A STUDY OF NON-HOMOGENEOUS ABSORBING MARKOV CHAINS

A Thesis

by

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ABSTRACT

A Study of Non-Homogeneous Absorbing Markov Chains. (December 1975)

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The objective of this thesis is to investigate general conditions which guarantee the existence of limits in non-homogeneous absorbing Markov chains. The major emphasis of this thesis will be a collection of theorems analogous to the classical results of Markov chain theory concerning the limiting behavior of Markov chains.

A first result generalizes the classical Markov chain result concerning the existence of $\lim_{k\to\infty} A^k$, where A is the transition matrix of the Markov chain. This result is generalized into a non-homogeneous Markov chain setting by allowing the transition matrix A to change at each step of time. This leads to the study of $\lim_{k\to\infty} A_1 \ldots A_k$. Conditions necessary for the existence of this limit are determined.

A second result generalizes the classical Markov chain result concerning the existence of

$$\lim_{k\to\infty} \frac{I + A + \ldots + A^{k-1}}{k}$$

where A is the transition matrix of the Markov chain. This result is also generalized to a non-homogeneous Markov chain result by allowing the transition matrix to change at each step of time. This leads to the study of

$$\lim_{k\to\infty} \frac{I + A_1 + \ldots + A_1 \ldots A_{k-1}}{k} \quad .$$

Conditions are provided guaranteeing the existence of this limit.

The results obtained in this thesis are then applied to problems usually resolved by classical Markov chain theory, but are actually more suited to a non-homogeneous Markov chain solution.

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CHAPTER I

INTRODUCTION

Definitions and Notations. The definitions and notations found in this thesis are consistent with those used in Gantmacher [2] and Hohn [5]. Exceptions to the above are the following.

All matrices considered herein are n x n and nonnegative where $n \geq 2$. For any matrix A_k, $a_{ij}^{(k)}$ will denote the ij^{th} entry of A_k. Likewise $x_i^{(k)}$ will denote the i^{th} entry in a vector x_k. Further, $|A|$ will denote the maximum absolute value of the entries of A and $r_i(A_k)$ will denote the i^{th} row sum of A_k. Finally, e^k will be used to denote the vector with all entries zero except the k^{th} entry which is one.

Background and Statement of the Problem. Probability theory is one of the most important and most studied of the applied mathematical fields. Probability study originated in the mid 1600's, with principle works attributed to Fermat, Pascal and Huygens, with later significant discoveries by Bernoulli, Gauss, Laplace and others. The study of Markov chains was begun in 1907 when Andrei Andreevich Markov investigated a sequence of dependent random variables [8]. The related sequences of probability matrices were called Markov chains. Further investigations established Markov chains as a great theoretical tool

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in probability study. In the hands of physicists, this theory was transformed into an active means for the study of natural processes, among these Brownian motion, population growth and games of chance. Economic as well as other applications were also found.

Markov's work dealt only with finite chains. The denumerable extension was developed by A. N. Kolmogorov in 1936. Further research, all dealing with the convergence of homogeneous Markov chains, was done by W. Feller, J. Hadamard and A. Y. Khinchin, all of whom made significant contributions to the theory.

Two classical results of Markov chain theory on the limiting behavior of Markov chains concerns the existence of

$$(1)\quad \lim_{k\to\infty} A^k$$

and

$$(2)\quad \lim_{k\to\infty} \frac{I+A+A^2+\ldots+A^{k-1}}{k}$$

where A is the transition matrix of the Markov chain. These two results can be generalized into a non-homogeneous Markov chain setting by allowing the transition matrix to change at each step of time. This leads to the non-homogeneous Markov chain problems considered in this thesis. These concern the existence of

$$(1)\quad \lim_{k\to\infty} A_1\ldots A_k$$

and

$$(2)\quad \lim_{k\to\infty} \frac{I+A_1+\ldots+A_1\ldots A_{k-1}}{k} .$$

In particular, in this thesis conditions are given which guarantee that

$$\lim_{k\to\infty} A_1 \cdots A_k = \begin{bmatrix} I & 0 \\ N & 0 \end{bmatrix}$$

and conditions are given which guarantee that

$$\lim_{k\to\infty} \frac{1}{k} (I+A_1+\ldots+A_1\ldots A_{k-1}) = \begin{bmatrix} I & 0 \\ N & 0 \end{bmatrix}$$

where N is some matrix.

