LOOP SPACES IN MOTIVIC HOMOTOPY THEORY

A Dissertation

by

MARVIN GLEN DECKER

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2006

Major Subject: Mathematics

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ABSTRACT

Loop Spaces in Motivic Homotopy Theory. (August 2006)

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In topology loop spaces can be understood combinatorially using algebraic theories. This approach can be extended to work for certain model structures on categories of presheaves over a site with functorial unit interval objects, such as topological spaces and simplicial sheaves of smooth schemes at finite type. For such model categories a new category of algebraic theories with a proper cellular simplicial model structure can be defined. This model structure can be localized in a way compatible with left Bousfield localizations of the underlying category of presheaves to yield a Motivic model structure for algebraic theories. As in the topological context, the model structure is Quillen equivalent to a category of loop spaces in the underlying category.

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CHAPTER I

INTRODUCTION

Objects in a category of spaces with algebraic structure can be understood as loop spaces; understanding the homotopy of loop spaces, therefore, is an avenue for understanding algebraic structure. In the topological setting, loop spaces have received a considerable amount of attention and many results exist describing not only the homological data of loop spaces but also techniques for recognizing when a topological spaces has the algebraic structure of a loop space. The development of similar techniques in categories other than topological spaces is the focus of this work; in particular we will focus on understanding loop spaces in categories of presheaves over a site with an interval as in [19].

Although many approaches exist in topology for recognizing loop spaces, a very functorial approach exploits Lawvere's algebraic theories[15]. Described in detail in [23], and more specific to loop space recognition in [4] and [3]; this approach relies on the development of a category of $\mathbf{T}$-algebras, where $\mathbf{T}$ is an algebraic theory, and imposing the structure of a model category. In this approach the homotopy of this category of algebras, $\mathbf{Alg_T}$, captures precisely the structure of topological spaces localized at maps which induce weak equivalence of loop spaces over a particular sphere object determining the choice of $\mathbf{T}$. The advantage of this approach in topology is that recognizing loop spaces can be accomplished with surprisingly limited conditions on topological spaces in contrast to the recognition machinery of May in [18], although certain combinatorial results about cohomology are lost.

This dissertation follows the style of Transactions of the American Mathematical Society.

The functorial nature of this technique of recognition suggests that it is possible to generalize to other categories of "spaces." The example of particular interest in this work, is the category of pointed simplicial sheaves over smooth schemes of finite type over a field k. We shall attempt to apply this machinery by way of a more general category of pointed simplicial presheaves over a small category with an interval, $\mathcal{C}$.

The first step is to understand precisely what should be the generalization of a **T**-algebra in the context of presheaves. A generalization already exists in the sense of a **T**-model, see [23], but this turns out to be too general to capture general data about loop spaces. We develop a notion of **T**-algebras for presheaves with a slightly more rigid construction than that of a **T**-model by understanding a **T**-algebra as a product preserving presheaf over an enlargement of the underling category $\mathcal{C}$. These **T**-algebras of presheaves have properties very similar to those of **T**-algebras of topological spaces and, so, seem to be the right generalization.

We give our category of **T**-algebras a model structure and demonstrate that it is a proper, simplicial, cellular model structure which permits homotopy localization arguments allowing us to apply the construction to categories of presheaves Quillen equivalent to Morel and Voevodsky's $\mathbb{A}^1$-homotopy theory as in [19]. We also offer an alternative development of these principles in a category of sheaves over a site.

Chapter III develops the homotopy of **T**-algebras to understand two relevant results. The first is:

Theorem I.1. *Let X be a cofibrant presheaf. Then for $\mathbf{T} = \mathbf{T}^A$, where A is cofibrant and the quotient of representables, and $\mathcal{C}$ is a site with an interval I, closed under finite coproducts, we have:*

$$\mathcal{U}(\mathcal{F}_{\mathbf{T}}(X)) \simeq \underline{Hom}(A, A \wedge \tilde{X}),$$

where $\simeq$ denotes an I-local weak equivalence and $\tilde{X}$ denotes a fibrant replacement of X.

This may be thought of as a generalization of May's Approximation theorem in [18] where a special type of free **T**-algebra replaces his little cubes operad.

The second result is:

Theorem I.2. *The functors $B_T : \textbf{Alg}_{\textbf{T}} \leftrightarrow \mathcal{LC} : \Omega^T$ form a Quillen equivalence.*

Here Ω^T is a functor sending a presheaf over $\mathcal{C}$ to one over the enlargment of $\mathcal{C}$ used to define **T**-algebras and B_T is its left adjoint. $\mathcal{LC}$ is a localization of the category of presheaves with respect to maps which induce week equivalences of these resulting **T**-algebras. This result, has as a corollary:

Theorem I.3. *Let $X \in SSet_{\bullet}^{\mathcal{C}^{op}}$, then X is weakly equivalent to a **A** loop space if and only if there exists $\tilde{X} \in \textbf{Alg}_{\textbf{T}}$ such that $\mathcal{U}(\tilde{X})$ is weakly equivalent to X.*

This is a precise generalization of the results of [4].

Although understanding whether or not a space is a loop space seems, in examples considered thus far, to be just as difficult as determining whether or not it has the structure of a **T**-algebra; this alternative characterization, which can be seen as having the action of a collection of presheaves, suggests that understanding loop spaces may be reduced to understanding the existence of algebraic actions determined by the presheaves acting on the space.

CHAPTER II

THE BOUSFIELD-KAN MODEL STRUCTURE

This chapter provides a brief introduction to the theory of Model Categories as well as some technical lemmata useful for working over general model categories. Special attention will be paid to a model structure for simplicial sets and more generally the projective, injective, and flasque model structure which yield universal ways to develop homotopy theories over a very general class of categories, $SSet_{\bullet}^{\mathcal{C}^{op}}$, where C is a small category.

A.   Introduction

Daniel Quillen developed the theory of Model Categories while investigating the role of simplicial objects in computing cohomology theories over categories closed under finite limits and containing sufficiently many projective objects. In his investigation he encountered the need for general formalism of arguments similar to arguments in Algebraic Topology. The formalism is described in detail in *Homotopical Algebra* [21].

Quillen exhibits that the conventional examples such as topological spaces, simplicial sets, and simplicial groups are all model categories, or categories sufficient for making sense of a homotopy theory. Historically, certain adaptations have been made to Quillen's original definition which have made certain properties of model categories more functorial and consequently have simplified constructions. Section B will provide modern definitions and examples of model categories and highlight general results useful in subsequent chapters.

