameter space, $P(\lambda) = (1-\lambda)P_i + \lambda P_f$. The algorithm [13] described in the dissertation computes a path from $P_i(x)$ to $p_g(x)$ which avoids the discriminant variety Σ_n. This means that we have an algorithm to move the mobile agents from the initial to final configuration avoiding collisions.

We can impose velocity and acceleration constraints on each mobile agent by reparametrizing P. The next main improvement would be to make sure that the agents avoid stationary obstacles. This is done by first finding the bounding disc for the agents at any given time. The bounding disc is a disc which contains all the roots of the polynomial $P(\lambda)$. This can be obtained by means of the following result [4]: All the roots of a polynomial , $a_n x^n + \ldots + a_0$ can be bounded within a disc of radius $r = 2\max_{k\in\{1,\ldots,n\}} |\frac{a_{n-k}}{a_n}|^{1/k}$. Once the bounding disc is found, we can use any standard methods of motion planning of a single agent for planning the motion of the bounding disc.

Chapter V describes all these algorithms in greater detail with illustrative examples. Since these results are among the first of its kind, there is a lot of scope to expand this idea. Extending this method to agents in three dimensional space is clearly one of the main open problems. One can also try to relax the treatment of agents as point objects. Given the size of the agents , research can be done to find complete algorithms which guarantee maintenance of a certain distance between the agents at all times.

CHAPTER II

NOTATION AND BACKGROUND

A. Preliminaries

In this section, we define the general class of multihomogeneous polynomials and describe some of their properties. Let $P_{n,k}$, denote the set of all homogeneous polynomials in n variables and degree k. Multihomogeneous polynomials are a natural extension of $P_{n,k}$ to the setting where the variable are divided into l blocks, with n_l variables in each block. Furthermore each block is homogeneous of degree k_l. More specifically if we set $N = (n_1, \ldots, n_l)$ and $K = (k_1, \ldots, k_l)$, then,

$$P_{N,K} = \otimes_{i=1}^{l} P_{n_i,k_i}$$

We can define the sums of squares and sums of powers of linear forms analogous to the $l = 1$ case.

$$Sq_{N,2K} = \left\{ f \in P_{N,K} \text{ such that, } f = \sum_{i=1}^{m} f_i^2, \text{ for some } f_i \in P_{N,K} \right\}$$

$$Lf_{N,2K} = \left\{ f \in P_{N,K} \text{ such that, } f = \sum_{i=1}^{m} \prod_{j=1}^{l} f_{ij}^{2k_i}, \text{ for some } f_{ij} \in P_{n_i,1} \right\}$$

Also, we have the non negative polynomials,

$$Pos_{N,2K} = \{f \in P_{N,K} \text{ such that, } f(x_1, \ldots, x_l) \geq 0, \ \forall (x_1 \ldots x_l) \in \mathbf{R}^{n_1} \times \ldots \times \mathbf{R}^{n_l}\}$$

We shall shortly demonstrate that $P_{N,K}$ can be viewed as a function on a suitable product of spheres. First the definition. For $N = (n_1, \ldots, n_l)$, we shall use the notation S^N to denote the product of spheres $S^{n_1-1}\dots S^{n_l-1}$, i.e.,

$$S^N := \prod_{i=1}^{l} S^{n_i-1}$$

Clearly, $P_{N,K}$ can be considered as a function on S^N, because if we know the value of $f \in P_{N,K}$ at some $v = (v_1, \dots, v_l) \in S^N$, the value at any $x = (x_1, \dots, x_l) \in \prod_{i=1}^{l} \mathbf{R^{n_i}}$ is determined. This is because there are λ_i such that $x_i = \lambda_i v_i$ for all $i = 1 \dots n$. Thus the value of f at x would simply be $\prod_{i=1}^{l} \lambda_i f(v)$. Let $\mathcal{N} = \sum_{i=1}^{l} n_i$. We can also consider $f \in P_{N,K}$ as a function on $S^{\mathcal{N}-1}$. However, $P_{N,K}$ is not invariant under the action of $SO(\mathcal{N})$, but it is invariant under the action of $\prod_{i=1}^{l} SO(n_i)$. This distinction is the crux of most of the arguments used in subsequent sections. Henceforth in this dissertation, we shall employ the products of spheres exclusively and also the product of $SO(n_i)$'s which we shall denote by $SO(N)$, i.e.,

$$SO(N) := \prod_{i=1}^{l} SO(n_i)$$

B. A natural inner product

Since we would be using a lot of convex geometry methods, it is very essential to work with a suitable inner product. Due to the fact that we will be exploiting the invariance of $P_{N,K}$ under $SO(N)$. We have the following lemma, which provides a way to get an inner product on $V_1 \otimes \dots \otimes V_n$, when we have inner products $\langle , \rangle_i$ on V_i.

Lemma 1. *[15] Given $v = v_1 \otimes \dots \otimes v_n$ and $w = w_1 \otimes \dots \otimes w_n$ in $V = V_1 \otimes \dots \otimes V_n$, we can construct the following inner product $\langle , \rangle$ on decomposable tensors and extend them via linearity to all elements of V.*

$$\langle v, w\rangle := \prod_{i=1}^{n} \langle v_i, w_i\rangle_i$$

Example 1. *For f , g in $P_{n,k}$ we have the following natural inner product which is invariant under the action of $SO(n)$.*

$$\langle f, g\rangle = \int_{S^{n-1}} fg d\sigma$$

Using Lemma 1 we can extend this to an inner product on $P_{N,K}$ that is invariant under the action of $SO(N)$.

$$\begin{aligned}\langle f_1 \otimes \ldots \otimes f_l, g_1 \otimes \ldots \otimes g_l\rangle &:= \prod_{i=1}^{l} \int_{S^{n_i-1}} f_i g_i d\sigma_i \\ &= \int_{\prod_{i=1}^{l} S^{n_i-1}} (f_1 \ldots f_l)(g_1 \ldots g_l) d\sigma\end{aligned}$$

C. Basics of convexity

For the basic definitions and concepts in convex geometry we refer the reader to the following excellent books [18], [19]. A delightful intuitive presentation is the excellent article by K.Ball [20]. We recall some of these in order to be as self contained as possible but we strongly encourage the reader to consult the above mentioned books.

Definition 1. *Let V be a real vector space. A set $A \subset V$ is called* convex, *provided for all $x, y \in A$, the interval,*

$$[x, y] = \{\alpha x + (1-\alpha)y : 0 \leq \alpha \leq 1\}$$

is contained in A. An empty set is convex by convention.

If in addition the vector space V is endowed with an inner product $\langle , \rangle$, then we can identify V^* with V by associating a $v \in V$ with every $l \in V^*$ as follows:

$$l_v(w) = \langle w, v \rangle$$

This leads us to the definition of a polar body of a convex set.

Definition 2. *Polar of a convex body Let $K \subset V$, be a convex set in a vector space V. We have the following definition of the polar of K, K°.*

$$K^\circ = \{l \in V^* : l(v) \leq 1, \forall v \in K\}$$

In case the vector space V is endowed with an inner product $\langle , \rangle$, then we can identify K° with a subset of V itself.

$$K^\circ = \{w \in V : \langle v, w \rangle \leq 1, \forall v \in K\}$$

Definition 3. *A subset K of a vector space V is called a cone, if $0 \in K$ and $\lambda x \in K$ for every $x \in K$ and $\lambda \geq 0$. Also K is called a convex cone if it is both convex and a cone or alternatively, $\alpha x + \beta y \in K$ for $\alpha, \beta \geq 0$ and for every $x \in K$.*

To describe some of the properties of Pos, Sq and Lf, we shall find it useful to describe a basis for $P_{N,K}$. First we shall introduce some compact notations to represent monomials in the multihomogeneous case. Let, $x_i = (x_{i1}, \ldots, x_{in_i}) \in \mathbf{R}^{n_i}$, for $i = 1, \ldots, l$ represent the l blocks of variables with corresponding homogeneous degrees k_i. We denote a monomial in the i^{th} block as follows,

$$x_i^{\alpha_i} := x_{i1}^{\alpha_{i1}} \ldots x_{in_i}^{\alpha_{in_i}}$$

where $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in_i}) \in \mathbf{Z}^{n_i}$ is the exponent vector such that $|\alpha_i| := \sum_{j=1}^{n_i} \alpha_{ij} = k_i$.

Now for $\mathbf{x} = (x_1, \ldots, x_l) \in \mathbf{R}^{n_1} \times \ldots \times \mathbf{R}^{n_l}$, and for a string of exponent vectors

$\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbf{Z}^{n_1} \times \ldots \times \mathbf{Z}^{n_l}$, we define,

$$\mathbf{x}^\alpha := x_1^{\alpha_1} \ldots x_l^{\alpha_l}$$

where $|\alpha_i| = k_i$ for $i = 1 \ldots l$. We also say that $|\alpha| = K$ when this condition holds. From now on we shall adopt the notation, $\mathbf{R}^N := \mathbf{R}^{n_1} \times \ldots \times \mathbf{R}^{n_l}$.

Now, a polynomial $f \in P_{N,K}$ can be expressed in the following form,

$$f := \sum_{|\alpha|=K} c_\alpha \mathbf{x}^\alpha$$

In fact x_α with $|\alpha| = K$ forms a basis for $P_{N,K}$. We shall call this the standard basis of $P_{N,2K}$. This leads us directly to the following lemma.

Lemma 2. *The dimension of $P_{N,K}$ is given by,*

$$Dim\,(P_{N,K}) = \prod_{i=1}^{l} \binom{n_i + k_i - 1}{k_i}$$

Lemma 3. *$Pos_{N,2K}, Sq_{N,2K}$ and $Lf_{N,2K}$ are full dimensional convex cones in $P_{N,2K}$.*

Proof. It is easy to show that Pos, Sq and Lf are convex cones. For example for $f, g \in Pos_{N,2K}$, clearly $\alpha f \geq 0$ for every $\alpha \geq 0$ and hence $\alpha f + \beta g \in Pos_{N,2K}$, for all $\alpha, \beta \geq 0$. For $f, g \in Sq_{N,2K}$, we have, $\alpha f + \beta g = \left(\sqrt{\alpha}\right)^2 f + \left(\sqrt{\beta}\right)^2 g \in Sq_{N,2K}$. We can construct a very similar argument for $Lf_{N,2K}$. To show closure is a little more involved and we shall prove it via the following lemmas. $\square$

Lemma 4. *The boundary $\partial Pos_{N,2K}$ of the cone of non negative polynomials is comprised of the set of non negative polynomials that attain zero at some point, i.e , $M = \{f \in Pos_{N,2K},$ such that $\exists x \neq 0 \in \mathbf{R}^N,$ with, $f(x) = 0\}$. We have,*

$$\partial Pos_{N,2K} = M$$

Proof. We shall first show that $M \subset \partial Pos_{N,2K}$. Let $f \in M, f \neq 0$ be such that

$f(y) = 0$ for some $y \in \mathbf{R}^N$. Let us write a non zero $f \in M$ in terms of the above mentioned standard basis of $P_{N,2K}$, that is let $f = \sum_{|\alpha|=2K} c_\alpha \mathbf{x}^\alpha$. Since f is non zero, there is some $\widetilde{\alpha}$, such that $y^{\widetilde{\alpha}} \neq 0$. Let $c = sign(y^{\widetilde{\alpha}})$. Let $f_i = -(c/n)y^{\widetilde{\alpha}} + f$. Consider the sequence $\{f_i\}_{i=1}^{\infty}$. Clearly $f_i \notin Pos_{N,2K}$ for all i and $f_i \to f$. Thus $f \in \partial Pos_{N,2K}$. As for the other direction, let us assume that there exists a sequence $\{f_i\}_{i=1}^{\infty}$ in $Pos_{N,2K}$ such that $f_i \to g$, where $g \notin M$. This implies that there is some $y \in \mathbf{R}^N$ such that $g(y) = c \leq 0$. Hence there is some $L \in \mathbf{N}$ such that $f_i(y) \leq c/2$, for all $i \geq L$. This is clearly a contradiction to f_i being in $Pos_{N,2K}$. $\square$

To show $Sq_{N,2K}$ is closed, we first note that any sequence $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_{ij}^2$ that converges to $f \in P_{N,2K}$ can be written as sum of squares whose coefficients are bounded. Hence we can find a subsequence in $\{f_{ij}\}_{j=1}^{\infty}$ that converges to f_i for each i. Thus we have $f = \sum_{i=1}^{\infty} f_i^2$. Hence Sq is closed. A similar argument works for $Lf_{N,2K}$.

Now we get back to proving that $Pos_{N,2K}$ is full dimensional. To do this, we shall show, that if $g \notin \partial Pos_{N,2K}$ and $f \in P_{N,2K}$, then there exists an $N_g \in \mathbf{N}$ such that $g - f/N \in Pos_{N,2K}$. First we note that since S^N is compact, g has a minimum on S^N, say $m_g > 0$. Also let m_f be the minimum of f. Then we see that we can set $N_g \geq m_g/m_f$. A similar argument works for $Lf_{N,2K}$. For $Sq_{N,2K}$, we refer the reader to [19].

Definition 4. *For a convex body K in a vector space V containing the origin in its interior, we define the gauge function G_K as follows:*

$$G_K(x) := \sup\{\lambda \geq 0 : \lambda x \in K\}$$

Lemma 5. *[20] Let K be a convex body in V with origin in its interior and let $\langle , \rangle$ be an inner product on V. Let S be the unit sphere in V and $d\mu$ the $SO(V)$ invariant* *measure on S. Then we have the following formula for the volume of K.*

$$\frac{\mathtt{Vol}(K)}{\mathtt{Vol}B_M} = \int_S G_K^{D_M}(x)d\mu$$

where D_M is the dimension of the vector space V and B_M is the unit ball in V.

Definition 5. *[18] For a convex body K in V, a point $v \in K$ is called an extreme point of K if $v = \alpha v_1 + \beta v_2$ with $\alpha, \beta > 0$, implies that v_1, v_2 are multiples of v.*

Lemma 6. *The extreme points of $Sq_{N,2K}$ are squares. That is, if f is an extreme point of $Sq_{N,2K}$, then $f = g^2$ for some $g \in Sq_{N,K}$.*

D. Slices of multihomogeneous polynomials

To compare the volumes of $Pos_{N,2K}$, $Sq_{N,2K}$ and $Lf_{N,2K}$ it would be useful to define slices of these objects with an appropriate hyperplane in $P_{N,2K}$. To this end we define two hyperplane sections and a special polynomial F:

$$L_{N,2K} := \{p \in P_{N,2K} | \int_{S^N} p d\sigma = 1\}$$

$$M_{N,2K} := \{p \in P_{N,2K} | \int_{S^N} p d\sigma = 0\}$$

$$F := \prod_{i=1}^{l} (x_{i1}^2 + \dots x_{in_1}^2)^{k_i}$$

Now we are in a position to describe the slices.

$$Pos'_{N,2K} = Pos_{N,2K} \cap L_{N,2K}$$

$$Sq'_{N,2K} = Sq_{N,2K} \cap L_{N,2K}$$

$$Lf'_{N,2K} = Lf_{N,2K} \cap L_{N,2K}$$

To use all the convexity results mentioned above, we need to work with convex bodies with origin in their interiors. Hence we shall translate $Pos^{'}, Sq^{'}$ and $Lf^{'}$ by F.

$$\widetilde{Pos}_{N,2K} := \{p \in M_{N,2K} | p + F \in Pos^{'}_{N,2K}\}$$

$$\widetilde{Sq}_{N,2K} := \{p \in M_{N,2K} | p + F \in Sq^{'}_{N,2K}\}$$

$$\widetilde{Lf}_{N,2K} := \{p \in M_{N,2K} | p + F \in Lf^{'}_{N,2K}\}$$

To take into account the effect of dimension on the volume, we shall define and use what is called the relative volume [15].