Finally the results of this thesis will be applied to problems usually arising in the theory of classical Markov chains, but which, in fact, are more appropriate to the theory of non-homogeneous Markov chains.

CHAPTER II

RESULTS CONCERNING LIMITS IN NON-HOMOGENEOUS ABSORBING MARKOV CHAINS

This chapter, which constitutes the theoretical portion of the thesis, will deal with the theory of non-homogeneous absorbing Markov chains. The chapter is divided into two sections. The first section deals with the limiting behavior of a non-homogeneous absorbing Markov chain. In the second section, that work is applied to evaluate a limit concerning the arithmetic mean of a non-homogeneous absorbing Markov chain.

Results Concerning Limits in Non-Homogeneous Absorbing Markov Chains. In this section, necessary conditions will be given that will guarantee that $\lim_{k\to\infty} A_1 \ldots A_k$ exists, where A_k is a stochastic matrix in an absorbing Markov chain for all k. A tool which will be employed in the development of these conditions is called the measure of full indecomposability [4] and is defined as follows.

Let $\mu(A) = \min_{|R|+|C|=n} \max_{\substack{i\in R \\ j\in C}} a_{ij}$, where R and C denote non-empty subsets of row and column indices, respectively, with $|N|$ being the number of elements in set N.

For example,

$$\text{if } A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 0 & 3 \\ 0 & 2 & 2 \end{bmatrix}, \text{ then } \mu(A) = 1$$

while

$$\text{if } A = \begin{bmatrix} 2 & 0 & 4 \\ 3 & 3 & 7 \\ 0 & 9 & 3 \end{bmatrix}, \text{ then } \mu(A) = 2.$$

The fundamental result given in this section is achieved through the following sequence of lemmas developing the properties of μ as related to the product of matrices.

Lemma 1. Suppose $A_1, A_2, \ldots, A_{n-1}$ is a sequence of matrices with $\mu(A_k) > 0$ for all k. Then, $A_{n-1} \ldots A_2 A_1 > 0$.

Proof. Let $A \geq 0$ be a matrix such that $\mu(A) > 0$. For any column vector $x \geq 0$ in R^n, define $|x|_+$ as the number of nonzero entries in x. Choose any column vector $x \geq 0$ such that x has at least one zero and at least one nonzero entry. Set $|x|_+ = s$, i.e. x has s nonzero entries. Then, there exists a permutation matrix P_1 which permutes the nonzero entries of x to the first s positions, i.e.

$$P_1 x = y = \begin{bmatrix} y_1 \\ \vdots \\ y_s \\ 0 \\ \vdots \\ 0 \end{bmatrix} .$$

Since P_1 is a permutation matrix, $P_1^t P_1 = I$. Thus, $Ax = AIx = (AP_1^t)(P_1 x) = AP_1^t y$. Now, choose a second permutation matrix P_2 such that

$$P_2 A P_1^t = \left(\begin{array}{c|c} B & - \\ \hline 0 & - \end{array} \right)$$

where B is r x s and B contains no zero rows. Thus, $P_2 Ax = (P_2 A P_1^t)y$. Clearly, $|y|_+ = |x|_+ = s$. Since B has at least one nonzero entry in each row, $|P_2Ax|_+ = |Ax|_+ = r$. Further, the zero-block of $P_2AP_1^t$ is $(n-r) \times s$. As we know by hypothesis that $\mu(A) > 0$, it follows that $n - r + s < n$. Therefore, $r > s$, i.e. $|x|_+ < |Ax|_+$.

