NUMERICAL SOLUTION TO DIFFERENTIAL EQUATIONS USING LEGENDRE POLYNOMIALS

A Thesis

By

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I. INTRODUCTION

This thesis will discuss a particular method that can be used for the numerical solution of ordinary differential equations. It strives to present this technique in an elementary manner and will give examples of the results worked out with the aid of a large scale computer.

The second chapter will serve to develop some of the more important properties of the Legendre polynomials, both in one and two variables. It is important to understand these polynomials in one variable, since those in two variables can be developed from the properties of the one variable class. Since Legendre polynomials play an important part in the evaluation of an integral by Gaussian quadrature, the last of chapter two will relate their use to this method of numerical integration. It will also discuss the numerical estimation of Fourier Coefficients using the above numerical integration technique.

Chapter three will formulate a procedure for raising a power series to a power and will illustrate its relation to the subject at hand, namely; the numerical solution of differential equations. The major portion of the third chapter, however, will serve to explain the manner in which an estimation to a numerical solution of a differential equation can be obtained using the principles outlined previously.

The numerical work of this investigation was performed on an IBM type 709 digital computer as indicated in the appendix. The work was written as a 709/7090 Fortran main program which called several 709/7090 Fortran subroutines, listings of which also appear in the appendix. The results from the computer appear in the end of the appendix and these results are used for comparison to the expected solutions.

The polynomial of degree n discussed in the following pages will be denoted by $P_n(x) = 1 + a_1x + a_2x^2 + \ldots. + a_nx^n$, where, for $n = 0$; $P_0(x) = 1$ for all values of x in the interval $[0,1]$.

Each definition, theorem, and some of the more important equations have been numbered. If, for example, the number (1.2.3) appears the one indicates the chapter, the two indicates the section of the chapter, and the three indicates the particular definition, theorem or equation in that section.

II. LENGENDRE POLYNOMIALS

2.1 Lengendre Coefficients

<u>Definition 2.1.1</u> The polynomials $P_n(x)$, $P_m(x)$, where $P_n(x)$ and $P_m(x)$ are continuous in the interval $[0,1]$, are said to be orthogonal over the interval $[0,1]$ provided

$$\int_0^1 P_m(x)P_n(x)dx \begin{cases} = 0 , & m \neq n \\ \neq 0 , & m = n \end{cases} . \tag{2.1.1}$$

Let us construct a polynomial $P_m(x)$ of degree m,

$$P_m(x) = 1 + a_1x + a_2x^2 + \ldots + a_mx^m \tag{2.1.2}$$

such that

$$\int_0^1 x^kP_m(x)dx = 0, \ k = 0,1,\ldots,m-1 . \tag{2.1.3}$$

We first multiply (2.1.2) by x^kdx giving

$$x^kP_m(x)dx = x^kdx + a_1x^{k+1}dx + a_2x^{k+2}dx + \ldots + a_mx^{k+m}dx$$

for $k = 0,1,\ldots,m-1$ and then integrate from $x=0$ to $x=1$, which gives

$$\int_0^1 x^k P_m(x)dx = \int_0^1 x^k dx + a_1 x^{k+1} dx + \dots + a_m x^{k+m} dx$$

where $k = 0,1,\dots,m-1$. (2.1.4)

Now making use of (2.1.3) and (2.1.4) we obtain m linear equations

$$\frac{1}{k+1} + \frac{a_1}{k+2} + \dots + \frac{a_m}{k+m+1} = 0$$

$$k = (0,1,\dots,m-1) \qquad (2.1.5)$$

from which the m coefficients a_1, a_2, ..., a_m can be determined. There are several methods for solving these equations to obtain the m coefficients desired, but one easy way is as follows:

$$\frac{1}{k+1} + \frac{a_1}{k+2} + \dots + \frac{a_m}{k+m+1} = \frac{Q(k)}{(k+1)(k+2)\dots(k+m+1)}$$

where $Q(k)$ is a polynomial in k of degree not higher than m. Since by (2.1.5) the polynomial $Q(k)$ must vanish for the m values of k, $k = 0,1,\dots,m-1$, $Q(k)$ can be written $Q(k) = Ck(k-1)(k-2)\dots(k-m+1)$ in which C is a constant. Then

$$\frac{1}{k+1} + \frac{a_1}{k+2} + \dots + \frac{a_m}{k+m+1} = \frac{Ck(k-1)(k-2)\dots(k-m+1)}{(k+1)(k+2)\dots(k+m+1)} .$$

Multiplying both sides of the above equation by k+1 gives

$$1 + \frac{a_1(k+1)}{k+2} + \dots + \frac{a_m(k+1)}{k+m+1} = \frac{Ck(k-1)(k-2)\dots(k-m+1)}{(k+2)(k+3)\dots(k+m+1)} .$$

Setting k = -1, we obtain

$$1 = \frac{C(-1)(-2)(-3)\ldots(-m)}{(1)(2)\ldots(m)} .$$

Therefore $\frac{C(-1)^m(m!)}{m!} = 1$

or $C = (-1)^m$.

Thus $Q(k) = (-1)^k k(k-1)(k-2)\ldots(k-m+1)$

and

$$\frac{1}{k+1} + \frac{a_1}{k+2} + \ldots + \frac{a_m}{k+m+1} = \frac{(-1)^m(k)(k-1)(k-2)\ldots(k-m+1)}{(k+1)(k+2)(k+3)\ldots(k+m+1)} .$$

To solve for a_i we need only to multiply the above equation by $(k+i+1)$ and set $k = -(i+1)$, $0 \leq i \leq m$. Each term on the left vanishes except a_i and

$$a_i = \frac{(-1)^m(-i-1)(-i-2)(-i-3)\ldots(-i-m)}{(-i)(-i+1)(-i+2)\ldots(-1)(1)(2)\ldots(-i+m)}$$

$$= \frac{(-1)^m(-1)^m(i+1)(i+2)\ldots(i+m)}{(-1)^i i!(m-i)!}$$

$$= \frac{(-1)^i(m+i)!}{i!m!} \cdot \frac{m!}{(m-i)!i!}$$

$$a_i = (-1)^i\binom{m+i}{i}\binom{m}{i} \quad . \tag{2.1.6}$$

We now put the values of the a's obtained from (2.1.6) into equation (2.1.2) and express the polynomial $P_m(x)$ in the form

$$P_m(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i}\binom{m+i}{i} x^i = \sum_{i=0}^{m} a_i x^i \quad .$$

This is the desired result which satisfies the equation (2.1.3).