Definition 6. *The relative volume $RVol$ of a convex body K in a D dimensional vector space V, with respect to the unit ball B in V is defined as,*

$$RVol := \left(\frac{\mathtt{Vol}K}{\mathtt{Vol}B}\right)^{1/D_V}$$

Example 2. *$\widetilde{Pos}_{N,2K}$ is a convex body in $M_{N,2K}$. The dimension of M is $D_M = \prod_{i=1}^{l} \binom{n_i+k_i-1}{k_i} - 1$, using 2. Thus we have, $RVol(\widetilde{Pos}_{N,2K}) = \left(\frac{\mathtt{Vol}\widetilde{Pos}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M}$, where B_M is the unit ball in $M_{N,2K}$.*

E. Multihomogeneous polynomials as linear functionals on group orbits

It is convenient to view $P_{N,2K}$ as a linear functional in some tensor product space. This would enable us to use the powerful results of Barvinok [16]. As always we have $N = (n_1, \ldots, n_l)$ and $K = (k_1, \ldots, k_l)$. Let us begin by considering the tensor product of $\mathbf{R}^{n_i}$'s.

$$T_{N,K} := (\mathbf{R}^{n_1})^{\otimes k_1} \otimes (\mathbf{R}^{n_2})^{\otimes k_2} \otimes \ldots \otimes (\mathbf{R}^{n_1})^{\otimes k_l}$$

We can think of $t \in T$ as an l dimensional array, each of whose elements is a multidimensional array indexed by k_i tuples, $\{1 \leq i_1 \leq i_2 \ldots i_{k_i} \leq n_i\}$. That is,

$$t_i = \{x_{i_1,\ldots,x_{k_i}} | 1 \leq i_1 \leq \ldots \leq k_i \leq n_i\}$$

Given an $x = (x_1, \ldots, x_l) \in \prod_{i=1}^{l} \mathbf{R}^{n_i}$, we have $x^{\otimes K}$ given by the following element in $T_{N,K}$, whose i^{th} multidimensional array is given by,

$$x^{\otimes K}_{i_1,\ldots,i_{ki}} = \{x_{i1}x_{i2}\ldots x_{ik_i} | 1 \leq i_1 \leq \ldots \leq k_i \leq n_i\}$$

Let, $Sym_K(T_{N,K})$ be the symmetric part of $T_{N,K}$ under $\mathfrak{S}(K) := \prod_{i=1}^{l} \mathfrak{S}(k_i)$, where $\mathfrak{S}(k_i)$ is the symmetric group of k_i objects. This means that $y \in Sym_K(T_{N,K})$ implies that, $y^{\otimes K}_{i_1,\ldots,i_{ki}} = y^{\otimes K}_{\sigma(i_1),\ldots,\sigma(i_{ki})}$ for every $\sigma \in \mathfrak{S}(k_i)$ and every i. Clearly, $x^{\otimes K}$ is in $Sym_K(T_{N,K})$.

Now choose $e \in S^N$ and let $w = e^{\otimes 2K}$. Then the orbit $\{gw | g \in SO(N)\}$ lies in the symmetric part of $T_{N,2K}$. Let $t = \int_{S^N} gw \, d\sigma$ be the center of the orbit. Then from [16] t is a multiple of F. Translating the orbit by shifting t to the origin we obtain the convex hull of the orbit of $v = w - t$.

$$B := \mathtt{conv}\{gv : g \in SO(N)\}$$

A multihomogeneous polynomial $f = \sum_{|\alpha|=2K} c_\alpha \mathbf{x}^\alpha \in P_{N,2K}$, viewed as mentioned above, as a function on the product of spheres S^N can be identified with the restriction onto the orbit $\{gw : g \in SO(N)\}$ of a linear functional $l : T_{N,2K} \to \mathbf{R}$, defined by coefficients c_α. Hence it is rather easy to see that the linear functionals on B are in one to one correspondence with the polynomials in $M_{N,2K}$. Furthermore the negative polar $-B^\circ$ can be identified with $\widetilde{Pos}_{N,2K}$.

We shall now state and explain the theorems of Barvinok alluded to in the above paragraph.

Theorem 7. *Barvinok's Theorem [16]*

Let G be a compact group acting on a finite dimensional vector space V. Let $v \in V$ be a point and let $l : V \to \mathbf{R}$ be a linear functional. Let us define

$$f : G \to \mathbf{R}, f(g) = l(gv), \forall g \in G \tag{2.1}$$

For $k > 0$, let d_k be the dimension of the subspace spanned by the orbit $\{gv^{\otimes k}, g \in G\}$ in $V^{\otimes k}$. In particular $d_k \leq \binom{dimV+k-1}{k}$. Then,

$$\|f\|_{2k} \leq max_{g \in G}|f(g)| \leq \left(d_k\right)^{1/2k} \|f\|_{2k} \tag{2.2}$$

This theorem enables to bound the sup norm of a function f by means of its $2k^{th}$ norms. We shall use this in our lower bound results.

Lemma 7. *Barvinok's Lemma [16]*

Let G be a compact group acting on a finite d-dimensional real vector space V endowed with a $G-$ invariant scalar product $\langle , \rangle$ and let $v \in V$ be a point. Let $S^{d-1} \subseteq V$ be the unit sphere endowed with the Haar probability measure dc. Then, for every positive integer k, we have,

$$\int_{S^{d-1}} \left(\int_G \langle c, gv \rangle^{2k} dg \right)^{1/2k} dc \leq \sqrt{\frac{2k\langle v, v\rangle}{d}}$$

F. More results from convexity

We shall first describe the Blaschke-Santalo inequality [21], [22], [18], [23]. This will prove quite useful in this dissertation, to transfer lower bound results into upper bound results. Let K be a convex set in an n dimensional vector space V, endowed with an inner product $\langle , \rangle$. We introduced the concept of a polar body above. This can be generalized to what is called the polar of K with respect to an arbitrary point

$z \in V$. Thus we have,

$$K^z := \{y + z : \langle y, x + z \rangle \leq 1 \ , \forall x \in K\}$$

Let,

$$p(K) = \inf\{(\mathtt{Vol}K^z)(\mathtt{Vol}K) : z \in int(K)\}$$

This infimum is reached for a unique point in V called the Santalo point of K, $s(K)$. The following is the Blaschke-Santalo inequality.

$$(\mathtt{Vol}K)(\mathtt{Vol}K^{s(K)}) \leq (\mathtt{Vol}B_M)^2$$

where B_M is the unit ball in V. This was proved by fairly technical arguments in [21]. Saint Raymond in 1981 [24] gave a simple proof of this for the special case of centrally symmetric convex bodies. A simple proof of a generalization of the Blaschke-Santalo inequality was provided by Meyer and Pajor [25] using Steiner symmetrization.

Definition 7. *The sup norm of $f \in P_{N,K}$ is defined as,*

$$\|f\|_\infty := \mathtt{sup}\{f(x) : x \in S^N\}$$

Definition 8. *The unit ball in $M_{N,2K}$ under the sup norm is as follows:*

$$B_\infty := \{f \in M_{N,2K} : \|f\|_\infty \leq 1\}$$

Lemma 8. *B_∞ is the intersection of $\widetilde{Pos}^\circ_{N,2K}$ with its negative polar, $-\widetilde{Pos}^\circ_{N,2K}$.*

Proof. $f \in \widetilde{Pos}^\circ_{N,2K}$, implies that $\|f\|_\infty \geq -1$. Therefore, $f \in -\widetilde{Pos}^\circ_{N,2K}$, implies that $\|f\|_\infty \leq -1$.Thus clearly we have the desired result. $\square$

Lemma 9. *For subsets A and B of a vector space V, the polar of $A \cap B$ is the convex* *hull of the polars of A and B. That is,*

$$(A \cap B)^{\circ} = \texttt{Conv}\{A^{\circ}, \cap B^{\circ}\}$$

Proof. $f \in \texttt{Conv}\{A^{\circ}, \cap B^{\circ}\}$, implies that there exist $f_1 \in A^{\circ}$ and $f_2 \in B^{\circ}$, such that, $f = \lambda f_1 + (1-\lambda) f_2$. Now $\langle f, g\rangle = \lambda\langle f_1, g\rangle + (1-\lambda)\langle f_2, g\rangle$. For $g \in A \cap B$, we have $\langle f_1, g\rangle \leq 1$ and $\langle f_2, g\rangle \leq 1$. This clearly means, $\langle f, g\rangle \leq 1$. □

We shall now introduce the concept of Minkowski addition of subsets of a vector space. This has widespread applications in many areas of convexity and we encourage the reader to consult [18] for more on this very useful topic.

Definition 9. *For subsets A, B of a vector space V, we define the Minkowski sum of A and B as follows:*

$$A \oplus B = \{v \in V : \exists x \in A, y \in B \text{ with, } v = x + y\}$$

Since a closed convex set is given by the intersection of its supporting half spaces, such a set can be conveniently described by the position of its support planes. Such a description is given by the support function. This is used the the proof of the upper bound for SOS polynomials.

Definition 10. *For a non empty closed convex set $K \subset V$, the support function $h(K, .) = h_K$, is defined by,*

$$h(K, y) := \sup\{\langle x, y\rangle : x \in K\}$$

The support plane is given by $H(K, y) := \{x \in V : \langle x, y\rangle = h(K, y)\}$. The support set $F(K, y)$ is nothing but the intersection of the support plane $H(K, y)$ with K.

The intuition of the support function is that it for each $v \in S(V)$ it gives the element of K that is furthest from v.

We shall end our section on convexity with an inequality which makes use of the celebrated Brunn-Minkowski inequality, namely the Rogers-Shephard inequality. The proof of this is beyond the scope of this dissertation. An useful reference is [18]. But first a definition.

Definition 11. *For a convex body $K \subset V$, the difference body $D(K)$ is defined by,*

$$D(K) := K - K = \{x \in V \,:\, K \cap (K + x) \neq \Phi\}$$

Lemma 10. *For $K \subset V$ in an n dimensional vector space, V, we have,*

$$\mathtt{Vol}(D(K)) \leq \binom{2n}{n} \mathtt{Vol}(K)$$

G. Exotic metrics on $P_{N,2K}$

1. The gradient metric

We shall now briefly discuss a couple of important metrics that would be used in our proofs later. Let us begin by defining a multigradient on $P_{N,2K}$ along the lines of Lemma 1. The idea is that given gradients ∇_i on $P_{ni,ki}$, we can tensor them together to obtain a gradient on $P_{N,2K}$.

$$\nabla := \otimes_{i=1}^{l} \nabla_i$$

where for every $f_i \in P_{n_i,k_i}$ we have $\nabla_i(f_i) = (\frac{\partial f_i}{\partial x_{i1}}, \ldots, \frac{\partial f_i}{\partial x_{in_i}})$ and $\nabla(f)$ would be $\otimes_{i=1}^{l} \nabla_i(f_i)$, when $f = \otimes_{i=1}^{l} f_i$ is a decomposable tensor. Using linearity like before, we can extend this to the whole of $P_{N,2K}$.

Definition 12. *For $f, g \in P_{N,2K}$, we define the gradient metric as follows:*

$$\langle f, g \rangle_G := \frac{1}{4^l \prod_{i=1}^{l} k_i^2} \int_{S(N)} \langle \nabla(f), \nabla(g) \rangle \; d\sigma$$

We have the following result of Kellogg [26]that is useful.

Lemma 11. *For $f \in P_{N,2K}$ we have,*

$$\|f\|_\infty \geq \|f\|_G$$

2. The differential metric on the space of multihomogeneous polynomials

We extend the differential metric of Blekherman's paper to the multihomogeneous case. First some more notation. Let $x_1 = (x_{11}, \ldots, x_{1n_1})$, $x_2 = (x_{21}, \ldots, x_{2n_2})$, …, $x_n = (x_{l1}, \ldots, x_{ln_l})$ denote the l sets of variables. Like before, $\mathbf{x} = (x_1, \ldots, x_l) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times \ldots \times \mathbf{R}^{n_l}$. We shall use the following compact notation to name the associated differential operators of monomials. Let $\alpha_1 = (\alpha_{11}, \ldots, \alpha_{1n_1})$, …, $\alpha_l = (\alpha_{l1}, \ldots, \alpha_{ln_l})$. Let $\alpha = (\alpha_1, \ldots, \alpha_l)$. Then,

$$\mathbf{x}^\alpha = \prod_{i=1}^{l} x_{i1}^{\alpha_{i1}} \ldots x_{in_i}^{\alpha_{in_i}}$$

and,

$$D_{\mathbf{x}^\alpha} := \bigotimes_{i=1}^{l} \frac{\partial^{\alpha_{i1}} \ldots \partial^{\alpha_{in_i}}}{\partial x_{i1}^{\alpha_{i1}} \ldots \partial x_{in_i}^{\alpha_{in_i}}}$$

Now for a form $f \in P_{N,2K}$,

$$f = \sum_{|\alpha|=2K} c_\alpha \mathbf{x}^\alpha$$

we can define an associated linear operator as follows:

$$D_f := \sum_{|\alpha|=2K} c_\alpha D_{\mathbf{x}}^\alpha$$

Now we finally get to defining the differential metric using a positive definite bilinear form using D_f.

$$\langle f, g \rangle_D := D_f(g)$$

H. Representation theory

We will need some simple concepts from representation theory for our proof of the upper bound for non negative polynomials. The basic reference [27] is more than adequate for our purpose. In particular we shall need the following lemma. If G_1 and G_2 are two groups and V_1 and V_2 are representations of G_1 and G_2, then the tensor product, $V_1 \otimes V_2$ is a representation of $G_1 \times G_2$, by $(g_1 \times g_2).(v_1 \otimes v_2) = g_1 v_1 \otimes g_2 v_2$. To distinguish this "external" tensor product from the "internal" tensor product when $G_1 = G_2$, we denote this by, $V_1 \odot V_2$.

Lemma 12. *If V_1 and V_2 are irreducible then $V_1 \odot V_2$ is also irreducible and every irreducible representation of $G_1 \times G_2$ arises this way.*

CHAPTER III

VOLUMES OF CERTAIN CONES OF BIHOMOGENEOUS POLYNOMIALS

A. Preliminaries

In this Chapter we shall obtain extensions of Blekherman's results [28](also see [29], [30] and [31]) to the bihomogeneous case. In this section we shall quickly summarize what the general notation introduced in Chapter II boils down to in the case the number of blocks $l = 2$. Thus bihomogeneous polynomials are an extension of the usual homogeneous polynomials where there are two sets of variables with n_1 variables in the first block and n_2 variables in the second block. In the notation of Chapter II, we have, $N = (n_1, n_2)$, $K = (k_1, k_2)$ and of course $l = 2$.

$$P_{N,K} = P_{n_1,k_1} \otimes P_{n_2,k_2}$$

We denote the two blocks by $x_1 = (x_{11}, \ldots, x_{1n_1})$ and $x_2 = (x_{21}, \ldots, x_{2n_2})$. We may also use $x = (x_1, \ldots, x_{n_1})$ and $y = (y_1, \ldots, y_{n_2})$, to denote the two blocks. The sums of squares and the sums of powers of linear forms reduce to the following,

$$Sq_{N,2K} = \left\{ f \in P_{N,K} \text{ such that, } f = \sum_{i=1}^{m} f_i^2, \text{ for some } f_i \in P_{N,K} \right\}$$

$$Lf_{N,2K} = \left\{ f \in P_{N,K} \text{ such that, } f = \sum_{i=1}^{m} \prod_{j=1}^{2} f_{ij}^{2k_i}, \text{ for some } f_{ij} \in P_{n_i,1} \right\}$$

Also like before, we have the non negative polynomials,

$$Pos_{N,2K} = \{ f \in P_{N,2K} \text{ such that, } f(x_1, x_2) \geq 0, \ \forall (x_1, x_2) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \}$$

We also use the notation $P_{(n_1,n_2),(k_1,k_2)}$ and so forth to denote $P_{N,K}$. As outlined in IIA, bihomogeneous polynomials can be considered as functions on the product of spheres $S^N = S^{n_1-1} \times S^{n_2-1}$. We have the action of $SO(N) = SO(n_1) \times SO(n_2)$ on $P_{N,K}$ by means of an orthogonal change of coordinates. We can similarly write down the $SO(n_1) \times SO(n_2)$ invariant inner product introduced in Chapter II, example 1. For $f_1 \otimes f_2, g_1 \otimes g_2 \in P_{N,K}$, we have,

$$\begin{aligned} \langle f_1 \otimes f_2, g_1 \otimes g_2 \rangle &= \int_{S^{n_1-1} \times S^{n_2-1}} (f_1 \otimes f_2)(g_1 \otimes g_2) \, d\sigma \\ &= \int_{S^{n_1-1}} f_1 g_1 \, d\sigma_1 \int_{S^{n_2-1}} f_2 g_2 \, d\sigma_2 \end{aligned} \tag{3.1}$$

where, $d\sigma$ is the rotation invariant probability measure on $S^{n_1-1} \times S^{n_2-1}$ and $d\sigma_i$ is the probability measure on S^{n_i-1} for $i = 1, 2$. The hyperplane sections described in IID become,

$$L_{N,2K} := \{p \in P_{N,2K} | \int_{S^{n_1-1} \times S^{n_2-1}} p \, d\sigma = 1\}$$

$$M_{N,2K} := \{p \in P_{N,2K} | \int_{S^{n_1-1} \times S^{n_2-1}} p \, d\sigma = 0\}$$

and the polynomial F, reduces to $(x_{11}^2 + \ldots + x_{1n_1}^2)^{k_1}(x_{21}^2 + \ldots + x_{2n_2}^2)^{k_2}$. The slices are given by, $Pos'_{N,2K} = Pos_{N,2K} \cap L_{N,2K}$, $Sq'_{N,2K} = Sq_{N,2K} \cap L_{N,2K}$, $Lf'_{N,2K} = Lf_{N,2K} \cap L_{N,2K}$ and the translates are,

$$\begin{aligned} \widetilde{Pos}_{N,2K} &= \{p \in M_{N,2K} : p + F \in Pos'_{N,2K}\} \\ \widetilde{Sq}_{N,2K} &= \{p \in M_{N,2K} : p + F \in Sq'_{N,2K}\} \\ \widetilde{Lf}_{N,2K} &= \{p \in M_{N,2K} : p + F \in Lf'_{N,2K}\} \end{aligned} \tag{3.2}$$

From lemma 2 we see that the dimension D_M of $M_{N,2K}$ is,

$$D_M = \binom{n_1 + 2k_1 - 1}{k_1} \binom{n_2 + 2k_2 - 1}{k_2} - 1$$

As explained in Chapter II, S_M will be the unit sphere in $M_{N,2K}$ and B_M the unit ball in $M_{N,2K}$. We shall now compute the gauge of $\widetilde{Pos}_{N,2K}$.