One recent addition to the understanding of model categories has been understanding localization of model structure. A model category is defined in such a way as to permit a formal inversion of a class of weak equivalences. Given a model structure

it is sometimes useful to enlarge the class of weak equivalences so that the formal inversion of the new larger class remains possible. In *Model Categories and Their Localizations*, [11], Hirschorne describes functorial localization of model categories which possess a sufficiently combinatorial model structure. These types of techniques are crucial in localized constructions such as Morel and Voevodsky's $\mathbb{A}^1$-homotopy theory as discussed in [19]. Section C recalls the basic definitions and existence results for localizations of model structures.

Simplicial sets are in some sense the most universal example of a model category. Applications of simplicial sets in homology and homotopy theory are well documented and understood. The basic results of simplicial sets and the definition of a useful model structure for the category of simplicial sets are recalled in Section D along with some discussion about universality of the simplicial model structure. Applications to understanding homotopy theory in more general model categories as well as some standard results useful to the localization techniques of Section C are also discussed. Although nearly all of the results in this section are classical, some attention will be paid to details which are particularly enlightening in the context of later chapters.

In *Universal Homotopy Theories*, [7], Daniel Dugger describes how one can use the Bousfield-Kan model structure of simplicial presheaves over a small category $\mathcal{C}$ to make sense of a sort of universal model category structure associated to any space. This model structure is particularly powerful in that it can be localized so as to recover important examples of model structure such as Morel and Voevodsky's $\mathbb{A}^1$-homotopy theory of schemes over a site[19]. The Bousfield-Kan structure is useful in other regards as well, it is constructed in such a way to preserve many of the extremely nice properties of the model structure on simplicial sets. Additionally, the Bousfield-Kan structure has combinatorial properties sufficient for generalized localization techniques, such as those discussed in Section C. One limitation of the

Bousfield-Kan structure is that certain motivic spheres, defined later, are not cofibrant. Another, equivalent, model structure one can consider to handle this difficulty is the Flasque Model structure of Daniel Isakson[13]. This model structure has the same weak equivalences and also localizes to the Morel-Voevodsky A^1-homotopy theory, but has the added advantage that certain spheres are already cofibrant. The Bousfield-Kan and Flasque structures will be described in Section E. Additionally, we shall consider the injective model structure for presheaves in this section, which will have relevance for proving properness results later.

Throughout these sections we will always assume that we are working over small categories. As a convention $\mathcal{C}$ will denote a general small category, $\mathcal{M}$ will denote a category with an associated model structure, and given a category $\mathcal{D}$ we will denote by $\mathcal{D}^{op}$ the opposite category where all arrows have been reversed. We shall pay special attention to the examples of topological spaces, Top; pointed topological spaces, $Top_\bullet$; simplicial sets, $SSet$; pointed simplicial sets, $SSet_\bullet$; presheaves of simplicial sets over a category $\mathcal{C}$, $SSet^{\mathcal{C}^{op}}$; and pointed simplicial presheaves over $\mathcal{C}$, $SSet_\bullet^{\mathcal{C}^{op}}$.
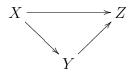
B. Model Categories

This section describes the basic definitions and results behind the theory of model categories as described by Quillen in [21]. A model category $\mathcal{M}$ is essentially a category with properties sufficient to create a new category $\mathbf{Ho}\mathcal{M}$ which contains the same objects but whose morphisms include formal inversions of a class of maps in $\mathcal{M}$ called weak equivalences. The best known example of this type of construction is the homotopy theory of pointed topological spaces with non-degenerate basepoint. Weak equivalences are maps which induce isomorphisms of homotopy groups and are homotopy invertible and so the homotopy model structure can be taken to be pointed

topological spaces with homotopy classes of maps as morphisms. The construction of a homotopy category for a general model category $\mathcal{M}$ is constructed by categorically modeling techniques from the topological situation. One defines a notion of homotopy between maps and then reduces to homotopy classes of maps between objects. In order to accomplish this one needs a category to satisfy some axioms allowing similar constructions as in topological spaces.

Definition II.1. Let $\mathcal{C}$ be a complete and cocomplete category with three classes of maps: weak equivalences, denoted W; fibrations, denoted by F; and cofibrations, C. Then $\mathcal{C}$ is a model category if the following axioms hold.

- For any commutative diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Z \\
 & \searrow \quad \nearrow & \\
 & Y &
\end{array}
$$

  if any two maps are in W then so is the third.

- W, C, and F are closed under retracts.

- For any commutative diagram of the form

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\
B & \longrightarrow & Y
\end{array}
$$

  where $i \in C$ and $p \in F$ if either i or p is also in W then $\exists h : B \to X$ such that the following diagram commutes

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
{\scriptstyle i}\downarrow & {\scriptstyle h}\nearrow & \downarrow{\scriptstyle p} \\
B & \longrightarrow & Y
\end{array}
$$

- Given a map $f : X \to Y$ there exist two functorial factorizations

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\
 & {\scriptstyle i}\searrow \quad \nearrow {\scriptstyle \tilde{p}} & \\
 & Z &
\end{array}
$$

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\
 & {\scriptstyle \tilde{i}}\searrow \quad \nearrow {\scriptstyle p} & \\
 & Z &
\end{array}
$$

with $p, \tilde{p} \in F$, $i, \tilde{i} \in C$, and $\tilde{i}, \tilde{p} \in W$.

We note that the later part of the definition differs slightly from Quillen's original formulation in that Quillen did not assume that the factorization of f was functorial. This assumption is true for a wide variety of typical model categories encountered in practice and simplifies certain constructions.

Remark II.2. Since $\mathcal{M}$ is complete and cocomplete, the empty diagram has both a limit and a colimit associated to it, so in particular $\mathcal{M}$ has initial and terminal objects.

It is helpful to adopt the following terminology for model categories.

Definition II.3. Let $\mathcal{M}$ be a model category.

1. An *acyclic fibration* or *trivial fibration* is a map $\rho \in F \cap W$.

2. An *acyclic cofibration* or *trivial cofibration* is a map $i \in C \cap W$.

3. A *fibrant* object in $\mathcal{M}$ is one such that the map to the final object is a fibration.

4. A *cofibrant* object is an object such that the map from the initial object is a cofibration.

5. A *cofibrant-fibrant* object is an object which is both fibrant and cofibrant.

Remark II.4. In practice, when giving a category the structure of a model category it is typical to define weak equivalences and either cofibrations or fibrations, and then define the remaining collection by a lifting property as in the following examples.

