AUTOMATA GROUPS

A Dissertation

by

YEVGEN MUNTYAN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2009

Major Subject: Mathematics

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ABSTRACT

Automata Groups. (May 2009)

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This dissertation is devoted to the groups generated by automata. The first part of the dissertation deals with L-presentations for such groups. We describe the sufficient condition for an essentially free automaton group to have an L-presentation. We also find the L-presentation for several other groups generated by three-state automata, and we describe the defining relations in the Grigorchuk groups G_ω. In case when the sequence ω is almost periodic these relations provide an L-presentation for the group G_ω. We also describe defining relations in the series of groups which contain Grigorchuk-Erschler group and the group of iterated monodromies of the polynomial $z^2 + i$.

The second part of the dissertation considers groups generated by 3-state automata over the alphabet of 2 letters and 2-state automata over the 3-letter alphabet. We continue the classification work started by the research group at Texas A&M University ([BGK$^+$07a, BGK$^+$07b]) and further reduce the number of pairwise non-isomorphic groups generated by 3-state automata over the 2-letter alphabet. We also study the groups generated by 2-state automata over the 3-letter alphabet and obtain a number of classification results for this class of group.

To Sveta and Max

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CHAPTER I

INTRODUCTION

Groups generated by automata (or simply automata groups) were introduced and studied by V.M. Glushkov and his students in 1960's [Glu61]. In his original paper V.M. Glushkov conjectured that automata groups may have relation to Burnside Problem. This was later confirmed by S.V. Aleshin (1972), V.I. Sushchansky (1979), R.I. Grigorchuk (1980), N.D. Gupta and S. Sidki (1983), who constructed finite automata which generate infinite torsion groups [Ale72, Sus79, Gri80, GS83]. Later R.I. Grigorchuk proved that the groups he constructed have intermediate growth between polynomial and exponential, providing a solution to Milnor Problem [Mil68] on intermediate growth and Day Problem [Day57] on amenability. These developments pushed the study of groups of automata in many directions: analysis [Gri84, Ers04], geometry [BGN03], probability [BV05, Ers04, AV05], dynamics [BG00], formal languages [HR06], etc.

This dissertation is devoted to several aspects of combinatorial theory of groups generated by finite automata. The class of groups generated by finite automata is extremely rich, filled with numerous groups with different remarkable properties, and is still largely unexplored. Examples of groups from this class are finite groups, free groups, free products of finite groups, linear groups $GL_n(\mathbb{Z})$ and $GL_n(\mathbb{Z}_m)$, as well as groups with exotic properties mentioned above.

One important class of automata groups is the class of contracting self-similar groups, which have nice algorithmic and geometric properties. For example, the strong contracting properties of Grigorchuk groups were used to prove that they have

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intermediate growth and as for today it is essentially the only known method to get upper estimates on the growth function of a group. A large class of contracting self-similar groups is represented by iterated monodromy groups of sub-hyperbolic rational functions [Nek05] which stand as an important link between combinatorial group theory and complex holomorphic dynamics. Contracting property is essential in computing L-presentations of groups, to which we devote a chapter of this dissertation.

Another important class of automata groups is the class of groups with branch structure. Branch groups arise as one of the three [Gri00] possible types of just-infinite groups (infinite groups for which all proper homomorphic images are finite). They also provide examples of minimal groups [GW04]. Class of branch groups contains groups with many extraordinary properties mentioned above, in particular the first Grigorchuk group $\mathcal{G}$ belongs to the class.

An important property of known so far branch groups is that they are not finitely presented. It is proved for regular branch groups, and it is believed to be true for all branch groups. On the other hand, it is often (always for regular branch groups) possible to find a recursive infinite presentation for such groups (L-presentations and endomorphic presentations) [BGv03]. We find L-presentations for several important groups and series of groups, and we study the conditions when it is possible to find such a presentation from the structure of the automaton which generates a group.

The second part of the dissertation considers the problem of classification of groups generated by 3-state automata over the alphabet of 2 letters and 2-state automata over the 3-letter alphabet. We continue the classification work started by the research group at Texas A&M University ([BGK$^+$07a, BGK$^+$07b]) and lay a foundation for analogous study of the groups generated by 2-state automata over the 3-letter alphabet. In our classification work we extensively used the *AutomGrp* computer algebra package developed at Texas A&M University [MS]. This package provides basic facilities for computations in groups generated by finite automata, as well as implementation of several important algorithms such as word problem algorithm and detecting whether the group is contracting. The package has been deposited to the repository of packages for the computer algebra system *GAP* [GAP], and is freely available for download at http://www.gap-system.org.

We refer the reader to [GNS00] for basic definitions and facts about automata groups.

CHAPTER II

L-PRESENTATIONS

As it was mentioned, many important examples of automata groups are not finitely presented, yet they have a finitely-defined recursive presentation, called *L-presentation*.

Definition II.1. *L-presentation* of a group G is a presentation $G = \langle X|R\rangle$ where the set of defining relations R can be written as

$$R = \bigcup_{n=0}^{\infty} \phi^n(R_0),$$

for some endomorphism ϕ of the free group $F(X)$ and a finite set of the base relations $R_0 \subset F(Z)$.

First example of an L-presentation was that of the Grigorchuk group found by I. Lysionok in 1985.

Theorem II.2 (Lysionok [Lys85])**.** *Grigorchuk group $\mathcal{G}$ has the presentation*

$$\mathcal{G} = \left\langle a, b, c, d \,\middle|\, \begin{array}{l} a^2 = b^2 = c^2 = d^2 = bcd = 1, \\ w_n^4 = (w_n w_{n+1})^4 = 1, n \geq 0 \end{array} \right\rangle, \tag{2.1}$$

where $(w_n)_{n\geq 0}$ is the sequence of words defined inductively by $w_0 = ad$ and $w_{n+1} = \phi(w_n)$, and the mapping ϕ is defined by

$$\phi(a) = aca, \phi(b) = d, \phi(c) = b, \phi(d) = c.$$

Remark II.3. The presentation (2.1) is not an L-presentation in the sense of the definition II.1 but it is easy to see that its set of defining relations coincides (after cyclical reduction) with that of the following L-presentation:

$$\mathcal{G} \cong \left\langle a, b, c, d | \phi^n(a^2), \phi^n(bcd), \phi^n((ad)^4), \phi^n((adacac)^4), n \geq 0 \right\rangle. \tag{2.2}$$

The definition II.1 could be naturally generalized to include cases similar to (2.1) but it is not necessary for our purposes.

Later R.I. Grigorchuk used this presentation to construct the example of a finitely presented amenable but not elementary amenable group ([Gri98]). Thus L-presentations, in addition to helping describe the algebraic structure of the group, allow us to construct an embedding of the group into a finitely presented group.

Another example of an L-presentation is that of the iterated monodromy group of the polynomial $z^2 + i$.

Theorem II.4 (Grigorchuk, Savchuk, Sunik [GSŠ07])**.** *Iterated monodromy group of the polynomial $z^2 + i$ has the L-presentation*

$$G = \left\langle a, b, c \;\middle|\; \begin{array}{c} \phi^n(a^2), \phi^n((ac)^4), \phi^n([c, ab]^2), \phi^n([c, bab]^2), \\ \phi^n([c, ababa]^2), \phi^n([c, ababab]^2), \phi^n([c, bababab]^2), n \geq 0 \end{array} \right\rangle,$$

where the mapping ϕ is defined by

$$\phi(a) = b, \phi(b) = c, \phi(c) = aba.$$

Natural question is: when is it possible to find and how to find a presentation of the group generated by the given finite automaton, and when does the group generated by the finite automaton have an L-presentation? The groups mentioned above are contracting, which is a sufficient condition for the branch word problem algorithm to work. On the other hand, there are known examples of L-presentations for non-contracting groups, for example the Lamplighter group, which possesses a weaker property: an element which represents identity transformation necessarily has 1 as its states at some level of the tree. We study other examples of automata for which the branch word problem algorithm works and find the conditions under which it is possible to obtain an L-presentation for the group using word substitutions which arise from the automaton structure.

A. Definitions and notations

Let $G = \langle a_1, \dots, a_n \rangle$ be a self-similar group of automorphisms of the tree $\mathcal{T}_d$ generated by the automaton

$$\mathcal{A} : \begin{cases} a_1 & = & (w_{11}, \dots, w_{1d})\pi_1, \\ & \dots & \\ a_n & = & (w_{n1}, \dots, w_{nd})\pi_n, \end{cases} \tag{2.3}$$

where $\pi_i \in S_d, i = 1, \dots, n$, $a_{ij} \in \{a_1, \dots, a_n\}$, and w_{ij} are group words over the alphabet $\{a_1, \dots, a_n\}$. Automaton $\mathcal{A}$ naturally induces an action of the free group $F = F(a_1, \dots, a_n)$ on the tree $\mathcal{T}_d$, as well as the homomorphism $\Psi_{\mathcal{A}}$ from the group F to the wreath product $F \wr S_d$ defined by

$$\Psi : \begin{cases} a_1 & \to & (w_{11}, \dots, w_{1d})\pi_1, \\ & \dots & \\ a_n & \to & (w_{n1}, \dots, w_{nd})\pi_n. \end{cases} \tag{2.4}$$

We will call this homomorphism the *wreath recursion* corresponding to the automaton $\mathcal{A}$. It is easy to see that the wreath recursion $\Psi_{\mathcal{A}}$ agrees with the canonical projection P of the group F onto the group G, i.e. the following holds.

Proposition II.5. *Let $P : F \ni a_i \to a_i \in G$ be the canonical projection of the group F onto the group G, and $P' : F \wr S_d \to G \wr S_d$ be the homomorphism induced by the projection P. Then the following is a commutative diagram of group homomorphisms*

$$\begin{array}{ccc} F & \xrightarrow{\Psi_{\mathcal{A}}} & F \wr S_d \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\ G & \xrightarrow[\psi]{} & G \wr S_d \end{array} \tag{2.5}$$

where $\psi : G \ni a_i \to (w_{i1}, \dots, w_{id})\pi_i \in G \wr S_d$ *is the embedding induced by the automaton* $\mathcal{A}$.

Proof. Obviously $a_i^{P\psi} = a_i^{\Psi_{\mathcal{A}}P'}$, therefore homomorphisms $P\psi$ and $\Psi_{\mathcal{A}}P'$ coincide since F is a free group. $\square$

The notion of a wreath recursion may be extended to certain intermediate factor groups of the group F of which G is a factor group. Let Ω be the kernel of the homomorphism P, and R be a normal subgroup of the group F such that $R \lhd \Omega \lhd F$. Then we have the group $\Gamma = F/R$ and a pair of the canonical projections $P_{F,\Gamma} : F \to \Gamma$, $P_{\Gamma,G} : \Gamma \to G$ which make the following diagram commutative.

$$\begin{array}{ccc} F & \xrightarrow{P} & G \\ {\scriptstyle P_{F,\Gamma}}\downarrow & \nearrow{\scriptstyle P_{\Gamma,G}} & \\ \Gamma & & \end{array}$$

Definition II.6. We call such a group Γ a *covering group* for the group G if the wreath recursion (2.3) is well-defined for Γ, i.e. if the map $\Psi : \{a_1, \dots, a_n\} \to \Gamma \wr S_d$ defined on the generators $\{a_1, \dots, a_n\}$ of the group Γ by the rules (2.4)

$$\Psi : \begin{cases} a_1 & \to & (w_{11}, \dots, w_{1d})\pi_1, \\ & \dots & \\ a_n & \to & (w_{n1}, \dots, w_{nd})\pi_n. \end{cases}$$

extends to a homomorphism $\Gamma \to \Gamma \wr S_d$ which makes the following diagram commutative,

$$\begin{array}{ccccc} F & \xrightarrow{P_{F,\Gamma}} & \Gamma & \xrightarrow{P_{\Gamma,G}} & G \\ {\scriptstyle \Psi_{\mathcal{A}}}\downarrow & & {\scriptstyle \Psi}\downarrow & & {\scriptstyle \psi}\downarrow \\ F \wr S_d & \xrightarrow[P'_{F,\Gamma}]{} & \Gamma \wr S_d & \xrightarrow[P'_{\Gamma,G}]{} & G \wr S_d \end{array}$$

where $P'_{F,\Gamma}$, $P'_{\Gamma,G}$ are homomorphisms induced by $P_{F,\Gamma}$, $P_{\Gamma,G}$ respectively. $P'_{F,\Gamma}P'_{\Gamma,G} = P'$.

The map Ψ in this case is called the *wreath recursion* for the group Γ. We call the pair (Γ, Ψ) a wreath recursion as well.

Remark II.7. The condition of the definition II.6 is equivalent to

$$\Psi_{\mathcal{A}}(R) < R \times \cdots \times R.$$

Obviously, the whole free group F is a covering group for G ($R = 1$), and G is a covering group for itself ($R = \Omega$).

Remark II.8. If $\Gamma = F/R$ is a finitely presented group then to check whether it is a covering group for G it is sufficient to check that $\Psi_{\mathcal{A}}(r) \in R \times \cdots \times R$ for every defining relation r of the group Γ.

Map Ψ induces a homomorphism $\Gamma \wr S_d \to (\Gamma \wr S_d) \wr S_d = \Gamma \wr S_d \wr S_d$ which in turn induces a homomorphism $\Psi_2 : \Gamma \to \Gamma \wr S_d \wr S_d$, and we obtain a sequence of homomorphisms

$$\Psi_k : \Gamma \to \Gamma \wr \underbrace{S_d \wr \cdots \wr S_d}_{k}, k \geq 1,$$

with the kernels

$$\Omega_k = \ker \Psi_k.$$

Obviously, $\Omega_k \subset \Omega_{k+1}$ and we can consider the limit of this sequence

$$\Omega_\infty = \bigcup_{k \geq 1} \Omega_k.$$

The group $\Gamma_\infty = \Gamma/\Omega_\infty$ is again a covering group for the group G, and it does or does not coincide with the group G, which depends on whether the wreath recursion (Γ, Ψ) has property BA, which is a weaker analog of the contracting property [GNS00].

Definition II.9. A wreath recursion (Γ, Ψ) has property BA if the following condition holds: for any $w \in F$ such that $P(w) = 1$ there exists integer k such that

$\Psi_k(P_{F,\Gamma}(w)) = (1, 1, \dots, 1)$.

Example II.10. Lamplighter group [GŻ01]. Let L be the *Lamplighter group* generated by the automaton

$$\begin{aligned} a &= (a, b)\sigma, \\ b &= (a, b), \end{aligned} \tag{2.6}$$

and $F = F(a, b)$ be the free two-generated group. Then the wreath recursion $\Psi_L : F \to F \wr S_2$ induced by (2.6) has property BA.

It is easy to see that if Γ has property BA then

$$\Omega = \left\langle R, P_{F,\Gamma}^{-1}(\Omega_\infty) \right\rangle,$$

i.e. the set of defining relations of the group G can be obtained as the union of the set of the defining relations of the group Γ and the set of the elements which generate the groups Ω_k. In other words, if $R = \langle r_i, i \in I \rangle^F$, $\Omega_k = \langle u_{kj}, j \in J_k \rangle^\Gamma$ then

$$G = \left\langle a_1, \dots, a_n | r_i, i \in I, u'_{kj}, k \geq 1, j \in J_k \right\rangle,$$

where u'_{kj} is a representative of the set $P_{F,\Gamma}^{-1}(u_{kj})$. Therefore, to find a presentation of the group G, it is enough to find a suitable covering group Γ with property BA and the generators of the groups Ω_k.

The group Γ acts on the tree $\mathcal{T}_d$ (maybe not faithfully), therefore it is natural to consider stabilizers of levels and vertices in the group Γ and the wreath products $\Gamma \wr S_d \wr \dots \wr S_d$. We have

$$\Omega_1 = \{ g \in \Gamma | \Psi(g) = (1, \dots, 1) \},$$

therefore Ω_1 coincides with the kernel of the restriction Ψ_H of the homomorphism Ψ

onto the subgroup

$$H = St_{\Gamma}(1) = \Psi^{-1}(St_{\Gamma \wr S_d}(1)) = \{g \in \Gamma | \Psi(g) = (g_1, \ldots, g_d)\},$$

and since $St_{\Gamma \wr S_d}(1)$ is isomorphic to the direct product Γ^d, Ψ_H may be viewed as a homomorphism from H to Γ^d, and the kernel of Ψ_H coincides with Ω_1.

If the group G acts on the binary tree, then $\Psi_H(H)$ is a subgroup of the direct product of two copies of the free group F_n. In our study an important role is played by *Mihailova normal form* for such subgroups [Mih58]:

Definition II.11. Let F_n be the free group with free generators $\{a_1, \ldots, a_n\}$, and H be a subgroup of the direct product $F_n \times F_n$. A *Mihailova normal form* of the group H is a system of generators $\{s_1, \ldots, s_n, u_1, \ldots, u_m\}$ of H such that

$$\begin{array}{rcl} s_1 & = & (a_1, \alpha(a_1)), \\ & \ldots & \\ s_n & = & (a_n, \alpha(a_n)), \\ u_1 & = & (1, v_1), \\ & \ldots & \\ u_m & = & (1, v_m), \end{array} \tag{2.7}$$

where $m \geq 0$, $v_i \in F_n \setminus \{1\}, i = 1, \ldots, m$, and α is an automorphism of the group F_n.

Obviously not every subgroup of $F_n \times F_n$ has a Mihailova normal form, a necessary condition for that is that the images of the canonical projections of the group H onto the components of the product $F_n \times F_n$ coincide with the group F_n. On the other hand, Mihailova normal form may not be unique when it exists, even modulo automorphisms of the free group F_n.

Even though the classic definition of Mihailova normal form concerns only free groups, it can be useful for any group G, and we will use the following natural definition:

Definition II.12. Let G be a group with non-trivial generators $\{a_1, \ldots, a_n\}$, and H be a subgroup of the direct product $G \times G$. A *Mihailova normal form* of the group H is a system of generators $\{s_1, \ldots, s_n, u_1, \ldots, u_m\}$ of H such that

$$\begin{aligned} s_1 &= (a_1, \alpha(a_1)), \\ &\ldots \\ s_n &= (a_n, \alpha(a_n)), \\ u_1 &= (1, v_1), \\ &\ldots \\ u_m &= (1, v_m), \end{aligned} \tag{2.8}$$

where $m \geq 0$, $v_i \in G \setminus \{1\}$, $i = 1, \ldots, m$, and α is an automorphism of the group G.

Remark II.13. Let F_n and H be as in the definition II.11, $G = \langle a_1, \ldots, a_n \rangle$, and H' be the image of the group H under the canonical homomorphism $F_n \times F_n \to G \times G$. If the automorphism α from (2.7) induces an automorphism of the group G then the Mihailova normal form (2.7) corresponds to the Mihailova normal form (2.8) of the group H'.

Definition II.14. *Diagonal* of a direct product of two copies of a group G is the set

$$\Delta(G \times G) = \{(g, g) | g \in G\}.$$

If α is an automorphism of the group G then the α*-diagonal* is the set

$$\Delta_\alpha(G \times G) = \{(g, \alpha(g)) | g \in G\}.$$

Remark II.15. Elements $s_1, \ldots, s_n$ in (2.8) generate the α-diagonal $\Delta_\alpha(G \times G) \cong G$, and if $m = 0$ then $H \cong G$.

B. Essentially free groups

Let $G = \langle a_1, \dots, a_n \rangle$ be the self-similar group generated by the automaton (2.3), F be the free group with generators $a_1, \dots, a_n$, and $H = St_{(F,\Psi)}(1)$. Suppose that the group H is generated by the set $\{s_1, \dots, s_n, u_1, \dots, u_m\}$ such that

$$\begin{array}{rcl} \Psi(s_1) & = & (a_1, \alpha(a_1)), \\ & \dots & \\ \Psi(s_n) & = & (a_n, \alpha(a_n)), \\ \Psi(u_1) & = & (1, 1), \\ & \dots & \\ \Psi(u_m) & = & (1, 1), \end{array} \tag{2.9}$$

where α is an automorphism of the group F. Then

$$\Omega_1 = \langle u_1, \dots, u_m \rangle^F. \tag{2.10}$$

Let ϕ be the endomorphism of the group F defined by

$$\phi : \begin{cases} a_1 & \to & s_1, \\ & \dots & \\ a_n & \to & s_n, \end{cases} \tag{2.11}$$

H is obviously invariant under ϕ and for any $g \in H$ we have

$$\Psi(\phi(g)) = (g, \alpha(g)). \tag{2.12}$$

Lemma II.16. *If the subgroups Ω_k are invariant under the action of the automorphism α then*

$$\Omega_{k+1} = \langle \Omega_k, \phi(\Omega_k) \rangle^F = \left\langle \phi^i(u_j), i = 0, 1, \dots, k, j = 1, \dots, m \right\rangle^F.$$

Proof. Induction on k. Let $k \geq 1$. From (2.12) it follows that $\phi(\Omega_k) \subset \Omega_k \times \Omega_k \subset \Omega_{k+1}$, and hence $\Omega_{k+1} \supset \langle \Omega_k, \phi(\Omega_k) \rangle^F$. On the other hand, if $g =\in \Omega_{k+1}$ then $\Psi(g) = (g_1, g_2)$ for some $g_1, g_2 \in \Omega_k$. $g \in H$, therefore g is a product of the elements s_i modulo Ω_1 and hence $\Psi(g) = (g_1, \alpha(g_1))$ and $g = \phi(g_1)$ modulo Ω_1, so $g \in \langle \phi(\Omega_k), \Omega_1 \rangle^F$. □

Subgroups Ω_k are normal in the group F, hence the condition of the Lemma II.16 is equivalent to the following:

$$\forall k > 0 : \alpha(\phi^{k-1}(u_i)) \in \Omega_k. \tag{2.13}$$

Therefore, the following proposition holds.

Proposition II.17. *Suppose that for the group G the wreath recursion (F, Ψ) has the property BA, and it satisfies (2.9) and (2.13). Then the mapping ϕ from (2.11) provides an L-presentation for the group G:*

$$G = \left\langle a_1, \ldots, a_n | \phi^k(u_i), k \geq 0, i = 1, \ldots, m \right\rangle.$$

Proof. From above it follows that $\left\langle \phi^k(u_i), k \geq 0, i = 1, \ldots, m \right\rangle^F = \Omega_\infty$, and since (F, Ψ) has property BA, we have $\Omega_\infty = \Omega$, i.e. $\left\{ \phi^k(u_i), k \geq 0, i = 1, \ldots, m \right\}$ is a set of defining relations of the group G. □

Proving inclusions (2.13) requires calculations in normal subgroups of a free group, which are hard in general. Instead, it may be enough to perform calculations in the group G itself. For the substitution ϕ to work, it is necessary that $\phi^k(u_i) = \alpha(\phi^k(u_i)) = 1$ in the group G. This property is essential:

Theorem II.18. *Suppose $\phi^k(u_i) = \alpha(\phi^k(u_i)) = 1$ in the group G, and let Γ_∞ be the group defined by generators and relations*

$$\Gamma_\infty = \left\langle a_1, \ldots, a_n | \phi^k(u_i) = \alpha(\phi^k(u_i)) = 1, k \geq 0, i = 1, \ldots, m \right\rangle.$$

Then

(i) *Group* Γ_∞ *is a covering group for the group* G*, i.e. the wreath recursion (2.4) induces a homomorphism* $\Gamma_\infty \to \Gamma_\infty \wr S_2$.

(ii) *If* Γ_∞ *has the property* BA *then* $\Gamma_\infty \cong G$.

Proof. (i) follows from the definition of Γ_∞ and (2.13). $\Gamma_\infty = F/\Omega_\infty$, therefore if Γ_∞ has the property BA then $\Omega_\infty = \Omega$ and $\Gamma_\infty = G$. □

By definition if Γ is a covering group for the group G (in particular if Γ is F) and it has property BA then the group Γ_∞ also has property BA and the condition (ii) of the theorem II.18 is satisfied, therefore we need not actually perform any calculations (which would be very hard, if not impossible) in the group Γ_∞ to use the theorem II.18 and the following holds.

Theorem II.19. *Suppose that the following conditions hold for the group* G *and the covering group* Γ.

(i) $\phi^k(u_i) = \alpha(\phi^k(u_i)) = 1$ *in the group* G,

(ii) $R \subset \Omega_\infty$, *where* R *is the kernel of the canonical homomorphism of the group* F *onto the group* Γ,

(iii) Γ *has the property* BA.

Then G *has the L-presentation*

$$G \cong \left\langle a_1, \dots, a_n | \phi^k(u_i) = 1, i = 1, \dots, m, k \geq 0 \right\rangle, \tag{2.14}$$

and α *induces an automorphism of the group* G.

Proof. Follows from theorem II.18. Condition (ii) ensures that the group Γ covers Γ_∞ and hence Γ_∞ has property BA. □

C. Grigorchuk groups G_ω

Let us consider the family of Grigorchuk groups G_ω [Gri83]. For every sequence $\omega = (\omega_1, \omega_2, \ldots) \in \{0,1,2\}^{\mathbb{N}}$ the group G_ω is generated by the tree automorphisms a_ω, b_ω, c_ω, d_ω which are defined recursively as follows:

$$\begin{aligned} a_\omega &= (1,1)\sigma, \\ b_\omega &= (a_{\omega'}^{\varepsilon_1}, b_{\omega'}), \\ c_\omega &= (a_{\omega'}^{\varepsilon_2}, c_{\omega'}), \\ d_\omega &= (a_{\omega'}^{\varepsilon_3}, d_{\omega'}), \end{aligned} \tag{2.15}$$

where $\omega' = (\omega_2, \omega_3, \ldots)$ is the shift of the sequence ω, and $\varepsilon_i \in \{0,1\}$ depend on ω_1:

$$\begin{aligned} (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1,1,0) \quad &\text{if} \quad \omega_1 = 0, \\ (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1,0,1) \quad &\text{if} \quad \omega_1 = 1, \\ (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0,1,1) \quad &\text{if} \quad \omega_1 = 2. \end{aligned}$$

Remark II.20. The first digit in the sequence ω corresponds to the letter x from b, c, d such that $x_\omega = (1, x_{\omega'})$, and we can identify digits $0, 1, 2$ with the letters d, c, b (in this order). We will use letters b, c, d as elements of the sequences ω when it is convenient, for instance we may say $\omega = dddd\ldots$ instead of $\omega = 0000\ldots$.