If we multiply equation (2.1.2) by $P_n(x)$ and integrate from $x = 0$ to $x = 1$, we have

$$\int_0^1 P_m(x)P_n(x)dx = \sum_{i=0}^{m} a_i \int_0^1 x^i P_n(x)dx \quad . \tag{2.1.7}$$

Now if $m \neq n$ all the integrals on the right vanish because of (2.1.3) so that

$$\int_0^1 P_m(x)P_n(x)dx = 0 \ , \quad (m \neq n) \quad .$$

If $m = n$ all the integrals on the right of (2.1.7) vanish except the last and (2.1.7) becomes

$$\int_0^1 P_m^2(x)dx = a_m \int_0^1 x^m P_m(x)dx$$

$$= a_m \int_0^1 x^m \sum_{i=0}^{m} a_i x^i dx$$

$$= a_m \sum_{i=0}^{m} a_i \int_0^1 x^m x^i dx$$

$$= a_m \left[\int_0^1 x^m dx + a_1 \int_0^1 x^{m+1} dx + \dots + a_m \int_0^1 x^{2m} dx \right]$$

$$= a_m \left[\frac{x^{m+1}}{m+1} + \frac{a_1 x^{m+2}}{m+2} + \dots + \frac{a_m x^{2m+1}}{2m+1} \right]_0^1$$

$$= a_m \left[\frac{1}{m+1} + \frac{a_1}{m+2} + \dots + \frac{a_m}{2m+1} \right]$$

$$= a_m \frac{(-1)^m (m)(m-1)(m-2)\dots(1)}{(m+1)(m+2)\dots(2m+1)}$$

$$= a_m \frac{(-1)^m m!m!}{(2m+1)!} = \frac{1}{2m+1} .$$

Hence, $$\int_0^1 P_m^{\,2}(x)dx = \frac{1}{2m+1} . \qquad (2.1.8)$$

From this we see that

$$\int_0^1 P_m(x)P_n(x)dx = 0 \ , \quad m \neq n$$

$$\neq 0 \ , \quad m = n \ ,$$

which satisfies equation (2.1.1) and hence the definition of orthogonality.

Definition 2.1.2 The polynomials $P_m(x)$ are the Legendre polynomials orthogonal over the interval $[0,1]$.

$$P_m(x) = \sum_{i=0}^{m} a_i x^i = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \binom{m+i}{i} x^i \ . \qquad (2.1.9)$$

The Legendre polynomials $P_n(x)$ over the interval $(0,1)$ have the following integral representation:[2]

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \left[(1-2x) + \{(1-2x)^2-1\}^{\frac{1}{2}} \cos\beta\right]^n d\beta \ . \qquad (2.1.10)$$

Theorem 2.1.1 The Legendre polynomials $P_n(x)$ are bounded over the region $[0,1]$, i.e. for $0 \leq x \leq 1$

$$|P_n(x)| \leq 1 \ . \qquad (2.1.11)$$

Proof:

Equation (2.1.10) may be written

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \left[(1-2x) + i\left\{1-(1-2x)^2\right\}^{\frac{1}{2}} \cos\beta\right]^n d\beta \ .$$

The integrand of the above equation may be written in terms of its real and imaginary parts $G + iG_1$, and thus

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} (G + iG_1)\, d\beta \ .$$

Then,

$$\left|P_n(x)\right| = \left|\frac{1}{\pi}\int_0^{\pi} (G+iG_1)\, d\beta\right| \leq \frac{1}{\pi}\int_0^{\pi} |G+iG_1|\, d\beta$$

$$= \frac{1}{\pi}\int_0^{\pi} \left|(1-2x) + i\left\{1-(1-2x)^2\right\}^{\frac{1}{2}} \cos\beta\right|^n d\beta \ .$$

Now

$$\left|(1-2x) + i\left\{1-(1-2x)^2\right\}^{\frac{1}{2}} \cos\beta\right|$$

$$= \left[(1-2x)^2 + \left\{1-(1-2x)^2\right\} \cos^2\beta\right]^{\frac{1}{2}}$$

$$= \left[(1-2x)^2 + \left\{1-(1-2x)^2\right\} (1-\sin^2\beta)\right]^{\frac{1}{2}}$$

$$= \left[\cos^2 \beta + (1-2x)^2 \sin^2 \beta\right]^{\frac{1}{2}} \text{ and}$$

$$\left[\cos^2 \beta + (1-2x)^2 \sin^2 \beta\right]^{\frac{1}{2}} \leq \left[\cos^2 \beta + \sin^2 \beta\right]^{\frac{1}{2}} = 1,$$

for $0 \leq x \leq 1$ and $0 \leq \beta \leq \pi$. Therefore

$$\left|P_n(x)\right| \leq \frac{1}{\pi}\int_0^{\pi} \left|(1-2x) + \left\{1-(1-2x)^2\right\}^{\frac{1}{2}} \cos\beta \right|^n d\beta$$

$$\leq \frac{1}{\pi}\int_0^{\pi} d\beta = 1 \text{ , for } 0 \leq x \leq 1 \text{ and } 0 \leq \beta \leq \pi \text{ .}$$

2.2 Legendre Product Polynomials in Two Variables

Definition 2.2.1 The Legendre product polynomial of degree n in two variables is given by

$$L_n(x,y) = C_{n,0}P_n(x)P_0(y) + C_{n-1,1}P_{n-1}(x)P_1(y)$$

$$+ \dots + C_{0,n}P_0(x)P_n(y)$$

$$= \sum_{i=0}^{n} C_{n-i,i}P_{n-i}(x)P_i(y) \quad . \tag{2.2.1}$$

where $P_{n-i}(x)$ and $P_i(y)$ are the Legendre polynomials defined previously.

It should be noted that if $F_{n-1}(x,y)$ is a polynomial in two variables of degree not higher than n-1 then

$$F_{n-1}(x,y) = b_{0,0} + b_{1,0}x + b_{0,1}y$$

$$+ \dots + b_{r-p,p}x^{r-p}y^{p} + b_{0,n-1}y^{n-1}$$

$$= \sum_{r=0}^{n-1} \sum_{p=0}^{r} b_{r-p,p}x^{r-p}y^{p} , \quad n \leq 1 \ .$$

<u>Theorem 2.2.1</u> If $F_{n-1}(x,y)$ is any polynomial of degree at most n-1, then

$$\int_0^1 \int_0^1 F_{n-1}(x,y)P_{n-k}(x)P_k(y)dxdy = 0, \ k = 0,1,2,\dots,n \ . \qquad (2.2.2)$$

Proof:

$$\int_0^1 \int_0^1 F_{n-1}(x,y)P_{n-k}(x)P_k(y)dxdy$$

$$= \int_0^1 \int_0^1 \sum_{r=0}^{n-1} \sum_{p=0}^{r} (b_{r-p,p}x^{r-p}y^{p})(P_{n-k}(x)P_k(y)dxdy)$$

$$= \sum_{r=0}^{n-1} \sum_{p=0}^{r} b_{r-p,p} \int_0^1 \int_0^1 x^{r-p} P_{n-k}(x) y^p P_k(y) dx dy .$$

Now since r-p < n-k or p < k then by (2.1.3) either

$$\int_0^1 x^{r-p} P_{n-k}(x) dx = 0 \quad \text{or} \int_0^1 y^p P_k(y) dy = 0, \quad r,p = 0,1,\ldots,n-1 .$$

Thus

$$\sum_{r=0}^{n-1} \sum_{p=0}^{r} b_{r-p,p} \int_0^1 x^{r-p} P_{n-k}(x) dx \int_0^1 y^p P_k(y) dy = 0, \ (r,p=0,1,\ldots n-1)$$

and therefore

$$\int_0^1 \int_0^1 F_{n-1}(x,y) P_{n-k}(x) P_k(y) dx dy = 0, \ k = 0,1,\ldots,n$$

which establishes the theorem.