Example II.5. We give *Top* the structure of a model category by defining weak equivalences as maps which induce isomorphisms on all homotopy groups, cofibrations as closed inclusions, and fibrations as all maps which have the right lifting property with respect to all acyclic cofibrations.

Example II.6. We give *SSet*, the category of simplicial sets, the structure of a model category by defining weak equivalences as those maps which induce weak equivalences under geometric realization, fibrations are defined to be Kan fibrations, and cofibrations are those maps which have the left lifting property with respect to trivial fibrations.

It is clear that cofibrations and fibrations are dual notions, so any property which holds for cofibrations has a dual statement which holds for fibrations. Standard results about model categories, then, are generally proved for either collection and then duality yields a corresponding statement about the other. One typical result of this nature, which will be useful later, is the following proposition.

Proposition II.7. *Let $\mathcal{M}$ be a model category.*

1. *Acyclic fibrations are characterized by having the right lifting property with respect to cofibrations.*

2. *Fibrations are characterized by having the right lifting property with respect to acyclic cofibrations.*

3. *Cofibrations are characterized by having the left lifting property with respect to acyclic fibrations.*

4. Acyclic cofibrations are characterized by having the left lifting property with respect to fibrations.

Proof. All proofs are very similar and the proofs for the third and fourth assertion are dual to the arguments for the second and first respectively, we only give the proof for the first statement. By the axioms defining a model category it is clear that an acyclic fibration satisfies the desired property. Suppose one has a map $f : X \to Y$ which has the right lifting property with respect to cofibrations. Consider a factorization as per axiom four of Definition II.1

$$X \xrightarrow{i_f} X' \xrightarrow{p_f} Y.$$

Where i_f is a cofibration and p_f is an acyclic fibration. We then have a lifting diagram

$$
\begin{array}{ccc}
X & = & X \\
\downarrow{\scriptstyle i_f} & \nearrow^{h} & \downarrow{\scriptstyle f} \\
X' & \xrightarrow{p_f} & Y
\end{array}
$$

and using h one has the retract diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{i_f} & X' & \xrightarrow{h} & X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p_f} & & \downarrow{\scriptstyle f} \\
Y & \longrightarrow & Y & \longrightarrow & Y
\end{array}
$$

showing f is the retract of an acyclic fibration and so is an acyclic fibration. $\square$

One other functorial property of model categories which we shall frequently make use of is the notion of fibrant and cofibrant replacement. Given that every map has a functorial factorization it is clear that the the canonical map from the initial object to an object X can be factored as a cofibration followed by a weak equivalence. If we denote the initial object as $\tilde{i}$ then the factorization is given by

$$\tilde{i} \to X_c \to X.$$

Such a factorization will be denoted simply by X_c and we will call this a cofibrant replacement for X. We make a dual construction for the canonical map over the final object, and we will denote this fibrant replacement by X_f.

The use of cofibrant and fibrant replacement can be used in conjunction with a notion of left and right homotopy to create an equivalence class on maps between objects in $\mathcal{M}$, which allows one to construct the homotopy category $\mathbf{Ho}(\mathcal{M})$.

Theorem II.8. *Let $\mathcal{M}$ be a model category, then there exists a category, denoted $\mathbf{Ho}(\mathcal{M})$, with a functor $H : \mathcal{M} \to \boldsymbol{Ho}(\mathcal{M})$ such that if $f \in \mathcal{M}(X,Y)$ is a weak equivalence then $H(f)$ is an isomorphism. Moreover, $\boldsymbol{Ho}(\mathcal{M})$ is universal with respect to this property.*

The proof of this result is a standard application of the ideas of formally localizing a category, the details of which are explained very well in [9], [12], and [11]. Although this is a beautiful type of construction, the details will not serve the results here and so are omitted.

Suppose one is given a functor $F : \mathcal{M} \to \mathcal{M}'$ of model categories $\mathcal{M}$ and $\mathcal{M}$'. An obvious question is whether or not F induces a functor $\tilde{F} : \mathbf{Ho}(\mathcal{M}) \to \mathbf{Ho}(\mathcal{M})$. If such a functor exists, it is called the derived functor of F. We will now try to understand properties of F which will guarantee the existence of such a derived functor and extend this to understand when adjoint functors induce adjoint functors on the derived categories. This will provide a context for the comparison of homotopy theories that will be crucial to subsequent discussion. Any standard reference on model categories provides a detailed account of the proofs to the results outlined here: A complete account can be found in Section 8.4 of [11].

Definition II.9. Suppose $F : \mathcal{M} \to \mathcal{M}'$ is a functor. We will call a pair (G, s) a homotopy factorization of F, if $G : \mathbf{Ho}(\mathcal{M}) \to \mathcal{M}'$ and s is a natural transformation

$s : G\gamma \to F$, where $\gamma : \mathcal{M} \to \mathbf{Ho}(\mathcal{M})$. We will call such a pair a left derived functor of F if the pair is terminal amongst all such pairs (here we are in the category with objects homotopy factorizations and with morphisms given by natural transformations which make the obvious diagrams commute). We define right derived functors for F dually.

An obvious question is what conditions guarantee the existence of such homotopy factorizations. Clearly, it should be sufficient that F take weak equivalences to isomorphisms up to natural transformation. In fact we have:

Proposition II.10. *Suppose $F : \mathcal{M} \to \mathcal{M}'$ sends weak equivalences between cofibrant objects to isomorphisms, then the the left derived functor of F exists. By duality, if F sends weak equivalences between fibrant objects to isomorphisms then the right derived functor of F exists. Denote these functors LF and RF respectively.*

Since the proof relies on an investigation of right and left homotopies, which have been omitted here, the proof, too, will be omitted. A readable proof can be found in [9] on page 41.

Remark II.11. It is now clear that we have sufficient criteria for understanding when $F : \mathcal{M} \to \mathcal{M}'$ induces a functor of homotopy categories, namely we consider the left derived functor of the composition $\gamma_{\mathcal{M}'} \circ F$, where $\gamma_{\mathcal{M}'} : \mathcal{M}' \to \mathbf{Ho}(\mathcal{M}')$.

From this remark we have a definition.

Definition II.12. The left derived functor of $\gamma_{\mathcal{M}'} \circ F$ described above will be called the total derived functor of F.

Using derived functors we have a meaningful way of comparing model structures over a category. One way of comparing model structures more precisely is to understand the existence of adjoint functors. Whereas this will be critical to future discussion, we introduce the following.