It is easy to see that the automorphism $a = a_\omega = \sigma$ does not depend on ω, so

$$\begin{array}{ccccccccc} & & & a_{x\omega} & = & (1,1)\sigma, & & & \\ b_{d\omega} & = & (a, b_\omega), & b_{c\omega} & = & (a, b_\omega), & b_{b\omega} & = & (1, b_\omega), \\ c_{d\omega} & = & (a, c_\omega), & c_{c\omega} & = & (1, c_\omega), & c_{b\omega} & = & (a, c_\omega), \\ d_{d\omega} & = & (1, d_\omega), & d_{c\omega} & = & (a, d_\omega), & d_{b\omega} & = & (a, d_\omega). \end{array} \tag{2.16}$$

We may write it in the following way:

$$\begin{aligned} a_{x\omega} &= (1,1)\sigma \\ x_{x\omega} &= (1, x_\omega), \\ y_{x\omega} &= (a, y_\omega), \\ z_{x\omega} &= (a, z_\omega), \end{aligned}$$

where $\{x, y, z\} = \{b, c, d\}$.

Obviously, $a_\omega^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = 1$ for any ω. It is also easy to see that $b_\omega c_\omega = d_\omega$. If the sequence ω is constant, e.g. if $\omega = 000\ldots$, then

$$\begin{aligned} &b_\omega = (a, b_\omega), \\ &c_\omega = (a, c_\omega) = b_\omega, \\ &d_\omega = (1, d_\omega) = 1, \end{aligned}$$

in which case the group $G_\omega = \langle a, b\rangle \cong D_\infty$, and groups $G_{\omega_1\omega_2\ldots\omega_k\omega}$ are subgroups of a wreath product $D_\infty \wr S_2 \wr \cdots \wr S_2$, which is a finite extension of a direct product of multiple copies of the group D_∞. We will not consider the case of almost constant sequences ω in the sequel, i.e. we will assume that for any $n \in \mathbb{N}$ there is $n' > n$ such that $\omega_{n'} \neq \omega_n$.

Assuming the sequence ω is not almost constant, we have that elements b_ω, c_ω, d_ω are of order 2, and they generate elementary 2-group of order 4. Moreover, wreath recursion (2.16) implies that $(ax_\omega)^4 = 1$ where $x \in \{b, c, d\}$ is such that $x_\omega = (1, x_{\omega'})$, and there is $y \in \{b, c, d\} \setminus \{x\}$ such that $4 < |ay_\omega| = 2^k < \infty$, $|az_\omega| > |ay_\omega|$, $z \in \{b, c, d\} \setminus \{x, y\}$, y corresponds to the first ω_k not equal to ω_1. In other words, if $\omega = \underbrace{xx\ldots x}_{n} y\ldots$ for some $\{x, y, z\} = \{b, c, d\}$ then

$$\Gamma_\omega = \left\langle a, x, y, z | a^2 = x^2 = y^2 = z^2 = xyz = 1, (ax)^4 = 1, (ay)^{2^{n+1}} = 1 \right\rangle. \tag{2.17}$$

az_ω in this case has finite order if and only if ω contains a letter corresponding to z, otherwise we obtain $z_\omega = (a, z_\omega)$ and $|az_\omega| = \infty$. Note that az always have infinite order in the group Γ_ω, regardless of the order of the element az_ω in G_ω. (We could add the relator $(az)^{|az_\omega|}$ to (2.17) but it would only make calculations more complicated).

Proposition II.21. *Let* $\omega \in \{b, c, d\}^{\mathbb{N}}$ *is not almost constant. Then for some* $\{x, y, z\} = \{b, c, d\}$ *and* $n > 2$ *exactly one of the following holds.*

(i) $\omega = \underbrace{xx\ldots xy}_{n}\ldots$. *Then*

$$\Gamma_\omega = \left\langle a, x, y, z | a^2 = x^2 = y^2 = z^2 = xyz = (ax)^4 = (ay)^{2^{n+1}} = 1 \right\rangle,$$

$$\Gamma_{\omega'} = \left\langle \bar{a}, \bar{x}, \bar{y}, \bar{z} | \bar{a}^2 = \bar{x}^2 = \bar{y}^2 = \bar{z}^2 = \bar{x}\bar{y}\bar{z} = (\bar{a}\bar{x})^4 = (\bar{a}\bar{y})^{2^n} = 1 \right\rangle,$$

(ii) $\omega = \underbrace{xy\ldots yx}_{n}\ldots$. *Then*

$$\Gamma_\omega = \left\langle a, x, y, z | a^2 = x^2 = y^2 = z^2 = xyz = (ax)^4 = (ay)^8 = 1 \right\rangle,$$

$$\Gamma_{\omega'} = \left\langle \bar{a}, \bar{x}, \bar{y}, \bar{z} | \bar{a}^2 = \bar{x}^2 = \bar{y}^2 = \bar{z}^2 = \bar{x}\bar{y}\bar{z} = (\bar{a}\bar{y})^4 = (\bar{a}\bar{x})^{2^n} = 1 \right\rangle,$$

(iii) $\omega = \underbrace{xy\ldots yz}_{n}\ldots$. *Then*

$$\Gamma_\omega = \left\langle a, x, y, z | a^2 = x^2 = y^2 = z^2 = xyz = (ax)^4 = (ay)^8 = 1 \right\rangle,$$

$$\Gamma_{\omega'} = \left\langle \bar{a}, \bar{x}, \bar{y}, \bar{z} | \bar{a}^2 = \bar{x}^2 = \bar{y}^2 = \bar{z}^2 = \bar{x}\bar{y}\bar{z} = (\bar{a}\bar{y})^4 = (\bar{a}\bar{z})^{2^n} = 1 \right\rangle.$$

Map Ψ_ω *from the set of generators of* Γ_ω *to* $\Gamma_{\omega'} \wr S_2$ *defined by*

$$\Psi_\omega(a) = (1, 1)\sigma,$$
$$\Psi_\omega(x) = (1, \bar{x}),$$
$$\Psi_\omega(y) = (\bar{a}, \bar{y}),$$
$$\Psi_\omega(z) = (\bar{a}, \bar{z}),$$

extends to a homomorphism $\Psi_\omega : \Gamma_\omega \to \Gamma_{\omega'} \wr S_2$.

Proof. The first part follows directly from the definition of the groups Γ_ω. To prove that Ψ_ω is a homomorphism we need to show that the defining relators in the group Γ_ω are mapped to the trivial element of the group $\Gamma_{\omega'}$.

In all three cases we have

$$\Psi_\omega(a)^2 = ((1,1)\sigma)^2 = (1,1),$$

$$\Psi_\omega(x)^2 = (1, \bar{x}^2) = (1,1),$$

$$\Psi_\omega(y)^2 = (\bar{a}^2, \bar{y}^2) = (1,1),$$

$$\Psi_\omega(z)^2 = (\bar{a}^2, \bar{z}^2) = (1,1),$$

$$\Psi_\omega(x)\Psi_\omega(y)\Psi_\omega(z) = (\bar{a}^2, \bar{x}\bar{y}\bar{z}) = (1,1),$$

$$(\Psi_\omega(a)\Psi_\omega(x))^4 = (\sigma(1,\bar{x}))^4 = (\bar{x}^2, \bar{x}^2) = (1,1).$$

For the last relator we have

(i) $(\Psi_\omega(a)\Psi_\omega(y))^{2^{n+1}} = (\sigma(\bar{a},\bar{y}))^{2^{n+1}} = ((\bar{a}\bar{y})^{2^n}, (\bar{y}\bar{a})^{2^n}) = (1,1)$,

(ii) $(\Psi_\omega(a)\Psi_\omega(y))^8 = (\sigma(\bar{a},\bar{y}))^8 = ((\bar{a}\bar{y})^4, (\bar{y}\bar{a})^4) = (1,1)$,

(iii) $(\Psi_\omega(a)\Psi_\omega(y))^8 = (\sigma(\bar{a},\bar{y}))^8 = ((\bar{a}\bar{y})^4, (\bar{y}\bar{a})^4) = (1,1)$.

□

We will find recursive presentations for G_ω using approach similar to that used in the section D. First we will find subgroups $\Omega^\omega = \ker \Psi^\omega$, then we will describe a substitution ϕ_ω which lifts Ω_1^ω to $\Omega_2^{\omega'}$, and then we will prove that obtained Ω_k^ω indeed provide us with a representation for the group G_ω. The sets Ω_k^ω and substitutions ϕ_ω will be defined recursively via $\Omega_k^{\omega'}$, and we will not be able to obtain explicit representations for groups G_ω in general (which is clear given that the sequence ω

may not be possible to describe), but we will obtain an L-presentation for the case of periodic sequences ω, as well as we will have a way to obtain those for the case of almost periodic sequences.

1. Base relations: Ω_1

Without loss of generality, let us assume that ω starts with one or more zeroes and then a two:

$$\omega = 0\omega' = 0\dots02\dots. \tag{2.18}$$

Let us denote $a_\omega, b_\omega, c_\omega, d_\omega$ by a, b, c, d, and $b_{\omega'}$, $c_{\omega'}$, $d_{\omega'}$ by $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ respectively. Also, let $\bar{G} = G_{\omega'} = \left\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} \right\rangle$, $\Gamma = \Gamma_\omega$, $\bar{\Gamma} = \Gamma_{\omega'}$, and let $\Psi = \Psi_\omega$ and $\bar{\Psi} = \Psi_{\omega'}$.

Let us find $\Omega_1 = \ker\Psi$. As in the section D we have

$$\ker\Psi < H = \langle b, c, d, aba, aca, ada \rangle < \Gamma,$$

and

$$\Psi(H) \cong \left\langle (\bar{a}, \bar{b}), (\bar{a}, \bar{c}), (1, \bar{d}), (\bar{b}, \bar{a}), (\bar{c}, \bar{a}), (\bar{d}, 1) \right\rangle < \bar{\Gamma} \times \bar{\Gamma}.$$

We have

$$K = D \times D < \Psi(H),$$

where $D = \left\langle \bar{d} \right\rangle^{\bar{\Gamma}}$, and $\Psi(H)$ is isomorphic to the semidirect product

$$\Psi(H) \cong A \ltimes K, \tag{2.19}$$

where $A = \left\langle (\bar{a}, \bar{b}), (\bar{b}, \bar{a}) \right\rangle = \langle (\bar{a}, \bar{c}), (\bar{c}, \bar{a}) \rangle$ is a dihedral group $D_{\min\left\{|\bar{a}\bar{b}|, |\bar{a}\bar{c}|\right\}}$ (at least one of $|\bar{a}\bar{b}|, |\bar{a}\bar{c}|$ is not ∞). According to (2.10), Ω_1 is generated as a normal subgroup of the group Γ by the preimages of the defining relators of the group $\Psi(H)$, so we need to find a presentation for the group $\Psi(H)$, and for that we need to find a presentation for the group D.

By assumption (2.18), we have $|\bar{a}\bar{b}| < |\bar{a}\bar{c}|$, in particular $4 \leq |\bar{a}\bar{b}| < \infty$. Then

$$A = \left\langle (\bar{a}, \bar{b}), (\bar{b}, \bar{a}) \right\rangle \cong D_{2^n},$$

$$D = \left\langle \bar{d} \right\rangle^{\bar{\Gamma}} = \left\langle \bar{d}^x | x \in \bar{\Gamma} \right\rangle = \left\langle \bar{d}^x | x \in \left\langle \bar{a}, \bar{b} \right\rangle \right\rangle$$

$|\bar{a}\bar{b}| = 2^n$ for some $n \geq 2$ and $\bar{b}\bar{d} = \bar{d}\bar{b}$, therefore

$$D = \langle \bar{d}^x | x \in \{t_1 = 1, t_2 = \bar{a}, t_3 = \bar{a}\bar{b}, t_4 = \bar{a}\bar{b}\bar{a}, t_5 = \bar{a}\bar{b}\bar{a}\bar{b}, \ldots, t_{2^n} = \underbrace{\bar{a}\bar{b}\ldots\bar{a}\bar{b}\bar{a}}_{2^n - 1}\} \rangle.$$

Let

$$\xi_i = \bar{d}^{t_i}, i = 1, \ldots, 2^n,$$

then the group D has representation

$$D = \left\langle \xi_1, \ldots, \xi_{2^n} | R_D, \xi_i^2, i = 1, \ldots, 2^n \right\rangle,$$

where R_D is some set of additional relations, which depends on the group $\bar{\Gamma}$.

$$K = D \times D = \left\langle \hat{\xi}_1, \ldots, \hat{\xi}_{2^n}, \tilde{\xi}_1, \ldots, \tilde{\xi}_{2^n} | \hat{\xi}_i^2 = \tilde{\xi}_j^2 = [\hat{\xi}_i, \tilde{\xi}_j] = \hat{r} = \tilde{r} = 1, 1 \leq i, j \leq 4, r \in R \right\rangle, \tag{2.20}$$

where

$$\hat{\xi}_i = (\xi_i, 1), \tilde{\xi}_j = (1, \xi_j),$$

$$\hat{r} = (r, 1) = r(\hat{\xi}_i), \tilde{r} = (1, r) = r(\tilde{\xi}_i) \text{ for } r = r(\xi_i) \in R_D.$$

Let $x = (\bar{a}, \bar{b}), y = (\bar{b}, \bar{a})$. We have

$$\xi_{2k-1}^{\bar{a}} = \xi_{2k}, k = 1, \dots, 2^{n-1}$$
$$\xi_{2k}^{\bar{b}} = \xi_{2k+1}, k = 1, \dots, 2^{n-1} - 1,$$
$$\xi_1^{\bar{b}} = \xi_1, \xi_{2^n}^{\bar{b}} = \xi_{2^n},$$
$$\hat{\xi}_i^x = (\xi_i^{\bar{a}}, 1), \hat{\xi}_i^y = (\xi_i^{\bar{b}}, 1),$$
$$\tilde{\xi}_i^y = (1, \xi_i^{\bar{a}}), \tilde{\xi}_i^x = (1, \xi_i^{\bar{b}}),$$

therefore x and y act as permutations

$$\begin{aligned} \pi &= (1,2)(3,4)\dots(2^n - 1, 2^n), \\ \rho &= (2,3)(4,5)\dots(2^n - 2, 2^n - 1) \end{aligned} \tag{2.21}$$

on the sets $\left\{\hat{\xi}_i\right\}$ and $\left\{\tilde{\xi}_i\right\}$:

$$\begin{aligned} \hat{\xi}_i^x &= \hat{\xi}_{\pi(i)}, \tilde{\xi}_i^x = \hat{\xi}_{\rho(i)}, \\ \hat{\xi}_i^y &= \hat{\xi}_{\rho(i)}, \tilde{\xi}_i^y = \hat{\xi}_{\pi(i)}, \end{aligned} \tag{2.22}$$

Therefore, defining relations of the group $\Psi(H)$ are those from (2.20) and (2.22). Equations (2.19), (2.20), (2.22) show that the group $\Psi(H)$ is generated by $\hat{\xi}_1, \tilde{\xi}_1, x, y$, and its defining relations are

$$R = \left\{x^2, y^2, (xy)^{2^n}, \hat{\xi}_1^2, \tilde{\xi}_1^2, [\hat{\xi}_i, \tilde{\xi}_j], \hat{r}, \tilde{r} | 1 \le i, j, \le 2^n, r \in R_D, z_1, z_2 \in \langle x, y \rangle \right\}.$$

Now, $\Omega_1 = \langle \Psi^{-1}(R) \rangle^{\Gamma}$. As a representative of the class $\Psi^{-1}(r), r \in R$ we can choose $\psi(r)$ where $\psi : F(x, y, \hat{\xi}_1, \tilde{\xi}_1) \to F = F(a, b, c, d)$ is defined by

$$\psi : \begin{cases} x & \to & b, \\ y & \to & b^a, \\ \hat{\xi}_1 & \to & d^a, \\ \tilde{\xi}_1 & \to & d = \psi(\hat{\xi}_1)^a, \end{cases} \tag{2.23}$$

(i.e. we choose simplest representatives according to (2.16)). It follows from (2.22), (2.21), (2.23) that

$$\{\psi(\hat{\xi}_i)|i = 1, \ldots, 2^n\} = \{d^{az}|z \in \langle b, b^a \rangle\},$$

and

$$\{\psi(\tilde{\xi}_i)|i = 1, \ldots, 2^n\} = \{d^{z}|z \in \langle b, b^a \rangle\}.$$

Hence, we have

$$\Omega_1 = \langle \psi(R) \rangle^\Gamma = \langle [d^{az_1}, d^{z_2}], \psi(\hat{r}), \psi(\tilde{r})|r \in R_D, z_1, z_2 \in \langle b, b^a \rangle \rangle^\Gamma. \tag{2.24}$$

Since z_1 and z_2 are independent and

$$[d^{az_1}, d^{z_2}] = [d^{az_1 z_2^{-1}}, d]^{z_2},$$
$$[d^{az}, d] = (z^{-1}adazd)^2 = [az, d]^2,$$

(2.24) can be written as

$$\Omega_1 = \left\langle [d, az]^2, \psi(\hat{r})|r \in R_D, z \in \langle b, b^a \rangle \right\rangle^{\Gamma_\omega}. \tag{2.25}$$

We have $(ab)^{2^{n+1}} = 1$, therefore group $\langle b, b^a \rangle$ is a subgroup of index 2 in the dihedral group $\langle a, b \rangle$ which consists of reduced words which contain an even number of letter a:

$$\langle b, b^a \rangle = \{1, b, aba, baba, abab, \ldots\},$$

so the set $a\langle b, b^a \rangle$ consists of all reduced words in $\langle a, b \rangle$ which contain an odd number of letter a:

$$a\langle b, b^a \rangle = \{a, ab, ba, bab, ababa, \ldots\}.$$

We also have that for any $z \in \Gamma$

$$[d, bz] = dd^{bz} = dd^z = [d, z],$$

$$[d, zb] = [d, z]^b,$$

and

$$[d, a]^2 = (da)^4 = 1,$$

therefore the set of relations of the form $[d, az]^2$ in (2.25) can be reduced to the set

$$U_1 = \left\{[d, a(ba)^{2k}]^2 = (da(ba)^{2k})^4, k = 1, \ldots, 2^{n-1} - 1\right\}.$$

Let us find the set R_D now. We have to consider two cases. One is when $(\bar{a}\bar{d})^4 = 1$ and another one is when $(\bar{a}\bar{b})^4 = 1$. These correspond to $\omega = 00\ldots$ and $\omega = 02\ldots$.

(I) $|\bar{a}\bar{d}| = 4$

We have $\omega = \underbrace{00\ldots02}_{n}\ldots$, $\bar{d} = (1, \tilde{d})$, $|\bar{a}\bar{b}| = 2^n$, $n > 2$, and

$$\Gamma = \left\langle a, b, c, d | a^2 = b^2 = c^2 = d^2 = bcd = (ad)^4 = (ab)^{2^{n+1}} = 1\right\rangle,$$

$$\bar{\Gamma} = \left\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} | \bar{a}^2 = \bar{b}^2 = \bar{c}^2 = \bar{d}^2 = \bar{b}\bar{c}\bar{d} = (\bar{a}\bar{d})^4 = (\bar{a}\bar{b})^{2^n} = 1\right\rangle.$$

$$D = \left\langle \bar{d}\right\rangle^{\bar{\Gamma}} = \left\langle \xi_i = \bar{d}^{\eta_i}, i = 1, \ldots, 2^n\right\rangle,$$

where

$$\eta_{2k-1} = (\bar{a}\bar{b})^k,$$

$$\eta_{2k} = (\bar{a}\bar{b})^k\bar{a}, k = 1, \ldots, 2^{n-1},$$

and its relations, apart from $\xi_i^2 = 1$, are

$$\begin{aligned} R_D = \{ \quad & (\xi_1\xi_3)^2, (\xi_3\xi_5)^2, \ldots, (\xi_{2^n-3}\xi_{2^n-1})^2, \\ & (\xi_2\xi_4)^2, (\xi_4\xi_6)^2, \ldots, (\xi_{2^n-2}\xi_{2^n})^2, \\ & (\xi_1\xi_2)^2, (\xi_{2^n-1}\xi_{2^n})^2\}. \end{aligned}$$

According to (2.21), (2.22),

$$\hat{\xi}_{2k+1} = \hat{\xi}^{xy}_{2k-1}, k = 1, \ldots, 2^{n-1} - 1,$$

$$\hat{\xi}_{2k+2} = \hat{\xi}^{yx}_{2k}, k = 1, \ldots, 2^{n-1} - 1,$$

therefore $\psi(R_D)$ gives us the following relations in the group $\Psi(H)$:

$$(\hat{\xi}_{2k-1}\hat{\xi}_{2k+1})^2 = (\hat{\xi}_1^{(xy)^{k-1}}\hat{\xi}_1^{(xy)^{k+1}})^2 \sim (\hat{\xi}_1\hat{\xi}_1^{xy})^2 \rightarrow (d^a d^{abb^a})^2 = (d^a d^{ababa})^2 \sim$$
$$(dd^{abab})^2 = (dd^{aba})^2 = (daba)^4,$$
$$(\hat{\xi}_{2k}\hat{\xi}_{2k+2})^2 = (\hat{\xi}_2^{(yx)^{k-1}}\hat{\xi}_2^{(yx)^{k+1}})^2 \sim (\hat{\xi}_2\hat{\xi}_2^{yx})^2 \rightarrow (d^{ab}a^{abb^a b})^2 = (d^{ab}d^{ababab})^2 \sim (dd^{abab})^2,$$
$$(\hat{\xi}_1\hat{\xi}_2)^2 = (\hat{\xi}_1\hat{\xi}_1^x) \rightarrow (d^a d^{ab})^2 \sim (dd^{aba})^2,$$
$$(\hat{\xi}_{2^n-1}\hat{\xi}_{2^n})^2 = (\hat{\xi}_{2^n-1}\hat{\xi}^x_{2^n-1})^2 = (\hat{\xi}_1^{(xy)^{2^{n-1}-1}}\hat{\xi}_1^{(xy)^{2^{n-1}-1}x})^2 \rightarrow (d^{a(bb^a)^{2^{n-1}-1}}d^{a(bb^a)^{2^{n-1}-1}b})^2 =$$
$$(d^{a(ba)^{2^n-2}}d^{a(ba)^{2^n-2}b})^2 = (d^{a(ba)^{2^n-2}}d^{(ab)^{2^n-1}})^2 \sim (dd^{(ab)^{2^n-1}(ab)^{2^n-2}a})^2 =$$
$$(dd^{(ab)^{2^{n+1}-3}a})^2 = (dd^{babab})^2 = (dd^{abab})^2 \sim (dd^{aba})^2,$$

and hence the formula (2.25) becomes

$$\Omega_1 = \left\langle (daba)^4, (da(ba)^{2k})^4, k = 1, \ldots, 2^{n-1} - 1 \right\rangle^{\Gamma}.$$

(II) $|\bar{a}\bar{d}| > 4$

In this case $|\bar{a}\bar{b}| = 4$, i.e. $\bar{b} = (1, \tilde{b})$, and formula (2.25) becomes

$$\Omega_1 = \left\langle (dababa)^4, \psi(\hat{r}) | r \in R_D \right\rangle^{\Gamma_\omega}.$$

We have two cases: $\bar{a}\bar{d}$ has finite order, i.e. $|\bar{a}\bar{d}| = 2^m$ for some $m > 2$, or $\bar{a}\bar{d}$ has infinite order.

(IIa) $|\bar{a}\bar{d}| < \infty$

We have $\omega = \underbrace{02\ldots20}_{m}$, $|\bar{a}\bar{d}| = 2^m$, $m > 2$, and

$$\Gamma = \left\langle a, b, c, d | a^2 = b^2 = c^2 = d^2 = bcd = (ab)^8 = (ad)^4 = 1 \right\rangle,$$

$$\bar{\Gamma} = \left\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} | \bar{a}^2 = \bar{b}^2 = \bar{c}^2 = \bar{d}^2 = \bar{b}\bar{c}\bar{d} = (\bar{a}\bar{b})^4 = (\bar{a}\bar{d})^{2^m} = 1 \right\rangle,$$

$$R_D = \left\{ (\xi_1\xi_2)^{2^{m-1}}, (\xi_1\xi_3)^{2^{m-1}}, (\xi_2\xi_4)^{2^{m-1}}, (\xi_3\xi_4)^{2^{m-1}} \right\}.$$

According to (2.21), (2.22), (2.23),

$$\begin{aligned}
\psi(\hat{\xi}_1\hat{\xi}_2) &= d^a d^{ab} \sim dd^{aba} = [d, aba] = (daba)^2, \\
\psi(\hat{\xi}_1\hat{\xi}_3) &= d^a d^{ababa} \sim dd^{abab} = [d, abab] \sim [d, aba], \\
\psi(\hat{\xi}_2\hat{\xi}_4) &= d^{ab} d^{ababab} \sim dd^{abab} \sim [d, aba], \\
\psi(\hat{\xi}_3\hat{\xi}_4) &= d^{ababa} d^{ababab} \sim dd^{abababababa} = dd^{babab} = [d, babab] \sim [d, baba] = [d, aba],
\end{aligned}$$

so

$$\Omega_1 = \left\langle (dababa)^4, (daba)^{2^m} \right\rangle^{\Gamma_\omega}.$$

(IIb) $|\bar{a}\bar{d}| = \infty$

Then $\omega = \underbrace{02\ldots21}_{m}$, $|\bar{a}\bar{c}| = 2^m$, $m > 2$,

$$\Gamma = \left\langle a, b, c, d | a^2 = b^2 = c^2 = d^2 = bcd = (ad)^4 = (ab)^8 = 1 \right\rangle,$$

$$\bar{\Gamma} = \left\langle \bar{a}, \bar{b}, \bar{c}, \bar{d} | \bar{a}^2 = \bar{b}^2 = \bar{c}^2 = \bar{d}^2 = \bar{b}\bar{c}\bar{d} = (\bar{a}\bar{b})^4 = (\bar{a}\bar{c})^{2^m} = 1 \right\rangle,$$

and

$$R_D = \left\{ (\xi_1\xi_2\xi_4\xi_3)^{2^{m-2}} \right\}.$$

According to (2.21), (2.22),

$$\psi(\xi_1\xi_2\xi_4\xi_3) = d^a d^{ab} d^{ababab} d^{ababa} = adabadabbababadababababababadababa = adabadbabadbababdababa \sim (caba)^4,$$

so

$$\Omega_1 = \left\langle (dababa)^4, (caba)^{2^m} \right\rangle^{\Gamma_\omega}.$$

Let us summarize the results of the calculations above.