Now, consider e^i for $i \in \{1,2,\ldots,n\}$. Clearly, $|e^i|_+ = 1$ for all i. Choose some arbitrary e^i. Then, $|A_1e^i|_+ > |e^i|_+ = 1$. Hence, by induction $|A_k\ldots A_1e^i|_+ > k$ for all $k < n$ so that $|A_{n-1}\ldots A_1e^i|_+ > n - 1$. Thus, $A_{n-1}\ldots A_1e^i > 0$ and since e^i was arbitrarily chosen, $A_{n-1}\ldots A_1 > 0$.

Lemma 2. Suppose A is a matrix with $\mu(A) > 0$. Then, there exists a matrix B such that

(i) $A \geq B$ and

(ii) $\min_{b_{ij}>0} b_{ij} = \mu(B) = \mu(A)$.

Proof. Construct a matrix B as follows: Set

$$b_{ij} = \begin{cases} a_{ij} & \text{if} \quad a_{ij} \geq \mu(A) \\ 0 & \text{otherwise} \end{cases} .$$

Clearly, $A \geq B \geq 0$ and hence $\mu(A) \geq \mu(B)$.

Now, consider an arbitrary submatrix L in B such that L is in rows with indices in R and in columns with indices in C, where $|R| + |C| = n$. Let L' be the submatrix in A with row indices in R and column indices in C. From the definition of $\mu(A)$, we can find some ℓ'_{ij} such that $\ell'_{ij} \geq \mu(A)$. Thus, $\ell_{ij} \geq \mu(A)$ and since L was arbitrarily chosen, $\mu(B) \geq \mu(A)$. Therefore, $\mu(B) = \mu(A)$.

Finally, consider $\min_{b_{ij}>0} b_{ij}$. By the construction of B, $\min_{b_{ij}>0} b_{ij} \geq \mu(A)$. As $\mu(B) = \mu(A)$, there is a $b_{ij} > 0$ so that $b_{ij} = \mu(A)$. Hence $\min_{b_{ij}>0} b_{ij} = \mu(B) = \mu(A)$.

Lemma 3. Suppose $A_1, A_2, \ldots, A_{n-1}$ is a sequence of matrices with $\mu(A_1)>0, \ldots, \mu(A_{n-1})>0$. Then, $\min_{i,j}(A_1A_2\ldots,A_{n-1})_{ij} \geq \mu(A_1)\ldots\mu(A_{n-1})$.

Proof. Consider matrices $A_1, A_2, \ldots, A_{n-1}$. Construct matrices $B_1B_2, \ldots, B_{n-1}$ as follows: Set

$$b_{ij}^{(k)} = \begin{cases} a_{ij}^{(k)} & \text{if } a_{ij}^{(k)} \geq \mu(A_k) \\ 0 & \text{otherwise} \end{cases}$$

so that by Lemma 2, $\mu(B_k) = \mu(A_k) > 0$ for all $k \leq n-1$. Let $C = B_1B_2\ldots B_{n-1}$.

Now, since

$$\min_{i,j} (B_1B_2)_{ij} = \min_{i,j} \sum_{i_1} b_{ii_1}^{(1)} b_{i_1j}^{(2)} ,$$

it follows by induction that

$$\min_{i,j} c_{ij} = \min_{i,j} (B_1\ldots B_{n-1})_{ij} = \min_{i,j} \sum_{i_1\ldots i_{n-2}} b_{ii_1}^{(1)}\ldots b_{i_{n-2}j}^{(n-1)}.$$

This fact, in conjunction with Lemma 1, implies

$$c_{ij} = (B_1\ldots B_{n-1})_{ij} = \sum_{i_1\ldots i_{n-2}} b_{ii_1}^{(1)}\ldots b_{i_{n-2}j}^{(n-1)} > 0.$$