Proposition II.13. *Let $F : \mathcal{M} \to \mathcal{M}' : G$ be an adjoint pair of functors between model categories, which satisfy the following equivalent conditions:*

1. *F preserves cofibrations and G preserves fibrations*

2. *F preserves cofibrations and acyclic cofibrations*

3. *G preserves fibrations and acyclic fibrations*

Then, the total derived functors of F and G exist and are adjoint.

Proof. We start by showing the equivalence of each of the conditions. By Proposition II.7, we may restate each of the conditions in terms of lifting diagrams. The fact that F and G are adjoint implies that the diagram

$$
\begin{array}{ccc}
F(A) & \longrightarrow & X \\
{\scriptstyle F(i)}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
F(B) & \longrightarrow & Y
\end{array}
$$

admits a lift if and only if the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & G(X) \\
{\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle G(p)} \\
B & \longrightarrow & G(Y)
\end{array}
$$

admits one, where i is an acyclic cofibration and p is a fibration. The equivalence of (1) and (3) are proved similarly. The conclusion of the proposition follows from the following lemma, which is useful on its own; the observation that if F sends acyclic cofibrations between cofibrant objects to weak equivalences, then $\gamma_{\mathcal{M}'} \circ F$ sends acyclic cofibrations between cofibrant objects to isomorphisms; Proposition II.10; and duality. $\qquad\square$

Lemma II.14. *If F takes acyclic cofibrations between cofibrant objects to weak equivalences, then F takes all weak equivalences between cofibrant objects to weak equivalences.*

Proof. This result follows from the fact that if $g : X \to Y$ is a weak equivalence between cofibrant objects in $\mathcal{M}$, then there is a functorial factorization of g as ji where i is a trivial cofibration and j is a trivial fibration that has a right inverse which is a trivial cofibration. This is due to K.S. Brown [6] and is proved as follows. Since X and Y are cofibrant, it's easy to show that $X \to X \amalg Y$ and $Y \to X \amalg Y$ are cofibrations. Factor the map $g \amalg 1_Y : X \amalg Y \to Y$ as an cofibration followed by an acyclic fibration as in

$$X \amalg Y \to Z \xrightarrow{j} Y.$$

Now, since g and j are weak equivalences we see that the induced map $X \to Z$ is a weak equivalence and the composition inducing $Y \to Z$ is the inverse to $Z \to Y$ required. □

We shall call adjoint functors satisfying the requirements of Proposition II.13 a *Quillen Pair*. Quillen funtors yield a pair of adjoint functors on the homotopy categories. We can say more.

Proposition II.15. *[12] Let (F, G) be a Quillen pair, if $f : F(X) \to Y$ is a weak equivalence if and only if $f^{\sharp} : X \to G(Y)$ is a weak equivalence for any cofibrant X and fibrant Y, then (F, G) induce an equivalence of homotopy categories. In this case we shall call (F, G) a* Quillen equivalence.

The proof of this essentially follows from the fact that morphisms in the homotopy category are determined by considering equivalence classes of maps between cofibrant and fibrant replacements of objects.

C.   Recognizing and Localizing Model Categories

This section explores definitions and results useful for recognizing special model structures and defines two types of localization procedures relevant for these structures. The special model structures explored here are more combinatorial in nature than the general structure described above and many classical examples of model categories satisfy these combinatorial properties. Existence results for localizations are also stated, as are some requisite definitions and constructions essential to localization. The constructions given will be critically important to later proofs.

We first recall the notion of a transfinite composition and a λ-sequence, as well as some auxiliary terms useful for functorial constructions.

Definition II.16. Let $\mathcal{C}$ be a category closed under colimits. A λ-sequence will be a functor $X : \lambda \to \mathcal{C}$ such that for every limit ordinal $\gamma < \lambda$ one has

$$colim_{\beta<\gamma}X_\beta \to X_\gamma$$

is an isomorphism. The *composition* of the λ-sequence will be the map $X_0 \to colim_{\beta<\lambda}X_\beta$.

Definition II.17. Let $\mathcal{C}$ be a category closed under colimits.

1. If I is a class of maps in $\mathcal{C}$ then a λ-sequence of maps in I is a λ-sequence in $\mathcal{C}$ such that each of the maps $X_\beta \to X_{\beta+1} \in I$.

2. If I is a class of maps in $\mathcal{C}$ then a *transfinite composition* in I is a map in $\mathcal{C}$ that is the composition of a λ-sequence in I.

Remark II.18. Using transfinite induction one can show that maps with a left lifting property are closed under transfinite composition.

Definition II.19. [11] Let $\mathcal{C}$ be a cocomplete category and let $\mathcal{D}$ be a subcategory of $\mathcal{C}$.

1. If κ is a cardinal, then an object W in $\mathcal{C}$ is *κ-small relative to* $\mathcal{D}$ if, for every regular cardinal $\lambda > \kappa$ and every λ-sequence in $\mathcal{D}$, the map of sets $colim_{\beta<\lambda}\mathcal{C}(W, X_\beta) \to \mathcal{C}(W, colim\beta < \lambda)$ is an isomorphism.

2. An object is *small relative to* $\mathcal{D}$ if it is κ-small relative to $\mathcal{D}$ for some cardinal κ, and it is small if it is small relative to $\mathcal{C}$.

Definition II.20. [11] Let $\mathcal{C}$ be a category and let I be a set of maps in $\mathcal{C}$.

1. The subcategory of I-injectives is the subcategory of maps that have the right lifting property (See II.1) with respect to every element of I.

2. The subcategory of I-cofibrations is the subcategory of maps that have the left lifting property (See II.1) with respect to all I-injectives. An object is I-injective if the map from the initial object to it is in I-cofibrations.

Definition II.21. [11] If $\mathcal{C}$ is a category that is closed under small colimits and I is a set of maps in $\mathcal{C}$, then

1. The subcategory of *relative I-cell complexes* is the subcategory of maps that can be constructed as a transfinite composition of pushouts of elements of I,

2. An object is an *I-cell complex* if the map to it from the initial object is a relative I-cell complex,

We may now define a cofibrantly generated model category.

Definition II.22. A cofibrantly generated model category is a model category $\mathcal{M}$ such that

1. There exists a set I of maps called generating cofibrations which permit the small object argument (i.e. the domains of the maps are small as in II.19) and such that trivial fibrations are characterized as having the right lifting property (II.1) with respect to all maps in I, and

2. There exists a set of maps J, called generating acyclic cofibrations, such that fibrations are characterized as having the right lifting property with respect to maps in J.