Lemma 13. *The gauge $G_{\widetilde{Pos}}$ of $\widetilde{Pos}_{N,2K}$ at any polynomial $f \in M_{N,2K}$ is given by,*

$$G_{\widetilde{Pos}_{N,2K}}(f) = |\inf_{v \in S^N} f(v)|^{-1}$$

Proof. From definition 4 we have,

$$G_{\widetilde{Pos}_{N,2K}}(f) = \sup\{\lambda > 0 : \lambda f \in \widetilde{Pos}_{N,2K}\}$$

We know from the definition of $\widetilde{Pos}_{N,2K}$ that if $g \in M_{N,2K}$, then g is in $\widetilde{Pos}_{N,2K}$ if $\inf_{v \in S^N} g(v) \geq -1$. Let $m_g = \inf_{v \in S^N} g(v)$. Clearly if $g \in M_{N,2K}$, then $m_g < 0$. Now $\inf_{v \in S^N} g/|m_g| \geq -1$ and $\inf_{v \in S^N} g/|m_g + \epsilon| \leq -1$ for every $\epsilon > 0$. □

B. A lower bound for the non negative multihomogeneous polynomials

The proof of the lower bound for $\mathtt{Vol}\widetilde{Pos}_{N,2K}$ can be broken down into the following steps, analogous to the approach adopted in [28]:

1. The volume taking into account the effect of higher dimensions is defined using the integral of gauge function of $\widetilde{Pos}_{N,2K}$ over the unit sphere in $M_{N,2K}$, S_M.

2. This is then manipulated to an integral involving the sup norm of bihomogeneous polynomials over S_M.

3. Using Barvinok's theorem, (Theorem 7) we bound $\|f\|_\infty$ by the k^{th} norm of f, $\|f\|_k$, for some suitably chosen $k \in \mathbf{N}$.

4. Using Barvinok's lemma, (Lemma 7) we then bound $\|f\|_k$ to obtain our result, which we shall state as our first theorem.

Theorem 8.

$$\left(\frac{\mathtt{Vol}\widetilde{Pos}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} \geq \frac{\beta}{\sqrt{max\{n_1\ln(2k_1+1), n_2\ln(2k_2+1)\}}}$$

where $\alpha = 9e^2$ *and* $\beta = \frac{1}{9e^2}$.

Since $\widetilde{Pos}_{N,2K}$ is a convex body with origin in its interior, we can use definition 4 and Lemma 26 to represent the relative volume of $\widetilde{Pos}_{N,2K}$. One can prove this using integration in polar coordinates. However we shall not present the proof. Instead the interested reader can consult [20]. We have,

$$\left(\frac{\mathtt{Vol}\widetilde{Pos}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} = \left(\int_{S_M} G^{D_M}_{\widetilde{Pos}_{N,2K}}\ d\mu\right)^{1/D_M}$$

We notice that we can apply Holder's theorem, since the right hand side is nothing but the D_M norm of G_P. Hence,

$$\begin{aligned}
\left(\frac{\mathtt{Vol}\widetilde{Pos}_{N,2K}}{\mathtt{Vol}B_M}\right) &\geq \left(\int_{S_M} G_P(f)\ d\mu\right) && \text{(by Holder's Inequality)} \quad (3.3)\\
&\geq \left(\int_{S_M} |\inf_{v\in S^N} f(v)|^{-1} d\mu\right) && \text{(by Lemma 26)}\\
&\geq \left(\int_{S_M} |\inf_{v\in S^N} f(v)| d\mu\right)^{-1} && \text{(by Jensen's Inequality)}
\end{aligned}$$

Finally, it is easy to observe that $\|f\|_\infty \geq |\inf_{v\in S^N} f(v)|$. Hence to lower bound the volume of non negative multihomogeneous polynomials we only have to estimate the integral of the sup norm over the unit sphere.

$$\left(\frac{\mathtt{Vol}\widetilde{Pos}_{N,2K}}{\mathtt{Vol}B_M}\right) \geq \left(\int_{S_M} \|f\|_\infty d\mu\right)^{-1}$$

We proceed by bounding the $\|f\|_\infty$ norm by $\|f\|_{2k}$ using Barvinok's results (Theorem 7). To apply Barvinok's theorem, we shall view an $f \in P_{N,2K}$ as the restriction of a linear functional on a particular vector space $T_{N,2K}$ to a $SO(n_1) \times SO(n_2)$ orbit in $T_{N,2K}$. The vector space $T_{N,2K}$ is described in section D.

Lemma 14. *Given a vector space $V = V_1 \times V_2$ and a group action of $G = G_1 \times G_2$ on V, we have a natural G action on $V^{\otimes K} := V_1^{\otimes k_1} \otimes V_2^{\otimes k_2}$. As usual $K = (k_1, k_2)$.*

Proof. For $g_1 \times g_2 \in G$, and decomposable tensor $(v_1 \otimes \ldots \otimes v_{k_1}) \otimes (w_1 \otimes \ldots \otimes w_{k_2}) \in V^{\otimes K}$, we set,

$$g_1 \times g_2((v_1 \otimes \ldots \otimes v_{k_1}) \otimes (w_1 \otimes \ldots \otimes w_{k_2})) = (g_1(v_1) \otimes \ldots \otimes g_1(v_{k_1})) \otimes (g_2(w_1) \otimes \ldots \otimes g_2(w_{k_2})).$$

We extend this to other elements of $V^{\otimes K}$ by appealing to linearity. $\square$

Example 3. *In our setting we have a natural action of $SO(N) = SO(n_1) \times SO(n_2)$ on $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$. Following the procedure in Lemma 14, we extend this to $T_{N,2K}$,*

$$T_{N,2K} = (\mathbf{R}^{n_1})^{\otimes 2k_1} \otimes (\mathbf{R}^{n_2})^{\otimes 2k_2}$$

We can think of $T_{N,2K}$ as an array, indexed by multi indices, that is $x_{(1i_1,\ldots,1i_{2k_1}),(2j_1,\ldots,1j_{2k_2})}$, where $1 \leq i_1, \ldots i_{2k_1} \leq n_1$ and $1 \leq j_1, \ldots j_{2k_2} \leq n_2$. Given an $x = (x_1, x_2) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$, we can think of $x^{\otimes K}$ as given by,

$$x_{(1i_1,\ldots,1i_{2k_1}),(2j_1,\ldots,1j_{2k_2})} = (x_{1i_1} \ldots x_{1i_{2k_1}})(x_{2j_1} \ldots x_{2j_{2k_2}}) \tag{3.4}$$

We notice that $x^{\otimes K}$ lies in $Sym_{2k_1}(\mathbf{R}^{n_1}) \otimes Sym_{2k_2}(\mathbf{R}^{n_2})$. A bihomogeneous polynomial $p \in P_{N,2K}$ of the form, as described in Section C

$$p = \sum_{|\alpha|=K} c_\alpha \mathbf{x}^\alpha$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^{n_1} \times \mathbf{N}^{n_2}$, with $|\alpha_1| = k_1$ and $|\alpha_2| = k_2$. We shall write this *out in all its gory detail just for fun,*

$$p = \sum_{|\alpha|=K} c_\alpha (x_{11}^{\alpha_{11}} \dots x_{1n_1}^{\alpha_{1n_1}})(x_{21}^{\alpha_{21}} \dots x_{2n_2}^{\alpha_{2n_2}}) \tag{3.5}$$

$$= \sum_{\substack{1\le i_1,\dots,i_{k_1}\le n_1 \\ 1\le j_1,\dots,j_{k_2}\le n_2}} \tilde{c}_{(1i_1,\dots,1i_{2k_1}),(2j_1,\dots,1j_{2k_2})}(x_{1i_1}\dots x_{1i_{2k_1}})(x_{2j_1}\dots x_{2j_{2k_2}}) \tag{3.6}$$

We conclude the last equation by comparing with equation 3.4. This basically means that we have, $p(x) = \langle \tilde{c}, x^{\otimes K}\rangle$. With $\tilde{c}$ as in equation 3.6, we have the following linear functional l_p on T,

$$l_p(v) = \langle \tilde{c}, v\rangle$$

for every $v \in T_{N,2K}$. Thus we have the following equivalence $p \leftrightarrow l_p$ between bi-homogeneous polynomials and linear functionals on G orbits on T. Let us take $x = (e_1, \tilde{e}_1) \in S^{n_1-1} \times S^{n_2-1}$. Then we have, for all $g \in G$,

$$p(gx) = \langle \tilde{c}, gx^{\otimes K}\rangle = l_p(gv)$$

This means that we have a group action of $G = SO(n_1) \times SO(n_2)$ on $T_{N,2K}$ and a linear functional on T. Define, $f : G \to \mathbf{R}$ as $f(g) = l_f(gx^{\otimes K})$. For $f \in S_M$, we have,

$$\|f\|_\infty = \sup_{g\in SO(n_1)\times SO(n_2-1)} f(gx) \tag{3.7}$$

$$= \sup_{g\in SO(n_1)\times SO(n_2-1)} l_f(gx^{\otimes K}) \tag{3.8}$$

Now we can apply Barvinok's theorem to bound, $\|f\|_\infty$ by $\|f\|_k$. For $k > 0$, let d_k be the dimension of the subspce spanned by the orbit, $\{g\left(x^{\otimes K}\right)^{\otimes k}\}$. Then from Theorem 7, we have,

$$\|f\|_{2k} \le \|f\|_\infty \le (d_k)^{1/2k}\,\|f\|_{2k}$$

It is easy to see that in our case,

$$d_k = \binom{n_1 + 2k_1k - 1}{2k_1k}\binom{n_2 + 2k_2k - 1}{2k_2k} \tag{3.9}$$

This leads us to the following upper bound for $\|f\|_\infty$.

$$\|f\|_\infty \leq \binom{n_1 + 2k_1k - 1}{2k_1k}^{1/2k_1}\binom{n_2 + 2k_2k - 1}{2k_2k}^{1/2k_2}\|f\|_k \tag{3.10}$$

We shall now use the following lemma to upper bound the combinatorial factor appearing in the above equation.

Lemma 15. *When $k \geq nln(m+1)$,*

$$\left(\frac{n + mk - 1}{mk}\right)^{1/2k} \leq \alpha$$

for some absolute constant α.

Proof. From [32] we have the following estimate for $\binom{a}{b}$,

$$\binom{a}{b} \leq \exp aH(\frac{b}{a})$$

where, $H(x) = x\ln(\frac{1}{x}) + (1-x)\ln(\frac{1}{1-x})$, where $0 \leq x \leq 1$. We note that $H(\delta) = H(1-\delta)$, when $0 \leq \delta \leq 0.5$. Also, $H(x)$ is decreasing in the interval $[0.5, 1]$. Now,

$$\frac{mk}{n + mk - 1} \geq 1 - \frac{1}{mk}$$

Let $\delta = \frac{n}{mk}$. Then with $b = mk$ and $a = n + mk - 1$, using the above inequality and the properties of $H(x)$, we see that,

$$H(\frac{b}{a}) \leq H(\delta)$$

Now expanding the entropy function formula and noting that when $0 \leq x \leq 0.5$,

$x^{1/x} \geq (1-x)^{(1/(1-x))}$, we get the following bound for $H(\delta)$:

$$H(\delta) = \frac{n}{mk}\ln(\frac{mk}{n}) + \left(1 - \frac{n}{mk}\right)\ln(\frac{1}{1-\frac{n}{mk}})$$
$$\leq \quad \frac{2n}{mk}\ln(\frac{mk}{n})$$

Therefore we get,

$$\binom{a}{b} \leq \exp\left\{mk\frac{2n}{mk}\ln(\frac{mk}{n})\right\}$$
$$\leq \quad \left(\frac{mk}{n}\right)^{2n}$$

Finally using $k \geq n\ln(m+1)$,

$$\binom{b}{a}^{1/2k} \leq \left(\frac{mn\ln(m+1)}{n}\right)^{\frac{n}{n\ln(m+1)}} \quad \leq \quad 3e$$

$\square$

Corollary 1. *When* $k \geq \max\{n_1\ln(m_1+1), n_2\ln(m_2+1)\}$,

$$\left(\frac{n+mk-1}{mk}\right)^{1/2k}\left(\frac{n+mk-1}{mk}\right)^{1/2k} \leq 9e^2$$

As promised above, using the Corollary 1 in Equation 4.3, and taking $k = max\{n_1\ln(2k_1+1), n_2\ln(2k_2+1)\}$, we obtain,

$$\|f\|_\infty \leq 3e^2\|f\|_k$$

Hence we now have,

$$\left(\int_{S_M}\|f\|_\infty d\mu\right) \leq \alpha\left(\int_{S_M}\|f\|_k d\mu\right)$$

To obtain a bound for $\|f\|_K$, we use Lemma 7 due to Barvinok [16].

Letting $c = l_f$, this lemma helps us to bound the integral of $\|f\|_k$ as follows:

$$\left(\int_{S_M} \|f\|_k d\mu\right) \leq \sqrt{\frac{k\langle v, v\rangle}{D_M}}$$

But since $\langle v, v\rangle = D_M$, we have

$$\left(\int_{S_M} \|f\|_\infty d\mu\right) \leq \alpha\sqrt{k}$$

Thus we finally prove that,

$$\left(\frac{\texttt{Vol}\widetilde{Pos}_{N,2K}}{\texttt{Vol}B_M}\right)^{1/D_M} \geq \frac{\beta}{\sqrt{max\{n_1 \ln(2k_1+1), n_2 \ln(2k_2+1)\}}}$$

where $\alpha = 9e^2$ and $\beta = \frac{1}{9e^2}$.

C. An upper bound on the volume of non negative multihomogeneous polynomials

Theorem 9.

$$\left(\frac{\widetilde{Pos}_{N,2K}}{\texttt{Vol}B_M}\right) \leq 4\left(\frac{2k_1^2}{4k_1^2+n_1-2}\right)^{1/2}\left(\frac{2k_2^2}{4k_2^2+n_2-2}\right)^{1/2}$$

The proof of the upper bound for $\widetilde{Pos}_{N,2K}$ can be broken down into the following steps,

1. Relate the volume of $\widetilde{Pos}_{N,2K}$ to that of its polar, $\widetilde{Pos}^\circ_{N,2K}$, using the Blaschke-Santalo inequality.