Lemma 2.2.1 The Legendre product polynomials are bounded over the region (0,1), i.e. for $0 < x < 1$

$$|P_{n-k}(x) P_k(y)| \leq 1 \quad . \qquad (2.2.3)$$

Proof:

Equation (2.2.3) may be written

$$|P_{n-k}(x)P_k(y)| = |P_{n-k}(x)|\ |P_k(y)| . \tag{2.2.4}$$

Now by theorem 2.1.1 we know that both members of the right-side of equation (2.2.4) are less than or equal to one, hence;

$$|P_{n-k}(x)P_k(y)| \le 1 .$$

2.3 Gauss's Formula for Numerical Integration.

A very good quadrature formula for finding the value of the definite integral

$$I = \int_a^b f(x)dx \tag{2.3.1}$$

where $f(x)$ is a known function but whose integral is to be evaluated numerically was derived by Gauss[4] and is based on Legendre polynomials.

If the definite integral (2.3.1) is to be computed from a given number of values of $f(x)$, the problem arises as just where should these values be taken so as to obtain a value of the integral with the greatest possible accuracy? The first step is to change the variable so that the interval of integration (a,b) with respect to x is from 0 to 1. This is accomplished by letting

$$x = a - (a-b)u$$

or

$$u = \frac{x-a}{-(a-b)} = \frac{a-x}{a-b} .$$

Then at $x = a$, we have

$$u = \frac{a-a}{a-b} = 0$$

and at x = b, we have

$$u = \frac{a-b}{a-b} = 1 .$$

The new value of f(x) is

$$f(x) = f\left[a-(a-b)u\right] = Q(u) .$$

Now since dx = -(a-b)du, the integral becomes

$$I = \int_a^b f(x)dx = (b-a)\int_0^1 Q(u)du .$$

It is desired to have a formula such that

$$I = \int_0^1 Q(u)du = A_1Q(u_1) + A_2Q(u_2) + \ldots + A_nQ(u_n) \qquad (2.3.2)$$

shall be as good an evaluation of the integral as possible. When f(x) is any polynomial of as high degree as possible (2.3.2) should be exact. A count of the constants available, n A's and n u's, totalling 2n, suggests that the highest degree of the polynomial Q(u) will probably be 2n-1. Hence we write

$$Q(u) = a_0 + a_1u + a_2u^2 + \ldots + a_{2n-1}u^{2n-1} \qquad (2.3.3)$$

Integrating (2.3.3) between the limits 0 and 1, we obtain

$$I = \int_0^1 Q(u)du = \left[a_0u + \frac{1}{2}a_1u^2 + \frac{1}{3}a_2u^3 + \dots + \frac{1}{2n}a_{2n-1}u^{2n}\right]_0^1$$

$$= a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \dots + \frac{1}{2n}a_{2n-1} . \qquad (2.3.4)$$

From (2.3.3) we also have

$$Q(u_1) = a_0 + a_1u_1 + a_2u_1^2 + \dots + a_{2n-1}u_1^{2n-1}$$

$$Q(u_2) = a_0 + a_1u_2 + a_2u_2^2 + \dots + a_{2n-1}u_2^{2n-1}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$Q(u_n) = a_0 + a_1u_n + a_2u_n^2 + \dots + a_{2n-1}u_n^{2n-1} .$$

Substituting in the above values of $Q(u_1),Q(u_2),\dots,Q(u_n)$ in (2.3.2), we obtain

$$I = A_1(a_0 + a_1u_1 + a_2u_1^2 + \dots + a_{2n-1}u_1^{2n-1})$$

$$+ A_2(a_0 + a_1u_2 + a_2u_2^2 + \dots + a_{2n-1}u_2^{2n-1})$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ A_n(a_0 + a_1u_n + a_2u_n^2 + \dots + a_{2n-1}u_n^{2n-1}) ,$$

or, rearranging,

$$\begin{aligned} I = {} & a_0(A_1 + A_2 + \dots + A_n) \\ & + a_1(A_1u_1 + A_2u_2 + \dots + A_nu_n) \\ & + a_2(A_1u_1{}^2 + A_2u_2{}^2 + \dots + A_nu_n{}^2) \\ & + \dots\dots\dots\dots \\ & + \dots\dots\dots\dots \\ & + a_{2n-1}(A_1u_1{}^{2n-1} + A_2u_2{}^{2n-1} + \dots + A_nu_n{}^{2n-1}) \ . \end{aligned} \qquad (2.3.5)$$

Since it is desired that (2.3.4) be identically the same as (2.3.5) for all values of a_i , the coefficients of a_i must be equal. Therefore we obtain the 2n equations

$$\begin{aligned} & A_1 + A_2 + A_3 + \dots + A_n = 1 \\ & A_1u_1 + A_2u_2 + A_3u_3 + \dots + A_nu_n = \tfrac{1}{2} \\ & A_1u_1{}^2 + A_2u_2{}^2 + A_3u_3{}^2 + \dots + A_nu_n{}^2 = \tfrac{1}{3} \\ & \dots\dots\dots\dots \\ & \dots\dots\dots\dots \\ & A_1u_1{}^{2n-1} + A_2u_2{}^{2n-1} + A_3u_3{}^{2n-1} + \dots + A_nu_n{}^{2n-1} = \tfrac{1}{2n} \ . \end{aligned} \qquad (2.3.6)$$

The solution of the above system of nonlinear equations would theoretically give the 2n values $u_1, u_2, \dots, u_n$ and $A_1, A_2, \dots, A_n$. However, (2.3.6) can be reduced to a system of linear equations in

A_i if we choose the values for u_i to be the zeros of the Legendre polynomials previously defined. [4] The n roots of $P_n(x) = 0$ are known to be real, distinct, and all in the interval from 0 to 1, therefore, (2.3.6) will always have a solution since the u's are all distinct.

Having found the A_i's and u_i's (2.3.2) can now be applied to find the value of a definite integral.

It should be noted here that the process of calculating the value of a definite double integral of a function of two variables is called numerical double integration and also mechanical cubature. A formula for this process has been derived which uses an interpolation function in terms of the differences of a function of two variables. However, this is not necessary since numerical double integration may be performed by a double application of a quadrature formula. The value of the double integral can thus be found by repeated application of Gauss's formula.