Thus, some $b_{ii_1}, \ldots, b_{i_{n-2}j} > 0$. Now, $b_{ii_1} \geq \mu(A_1), \ldots, b_{i_{n-2}j} \geq \mu(A_{n-1})$ so that $c_{ij} \geq \mu(A_1)\mu(A_2)\ldots\mu(A_{n-1})$. Hence, $\min_{i,j} c_{ij} \geq \mu(A_1)\ldots\mu(A_{n-1})$. Finally, as $A_k \geq B_k$, $(A_1\ldots A_{n-1})_{ij} \geq (B_1\ldots B_{n-1})_{ij}$ for all i and j so that

$$\min_{i,j} (A_1\ldots A_{n-1})_{ij} \geq \min_{i,j} (B_1\ldots B_{n-1})_{ij} \geq \mu(A_1)\ldots\mu(A_{n-1}).$$

By applying these three lemmas, conditions will now be given that guarantee that a sequence of matrices $A_1,\ldots,A_k,\ldots$ is such that $\lim_{k\to\infty} A_1\ldots A_k = 0$. The conditions will only require certain conditions imposed on the $\mu(A_k)$ and the row sums of A_k, for each k.

Theorem 1. Suppose $A_1,\ldots,A_{n-1},\ldots$ is a sequence of matrices such that $\mu(A_1) \geq \mu,\ldots,\mu(A_k) \geq \mu,\ldots$ where $\mu > 0$,

(i) $\max_i r_i(A_k) \leq 1$ and

(ii) there exists $\delta_1, 0 < \delta_1 < 1$, such that for all k there exists i(k) such that $r_{i(k)}(A_k) \leq \delta$.

Then $\lim_{k\to\infty} A_1\ldots A_k = 0$.

Proof. Consider $A_1\ldots A_k\ldots$. Choose $B_1 = (A_1\ldots A_{n-1})$, $B_2 = (A_n\ldots A_{2(n-1)}),\ldots,B_s = \prod_{s(n-1)+1}^{(s+1)(n-1)} A_k$. By Lemma 3, $b_{ij}^{(s)} \geq \mu^{n-1}$ for each i,j and s.

Consider the product $B_1\ldots B_k$. Let $B_sB_{s+1} = C$. Then,

$$\sum_{k=1}^{n} c_{ik} = \sum_{k=1}^{n}\left(\sum_{t=1}^{n} b_{it}^{(s)} b_{tk}^{(s+1)}\right) = \sum_{t=1}^{n}\sum_{k=1}^{n} b_{it}^{(s)} b_{tk}^{(s+1)}$$

$$= \sum_{t=1}^{n} b_{it}^{(s)} r_t(B_{s+1})$$

$$\leq \min_i b_{it}^{(s)} \quad \delta + 1 - \min_i b_{it}^{(s)}$$

$$= \mu^{n-1} \delta + 1 - \mu^{n-1}$$

$$= 1 - \mu^{n-1}(1-\delta) < 1.$$

Hence, $r_k[B_s B_{s+1}] \leq 1 - \mu^{n-1}(1-\delta) < 1$ for all k. Set $[1-\mu^{n-1}(1-\delta)] = \ell$.

Thus, $\max_i r_i[B_s B_{s+1}] \leq \ell < 1$.

Now, $r_i(XY) = r_i\left(\sum_{t=1}^{n} X_{it} Y_{tk}\right) = \sum_{k=1}^{n}\sum_{t=1}^{n} X_{it} Y_{tk}$

$$= \sum_{k=1}^{n} X_{ik} r_k(Y) \leq r_i(X) \max_i r_i(Y) \quad .$$

Since X and Y were arbitrarily chosen, this holds for any X and Y.

By applying these results, we have

$$r_i(B_{s+1} \cdots B_{s+2m}) \leq \max_i r_i(B_{s+1} B_s) \cdots \max_i r_i(B_{s+2m-1} B_{s+2m})$$

$$\leq \ell \cdot \ell \cdots \ell = \ell^m \text{ for all } i.$$

Now, let m approach ∞. As $m \to \infty$, $\ell^m \to 0$. Hence, for all i, $r_i(B_1 \ldots B_m) \to 0$ and $\lim_{k\to\infty} B_1 \ldots B_k = 0$. Therefore, $\lim_{k\to\infty} A_1 \ldots A_k = 0$.