2. Obtain a relation between the polar of the unit ball in the sup norm, B°_∞ and $\widetilde{Pos}^\circ_{N,2K}$.

3. Introduce gradient metric to upper bound B°_∞ by the unit ball in the gradient metric, B_G.

4. Finally bound the ratio of B_G to B_M using arguments from representation theory.

We begin by defining the polar $\widetilde{Pos}^{\circ}_{N,2K}$, of $\widetilde{Pos}_{N,2K}$, using definition 2.

$$\widetilde{Pos}^{\circ}_{N,2K} = \{f \in M_{N,2K} : \langle f, g\rangle \leq 1 \ , \forall g \in \widetilde{Pos}_{N,2K}\}$$

Since $\tilde{Pos}_{N,2K}$ is fixed by $SO(n_1) \times SO(n_2)$ and origin is the only point in $M_{N,2K}$ fixed by $SO(n_1) \times SO(n_2)$. From Chapter II we have that the Santalo point of a convex body is unique. Hence the origin is the Santalo point of $\tilde{Pos}_{N,2K}$. Using Blaschke-Santalo inequality, we get the following:

$$\left(\mathtt{Vol}\widetilde{Pos}_{N,2K}\right)\left(\mathtt{Vol}\widetilde{Pos}^{\circ}_{N,2K}\right) \leq \left(\mathtt{Vol}B_M\right)^2$$

Therefore it would suffice to show that,

$$\left(\frac{\mathtt{Vol}\widetilde{Pos}^{\circ}_{N,2K}}{\mathtt{Vol}B_M}\right) \geq \frac{1}{4}\left(\frac{2k_1^2}{4k_1^2+n_1-2}\right)^{1/2}\left(\frac{2k_2^2}{4k_2^2+n_2-2}\right)^{1/2}$$

We define the unit ball in the sup-norm as follows,

$$B_\infty = \{f \in M_{N,2K} \mid \|f\|_\infty \leq 1\}$$

Now B_∞ is the intersection of $\widetilde{Pos}_{N,2K}$ with $-\widetilde{Pos}^{\circ}_{N,2K}$:

$$B_\infty = \widetilde{Pos}_{N,2K} \cap -\widetilde{Pos}^{\circ}_{N,2K}$$

From Lemma 9, we see that,

$$B_\infty^\circ = \mathtt{Conv}\{\widetilde{Pos}^{\circ}_{N,2K} \ , \ -\widetilde{Pos}^{\circ}_{N,2K}\} \subset \widetilde{Pos}^{\circ}_{N,2K} \oplus -\widetilde{Pos}^{\circ}_{N,2K}$$

We now apply Rogers and Shephard theorem [18] in convex geometry to get a bound on the polar of the sup norm unit ball.

$$\mathtt{Vol} B_\infty^\circ \leq \binom{2D_M}{D_M} \mathtt{Vol} \widetilde{Pos}_{N,2K}^\circ$$

Lemma 16. *For $n > 0$, we have,*

$$\binom{2n}{n} \leq 4^n$$

Proof. The left hand side is the coefficient of the x^n term in the expansion of $(1+x)^{2n}$. Taking $x = 1$, we clearly have, $\binom{2n}{n} \leq (1+1)^{2n} = 4^n$. □

From Lemma 16 it follows that,

$$\left(\frac{\mathtt{Vol} B_\infty^\circ}{\mathtt{Vol} \widetilde{Pos}_{N,2K}^\circ} \right)^{1/D_M} \geq \frac{1}{4}$$

This reduces the proof of the upper bound to,

$$\left(\frac{\mathtt{Vol} B_\infty^\circ}{\mathtt{Vol} B_M} \right)^{1/D_M} \geq \left(\frac{4k_1^2 + n_1 - 2}{2k_1^2} \right)^{1/2} \left(\frac{4k_2^2 + n_2 - 2}{2k_2^2} \right)^{1/2}$$

We now bound the infinity ball using the gradient metric introduced in Lemma 12. For $f \in M_{N,2K}$, which is decomposable, say $f = f_1 \otimes f_2$, we have,

$$\nabla f = \left(\frac{\partial f_1}{\partial x_1}, \dots, \frac{\partial f_1}{\partial x_{n_1}} \right) \otimes \left(\frac{\partial f_2}{\partial y_1}, \dots, \frac{\partial f_2}{\partial y_{n_2}} \right)$$

From Lemma 12, the gradient metric of f as above would be,

$$\langle f, f \rangle_G = \frac{1}{16 k_1^2 k_2^2} \int_{S^{n_1-1} \times S^{n_2-1}} \left(\left(\frac{\partial f_1}{\partial x_1} \right)^2 + \dots + \left(\frac{\partial f_1}{\partial x_{n_1}} \right)^2 \right) \left(\left(\frac{\partial f_2}{\partial y_1} \right)^2 + \dots + \left(\frac{\partial f_2}{\partial y_{n_2}} \right)^2 \right) d\sigma$$

Let B_G be the unit ball in the gradient metric and the corresponding norm $\|f\|_G$.

From Kellog's lemma, (Lemma 11),

$$B_\infty \subseteq B_G$$

Polarity reverses inclusion and so,

$$B_G^\circ \subseteq B_\infty^\circ \tag{3.11}$$

$$\mathtt{Vol} B_G^\circ = \frac{(\mathtt{Vol} B_M)^2}{\mathtt{Vol} B_G} \qquad \text{(Using the Blaschke-Santalo Inequality)} \tag{3.12}$$

Consequently, we have, $\mathtt{Vol} B_\infty^\circ \geq \frac{(\mathtt{Vol} B_M)^2}{\mathtt{Vol} B_G}$ and hence,

$$\frac{\mathtt{Vol} B_\infty^\circ}{\mathtt{Vol} B_M} \geq \frac{\mathtt{Vol} B_M}{\mathtt{Vol} B_G}$$

Thus, we are left with proving the following:

Lemma 17.

$$\left(\frac{\mathtt{Vol} B_M}{\mathtt{Vol} B_G}\right)^{1/D_M} \geq \left(\frac{4k_1^2 + n_1 - 2}{2k_1^2}\right)^{1/2} \left(\frac{4k_2^2 + n_2 - 2}{2k_2^2}\right)^{1/2}$$

Proof. It is enough that we show the following is true for all $f \in M_{N,2K}$.

$$\langle f, f \rangle \geq \left(\frac{4k_1^2 + n_1 - 2}{2k_1^2}\right)^{1/2} \left(\frac{4k_2^2 + n_2 - 2}{2k_2^2}\right)^{1/2} \langle f, f \rangle_G$$

By the invariance of both inner products under the action of $SO(n_1) \times SO(n_2)$, it is enough to prove the lemma in the irreducible components of the representation. We know that(Lemma 12), the irreducible components are $H_{n_1,2l_1} \otimes H_{n_2,2l_2}$ for $0 \leq l_1 \leq k_1$ and $0 \leq l_2 \leq k_2$. And,

$$H_{n,2l} = \{f \in P_{n,2k} \mid f = (x_1^2 + \ldots + x_n^2)^{k-l} h, h \in P_{n,2l}\}$$

If f is a harmonic form of degree $2d$ in n variables, Stokes' formula gives us,

$$\langle f, f \rangle_G = \frac{2d}{4d + n - 2} \langle f, f \rangle_G$$

Also, when $f = (x_1^2 + \ldots + x_n^2)^{k-d}h$, where h is a harmonic form of degree $2d \leq 2k$, it is easy to check that,

$$\langle f, f\rangle_G = \frac{d^2}{k^2}\langle h, h\rangle_G + \frac{k^2 - d^2}{k^2}\langle h, h\rangle$$

We now obtain the following similar results when $f_1 = (x_1^2 + \ldots + x_{n1}^2)^{k_1-d_1}h_1$ and $f_2(y_1^2 + \ldots + y_{n1}^2)^{k_2-d_2}h_2$. We notice that,

$$\begin{aligned}
\langle f_1f_2, f_1f_2\rangle_G &= \langle f_1, f_1\rangle_G\langle f_2, f_2\rangle_G \\
&= \left(\frac{d_1^2}{k_1^2}\langle h_1, h_1\rangle_G + \frac{k_1^2 - d_1^2}{k_1^2}\langle h_1, h_1\rangle\right) \\
&\quad \left(\frac{d_2^2}{k_2^2}\langle h_2, h_2\rangle_G + \frac{k_2^2 - d_2^2}{k_2^2}\langle h_2, h_2\rangle\right) \\
&= \left(\frac{2d_1^2 + d_1(n_1 - 2) + 2k_1^2}{2k_1^2}\right)\left(\frac{2d_2^2 + d_2(n_2 - 2) + 2k_2^2}{2k_2^2}\right)\langle f_1f_2, f_1f_2\rangle \\
&\leq \left(\frac{4k_1^2 + n_1 - 2}{2k_1^2}\right)\left(\frac{4k_1^2 + n_1 - 2}{2k_1^2}\right)\langle f_1f_2, f_1f_2\rangle
\end{aligned}$$

The last step follows since the minimum clearly occurs when $d_1 = d_2 = 1$. This proves the lemma. □

D. Upper bound for bihomogeneous SOS polynomials

Throughout this Chapter we shall assume $N = (n_1, n_2)$ and $K = (k_1, k_2)$, and hence $2K = (2k_1, 2k_2)$. We can outline the steps involved in computing the lower bound of multihomogeneous polynomials as follows:

1. Bound the volume of SOS polynomials by the average width using results from convexity theory [18].

2. Express the average width in terms of an integral involving the support function of SOS polynomials.

3. Bound the support function by a max norm.

4. Use Barvinok's method to bound the integral of the max norm by a high L^{2p} norm.

We have the following bound for the volume of multihomogeneous SOS polynomials from the Uryshon's Inequality [18].

$$\left(\frac{\mathtt{Vol}\widetilde{Sq}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} \leq \frac{W_{\widetilde{Sq}}}{2}$$

Here $W_{\widetilde{Sq}}$ is the average width of $\widetilde{Sq}$ and is given by,

$$W_{\widetilde{Sq}} = 2\int_{S_M} L_{\widetilde{Sq}} d\mu$$

where $L_{\widetilde{Sq}}$ is the support function of $\widetilde{Sq}_{N,2K}$ which can be computed by the following formula:

$$L_{\widetilde{Sq}}(f) = \max_{g\in\widetilde{Sq}} \langle f, g\rangle$$

Thus we can obtain a lower bound for the volume of multihomogeneous SOS polynomials by bounding their average width $W_{\widetilde{Sq}}$. The extreme points in $Sq_{N,2K}$ are clearly perfect squares. Hence the since $\widetilde{Sq}$ is a translation of Sq by $(x_1^2+\ldots+x_{n_1}^2)^{k_1}(y_1^2+\ldots+y_{n_2}^2)^{k_2}$, the extreme points in $\widetilde{Sq}$ are given as below.

$$g^2 - (x_1^2+\ldots+x_{n_1}^2)^{k_1}(y_1^2+\ldots+y_{n_2}^2)^{k_2}$$
$$\text{where } g \in P_{N,K} \text{ and } \int_{S^{n_1-1}\times S^{n_2-1}} g^2 d\sigma = 1$$

For $f \in M_{N,2K}$,

$$\langle f, (x_1^2+\ldots+x_{n_1}^2)^{k_1}(y_1^2+\ldots+y_{n_2}^2)^{k_2}\rangle = \int_{S^{n_1-1}\times S^{n_2-1}} f d\sigma = 0$$

Hence the expression for the support function simplifies to,

$$L_{\widetilde{Sq}}(f) = \max_{g \in S_{P_{N,K}}} \langle f, g^2 \rangle$$

Now we introduce a quadratic form on $P_{N,K}$ whose norm bounds $L_{\widetilde{Sq}}$.

$$H_f(g) = \langle f, g^2 \rangle \text{for } g \in P_{N,K}$$

Now,

$$L_{\widetilde{Sq}}(f) \leq \|H_f\|_\infty$$

We can now use Barvinok's theorem to bound $\|H_f\|_\infty$ by a high L^{2p} norm of H_f. Since H_f is a form of degree 2 on the vector space $P_{N,K}$ of dimension $D_{N,K}$ we get,

$$\|H_f\|_\infty \leq 2\sqrt{3}\|H_f\|_{2D_{N,K}}$$

We now proceed as in the case of non negative multihomogeneous polynomials, using Hõlder's inequality to estimate the integral of $\|H_f\|_\infty$.

$$\int_{S_M} L_{\widetilde{Sq}} d\mu \leq \left(\int_{S_M} \int_{S_{P_{N,K}}} \langle f, g^2 \rangle^{2D_{N,K}} d\sigma(g) d\mu(f) \right)^{1/2D_{N,K}}$$

Since the inner integral depends only on the projection of g^2 into $M_{N,2K}$, we have,

$$\int_{S_M} \langle f, g^2 \rangle^{2D_{N,K}} d\mu(f) \leq \|g^2\|_2^{2D_{N,K}} \int_{S_M} \langle f, p \rangle^{2D_{N,K}} d\mu(f)$$

for any $p \in S_M$.

We can compute the second integral easily due to the invariance of the inner product under $SO(n_1) \times SO(n_2)$. See for example [33] and [16].

$$\int_{S_M} \langle f, p \rangle^{2D_{N,K}} d\mu(f) = \frac{\Gamma(D_{N,K} + \frac{1}{2})\Gamma(\frac{1}{2}D_M)}{\sqrt{\Pi}\Gamma(D_{N,K} + \frac{1}{2}D_M)}$$

Furthermore, Duoandikoetxea [34](also see [35] has shown that for $g \in S_{P_{n_1,k_1}}$, $\|g^2\|_2 \leq 4^{2k_1}$. This implies that for $g \in P_{N,K}$, $\|g^2\|_2 \leq 4^{2k_1}4^{2k_2}$

Combining these two results, we have,

$$\int_{S_M} L_{\widetilde{Sq}}(f)d\mu \leq 4^{2k_1}4^{2k_2}\left(\frac{\Gamma(D_{N,K}+\frac{1}{2})\Gamma(\frac{1}{2}D_M)}{\sqrt{\Pi}\Gamma(D_{N,K}+\frac{1}{2}D_M)}\right)^{\frac{1}{D_{N,K}}}$$

Abramowitz and Stegun [36] list the following inequality for the Gamma function,

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \leq \frac{1}{n^{b-a}} \text{ , for } b-a \geq 0, a \geq 0, n \in \mathbf{N}$$

Using this we obtain,

$$\left(\frac{\Gamma(\frac{1}{2}D_M)}{\Gamma(D_{N,K}+\frac{1}{2}D_M)}\right)^{\frac{1}{2D_{N,K}}} \leq \sqrt{\frac{2}{D_M}}$$

$$\left(\frac{\Gamma(\frac{1}{2}+D_{N,K})}{\sqrt{\pi}}\right)^{\frac{1}{2D_{N,K}}} \leq \sqrt{D_{N,K}}$$

This implies,

$$\int_{S_M} L_{\widetilde{Sq}}(f)d\mu \leq 4^{2k_1}4^{2k_2}2\sqrt{3}\sqrt{\frac{2D_{N,K}}{D_M}}$$

Thus we get the following upper bound for the volume on SOS multihomogeneous polynomials.

Theorem 10. *Upper bound on the volume of SOS multihomogeneous polynomials*

$$\left(\frac{\mathtt{Vol}\widetilde{Sq}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} \leq 4^{2k_1}4^{2k_2}\sqrt{24}\sqrt{\frac{D_{N,K}}{D_M}} \tag{3.13}$$

$$\textit{where,} D_{N,K} = \binom{n_1+k_1-1}{k_1}\binom{n_2+k_2-1}{k_2} \tag{3.14}$$

$$\textit{and } D_M = \binom{n_1+2k_1-1}{2k_1}\binom{n_1+2k_1-1}{2k_1} - 1 \tag{3.15}$$

We can further simplify the above expression to get,

$$\left(\frac{\mathtt{Vol}\widetilde{Sq}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} \leq 4^{2k_1}4^{2k_2}\sqrt{24}\frac{(2k_1)!(2k_2)!}{k_1!k_2!}n_1^{-k_1/2}n_2^{-k_2/2}$$

E. The differential metric on the space of bihomogeneous polynomials

We would need to switch to a new metric during the course of the proof of the lower bound for multihomogeneous polynomials. We extend the differential metric of Blekherman's paper to the multihomogeneous case. First some more notation. Let $\mathbf{x} = (x_1, \ldots, x_{n_1})$ and $\mathbf{y} = (y_1, \ldots, y_{n_2})$ denote the two sets of variables. We shall use the following compact notation to name monomials and their associated differential operators. Let $\alpha = (i_1, \ldots, i_{n_1})$ and $\beta = (j_1, \ldots, j_{n_2})$. Then,

$$\mathbf{x}^\alpha = x_1^{i_1} \ldots x_{n_1}^{i_{n_1}} \text{ and } \mathbf{y}^\beta = y_1^{j_1} \ldots y_{n_2}^{j_{n_2}}$$

and,

$$D_{\mathbf{x}^\alpha} := \frac{\partial^{i_1} \ldots \partial^{i_{n_1}}}{\partial x_1^{i_1} \ldots \partial x_{n_1}^{i_{n_1}}} \text{ and } D_{\mathbf{y}^\beta} := \frac{\partial^{j_1} \ldots \partial^{j_{n_2}}}{\partial y_1^{j_1} \ldots \partial y_{n_2}^{j_{n_2}}}$$

Now for a form $f \in P_{N,2K}$,

$$f = \sum_{\substack{\alpha=(i_1,\ldots,i_{n_1}),|\alpha|=2k_1 \\ \beta=(j_1,\ldots,j_{n_2}),|\beta|=2k_2}} c_{\alpha\beta}\mathbf{x}^\alpha\mathbf{y}^\beta$$

we can define an associated linear operator as follows:

$$D_f := \sum_{\substack{\alpha=(i_1,\ldots,i_{n_1}),|\alpha|=2k_1 \\ \beta=(j_1,\ldots,j_{n_2}),|\beta|=2k_2}} c_{\alpha\beta}D_{\mathbf{x}^\alpha}D_{\mathbf{y}^\beta}$$

Now we finally get to defining the differential metric using a positive definite bilinear form using D_f.

$$\langle f, g\rangle_D := D_f(g)$$

The reason for defining the differential metric is that a certain linear operator $T : P_{N,2K} \rightarrow P_{N,2K}$ maps the dual cone in the usual metric into the dual cone in the differential metric. It will be shown later that the dual cone of the semidefinite forms under the differential metric is contained in the cone of semi definite forms. This can be used to transfer the upper bound result into a lower bound result.