Proposition II.22. *Let $\omega \in \{b,c,d\}^{\mathbb{N}}$ (see Proposition II.21). Then for some $\{x,y,z\} = \{b,c,d\}$ and $n > 2$ exactly one of the following holds.*

(i) $\omega = \underbrace{xx\ldots xy}_{n}\ldots$. *Then* $\Omega_1^\omega = \left\langle (xaya)^4, (xa(ya)^{2k})^4, k = 1,\ldots,2^{n-1}-1 \right\rangle^{\Gamma_\omega}$.

(ii) $\omega = \underbrace{xy\ldots yx}_{n}\ldots$. *Then* $\Omega_1^\omega = \left\langle (xayaya)^4, (xaya)^{2^n} \right\rangle^{\Gamma_\omega}$.

(iii) $\omega = \underbrace{xy\ldots yz}_{n}\ldots$. *Then* $\Omega_1^\omega = \left\langle (xayaya)^4, (zaya)^{2^n} \right\rangle^{\Gamma_\omega}$.

In all three cases the group Ω_1^ω contains element $(xayaya)^4$.

2. Substitutions ϕ_ω

Let the sequence ω start with $x \in \{b,c,d\}$: $\omega = x\omega'$. Then the map Ψ_ω acts as follows:

$$\begin{aligned} x &\to (1,\bar{x}), \quad axa \to (\bar{x},1),\\ y &\to (\bar{a},\bar{y}), \quad aya \to (\bar{y},\bar{a}),\\ z &\to (\bar{a},\bar{z}), \quad aza \to (\bar{z},\bar{a}). \end{aligned} \tag{2.26}$$

Let the map $\phi_\omega : F(\bar{a},\bar{b},\bar{c},\bar{d}) \to F(a,b,c,d)$ be defined by

$$\begin{aligned} \phi_\omega(\bar{x}) &= x \quad (\longrightarrow (1,\bar{x})),\\ \phi_\omega(\bar{y}) &= y \quad (\longrightarrow (\bar{a},\bar{y})),\\ \phi_\omega(\bar{z}) &= z \quad (\longrightarrow (\bar{a},\bar{z})),\\ \phi_\omega(\bar{a}) &= aya \quad (\longrightarrow (\bar{y},\bar{a})). \end{aligned}$$

Proposition II.23. $\phi_\omega(\Omega_1^{\omega'}) = \Omega_2^\omega$ *and* $\Psi_\omega(\Omega_2^\omega) = \Omega_1^{\omega'} \times \Omega_1^{\omega'}$.

Proof. Let $U = \phi_\omega(\Omega_1^{\omega'})$. Obviously, $\Omega_2^\omega = \Psi_\omega^{-1}(\Omega_1^{\omega'} \times \Omega_1^{\omega'})$. Together with any element $(1, g), g \in \Gamma_{\omega'}$ the group $\Psi_\omega(\Gamma_\omega)$ contains the element $(g, 1)$, and it follows from (2.26) that together with any element $(1, g), g \in \Gamma_{\omega'}$ the group $\Psi_\omega(\Gamma_\omega)$ contains the subgroup $(1, \langle g \rangle^{\Gamma_{\omega'}})$. Therefore, it is enough to prove that U contains all elements $(1, u), u \in \Omega_1^{\omega'}$, and this is true if and only if U contains all elements $(1, u_i)$ where u_i are generators of the subgroup $\Omega_1^{\omega'}$ as a normal subgroup: $\Omega_1^{\omega'} = \langle u_i, i = 1, \ldots, k \rangle^{\Gamma_{\omega'}}$, i.e. the generators from Proposition II.22.

By construction, the second coordinate of $\Psi_\omega(\phi_\omega(u))$ in $\Gamma_{\omega'} \times \Gamma_{\omega'}$ equals u, and it is enough to check that the first coordinate is the identity element. In other words, we need to check that $\phi'_\omega(u_i) = 1$, where ϕ'_ω is defined by

$$\begin{aligned} \phi'_\omega(\bar{x}) &= 1, \\ \phi'_\omega(\bar{y}) &= \bar{a}, \\ \phi'_\omega(\bar{z}) &= \bar{a}, \\ \phi'_\omega(\bar{a}) &= \bar{y}. \end{aligned}$$

As before, we have three possible cases.

(i) $\omega = \underbrace{xx\ldots xy}_{n}\ldots$, $n > 2$. Then

$$\Gamma_\omega = \left\langle a, x, y, z | a^2 = x^2 = y^2 = z^2 = xyz = (ax)^4 = (ay)^{2^{n+1}} = 1 \right\rangle,$$

$$\Gamma_{\omega'} = \left\langle \bar{a}, \bar{x}, \bar{y}, \bar{z} | \bar{a}^2 = \bar{x}^2 = \bar{y}^2 = \bar{z}^2 = \bar{x}\bar{y}\bar{z} = (\bar{a}\bar{x})^4 = (\bar{a}\bar{y})^{2^n} = 1 \right\rangle.$$

(ia) $n > 3$. Then

$$\Omega_1^{\omega'} = \left\langle (\bar{x}\bar{a}\bar{y}\bar{a})^4, (\bar{x}\bar{a}(\bar{y}\bar{a})^{2k})^4, k = 1, \ldots, 2^{n-2} - 1 \right\rangle^{\Gamma_{\omega'}},$$

$$\phi'_\omega((\bar{x}\bar{a}\bar{y}\bar{a})^4) = (\bar{y}\bar{a}\bar{y})^4 = 1,$$

$$\phi'_\omega((\bar{x}\bar{a}(\bar{y}\bar{a})^{2k})^4) = (\bar{y}(\bar{a}\bar{y})^{2k})^4 = (\bar{y}^{(\bar{a}\bar{y})^k})^4 = 1.$$

(ib) $n = 3$, $\omega = x\,\underbrace{xy\dots yx}_{m}\dots$, $m > 2$. Then

$$\Omega_1^{\omega'} = \left\langle (\bar{x}\bar{a}\bar{y}\bar{a}\bar{y}\bar{a})^4, (\bar{x}\bar{a}\bar{y}\bar{a})^{2^m} \right\rangle^{\Gamma_{\omega'}}.$$

$$\phi'_\omega((\bar{x}\bar{a}\bar{y}\bar{a}\bar{y}\bar{a})^4) = (\bar{y}\bar{a}\bar{y}\bar{a}\bar{y})^4 = 1,$$

$$\phi'_\omega((\bar{x}\bar{a}\bar{y}\bar{a})^{2^m}) = (\bar{y}\bar{a}\bar{y})^{2^m} = 1.$$

(ic) $n = 3$, $\omega = x\,\underbrace{xy\dots yz}_{m}\dots$, $m > 2$. Then

$$\Omega_1^{\omega'} = \left\langle (\bar{x}\bar{a}\bar{y}\bar{a}\bar{y}\bar{a})^4, (\bar{z}\bar{a}\bar{y}\bar{a})^{2^m} \right\rangle^{\Gamma_{\omega}}.$$

$$\phi'_\omega((\bar{x}\bar{a}\bar{y}\bar{a}\bar{y}\bar{a})^4) = (\bar{y}\bar{a}\bar{y}\bar{a}\bar{y})^4 = 1,$$

$$\phi'_\omega((\bar{z}\bar{a}\bar{y}\bar{a})^{2^m}) = (\bar{a}\bar{y}\bar{a}\bar{y})^{2^m} = (\bar{a}\bar{y})^{2^{m+1}} = 1 \ (m \geq 3 \text{ and } (\bar{a}\bar{y})^8 = 1),$$

(ii) $\omega = \underbrace{xy\dots yx}_{n}\dots$. Then

$$\Gamma_\omega = \left\langle a, x, y, z | a^2 = x^2 = y^2 = z^2 = xyz = (ax)^4 = (ay)^8 = 1 \right\rangle,$$

$$\Gamma_{\omega'} = \left\langle \bar{a}, \bar{x}, \bar{y}, \bar{z} | \bar{a}^2 = \bar{x}^2 = \bar{y}^2 = \bar{z}^2 = \bar{x}\bar{y}\bar{z} = (\bar{a}\bar{y})^4 = (\bar{a}\bar{x})^{2^n} = 1 \right\rangle.$$

(iia) $n > 3$. Then

$$\Omega_1^{\omega'} = \left\langle (\bar{y}\bar{a}\bar{x}\bar{a})^4, (\bar{y}\bar{a}(\bar{x}\bar{a})^{2k})^4, k = 1, \dots, 2^{n-2} - 1 \right\rangle^{\Gamma_{\omega'}},$$

$$\phi'_\omega((\bar{y}\bar{a}\bar{x}\bar{a})^4) = (\bar{a}\bar{y}\bar{y})^4 = 1,$$

$$\phi'_\omega((\bar{y}\bar{a}(\bar{x}\bar{a})^{2k})^4) = (\bar{a}\bar{y}(\bar{y})^{2k})^4 = (\bar{a}\bar{y})^4 = 1.$$

(iib) $n = 3$, $\omega = x\,\underbrace{yx\dots xy}_{m}\dots$. Then

$$\Omega_1^{\omega'} = \left\langle (\bar{y}\bar{a}\bar{x}\bar{a}\bar{x}\bar{a})^4, (\bar{y}\bar{a}\bar{x}\bar{a})^{2^m} \right\rangle^{\Gamma_{\omega'}},$$

$$\phi'_\omega((\bar{y}\bar{a}\bar{x}\bar{a}\bar{x}\bar{a})^4) = (\bar{a}\bar{y}\bar{y}\bar{y})^4 = (\bar{a}\bar{y})^4 = 1,$$

$$\phi'_\omega((\bar{y}\bar{a}\bar{x}\bar{a})^{2^m}) = (\bar{a}\bar{y}\bar{y})^{2^m} = 1.$$

(iic) $n = 3$, $\omega = x\underbrace{yx\dots xz}_{m}\dots$. Then

$$\Omega_1^{\omega'} = \left\langle (\bar{y}\bar{a}\bar{x}\bar{a}\bar{x}\bar{a})^4, (\bar{z}\bar{a}\bar{x}\bar{a})^{2^m} \right\rangle^{\Gamma_{\omega'}},$$

$$\phi'_\omega((\bar{y}\bar{a}\bar{x}\bar{a}\bar{x}\bar{a})^4) = (\bar{a}\bar{y}\bar{y}\bar{y})^4 = (\bar{a}\bar{y})^4 = 1,$$

$$\phi'_\omega((\bar{z}\bar{a}\bar{x}\bar{a})^{2^m}) = (\bar{a}\bar{y}\bar{y})^4 = 1.$$

(iii) $\omega = \underbrace{xy\dots yz}_{n}\dots$. Then

$$\Gamma_\omega = \left\langle a, x, y, z | a^2 = x^2 = y^2 = z^2 = xyz = (ax)^4 = (ay)^8 = 1 \right\rangle,$$

$$\Gamma_{\omega'} = \left\langle \bar{a}, \bar{x}, \bar{y}, \bar{z} | \bar{a}^2 = \bar{x}^2 = \bar{y}^2 = \bar{z}^2 = \bar{x}\bar{y}\bar{z} = (\bar{a}\bar{y})^4 = (\bar{a}\bar{z})^{2^n} = 1 \right\rangle.$$

(iiia) $n > 3$. Then

$$\Omega_1^{\omega'} = \left\langle (\bar{y}\bar{a}\bar{z}\bar{a})^4, (\bar{y}\bar{a}(\bar{z}\bar{a})^{2k})^4, k = 1, \dots, 2^{n-2} - 1 \right\rangle^{\Gamma_{\omega'}},$$

$$\phi'_\omega((\bar{y}\bar{a}\bar{z}\bar{a})^4) = (\bar{a}\bar{y}\bar{a}\bar{y})^4 = (\bar{a}\bar{y})^8 = 1,$$

$$\phi'_\omega((\bar{y}\bar{a}(\bar{z}\bar{a})^{2k})^4) = (\bar{a}\bar{y}(\bar{a}\bar{y})^{2k})^4 = (\bar{a}\bar{y})^{8k+4} = 1.$$

(iiib) $n = 3$, $\omega = x\underbrace{yz\dots zy}_{m}\dots$. Then

$$\Omega_1^{\omega'} = \left\langle (\bar{y}\bar{a}\bar{z}\bar{a}\bar{z}\bar{a})^4, (\bar{y}\bar{a}\bar{z}\bar{a})^{2^m} \right\rangle^{\Gamma_{\omega'}},$$

$$\phi'_\omega((\bar{y}\bar{a}\bar{z}\bar{a}\bar{z}\bar{a})^4) = (\bar{a}\bar{y}\bar{a}\bar{y}\bar{a}\bar{y})^4 = (\bar{a}\bar{y})^12 = 1,$$

$$\phi'_\omega((\bar{y}\bar{a}\bar{z}\bar{a})^{2^m}) = (\bar{a}\bar{y}\bar{a}\bar{y})^{2^m} = (\bar{a}\bar{y})^{2^{m+1}} = 1.$$

(iiic) $n = 3$, $\omega = x\underbrace{yz\ldots zx}_{m}\ldots$. Then

$$\Omega_1^{\omega'} = \left\langle (\bar{y}\bar{a}\bar{z}\bar{a}\bar{z}\bar{a})^4, (\bar{x}\bar{a}\bar{z}\bar{a})^{2^m} \right\rangle^{\Gamma_{\omega'}},$$

$$\phi'_\omega((\bar{y}\bar{a}\bar{z}\bar{a}\bar{z}\bar{a})^4) = (\bar{a}\bar{y}\bar{a}\bar{y}\bar{a}\bar{y})^4 = (\bar{a}\bar{y})^12 = 1,$$

$$\phi'_\omega((\bar{x}\bar{a}\bar{z}\bar{a})^{2^m}) = (\bar{y}\bar{a}\bar{y})^{2^m} = 1.$$

□

Theorem II.24. *Substitution rules ϕ_ω constructed above provide recursive presentations for groups G_ω with non-almost-constant ω. Namely,*

$$G_\omega \cong \left\langle a, b, c, d \middle| \bigcup_{k=1}^{\infty} U_k^\omega \right\rangle, \tag{2.27}$$

where U_1^ω is as described in the Proposition II.22, and consequent sets of relators U_k^ω are obtained recursively by the rule

$$U_{k+1}^\omega = \phi_\omega(U_k^{\omega'}). \tag{2.28}$$

All groups G_ω with non-almost-constant ω are infinitely presented.

Proof. Proof is straightforward induction on k. Case $k = 1$ is the proposition II.21, and $k > 1$ is done in the same way as in the section D. Let us show that the groups G_ω have the property analogous to property BA of self-similar groups. Namely, we need to prove that if an element $g \in \Gamma_\omega$ represents the trivial element in the group G_ω then $\Psi_k^\omega(g) = 1$ for some $k \geq 0$. For this it is enough to prove that any word of the length $l \geq 2$ contracts to strictly shorter words on the next level. That follows from the following simple equalities.

$$\Psi(a_\omega b_\omega) = (b_{\omega'}, \beta_{\omega'})\sigma,$$
$$\Psi(a_\omega c_\omega) = (c_{\omega'}, \gamma_{\omega'})\sigma,$$
$$\Psi(a_\omega d_\omega) = (d_{\omega'}, \delta_{\omega'})\sigma,$$
$$\Psi(b_\omega a_\omega) = (\beta_{\omega'}, b_{\omega'})\sigma,$$
$$\Psi(c_\omega a_\omega) = (\gamma_{\omega'}, c_{\omega'})\sigma,$$
$$\Psi(d_\omega a_\omega) = (\delta_{\omega'}, d_{\omega'})\sigma,$$
$$a_\omega^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = 1,$$
$$b_\omega c_\omega = c_\omega b_\omega = d_\omega,$$
$$c_\omega d_\omega = d_\omega c_\omega = b_\omega,$$
$$b_\omega d_\omega = d_\omega b_\omega = c_\omega,$$

where β, γ, δ are 1 or a'_ω. □

D. Grigorchuk-Erschler group and $IMG(z^2 + i)$

Consider the following two automata.

$$\mathcal{A}_1 : \begin{cases} a &= (1,1)\sigma, \\ b &= (a,c), \\ c &= (1,b), \end{cases}$$

$$\mathcal{A}_2 : \begin{cases} a &= (1,1)\sigma, \\ b &= (c,a), \\ c &= (1,b). \end{cases}$$

The automaton $\mathcal{A}_1$ generates the Grigorchuk-Erschler group [Ers04], and the automaton $\mathcal{A}_2$ generates the group isomorphic to the group of iterated monodromies of the complex polynomial z^2+i [GSŠ07]. L-presentation of the group $IMG(z^2+i)$ was found in the cited paper, and here we will find the L-presentation of the Grigorchuk-Erschler group. Moreover, we will find recursively defined presentations for the whole family of groups to which the above two groups belong.

Let Λ be the set of all infinite sequences of digits $\{1, 2\}$,

$$\Lambda = \{1,2\}^{\omega} = \{\lambda = (\lambda_1\lambda_2\dots)|\lambda_i = 1,2, i = 1,2,\dots\},$$

To each sequence $\lambda = (\lambda_1\lambda_2\lambda_3\dots) \in \Lambda$ we assign the group $\mathcal{K}_\lambda < Aut\,\mathcal{T}_2$ generated by the elements $a_\lambda, b_\lambda, c_\lambda$, which are defined recursively by

$$\begin{aligned} a_\lambda &= (1,1)\sigma, \\ b_\lambda &= (x_{\lambda'}, y_{\lambda'}), \\ c_\lambda &= (1, b_{\lambda'}), \end{aligned}$$

where $x = c, y = a$ if $\lambda_1 = 1$ and $x = a, y = c$ if $\lambda_1 = 2$, and λ' is the tail of the sequence λ: $\lambda' = (\lambda_2\lambda_3\dots)$. In other words, we have

$$\begin{aligned} a_{1\lambda'} &= (1,1)\sigma, & a_{2\lambda'} &= (1,1)\sigma, \\ b_{1\lambda'} &= (c_{\lambda'}, a_{\lambda'}), & b_{2\lambda'} &= (a_{\lambda'}, c_{\lambda'}), \\ c_{1\lambda'} &= (1, b_{\lambda'}), & c_{2\lambda'} &= (1, b_{\lambda'}). \end{aligned} \tag{2.29}$$

It is easy to see that the group $\mathcal{K}_{222\dots}$ is the Grigorchuk-Erschler group, and $\mathcal{K}_{111\dots}$ is $IMG(z^2 + i)$.

For any $\lambda \in \Lambda$ we have the following:

$$a_\lambda^2 = b_\lambda^2 = c_\lambda^2 = (a_\lambda c_\lambda)^4 = 1,$$

therefore every group $\mathcal{K}_\lambda$ is a factor group of the group

$$\Gamma = \left\langle a, b, c | a^2 = b^2 = c^2 = (ac)^4 = 1 \right\rangle.$$

Group Γ is a covering group for each group $\mathcal{K}_\lambda$, i.e. for every $\lambda \in \Lambda$ the map

$\Psi^{\Gamma,\lambda} : \Gamma \to \Gamma \wr S_2$ defined by

$$\begin{aligned} \Psi^{\Gamma,\lambda}(a) &= (1,1)\sigma, \\ \Psi^{\Gamma,\lambda}(b) &= (x,y), \\ \Psi^{\Gamma,\lambda}(c) &= (1,b), \end{aligned}$$

extends to a homomorphism. Indeed, if F is the free group with generators a, b, c and $\Gamma = F/R$, $R = \langle a^2, b^2, c^2, (ac)^4 \rangle^F$, then we have

$$\begin{aligned} \Psi^{\Gamma,\lambda}(a^2) &= (1,1), \\ \Psi^{\Gamma,\lambda}(b^2) &= (x^2,y^2), \\ \Psi^{\Gamma,\lambda}(c^2) &= (1,b^2), \\ \Psi^{\Gamma,\lambda}((ac)^4) &= (b^2,b^2), \end{aligned}$$

and hence $\Psi^{\Gamma,\lambda}(R) \subset R \times R$.

As we will see below, in one special case we need to use the more restricted covering group. Namely, let $\lambda = 2222\dots$. Then in the group $\mathcal{K}_\lambda$ we have $(bc)^2 = (1, (cb)^2), (cb)^2 = (1, (bc)^2)$, i.e. $(bc)^2 = 1$, and we will see that this relation is required for our wreath recursion to have the property BA. Let Γ' be the group Γ with this extra relation added, i.e.

$$\Gamma' = \left\langle a, b, c | a^2 = b^2 = c^2 = (ac)^4 = (bc)^2 = 1 \right\rangle,$$

and let $\Psi^{\Gamma,\Gamma'}$ be the map from Γ to $\Gamma' \wr S_2$ defined by

$$\begin{aligned} \Psi^{\Gamma,\Gamma'}(a) &= (1,1)\sigma, \\ \Psi^{\Gamma,\Gamma'}(b) &= (c,a), \\ \Psi^{\Gamma,\Gamma'}(c) &= (1,b), \end{aligned}$$

and let $\Psi^{\Gamma'} : \Gamma' \to \Gamma'$ be the analogous map defined for the group Γ':

$$\begin{aligned}\Psi^{\Gamma'}(a) &= (1,1)\sigma,\\ \Psi^{\Gamma'}(b) &= (a,c),\\ \Psi^{\Gamma'}(c) &= (1,b),\end{aligned}$$

$(\Gamma, \Psi^{\Gamma,\Gamma'})$ and $(\Gamma', \Psi^{\Gamma'})$ are wreath recursions. Indeed, the calculations for these are the same as above for the group Γ, and the only relator we have to check is $(bc)^2$:

$$\Psi^{\Gamma'}((bc)^2) = (1,(cb)^2) = (1,1) \text{ in } \Gamma'.$$

So, as with the Grigorchuk groups G_ω, let us define for every λ the covering group Γ_λ and the wreath recursion map $\Psi^\lambda : \Gamma_\lambda \to \Gamma_{\lambda'} \wr S_2$:

$$\Gamma_\lambda = \begin{cases} \Gamma' \text{ if } \lambda = 222\ldots, \\ \Gamma \text{ otherwise }, \end{cases}$$

$$\Psi^\lambda : \begin{cases} a \to (1,1)\sigma, \\ b \to (x,y), \\ c \to (1,b), \end{cases}$$

By above, $(\Gamma_\lambda, \Psi^\lambda)$ are wreath recursions, and for every λ we obtain a series of maps $\Psi^\lambda_n : \Gamma_\lambda \to \Gamma_{\lambda^{(n)}} \wr S_2 \wr \cdots \wr S_2$ (where $\lambda^{(n)}$ denotes n-th iteration of the shift of the sequence λ).

To find the presentations for the groups $\mathcal{K}_\lambda$, we will proceed in several steps. First we will find the kernels Ω^λ_1 of the maps Ψ^λ, then we will show that consequent sets of relations Ω^λ_{k+1} can be obtained from Ω^λ_k using appropriate substitutions π_λ, and finally we will show that the sets Ω^λ_k indeed provide the presentation for the group $\mathcal{K}_\lambda$.

1. Calculation of Ω_1^λ

In this subsection we fix $\lambda \in \Lambda$ and use the notations Ψ and Ω_k instead of Ψ^λ and Ω_k^λ when it is convenient. Let $\Omega_1 = \Omega_1^\lambda$ be the kernel of the homomorphism $\Psi = \Psi^\lambda$:

$$\Omega_1 = \ker \Psi = \{g \in \Gamma_\lambda | \Psi(g) = (1,1)\}.$$

We have two possible cases, depending on the first digit of the sequence λ.

$$\Psi^\lambda(a) = (1,1)\sigma,$$

and

$$\Psi^{1\lambda'} : \begin{cases} b & \rightarrow & (c,a) \\ c & \rightarrow & (1,b) \end{cases}, \Psi^{2\lambda'} : \begin{cases} b & \rightarrow & (a,c) \\ c & \rightarrow & (1,b) \end{cases}.$$

Let $H = St_\Gamma(1)$. Let us describe the group $\Psi(H)$. Since $\Omega_1 < H$, Ω_1 is generated as a normal subgroup by the preimages of the relators of the group $\Psi(H)$. H is generated by the elements b, c, b^a, c^a.

$$\Psi(H) = \langle (a,c),(c,a),(1,b),(b,1)\rangle < \Gamma_{\lambda'} \times \Gamma_{\lambda'}.$$

Hence we get

$$B \times B \trianglelefteq \Psi(H),$$

where $B = \langle b\rangle^{\Gamma_{\lambda'}}$. Furthermore, $\Psi(H)/B \times B \cong \langle (a,c),(c,a)\rangle \cong D_4$ (regardless of whether the group $\Gamma_{\lambda'}$ is the group Γ or Γ'). Therefore

$$\Psi(H) \cong (B \times B) \rtimes D_4.$$

Let us find the representation of the group B. We have two possibilities. First, $\lambda' = 222\ldots$ and $\Gamma_{\lambda'} = \Gamma' = \langle a,b,c | a^2 = b^2 = c^2 = (bc)^2 = (ac)^4 = 1\rangle$, and the second, when $\lambda' \neq 222\ldots$ and $\Gamma_{\lambda'} = \Gamma = \langle a,b,c | a^2 = b^2 = c^2 = (ac)^4 = 1\rangle$. Let us consider these two cases separately.

(I) $\lambda' \neq 222\ldots$.

Let

$$\xi_z = b^z, z \in D = \langle a, c\rangle \cong D_4.$$

Obviously B is generated by the elements ξ_z, $z \in D$. $\Gamma = \langle b\rangle * D$, hence B is a free product of the subgroups of order 2 generated by the elements ξ_z, i.e.

$$B = \left\langle \xi_z | \xi_z^2 = 1 \right\rangle_{z \in D}.$$

Therefore $B \times B$ is generated by the elements $\tilde{\xi}_z = (\xi_z, 1)$ and $\hat{\xi}_z = (1, \xi_z)$, $z \in D$, and its presentation is

$$B \times B = \left\langle \tilde{\xi}_z, \hat{\xi}_t | \tilde{\xi}_z^2 = \hat{\xi}_t^2 = [\tilde{\xi}_z, \hat{\xi}_t] = 1 \right\rangle_{z,t \in D}.$$

The action of the group D_4 generated by the elements $x = (a, c)$ and $y = (c, a)$ on the group $B \times B$ is defined as follows:

$$\begin{aligned} \tilde{\xi}_z^x &= (\xi_z, 1)^{(a,c)} = (b^{za}, 1) = (\xi_{za}, 1) = \tilde{\xi}_{za}, \\ \tilde{\xi}_z^y &= (\xi_z, 1)^{(c,a)} = (b^{zc}, 1) = (\xi_{zc}, 1) = \tilde{\xi}_{zc}, \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} \hat{\xi}_t^x &= (1, \xi_t)^{(a,c)} = (1, b^{tc}) = (1, \xi_{tc}) = \hat{\xi}_{tc}, \\ \hat{\xi}_t^y &= (1, \xi_t)^{(c,a)} = (1, b^{ta}) = (1, \xi_{ta}) = \hat{\xi}_{ta}. \end{aligned} \tag{2.31}$$

Relations (2.30) and (2.31) show that $\Psi(H) = \left\langle \tilde{\xi}_1, \hat{\xi}_1, x, y \right\rangle$ and its presentation is

$$\Psi(H) = \left\langle \tilde{\xi}_1, \hat{\xi}_1, x, y \;\middle|\; \begin{array}{c} \tilde{\xi}_1^2 = \hat{\xi}_1^2 = x^2 = y^2 = (xy)^4 = 1, \\ [\tilde{\xi}_1^p, \hat{\xi}_1^q] = 1, p, q \in \langle x, y\rangle \end{array} \right\rangle \tag{2.32}$$

(II) $\lambda' = 222\ldots$.