The value of the double integral may be approximated by applying to each horizontal row any quadrature formula. Then, to the results thus obtained for the rows, again apply a similar formula. Hence

$$J = \int_0^1 \int_0^1 g(x,y)\,dx\,dy$$

$$= B_1 g(x_1,y_1) + B_2 g(x_2,y_1) + \dots + B_n g(x_n,y_1) = G(y_1)$$

$$+ B_1 g(x_1,y_2) + B_2 g(x_2,y_2) + \dots + B_n g(x_n,y_2) = G(y_2)$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ B_1 g(x_1,y_n) + B_2 g(x_2,y_n) + \dots + B_n g(x_n,y_n) = G(y_n)$$

then

$$J = \int_0^1 \int_0^1 g(x,y)\,dx\,dy = B_1 G(y_1) + B_2 G(y_2) + \dots + B_n G(y_n) .$$

On the basis of the facts now available an arbitrary function defined on the interval (0,1) can be formally expanded in a series of Legendre polynomials.

<u>Definition 2.3.1</u> The Fourier coefficients $C_{i,j}$ of any integrable function $F(x,y)$ associated with the system of orthogonal functions $P_i(x)$ and $P_j(y)$ is given by

$$C_{i,j} = \frac{\int_0^1 \int_0^1 P_i(x)P_j(y)F(x,y)\,dx\,dy}{(\frac{1}{2i+1}) \quad (\frac{1}{2j+1})}$$

These coefficients may be arrived at in the following manner:

Let the expansion be

$$F(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{i,j}\, P_i(x)P_j(y) \tag{2.3.7}$$

if we now multiply by $P_m(x)P_n(y)$ we obtain

$$F(x,y)P_m(x)P_n(y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{i,j}P_i(x)\; P_j(y)P_m(x)P_n(y) \;.$$

Assuming uniform convergence and integrating term by term from 0 to 1 each integral on the right is zero, because of the orthogonality of the P's, except the ones where i=m and j=n giving

$$\int_0^1 \int_0^1 F(x,y)P_m(x)P_n(y)dxdy = \int_0^1 \int_0^1 C_{m,n}P_m{}^2(x)P_n{}^2(y)dxdy$$

when the integral is evaluated by means of (2.1.8) the whole right-hand member reduces to

$$\int_0^1 \int_0^1 C_{m,n}P_m{}^2(x)P_n{}^2(y)dxdy = \frac{C_{m,n}}{(2m+1)(2n+1)}$$

so that the general coefficient in the series form is given by

$$C_{m,n} = \frac{\int_0^1 \int_0^1 F(x,y)P_m(x)P_n(y)dxdy}{(\frac{1}{2m+1}) \quad (\frac{1}{2n+1})} \quad . \tag{2.3.8}$$

III. OBTAINING A SOLUTION TO A DIFFERENTIAL EQUATION

3.1 Raising a Power Series to a Power

Let $\sum_{i=0}^{\infty} b_i(t-t_0)^i$ be any Taylor's series which has a positive radius of convergence and for which $b_0 = 1$. For brevity, we may replace $(t-t_0)$ by z and let f(z) denote the analytic function determined by the Taylor's series above. Now let Log w denote the principal value of the natural logarithm of the complex number w, and let λ be any complex number. We may then define $\left[f(z)\right]^{\lambda}$ as $\exp\left(\lambda \text{ Log}\left[f(z)\right]\right)$ and the function so defined has usual exponential properties and is analytic in a neighborhood of z = 0. For the power series expansion pertaining to $\left[f(z)\right]^{\lambda}$ we write

$$g(z, \lambda) = \sum_{i=0}^{\infty} P_i(\lambda) z^i . \qquad (3.1.1)$$

In particular

$$g(z,1) = \sum_{i=0}^{\infty} P_i(1) z^i = \sum_{i=0}^{\infty} b_i z^i \qquad (3.1.2)$$

where $P_0(\lambda) = b_0 = 1$.

The objective is the determination of the coefficients $P_i(\lambda)$. These are the coefficients of the power series after being raised to the power λ, as functions of the b_i. Results in this direction are found in [6] . One such recursive relation given therein is

$$P_i(\lambda) = \frac{1}{i} \sum_{j=1}^{i} b_j(j\lambda + j - i)P_{i-j}(\lambda) \quad (3.1.3)$$

Example: Suppose we wish to calculate f(x), where $f(x) = (1 + 2x)^2$. We may proceed as follows:

$$P_0(2) = 1.0$$

$$P_1(2) = \frac{1}{1}\left[b_1(2)P_0(2)\right] = 2b_1$$

$$P_2(2) = \frac{1}{2}\left[b_1(1)(2b_1) + b_2(4)(1)\right] = b_1^2 + 2b_2$$

$$P_3(2) = \frac{1}{3}\left[b_1(0)(b_1^2 + 2b_2) + b_2(3)(2b_1) + b_3(b)(1)\right] .$$

Now since $b_0 = 1$, $b_1 = 2$, $b_2 = 0$, and $b_3 = 0$ we obtain

$$P_0(2) = 1.0,\ P_1(2) = 4,\ P_2(2) = 4 ,$$

and all others will be zero. Hence,

$$f(x) = \sum_{i=0}^{\infty} P_i(\lambda)x^i$$

$$= 1 + 4x + 4x^2 .$$

3.2 Method of Solution

It is the purpose of this investigation to study the feasibility of using orthogonal polynomials, in this case Legendre polynomials, to obtain numerical solutions to differential systems which can be written in the form

$$y' = F(x,y) \qquad (3.2.1)$$

$$y(0) = 1 \; .$$

One such method can now be described based on the properties discussed to this point and one assumption. The assumption being that the solution to the differential equation will have the form

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

where $a_0 = 1$. (3.2.2)

In definition 2.3.1 the Fourier coefficients $C_{i,j}$ of any arbitrary function $F(x,y)$ associated with the system of orthogonal polynomials $P_i(x)$ and $P_j(y)$ were arrived at by expanding $F(x,y)$ in a series of Legendre polynomials. This expansion was

$$F(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{i,j} \, P_i(x) \, P_j(y) \; . \qquad (3.2.3)$$

Since we are interested in obtaining numerical solutions to differential systems of the form

$$y' = F(x,y)$$

and F(x,y) can be expressed by (3.2.3) we may then write

$$y' = F(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{i,j}\ P_i(x)\ P_j(y) \ .$$

Also since the solution to the differential equation was assumed to be

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \ , \quad (a_0 = 1)$$

we can write

$$y' = a_1 + 2a_2 x + \dots + na_n x^{n-1} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} C_{i,j}\ P_i(x)\ P_j(y) \ . \tag{3.2.4}$$

For the polynomial in y, i.e. $P_j(y)$, we replace y by (3.2.2) to get a polynomial in x alone on the right hand side of the above equation.

Having obtained a method for estimating $C_{i,j}$ and generating the polynomials $P_i(x)$ and $P_j(y)$, (2.3.8 and 2.1.9 respectively), we can now obtain a numerical solution to $y' = F(x,y)$.