This result finds its use in the following corollary which provides conditions guaranteeing the existence of a limit of a non-homogeneous absorbing Markov chain.

Corollary 1. Suppose $C_1, C_2, \ldots, C_k, \ldots$ is a sequence of stochastic matrices and

$$C_k = \begin{bmatrix} I & 0 \\ B_k & A_k \end{bmatrix}$$

for all k.

Further suppose $\lim_{k\to\infty} A_1 \ldots A_k = 0$. Then, $\lim_{k\to\infty} C_1 \ldots C_k$ exists.

Proof. Set the product

$$C_1 \ldots C_k = \begin{bmatrix} I & 0 \\ L_k & A_1 \ldots A_k \end{bmatrix} .$$

As $\lim_{k\to\infty} A_1 \ldots A_k = 0$, to show $\lim_{k\to\infty} C_1 \ldots C_k$ exists it is only necessary to prove L_k has a limit. Note that

$$L_2 = B_1 + A_1 B_2$$

$$L_3 = B_1 + A_1 B_2 + A_1 A_2 B_3$$

$$\vdots$$

$$L_k = B_1 + A_1 B_2 + \ldots + A_1 \ldots A_{k-1} B_k .$$

Hence, $\ell_{ij}^{(1)} \leq \ell_{ij}^{(2)} \leq \ldots \leq \ell_{ij}^{(k)}$ for all i and j. Thus, the sequence $L_1, \ldots L_k, \ldots$ is monotonically increasing componentwise.

We know C_k is stochastic for each k. Further, $C_1 C_2$ is stochastic and hence by induction $C_1 \ldots C_k$ is stochastic. Thus, 1 is an upper bound for $\ell_{ij}^{(k)}$, i.e. $\ell_{ij}^{(k)} \leq 1$, for all i, j and k. Therefore, $\lim_{k\to\infty} \ell_{ij}^{(k)}$ exists and hence $\lim_{k\to\infty} L_k$ exists, which gives the result of the corollary.

From this corollary, it is seen that

$$\lim_{k\to\infty} C_1 \ldots C_k = \begin{bmatrix} I & 0 \\ N & 0 \end{bmatrix}$$

where N is some matrix. However, by reviewing the proof of the corollary, a recipe for the calculation of the matrix N is not achievable. In fact, for the general problem, it is doubtful that a recipe exists. Some relief for this problem, however, can be obtained by using the work in the proof of Theorem 1. From this work, it is seen that given $\varepsilon > 0$ a k can be calculated so that

$$\left| C_1 \ldots C_k - \begin{bmatrix} I & 0 \\ N & 0 \end{bmatrix} \right| < \varepsilon \quad .$$

Further, this k depends on μ so that μ measures the rate of convergence of $C_1 \ldots C_k$ to $\begin{bmatrix} I & 0 \\ N & 0 \end{bmatrix}$.

This then concludes this section. In the next section, the results of this section will be applied to a study of the limit of the arithmetic mean of a non-homogeneous absorbing Markov chain.

Results Concerning the Limit of the Arithmetic Mean of a Non-Homogeneous Absorbing Markov Chain. It is the intent of this section to provide necessary conditions for the existence of the limit of the arithmetic mean of a non-homogeneous absorbing Markov chain. The result is based on the following theorem.