The group B has the following presentation:

$$B = \left\langle \xi_1, \xi_a, \xi_{ac}, \xi_{aca} | \xi_z^2 = 1, z \in \{1, a, ac, aca\} \right\rangle,$$

where $\xi_z = b^z$. Therefore $B \times B$ is generated by the elements $\tilde{\xi}_z = (\xi_z, 1)$ and $\hat{\xi}_z = (1, \xi_z)$, $z \in \{1, a, ac, aca\}$, and its presentation is

$$B \times B = \left\langle \tilde{\xi}_z, \hat{\xi}_t | \tilde{\xi}_z^2 = \hat{\xi}_t^2 = [\tilde{\xi}_z, \hat{\xi}_t] = 1 \right\rangle_{z,t \in \{1,a,ac,aca\}}.$$

The action of the group D_4 generated by the elements $x = (a, c)$ and $y = (c, a)$ on the group $B \times B$ is defined as follows:

$$\begin{aligned} \tilde{\xi}_z^x &= (\xi_z, 1)^{(a,c)} = (b^{za}, 1) = (\xi_{za}, 1) = \tilde{\xi}_{za}, \\ \tilde{\xi}_z^y &= (\xi_z, 1)^{(c,a)} = (b^{zc}, 1) = (\xi_{zc}, 1) = \tilde{\xi}_{zc}, \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} \hat{\xi}_t^x &= (1, \xi_t)^{(a,c)} = (1, b^{tc}) = (1, \xi_{tc}) = \hat{\xi}_{tc}, \\ \hat{\xi}_t^y &= (1, \xi_t)^{(c,a)} = (1, b^{ta}) = (1, \xi_{ta}) = \hat{\xi}_{ta}. \end{aligned} \tag{2.34}$$

We have

$$b^c = b,$$

$$b^{aca \cdot c} = b^{caca} = b^{aca},$$

and therefore x and y act on the set $\left\{\tilde{\xi}_z, \hat{\xi}_t\right\}$ as the following permutations:

$$x = (\tilde{\xi}_1, \tilde{\xi}_a)(\tilde{\xi}_{ac}, \tilde{\xi}_{aca})(\hat{\xi}_a, \hat{\xi}_{ac}),$$

$$y = (\tilde{\xi}_a, \tilde{\xi}_{ac})(\hat{\xi}_1, \hat{\xi}_a)(\hat{\xi}_{ac}, \hat{\xi}_{aca}).$$

As in the case $\lambda' \neq 222\ldots$, relations (2.33) and (2.34) show that $\Psi(H) =$

$\left\langle \tilde{\xi}_1, \hat{\xi}_1, x, y \right\rangle$ and its presentation is

$$\Psi(H) = \left\langle \tilde{\xi}_1, \hat{\xi}_1, x, y \;\middle|\; \begin{array}{c} \tilde{\xi}_1^2 = \hat{\xi}_1^2 = x^2 = y^2 = (xy)^4 = 1, \\ [\tilde{\xi}_1^p, \hat{\xi}_1^q] = 1, p, q \in \langle x, y \rangle \end{array} \right\rangle \tag{2.35}$$

The kernel of Ψ is generated as a normal subgroup by preimages of the relators in $\Psi(H)$. We have

$$\Psi(c) = \hat{\xi}_1, \Psi(aca) = \tilde{\xi}_1, \tag{2.36}$$

and

$$\Psi(aba) = x, \Psi(b) = y \tag{2.37}$$

or

$$\Psi(b) = x, \Psi(aba) = y \tag{2.38}$$

if $\lambda_1 = 1$ or $\lambda_1 = 2$ respectively.

- If $\lambda_1 = 1$, then $\Psi(aba) = x, \Psi(b) = y$. We have

$$x^2 \leftarrow (aba)^2 = 1, y^2 \leftarrow b^2 = 1, (xy)^4 \leftarrow (abab)^4 = (ab)^8.$$

- If $\lambda_1 = 2$, then $\Psi(b) = x, \Psi(aba) = y$. We have

$$x^2 \leftarrow b^2 = 1, y^2 \leftarrow (aba)^2 = 1, (xy)^4 \leftarrow (baba)^4 = (ba)^8.$$

In either case we obtain the single relator $w_1 = (ab)^8$ from the relators $x^2, y^2, (xy)^4$.

The relators $\tilde{\xi}_1^2, \hat{\xi}_1^2$ give us c^2 and $(aca)^2$, therefore all remaining generators of the group Ω_1 come from the relators

$$[\tilde{\xi}_1^p, \hat{\xi}_1^q], p, q \in \langle x, y \rangle . \tag{2.39}$$

From (2.36, 2.37, 2.38) we obtain that modulo conjugation and taking inverses all generators of Ω_1 obtained from (2.39) are

$$[c, c^{ta}], t \in \langle b, aba \rangle \,. \tag{2.40}$$

c is an involution, hence we have $[c, c^v] = (cc^v)^2 = [c, v]^2$. Also, we can use the fact that $w_1 = (ab)^8 \in \Omega_1$ and therefore we obtain

$$\Omega_1 = \left\langle (ab)^8, [c, a]^2, [c, ta]^2 | t \in \langle b, aba \rangle \right\rangle^\Gamma =$$

$$\left\langle (ab)^8, [c, ab]^2, [c, ba]^2, [c, bab]^2, [c, ababa]^2, [c, ababab]^2, [c, bababa]^2, [c, bababab]^2 \right\rangle^\Gamma .$$

Now,

$$[c, a]^2 = (ca)^4 = 1,$$

$$[c, ba] = cabcba \sim bacabc = (cbacab)^{-1} = [c, ab]^{-1},$$

$$[c, bababa] = cababab cbababa \sim bababacabababc = (cbababacababab)^{-1} = [c, ababab]^{-1},$$

therefore we get $\Omega_1 = \langle U_1 \rangle^\Gamma$ where

$$U_1 = \left\{ (ab)^8, [c, ab]^2, [c, bab]^2, [c, ababa]^2, [c, ababab]^2, [c, bababab]^2 \right\} .$$

2. Ω_k^λ

Let us find the substitution ϕ_λ which transforms Ω_k^λ to Ω_{k+1}^λ. From (2.29) we have

$$\begin{array}{rcl@{\qquad}rcl} & \lambda_1 = 1 & & & \lambda_1 = 2 & \\ \Psi(b) & = & (c, a), & \Psi(b^a) & = & (c, a), \\ \Psi(c) & = & (1, b), & \Psi(c) & = & (1, b), \\ \Psi(b^a) & = & (a, c), & \Psi(b) & = & (a, c), \end{array}$$

so it is natural to put

$$\phi_{1\lambda'}:\begin{cases} a\to b,\\ b\to c,\\ c\to aba,\end{cases}\qquad \phi_{2\lambda'}:\begin{cases} a\to aba,\\ b\to c,\\ c\to b.\end{cases}$$

Obviously, $\Psi(\phi_\lambda(w))=(w',w)$ where w' is an element of the group $\langle a_{\lambda'},c_{\lambda'}\rangle$. Now, for any $w\in U_1^\lambda$ the element w' is a homomorphic image of w, and therefore it is a fourth power or a square of a commutator. In either case w' will be trivial because $\langle a_{\lambda'},c_{\lambda'}\rangle\cong D_4$. Therefore, we have $\Psi(\phi_\lambda(U_1^\lambda))\subset 1\times\Omega_1^{\lambda'}\subset\Psi(\Omega_2^\lambda)$. Group Ω_2^λ is normal in Γ_λ, hence $\Psi(\Omega_2^\lambda)=\Psi((\Omega_2^\lambda)^a)\supset\Omega_1^{\lambda'}\times 1$, and hence $\Psi(\Omega_2^\lambda)\supset\Omega_1^{\lambda'}\times\Omega_1^{\lambda'}$. On the other hand, $\Psi(\Omega_2^\lambda)\subset\Omega_1^{\lambda'}\times\Omega_1^{\lambda'}$ by definition, hence $\Psi(\Omega_2^\lambda)=\Omega_1^{\lambda'}\times\Omega_1^{\lambda'}$ and

$$\Omega_2^\lambda=\left\langle\ker\Psi,\Psi^{-1}(\Omega_1^{\lambda'}\times\Omega_1^{\lambda'})\right\rangle^{\Gamma_\lambda}=\left\langle U_1^\lambda,\phi_\lambda(U_1^\lambda)\right\rangle^{\Gamma_\lambda}.$$

Using induction on n we obtain in the similar way that

$$\Omega_n^\lambda=\left\langle U_1^\lambda,U_2^\lambda,\ldots,U_n^\lambda\right\rangle^{\Gamma_\lambda},$$

where U_k^λ are defined recursively by

$$U_{k+1}^\lambda=\phi_\lambda(U_k^{\lambda'}),k\geq 1,$$

$$U_1^\lambda=U=\left\{(ab)^8,[c,ab]^2,[c,bab]^2,[c,ababa]^2,[c,ababab]^2,[c,bababab]^2\right\}.$$

Let us show that if an element of Γ_λ represents a trivial element of the group $\mathcal{K}_\lambda$ then it is mapped to the trivial element by Ψ_k^λ for some $k\geq 1$. For this it is enough to show that for any word $g\in\Gamma_\lambda$ such that $|g|\geq 2$ there exists $k\geq 1$ such that all sections of g on the level k will be strictly shorter then the word g. Let us consider the second level. We have

$$\begin{aligned}\Psi(a) &= (1,1)\sigma\\ \Psi(b) &= (c,a) \qquad \text{if } \lambda = 1\lambda',\\ \Psi(c) &= (1,b)\end{aligned}$$

$$\begin{aligned}\Psi(a) &= (1,1)\sigma\\ \Psi(b) &= (a,c) \qquad \text{if } \lambda = 2\lambda',\\ \Psi(c) &= (1,b)\end{aligned}$$

hence

- If $\lambda = 11\lambda''$ then

$$\Psi_2(a) = (1,1,1,1)(13)(24),$$
$$\Psi_2(b) = (1,b,1,1)(34),$$
$$\Psi_2(c) = (1,1,c,a),$$
$$\Psi_2(a^2) = 1, \Psi_2(b^2) = 1, \Psi_2(c^2) = 1,$$
$$\Psi_2(ab) = (1,1,1,b)(1423),$$
$$\Psi_2(ba) = (1,b,1,1)(1324),$$
$$\Psi_2(ac) = (c,a,1,1)(13)(24),$$
$$\Psi_2(ca) = (1,1,c,a)(13)(24),$$
$$\Psi_2(bc) = (1,b,a,c)(34),$$
$$\Psi_2(cb) = (1,b,c,a)(34),$$

- If $\lambda = 12\lambda''$ then

$$\Psi_2(a) = (1,1,1,1)(13)(24),$$
$$\Psi_2(b) = (1,b,1,1)(34),$$
$$\Psi_2(c) = (1,1,a,c),$$
$$\Psi_2(a^2) = 1, \Psi_2(b^2) = 1, \Psi_2(c^2) = 1,$$

$$\Psi_2(ab) = (1,1,1,b)(1423),$$
$$\Psi_2(ba) = (1,b,1,1)(1324),$$
$$\Psi_2(ac) = (a,c,1,1)(13)(24),$$
$$\Psi_2(ca) = (1,1,a,c)(13)(24),$$
$$\Psi_2(bc) = (1,b,c,a)(34),$$
$$\Psi_2(cb) = (1,b,a,c)(34),$$

- If $\lambda = 21\lambda''$ then

$$\Psi_2(a) = (1,1,1,1)(13)(24),$$
$$\Psi_2(b) = (1,1,1,b)(12),$$
$$\Psi_2(c) = (1,1,c,a),$$
$$\Psi_2(a^2) = 1, \Psi_2(b^2) = 1, \Psi_2(c^2) = 1,$$
$$\Psi_2(ab) = (1,b,1,1)(1324),$$
$$\Psi_2(ba) = (1,1,1,b)(1423),$$
$$\Psi_2(ac) = (c,a,1,1)(13)(24),$$
$$\Psi_2(ca) = (1,1,c,a)(13)(24),$$
$$\Psi_2(bc) = (1,1,c,ba)(12),$$
$$\Psi_2(cb) = (1,1,c,ab)(12),$$

- If $\lambda = 22\lambda''$ then

$$\Psi_2(a) = (1,1,1,1)(13)(24),$$
$$\Psi_2(b) = (1,1,1,b)(12),$$
$$\Psi_2(c) = (1,1,a,c),$$
$$\Psi_2(a^2) = 1, \Psi_2(b^2) = 1, \Psi_2(c^2) = 1,$$

$$\Psi_2(ab) = (1, b, 1, 1)(1324),$$
$$\Psi_2(ba) = (1, 1, 1, b)(1423),$$
$$\Psi_2(ac) = (a, c, 1, 1)(13)(24),$$
$$\Psi_2(ca) = (1, 1, a, c)(13)(24),$$
$$\Psi_2(bc) = (1, 1, a, bc)(12),$$
$$\Psi_2(cb) = (1, 1, a, cb)(12),$$

Now, if λ does not start with two 2's, then the sections of g on the third level will be shorter than g (worst case is the pairs bc and cb when $\lambda = 21\ldots$, when ab and ba from the second level contract to a single letter on the next level). Moreover, if $\lambda \neq 2$ then the sections of g will necessarily become shorter on the level $k+2$ where k is the first occurrence of 1 in the sequence λ. The only case when the shortening doesn't happen is when $\lambda = 222\ldots$, but in that case we can add bc to the system of generators and obtain the following:

$$\Psi_2(a \cdot bc) = (a, bc, 1, 1)(1324),$$
$$\Psi_2(b \cdot bc) = c = (1, 1, a, c),$$
$$\Psi_2(c \cdot bc) = b = (1, 1, 1, b)(12),$$
$$\Psi_2(bc \cdot a) = (1, 1, a, bc)(1423),$$
$$\Psi_2(bc \cdot b) = c = (1, 1, a, c),$$
$$\Psi_2(bc \cdot c) = b = (1, 1, 1, b)(12),$$

i.e. contraction happens on the second level.

Therefore, we obtain

Theorem II.25.

$$\mathcal{K}_\lambda \cong \left\langle a, b, c \middle| \bigcup_{k=1}^{\infty} U_k^\lambda \right\rangle,$$

where

$$U_1^{\lambda} = \left\{a^2, (ac)^4, [c, ab]^2, [c, bab]^2, [c, ababa]^2, [c, ababab]^2, [c, bababab]^2\right\}$$

for $\lambda \neq 222\ldots$, *and*

$$U_1^{222\ldots} = \left\{a^2, (bc)^2, (ac)^4, [c, ab]^2, [c, bab]^2, [c, ababa]^2, [c, ababab]^2, [c, bababab]^2\right\},$$

and

$$U_{k+1}^{\lambda} = \phi_{\lambda}(U_k^{\lambda'}), k \geq 1,$$

and the endomorphisms ϕ_λ *of the free group* $F(a, b, c)$ *are defined as follows,*

$$\phi_{1\lambda'} : \begin{cases} a \to b, \\ b \to c, \\ c \to aba, \end{cases} \qquad \phi_{2\lambda'} : \begin{cases} a \to aba, \\ b \to c, \\ c \to b. \end{cases}$$

Proof. Above we obtained

$$U_1^{\lambda} = \left\{a^2, b^2, c^2, (ac)^4, (ab)^8, [c, ab]^2, [c, bab]^2, [c, ababa]^2, [c, ababab]^2, [c, bababab]^2\right\}$$

for $\lambda \neq 222\ldots$, and

$$U_1^{222\ldots} = \left\{a^2, b^2, c^2, (bc)^2, (ac)^4, (ab)^8, [c, ab]^2, [c, bab]^2, [c, ababa]^2, [c, ababab]^2, [c, bababab]^2\right\}.$$

We can reduce these sets by eliminating b^2, c^2, and $(ab)^8$ because

$$\phi_{1\lambda'}(a^2) = b^2, \phi_{1\lambda'}(b^2) = c^2,$$
$$\phi_{1\lambda'}((ac)^4) = (baba)^4 = (ba)^8 = ((ab)^8)^{-1},$$
$$\phi_{2\lambda'}(a^2) = ab^2a, \phi_{2\lambda'}(b^2) = c^2,$$
$$\phi_{2\lambda'}((ac)^4) = (abab)^4 = (ab)^8.$$

□

Corollary II.26. *The group $IMG(z^2+i)$ has the following L-presentation:*

$$IMG(z^2+i) \cong \left\langle a,b,c | \phi^k(r), k \geq 0, r \in R \right\rangle,$$

where

$$R = \left\{a^2, (ac)^4, [c,ab]^2, [c,bab]^2, [c,ababa]^2, [c,ababab]^2, [c,bababab]^2\right\},$$

and the endomorphism ϕ is defined by

$$\phi : \begin{cases} a \to b, \\ b \to c, \\ c \to aba. \end{cases}$$

This presentation coincides with that found in [GSŠ07].

Corollary II.27. *The Grigorchuk-Erschler group $\mathcal{GE}$ has the following L-presentation:*

$$\mathcal{GE} \cong \left\langle a,b,c | \phi^k(r), k \geq 0, r \in R \right\rangle,$$

where

$$R = \left\{a^2, (bc)^2, (ac)^4, [c,ab]^2, [c,bab]^2, [c,ababa]^2, [c,ababab]^2, [c,bababab]^2\right\},$$

and the endomorphism ϕ is defined by

$$\phi : \begin{cases} a \to aba, \\ b \to c, \\ c \to b. \end{cases}$$

E. One four-state group

Let G be the group generated by the automaton

$$\begin{cases} a &= \sigma, \\ b &= (b,c), \\ c &= (1,a). \end{cases}$$

Theorem II.28. *Group G has the following L-presentation:*

$$G \cong \langle a,b,c | \phi^n(r), r \in R, n \geq 0 \rangle ,$$

where

$$R = \{a^2, (ac)^4, [c,ab]^2, [c,bab]^2, [c,ababa]^2, [c,ababab]^2, [c,bababab]^2\},$$

and ϕ is given by

$$\phi : \begin{cases} a \to c, \\ b \to aba, \\ c \to b. \end{cases}$$

Proof. We have the following in the group G:

$$a^2 = b^2 = c^2 = (ac)^4 = 1,$$

therefore group G is a factor group of the group

$$\Gamma = \left\langle a,b,c | a^2 = b^2 = c^2 = (ac)^4 = 1 \right\rangle .$$

Group Γ is a covering group for the group G, i.e. the map $\Psi : \Gamma \to \Gamma \wr S_2$ defined by

$$\begin{aligned} \Psi(a) &= (1,1)\sigma, \\ \Psi(b) &= (b,c), \\ \Psi(c) &= (1,a), \end{aligned}$$

extends to a homomorphism. Indeed, if F is the free group with generators a, b, c and $\Gamma = F/R$, $R = \langle a^2, b^2, c^2, (ac)^4 \rangle^F$, then

$$\begin{aligned} \Psi(a^2) &= (1,1), \\ \Psi(b^2) &= (b^2, c^2), \\ \Psi(c^2) &= (1, a^2), \\ \Psi((ac)^4) &= (a^2, a^2), \end{aligned}$$

and hence $\Psi(R) \subset R \times R$.

Let us find the presentation for the groups G. Let Ω_1 be the kernel of the homomorphism Ψ:

$$\Omega_1 = \ker \Psi = \{ g \in \Gamma | \Psi(g) = (1,1) \}.$$

We have

$$\Psi : \begin{cases} a & \to & (1,1)\sigma, \\ b & \to & (b,c) \\ c & \to & (1,a) \end{cases} .$$

Let $H = St_\Gamma(1) = \langle b, c, b^a, c^a \rangle$. Then $\Psi(c) = (1,a)$ and $\Psi(b) = (b,c)$ respectively, and

$$\Psi(H) = \langle (b,c), (c,b), (1,a), (a,1) \rangle < \Gamma \times \Gamma.$$

Hence we get

$$A \times A \trianglelefteq \Psi(H),$$

where $A = \langle a \rangle^{\Gamma}$. Furthermore, $\Psi(H)/A \times A \cong \langle (b,c),(c,b) \rangle \cong D_\infty$. Therefore

$$\Psi(H) \cong (A \times A) \rtimes D_\infty.$$

Let us find the representation of the group A. $\Gamma \cong \langle a, c \rangle * \langle b \rangle$, therefore A is a free product of its subgroups $A_x = \left\{ g^x | g \in \langle a \rangle^{\langle a,c \rangle} \right\}$ for all $x \in B_C$ where $B_C \subset \langle b, c \rangle$ is the set of all elements of the group $\langle b, c \rangle$ which do not start with c. Hence its presentation is

$$A \cong \left\langle \xi_k, \eta_k, k = 0, 1, 2, \ldots | \xi_k^2 = \eta_k^2 = (\xi_k \eta_k)^2 = 1, k = 0, 1, 2, \ldots \right\rangle,$$

where $\xi_0 = \xi = a$, $\eta_0 = \eta = cac$, $\alpha_k = \alpha_{\{\gamma_k\}} = \alpha^{\gamma_k}$, $\gamma_{2k} = (bc)^k$, $\gamma_{2k+1} = (bc)^k b$, and

$$A \times A \cong \left\langle \tilde{\xi}_k, \tilde{\eta}_k, \hat{\xi}_m, \hat{\eta}_m, k, m = 0, 1, 2, \ldots \left| \begin{array}{l} \tilde{\xi}_k^2 = \tilde{\eta}_k^2 = (\tilde{\xi}_k \tilde{\eta}_k)^2 = 1, \\ \hat{\xi}_m^2 = \hat{\eta}_m^2 = (\hat{\xi}_m \hat{\eta}_m)^2 = 1, \\ [\tilde{*}_k, \hat{*}_m] = 1, k, m = 0, 1, 2, \ldots \end{array} \right. \right\rangle,$$

where $\tilde{\alpha}_k = (\alpha_k, 1), \hat{\alpha_m} = (1, \alpha_m), \alpha \in \{\xi, \eta\}$. Let us compute the action of the group $\langle (b,c),(c,b) \rangle$ on $A \times A$. Let $x = (b,c)$ and $y = (c,b)$. Then

$$\tilde{\alpha}_k^x = (\alpha_k, 1)^{(b,c)} = (\alpha_k^b, 1) = (\alpha^{\gamma_k b}, 1) = \tilde{\alpha}_{\{\gamma_k b\}},$$

$$\tilde{\alpha}_k^y = (\alpha_k, 1)^{(c,b)} = (\alpha_k^c, 1),$$

$$\hat{\alpha}_k^x = (1, \alpha_k)^{(b,c)} = (1, \alpha_k^c),$$

$$\hat{\alpha}_k^y = (1, \alpha_k)^{(c,b)} = (1, \alpha_k^b) = (1, \alpha^{\gamma_k b}) = \hat{\alpha}_{\{\gamma_k b\}}.$$

If $k \geq 1$ then $\alpha_{2k}^c = \alpha_{\{(bc)^k\}}^c = \alpha_{2k-1}$; $\xi_0^c = \xi^c = a^c = \eta_0$, hence x and y act on the sets $\left\{ \tilde{\xi}_k, \tilde{\eta}_k, k \geq 0 \right\}$ and $\left\{ \hat{\xi}_k, \hat{\eta}_k, k \geq 0 \right\}$ as the following permutations of order 2:

$$\begin{array}{ccccccccccccccc} \ldots & \leftrightarrow & \eta_3 & & \eta_2 & \leftrightarrow & \eta_1 & & \eta_0 & \leftrightarrow & \xi_0 & & \xi_1 & \leftrightarrow & \xi_2 & & \xi_3 & \ldots \\ \ldots & & \eta_3 & \leftrightarrow & \eta_2 & & \eta_1 & \leftrightarrow & \eta_0 & & \xi_0 & \leftrightarrow & \xi_1 & & \xi_2 & \leftrightarrow & \xi_3 & \ldots \end{array}$$

Therefore, the group $\Psi(H)$ is generated by the elements $x, y, \tilde{\xi}_0, \tilde{\eta}_0, \hat{\xi}_0, \hat{\eta}_0$ and its presentation is

$$\Psi(H) \cong \left\langle x, y, \tilde{\xi}_0, \tilde{\eta}_0, \hat{\xi}_0, \hat{\eta}_0 \;\middle|\; \begin{array}{c} x^2 = y^2 = 1, \\ \tilde{\xi}_0^2 = \tilde{\eta}_0^2 = (\tilde{\xi}_0\tilde{\eta}_0)^2 = 1, \hat{\xi}_0^2 = \hat{\eta}_0^2 = (\hat{\xi}_0\hat{\eta}_0)^2 = 1, \\ [\tilde{*}_k, \hat{*}_m] = 0, k, m \geq 0 \end{array} \right\rangle. \tag{2.41}$$

Group Ω_1 is generated as a normal subgroup of the group Γ by the preimages of the relators of the group $\Psi(H)$. We have

$$b \longrightarrow (b, c) = x,$$

$$aba \longrightarrow (c, b) = y,$$

$$aca \longrightarrow (a, 1) = \tilde{\xi}_0,$$

$$c \longrightarrow (1, a) = \hat{\xi}_0.$$

From (2.41) we obtain

$$x^2 \rightarrowtail b^2 = 1,$$

$$y^2 \rightarrowtail (aba)^2 = 1,$$

$$\tilde{\xi}_0^2 \rightarrowtail (aca)^2 = 1,$$

$$\tilde{\eta}_0^2 \rightarrowtail ((aca)^{aba})^2 = 1,$$

$$(\tilde{\xi}_0\tilde{\eta}_0)^2 \rightarrowtail (acaabaacaaba)^2 = (acba)^4 \sim (bc)^4,$$

$$\hat{\xi}_0^2 \rightarrowtail c^2 = 1,$$

$$\hat{\eta}_0^2 \rightarrowtail (c^b)^2 = 1,$$

$$(\hat{\xi}_0\hat{\eta}_0)^2 \rightarrowtail (cbcb)^2 \sim (bc)^4.$$

Now let us look at the commutators in (2.41). We have

$$x \rightarrowtail b,$$

$$y \rightarrowtail aba,$$

$$\tilde{\xi}_{\{w(x,y)\}} \rightarrowtail (c^a)^{w(b,aba)=c^{aw(b,aba)}},$$

$$\tilde{\eta}_{\{w(x,y)\}} \rightarrowtail (c^b a)^{w(b,aba)} = c^{aw'(b,aba)},$$

$$\hat{\xi}_{\{v(x,y)\}} \rightarrowtail c^{v(aba,b)},$$

$$\hat{\eta}_{\{v(x,y)\}} \rightarrowtail (c^b)^{v(aba,b)} = c^{v'(b,aba)},$$

therefore we obtain the following:

$$\Omega_1 = \left\langle (bc)^4, [c^{az}, c^t] | z, t \in \langle b, aba \rangle \right\rangle^{\Gamma}.$$

As in the previous subsection, we have $\Omega_1 = \langle U_1 \rangle^{\Gamma}$ where

$$U_1 = \left\{ (bc)^4, [c, ab]^2, [c, bab]^2, [c, ababa]^2, [c, ababab]^2, [c, bababab]^2 \right\}.$$

Now let us find the substitution ϕ which transforms Ω_k to Ω_{k+1}. We have

$$\begin{aligned} \Psi(c) &= (1, a), \\ \Psi(b^a) &= (c, b), \\ \Psi(b) &= (b, c), \end{aligned}$$

so it is natural to define

$$\phi : \begin{cases} a \to c, \\ b \to aba, \\ c \to b. \end{cases}$$

$(bc)^4 = 1$ in G, therefore $\Psi(\phi(U_1)) \subset 1 \times \Omega_1$ and $\Psi(\phi(\Omega_1)) = \Omega_1 \times \Omega_1$. As above, we obtain that $\Omega_n = \left\langle \phi^k(U_1), k = 0, 1, \ldots, n-1 \right\rangle$. All we have left to prove is that the wreath recursion (Γ, Ψ) has property BA. Indeed, we have

$$\Psi_2(a) = (1, 1, 1, 1)(13)(24),$$
$$\Psi_2(b) = (b, c, 1, a),$$
$$\Psi_2(c) = (1, 1, 1, 1)(34),$$

hence

$$\Psi_2(ab) = (1, a, b, c)(13)(24),$$
$$\Psi_2(ac) = (1, 1, 1, 1)(1423),$$
$$\Psi_2(ba) = (b, c, 1, a)(13)(24),$$
$$\Psi_2(bc) = (b, c, 1, a)(34),$$
$$\Psi_2(ca) = (1, 1, 1, 1)(1324),$$
$$\Psi_2(cb) = (b, c, a, 1)(34),$$

i.e. any element of the group Γ is going to be contracted to the elements $\{1, a, b, c\}$ on some level of the tree. Since no elements from $\{a, b, c\}$ are trivial in G, any element of the group Γ which represents the trivial element of the group G is necessarily contracted to 1s on some level of the tree, i.e. the wreath recursion (Γ, Ψ) does have property BA. $\square$

CHAPTER III

CLASSIFICATION OF GROUPS GENERATED BY SMALL AUTOMATA

A fundamental problem of the theory of automata groups is the connection between the structure of an automaton and the properties of the group it generates. As we could see, finite automata generate very different groups, from finite abelian groups to free groups and groups of intermediate growth. But even small automata (i.e. automata with a small number of states) provide a huge number of groups with vastly different properties. It is natural to study groups generated by small automata, which often are elementary building blocks of bigger groups. Two important characteristics of a finite automaton are its number of states n and the cardinality d of the alphabet, and the pair (n, d) is a natural measure of complexity of the automaton.