To illustrate the method an example follows:

For simplicity we will carry the indicies i and j in (3.2.4) through 2 only. We then have

$$y' = a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} = \sum_{i=0}^{2} \sum_{j=0}^{2} C_{i,j}\, P_i(x)\, P_j(y)$$

or

$$a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1}$$

$$= \sum_{i=0}^{2} \left[C_{i,0}\, P_i(x)\, P_0(y) + C_{i,1}\, P_i(x)\, P_1(y) + C_{i,2}\, P_i(x)\, P_2(y)\right]$$

$$= C_{0,0}\, P_0(x)\, P_0(y) + C_{0,1}\, P_0(x)\, P_1(y) + C_{0,2}\, P_0(x)\, P_2(y)$$

$$+ C_{1,0}\, P_1(x)\, P_0(y) + C_{1,1}\, P_1(x)\, P_1(y) + C_{1,2}\, P_1(x)\, P_2(y)$$

$$+ C_{2,0}\, P_2(x)\, P_0(y) + C_{2,1}\, P_2(x)\, P_1(y) + C_{2,2}\, P_2(x)\, P_2(y)\ .$$

Using the recursive formula (2.1.9) for $P_i(x)$ and $P_j(y)$, we obtain

$$a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1}$$

$$= C_{0,0}(1)(1) + C_{0,1}(1)(1-2y) + C_{0,2}(1)(1-6y+6y^2)$$

$$+ C_{1,0}(1-2x)(1) + C_{1,1}(1-2x)(1-2y) + C_{1,2}(1-2x)(1-6y+6y^2)$$

$$+ C_{2,0}(1-6x+6x^2)(1) + C_{2,1}(1-6x+6x^2)(1-2y) + C_{2,2}(1-6x+6x^2)(1-6y+6y^2)$$

or, multiplying we have

$$a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1}$$

$$= C_{0,0} + C_{0,1} - 2C_{0,1}y + C_{0,2} - 6C_{0,2}y + 6C_{0,2}y^2$$

$$+ C_{1,0} - 2C_{1,0}x + C_{1,1} - 2C_{1,1}x - 2C_{1,1}y + 4C_{1,1}xy$$

$$+ C_{1,2} - 6C_{1,2}y + 6C_{1,2}y^2 - 2C_{1,2}x + 12C_{1,2}xy - 12C_{1,2}xy^2$$

$$+ C_{2,0} - 6C_{2,0}x + 6C_{2,0}x^2 + C_{2,1} - 6C_{2,1}x + 6C_{2,1}x^2$$

$$- 2C_{2,1}y + 12C_{2,1}xy - 12C_{2,1}x^2y + C_{2,2} - 6C_{2,2}x$$

$$+ 6C_{2,2}x^2 - 6C_{2,2}y + 36C_{2,2}xy - 36C_{2,2}x^2y + 6C_{2,2}y^2$$

$$- 36C_{2,2}xy^2 + 36C_{2,2}x^2y^2$$

or, rearranging,

$$a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1}$$

$$= (C_{0,0} + C_{0,1} + C_{0,2} + C_{1,0} + C_{1,1} + C_{1,2} + C_{2,0} + C_{2,1} + C_{2,2})$$

$$- (2C_{1,0} + 2C_{1,1} + 2C_{1,2} + 6C_{2,0} + 6C_{2,1} + 6C_{2,2})\ x$$

$$+ (6C_{2,0} + 6C_{2,1} + 6C_{2,2})\ x^2$$

$$- (2C_{0,1} + 6C_{0,2} + 2C_{1,1} + 6C_{1,2} + 2C_{2,1} + 6C_{2,2})\ y$$

$$+ (6C_{0,2} + 6C_{1,2} + 6C_{2,2})\ y^2$$

$$+ (4C_{1,1} + 12C_{1,2} + 12C_{2,1} + 36C_{2,2})\ xy$$

$$- (12C_{2,1} + 36C_{2,2})\ x^2y - (12C_{1,2} + 36C_{2,2})\ xy^2$$

$$+ (36C_{2,2})\ x^2y^2.$$

Substituting (3.2.2) for y, we obtain

$$a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1}$$

$$= (C_{0,0} + C_{0,1} + C_{0,2} + C_{1,0} + C_{1,1} + C_{1,2} + C_{2,0} + C_{2,1} + C_{2,2})$$

$$- (2C_{1,0} + 2C_{1,1} + 2C_{1,2} + 6C_{2,0} + 6C_{2,1} + 6C_{2,2})\ x$$

$$+ (6C_{2,0} + 6C_{2,1} + 6C_{2,2})\ x^2$$

$$- (2C_{0,1} + 6C_{0,2} + 2C_{1,1} + 6C_{1,2} + 2C_{2,1} + 6C_{2,2})(a_0 + a_1x + \ldots + a_nx^n)$$

$$+ (6C_{0,2} + 6C_{1,2} + 6C_{2,2})(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n)^2$$

$$+ (4C_{1,1} + 12C_{1,2} + 12C_{2,1} + 36C_{2,2})(a_0x + a_1x^2 + a_2x^3 + \ldots + a_nx^{n+1})$$

$$- (12C_{2,1} + 36C_{2,2})(a_0x^2 + a_1x^3 + a_2x^4 + \dots + a_nx^{n+2})$$

$$- (12C_{1,2} + 36C_{2,2})(x)(a_0 + a_1x + a_2x^2 + \dots + a_nx^n)^2$$

$$+ (36C_{2,2})(x^2)(a_0 + a_1x + a_2x^2 + \dots + a_nx^n)^2 .$$

To arrive at the desired polynomial answer we need only to equate coefficients to find the a_i's, $(i=1,2,\dots n)$, hence

$a_0 = 1$ (given) .

$$a_1 = (C_{0,0} + C_{0,1} + C_{0,2} + C_{1,0} + C_{1,1} + C_{1,2} + C_{2,0} + C_{2,1} + C_{2,2})$$

$$- (2C_{0,1} + 6C_{0,2} + 2C_{1,1} + 6C_{1,2} + 2C_{2,1} + 6C_{2,2})\ a_0$$

$$+ (6C_{0,2} + 6C_{1,2} + 6C_{2,2})\ a_0^2 .$$

$$2a_2 = - (2C_{1,0} + 2C_{1,1} + 2C_{1,2} + 6C_{2,0} + 6C_{2,1} + 6C_{2,2})$$

$$- (2C_{0,1} + 6C_{0,2} + 2C_{1,1} + 6C_{1,2} + 2C_{2,1} + 6C_{2,2})\ a_1$$

$$+ (6C_{0,2} + 6C_{1,2} + 6C_{2,2})\, 2a_0a_1$$

$$+ (4C_{1,1} + 12C_{1,2} + 12C_{2,1} + 36C_{2,2})\, a_0$$

$$- (12C_{1,2} + 36C_{2,2})\, a_0^2 .$$

⋮

etc.

By continuing the process all the a_i's , $(i=1,2,\ldots n)$, can be found, and hence the polynomial answer $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ may be calculated.

The greatest error in the above example comes from truncation in that the indicies i and j were carried through 2 only. If we had carried these indicies through a greater number, say 100, we would have eliminated some of this error and could expect a better estimation of the a_i's.