Example II.23. Let *Top* be the category of topological spaces. Top has the structure of a cofibrantly generated model category where the generating cofibrations are the geometric realizations of all inclusions $\partial\Delta[n] \to \Delta[n]$ for $n \geq 0$ and the generating acyclic cofibrations are the geometric realizations of inclusions $\Delta[n, k] \to \Delta[n]$ for $n \geq 1$, where $\Delta[n, k]$ denotes the kth hat complex.

Cofibrantly generated model categories have nice combinatorial properties, in particular one can describe all cofibrations as retracts of I-cell complexes and fibrant/cofibrant replacement can also be described in terms of cellular complexes. Although we will not prove these results in generality here, we will exploit techniques from the proof of these facts in later sections. We now state a result useful for identifying cofibrantly generated model categories.

Theorem II.24. *[12] Let $\mathcal{M}$ be a category closed under small limits and colimits and let W be a class of maps closed under retracts which satisfies the "2 of 3" axiom. If I and J are sets of maps in $\mathcal{M}$ such that*

1. *Both I and J permit the small object argument,*

2. *Every J-cofibration is an I-cofibration and an element of W,*

3. *Every I-injective is both a J-injective and an element of W, and*

4. *One of the following holds:*

   (a) *A map that is both an I-cofibration and an element of W is a J-cofibration, or*

   (b) *A map that is both a J-injective and an element of W is an I-injective,*

then there is a cofibrantly generated model structure on $\mathcal{M}$ with W the weak equivalences F the J-injectives and C the I-cofibrations. Here I is the set of generating cofibrations and J the set of generating trivial cofibrations.

Localizing model categories to enlarge the class of weak equivalences is central to the development of important model structures such as $\mathbb{A}^1$-homotopy theory[19]. Two approaches to localization include left or right Bousfield localization. To describe these we introduce two more definitions.

Definition II.25. Let $\mathcal{M}$ be a model category with objects X and Y.

1. A *left homotopy function complex* from X to Y is a triple $(\tilde{X}, \check{Y}, M(\tilde{X}, \check{Y})$ where

   - $\tilde{X}$ is a comsimplicial resolution of X (a cosimplicial resolution is a cofibrant approximation in the Reedy model category structure on $\mathcal{M}^\Delta$; for details see [12]).

   - $\check{Y}$ is fibrant approximation to Y.

   - $M(\tilde{X}, \check{Y})$ is the canonical simplicial set induced by the cosimplicial structure on $\tilde{X}$.

2. A *right homotopy function complex* is dual to a left homotopy function complex.

3. A *homotopy function complex* is either a right or left homotopy function complex.

Definition II.26. Let $\mathcal{M}$ be a model category and let C_w be a class of maps in $\mathcal{M}$.

1. (a) An object W of $\mathcal{M}$ is C_w-local if W is fibrant and for every element f : $A \rightarrow B$ of C_w the induced map of homotopy function complexes f^* : $map(B, W) \rightarrow map(A, W)$ is a weak equivalence of homotopy function complexes. If C_w consists of the single map $f : A \rightarrow B$ then a C_w-local object will be called f-local. If C_w consists of the single map from the initial object of $\mathcal{M}$ to an object A then a C_w-local object will also be called A-local or A-null.

   (b) A map $g : X \rightarrow Y$ in $\mathcal{M}$ is a C_w-local equivalence if for every C_w-local object W, the induced map of homotopy function complexes g^* : $map(Y, W) \rightarrow map(X, W)$ is a weak equivalence. f-local equivalences and A-local equivalences are defined analogously to the above definitions.

2. (a) An object W is C_w-colocal if W is cofibrant and for every element f : $A \rightarrow B$ of C_w the induced map of homotopy function complexes f_* : $map(W, A) \rightarrow map(W, B)$ is a weak equivalence.

   (b) A map is a $g : X \rightarrow Y$ in $\mathcal{M}$ is a C_w-colocal equivalence if for every C_w-colocal object W the induced map of homotopy function complexes $g_* : map(W, X) \rightarrow map(W, Y)$ is a weak equivalence.

We may now define *Bousfield Localizations*.

Definition II.27. [11] Let $\mathcal{M}$ be a model category C_w a class of maps in $\mathcal{M}$.

1. The *left Bousfield localization* of $\mathcal{M}$ with respect to C_w, if it exists, is a model category structure $L_{C_w}\mathcal{M}$ such that

   (a) The class of weak equivalences of $L_{C_w}\mathcal{M}$ equals the class of C_w-local equivalences in $\mathcal{M}$.

   (b) The class of cofibrations of $L_{C_w}\mathcal{M}$ is the class of cofibrations of $\mathcal{M}$, and

(c) The class of fibrations of $L_{C_w}\mathcal{M}$ is the class of maps with the right lifting property with respect to the class of cofibrations and C_w-local equivalences.

2. The *right Bousfield localization* of $\mathcal{M}$ with respect to C_w, if it exists, is a model category structure $R_{C_w}\mathcal{M}$ such that

   (a) The class of weak equivalences of $R_{C_w}\mathcal{M}$ equals the class of C_w-colocal weak equivalences of $\mathcal{M}$,

   (b) The class of fibrations of $R_{C_w}\mathcal{M}$ equals the class of fibrations of $\mathcal{M}$, and

   (c) The class of cofibrations for $R_{C_w}\mathcal{M}$ are determined by those with the left lifting property.

The existence of the localizations depends on two more definitions, which we now state.
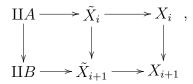
Definition II.28. We shall say that a a model category $\mathcal{M}$ is *left proper* if the class of weak equivalences are closed under pushouts along cofibrations. We shall say that it is *right proper* if the class of weak equivalences is closed under pullbacks along fibrations.

Definition II.29. A *cellular model category* is a cofibrantly generated model category for which there are sets I,generating cofibrations, and J, generating acyclic cofibrations, such that:

1. If W is either the domain or codomain of an element of I, whenever one has a diagram

$$
\begin{array}{ccc}
& & X \\
& & \downarrow f \\
W & \longrightarrow & Y
\end{array}
$$

   where f is a relative I-cell complex, the map $W \to Y$ factors through one of the form $\tilde{f} : X \to \tilde{Y}$, which is a I-cell complex such that for each term of the

transfinite composition of $\tilde{f}$ one has commutative diagrams:

$$
\begin{array}{ccccc}
\amalg A & \longrightarrow & \tilde{X}_i & \longrightarrow & X_i \\
\downarrow & & \downarrow & & \downarrow \\
\amalg B & \longrightarrow & \tilde{X}_{i+1} & \longrightarrow & X_{i+1}
\end{array} \quad ,
$$

where the left most vertical arrows are coproducts of maps in I,(In this case we say that W is *compact* relative to I)

2. The domains of elements of of J are small relative to I, and

3. The cofibrations are effective monomorphisms, that is if $f : A \to B$ is a cofibration then f is the equalizer to the diagram

$$
B \rightrightarrows B \amalg_A B.
$$

Since cellular model categories will be important to our discussion, a recognition principle is stated here.