For $v \in S^{n_1-1} \times S^{n_2-1}$ such that $v = (v_1, v_2)$ we will use v^{2K} where $K = (k_1, k_2)$ to denote the following form in $P_{N,2K}$,

$$v^{2K} = (v_{11}x_1 + \ldots + v_{1n_1}x_{n_1})^{2k_1}(v_{21}y_1 + \ldots + v_{2n_2}y_{n_2})^{2k_2}$$

Now we can define the linear operator T mentioned in the previous paragraph. For $f \in P_{N,2K}$, we have,

$$T(f) = \int_{S^{n_1-1}\times S^{n_2-1}} f(v)v^{2K}d\sigma_1 d\sigma_2$$

The reason T maps the dual cone in the usual metric into the dual cone in the differential metric is due to the following lemma:

Lemma 18. *There is the following relationship between the differential and the L^2 metric.*

$$\langle Tf, g\rangle_D = (2k_1!)(2k_2!)\langle f, g\rangle$$

Proof. For $f = f_1 \otimes f_2, g = g_1 \otimes g_2 \in P_{N,2K}$,

$$\begin{aligned}\langle Tf, g\rangle_D &= \langle \int_{S^{n_1-1}} f_1(v_1)v_1^{2k_1}d\sigma_1 \int_{S^{n_2-1}} f_2(v_2)v_2^{2k_2}d\sigma_2, g_1 \otimes g_2\rangle_D \\ &= \int_{S^{n_1-1}} \langle f(v_1)v_1^{2k_1}, g_1\rangle_D d\sigma_1 \int_{S^{n_2-1}} \langle f(v_2)v_2^{2k_2}, g_2\rangle_D d\sigma_2\end{aligned}$$

Now,

$$\langle v_i^{2k_i}, g_i\rangle_D = (2k_i)!g_i(v_i) \text{ , for } i \in 1, 2$$

and so,

$$\begin{aligned}\langle Tf, g\rangle_D &= (2k_1)!(2k_2!)\int_{S^{n_1-1}} f(v_1)g(v_1)d\sigma_1 \int_{S^{n_2-1}} f(v_2)g(v_2)d\sigma_2 \\ &= (2k_1!)(2k_2!)\langle f, g\rangle\end{aligned}$$

$\square$

We shall now describe the important property of the operator T. Let L be a full dimensional cone. Let $(x_1^2+\ldots+x_{n_1}^2)^{2k_1}(y_1^2+\ldots+y_{n_2}^2)^{2k_2}$ be in the interior of L. We recall $M_{N,2K}$ is the set of all forms in $P_{N,2K}$ whose integral is zero. We translate L by $(x_1^2+\ldots+x_{n_1}^2)^{2k_1}(y_1^2+\ldots+y_{n_2}^2)^{2k_2}$ as follows,

$$\widetilde{L} = \{f \in M_{N,2K} | f + (x_1^2+\ldots+x_{n_1}^2)^{2k_1}(y_1^2+\ldots+y_{n_2}^2)^{2k_2} \in L\}$$

Let L_i^* and L_d^* be the duals of L in the L^2 and the differential metric respectively.

$$L_i^* = \{f \in P_{N,2K} | \langle f, g\rangle \geq 0, \forall g \in L\}$$

$$L_d^* = \{f \in P_{N,2K} | \langle f, g\rangle_D \geq 0, \forall g \in L\}$$

Since $(x_1^2+\ldots+x_{n_1}^2)^{2k_1}(y_1^2+\ldots+y_{n_2}^2)^{2k_2}$ lies in the interior of both L_i^* and L_d^* we can define $\widetilde{L_i^*}$ and $\widetilde{L_d^*}$ exactly analogous to $\widetilde{L}$. From Lemma 18 it is clear that T maps L_i^* to L_d^*,

$$T(L_i^*) = L_d^*$$

Since, T commutes with the action of $SO(n_1)\times SO(n_2)$, T acts by contraction in each irreducible subspace of $P_{N,2K}$. From [28], we have,

$$T((x_1^2+\ldots+x_{n_1}^2)^{2k_1}(y_1^2+\ldots+y_{n_2}^2)^{2k_2}) = c(x_1^2+\ldots+x_{n_1}^2)^{2k_1}(y_1^2+\ldots+y_{n_2}^2)^{2k_2}$$

And c is computed as,

$$c = \int_{S^{n_1-1}} x_1^{2k_1} d\sigma_1 \int_{S^{n_2-1}} y_1^{2k_2} d\sigma_2 = \frac{\Gamma(\frac{2k_1+1}{2})\Gamma(\frac{n_1}{2})}{\sqrt{\pi}\Gamma(\frac{n_1+2k_1}{2})} \frac{\Gamma(\frac{2k_2+1}{2})\Gamma(\frac{n_2}{2})}{\sqrt{\pi}\Gamma(\frac{n_2+2k_2}{2})}$$

Therefore $(1/c)T$ is a contraction operator on each irreducible subspace of $P_{N,2K}$ and the change in volume $\left(\frac{\mathtt{Vol}\widetilde{L_d^*}}{\mathtt{Vol}\widetilde{L_i^*}}\right)^{1/D_M}$, is bounded by the largest contraction coefficient:

$$\left(\frac{\mathtt{Vol}\widetilde{L_d^*}}{\mathtt{Vol}\widetilde{L_i^*}}\right)^{1/D_M} \geq \frac{k_1!\Gamma(k_1+n_1/2)}{\Gamma(2k_1+n_1/2)} \frac{k_2!\Gamma(k_2+n_2/2}{\Gamma(2k_2+n_2/2)}$$

Like before we can bound the ration of gamma functions,

$$\frac{\Gamma(k+n/2)}{\Gamma(2k+n/2)} \geq \frac{k!}{(n/2+k)^k}$$

Combining these we have the following useful lemma,

Lemma 19.
$$\left(\frac{\mathtt{Vol}\widetilde{L_d^*}}{\mathtt{Vol}\widetilde{L_i^*}}\right)^{1/D_M} \geq \frac{k_1!}{(n_1/2+k_1)^{k_1}} \frac{k_2!}{(n_2/2+k_2)^{k_2}}$$

We now take the next big step towards obtaining the lower bound via the following lemma.

Lemma 20. *The dual cone to the cone of multihomogeneous SOS polynomials in the differential metric, namely,* Sq_D^* *is contained in the cone of multihomogeneous SOS polynomials,* $Sq_{N,2K}$.

Proof. We recall that,

$$Sq_D^* = \{f \in P_{N,2K} | \langle f, g\rangle_D \geq 0, \forall g \in Sq_{N,2K}\}$$

Now, for $f \in Sq_D^*$, we can associate the following quadratic form H_f in $P_{N,K}$, for any $p \in P_{N,K}$,

$$H_f(p) = \langle p^2, f\rangle_D$$

We shall now proceed to show that $H_f = H_{\sum q^2}$, thereby proving that $f \in Sq_{N,2K}$. To each quadratic form A in $P_{N,K}$, there is a corresponding symmetric matrix M_A. Let W denote the vector space of all quadratic forms on $P_{N,K}$. Then H_f can be written as the sum of rank one forms A_q of the form,

$$A_q(p) = \langle p, q\rangle_D^2$$

and,

$$H_f = \sum A_q, \text{ for some, } q \in P_{N,K}$$

Let V be the subspace of W given by the linear span of H_f where $f \in P_{N,2K}$. $\mathbf{P}$ is the orthogonal projection of W onto V, then,

$$\mathbf{P}(A_q) = \binom{2k_1}{k_1}^{-1}\binom{2k_2}{k_2}^{-1} H_{q^2}$$

It suffices to show $A_q - \binom{2k_1}{k_1}^{-1}\binom{2k_2}{k_2}^{-1} H_{q^2}$ is orthogonal to $H_{v^{2K}}$ since forms of this type span V.

$$H_{v^{2K}}(p) = (2k_1!)(2k_2!)p(v)^{2K} = \binom{2k_1}{k_1}\binom{2k_2}{k_2} A_{v^{2K}}(p)$$

Hence,

$$\begin{aligned}\langle A_q - \binom{2k_1}{k_1}^{-1}\binom{2k_2}{k_2}^{-1} H_{q^2}, H_{v^{2K}}\rangle &= H_{v^{2K}}(q) - \langle H_{q^2}, A_{v^{2K}}\rangle \\ &= H_{v^{2K}}(q) - H_{q^2}(v^K) = 0\end{aligned}$$

Applying $\mathbf{P}$ to both sides,

$$H_f = \mathbf{P}\Big(\sum A_q\Big) = \sum \binom{2k_1}{k_1}^{-1}\binom{2k_2}{k_2}^{-1} H_{q^2} = \binom{2k_1}{k_1}^{-1}\binom{2k_2}{k_2}^{-1} H_{\sum q^2}$$

$\square$

F. Lower bound for SOS bihomogeneous polynomials

We prove the lower bound by first noting that as a result of Lemma 20 we have,

$$\widetilde{Sq}_d^* \subseteq \widetilde{Sq}_{N,2K}$$

This gives us,

$$\frac{\mathtt{Vol}\widetilde{Sq}_{N,2K}}{\mathtt{Vol}B_M} \geq \frac{\mathtt{Vol}\widetilde{Sq}_d^*}{\mathtt{Vol}B_M}$$

Hence we are done if we find an upper bound for the right-hand side of that above equation. To do that, we shall obtain inequalities involving the relative sizes of $\widetilde{Sq}_d^*$ and $\widetilde{Sq}_i^*$ and also that of $\widetilde{Sq}_i^*$ and B_M. We begin by observing that, $(x_1^2 + \ldots + x_{n_1}^2)^{2k_1}(y_1^2 + \ldots + y_{n_2}^2)^{2k_2}$ lies in the interior of $\widetilde{Sq}_d^*$, by Lemma 6. Therefore applying Lemma 20,

$$\left(\frac{\mathtt{Vol}\widetilde{Sq}_d^*}{\mathtt{Vol}\widetilde{Sq}_i^*}\right)^{1/D_M} \geq \frac{k_1!}{(n_1/2+k_1)^{k_1}}\frac{k_2!}{(n_2/2+k_2)^{k_2}}$$

For the part involving $\widetilde{Sq}_i^*$, we start by defining the unit ball in the sq norm, namely B_{sq}.

$$B_{sq} = \{f \in M_{N,2K} : \|f\|_{sq} \leq 1\}$$

Let $G_{B_{sq}}$ be the gauge of B_{sq}. From Lemma 5, we obtain,

$$\frac{\mathtt{Vol}B_{sq}}{\mathtt{Vol}B_M} = \int_{S_M} G_{B_{sq}}\, d\mu$$

It is not very difficult to determine the gauge of B_{sq}. Indeed we have the following lemma.

Lemma 21. *For $f \in S_M$, the gauge of B_{sq} is given by,*

$$G_{B_{sq}}(f) = \|f\|_{sq}^{-1}$$

Proof. $g \in \partial B_{sq}$ implies that $\|g\|_{sq} =$. Thus for $f \in S_M$, $\lambda f \in B_{sq}$ means $\lambda\|f\|_{sq} = 1$.

This inturn leads us to our lemma. □

We shall go through the same process as in Section IIIC to bound the volume of B_{sq}.

$$\frac{\mathtt{Vol}B_{sq}}{\mathtt{Vol}B_M} = \int_{S_M} G_{B_{sq}}\, d\mu \tag{3.16}$$

$$= \int_{S_M} \|f\|_{sq}^{-1}\, d\mu \quad \text{(by Lemma 31)} \tag{3.17}$$

$$= \left(\int_{S_M} \|f\|_{sq}\, d\mu\right)^{-1} \quad \text{(by Jensen's inequality)} \tag{3.18}$$

$$\geq \frac{1}{4^{2k_1}4^{2k_2}\sqrt{24}} \frac{k_1!k_2!}{(2k_1)!(2k_2)!} n_1^{k_1/2} n_2^{k_2/2} \tag{3.19}$$

The last equality follows from Theorem 13, where we essentially had,

$$\left(\int_{S_M} \|f\|_{sq}\, d\mu\right) \leq 4^{2k_1}4^{2k_2}\sqrt{24}\frac{(2k_1)!(2k_2)!}{k_1!k_2!} n_1^{-k_1/2} n_2^{-k_2/2}$$

Lemma 22.

$$B_{sq} = \widetilde{Sq}^{\circ}_{N,2K} \cap -\widetilde{Sq}^{\circ}_{N,2K}$$

Lemma 23.

$$\widetilde{Sq}^{\circ}_{N,2K} = -\widetilde{Sq}^{*}_{i}$$

where, $\widetilde{Sq}^{*}_{i}$ is the dual cone of $Sq_{N,2K}$ in the differential metric.

From the above two lemmas we get,

$$\left(\frac{\widetilde{Sq}^{*}_{i}}{B_M}\right)^{1/D_M} \geq \frac{1}{4^{2k_1}4^{2k_2}\sqrt{24}} \frac{k_1!k_2!}{(2k_1)!(2k_2)!} n_1^{k_1/2} n_2^{k_2/2}$$

Combining this with our bound for $\left(\frac{\mathtt{Vol}\widetilde{Sq}^{*}_{d}}{\mathtt{Vol}\widetilde{Sq}^{*}_{i}}\right)^{1/D_M}$, we finally get,

$$\frac{\mathtt{Vol}\widetilde{Sq}^{*}_{d}}{\mathtt{Vol}B_M} \geq \frac{1}{4^{2k_1}4^{2k_2}\sqrt{24}} \frac{k_1!k_2!}{(2k_1)!(2k_2)!} n_1^{k_1/2} n_2^{k_2/2} \frac{k_1!}{(n_1/2+k_1)^{k_1}} \frac{k_2!}{(n_2/2+k_2)^{k_2}}$$

G. Sums of powers of linear forms

1. Lower bound

Theorem 11.

$$\left(\frac{\mathtt{Vol}\widetilde{Lf}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} \geq \frac{1}{4}\left(\frac{4k_1^2+n_1-2}{2k_1^2}\right)^{1/2}\left(\frac{4k_2^2+n_2-2}{2k_2^2}\right)^{1/2}$$
$$\left(\frac{k_1!}{(n_1/2+2k_1)_1^k}\right)\left(\frac{k_2!}{(n_2/2+2k_2)_2^k}\right)$$

We start with the following observation,

Lemma 24.

$$Lf_{N,2K} = Pos_d^*$$

That is, the sum of powers of linear forms is dual to the non negative polynomials in the differential metric.