IV. CONCLUSION

The method just described was tested on the equations listed below. Comparisons of the estimated value, Y (EST), and of the theoretical value, Y, of these equations are given in the Appendix.

Equation No. 1

$$y' = xy - x^5 + 2.0x^4 + 2.75x^3 - 5.75x^2 + 1.50x - .25 \quad .$$

The solution is

$$y = 1.0 - .25x + 1.25x^2 - 2.0x^3 + 1.0x^4 \quad .$$

Equation No. 2

$$y' = ye^{-x} - y^2 - y \quad .$$

The solution is

$$y = e^{-x} \quad .$$

Equation No. 3

$$y' = x\sqrt{1-y^2} - x \sin x - \sin x \quad .$$

The solution is

$$y = \cos x \quad .$$

Equation No. 4

$$y' = xy^{1.5} - xy \cos x - \sin 2x \quad .$$

The solution is

$$y = \cos^2 x \quad .$$

The comparisons shown in the Appendix are those obtained using the IBM 709/7090 FORTRAN program which is listed just prior to these comparisons. The results thus obtained are subject to machine error, that is, round-off and truncation error. Keeping this in mind, the results obtained were very good. It should also be noted that values of x lie within the range 0 to 1 so as to stay within the limits of the theory.

Comments are periodically placed throughout the program listing to aid the user in understanding the program. Two subroutines called for by the main program, Subroutine SOLEQT and Subroutine ERASE, are library subroutines found at the Texas A. and M. College Data Processing Center and can be obtained upon request. The purpose of Subroutine ERASE is to set designated elements of an array to zero. The purpose of Subroutine SOLEQT is to find the solution to a system of simultaneous equations.