Theorem II.30. *[11] If $\mathcal{M}$ is a model category, then $\mathcal{M}$ is a cellular model category if there are sets of maps I and J such that*

1. *A map is a trivial fibration if and only if it has the right lifting property with respect to all maps in I,*

2. *A map is a fibration if and only if it has the right lifting property with respect to all maps in J,*

3. *The domains and codomains of I are compact relative to I,*

4. *The domains of the elements of J are small relative to I, and*

5. *Relative I-cell complexes are effective monomorphisms.*

We may finally state our existence result for Bousfield localization.

Theorem II.31. *[11] Suppose $\mathcal{M}$ is a proper (right and left proper) cellular model category with C_w a set of maps in $\mathcal{M}$. Then*

- *The left Bousfield localization of $\mathcal{M}$ with respect to C_w exists and,*

  1. *The fibrant objects of $L_{C_w}\mathcal{M}$ are the C_w-local objects,*

  2. *$L_{C_w}\mathcal{M}$ is left proper and cellular,*

  3. *$L_{C_W}\mathcal{M}$ is simplicial whenever $\mathcal{M}$ is.*

- *The right Bousfield localization of $\mathcal{M}$ with respect to C_w exists and,*

  1. *The cofibrant objects of $R_{C_w}\mathcal{M}$ are the C_w-local objects,*

  2. *$R_{C_w}\mathcal{M}$ is right proper, and*

  3. *$R_{C_w}\mathcal{M}$ is a simplicial model category whenever $\mathcal{M}$ is.*

Remark II.32. The existence of left Bousfield localizations only requires that cellular model structure be left proper. An analagous statement does not hold for right Bousfield localization.

D. Simplicial Sets and Simplicial Model Categories

With the basic tools of model categories at our disposal we look to the two standard examples of model categories: topological spaces and simplicial sets. In some sense the category of simplicial sets is the universal example of a model category in that homotopy structure in a general model category may be thought of simplicially via Reedy model structure, which we will infrequently refer to. Here we recall the basic ideas of the category of simplicial sets and define a model structure on topological spaces such that the homotopy categories are equivalent, i.e. there is a Quillen equivalence between the two model structures. We will conclude by defining simplicial model categories.

We begin with notation and two definitions.

Notation II.33. $\Delta[n]$ will denote the standard n-simplex with $\partial\Delta[n]$ its boundary. $1_n \in \Delta[n]_n$ will denote the unique element of $\Delta[n]_n$, ∂_k will denote the kth face map, and s_k will denote the kth degeneracy map. $\Delta[n,k]$ will denote the colimit of all simplicial subsets of $\Delta[n]$ not containing $\partial_k(1_n)$.

Definition II.34. We shall call $f : X \to Y$ a Kan fibration if f has the right lifting property with respect to all maps of the form $\Delta[n,k] \hookrightarrow \Delta[n]$ for $n \geq 0$ and $0 \leq k \leq n$.

Definition II.35. The model structure on $SSet$ will be given by the following three classes of maps.

1. $\mathcal{W}$: weak equivalences will consist of maps that induce weak equivalences of topological spaces under geometric realization.

2. $\mathcal{F}$: will denote all Kan fibrations.

3. $\mathcal{C}$: will denote all maps having the left lifting property with respect to $\mathcal{W} \cap \mathcal{F}$.

That the category of simplicial sets is a model category is non-trivial, the reader is referred to Chapter 3 of [12] for an elegant approach to proving this fact. In fact, simplicial sets form a proper cellular model category. For properness see [10] and for the cellular structure see [11]. As a model category, simplicial sets are particularly nice in that they have an "internal function" object.

Definition II.36. Let X and Y be two simplicial sets. Denote by $\underline{Hom}(X,Y)$ the simplicial set whose n^{th} simplex is given by $Hom_{SSet}(X \times \Delta_n, Y)$.

Among the convenient properties of simplicial sets is the following:

Proposition II.37. *Let X, Y, and K be simplicial sets. then the following are true*

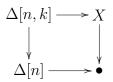1. $\underline{Hom}(X,Y)$ *is an internal function object, i.e.,*

$$\underline{Hom}(X \times K, Y) \approx \underline{Hom}(K, \underline{Hom}(X,Y))$$

2. *If* $i : A \to B$ *is a cofibration and* $f : X \to Y$ *is a fibration then*

$$\underline{Hom}(B, X) \to \underline{Hom}(A, X) \times_{\underline{Hom}(B,Y)} \underline{Hom}(B,Y)$$

is a fibration which is acyclic if either i *or* f *is.*

The category of simplicial sets also admits a nice fibrant replacement functor Ex^∞. This is the transfinite composition of the the functor Ex which is right adjoint to normal subdivision. Geometrically, one can think of Ex as completing $\Delta[n,k]$ subcomplexes of a simplicial set X so that there exists a lift to the diagram

$$
\begin{array}{ccc}
\Delta[n,k] & \longrightarrow & X \\
\downarrow & & \downarrow \\
\Delta[n] & \longrightarrow & \bullet
\end{array}
$$

Iterative composition obviously yields a fibrant object since every $\Delta[n,k]$, as a finite simplicial set must factor through some iteration $Ex^m(X)$ and so is completed in $Ex^{m+1}(X)$. One can prove:

Proposition II.38. *[10] The functor* Ex^∞ *preserves*

1. *The simplicial set* $\Delta[0]$

2. *Simplicial homotopies*

3. *Simplicial homotopy equivalences*

4. *Kan fibrations*

Model Categories which share some properties with the category of simplicial sets are abundant, we recall the definition of a simplicial model category.

Definition II.39. $\mathcal{M}$ is a simplicial category if $\mathcal{M}$ is a category together with:

1. For every two objects X and Y of $\mathcal{M}$ a simplicial set $Map(X, Y)$ (function complex).

2. For every three objects X, Y, and Z a composition map:

$$c_{X,Y,Z} : Map(Y, Z) \times Map(X, Y) \to Map(X, Z)$$

3. For every object X of $\mathcal{M}$ a map of simplicial sets $I_X : * \to Map(X, X)$.

4. For every two objects X and Y of $\mathcal{M}$ an isomorphism $Map_0(X, Y) \simeq \mathcal{M}(X, Y)$ which commutes with composition.

such that $*$ acts as a left and right unit under composition, and composition is associative.