Proof. $f \in Pos_d^*$ implies, $\langle f, g\rangle_D \geq 0$, for all $g \in Pos_{N,2K}$. Letting $f = v^{2K}$ we find using Lemma 20, that

$$\langle v^{2K}, g\rangle_D = (2k_1)!(2k_2)!g(v)$$

This implies that if $f = v^{2K}$, $\langle f, g\rangle_D \geq 0$, for all $g \in Pos_{N,2K}$. Hence, $Pos_d^* \subseteq Lf_{N,2K}$. Now if we take some $g \notin Pos_{N,2K}$, the right hand side will be negative for some value of v and hence we have, $Lf_{N,2K} \subseteq Pos_d^*$. $\square$

Thus we have,

$$\frac{\mathtt{Vol}\widetilde{Lf}_{N,2K}}{\mathtt{Vol}B_M} = \frac{\mathtt{Vol}\widetilde{Pos}_d^*}{\mathtt{Vol}B_M}$$

Lemma 25.

$$\widetilde{Pos}^\circ = -\widetilde{Pos}_i^*$$

Proof. Let us recall the definition of $\widetilde{Pos}^{\circ}$.

$$\widetilde{Pos}^{\circ} = \{f \in M_{N,2K} : \langle f, g\rangle \leq 1 \ , \forall g \in M_{N,2K}\}$$

We also have,

$$-Pos_i^* = \{f \in P_{N,2K} : \langle f, g\rangle \leq 0 \ , \forall g \in P_{N,2K}\}$$

Now it is easy to see that the usual tilde operation gives us the desired result. $\square$

Hence we have,

$$\begin{aligned}
\left(\frac{\mathtt{Vol}\widetilde{L}f_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} &\geq \left(\frac{\mathtt{Vol}\widetilde{Pos}^{\circ}}{\mathtt{Vol}B_M}\right)^{1/D_M} \left(\frac{\mathtt{Vol}\widetilde{Pos}_i^*}{\mathtt{Vol}\widetilde{Pos}_d^*}\right)^{1/D_M} \\
&\geq \frac{1}{4}\left(\frac{4k_1^2+n_1-2}{2k_1^2}\right)^{1/2}\left(\frac{4k_2^2+n_2-2}{2k_2^2}\right)^{1/2} \\
&\left(\frac{\mathtt{Vol}\widetilde{Pos}_i^*}{\mathtt{Vol}\widetilde{Pos}_d^*}\right)^{1/D_M} \\
&\geq \frac{1}{4}\left(\frac{4k_1^2+n_1-2}{2k_1^2}\right)^{1/2}\left(\frac{4k_2^2+n_2-2}{2k_2^2}\right)^{1/2} \\
&\left(\frac{k_1!}{(n_1/2+2k_1)_1^k}\right)\left(\frac{k_2!}{(n_2/2+2k_2)_2^k}\right)
\end{aligned}$$

This concludes the proof of our theorem.

CHAPTER IV

EXTENSION TO THE GENERAL MULTIHOMOGENEOUS CASE

A. Preliminaries

In this chapter, we shall extend the results of the previous chapter to the general multihomogeneous case. This is by and large straight forward and hence we shall be concise to avoid too much repetition. Throughout this chapter, $N = (n_1, \ldots, n_l)$ and $K = (k_1, \ldots, k_l)$.

From lemma 2 we see that the dimension D_M of $M_{N,2K}$ is,

$$D_M = \prod_{i=1}^{l} \binom{n_i + 2k_i - 1}{k_1}$$

Like before, S_M will be the unit sphere in $M_{N,2K}$ and B_M the unit ball in $M_{N,2K}$. We shall now compute the gauge of $\widetilde{Pos}_{N,2K}$.

Lemma 26. *The gauge $G_{\widetilde{Pos}}$ of $\widetilde{Pos}_{N,2K}$ at any polynomial $f \in M_{N,2K}$ is given by,*

$$G_{\widetilde{Pos}_{N,2K}}(f) = |\inf_{v \in S^N} f(v)|^{-1}$$

Proof. From definition 4 we have,

$$G_{\widetilde{Pos}_{N,2K}}(f) = \sup\{\lambda > 0 : \lambda f \in \widetilde{Pos}_{N,2K}\}$$

We know from the definition of $\widetilde{Pos}_{N,2K}$ that if $g \in M_{N,2K}$, then g is in $\widetilde{Pos}_{N,2K}$ if $\inf_{v \in S^N} g(v) \geq -1$. Let $m_g = \inf_{v \in S^N} g(v)$. Clearly if $g \in M_{N,2K}$, then $m_g < 0$. Now $\inf_{v \in S^N} g/|m_g| \geq -1$ and $\inf_{v \in S^N} g/|m_g + \epsilon| \leq -1$ for every $\epsilon > 0$. □

B. A lower bound for the general non negative multihomogeneous polynomials

The proof follows that of the bihomogeneous case closely.

Theorem 12.

$$\left(\frac{\mathtt{Vol}\widetilde{Pos}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} \geq \frac{\beta}{\sqrt{\max_{i\in\{1,\ldots,l\}}\{n_i \ln(2k_i+1)\}}}$$

where $\alpha = 9e^2$ *and* $\beta = \frac{1}{9e^2}$.

Like before the first step is to express $\mathtt{Vol}\widetilde{Pos}_{N,2K}$ in terms of the integral of its gauge function. We have,

$$\left(\frac{\mathtt{Vol}\widetilde{Pos}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} = \left(\int_{S_M} G_{\widetilde{Pos}_{N,2K}}^{D_M}\ d\mu\right)^{1/D_M}$$

We notice that we can apply Holder's theorem, since the right hand side is nothing but the D_M norm of G_P. Hence,

$$\begin{aligned}
\left(\frac{\mathtt{Vol}\widetilde{Pos}_{N,2K}}{\mathtt{Vol}B_M}\right) &\geq \left(\int_{S_M} G_P(f)\ d\mu\right) && \text{(by Holder's Inequality)} \qquad (4.1)\\
&\geq \left(\int_{S_M} |\inf_{v\in S^N} f(v)|^{-1} d\mu\right) && \text{(by Lemma 26)}\\
&\geq \left(\int_{S_M} |\inf_{v\in S^N} f(v)| d\mu\right)^{-1} && \text{(by Jensen's Inequality)}
\end{aligned}$$

Finally, it is easy to observe that $\|f\|_\infty \geq |\inf_{v\in S^N} f(v)|$. Hence to lower bound the volume of non negative multihomogeneous polynomials we only have to estimate the integral of the sup norm over the unit sphere.

$$\left(\frac{\mathtt{Vol}\widetilde{Pos}_{N,2K}}{\mathtt{Vol}B_M}\right) \geq \left(\int_{S_M} \|f\|_\infty d\mu\right)^{-1}$$

We can bound $\|f\|_\infty$ norm by $\|f\|_{2k}$ using Barvinok's theorem. To apply this in the general setting, we shall generalize $T_{N,2K}$ to the multihomogeneous case.

$$T_{N,2K} = \bigotimes_{i=1}^{l} (\mathbf{R}^{n_i})^{\otimes 2k_i}$$

Now we can apply Barvinok's theorem to bound, $\|f\|_\infty$ by $\|f\|_k$. For $k > 0$, let

d_k be the dimension of the subspce spanned by the orbit, $\{g\left(x^{\otimes K}\right)^{\otimes k}\}$. Then from Theorem 7, we have,

$$\|f\|_{2k} \leq \|f\|_\infty \leq (d_k)^{1/2k}\|f\|_{2k}$$

In the general case,

$$d_k = \prod_{i=1}^{l} \binom{n_i + 2k_i k - 1}{2k_i k} \tag{4.2}$$

This leads us to the following upper bound for $\|f\|_\infty$.

$$\|f\|_\infty \leq \prod_{i=1}^{l} \binom{n_i + 2k_i k - 1}{2k_i k} \|f\|_k \tag{4.3}$$

We shall now appeal to Lemma 15 to conclude,

$$\|f\|_\infty \leq 3e^2\|f\|_k$$

Using Lemma 7,

$$\left(\int_{S_M} \|f\|_k d\mu\right) \leq \sqrt{\frac{k\langle v, v\rangle}{D_M}}$$

But since $\langle v, v\rangle = D_M$, we have

$$\left(\int_{S_M} \|f\|_\infty d\mu\right) \leq \alpha\sqrt{k}$$

Thus we obtain the following bound in the general case:

$$\left(\frac{\mathtt{Vol}\widetilde{Pos}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} \geq \frac{\beta}{\sqrt{\max_{i=1}^{l}\{n_i \ln(2k_i + 1)\}}}$$

where $\alpha = 9e^2$ and $\beta = \frac{1}{9e^2}$.

C. An upper bound on the volume of non negative multihomogeneous polynomials

Theorem 13.

$$\left(\frac{\widetilde{Pos}_{N,2K}}{\mathtt{Vol}B_M}\right) \leq 4\prod_{i=1}^{l}\left(\frac{2k_i^2}{4k_i^2 + n_i - 2}\right)^{1/2}$$

The Blaschke-Santalo inequality generalizes to the general setting and we get the following:

$$\left(\mathtt{Vol}\widetilde{Pos}_{N,2K}\right)\left(\mathtt{Vol}\widetilde{Pos}^{\circ}_{N,2K}\right) \leq \left(VolB_M\right)^2$$

Therefore it would suffice to show that,

$$\left(\frac{\mathtt{Vol}\widetilde{Pos}^{\circ}_{N,2K}}{\mathtt{Vol}B_M}\right) \geq \frac{1}{4}\prod_{i=1}^{l}\left(\frac{2k_i^2}{4k_i^2+n_i-2}\right)^{1/2}$$

We recall that,

$$B_\infty = \{f \in M_{N,2K} \mid \|f\|_\infty \leq 1\}$$

and that,

$$B_\infty = \widetilde{Pos}_{N,2K} \cap -\widetilde{Pos}^{\circ}_{N,2K}$$

From Lemma 9, and applying Rogers and Shephard theorem,

$$\mathtt{Vol}B^\circ_\infty \leq \binom{2D_M}{D_M}\mathtt{Vol}\widetilde{Pos}^{\circ}_{N,2K}$$

From Lemma 16 it follows that,

$$\left(\frac{\mathtt{Vol}B^\circ_\infty}{\mathtt{Vol}\widetilde{Pos}^{\circ}_{N,2K}}\right)^{1/D_M} \geq \frac{1}{4}$$

This reduces the proof of the upper bound to,

$$\left(\frac{\mathtt{Vol}B^\circ_\infty}{\mathtt{Vol}B_M}\right)^{1/D_M} \geq \prod_{i=1}^{l}\left(\frac{4k_i^2+n_i-2}{2k_i^2}\right)^{1/2}$$

We now bound the infinity ball using the gradient metric introduced in Lemma 12. For $f \in M_{N,2K}$, which is decomposable, say $f = \otimes_{i=1}^{l} f_i$, we have the following generalization of the graident metric ,

From Lemma 12, the gradient metric of f as above would be,

$$\langle f, f\rangle_G = \frac{1}{16\prod_{i=1}^{l} k_i^2}\int_{S^N}\prod_{i=1}^{l}\left(\left(\frac{\partial f_i}{\partial x_{i1}}\right)^2 + \ldots + \left(\frac{\partial f_i}{\partial x_{in_1}}\right)^2\right)\, d\sigma$$

Let B_G be the unit ball in the gradient metric and the corresponding norm $\|f\|_G$. From Kellog's lemma, (Lemma 11),

$$B_\infty \subseteq B_G$$

Polarity reverses inclusion and so,

$$B_G^\circ \subseteq B_\infty^\circ \tag{4.4}$$

$$\mathtt{Vol} B_G^\circ = \frac{(\mathtt{Vol} B_M)^2}{\mathtt{Vol} B_G} \qquad \text{(Using the Blaschke-Santalo Inequality)} \tag{4.5}$$

Consequently, we have, $\mathtt{Vol} B_\infty^\circ \geq \frac{(\mathtt{Vol} B_M)^2}{\mathtt{Vol} B_G}$ and hence,

$$\frac{\mathtt{Vol} B_\infty^\circ}{\mathtt{Vol} B_M} \geq \frac{\mathtt{Vol} B_M}{\mathtt{Vol} B_G}$$

Thus, we are left with proving the following:

Lemma 27.

$$\left(\frac{\mathtt{Vol} B_M}{\mathtt{Vol} B_G}\right)^{1/D_M} \geq \prod_{i=1}^{l} \left(\frac{4k_i^2 + n_i - 2}{2k_i^2}\right)^{1/2}$$

Proof. The proof follows that of the bihomogeneous case. We note the invariance of both inner products under the action of $SO(N)$, it is enough to prove the lemma in the irreducible components of the representation. In this setting, the irreducible components are $\otimes_{i=1}^{l} H_{n_i,2l_i}$ for $0 \leq l_i \leq k_i$. And,

$$H_{n,2l} = \{f \in P_{n,2k} \mid f = (x_1^2 + \ldots + x_n^2)^{k-l} h, h \in P_{n,2l}\}$$

We finally notice that,

$$\langle f, f\rangle_G \leq \prod_{i=1}^{l} \left(\frac{4k_i^2 + n_i - 2}{2k_i^2}\right) \langle f, f\rangle$$

The last step follows since the minimum clearly occurs when $d_i = 1$. This proves the lemma. $\square$

D. Upper bound for multihomogeneous SOS polynomials

We still have the following bound for the volume of multihomogeneous SOS polynomials from the Uryshon's Inequality [18].

$$\left(\frac{Vol\widetilde{\widetilde{Sq}}_{N,2K}}{VolB_M}\right)^{1/D_M} \leq \frac{W_{\widetilde{Sq}}}{2}$$

Bounding their average width $W_{\widetilde{Sq}}$ by the support function $L_{\widetilde{Sq}}$, and using the following simplification, we have,

$$L_{\widetilde{Sq}}(f) = \max_{g \in S_{P_{N,K}}} \langle f, g^2 \rangle$$

We shall now bound the support function as follows:

$$H_f(g) = \langle f, g^2 \rangle \text{for } g \in P_{N,K}$$

Now,

$$L_{\widetilde{Sq}}(f) \leq \|H_f\|_\infty$$

We can now use Barvinok's theorem to bound $\|H_f\|_\infty$ by a high L^{2p} norm of H_f. Since H_f is a form of degree 2 on the vector space $P_{N,K}$ of dimension $D_{N,K}$ we get,

$$\|H_f\|_\infty \leq 2\sqrt{3}\|H_f\|_{2D_{N,K}}$$

We now proceed as in the case of non negative multihomogeneous polynomials, using Hõlder's inequality to estimate the integral of $\|H_f\|_\infty$.

$$\int_{S_M} L_{\widetilde{Sq}} d\mu \leq \left(\int_{S_M} \int_{S_{P_{N,K}}} \langle f, g^2 \rangle^{2D_{N,K}} d\sigma(g) d\mu(f) \right)^{1/2D_{N,K}}$$

Since the inner integral depends only on the projection of g^2 into $M_{N,2K}$, we have,

$$\int_{S_M} \langle f, g^2 \rangle^{2D_{N,K}} d\mu(f) \leq \|g^2\|_2^{2D_{N,K}} \int_{S_M} \langle f, p \rangle^{2D_{N,K}} d\mu(f)$$

for any $p \in S_M$.

We can compute the second integral easily due to the invariance of the inner product under $SO(N)$.

$$\int_{S_M} \langle f, p \rangle^{2D_{N,K}} d\mu(f) = \frac{\Gamma(D_{N,K} + \frac{1}{2})\Gamma(\frac{1}{2}D_M)}{\sqrt{\Pi}\Gamma(D_{N,K} + \frac{1}{2}D_M)}$$

Using, the results of Duoandikoetxea [34], we have,

$$\int_{S_M} L_{\widetilde{Sq}}(f) d\mu \leq \prod_{i=1}^{l} 4^{2k_i} \left(\frac{\Gamma(D_{N,K} + \frac{1}{2})\Gamma(\frac{1}{2}D_M)}{\sqrt{\Pi}\Gamma(D_{N,K} + \frac{1}{2}D_M)} \right)^{\frac{1}{D_{N,K}}}$$

We also have,

$$\left(\frac{\Gamma(\frac{1}{2}D_M)}{\Gamma(D_{N,K} + \frac{1}{2}D_M)} \right)^{\frac{1}{2D_{N,K}}} \leq \sqrt{\frac{2}{D_M}}$$

$$\left(\frac{\Gamma(\frac{1}{2} + D_{N,K})}{\sqrt{\pi}} \right)^{\frac{1}{2D_{N,K}}} \leq \sqrt{D_{N,K}}$$

This implies,

$$\int_{S_M} L_{\widetilde{Sq}}(f) d\mu \leq \prod_{i=1}^{l} 4^{2k_i} \sqrt{3} \sqrt{\frac{2D_{N,K}}{D_M}}$$

Thus we get the following upper bound for the volume on SOS multihomogeneous polynomials.