Theorem 2. Suppose $C_1, \ldots, C_k, \ldots$ is a sequence of stochastic matrices such that

$$C_k = \begin{bmatrix} I & 0 \\ B_k & A_k \end{bmatrix}$$

for all k. Suppose further that $\lim_{k\to\infty} A_1 \ldots A_k = 0$. Then,

$$\lim_{k\to\infty} \frac{1}{k} (C_1 + C_1C_2 + \ldots + C_1 \ldots C_k) = \lim_{k\to\infty} C_1 \ldots C_k \ .$$

Proof. As in Corollary 1, set

$$C_1 = \begin{bmatrix} I_1 & 0_1 \\ L_1 & A_1 \end{bmatrix}, \ C_1C_2 = \begin{bmatrix} I_2 & 0_2 \\ L_2 & A_1A_2 \end{bmatrix}, \ldots, C_1 \ldots C_k$$

$$= \begin{bmatrix} I_k & 0_k \\ L_k & A_1 \ldots A_k \end{bmatrix} .$$

We will examine the expression $\frac{1}{k}(C_1 + \ldots + C_1 \ldots C_k)$ blockwise.

Consider first the identity blocks, which give

$$\frac{1}{k}(I_1 + \ldots + I_k) = (\frac{1}{k})(k)(I) = I \ .$$

Likewise, the zero blocks yield

$$\frac{1}{k}(0_1 + \ldots + 0_k) = (\frac{1}{k})(0) = 0.$$

Now, consider the A blocks. By hypothesis, given $\varepsilon > 0$ there exists $R > 0$ such that for $R \le k$, $|A_1 \ldots A_k| \le \varepsilon$. Thus,

$$\left|\frac{A_1+\ldots+A_1\ldots A_k}{k}\right| \leq \frac{|A_1|}{k} + \ldots + \left|\frac{A_1\ldots A_{R-1}}{k}\right| + \left|\frac{A_1\ldots A_R}{k}\right| + \ldots + \left|\frac{A_1\ldots A_k}{k}\right|$$

$$\leq \frac{R-1}{k} + \varepsilon\left(\frac{k-R+1}{k}\right) .$$

Hence, for k sufficiently large

$$\left|\frac{A_1+\ldots+A_1\ldots A_k}{k}\right| \leq 2\,\varepsilon .$$

Since ε was arbitrary, $\lim_{k\to\infty} \frac{1}{k}\,(A_1+\ldots+A_1\ldots A_k) = 0$.

Finally, consider the L blocks. We know from Corollary 1 that $L_k \to N$ for some matrix N. Let $\varepsilon > 0$. Choose M such that for $M \leq k$, we have

$$|L_k - N| \leq \varepsilon$$

and thus

$$N - \varepsilon\, P \leq L_k \leq N + \varepsilon\, P$$

where $p_{ij} = 1$ for all i and j. Hence,

$$\frac{L_1+\ldots+L_{M-1}+(N-\varepsilon P)+\ldots+(N-\varepsilon P)}{k} \leq \frac{L_1+\ldots+L_k}{k}$$

$$\leq \frac{L_1+\ldots+L_{M-1}+(N+\varepsilon P)+\ldots+(N+\varepsilon P)}{k}$$

and

$$\frac{L_1+\ldots+L_{M-1}}{k} + \frac{k-M+1}{k}(N-\varepsilon P) \leq \frac{L_1+\ldots+L_k}{k} \leq \frac{L_1+\ldots+L_{M-1}}{k} + \frac{k-M+1}{k}(N+\varepsilon P) \quad .$$

Hence, for sufficiently large k,

$$N - 2 \varepsilon P \leq \frac{1}{k}(L_1+\ldots+L_k) \leq N + 2 \varepsilon P.$$

As ε was arbitrary, $\lim_{k\to\infty} \frac{1}{k}(L_1+\ldots+L_k) = N$. Therefore,

$$\lim_{k\to\infty} \frac{C_1+\ldots+C_1\ldots C_k}{k} = \begin{bmatrix} I & 0 \\ N & 0 \end{bmatrix} \quad .$$

The existence of the arithmetic mean of a non-homogeneous absorbing Markov chain can now be established.