Definition II.40. We shall call a simplicial category $\mathcal{M}$ a simplicial model catgegory if $\mathcal{M}$ is a model category such that the following two axioms hold.

1. For every two objects X and Y of $\mathcal{M}$, and every simplicial set K, there are objects $X \otimes K$ and Y^K such that the are isomorphisms of simplicial sets

$$Map(X \otimes K, Y) \simeq Map(K, Map(X, Y)) \simeq Map(X, Y^K).$$

2. If $i : A \to B$ is a cofibration in $\mathcal{M}$ and $p : X \to Y$ is a fibration $\mathcal{M}$, then the induced map of simplicial sets

$$Map(B, X) \to Map(A, X) \times_{Map(A,Y)} Map(B, Y)$$

is a fibration which is a weak equivalence whenever i or p is.

E.   Model Structures for Presheaves

One problem involved in working with the category of schemes is that it fails the axiom of cocompleteness, for example one cannot readily take quotients. This categorical deficiency makes it impossible to give Sch/k the structure of a model category and so prevents direct application of the techniques of homotopy theory. Clearly, if one has any hope of developing a homotopy theory on Sch/k one must find a way to naturally cocomplete the category and then develop a homotopy theory on the new category. Ideally, this cocomplete model of Sch/k must preserve the essential geometry of Sch/k in order to be useful. To understand how one might obtain such a co-completion, we turn to a more general categorical discussion.

Let $\mathcal{C}$ be an arbitrary small category, closed under finite limits. It is well known that one universal way to formally add colimits to the $\mathcal{C}$ is to work with the category $PreShv(\mathcal{C})$ or $Set^{\mathcal{C}^{op}}$, the category of contravariant functors from $\mathcal{C}$ into the category of sets. By defining colimits fiberwise, one sees that $PreShv(\mathcal{C})$ is indeed cocomplete, and for the same reason complete. Moreover, there exists a natural embedding $y :$ $\mathcal{C} \to PreShv(\mathcal{C})$, called the Yoneda embedding such that one has the following Yoneda Lemma.

Lemma II.41. *Let $\mathcal{C}$ be a small category and y be the Yoneda embedding defined by*

$$y(C) = Hom(-, C)$$

where $C \in \mathcal{C}$ and for $B \in \mathcal{C}$ one has $Hom(-, C)(B) = Hom_{\mathcal{C}}(B, C)$. Then one has a bijection

$$Hom_{PreShv(\mathcal{C})}(y(C), X) \simeq X(C)$$

$\forall X \in PreSh(\mathcal{C})$.

The proof of the Yoneda Lemma is a standard argument in category theory and

a proof can be found in most references for category theory, see for example [16]. An immediate corollary justifies the usefulness of the category of presheaves over $\mathcal{C}$.

Corollary II.42. *The Yoneda embedding is a full and faithful functor.*
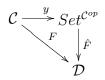
Proof. The proof is tautological. One has bijections

$$Hom_{Set^{\mathcal{C}^{op}}}(y(C), y(D)) \simeq y(D)(C)$$

$$\simeq Hom_{\mathcal{C}}(C, D)$$

for $C, D \in \mathcal{C}$. $\qquad\qquad\square$

By Corollary II.42 we may regard the category $Set^{\mathcal{C}^{op}}$ as an enlargement of the original category $\mathcal{C}$ to include colimits and all limits. In fact, it is the universal example of such a category in the following sense:

Proposition II.43. *Let $\mathcal{D}$ be a cocomplete category with a functor $F : \mathcal{C} \to \mathcal{D}$, then there exists a factorization*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\ y\ } & Set^{\mathcal{C}^{op}} \\ & {}_{F}\searrow & \downarrow{}^{\hat{F}} \\ & & \mathcal{D} \end{array}$$

The proof of this makes use of the following lemma.

Lemma II.44. *Every presheaf is the colimit of sheaves of the form $y(U)$ for $U \in \mathcal{C}$.*

Proof. To see this one first defines the comma category of a presheaf P, denoted $\int P$ with objects all pairs (C, p), with $C \in \mathcal{C}$ and $p \in P(C)$, and morphisms $u : C \to C'$ with $p = p'u$. There is an obvious projection map $\pi_P : \int P \to \mathcal{C}$ given by $(C, p) \to p$. One checks that P is the colimit of the composition $y\pi_P$ over $\int P$. The map from the colimit into P is self-evident since one maps an element $f \in y\pi_P(C, p)(D)$ to $P(D)$ according to the composition of the map $p \in P(C) \simeq Hom(y(C), P)$ with f. One

checks easily this is natural and surjective. To see this is injective, write $y\pi_P(H, q)$ as H_q then $f : D \to H_q$ being sent to $p \in \mathcal{D}$ shows that $id : D \to D_p$ are identified in the colimit and so the map is also injective as a map of presheaves. $\square$

The proposition now follows easily.

Proof. Given $F : \mathcal{C} \to \mathcal{D}$ one constructs $\tilde{F}$ according to $\tilde{F}(P) = colim_{\int P} F(\pi_P((C, p)))$. $\square$

We now consider a model structure on presheaves. The seminal reference for simplicial model structures for presheaves is Jardine's *Simplicial Presheaves* [14]. Several model structures are considered but the basic idea is to first embed $Set^{\mathcal{C}^{op}}$ into $SSet^{\mathcal{C}^{op}}$ according to the constant simplicial functor and then construct the model structure on $SSet^{\mathcal{C}^{op}}$. Jardine considered injective and projective model structures, including some intermediate between these. The projective structure has the advantage of several universal properties with easily recognizable fibrant maps.

Definition II.45. The *Bousfield-Kan* or *projective* model structure on $SSet^{\mathcal{C}^{op}}$ will consist of the following data:

- Weak equivalences will be all fiberwise weak equivalences of simplicial sets.

- Fibrations will be fiberwise Kan fibrations

- Cofibration will be all maps with the left lifting property with respect to acyclic fibrations.

Remark II.46. The *injective* model structure on presheaves is defined similarly with cofibrations being all objectwise injections.

It is known that the Bousfield-Kan and injective model structures have many nice properties, expressed in the following theorem.