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VI. APPENDICES

```
C
C
C     THIS PROGRAM WILL SOLVE DIFFERENTIAL EQUATIONS OF THE FORM YPRIME=F(X,Y)
C
C     THE FOLLOWING BOUNDARY CONDITION MUST HOLD   Y(0)=1.0
C
C     N EQUALS THE DEGREE OF THE LEGENDRE POLY. WHOSE ROOTS ARE USED IN THE
C
C     GAUSSIAN QUADRATURE FORMULA, THE DEGREE OF THE QUADRATURE FORMULA
C
C     WILL BE 2N-1.
C
C     NUMDIE IS EQUAL TO THE NUMBER OF DIFFERENTIAL EQUATIONS TO BE SOLVED.
C
C     THE DIFFERENTIAL EQUATION TO BE SOLVED IS PLACED IN SUBROUTINE FUNSUB.
C
C     THE PROGRAM WILL SOLVE UP TO SIX DIFFERENTIAL EQUATIONS IN ONE
C
C     RUN AS IS, BUT IS CAPABLE OF SOLVING MORE BY ALTERING SUBROUTINE
C
C     FUNSUB.
C
C     NOP IS EQUAL TO THE NUMBER OF POINTS AT WHICH THE SOLUTION IS TO
C
C     BE EVALUATED.
C
C     XXX(I) IS EQUAL TO THE POINT AT WHICH THE SOLUTION IS TO BE EVALUATED.
C
C
C
      DIMENSION ICOL(21),IROW(21),XXX(15),YEST(15)
D     DIMENSION A(15),ROOT(10),B(10),P(15),C(10,10),PX(15),PY(15),AR(10,
     110),ARR(10,10,100),COE(11),POWY(9,10),SUM(10),F(10,10)
      COMMON A,ROOT,N,B,ARR,ICOL,IROW,PY,AR,C,F,COE,POWY,SUM,P,PX,KEY,XX
     1X,YEST
      KEY=1
    6 INPUT,800,N,NUMDIE,NOP
```

```
      INPUT,678,(XXX(I),I=1,NOP)
      CALL ERASE (B,B(30))
      NP1=N+1
C
C     GENERATING  THE LEGENDRE COEFFICIENTS
C
D     A(1)=1.0
      DO 29 K=2,NP1
D     E=N+K-1
D     C=N-K+2
D     D=(K-1)**2
D  29 A(K)=-(E*C*A(K-1))/D
      XN=N
      XMIN=0.0
      XMAX=1.0
      DELTAX=(1.0/(10.0*XN**2))
C
C     HERE WE GO FINDING ROOTS (MAYBE)
C
      CALL PROOT(XMIN,XMAX,DELTAX)
      DO 10 I=1,N
D     XQ=I
D  10 F(I,NP1)=1./XQ
      DO 20 K=1,N
      DO 20 J=1,N
D     KK=K-1
D  20 F(K,J)=ROOT(J)**(KK)
      CALL SOLEQT(F,IROW,ICOL,N,10,1.0E-05,DET)
      DO 40 I=1,N
D  40 B(I)=F(I,NP1)
C
C     LETS FIND THE C(I,J)    (STARTING DOUBLE INTEGRATION)
C
   45 DO 30 I=1,NP1
      DO 25 J=1,NP1
D     CALL GFUNCT(I,J,ANS)
D     XI=I
```

```
D       XJ=J
D       STA=1./(2.*XI-1.)
D       STO=1./(2.*XJ-1.)
D       C(I,J)=ANS/(STA*STO)
        IF(ABSF(C(I,J))-.001)987,25,25
D 987   C(I,J)=0.
   25   CONTINUE
   30   CONTINUE
        CALL ERASE (AR,AR(200))
        DO 7770 L=1,10
D       CALL PXCOE(L,PX)
        M=L
        DO 9990 NO=1,M
D       AR(M,NO)=PX(NO)
 9990   CONTINUE
 7770   CONTINUE
        CALL ERASE (ARR,ARR(20000))
C
C       HERE WE ARE BUILDING A THREE DIMENSIONAL ARRAY  (OH BOY)
C
        ISAVE=1
        CALL ERASE (SUM,SUM(20))
        DO 1000 K=1,100
        L=K
        IH=XMODF(L+10,10)
        IF(IH) 1090,1100,1090
 1100   IH=10
 1090   DO 1110 NU=1,IH
D       PY(NU)=AR(IH,NU)
 1110   CONTINUE
        IF(XMODF(L+10,10)-1) 1080,1070,1080
 1070   IHOLD=L/10 + 1
D       CALL PXCOE(IHOLD,PX)
 1080   IF(XMODF(L+10,10)) 1000,1030,1040
 1030   IRO =L/10
        GO TO 1050
 1040   IRO =L/10 + 1
```

```
D1050 XOO=C(IRO,ISAVE)
      DO 1010 I=1,IRO
      DO 1020 J=1,ISAVE
D     ARR(I,J,K)=PX(I)*PY(J)*XOO
D     SUM(I)=ARR(I,J,K)+SUM(I)
 1020 CONTINUE
 1010 CONTINUE
      ISAVE=XMODF(ISAVE+11,10)
      IF(ISAVE) 1000,1060,1000
 1060 ISAVE=10
 1000 CONTINUE
C
C     THREE  DIMENSIONAL ARRAY DONE    (GOODY)
C
      CALL ERASE (POWY,POWY(180))
C
C     START RAISING POWER SERIES TO A POWER AND EQUATING COEFFICIENTS
C
D     COE(1)=1.0
D     COE(2)=SUM(1)
D     POWY(1,2)=SUM(1)
      DO 500 I=1,9
D     POWY(I,1)=1.0
  500 CONTINUE
      DO 600 IJ=2,10
      IST=IJ-1
      DO 520 IK=2,9
      LAMBDA=IK+1
      DO 530 IL=1,IST
      IND=IST+1-IL
D     XNT=(IL*LAMBDA-IST)
D     POWY(IK,IJ)=POWY(IK,IJ)+COE(IL+1)*XNT*POWY(IK,IND)
  530 CONTINUE
D     XNX=IST
D     POWY(IK,IJ)=POWY(IK,IJ)/XNX
  520 CONTINUE
      ISTA=IJ+1
```

```
      LOOPA=IJ-1
      KA=ISTA
D     SUMB=SUM(IJ)
      DO 540 I=1,LOOPA
      KA=KA-1
      DO 550 J=2,10
      L=J-1
D     XAB=POWY(L,KA)
      DO 560 K=1,100
D     SUMB=SUMB+ARR(I,J,K)*XAB
  560 CONTINUE
  550 CONTINUE
  540 CONTINUE
D     XNO=IJ
D     COE(ISTA)=SUMB/XNO
C
C     ARRAY COE CONTAINS COEFFICIENTS TO THE POLYNOMIAL ANSWER (FINISHED)
C
      IF(IJ-10)570,600,570
D 570 POWY(1,ISTA)=COE(ISTA)
  600 CONTINUE
D     POE=0.0
      DO 651 J=1,NOP
      POE=XXX(J)
      YEST(J)=FOFZ(POE)
  651 CONTINUE
      OUTPUT,497
      DO 456 K=1,NOP
      OUTPUT,457,XXX(K),YEST(K)
  456 CONTINUE
      KEY=KEY+1
      IF (KEY-NUMDIE) 45,45,6
  457 FORMAT(1H0,7HAT X = F12.8,3X,28HTHE VALUE OF THE SOLUTION = F12.8)
  497 FORMAT(1H1)
  678 FORMAT(15F4.4)
  800 FORMAT(3I3)
      END
```

```
C     SUBROUTINE PROOT FINDS THE POS. ROOTS OF A POLY. OF DEGREE N
      SUBROUTINE PROOT(XMIN,XMAX,DELTAX)
      DIMENSION ICOL(21),IROW(21),XXX(15),YEST(15)
D     DIMENSION A(15),ROOT(10),B(10),P(15),C(10,10),PX(15),PY(15),AR(10,
     110),ARR(10,10,100),COE(11),POWY(9,10),SUM(10),F(10,10)
      COMMON A,ROOT,N,B,ARR,ICOL,IROW,PY,AR,C,F,COE,POWY,SUM,P,PX,KEY,XX
     1X,YEST
      Z=XMIN
      L=1
      POO=FOFX(Z)
      IF(POO)5,20,5
    5 Z=Z+DELTAX
      IF(XMAX-Z)150,7,7
  150 RETURN
    7 COO=FOFX(Z)
      IF(POO*COO)10,21,35
   35 POO=COO
      GO TO 5
   20 ROOT(L)=Z
      GO TO 22
   21 ROOT(L)=Z
   22 L=L+1
      Z=Z+0.001
      POO=FOFX(Z)
      GO TO 5
   10 X1=Z-DELTAX
      X2=Z
D     X3=X1
D     DEL=.0001
      X=SIGNF(1.0,FOFX(X1))
D  17 X3=X3+DEL
      Y=FOFX(X3)
      IF(ABSF(Y)-.000001)15,15,16
   16 IF(X*Y)18,15,17
   18 DEL=-DEL/10.
      IF (ABSF(DEL)-1.0E-15)15,15,99
   99 X=-X
```

```
      GO TO 17
D  15 ROOT(L)=X3
      POO=FOFX(Z)
      L=L+1
      GO TO 5
      END
```

```
C     FUNCTION FOFX FINDS THE VALUE OF A POLY. OF DEGREE N.  A(1)=CONST.
      FUNCTION FOFX(X)
      DIMENSION ICOL(21),IROW(21),XXX(15),YEST(15)
D     DIMENSION A(15),ROOT(10),B(10),P(15),C(10,10),PX(15),PY(15),AR(10,
     110),ARR(10,10,100),COE(11),POWY(9,10),SUM(10),F(10,10)
      COMMON A,ROOT,N,B,ARR,ICOL,IROW,PY,AR,C,F,COE,POWY,SUM,P,PX,KEY,XX
     1X,YEST
      N=N
D     SUM=0.0
      NP1=N+1
      DO 40 J=1,NP1
      K=NP1-J+1
D     XT=X
D  40 SUM=SUM*XT+A(K)
D     FOFX=SUM
      RETURN
      END
```

```
C      SUBROUTINE GFUNCT FINDS  C(I,J)  BY EVALUATING A DOUBLE INTEGRAL
       SUBROUTINE GFUNCT(I,J,ANS)
       DIMENSION ICOL(21),IROW(21),XXX(15),YEST(15)
D      DIMENSION A(15),ROOT(10),B(10),P(15),C(10,10),PX(15),PY(15),AR(10,
      110),ARR(10,10,100),COE(11),POWY(9,10),SUM(10),F(10,10)
       COMMON A,ROOT,N,B,ARR,ICOL,IROW,PY,AR,C,F,COE,POWY,SUM,P,PX,KEY,XX
      1X,YEST
       KEY=KEY
       N=N
D      ANS=0
       DO 12 M1=1,N
D      ANS1=0
D      Y=ROOT(M1)
D      Z2=FOFY(Y,J)
       DO 10 M=1,N
D      X=ROOT(M)
D      Z1=FOFY(X,I)
D      CALL FUNSUB(X,Y,KEY,ANS2)
D   10 ANS1=ANS1+(B(M)*Z1*Z2*ANS2)
D   12 ANS=ANS+B(M1)*ANS1
       RETURN
       END
```

```
C     FUNCTION FOFY GENERATES THE COEFFICIENTS OF THE LEGENDRE POLY. OF DEGREE
C     I AND FINDS THE VALUE OF THE POLY. AT THE POINT X
      FUNCTION FOFY(X,I)
      DIMENSION ICOL(21),IROW(21),XXX(15),YEST(15)
D     DIMENSION A(15),ROOT(10),B(10),P(15),C(10,10),PX(15),PY(15),AR(10,
     110),ARR(10,10,100),COE(11),POWY(9,10),SUM(10),F(10,10)
      COMMON A,ROOT,N,B,ARR,ICOL,IROW,PY,AR,C,F,COE,POWY,SUM,P,PX,KEY,XX
     1X,YEST
D     FOFY=1.
      IF(I-1)15,20,15
D  15 P(1)=1.0
      DO 16 K=2,I
D     R=I+K-2
D     S=I-K+1
D     T=(K-1)**2
D  16 P(K)=-(R*S*P(K-1))/T
D     FOFY=0.
      DO 18 L=1,I
      LL=I-L+1
D     XD=X
D  18 FOFY=FOFY*XD+P(LL)
   20 RETURN
      END
```

```
C      SUBROUTINE PXCOE GENERATES THE COEFFICIENTS OF THE LEGENDRE POLY.
C      OF DEGREE IHOLD
       SUBROUTINE PXCOE(IHOLD,PX)
       DIMENSION ICOL(21),IROW(21),XXX(15),YEST(15)
D      DIMENSION A(15),ROOT(10),B(10),P(15),C(10,10),PX(15),PY(15),AR(10,
      110),ARR(10,10,100),COE(11),POWY(9,10),SUM(10),F(10,10)
       COMMON A,ROOT,N,B,ARR,ICOL,IROW,PY,AR,C,F,COE,POWY,SUM,P,PX,KEY,XX
      1X,YEST
D      PX(1)=1.0
       IF(IHOLD-1)15,20,15
    15 DO 16 K=2,IHOLD
D      R=IHOLD+K-2
D      S=IHOLD-K+1
D      T=(K-1)**2
D   16 PX(K)=-(R*S*PX(K-1))/T
    20 RETURN
       END
```

```
C     FUNCTION FOFZ FINDS THE VALUE OF THE POLY. ANS AT XXX(I)
      FUNCTION FOFZ(POE)
      DIMENSION ICOL(21),IROW(21),XXX(15),YEST(15)
D     DIMENSION A(15),ROOT(10),B(10),P(15),C(10,10),PX(15),PY(15),AR(10,
     110),ARR(10,10,100),COE(11),POWY(9,10),SUM(10),F(10,10)
      COMMON A,ROOT,N,B,ARR,ICOL,IROW,PY,AR,C,F,COE,POWY,SUM,P,PX,KEY,XX
     1X,YEST
D     SSUM=0.0
D     XPOE=POE
      DO 42 J1=1,11
      K=11-J1+1
D  42 SSUM=SSUM*XPOE+COE(K)
D     FOFZ=SSUM
      RETURN
      END
```

```
C       SUBROUTINE FUBSUB FIND THE VALUE OF THE DIFFERENTIAL EQUATION TO BE
C       SOLVED AT THE POINTS X AND Y
        SUBROUTINE FUNSUB(X,Y,KEY,ANS2)
        KEY=KEY
        GO TO (101,102,103,104,105,106),KEY
D 101   ANS2=X*Y-X**5+2.*X**4+(11./4.)*X**3-(23./4.)*X**2+(3./2.)*X-.25
        GO TO 107
D 102   ANS2=(Y*(EXPF(-X)))-(Y**2.0)-(Y)
        GO TO 107
D 103   ANS2=X*SQRTF(1.0-Y**2)-(X*SINF(X))-SINF(X)
        GO TO 107
D 104   ANS2=X*(Y**1.5)-(X*Y)*COSF(X)-SINF(2.0*X)
        GO TO 107
  105   CONTINUE
  106   CONTINUE
  107   RETURN
        END
```