Corollary 2. Suppose $C_1,\ldots,C_k,\ldots$ is a sequence of stochastic matrices such that

$$C_k = \begin{bmatrix} I & 0 \\ B_k & A_k \end{bmatrix}$$

for all k. Suppose further that $\lim_{k\to\infty} A_k = 0$. Then,

$$\lim_{k\to\infty} \frac{I+C_1+\ldots+C_1\ldots C_{k-1}}{k} = \lim_{k\to\infty} C_1C_2\ldots C_k \; .$$

Proof. The proof follows by noting that

$$\lim_{k\to\infty} \frac{I+\ldots+C_1\ldots C_{k-1}}{k} = \lim_{k\to\infty} \frac{I}{k} + \lim_{k\to\infty} \left(\frac{k-1}{k}\right) \frac{C_1+\ldots+C_1\ldots C_{k-1}}{k-1} = \lim_{k\to\infty} C_1\ldots C_k \; .$$

This then concludes our theoretical work on non-homogeneous absorbing Markov chains. In the following chapter, the use of these results will be shown by applying this work in the solution of several practical problems.

CHAPTER III

APPLICATIONS

The main objective of this chapter is to provide several applications of non-homogeneous absorbing Markov chain theory. It is the intent to apply the results of Chapter II to problems which have in the past been resolved by classical Markov chain theory, but which in fact are more of the non-homogeneous Markov chain variety.

The initial problem concerns actual problems incurred in space flight. To describe this problem simply, suppose three light bulbs are utilized in the lighting system of a lunar module. As there is no room for excess baggage, no replacement bulbs are brought on the trip, even though the bulbs may burn out, thus leaving the module in darkness. Hence, there is a need for computing the probability of the lights continuing to burn throughout the flight period. Time is a factor in this problem as the filaments become thinner as time passes, thus the probability of the lights burning out increases with time.

Let the k^{th} transition matrix for this non-homogeneous absorbing Markov chain be

$$C_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .4\left(\frac{2k}{k+1}\right) & .6\left(\frac{k+5}{3k+3}\right) & 0 & 0 \\ .3\left(\frac{2k+1}{3k+3}\right) & .3\left(\frac{6k+1}{3k+3}\right) & .4\left(\frac{k+4}{2k+2}\right) & 0 \\ .1\left(\frac{2k+3}{k+1}\right) & .2\left(\frac{k+2}{k+1}\right) & .3\left(\frac{4k+1}{3k+3}\right) & .4\left(\frac{k+2}{2k+2}\right) \end{bmatrix}$$

for all k, where

$c_{11}^{(k)}$ = probability zero lights are on, given zero lights were on the previous hour,

$c_{12}^{(k)}$ = probability one light is on, given zero lights were on the previous hour,

$c_{13}^{(k)}$ = probability two lights are on, given zero lights were on the previous hour,

$c_{14}^{(k)}$ = probability three lights are on, given zero lights were on the previous hour,

$c_{21}^{(k)}$ = probability zero lights are on, given one light was on the previous hour,

$c_{22}^{(k)}$ = probability one light is on, given one light was on the previous hour,

$c_{23}^{(k)}$ = probability two lights are on, given one light was on the previous hour,

$c_{24}^{(k)}$ = probability that three lights are on, given one light was on the previous hour,

$c_{31}^{(k)}$ = probability zero lights are on, given two lights were on the previous hour,

$c_{32}^{(k)}$ = probability one light is on, given two lights were on the previous hour,

$c_{33}^{(k)}$ = probability two lights are on, given two lights were on the previous hour,

$c_{34}^{(k)}$ = probability three lights are on, given two lights were on the previous hour,

$c_{41}^{(k)}$ = probability zero lights are on, given three lights were on the previous hour,

$c_{42}^{(k)}$ = probability one light is on, giventhree lights were on the previous hour,

$c_{43}^{(k)}$ = probability two lights are on, given three lights were on the previous hour,

$c_{44}^{(k)}$ = probability three lights are on, given three lights were on the previous hour.