Automata groups of complexity $(2, 2)$ have been described in [GNS00], there are only six groups in the class, namely, the trivial group, $\mathbb{Z}_2$, $\mathbb{Z}\times\mathbb{Z}$, $\mathbb{Z}$, the infinite dihedral group D_∞, and the lamplighter group $\mathbb{Z}\wr\mathbb{Z}_2$. The situation drastically changes when the complexity increases: $(3, 2)$ groups, which were studied by the research group at Texas A&M University [BGK$^+$07a, BGK$^+$07b], include groups as simple as abelian 2-groups, free abelian groups $\mathbb{Z}$ and $\mathbb{Z}^2$, and as complex groups as the Basilica group [GŻ02], iterated monodromy groups of several complex rational functions, Baumslag-Solitar groups $BS(1, 3)$ and $BS(1, -3)$ [Bv06], and a free non-abelian group with 3 generators [VV07].

We continue the work on the classification of the groups in the $(3, 2)$ class. We further reduce the number of possible non-isomorphic groups generated by $(3, 2)$ automata, and establish some connections between the groups in the class. Another part of our research is the studying groups in the class $(2, 3)$, which is a non-binary analog of the class $(3, 2)$, i.e. the class of minimal "interesting" automata over the alphabet with more than two letters.

A. Groups of complexity $(3,2)$

Theorem III.1. *There are no more than 115 pairwise non-isomorphic groups in the class of* $(3,2)$ *groups.*

Theorem III.2. *Group generated by the automaton*

$$\begin{aligned} a &= (c,c)\sigma, \\ b &= (a,b), \\ c &= (b,a) \end{aligned}$$

is a free product of its cyclic groups generated by a, b, *and* c.

Proposition III.3. *The group* G_{2193} *generated by the automaton*

$$\begin{aligned} a &= (c,b)\sigma, \\ b &= (a,a)\sigma, \\ c &= (a,a), \end{aligned}$$

contains the Lamplighter group as a subgroup of index 2.

Proof. Let

$$x = ca^{-1}, y = ab^{-1}, z = a^{-1}c, t = b^{-1}a,$$

$$s = (s,s)\sigma,$$

$$K = \langle x, z\rangle,$$

$$P = \langle x,y,z,t\rangle = \langle x,z,\sigma\rangle = \langle K,\sigma\rangle = \langle x,s\rangle = \langle x,z,s\rangle = \langle K,s\rangle,$$

then

$$\begin{aligned} x &= (y,x^{-1})\sigma, \\ y &= (x,y^{-1}), \\ z &= (t,z^{-1})\sigma, \\ t &= (t^{-1},z), \end{aligned}$$

and

$$s = x^2zy \in G, \sigma = xy \in G,$$

$$K \lhd P \overset{2}{\lhd} G,$$

$$|x| = |y| = |z| = |t| = \infty,$$

P is the normal subgroup of elements of length 2 in G.

$\langle x, y\rangle = \langle x, \sigma\rangle$ and $\langle z, t\rangle = \langle z, \sigma\rangle$ are lamplighters, and they are conjugated in G by element s:

$$x^s = z^{-1}, y^s = t^{-1}.$$

$$a^4 = b^4 = c^4 = (xz)^2 = 1,$$

$$b^2 = c^2,$$

We have the following:

$$[s^u, s^v] = 1, u, v \in K,$$

$$[\sigma^u, \sigma^v] = 1, u, v \in K,$$

i.e. P is a lamplighter. K is the subgroup of index 2 in the lamplighter group P, which corresponds to the set of even-number lamp configurations. □

Proposition III.4. $G_{957} \cong G_{939}$

Proof. We have $bc^{-2} = (bc^{-2}, ca^{-2})$, $ca^{-2} = (ca^{-1}c^{-1}, ca^{-2})$, $ca^{-1}c^{-1} = (ca^{-2}, ca^{-1}c^{-1})\sigma$, which is an automaton equivalent to automaton 939. bc^{-2}, ca^{-2}, and $ca^{-1}c^{-1}$ generate the whole group G_{957}, therefore $G_{957} \cong G_{939}$. □

Proposition III.5. $G_{966} \cong G_{740}$

Proof. We have $ab^{-1}a = (a, a), a = (c, a)(1, 2), c = (c, a)$, which is an automaton equivalent to automaton 740. $ab^{-1}a$, a, and c generate the whole group G_{966}, therefore $G_{966} \cong G_{740}$. □

Proposition III.6. $G_{741} = G_{2199}$

Proof. We have the following wreath recursions: $a = (c,a)\sigma, b = (b,a), c = (a,a)$ in G_{741}, and $a = (c,a)\sigma, b = (b,a)\sigma, c = (a,a)$ in G_{2199}. Obviously states a and c are the same in these two automata, so they are subautomata of the following:

$$a = (c,a)\sigma, c = (a,a), b_1 = (b_1,a), b_2 = (b_2,a)\sigma,$$

where b_1 is the element b in G_{741}, and b_2 is b in G_{2199}. $b_1 b_2 = a^{-1}$, therefore G_{741} and G_{2199} coincide as subgroups of the group of automorphisms of the binary tree. □

Proposition III.7. $G_{2361} \cong G_{939}$

Proof. We have $ca^{-1}b = (c, ca^{-1}b), c = (c,a), a = (c,a)\sigma$, which is an automaton equivalent to automaton 939. $ca^{-1}b$, c, and a generate the whole group G_{2361}, therefore $G_{2361} \cong G_{939}$. □

Proposition III.8. $G_{2365} \cong G_{939}$

Proof. We have $a^{-1}c^{-1}b = (c^{-1}, a^{-1}c^{-1}b)$, $c^{-1} = (c^{-1}, a^{-1})$, $a^{-1} = (c^{-1}, a^{-1})\sigma$, which is an automaton equivalent to automaton 939. $a^{-1}c^{-1}b$, c^{-1}, and a^{-1} generate the whole group G_{2365}, therefore $G_{2365} \cong G_{939}$. □

Proposition III.9. $G_{2395} \cong G_{937}$

Proof. We have $cb^{-1}a = (cb^{-1}a, ac^{-1}a)$, $ac^{-1}a = (a, ac^{-1}a)$, $a = (a,a)\sigma$, which is an automaton equivalent to automaton 937. $cb^{-1}a$, $ac^{-1}a$, and a generate the whole group G_{2395}, therefore $G_{2395} \cong G_{937}$. □

Proposition III.10. $G_{2401} \cong G_{920}$

Proof. We have $c^{-1}ab^{-1} = (c^{-1}ac^{-1}, c^{-1}ab^{-1})$, $c^{-1}ac^{-1} = (c^{-1}, c^{-1}ac^{-1})\sigma$, $c^{-1} = (c^{-1}, c^{-1}ac^{-1})$, which is an automaton equivalent to the automaton 920. $c^{-1}ab^{-1}$, $c^{-1}ac^{-1}$, and c^{-1} generate the whole group G_{2401}, therefore $G_{2401} \cong G_{920}$. □

B. Groups of complexity $(2, 3)$

Let us consider the groups generated by automata with two states a, b over the alphabet with three letters $\{1, 2, 3\}$. We will use the terminology analogous to that from [BGK+07a, BGK+07b]. Let us describe first the numeration system.

Given automaton $\mathcal{A}$ with the set of states $\{a, b\}$ over 3-letter alphabet $X = \{1, 2, 3\}$

$$\mathcal{A}: \begin{cases} a = (a_1, a_2, a_3)\pi, \\ b = (b_1, b_2, b_3)\rho, \end{cases}$$

we can assign to it the unique tuple $(a_1, a_2, a_3, b_1, b_2, b_3, \pi, \rho)$. By arranging these in some order, we obtain the ordering on the set of all three-state automata over the two-letter alphabet. Here we use the ordering analogous to that used in [BGK+07a, BGK+07b]. Namely, we say that $(a_1, a_2, a_3, b_1, b_2, b_3, \pi, \rho) < (a'_1, a'_2, a'_3, b'_1, b'_2, b'_3, \pi', \rho')$ if $(\rho, \pi, b_3, b_2, b_1, a_3, a_2, a_1) <_{lex} (\rho', \pi', b'_3, b'_2, b'_1, a'_3, a'_2, a'_1)$, where $<_{lex}$ is the lexicographic ordering induced by $a < b$ and the following ordering on the symmetric group S_3: $() < (12) < (13) < (23) < (123) < (132)$ (the latter is arbitrary, we only want $() < (12)$ to be consistent with the classification of the $(3, 2)$ groups). In this way we obtain the following ordering on the set of $(2,3)$ automata:

$$\begin{aligned}
\mathcal{A}_1: &\quad a=(a,a,a), b=(a,a,a),\\
\mathcal{A}_2: &\quad a=(b,a,a), b=(a,a,a),\\
\mathcal{A}_3: &\quad a=(a,b,a), b=(a,a,a),\\
\mathcal{A}_4: &\quad a=(b,b,a), b=(a,a,a),\\
&\quad \dots\\
\mathcal{A}_{64}: &\quad a=(b,b,b), b=(b,b,b),\\
\mathcal{A}_{65}: &\quad a=(a,a,a)(12), b=(a,a,a),\\
&\quad \dots\\
\mathcal{A}_{2304}: &\quad a=(b,b,b)(132), b=(b,b,b)(132).
\end{aligned}$$

As with $(3,2)$ groups, we eliminate isomorphic automata, and we also eliminate reducible automata (which are necessarily one-state automata generating trivial group, S_2, or S_3). Let $\mathcal{M}_n$ be the group generated by the automaton number n in the class $(2,3)$.

Theorem III.11.

- *There are* 139 *non-isomorphic non-reducible automata with two states on the alphabet with three letters.* 30 *of them generate finite groups,* 108 *generate infinite groups, and it is unknown whether* $\mathcal{M}_{675}$ *is finite (it is probably infinite).*
- *The finite groups generated by* $(2,3)$ *automata are:* 1*,* D_6*,* $S_3\times C_3$*,* $S_3\times S_3$*,* S_3*,* $\mathbb{Z}_2$*,* $\mathbb{Z}_2\times\mathbb{Z}_2$*,* $\mathbb{Z}_3$*,* $\mathbb{Z}_3\times\mathbb{Z}_3$*.*
- *Infinite abelian groups in the class are* $\mathbb{Z}$ *and* $\mathbb{Z}^2$*.*
- *There are no more than* 78 *non-isomorphic infinite non-abelian groups generated by automata in the class* $(2,3)$*. Among them there is the Lamplighter group and the infinite dihedral group* D_∞*.*

- *There are no infinite torsion groups in the class of* $(2,3)$ *groups. There are no non-abelian free groups in the class of* $(2,3)$ *groups.*

Proof of this theorem consists of studying every individual automaton from the list, below we provide the proofs of non-trivial facts for automata which do not generate finite groups. Significant part of the information we obtained was produced using the *AutomGrp* package [MS] for computer algebra system *GAP*, developed by the author and his fellow graduate student Dmytro Savchuk.

- **Automaton 66**:

  Group $\mathcal{M}_{66}$ is abelian. Image of the canonical projection onto $Aut\,\mathcal{T}_2$ is the infinite cyclic group generated by the automaton

  $$\alpha = (\beta, \alpha)(12), \beta = (\alpha, \alpha).$$

  In this group we have $\alpha^2\beta = 1$, therefore only possible relations in the group $\mathcal{M}_{66}$ are of the form $a^{2k}b^k = 1$, but then

  $$a^{2k}b^k = (ab, ab, a^2)^k(a^k, a^k, a^k) = (a^{2k}b^k, a^{2k}b^k, a^3k) = 1,$$

  and $a^k = 1$ which is possible only when $k = 0$. Therefore $\mathcal{M}_{66}$ is a two-generated free abelian group.

- **Automaton 70**:

  Similar to 66. The same projection, the difference is at the third coordinate: if $a^{2k}b^k = 1$ then

  $$a^{2k}b^k = (ab, ab, b^2)^k(a^k, a^k, a^k) = (a^{2k}b^k, a^{2k}b^k, a^kb^2k) = 1,$$

  and $a^kb^2k = 1$ which is possible only when $k = 0$. Therefore $\mathcal{M}_{70}$ is a two-generated free abelian group.

- **Automaton 73**:

  Image of the canonical homomorphism into $Aut\,\mathcal{T}_2$ is the infinite dihedral group generated by

$$\begin{array}{rcl} a & = & (a,a)\sigma \\ b & = & (b,a) \end{array},$$

  therefore $\mathcal{M}_{73}$ is infinite, and hence it is D_∞, since it is generated by two involutions.

- **Automaton 74**:

  Image of the canonical homomorphism into $Aut\,\mathcal{T}_2$ is the lamplighter group L generated by

$$\begin{array}{rcl} a & = & (b,a)\sigma \\ b & = & (b,a) \end{array},$$

  and $\mathcal{M}_{74}$ satisfies the defining relations of the group L: let $\sigma = b^{-1}a = (12)$, then

$$\sigma^2 = 1,$$

$$(\sigma b^n \sigma b^{-n})^2 = (a^n b^{-n}, b^n a^{-n}, 1)^2 = 1$$

  (it is trivial because it is so in the lamplighter, and on the third coordinate we have 1 in all states of the resulting automaton). Therefore, it is the lamplighter group.

  It is not contracting because of the third coordinate: $a^n = (\cdot,\cdot,a^n)(12)^n$, and a has infinite order.

- **Automaton 76**:

  Same as 73.

- **Automaton 77**:

Same as 73.

- **Automaton 78**:

Similar to 74. The only difference is that in the relations the third coordinate is not identity (but still is trivial, which is easy to see).

It is not contracting because of the third coordinate: $a^n = (\cdot, \cdot, b^n)(12)^n$, $b^n = (\cdot, \cdot, a^n)$.

- **Automaton 80**:

Same as 73.

- **Automaton 90**:

Group $\mathcal{M}_{90}$ is abelian. Image of the canonical projection onto $Aut\,\mathcal{T}_2$ is the infinite cyclic group generated by the adding machine

$$t = (1, t)(12).$$

Therefore the only possible relations in the group $\mathcal{M}_{66}$ are of the form $b^k = 1$, but then

$$b^k = (b^k, b^k, a^k) = 1$$

and $a^k = 1$, which is possible only when $k = 0$. Therefore $\mathcal{M}_{90}$ is a two-generated free abelian group.

- **Automaton 94**:

Same as 90.

- **Automaton 98**:

Same as 90.

- **Automaton 102**:

Similar to 66. The same projection, the difference is at the third coordinate: if $a^{2k}b^k = 1$ then

$$a^{2k}b^k = (ab, ab, b^2)^k(a^k, a^k, b^k) = (a^{2k}b^k, a^{2k}b^k, b^3k) = 1,$$

and $b^3k = 1$ which is possible only when $k = 0$. Therefore $\mathcal{M}_{102}$ is a two-generated free abelian group.

- **Automaton 105**:

  Same as 73.

- **Automaton 106**:

  Same as 78.

- **Automaton 108**:

  Same as 73.

- **Automaton 109**:

  Same as 73.

- **Automaton 110**:

  Same as 74.

- **Automaton 112**:

  Same as 73.

- **Automaton 260**:

  a and b have infinite order - proof is straightforward computations using the fact that the group is abelian.

- **Automaton 268**:

  For $x = b^{-1}aba^{-1}ba^{-1}b^{-1}a = (1, a^{-1}bab^{-1}ab^{-1}, a^{-1}b)$ the commutator $[[x, x^a], x^{a^{-1}}]$

is a relator for the group $\mathcal{M}_{268}$.

Let $x = ab^{-1}$. Then $x = (x^{-1}, x^{-1}, x)(1,2,3)$, $x^3 = (x^{-1}, x^{-1}, x^{-1})$, and therefore x has infinite order.

- **Automaton 270**:

  For $x = a^{-1}ba^{-2}b^2 = (1, b^{-1}a, b^{-2}a^2)$ the commutator $[[x, x^a], x^{a^{-1}}]$ is a relator for the group $\mathcal{M}_{270}$.

  Let $x = ab^{-1}$. Then $x = (x^{-1}, 1, 1)(1,2,3)$, $x^3 = (x^{-1}, x^{-1}, x^{-1})$, and therefore x has infinite order.

- **Automaton 282**:

  For $x = bab^{-1}a^{-1} = (1, ba^{-1}, a^2b^{-1}a^{-1})$ the commutator $[[x, x^a], x^{a^{-1}}]$ is a relator for the group $\mathcal{M}_{282}$.

  $a^{-1}b = (1, 1, a^{-1}b)(132)$, therefore $a^{-1}b$ has infinite order.

- **Automaton 283**:

  For $x = a^{-1}b^{-1}a^2ba^{-2}bab^{-1} = (1, a^{-1}b^{-1}abab^{-1}a^{-1}bab^{-1}, b^{-1}aba^{-1}ba^{-1})$ the commutator $[[x, x^a], x^{a^{-1}}]$ is a relator for the group $\mathcal{M}_{283}$.

  $a^{-1}b = (1, a^{-1}b, 1)(132)$, therefore $a^{-1}b$ has infinite order.

- **Automaton 288**:

  $a^{-1}b = (b^{-1}a, 1, 1)(132)$, therefore $a^{-1}b$ has infinite order.

- **Automaton 514**:

  For $x = b^3a^{-2}b^{-1} = (1, a^2b^{-1}a^{-1}, a^3b^{-1}a^{-2})$ the commutator $[[x, x^a], x^b]$ is a relator for the group $\mathcal{M}_{514}$.

  $a^{-1}b = (1, 1, b^{-1}a)(132) = (1, 1, (a^{-1}b)^{-1})(132)$, therefore $a^{-1}b$ has infinite order.

- **Automaton 515**:

  It is infinite, and it is generated by two involutions, hence it is D_∞.

- **Automaton 516**:

  For $x = ab^2a^{-1}b^{-2} = (ba^2b^{-1}a^{-2}, ba^2b^{-1}a^{-2}, 1)$ the commutator $[[x, x^a], x^{ab}]$ is a relator for the group $\mathcal{M}_{516}$.

  $a^{-1}b = (1, b^{-1}a, b^{-1}a)(132) = (1, (a^{-1}b)^{-1}, (a^{-1}b)^{-1})(132)$, therefore $a^{-1}b$ has infinite order.

- **Automaton 518**:

  We have $x = (x^{-1}, 1, x^{-1})(132)$ where $x = ab$. Then $x^3 = (x^{-2}, x^{-2}, x^{-2})$, $x^{-2} = (x, x, x)(123)$, therefore x has infinite order, and $\mathcal{M}_{518}$ is D_∞.

- **Automaton 521**:

  $a^{-1}b = (1, 1, a^{-1}b)(132)$, therefore $a^{-1}b$ has infinite order.

- **Automaton 524**:

  For $x = a^2b^{-2} = (1, ba^{-1}, aba^{-2})$ the commutator $[[x, x^a], x^b]$ is a relator for the group $\mathcal{M}_{524}$.

  $a^{-1}b = (1, b^{-1}a, 1)(132) = (1, (a^{-1}b)^{-1}, 1)(132)$, therefore $a^{-1}b$ has infinite order.

- **Automaton 526**:

  For $x = ba^2b^{-1}ab^{-2}a^{-1} = (1, aba^{-2}, ab^2a^{-1}ba^{-1}b^{-2})$ the commutator $[[x, x^a], x^b]$ is a relator for the group $\mathcal{M}_{526}$.

  $a^{-1}b = (b^{-1}a, 1, 1)(132) = ((a^{-1}b)^{-1}, 1, 1)(132)$, therefore $a^{-1}b$ has infinite order.

- **Automaton 527**:

  Dual group of $\mathcal{M}_{527}$ is spherically transitive, and for any $k > 0$ there is a word w of length k in $\langle a, b\rangle$ such that $w \neq 1$ in $\mathcal{M}_{527}$. Therefore the semigroup generated by a, b is free, in particular a and b have infinite order.

  $\mathcal{M}_{527}$ is not contracting because $a^n = (\cdot, b^n, \cdot)(13)^n$ and $b^n = (\cdot, \cdot, a^n)(12)^n$ and a and b have infinite order.

  For $x = a^{-1}b$ we have $x = (x^{-1}, x^{-1}, x)(132)$, $x^3 = (x^{-1}, x^{-1}, x^{-1})$, therefore x has infinite order.

- **Automaton 537**:

  Same as 518.

- **Automaton 538**:

  For $x = ab^2a^{-1}ba^{-2}b^{-1} = (1, ab^2a^{-1}ba^{-1}b^{-2}, aba^{-2})$ the commutator $[[x, x^a], x^b]$ is a relator for the group $\mathcal{M}_{538}$.

  $a^{-1}b = (1, a^{-1}b, 1)(132)$, therefore $a^{-1}b$ has infinite order.

- **Automaton 539**:

  We have $x = (1, 1, x)(132)$ where $x = ab$. x has infinite order, and therefore $\mathcal{M}_{539}$ is D_∞.

- **Automaton 545**:

  Same as 539.

- **Automaton 550**:

  We have $x = (1, 1, x^{-1})(132)$ where $x = ab$. x has infinite order, and therefore $\mathcal{M}_{550}$ is D_∞.

- **Automaton 554**:

  For $x = a^2b^{-2} = (1, a^2b^{-1}a^{-1}, ab^{-1})$ the commutator $[[x, x^a], x^b]$ is a relator for the group $\mathcal{M}_{554}$.

  $a^{-1}b = (a^{-1}b, 1, 1)(132)$, therefore $a^{-1}b$ has infinite order.

- **Automaton 557**:

  $a^{-1}b = (1, 1, a^{-1}b)(132)$, therefore $a^{-1}b$ has infinite order.

- **Automaton 643**:

  For $x = a^{-1}ba^{-1}b = (b^{-1}a, 1, b^{-1}a)$ the commutator $[[x, x^a], x^b]$ is a relator for the group $\mathcal{M}_{643}$.

- **Automaton 644**:

  For $x = b^2ab^{-2}a^{-1} = (a^2ba^{-2}b^{-1}, a^2ba^{-2}b^{-1}, 1)$ the commutator $[[x, x^a], x^{a^{-1}}]$ is a relator for the group $\mathcal{M}_{644}$.

- **Automaton 655**:

  Free semigroup and non-contracting - same as 527.

- **Automaton 666**:

  For $x = b^3a^{-1}b^{-1}a^{-1} = (1, b^3a^{-3}, a^2b^{-1}a^{-1})$ the commutator $[[x, x^a], x^{a^{-1}}]$ is a relator for the group $\mathcal{M}_{666}$.

- **Automaton 676**:

  Free semigroup - same as 527.

  $b^2 = (a^2, a^2, b^2)$ and b has infinite order, therefore $\mathcal{M}_{676}$ is not contracting.