RESULTS OF EQUATION 1

X	Y	Y(EST)	DIFF
0.060	0.98908094	0.98908098	-0.00000004
0.120	0.98475135	0.98475141	-0.00000006
0.180	0.98488574	0.98488583	-0.00000008
0.240	0.98766975	0.98766982	-0.00000007
0.300	0.99159998	0.99160003	-0.00000004
0.360	0.99548414	0.99548415	-0.00000001
0.420	0.99844094	0.99844091	0.00000003
0.480	0.99990015	0.99990010	0.00000005
0.540	0.99960254	0.99960249	0.00000005
0.600	0.99759998	0.99759994	0.00000004
0.660	0.99425535	0.99425533	0.00000002
0.720	0.99024255	0.99024256	-0.00000001
0.780	0.98654655	0.98654658	-0.00000003
0.840	0.98446336	0.98446340	-0.00000004
0.900	0.98559999	0.98560005	-0.00000006

RESULTS OF EQUATION 2

X	Y	Y(EST)	DIFF
0.060	0.94176455	0.94175420	0.00001035
0.120	0.88692044	0.88691451	0.00000593
0.180	0.83527021	0.83527242	-0.00000221
0.240	0.78662787	0.78663642	-0.00000855
0.300	0.74081823	0.74082926	-0.00001103
0.360	0.69767633	0.69768605	-0.00000972
0.420	0.65704682	0.65705276	-0.00000594
0.480	0.61878340	0.61878511	-0.00000171
0.540	0.58274826	0.58274823	0.00000004
0.600	0.54881164	0.54881709	-0.00000545
0.660	0.51685134	0.51687832	-0.00002699
0.720	0.48675226	0.48683429	-0.00008203
0.780	0.45840602	0.45861054	-0.00020453
0.840	0.43171053	0.43216848	-0.00045796
0.900	0.40656966	0.40752560	-0.00095594

RESULTS OF EQUATION 3

X	Y	Y(EST)	DIFF
0.060	0.99820054	0.99836136	-0.00016082
0.120	0.99280863	0.99324771	-0.00043908
0.180	0.98384369	0.98446665	-0.00062296
0.240	0.97133797	0.97197673	-0.00063876
0.300	0.95533648	0.95585214	-0.00051565
0.360	0.93589682	0.93624658	-0.00034976
0.420	0.91308894	0.91338383	-0.00029489
0.480	0.88699493	0.88760143	-0.00060650
0.540	0.85770869	0.85945968	-0.00175099
0.600	0.82533562	0.82989594	-0.00456031
0.660	0.78999224	0.80034989	-0.01035766
0.720	0.75180574	0.77270418	-0.02089844
0.780	0.71091355	0.74877207	-0.03785852
0.840	0.66746283	0.72891580	-0.06145296
0.900	0.62160997	0.70918979	-0.08757982

RESULTS OF EQUATION 4

X	Y	Y(EST)	DIFF
0.060	0.99640431	0.99641249	-0.00000818
0.120	0.98566898	0.98569091	-0.00002193
0.180	0.96794841	0.96797389	-0.00002548
0.240	0.94349744	0.94351312	-0.00001568
0.300	0.91266779	0.91266415	0.00000364
0.360	0.87590285	0.87587763	0.00002522
0.420	0.83373141	0.83369078	0.00004063
0.480	0.78676000	0.78671946	0.00004054
0.540	0.73566418	0.73565276	0.00001142
0.600	0.68117888	0.68125372	-0.00007483
0.660	0.62408773	0.62437318	-0.00028545
0.720	0.56521186	0.56598778	-0.00077592
0.780	0.50539806	0.50727769	-0.00187962
0.840	0.44550663	0.44976686	-0.00426023
0.900	0.38639895	0.39555581	-0.00915685