Using the results developed in Chapter II, we can predict the long-run behavior of these light bulbs. Clearly, the light bulbs will eventually all burn out and

$$\lim_{k\to\infty} C_1 \ldots C_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ n_1 & 0 & 0 & 0 \\ n_2 & 0 & 0 & 0 \\ n_3 & 0 & 0 & 0 \end{bmatrix}$$

where $n_3 = 1$ is the probability that all lights are burned out in the long run. However, space flight time is not infinite, and the time of flight is usually known. Suppose the flight time is 100 hours. Then, a computer can be used to compute $C_1 \ldots C_{100}$, which will contain $\ell_{31}^{(100)}$ the probability that all of the lights will be burned out after the 100 hours in flight.

A second problem concerns a physiological disturbance. For this, assume there is a small tar particle trapped in the lungs of an individual. For the particle to be removed from his body, it must pass through his throat into the atmosphere. Once it escapes into the atmosphere, we will assume it cannot return into his body. We will also assume that his lungs will grow weaker at each passing exhalation, the unit of time utilized in these experiments, thus making the problem time dependent. Let the k^{th} transition matrix for this non-homogeneous absorbing Markov chain be

$$C_k = \begin{bmatrix} 1 & 0 & 0 \\ .4\left(\frac{k+2}{2k+2}\right) & .6\left(\frac{3k+1}{3k+3}\right) & .2\left(\frac{k+2}{k+1}\right) \\ 0 & .01\left(\frac{k+55}{k+1}\right) & .99\left(\frac{22k+1}{22k+22}\right) \end{bmatrix}$$

where

$c_{11}^{(k)}$ = probability tar is in air, given it was in air at previous exhalation,

$c_{12}^{(k)}$ = probability tar is in throat, given it was in air at previous exhalation,

$c_{13}^{(k)}$ = probability tar is in lungs, given it was in air at previous exhalation,

$c_{21}^{(k)}$ = probability tar is in air, given it was in throat at previous exhalation,

$c_{22}^{(k)}$ = probability tar is in throat, given it was in throat at previous exhalation,

$c_{23}^{(k)}$ = probability tar is in lungs, given it was in throat at previous exhalation,

$c_{31}^{(k)}$ = probability tar is in air, given it was in lungs at previous exhalation,

$c_{32}^{(k)}$ = probability that tar is in throat, given it was in lungs at previous exhalation,

$c_{33}^{(k)}$ = probability that tar is in lungs, given it was in lungs at previous exhalation.

In this example, Corollary 1 yields

$$\lim C_1 \ldots C_k = \begin{bmatrix} 1 & 0 & 0 \\ n_1 & 0 & 0 \\ n_2 & 0 & 0 \end{bmatrix}$$

where n_1 is the probability of the particle going from the throat into the atmosphere and n_2 is the probability of the particle going from the lungs into the atmosphere in the long run. Since $C_1 \ldots C_k$ is stochastic, $n_1 = n_2 = 1$. Therefore, the tar particle will eventually be in the atmosphere.

These are but two applications indicating the use of non-homogeneous absorbing Markov chain theory. Countless others also exist. For example, in games of chance, in mass transportation, as well as in economics. Hence, while non-homogeneous absorbing Markov chains have great theoretical value, there is also a practical viewpoint which is of interest.

CHAPTER IV

SUUMARY AND CONCLUSION

The work contained in this thesis provides conditions for the existence of limits in non-homogeneous absorbing Markov chains The two primary results yield conditions which insure the existence of the limits;

(1) $\lim_{k\to\infty} A_1 \dots A_k$

(2) $\lim_{k\to\infty} \frac{I+A_1+\dots+A_1\dots A_{k-1}}{k}$,

where each A_k is stochastic. In addition, several practical applications of non-homogeneous absorbing Markov chains have been described.

Further investigation of the theory of non-homogeneous absorbing Markov chains will no doubt be undertaken. This work can attempt to generalize the results in this thesis. Further, as this thesis has dealt only with finite Markov chains, the extension of the results of this thesis to infinite chains is also a goal.

Finally, little work has been done in the area of providing applications for non-homogeneous Markov chain theory and practically none are given for the absorbing variety. This area can also be expanded. It seems that uses can be found in countless occurrences, as indicated in Chapter III.

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