- **Automaton 679**:

  For $x = a^{-1}ba^{-1}b = (b^{-1}a, 1, b^{-1}a)$ the commutator $[[x, x^a], x^{a^{-1}}]$ is a relator for the group $\mathcal{M}_{679}$.

- **Automaton 683**:

  Free semigroup - same as 527.

  $b^2 = (ba, ab, b^2)$ and b has infinite order, therefore $\mathcal{M}_{683}$ is not contracting.

- **Automaton 690**:

  Free semigroup - same as 527.

  $b^2 = (ab, ba, b^2)$ and b has infinite order, therefore $\mathcal{M}_{690}$ is not contracting.

- **Automaton 703**:

  $\mathcal{M}_{703}$ is generated by two involutions b and $a^{-1}b$ and it is infinite. Therefore $\mathcal{M}_{703}$ is the infinite dihedral group.

- **Automaton 1858**:

  For $x = ab^{-1}aba^{-2} = (1, a^2b^{-1}a^{-1}, ba^{-1})$ the commutator $[[x, x^a], x^{a^{-1}}]$ is a relator for the group $\mathcal{M}_{1858}$.

- **Automaton 1860**:

  For $x = b^{-1}a^2ba^{-2} = (1, a^{-1}ba^2b^{-2}, a^{-1}b^2ab^{-1}a^{-1})$ the commutator $[[x, x^a], x^{a^{-1}}]$ is a relator for the group $\mathcal{M}_{1860}$.

CHAPTER IV

CONCLUSION

In the first part of the dissertation we described the sufficient condition for an essentially free automaton group to have an L-presentation, suitable for actual computations in finite automata groups. We found the L-presentation for several groups generated by three-state automata, and we described the defining relations in the Grigorchuk groups G_ω and in the series of groups which contain Grigorchuk-Erschler group and the group of iterated monodromies of the polynomial $z^2 + i$. In case when these groups are generated by finite automata, the relations found constitute L-presentations.

In the second part of the dissertation we made further progress in the classification of the groups generated by 3-state automata acting on binary trees, and we laid the foundation for the classification of the groups generated by 2-state automata over the 3-letter alphabet. This is a part of the classification work of the research group at Texas A&M University.

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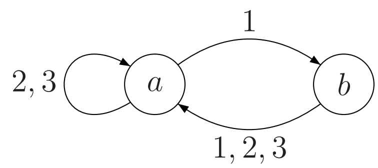
APPENDIX A

(2, 3) AUTOMATA

Here we provide the information about each (potentially) infinite group from the class $(2,3)$. We identify the group, specify whether it is finite, abelian, contracting, self-replicating, and torsion-free when it is known, otherwise we omit these data or list "?". For some groups we also provide additional data, such as short relations or short elements of infinite order. Also, for every group we provide the histograms of spectra of the discrete Laplace operator on Schreier graphs on the sixth level of the tree $\mathcal{T}_3$.

Automaton number 66

$$a = (b, a, a)(12)$$
$$b = (a, a, a)$$



Group: $\mathbb{Z}^2$

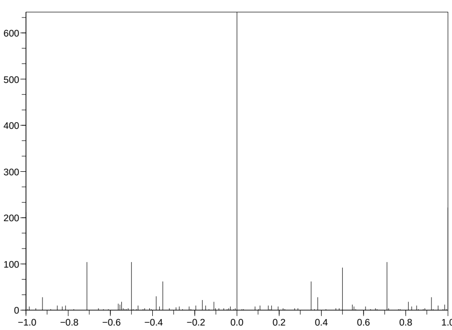
Abelian: *yes*

Finite: *no*

Contracting: *?*

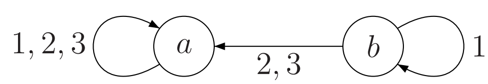
Self-replicating: *no*

Torsion-free: *yes*



Automaton number 73

$$a = (a, a, a)(12)$$
$$b = (b, a, a)$$



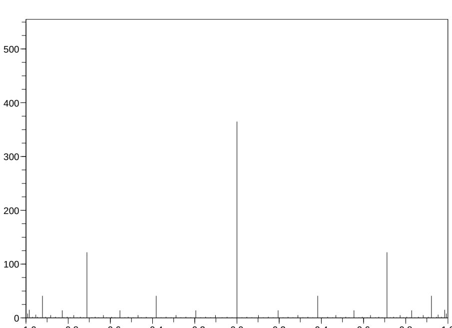
Group: D_∞

Abelian: *no*

Finite: *no*

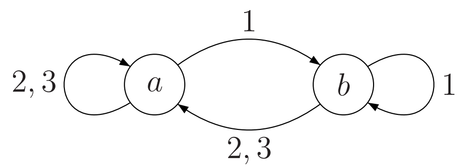
Contracting: *yes*

Self-replicating: *no*



Automaton number 74

$$a = (b, a, a)(12)$$
$$b = (b, a, a)$$



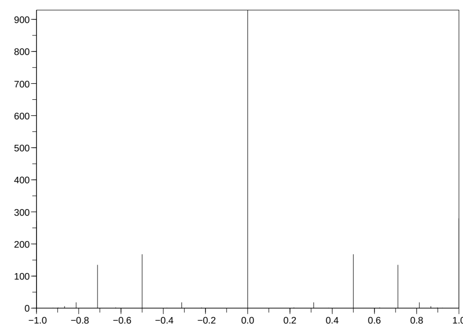
Group: Lamplighter

Abelian: *no*

Finite: *no*

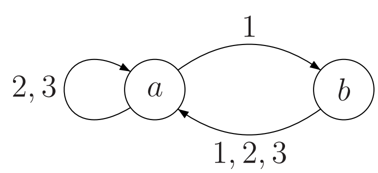
Contracting: *no*

Self-replicating: *no*



Automaton number 258

$$a = (b, a, a)(123)$$
$$b = (a, a, a)$$



Group: $\mathbb{Z}$

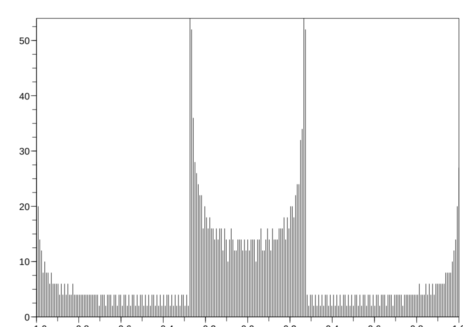
Abelian: *yes*

Finite: *no*

Contracting: *yes*

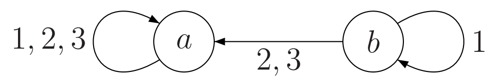
Self-replicating: *yes*

Torsion-free: *yes*
Notes: $a^3b = 1$



Automaton number 265

$$a = (a, a, a)(123)$$
$$b = (b, a, a)$$



Group:

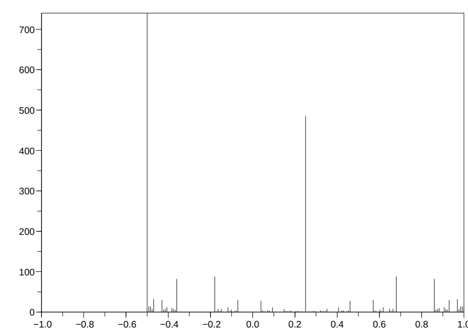
Abelian: *no*

Finite: *no*

Contracting: *yes*

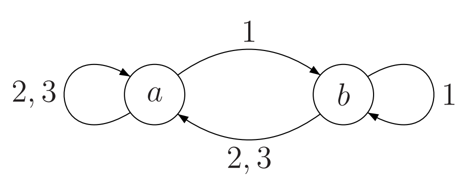
Self-replicating: *yes*

Notes: $|ab| = \infty$



Automaton number 266

$$a = (b, a, a)(123)$$
$$b = (b, a, a)$$



Group:

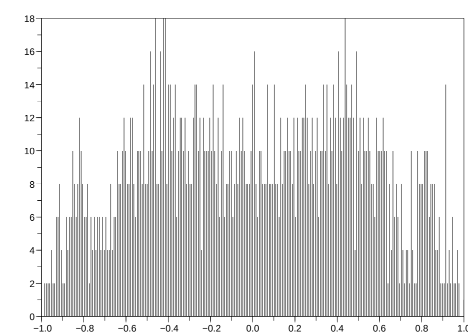
Abelian: *no*

Finite: *no*

Contracting: *?*

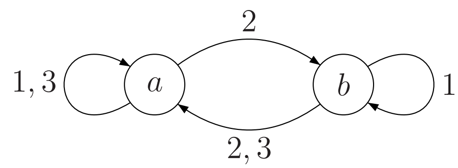
Self-replicating: *yes*

Torsion: *?*



Automaton number 267

$$a = (a,b,a)(123)$$
$$b = (b,a,a)$$



Group:

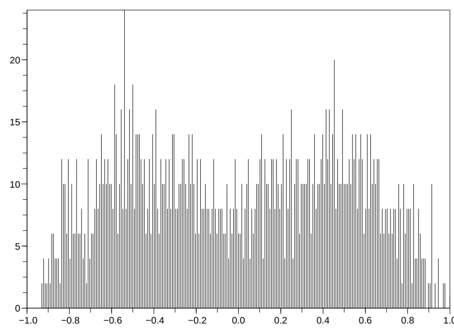
Abelian: *no*

Finite: *no*

Contracting: *?*

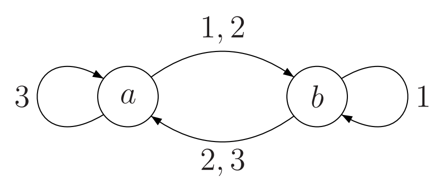
Self-replicating: *yes*

Torsion: *?*



Automaton number 268

$$a = (b,b,a)(123)$$
$$b = (b,a,a)$$



Group:

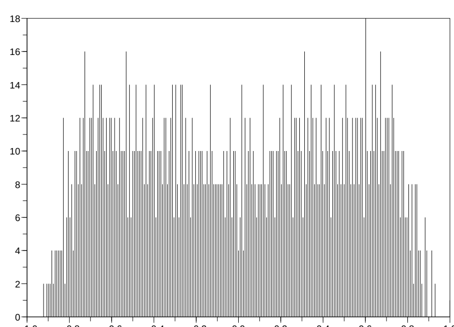
Abelian: *no*

Finite: *no*

Contracting: *no*

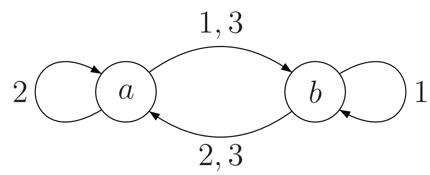
Self-replicating: *yes*

Torsion-free: *?*
Notes: no relations of length ≤ 7, not free, $|ab^{-1}| = \infty$



Automaton number 270

$$a = (b, a, b)(123)$$
$$b = (b, a, a)$$



Group:

Abelian: *no*

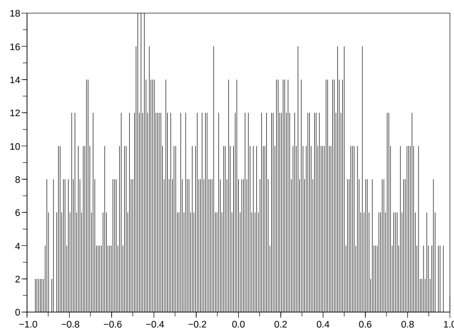
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

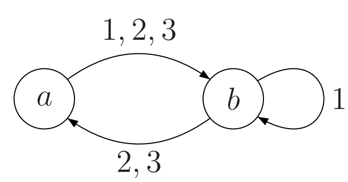
Torsion-free: *?*

Notes: not free, $|ab^{-1}| = \infty$



Automaton number 272

$$a = (b, b, b)(123)$$
$$b = (b, a, a)$$



Group:

Abelian: *no*

Finite: *no*

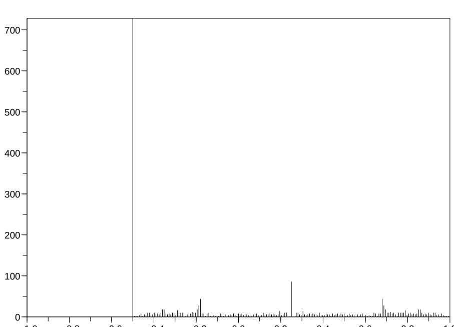
Contracting: *?*

Self-replicating: *yes*

Torsion: *?*

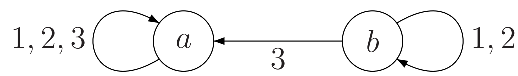
Torsion-free: *?*

Notes: not $\langle a \rangle * \langle b \rangle$



Automaton number 281

$$a = (a, a, a)(123)$$
$$b = (b, b, a)$$



Group:

Abelian: *no*

Finite: *no*

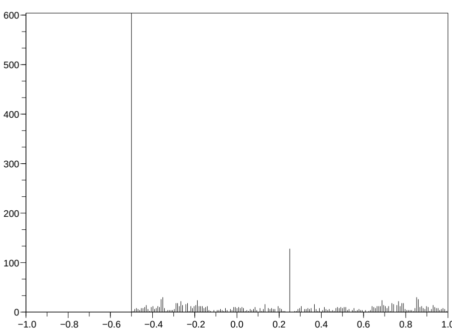
Contracting: *?*

Self-replicating: *yes*

Torsion: *?*

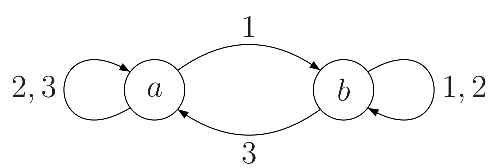
Torsion-free: *?*

Notes: not $\langle a \rangle * \langle b \rangle$



Automaton number 282

$$a = (b, a, a)(123)$$
$$b = (b, b, a)$$



Group:

Abelian: *no*

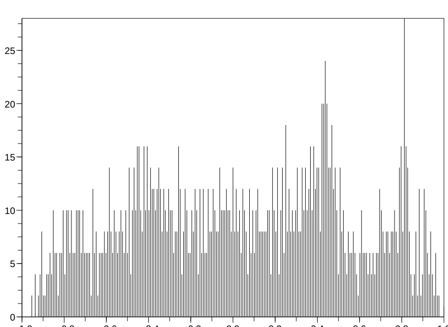
Finite: *no*

Contracting: *no*

Self-replicating: *yes*

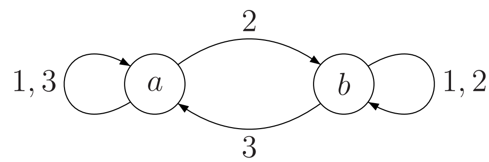
Torsion-free: *?*

Notes: not free, $|a^{-1}b| = \infty$



Automaton number 283

$$a = (a, b, a)(123)$$
$$b = (b, b, a)$$



Group:

Abelian: *no*

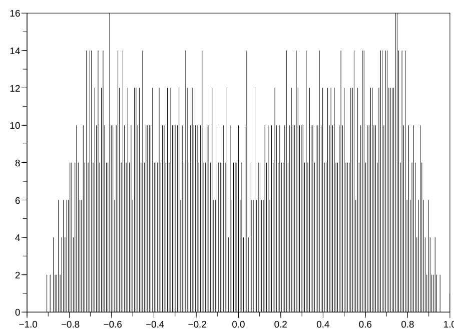
Finite: *no*

Contracting: *no*

Self-replicating: *yes*

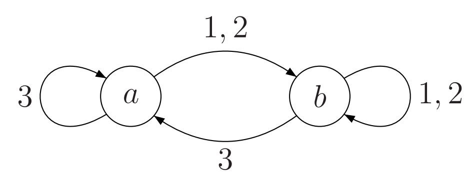
Torsion-free: *?*

Notes: not free, $|a^{-1}b| = \infty$



Automaton number 284

$$a = (b, b, a)(123)$$
$$b = (b, b, a)$$



Group:

Abelian: *no*

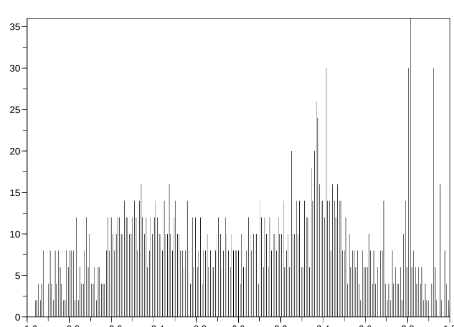
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

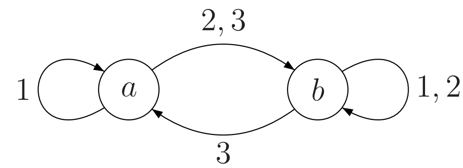
Torsion: *?*

Torsion-free: *?*



Automaton number 287

$$a = (a, b, b)(123)$$
$$b = (b, b, a)$$



Group:

Abelian: *no*

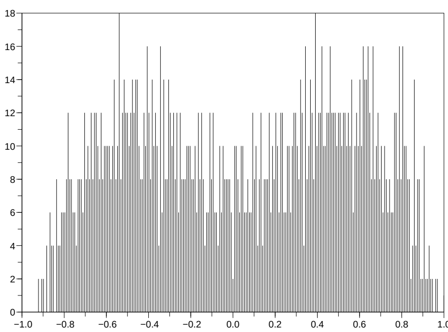
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

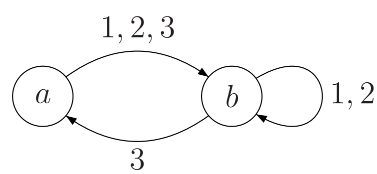
Torsion-free: *?*

Notes: $|a^{-2}b^2| = \infty$



Automaton number 288

$$a = (b, b, b)(123)$$
$$b = (b, b, a)$$



Group:

Abelian: *no*

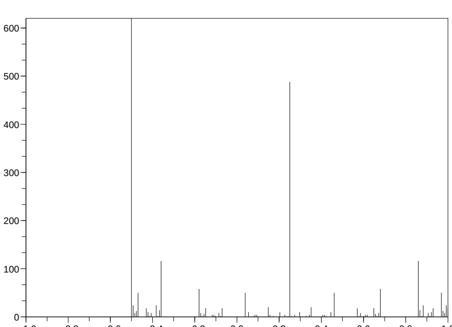
Finite: *no*

Contracting: *yes*

Self-replicating: *yes*

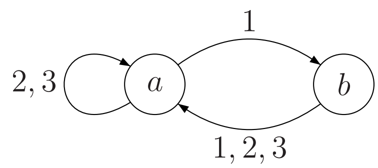
Torsion-free: *?*

Notes: $|ab| = \infty$, $|a^{-1}b| = \infty$



Automaton number 514

$$a = (b, a, a)(13)$$
$$b = (a, a, a)(12)$$



Group:

Abelian: *no*

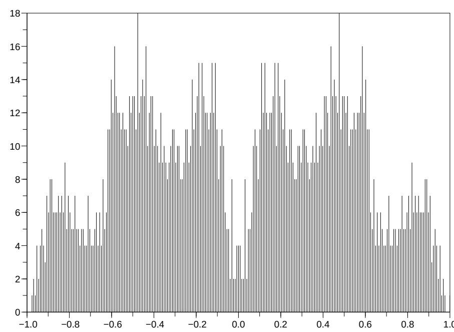
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

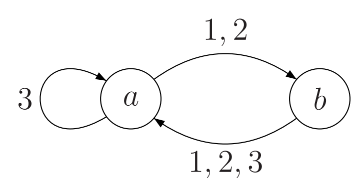
Torsion-free: *?*

Notes: not free, $|a^{-1}b| = \infty$



Automaton number 516

$$a = (b, b, a)(13)$$
$$b = (a, a, a)(12)$$



Group:

Abelian: *no*

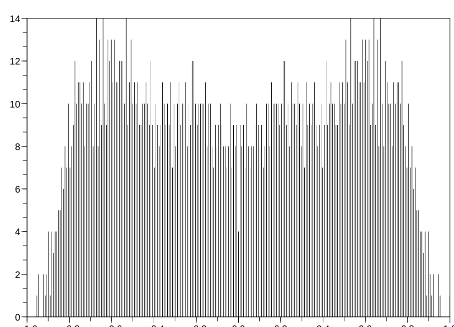
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

Torsion-free: *?*

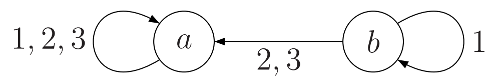
Notes: not free, $|a^{-1}b| = \infty$



Automaton number 521

$$a = (a, a, a)(13)$$

$$b = (b, a, a)(12)$$



Group:

Abelian: *no*

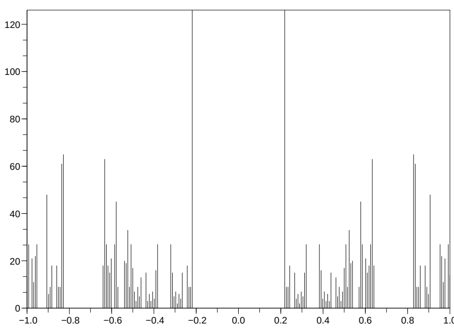
Finite: *no*

Contracting: *yes*

Self-replicating: *yes*

Torsion-free: *?*

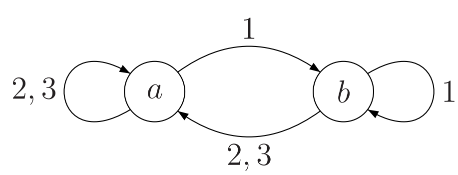
Notes: $|a^{-1}b| = \infty$



Automaton number 522

$$a = (b, a, a)(13)$$

$$b = (b, a, a)(12)$$



Group:

Abelian: *no*

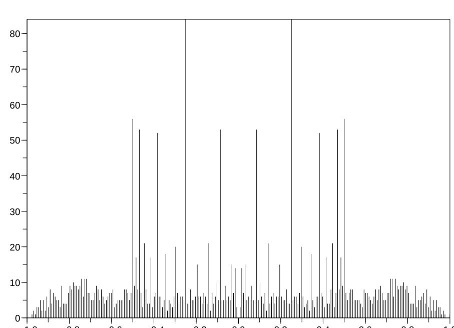
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

Torsion: *?*

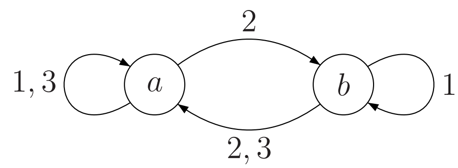
Torsion-free: *?*



Automaton number 523

$a = (a, b, a)(13)$

$b = (b, a, a)(12)$



Group:

Abelian: *no*

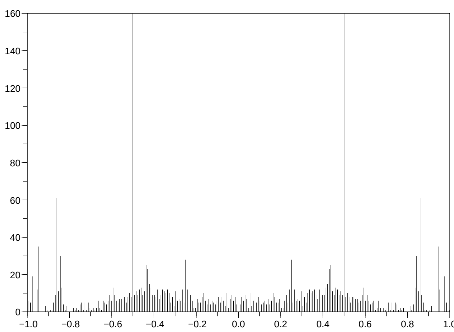
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

Torsion: *?*

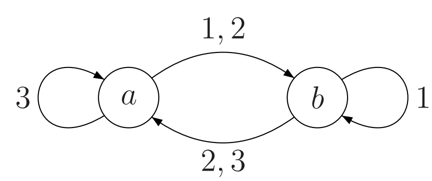
Torsion-free: *?*



Automaton number 524

$a = (b, b, a)(13)$

$b = (b, a, a)(12)$



Group:

Abelian: *no*

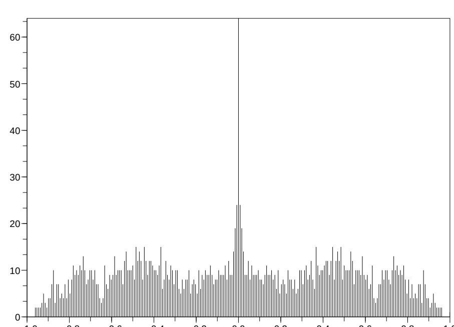
Finite: *no*

Contracting: *no*

Self-replicating: *yes*

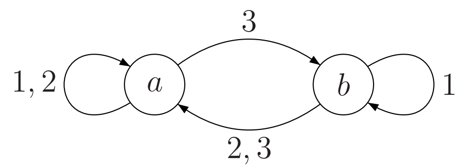
Torsion-free: *?*

Notes: not free, $|a^{-1}b| = \infty$



Automaton number 525

$$a = (a, a, b)(13)$$
$$b = (b, a, a)(12)$$



Group:

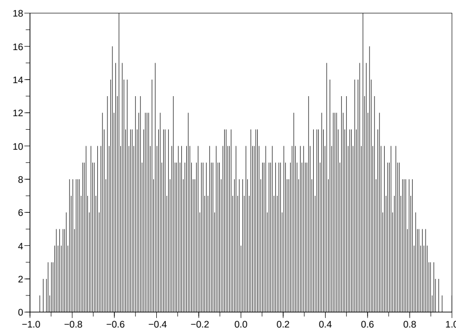
Abelian: *no*

Finite: *no*

Contracting: *?*

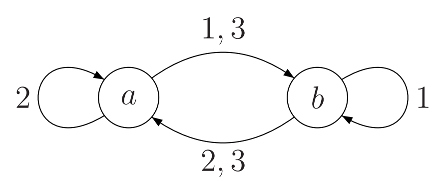
Self-replicating: *yes*

Torsion-free: *?*
Notes: $|a^{-2}b^2| = \infty$



Automaton number 526

$$a = (b, a, b)(13)$$
$$b = (b, a, a)(12)$$



Group:

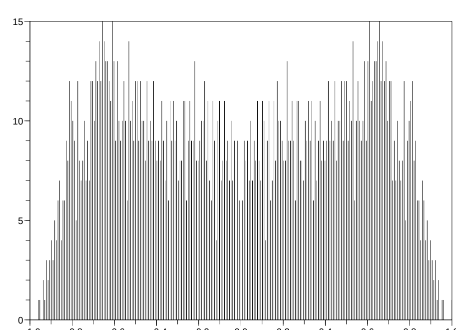
Abelian: *no*

Finite: *no*

Contracting: *no*

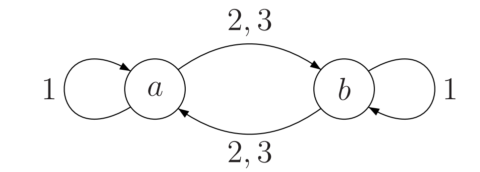
Self-replicating: *yes*

Torsion-free: *?*
Notes: not free, $|a^{-1}b| = \infty$



Automaton number 527

$$a = (a, b, b)(13)$$
$$b = (b, a, a)(12)$$



Group:

Abelian: *no*

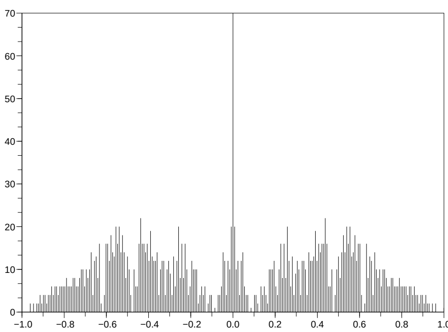
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

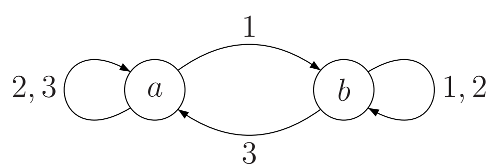
Torsion-free: *?*

Notes: free semigroup, $|a^{-1}b| = \infty$



Automaton number 538

$$a = (b, a, a)(13)$$
$$b = (b, b, a)(12)$$



Group:

Abelian: *no*

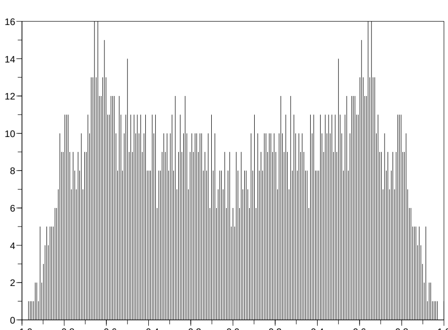
Finite: *no*

Contracting: *no*

Self-replicating: *yes*

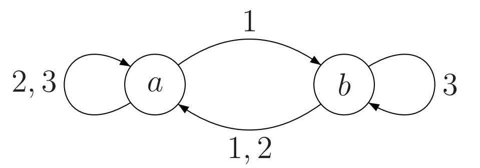
Torsion-free: *?*

Notes: not free, $|a^{-1}b| = \infty$



Automaton number 546

$$a = (b, a, a)(13)$$
$$b = (a, a, b)(12)$$



Group:

Abelian: *no*

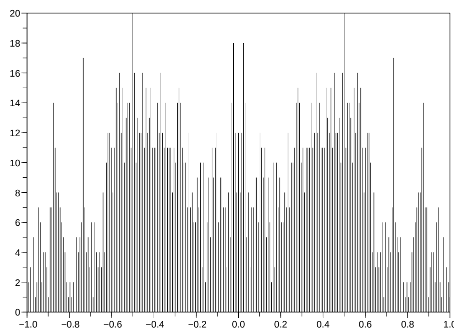
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

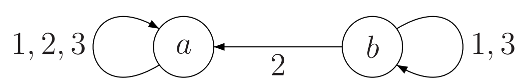
Torsion: *?*

Torsion-free: *?*



Automaton number 553

$$a = (a, a, a)(13)$$
$$b = (b, a, b)(12)$$



Group:

Abelian: *no*

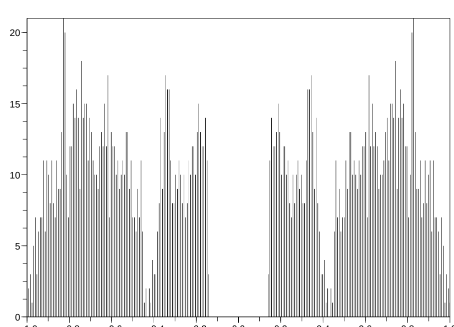
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

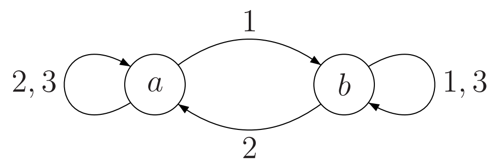
Torsion: *?*

Torsion-free: *?*



Automaton number 554

$$a = (b, a, a)(13)$$
$$b = (b, a, b)(12)$$



Group:

Abelian: *no*

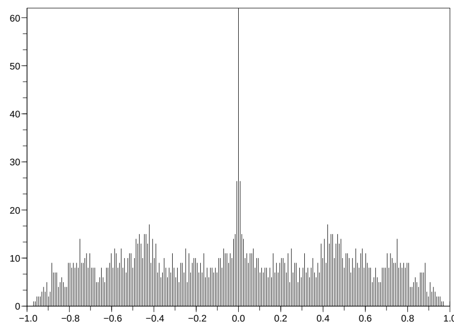
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

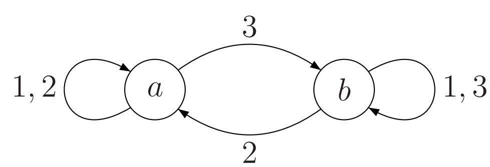
Torsion-free: *?*

Notes: not free, $|a^{-1}b| = \infty$



Automaton number 557

$$a = (a, a, b)(13)$$
$$b = (b, a, b)(12)$$



Group:

Abelian: *no*

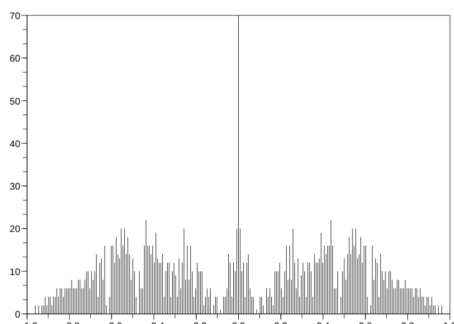
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

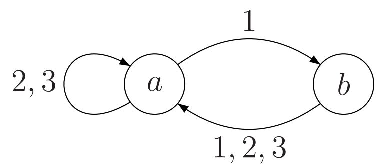
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 642

$$a = (b, a, a)(123)$$
$$b = (a, a, a)(12)$$



Group:

Abelian: *no*

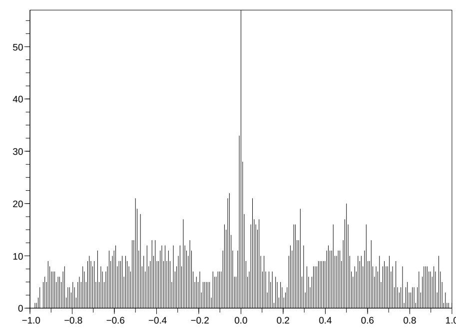
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

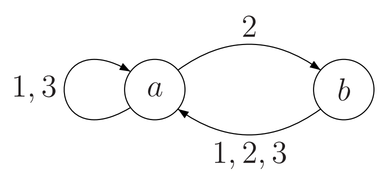
Torsion-free: *?*

Notes: $|a^{-2}b^2| = \infty$



Automaton number 643

$$a = (a, b, a)(123)$$
$$b = (a, a, a)(12)$$



Group:

Abelian: *no*

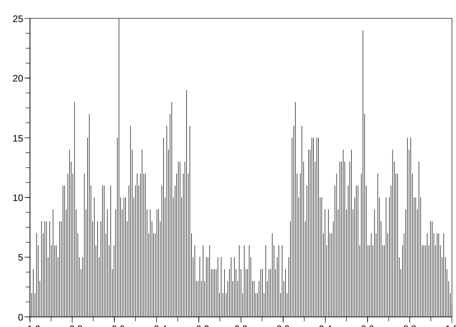
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

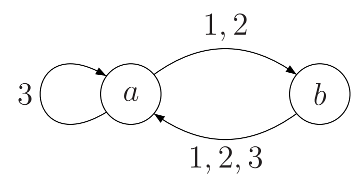
Torsion-free: *?*

Notes: not free, $|a^{-1}b| = \infty$



Automaton number 644

$$a = (b, b, a)(123)$$
$$b = (a, a, a)(12)$$



Group:

Abelian: *no*

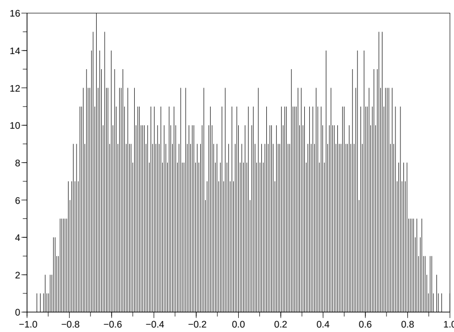
Finite: *no*

Contracting: *no*

Self-replicating: *yes*

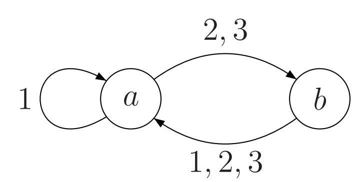
Torsion-free: *?*

Notes: not free, $|a^{-1}b| = \infty$



Automaton number 647

$$a = (a, b, b)(123)$$
$$b = (a, a, a)(12)$$



Group:

Abelian: *no*

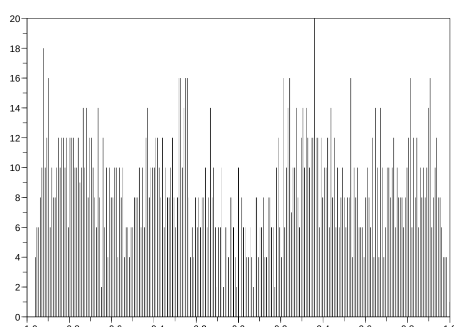
Finite: *no*

Contracting: *?*

Self-replicating: *no*

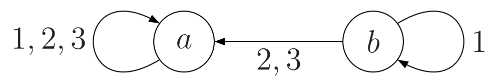
Torsion-free: *?*

Notes: $|a^{-4}b^{4}| = \infty$



Automaton number 649

$$a = (a, a, a)(123)$$
$$b = (b, a, a)(12)$$



Group:

Abelian: *no*

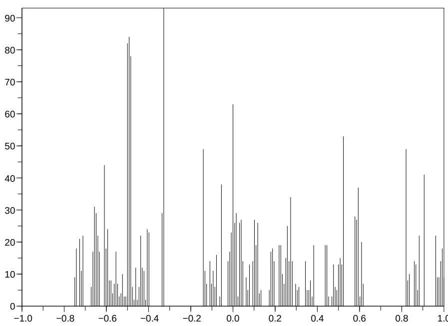
Finite: *no*

Contracting: *yes*

Self-replicating: *yes*

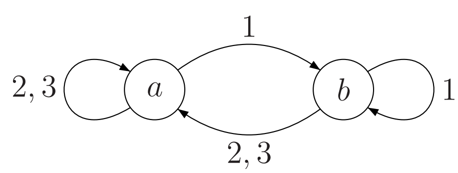
Torsion-free: *?*

Notes: $|a^{-2}b^2| = \infty$



Automaton number 650

$$a = (b, a, a)(123)$$
$$b = (b, a, a)(12)$$



Group:

Abelian: *no*

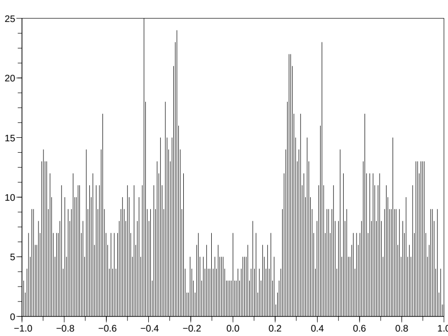
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

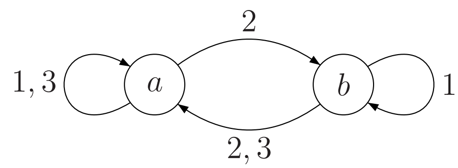
Torsion: *?*

Torsion-free: *?*



Automaton number 651

$$a = (a, b, a)(123)$$
$$b = (b, a, a)(12)$$



Group:

Abelian: *no*

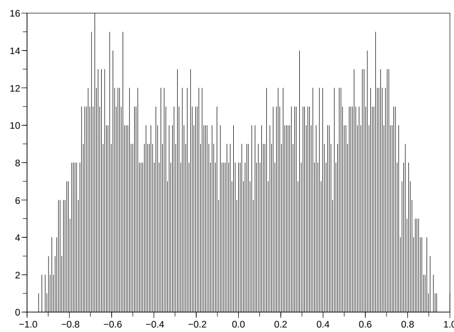
Finite: *no*

Contracting: *no*

Self-replicating: *yes*

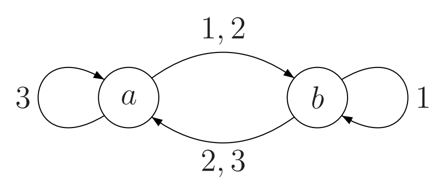
Torsion-free: *?*

Notes: $|a^{-1}b^1| = \infty$



Automaton number 652

$$a = (b, b, a)(123)$$
$$b = (b, a, a)(12)$$



Group:

Abelian: *no*

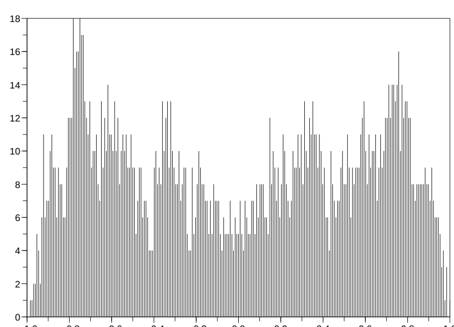
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

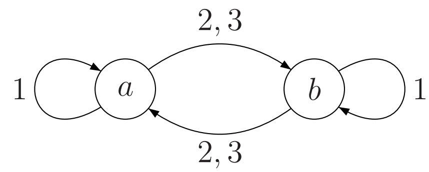
Torsion-free: *?*

Notes: $|a^{-1}b^1| = \infty$



Automaton number 655

$$a = (a, b, b)(123)$$
$$b = (b, a, a)(12)$$



Group:

Abelian: *no*

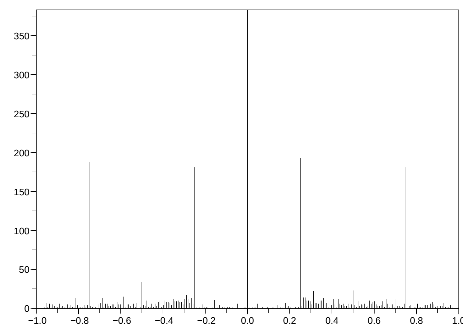
Finite: *no*

Contracting: *?*

Self-replicating: *?*

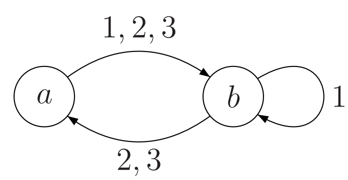
Torsion-free: *?*

Notes: free semigroup



Automaton number 656

$$a = (b, b, b)(123)$$
$$b = (b, a, a)(12)$$



Group:

Abelian: *no*

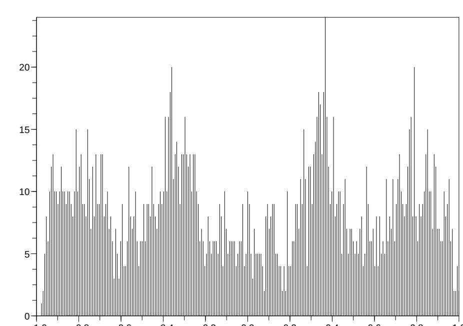
Finite: *no*

Contracting: *?*

Self-replicating: *?*

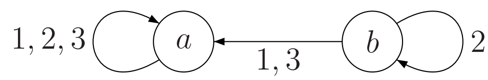
Torsion-free: *?*

Notes: $|a^{-3}b^3| = \infty$



Automaton number 657

$$a = (a, a, a)(123)$$
$$b = (a, b, a)(12)$$



Group:

Abelian: *no*

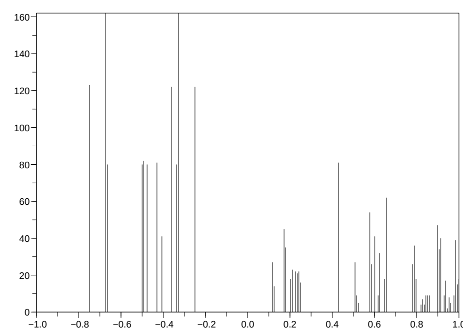
Finite: *no*

Contracting: *yes*

Self-replicating: *yes*

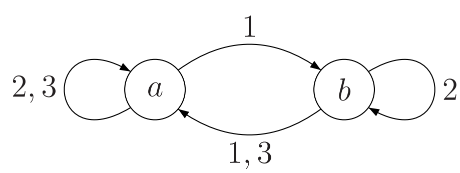
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 658

$$a = (b, a, a)(123)$$
$$b = (a, b, a)(12)$$



Group:

Abelian: *no*

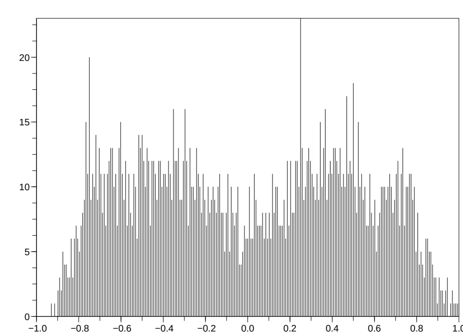
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

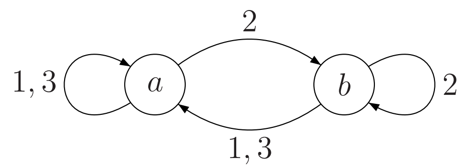
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 659

$$a = (a, b, a)(123)$$
$$b = (a, b, a)(12)$$



Group:

Abelian: *no*

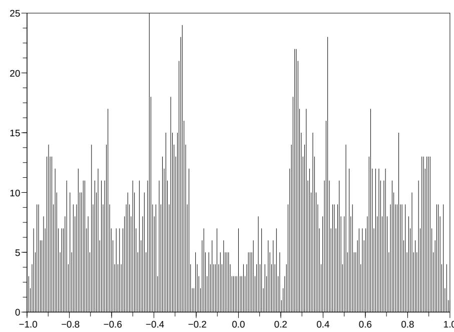
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

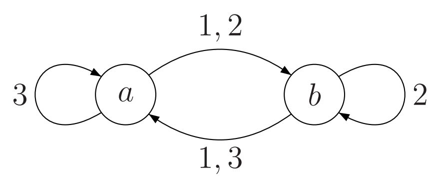
Torsion: *?*

Torsion-free: *?*



Automaton number 660

$$a = (b, b, a)(123)$$
$$b = (a, b, a)(12)$$



Group:

Abelian: *no*

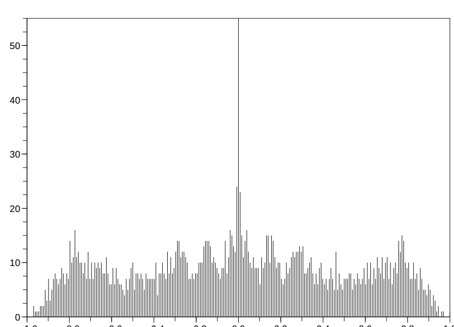
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

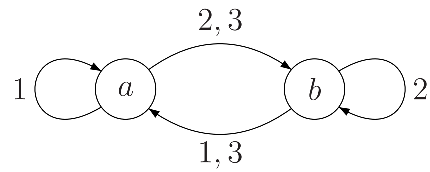
Torsion-free: *?*

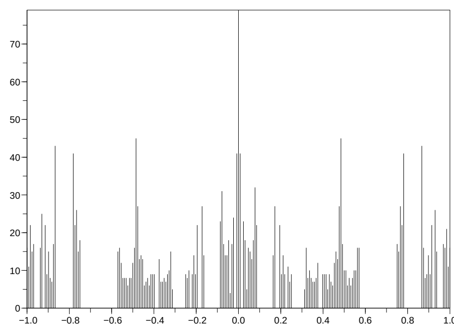
Notes: $|a^{-2}b^2| = \infty$



Automaton number 663

$$a = (a, b, b)(123)$$
$$b = (a, b, a)(12)$$





Group:

Abelian: *no*

Finite: *no*

Contracting: *yes*

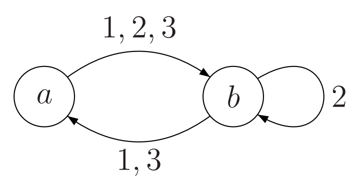
Self-replicating: *yes*

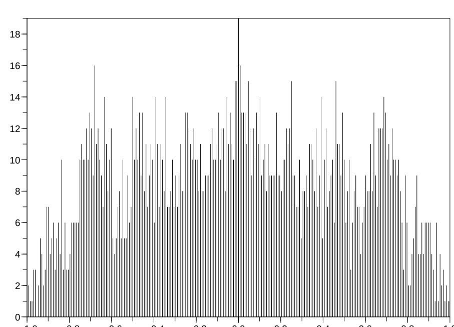
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$

Automaton number 664

$$a = (b, b, b)(123)$$
$$b = (a, b, a)(12)$$





Group:

Abelian: *no*

Finite: *no*

Contracting: *?*

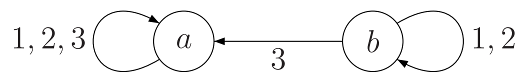
Self-replicating: *yes*

Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$

Automaton number 665

$$a = (a, a, a)(123)$$
$$b = (b, b, a)(12)$$



Group:

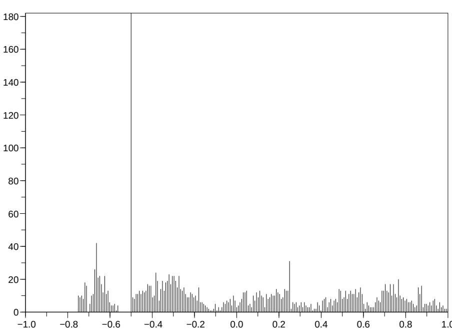
Abelian: *no*

Finite: *no*

Contracting: *?*

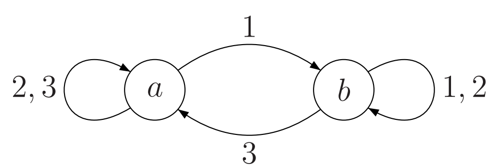
Self-replicating: *yes*

Torsion-free: *?*
Notes: $|a^{-1}b| = \infty$



Automaton number 666

$$a = (b, a, a)(123)$$
$$b = (b, b, a)(12)$$



Group:

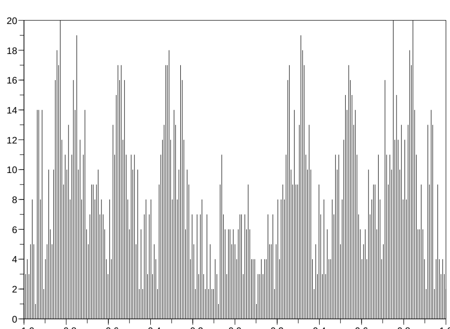
Abelian: *no*

Finite: *no*

Contracting: *?*

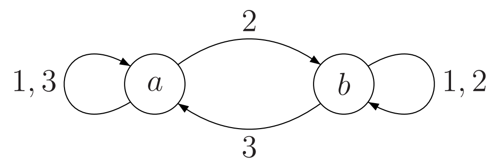
Self-replicating: *yes*

Torsion-free: *?*
Notes: not free, $|a^{-1}b| = \infty$



Automaton number 667

$$a = (a, b, a)(123)$$
$$b = (b, b, a)(12)$$



Group:

Abelian: *no*

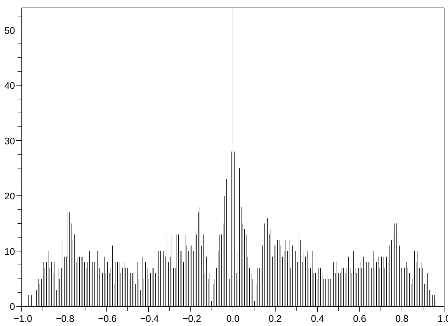
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

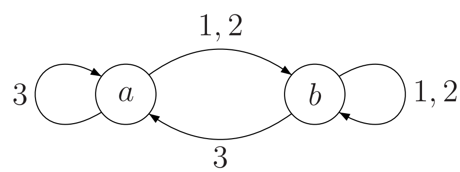
Torsion-free: *?*

Notes: $|a^{-2}b^2| = \infty$



Automaton number 668

$$a = (b, b, a)(123)$$
$$b = (b, b, a)(12)$$



Group:

Abelian: *no*

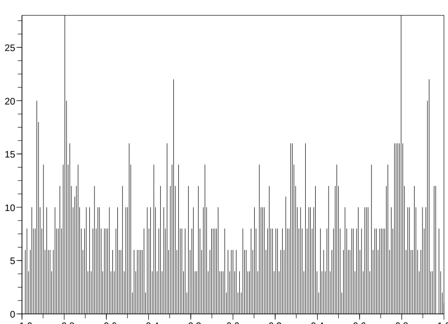
Finite: *no*

Contracting: *?*

Self-replicating: *no*

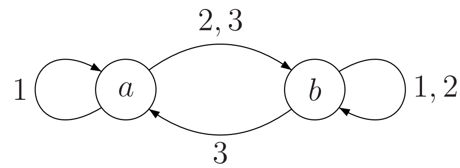
Torsion: *?*

Torsion-free: *?*



Automaton number 671

$$a = (a, b, b)(123)$$
$$b = (b, b, a)(12)$$



Group:

Abelian: *no*

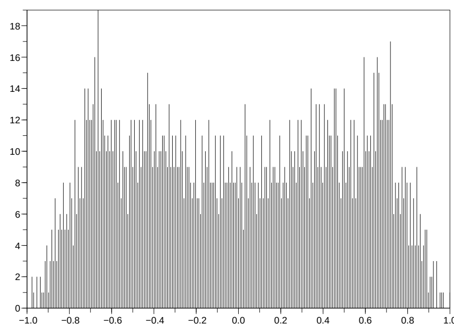
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

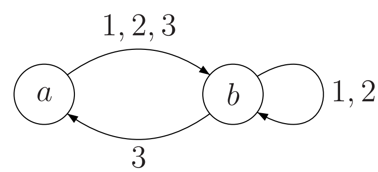
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 672

$$a = (b, b, b)(123)$$
$$b = (b, b, a)(12)$$



Group:

Abelian: *no*

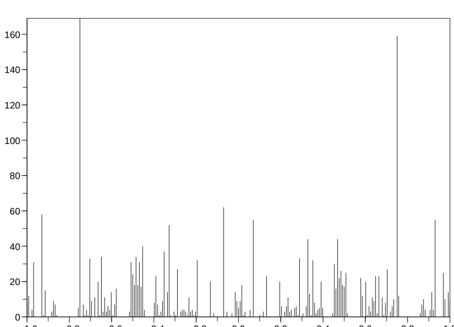
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