Theorem 14. *Upper bound on the volume of SOS multihomogeneous polynomials*

$$\left(\frac{\mathtt{Vol}\widetilde{Sq}_{N,2K}}{\mathtt{Vol}B_M}\right)^{1/D_M} \leq \prod_{i=1}^{l} 4^{2k_i}\sqrt{24}\sqrt{\frac{D_{N,K}}{D_M}} \tag{4.6}$$

$$\textit{where,} D_{N,K} = \prod_{i=1}^{l} \binom{n_i + k_i - 1}{k_i} \tag{4.7}$$

$$\textit{and } D_M = \prod_{i=1}^{l} \binom{n_i + 2k_i - 1}{2k_i} \tag{4.8}$$

We can further simplify the above expression to get,

$$\left(\frac{Vol\widetilde{Sq}_{N,2K}}{VolB_M}\right)^{1/D_M} \leq \prod_{i=1}^{l} 4^{2k_i}\sqrt{24}\frac{(2k_i)!}{k_i!}n_i^{-k_i/2}$$

E. Generalized differential metric

Let $\alpha = (\alpha_1, \ldots, \alpha_{n_1}) \in \mathbb{N}^l$. Then,

$$\mathbf{x}^\alpha = x_1^{\alpha_1} \ldots x_{n_1}^{n_1}$$

and,

$$D_{\mathbf{x}^\alpha} := \frac{\partial^{\alpha_1} \ldots \partial^{\alpha_{n_1}}}{\partial x_1^{\alpha_1} \ldots \partial x_{n_1}^{\alpha_{n_1}}}$$

Now for a form $f \in P_{N,2K}$,

$$f = \sum_{\alpha_i = (\alpha_{i_1}, \ldots, \alpha_{i_{n_1}}), |\alpha_i| = 2k_i} c_{\alpha_1 \ldots \alpha_l} \prod_{i=1}^{l} \mathbf{x}^{\alpha_i}$$

we can define an associated linear operator as follows:

$$D_f := \sum_{\alpha_i = (\alpha_{i_1}, \ldots, \alpha_{i_{n_1}}), |\alpha_i| = 2k_i} \prod_{i=1}^{l} D_{\mathbf{x}^{\alpha_i}}$$

Now we finally get to defining the differential metric using a positive definite bilinear form using D_f.

$$\langle f, g\rangle_D := D_f(g)$$

The linear operator T can be written as follows. For $f \in P_{N,2K}$, we have,

$$T(f) = \int_{S^N} f(v)v^{2K} d\sigma$$

The reason T maps the dual cone in the usual metric into the dual cone in the differential metric is due to the following lemma, which in the multihomogeneous case is:

Lemma 28. *There is the following relationship between the differential and the L^2 metric.*

$$\langle Tf, g\rangle_D = \prod_{i=1}^{l}(2k_i!)\langle f, g\rangle$$

Let L_i^* and L_d^* be the duals of L in the L^2 and the differential metric respectively.

$$L_i^* = \{f \in P_{N,2K} | \langle f, g\rangle \geq 0, \forall g \in L\}$$

$$L_d^* = \{f \in P_{N,2K} | \langle f, g\rangle_D \geq 0, \forall g \in L\}$$

Since, T commutes with the action of $SO(N)$, T acts by contraction in each irreducible subspace of $P_{N,2K}$.

Combining these we have the following lemma,

Lemma 29.

$$\left(\frac{Vol\widetilde{L_d^*}}{Vol\widetilde{L_i^*}}\right)^{1/D_M} \geq \prod_{i=1}^{l} \frac{k_i!}{(n_i/2 + k_i)^{k_i}}$$

And this leads us to,

Lemma 30. *The dual cone to the cone of multihomogeneous SOS polynomials in the differential metric, namely, Sq_D^* is contained in the cone of multihomogeneous SOS* *polynomials,* $Sq_{N,2K}$.

F. Lower bound for SOS multihomogeneous polynomials

We prove the lower bound by first noting that as a result of Lemma 20 we have,

$$\widetilde{Sq}_d^* \subseteq \widetilde{Sq}_{N,2K}$$

This gives us,

$$\frac{\mathtt{Vol}\widetilde{Sq}_{N,2K}}{\mathtt{Vol}B_M} \geq \frac{\mathtt{Vol}\widetilde{Sq}_d^*}{\mathtt{Vol}B_M}$$

Hence we are done if we find an upper bound for the right-hand side of that above equation. Therefore applying Lemma 30,

$$\left(\frac{\mathtt{Vol}\widetilde{Sq}_d^*}{\mathtt{Vol}\widetilde{Sq}_i^*}\right)^{1/D_M} \geq \prod_{i=1}^{l} \frac{k_i!}{(n_i/2+k_i)^{k_i}}$$

For the part involving $\widetilde{Sq}_i^*$, we have like before B_{sq}.

$$B_{sq} = \{f \in M_{N,2K} : \|f\|_{sq} \leq 1\}$$

Let $G_{B_{sq}}$ be the gauge of B_{sq}. From Lemma 5, we obtain,

$$\frac{\mathtt{Vol}B_{sq}}{\mathtt{Vol}B_M} = \int_{S_M} G_{B_{sq}}\, d\mu$$

It is not very difficult to determine the gauge of B_{sq}. Indeed we have the following lemma.

Lemma 31. *For $f \in S_M$, the gauge of B_{sq} is given by,*

$$G_{B_{sq}}(f) = \|f\|_{sq}^{-1}$$

We shall go through the same process as in Section IIIC to bound the volume of

B_{sq}.

$$\frac{\mathtt{Vol} B_{sq}}{\mathtt{Vol} B_M} \geq \prod_{i=1}^{l} \frac{1}{4^{2k_i}\sqrt{24}} \frac{k_i!}{(2k_i)!} n_i^{k_i/2} \tag{4.9}$$

Lemma 32.

$$B_{sq} = \widetilde{Sq}^{\circ}_{N,2K} \cap -\widetilde{Sq}^{\circ}_{N,2K}$$

Lemma 33.

$$\widetilde{Sq}^{\circ}_{N,2K} = -\widetilde{Sq}^{*}_{i}$$

where, $\widetilde{Sq}^{*}_{i}$ *is the dual cone of* $Sq_{N,2K}$ *in the differential metric.*

From the above two lemmas we get,

$$\left(\frac{\widetilde{Sq}^{*}_{i}}{B_M}\right)^{1/D_M} \geq \prod_{i=1}^{l} \frac{1}{4^{2k_i}\sqrt{24}} \frac{k_i!}{(2k_i)!} n_i^{k_i/2}$$

Combining this with our bound for $\left(\frac{\mathtt{Vol}\widetilde{Sq}^{*}_{d}}{\mathtt{Vol}\widetilde{Sq}^{*}_{i}}\right)^{1/D_M}$, we finally get,

$$\frac{\mathtt{Vol}\widetilde{Sq}^{*}_{d}}{\mathtt{Vol} B_M} \geq \prod_{i=1}^{l} \frac{1}{4^{2k_i}\sqrt{24}} \frac{k_i!}{(2k_i)!} n_i^{k_i/2} \frac{k_i!}{(n_i/2 + k_i)^{k_i}}$$

CHAPTER V

A HOMOTOPY APPROACH FOR THE MOTION COORDINATION OF A GROUP OF MOBILE AGENTS

A. Introduction

Autonomous mobile agents have been gaining a lot of attention in recent years because of their potential applications in military operations, automated factories and automated highways. It is widely believed that cooperative mobile robots will have a number of civilian and defense applications in addition to the ones listed above. In such scenarios, studying the motion planning algorithms becomes of paramount importance. The following topics generally fall under the umbrella of motion planning algorithms:

1. Algorithms which enable a group of n mobile agents to change position and formation

2. Algorithms which enable the mobile agents to avoid obstacles so as to negotiate through the amibient environment.

In this chapter we summarize our recent works in this area [13]. The listed references listed give an applied treatment, whereas in this thesis we shall adopt a more mathematical approach in an attempt to clarify the underlying mathematics. The motivation for this work has been to provide a complete numerical algorithm for the problem of pattern change in two dimensions, with central planning and hence with limited communication among the agents. Although we have not explored ways to optimize our solution, we believe that our work is among the first to transfer the motion planning problem to a simple root finding exercise.

Most of the current research that has been done in this area uses the composite configuration approach or decoupled planning approach [37], [38], [39]. These approaches do not often capture the conditions for the combined motion of the agents as succinctly as out polynomial space approach. Another common approach adopted in current literature is the distributed motion planning method [40], [41], [42], [43], [44], [45]. However this approach is limited to motion planning to achieve a limited number of patterns. Using the method in our work motion planning can be tailored to obtain any pattern starting from an arbitrary pattern. We shall also extend the algorithm described in the initial sections in combination with the framework proposed in [46] to avoid pre specified obstacles in two dimensions.

We shall now quickly summarize our approach. We represent the initial configuration of a group of n mobile agents by means of the roots of a polynomial of degree n, say P_i and the final configuration by roots of a polynomial P_f. In order to get from the initial configuration to the final configuration, we deform the coefficients of P_i to that of P_f. The condition to avoid collision is given by requiring the intermediate polynomial to have distinct roots throughout the deformation. The set of all polynomials of degree n having atleast one multiple root is called the discriminant variety Σ_n. It is known that the complement of the discriminant variety is connected in $\mathbb{C}$ [17]. Hence, there is always a path from P_i to P_f which avoids the discriminant variety, assuring that the agents do not collide. A parametric representation for Σ_n is described in [14]. Using this we can obtain a certificate that our path lies entirely in the complement of Σ_n. This method requires computation of roots of a polynomial of degree n at each time step during the deformation. We use Newton Raphson method for computing the roots. Due to the fact that the deformation changes the coefficients only slightly at each step, the roots of the previous polynomial would provide us with a very good guess at each time step for the Newton Raphson method to converge effectively.

B. Description of the motion planning algorithm

1. Assumptions

The key assumptions are:

1. The mobile agents move in a two dimensional space.

2. The agents are represented as point objects, that is having no girth.

Although our algorithm only guarantees that the point objects do not collide at any given time, it is possible to handle finite sized objects as well. This is made possible by means of a result which lower bounds the minimum distance between roots of a polynomial of degree n [47]. We shall discuss this in greater detail in the final section of this chapter where we describe obstacle avoidance.

We now establish some notation and definitions. We consider n agents or objects R_1, R_2, …, R_n. By a "pattern" we essentially refer to the set of coordinates of the n objects in a particular local coordinate system(see Figure 1). We allow for the case where the coordinate system itself is undergoing some translation.

Definition 13. *Discriminant variety The discriminant variety Σ_n is the set of all polynomials with coefficients in $\mathbb{C}$ of degree n with multiple roots.*

$$\Sigma_n := \{f \in P_n \,:\, f \textit{has multiple roots}\}$$

where, P_n is the set of all polynomials of degree, n. Hence, $P_n - \Sigma_n$ represents the set of all polynomials of degree n with distinct roots.

According to [17] the complement of the discriminant variety is connected. Hence given two polynomials in the complement of the discriminant variety we can find a path connecting them, that lies entirely in the complement of the discriminant variety.

Lemma 34. *[17] The complement of the discriminant variety is connected.*

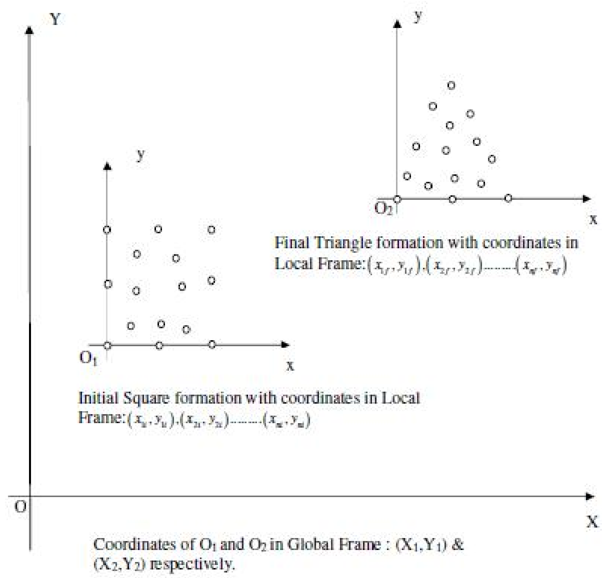

Fig. 1. Local and global coordinates

In our algorithm we represent the position of the agents with respect to some local coordinate system L in which the motion planning is carried out, by means of roots of polynomials. Given an initial pattern $Q_i = \{(x_{1i}, y_{1i}), \ldots, (x_{ni}, y_{ni})\}$, we can associate an initial polynomial P_i to Q_i as follows:

$$P_i := (x - (x_{1i} + y_{1i}\imath) \ldots (x - (x_{ni} + y_{ni}\imath))$$

Also, we can expand the above expression and write this in a more familiar coefficient notation as,

$$P_i = a_{oi} + a_{1i}x + \ldots + a_{ni}x^n$$

Similarly for the final pattern $Q_f = \{(x_{1f}, y_{1f}), \ldots, (x_n f, y_{nf})\}$, we can associate a final polynomial P_f to Q_f as follows:

$$P_f := (x - (x_{1f} + y_{1f}\imath) \ldots (x - (x_{nf} + y_{nf}\imath))$$

and like before we can expand and write this in the more familiar coefficient notation as,

$$P_f = a_{of} + a_{1f}x + \ldots + a_{nf}x^n$$

Now since we know that in the initial and final patterns, the agents do not collide, we clearly have that the roots of both these polynomials P_i and P_f are distinct. Hence we have, $P_i, P_f \in \Sigma - P_n$. From Lemma 34 we know that this set is connected. Let T be the time in which we need to deform pattern Q_i to Q_f in the local frame L. We shall consider a particularly simple deformation of P_i into P_f and describe a procedure to ensure that the polynomials arising in the deformation are always in $\Sigma_n - P_n$.

Definition 14. *Straight Line Deformation*

A straight line deformation of P_i to P_f is given by the following parametrization:

$$P(\lambda) := (1 - \lambda)P_i + \lambda P_f, \textit{where, } \lambda \in [0, 1]$$

Here we have reparametrized $[0, T]$ to $[0, 1]$, using $\lambda = t/T$, for $t \in [0, T]$.

We can use the results in [14] which describe a way to parametrize Σ_n ,to verify whether $P(\lambda) \in \Sigma_n - P_n$, for all $\lambda \in [0, 1]$. First we note that if a polynomial $p = a_0 + a_1x + \ldots + a_nx^n$ has multiple roots, so does any multiple of p. Thus we can represent each point in Σ_n, as a point in $P(\mathbb{C}^{n+1})$. We shall now describe the Horn parametrization of Σ_n.