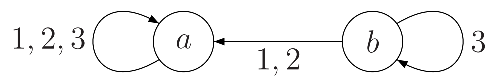
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 673

$$a = (a, a, a)(123)$$
$$b = (a, a, b)(12)$$



Group:

Abelian: *no*

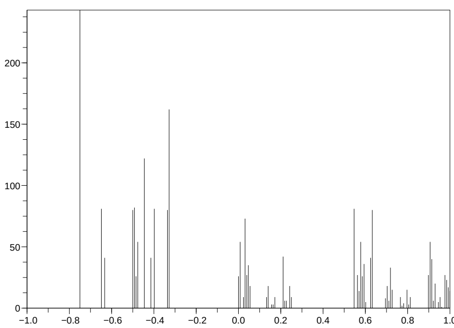
Finite: *no*

Contracting: *yes*

Self-replicating: *yes*

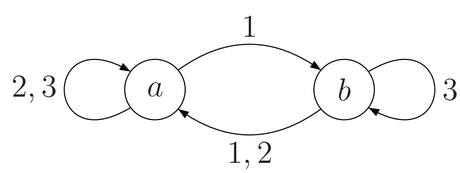
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 674

$$a = (b, a, a)(123)$$
$$b = (a, a, b)(12)$$



Group:

Abelian: *no*

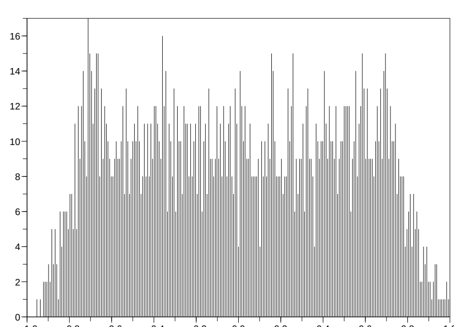
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

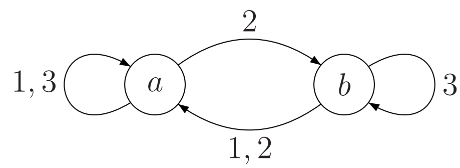
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 675

$$a = (a, b, a)(123)$$
$$b = (a, a, b)(12)$$



Group:

Abelian: *no*

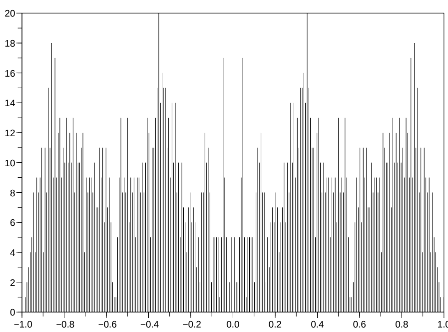
Finite: *?*

Contracting: *?*

Self-replicating: *no*

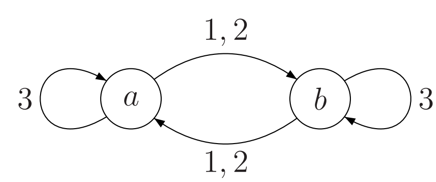
Torsion: *?*

Torsion-free: *?*



Automaton number 676

$$a = (b, b, a)(123)$$
$$b = (a, a, b)(12)$$



Group:

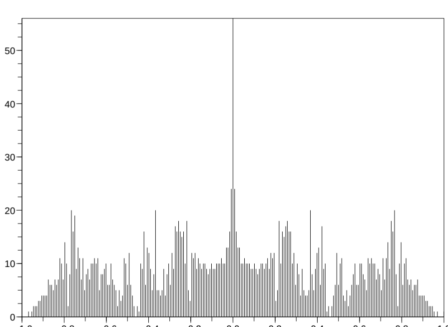
Abelian: *no*

Finite: *no*

Contracting: *no*

Self-replicating: *yes*

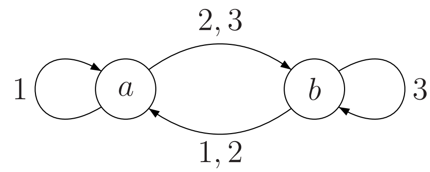
Torsion-free: *?*
Notes: free semigroup



Automaton number 679

$a = (a, b, b)(123)$

$b = (a, a, b)(12)$



Group:

Abelian: *no*

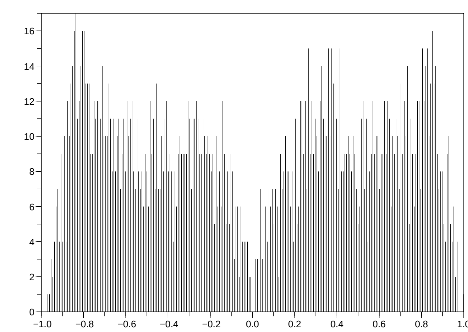
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

Torsion-free: *?*

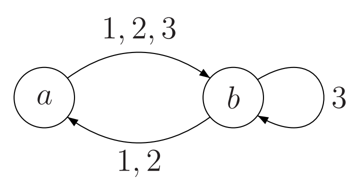
Notes: not free, $|a^{-1}b| = \infty$



Automaton number 680

$a = (b, b, b)(123)$

$b = (a, a, b)(12)$



Group:

Abelian: *no*

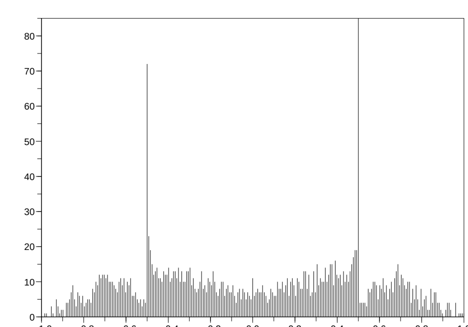
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

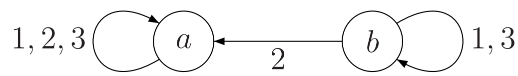
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 681

$$a = (a, a, a)(123)$$
$$b = (b, a, b)(12)$$



Group:

Abelian: *no*

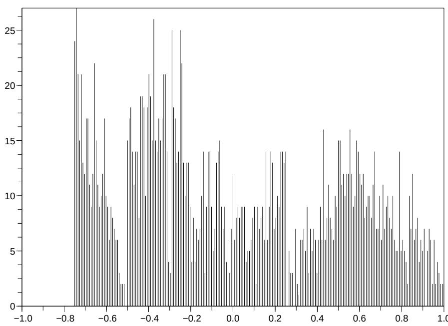
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

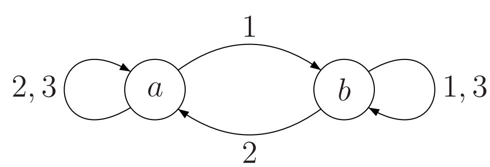
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 682

$$a = (b, a, a)(123)$$
$$b = (b, a, b)(12)$$



Group:

Abelian: *no*

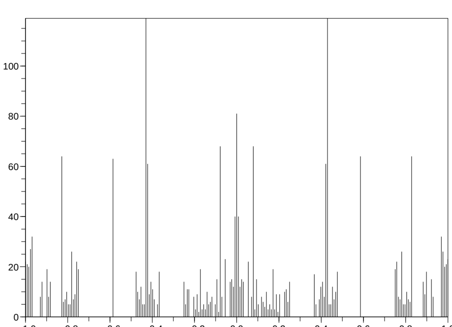
Finite: *no*

Contracting: *yes*

Self-replicating: *yes*

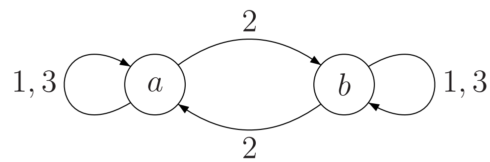
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 683

$$a = (a, b, a)(123)$$
$$b = (b, a, b)(12)$$



Group:

Abelian: *no*

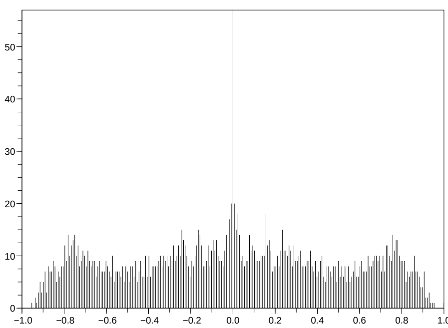
Finite: *no*

Contracting: *no*

Self-replicating: *yes*

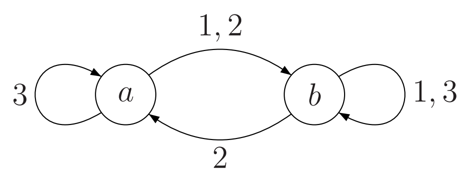
Torsion-free: *?*

Notes: free semigroup, $|a^{-2}b^2| = \infty$



Automaton number 684

$$a = (b, b, a)(123)$$
$$b = (b, a, b)(12)$$



Group:

Abelian: *no*

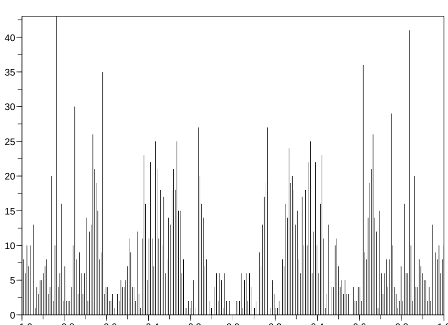
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

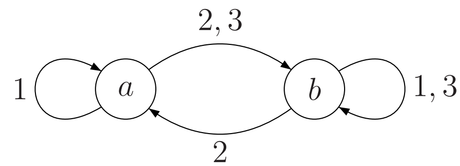
Torsion: *?*

Torsion-free: *?*



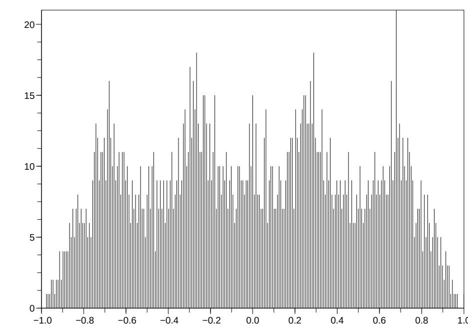
Automaton number 687

$$a = (a, b, b)(123)$$
$$b = (b, a, b)(12)$$



Group:

Abelian: *no*



Finite: *no*

Contracting: *?*

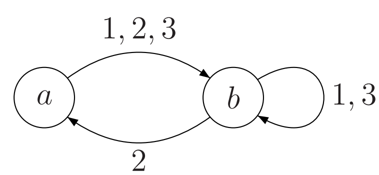
Self-replicating: *yes*

Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$

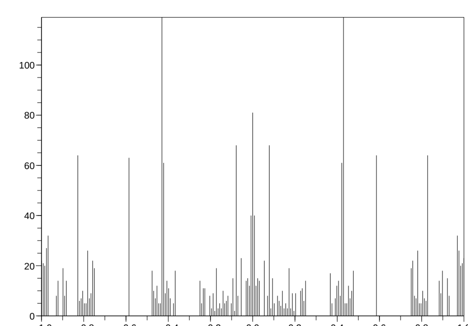
Automaton number 688

$$a = (b, b, b)(123)$$
$$b = (b, a, b)(12)$$



Group:

Abelian: *no*



Finite: *no*

Contracting: *yes*

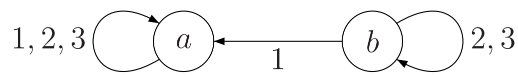
Self-replicating: *yes*

Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$

Automaton number 689

$$a = (a, a, a)(123)$$
$$b = (a, b, b)(12)$$



Group:

Abelian: *no*

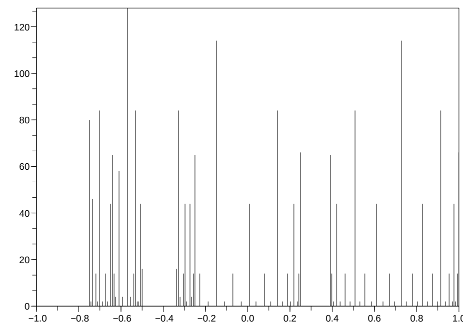
Finite: *no*

Contracting: *?*

Self-replicating: *no*

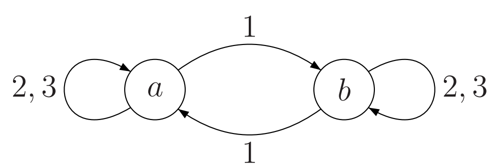
Torsion-free: *?*

Notes: $|a^{-3}b^3| = \infty$



Automaton number 690

$$a = (b, a, a)(123)$$
$$b = (a, b, b)(12)$$



Group:

Abelian: *no*

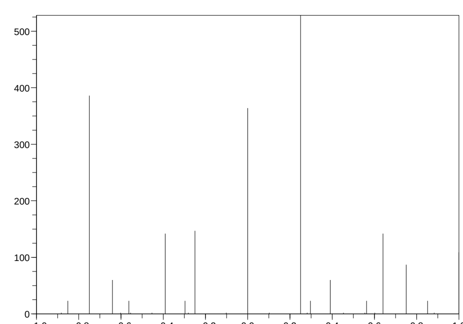
Finite: *no*

Contracting: *no*

Self-replicating: *?*

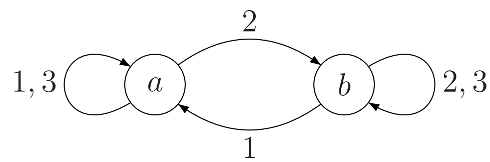
Torsion-free: *?*

Notes: free semigroup



Automaton number 691

$$a = (a, b, a)(123)$$
$$b = (a, b, b)(12)$$



Group:

Abelian: *no*

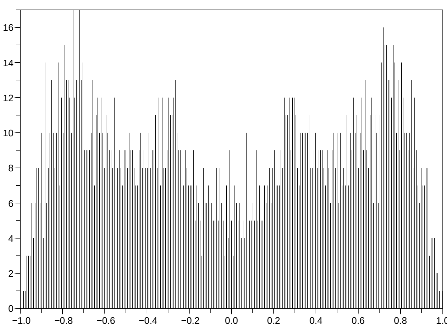
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

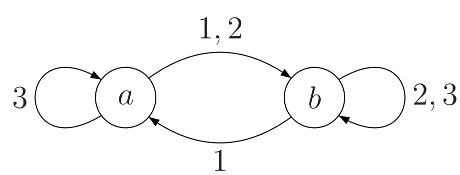
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 692

$$a = (b, b, a)(123)$$
$$b = (a, b, b)(12)$$



Group:

Abelian: *no*

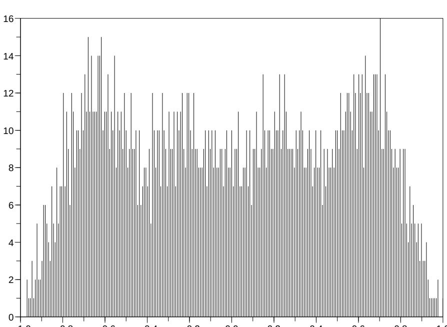
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

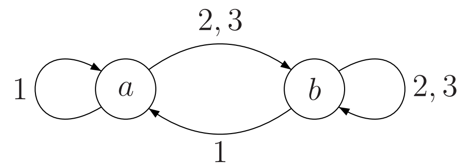
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 695

$$a = (a, b, b)(123)$$
$$b = (a, b, b)(12)$$



Group:

Abelian: *no*

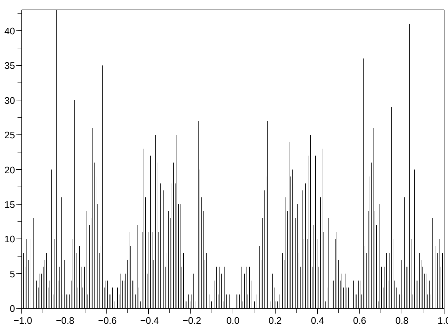
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

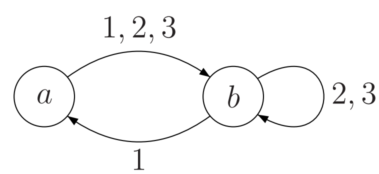
Torsion: *?*

Torsion-free: *?*



Automaton number 696

$$a = (b, b, b)(123)$$
$$b = (a, b, b)(12)$$



Group:

Abelian: *no*

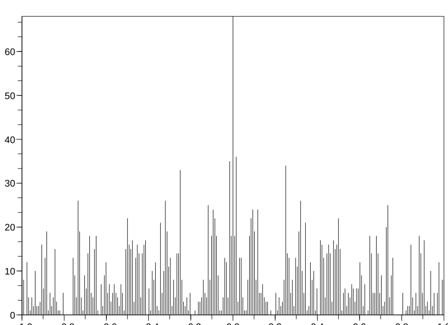
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

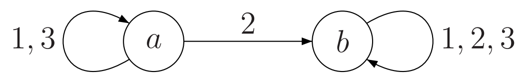
Torsion-free: *?*

Notes: $|a^{-2}b^2| = \infty$



Automaton number 699

$$a = (a, b, a)(123)$$
$$b = (b, b, b)(12)$$



Group:

Abelian: *no*

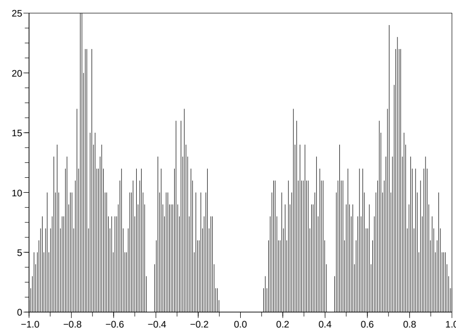
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

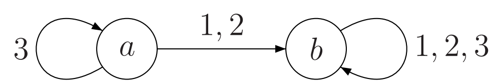
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 700

$$a = (b, b, a)(123)$$
$$b = (b, b, b)(12)$$



Group:

Abelian: *no*

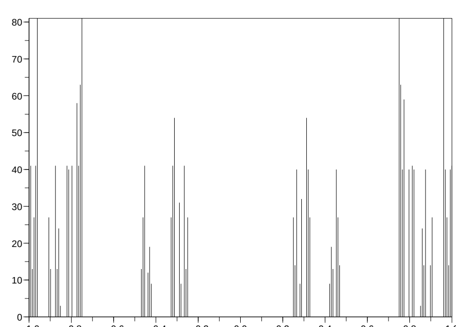
Finite: *no*

Contracting: *yes*

Self-replicating: *yes*

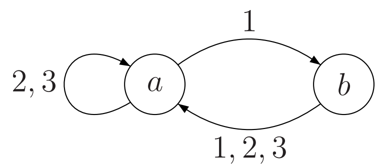
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 1858

$$a = (b, a, a)(132)$$
$$b = (a, a, a)(123)$$



Group:

Abelian: *no*

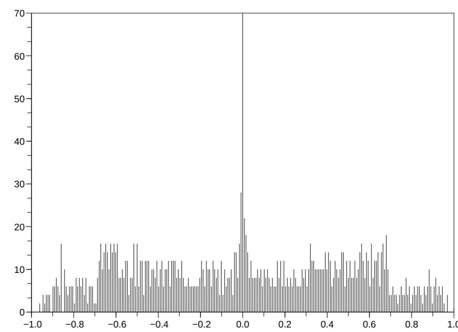
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

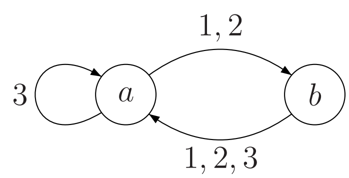
Torsion-free: *?*

Notes: not free, $|a^{-1}b| = \infty$



Automaton number 1860

$$a = (b, b, a)(132)$$
$$b = (a, a, a)(123)$$



Group:

Abelian: *no*

Finite: *no*

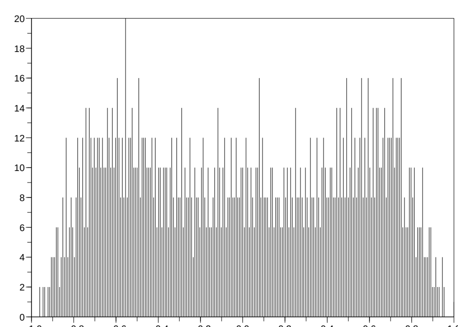
Contracting: *?*

Self-replicating: *yes*

Torsion: *?*

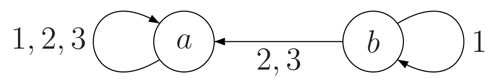
Torsion-free: *?*

Notes: not free



Automaton number 1865

$$a = (a, a, a)(132)$$
$$b = (b, a, a)(123)$$



Group:

Abelian: *no*

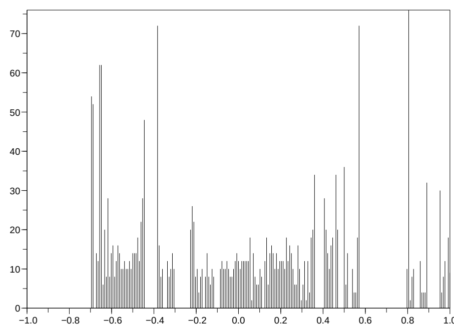
Finite: *no*

Contracting: *yes*

Self-replicating: *yes*

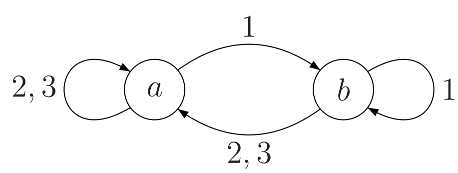
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 1866

$$a = (b, a, a)(132)$$
$$b = (b, a, a)(123)$$



Group:

Abelian: *no*

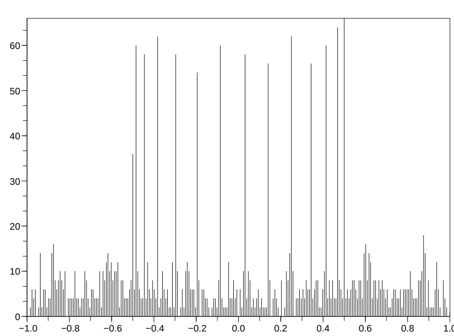
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

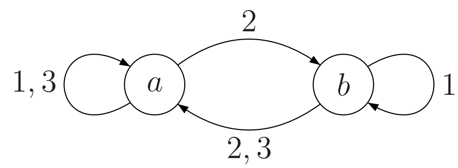
Torsion: *?*

Torsion-free: *?*



Automaton number 1867

$$a = (a, b, a)(132)$$
$$b = (b, a, a)(123)$$



Group:

Abelian: *no*

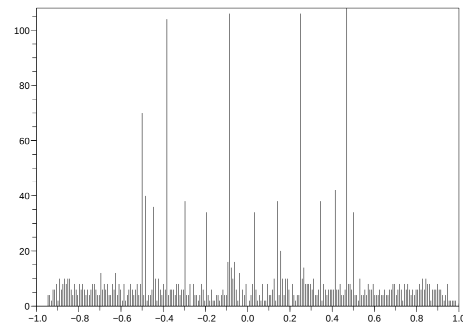
Finite: *no*

Contracting: *?*

Self-replicating: *?*

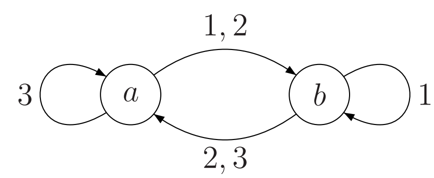
Torsion-free: *?*

Notes: $|a^{-6}b^6| = \infty$



Automaton number 1868

$$a = (b, b, a)(132)$$
$$b = (b, a, a)(123)$$



Group:

Abelian: *no*

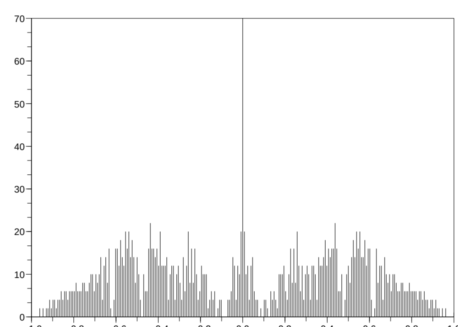
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

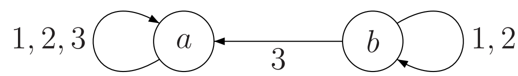
Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



Automaton number 1881

$$a = (a, a, a)(132)$$
$$b = (b, b, a)(123)$$



Group:

Abelian: *no*

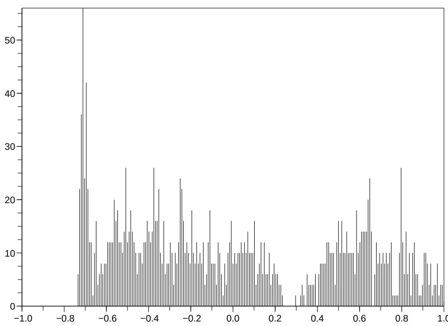
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

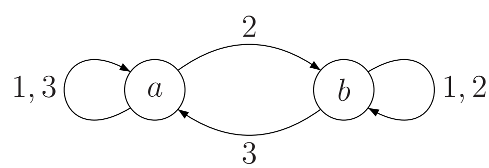
Torsion: *?*

Torsion-free: *?*



Automaton number 1883

$$a = (a, b, a)(132)$$
$$b = (b, b, a)(123)$$



Group:

Abelian: *no*

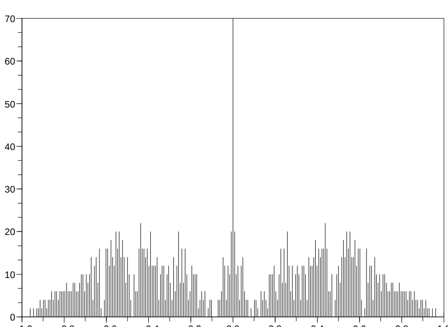
Finite: *no*

Contracting: *?*

Self-replicating: *yes*

Torsion-free: *?*

Notes: $|a^{-1}b| = \infty$



VITA

Yevgen Muntyan was born in Khmelnitsky, Ukraine. He received his B.A. degree and M.A.S. degree in Mathematics in June 2001 and June 2002, respectively, from National Taras Shevchenko University of Kyiv. He received his Ph.D. degree in mathematics from Texas A&M University in May 2009. His research interests include automata and self-similar groups, computational group theory, computer algebra.

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