Lemma 35. *Horn uniformization Let A be the following matrix,*

$$A := \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & n \end{pmatrix}$$

Let K be the kernel of A in $\mathbb{C}^{n+1}$. The discriminant variety is parametrized as follows:

$$\Sigma_n = \{[\tau_1 x_0 : \tau_1\tau_2 x_1 : \tau_1\tau_2^2 x_2 : \dots : \tau_1\tau_2^n x_n] \,:\, \tau_1, \tau_2 \in (\mathbb{C}^*)^2, (x_0, x_1, \dots, x_n) \in K\}$$

We shall now find a basis for K. For $(x_0, x_1, \dots, x_n) \in K$, we have,

$$x_0 + x_1 + \dots + x_n = 0$$

$$x_1 + 2x_2 + \dots + nx_n = 0$$

Using these two equation, we can conclude that the following $(n-1)$ vectors form a basis for K.

$$v_2 = (1 \;\; -2 \; 1 \; 0 \; \dots 0)$$

$$v_3 = (2 \;\; -3 \; 0 \; 1 \; \dots 0)$$

$$\vdots$$

$$v_n = (n-1 \;\; -n \; 0 \; \dots 1)$$

Let π_0, π_1, π_n denote the projections in $\mathbb{C}^{n+1}$ along $(1, \dots, 0), \dots, (0, \dots, 1)$ respectively. Then from the above we see that the parametrization of the discriminant reduces to,

$$\Sigma_n = \{[\tau_1(\pi_0(x_2v_0 + \ldots x_nv_n) : \tau_1\tau_2(\pi_1(x_2v_0 + \ldots x_nv_n) : \tau_1\tau_2^2(\pi_2(x_2v_0 + \ldots x_nv_n) : \tag{5.1}$$

$$\ldots : \tau_1\tau_2^n(\pi_n(x_2v_0 + \ldots x_nv_n)] : \tau_1, \tau_2 \in (\mathbb{C}^*)^2, (x_2, x_3, \ldots, x_n) \in \mathbb{C}^{n-2}\} \tag{5.2}$$

$$= \{[\tau_1(\pi_0(x_2w_0 + \ldots x_nw_n) : \tau_1(\pi_1(x_2w_0 + \ldots x_nw_n) : \tau_1(\pi_2(x_2w_0 + \ldots x_nw_n) : \tag{5.3}$$

$$\ldots : \tau_1(\pi_n(x_2w_0 + \ldots x_nw_n)] : \tau_1, \tau_2 \in (\mathbb{C}^*)^2, (x_2, x_3, \ldots, x_n) \in \mathbb{C}^{n-2}\} \tag{5.4}$$

where,

$$\begin{aligned} w_2 &= (1 \ \ -2\tau_2 \ \tau_2^2 \ 0 \ \ldots 0) \\ w_3 &= (2 \ \ -3\tau_2 \ 0 \ \tau_2^3 \ \ldots 0) \\ &\vdots \\ w_n &= (n-1 \ \ -n\tau_2 \ 0 \ \ldots \tau_2^n) \end{aligned}$$

Lemma 36. *A polynomial* $p = a_0 + a_1x + \ldots + a_nx^n$*, lies in* Σ_n *if and only if,* $(a_0 \ a_1 \ \ldots a_n)$ *lies in the span of* $w_2, \ldots, w_n$*.*

Proof. This is clear from Equation 5.1 □

Lemma 37. *We can find two vectors,* s_1 *and* s_2 *such that* $p = a_0 + a_1x + \ldots + a_nx^n$*, lies in* Σ_n *if and only if,* $\langle P, s_1 \rangle = 0$ *and* $\langle P, s_2 \rangle = 0$*.*

Proof. If we find two linearly independent vectors orthogonal to $w_2, \ldots, w_n$, we would have our result. Since the dimension of the vector space we are considering is $n+1$, the subspace orthogonal to the one spanned by $w_2, \ldots, w_n$ is 2 dimensional. We shall now explicitly write down two such vectors. Since, s_i is orthogonal to $w_2, \ldots w_n$, we have,

$$
\begin{aligned}
s_0^i - 2\tau_2 s_1^i + \ldots + \tau_2^2 s_2^i &= 0 \\
2s_0^i - 3\tau_2 s_1^i + \ldots + \tau_2^3 s_3^i &= 0 \\
&\vdots \\
(n-1)s_0^i - n\tau_2 s_1^i + \ldots + \tau_2^n s_n^i &= 0
\end{aligned}
$$

where s_j^i is the j^{th} component of s^i, for $i = 1, 2$ and $j = 0, 1, \ldots, n$. Letting $s_0^1 = 0, s_1^1 = 1$ and $s_0^2 = 1, s_1^2 = 0$, we have our desired two vectors.

$$
\begin{aligned}
s_1 &= (0\ 1\ 2/\tau_2\ 3/\tau_2^2\ \ldots n/\tau_2^{n-1}) \\
s_2 &= (1\ 0\ -1/\tau_2^2\ -2/\tau_2^3\ \ldots -(n-1)/\tau_2^n)
\end{aligned}
$$

□

Theorem 15. *There exists a continuous path $\Lambda : [0,1] \to \mathbb{C}$, such that $P(\lambda) = (1-\lambda)P_i + \lambda P_f$ is not in Σ_n for every $\lambda \in Image(\Lambda)$.*

Proof. Using the above mentioned vectors s_1 and s_2, we construct two polynomial equations, corresponding to $\langle P(\lambda), s_1 1\rangle = 0$ and $\langle P(\lambda), s_2\rangle = 0$

We can easily eliminate λ from one of the equations, because it is a linear term. The polynomial resulting from the elimination is a polynomial in τ_2 alone. Each of the solutions for τ_2 gives a corresponding value for λ. Thus there are only finitely many solutions for λ. Hence we can always find a path $\Lambda : [0,1] \to \mathbb{C}$ which avoids all the above values of λ. If the value of λ do not lie in $[0,1]$ then the simple straight line path $\Lambda : [0,1] \to [0,1]$ does the job. □

Once it has been verified that the straight line path between polynomials avoids the discriminant variety,the roots of the polynomial can be found out at each step to find the position of each mobile agent in local frame with the current local frame having undergone a translation from the initial frame. In other words each mobile agent is translated by the same amount with deformation caused by homotopy of the polynomial. The planning for translation can be done as in [20]. Given the initial and final polynomial to each mobile agent, its initial position and the velocity of translation,using Newton Raphson the mobile agents can calculate their position in the next time step in a distributed manner. Newton Raphson method can be used as we have a good initial guess at each time step for calculating the roots. The results of the simulations are plotted in Figure 2 and Figure 3

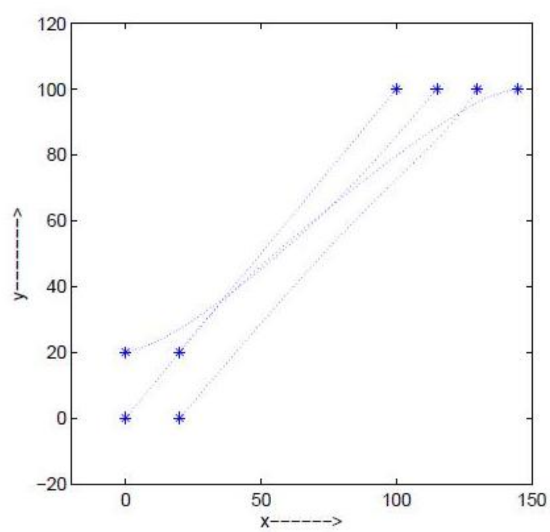

Fig. 2. Square to line formation

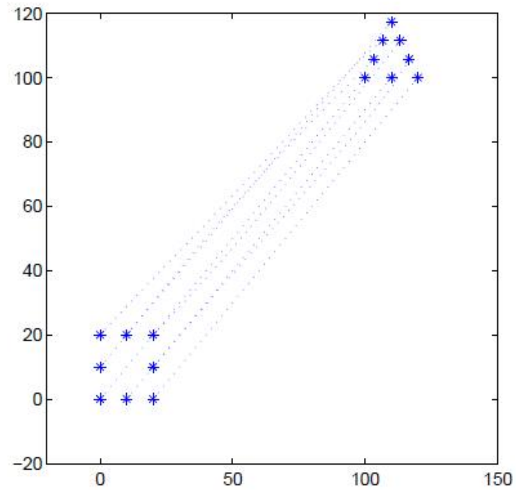

Fig. 3. Square to triangle formation

C. Imposing velocity and acceleration constraints

We can impose velocity and acceleration constraints on each mobile agent by reparameterizing P. Suppose we have a path $\Lambda : [0,1] \to [0,1]$, such that $P(\lambda) \notin \Sigma_n$ for all $\lambda \in [0,1]$. Let $x_i(\lambda)$ denote the position of the i^{th} agent. Then its velocity is given by, $dx_i/d\lambda$ and the speed would be, $|dx_i/d\lambda|$. Given a constraint on the speed, γ, our approach would be to reparametrize Λ with $f : [0,1] \to [0,1]$ so as to maintain the new speed, $|dx_i(\lambda(f)))/df|$ below γ. Since $|dx_i/df| = |dx_i/d\lambda| \times |d\lambda/df|$, we can set $|d\lambda/df|$ in such a way to keep $|dx_i/df|$ below γ. For instance, setting $|d\lambda/df| < 0.9 \times \gamma/|d\lambda/df|$ would achieve our goal. Figures 4 and 5 illustrate this.

The sparsely spaced dots in Figure 4 indicate high velocities which are kept within bounds using velocity constraint as shown in Figure 5.

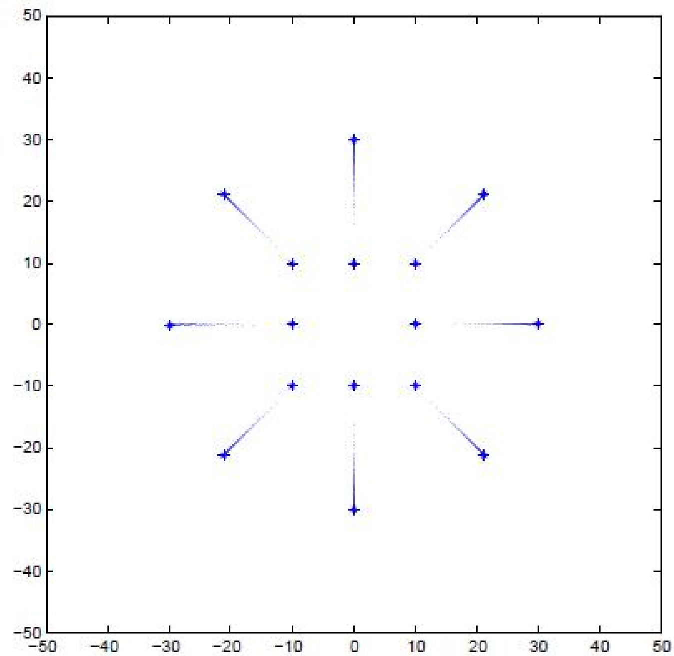

Fig. 4. Without velocity constraint

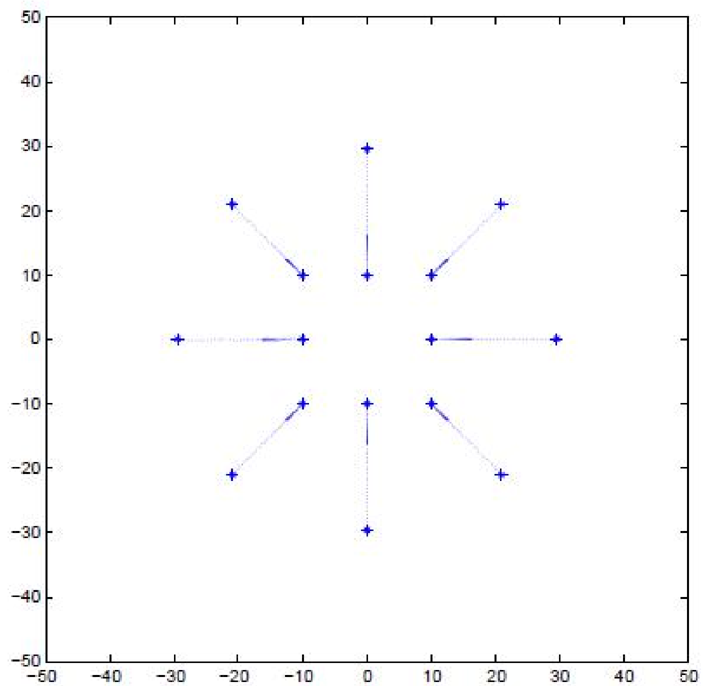

Fig. 5. With velocity constraint

D. Illustrative example

We shall present an example which has four mobile agents arranged in a square initial pattern, which needs to be deformed into a line.

1. Initial pattern, $Q_i = \{(0,0),(20,0),(0,20),(20,20)\}$.

   Final pattern $Q_f = \{(0,0),(15,0),(30,0),(45,0)\}$.

2. Hence, we have the initial and final polynomials as follows:

$$P_i = (x-0)(x-20)(x-20\imath)(x-20-20\imath)$$

$$P_f = (x-0)(x-15)(x-20)(x-45)$$

3. Using a straight line deformation, $P(\lambda) = (1-\lambda)P_i + \lambda P_f$. The following steps will verify that $P(\lambda) \notin \Sigma_n$, for all $\lambda \in [0,1]$.

4. We have,

$$A := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

5. We have the following basis for the kernel of A.

$$v_2 = (1 \ -2 \ 1 \ 0 \ 0)$$

$$v_3 = (2 \ -3 \ 0 \ 1 \ 0)$$

$$v_4 = (3 \ -4 \ 0 \ 0 \ 1)$$

6. The discriminant variety is parametrized as ,

$$\Sigma_n = \{[\tau_1(\pi_0(x_2w_0 + \dots x_4w_4) : \tau_1(\pi_1(x_2w_0 + \dots x_4w_4) : \tau_1(\pi_2(x_2w_0 + \dots x_4w_4) :$$
$$\dots : \tau_1(\pi_n(x_2w_0 + \dots x_4w_4)] : \tau_1, \tau_2 \in (\mathbb{C}^*)^2, (x_2, x_3, x_4) \in \mathbb{C}^3\}$$

where we have,

$$w_2 = (1 \quad -2\tau_2 \quad \tau_2^2 \quad 0 \quad 0)$$

$$w_3 = (2 \quad -3\tau_2 \quad 0 \quad \tau_2^3 \quad 0)$$

$$w_4 = (3 \quad -4\tau_2 \quad 0 \quad 0 \quad \tau_2^4)$$

7. The following two vectors are orthogonal to w_2, w_3 and w_4.

$$s_1 = \quad (0 \quad 1 \quad 2/\tau_2 \quad 3/\tau_2^3 \quad 4/\tau_2^3)$$

$$s_2 = \quad (1 \quad 0 \quad -1/\tau_2 \quad -2/\tau_2^3 \quad -(n-1)/\tau_2^4)$$

8. Using, $\langle P(\lambda), s_1\rangle = 0$ and $\langle P(\lambda), s_2\rangle = 0$ and solving for λ we get,

$$\lambda = \{0.6384 + 0.2690\imath, -0.0111 + 0.1943\imath, 0.0149 + 0.0152\imath, \\ 0.0040 - 0.5881\imath, 0.2441 - 0.6700\imath\}$$

Clearly these values do not lie in $[0, 1]$. Hence $P(\lambda)$ does not intersect the discriminant variety Σ_n for any $\lambda \in [0, 1]$. The simulations shown in Figure 2 also show that the agents do not collide while they are deformed from the initial to the final pattern.

E. Discussion

Paths were generated for groups of mobile agents for different initial and final shapes. Also the velocity and acceleration were kept under bounds by reparameterization which was done numerically using lookup tables. Even though the paths intersect they do so at different time steps. The paths generated are smooth as expected and are mostly non-linear.

Since this is a result which is first of its kind there is a lot of scope to extend the idea. Given the size of the agents research can go into finding complete algorithms which guarantee the maintenance of a certain distance between the agents at all times. This would require moving in a sub space of the discriminant variety space in which minimum distance between the roots is the sum of the radii of the largest mobile agents. Research can also be done to find the probability of avoidance of the discriminant variety using the straight line interpolation. Also if a generalized method to find all paths parameterized in time in the complement of the discriminant variety space is found research can go into finding the optimal path. Another interesting idea is to study the paths in polynomial space which ensure that the mobile agents do not wander too far off from the group [48]. In other words the idea of bounding the size of the formation at each time instant using this method will be explored. Also the 3-D extension of the method remains a significant open problem.

CHAPTER VI

CONCLUSION

Much recent work in optimization and algorithmic real algebraic geometry has arisen from the fact that deciding whether a polynomial is SOS can be done efficiently via SDP [8]. In particular, for certain n-variate degree k polynomials, it was shown in [8] that one could approximate their real minima within $nO(1)$ arithmetic operations via SOS and SDP. This is in sharp contrast to the $kO(n)$ complex- ity bounds coming from the best known algorithms from real algebraic geometry [29]. However, for an approach via SDP to be practical, one obviously needs to know how often nonnegative polynomials are in fact SOS. In one variable, nonnegative polynomials are actually always SOS , so one can then safely use SDP to decide nonnegativity and even decide the existence of real roots. However, since the classical technique of Sturm-Habicht sequences is already known to have complexity near-linear in the degree [30], the potential complexity savings of SDP over Sturm-Habicht are not clear. Whether SDP can provide a significant gain in speed for larger n, for a large fraction of inputs, is thus an important question. Similarly, many algebraic algorithms lack provable speed-ups when the input polynomials are sparse or have structured Newton polytopes, and thus one should also ask if SDP can provides speed gains in these settings as well.

Let us state clearly that while no current bounds (including our own) adequately describe classes of multigraded polynomial where $\Sigma_{2K,N}$ occupies a provably large fraction of $P_{2K,N}$ our results are at least a first step toward incorporating Newton polytopes and sparsity in the quantitative study of $P_{n,k}$ and $\Sigma_{n,k}$. In particular, we can at least point out new families where there are significantly more nonnegative polynomials than sums of squares.

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