Theorem II.47. *[5] The Bousfield-Kan and injective structures are simplicial, cellular, proper model structures.*

Remark II.48. The class of generating cofibrations for the Bousfield-Kan model structure can easily be seen to consist of maps of the form $\partial\Delta[n] \times U \to \Delta[n] \times U$ where $U \in \mathcal{C}$ and $n \geq 0$. $\Delta[n]$ is a simplicial object in the category of $SSet^{\mathcal{C}^{op}}$ given by the constant simplicial presheaf $\Delta[n](C) := \Delta[n]$. Generating trivial cofibrations can be defined $\Delta[n, k] \times U \to \Delta[n] \times U$ for all $U \in \mathcal{C}$.

In particular, we note that the Bousfield-Kan model category admits both left and right Bousfield localizations. The Bousfield-Kan structure also guarantees that every representable presheaf is cofibrant trivially (this follows from the fact that the fiberwise fibrations are surjective and the Yoneda lemma). Unfortunately, cofibrant in this sense refers to being cofibrant over the initial object in $SSet^{\mathcal{C}^{op}}$ which is the empty presheaf. If one considers the category of pointed presheaves, $SSet^{\mathcal{C}^{op}}_{\bullet}$; with representable objects given by taking the disjoint union of $y(U)$ with a basepoint, denoted $\bullet$, one still has that $U \amalg \bullet$ is cofibrant. If, however, $\bullet$ is itself a representable presheaf and one has a distinguished map $\bullet \to U$, there is no guarantee that U with this map as its basepoint is cofibrant. An example of such a situation is in the category of schemes of finite type over a field k, and the presheaf represented by the projective line pointed at ∞.

In order to circumvent this difficulty, one may appeal to Isaksen's flasque model structure [13]. One starts by defining a collection of generating acyclic cofibrations. Let $\cup U_i$ denote the categorical coproduct of representable presheaves U_i. Given a collection of monics in $\mathcal{C}$, say $\{U_i \hookrightarrow X\}$, one has an induced map $f : \cup U_i \to X$ in

$SPreSh(\mathcal{C})$. Given any $g : \Delta[n, k] \hookrightarrow \Delta[n]$, we construct the pushout of

$$\cup_i U_i \times \Delta[n, k] \longrightarrow \cup_i U_i \times \Delta[n] \ ,$$
$$\downarrow$$
$$X \times \Delta[n, k]$$

where the maps are the obvious inclusions, and take as a generating acyclic cofibration the induced map from the pushout to $X \times \Delta[n]$. The class of all trivial generating trivial cofibrations consists of all such maps and generating cofibrations are taken as the analogous constructions over the maps $\partial \Delta[n] \to \Delta[n]$.

Definition II.49. Define a *flasque* fibration to be a map which has the right lifting property with respect to all generating trivial cofibrations defined above.

Definition II.50. The *flasque model structure* on $SSet^{\mathcal{C}^{op}}$ is given by
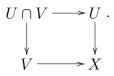
- Weak equivalences are fiberwise weak equivalences of simplicial sets.

- Fibrations are *flasque* fibrations.

- Cofibrations are all maps with right lifting property with respect to acyclic fibrations.

Theorem II.51. *[13]*

1. *The* flasque Model Structure *on $SSet^{\mathcal{C}^{op}}$ is a proper cellular model structure.*

2. *The identity functor is a left Quillen equivalence from the Bousfield-Kan structure to the Flasque structure.*

3. *If $\mathcal{C}$ contains finite products, then the model structure of (1) is simplicial.*

One drawback of working in the category of simplicial presheaves stems from the following remark.

Remark II.52. Suppose U and V are in $\mathcal{C}$ such that the following pushout diagram exists in $\mathcal{C}$

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array} \ .$$

Then this is not a pushout diagram in $Set^{\mathcal{C}op}$ (or $SSet^{\mathcal{C}op}$ for that matter) unless the following diagram is a pullback diagram for every presheaf P:

$$\begin{array}{ccc} P(X) & \longrightarrow & P(U) \\ \downarrow & & \downarrow \\ P(V) & \longrightarrow & P(U \cup V) \end{array} \ .$$

Since there is no reason to believe that the presheaf diagram in the remark would be a pullback, one observes that the colimits that naturally exist in $\mathcal{C}$ are unlikely to coincide with pushouts under the Yoneda embedding in $SSet^{mathcalC op}$, and so one loses information about the underlying category in the passage to presheaves. One way to address this problem is to define some collection of colimit diagrams in $\mathcal{C}$ satisfying the properties of a Grothendieck basis [17], and passing to the category of sheaves. This is Morel and Voevodsky's approach in [19]. An alternative is to localize the homotopy category on presheaves so that the sheaves represented by pushouts of objects in $\mathcal{C}$ are weakly equivalent to the pushouts of the the representable presheaves determined by the objects in $\mathcal{C}$. Either approach relies first on specifying a set collection of colimit diagrams to preserve. We focus on the localization of presheaves, which although less geometrically intuitive, is more natural in the context of the general categorical constructions relevant to this work.

A particular example of this type of construction can be found in the context of presheaves of Sm_k. There is a technical difficulty in that Sm_k is not a small category, however, for now we will overlook this detail commenting only that it is equivalent

to a small category to be described in subsequent sections.

Definition II.53. [19] Let U, V, and X be smooth schemes of finite type over k. We shall say the square

$$
\begin{array}{ccc}
U \times_X V & \longrightarrow & V \\
\downarrow & & \downarrow{\scriptstyle p} \\
U & \xrightarrow{\ j\ } & X
\end{array}
$$

is a Nisnevich square if it is cartesian, p is etale, and j is an open embedding with $j : p^{-1}(X - U) \to X - U$ an isomorphism.

Remark II.54. A presheaf is a sheaf over the completely decomposed topology of [20] if and only if every Nisnevich square is cartesian under the image of the presheaf[19].

Definition II.55. Given a proper cellular model category of presheaves over Sch/k, we define the local Nisnevich structure to be the left Bousfield localization with respect to all Nisnevich squares. By this we mean to include in the class C_w, for every distinguished square, the evident map from the pushout over U and V into X.

Remark II.56. We shall call the localizations of the Bousfield-Kan and Flasque structures the local Bousfield-Kan and local Flasque structures respectively. Both exist by the results in section C. Both are Quillen equivalent and are Quillen equivalent to Voevodsky and Morel's $(Sch/k)_{Nis}$, homotopy category of simplicial sheaves with the Nisnevich topology by [7].

Among the more powerful innovations of [19] was the idea of localizing $(Sch/k)_{Nis}$ with respect to maps of the form $X \to X \times \mathbb{A}^1$ induced by the inclusion $Spec(k) \to \mathbb{A}^1$, this has the effect of making the affine line homotopically similar in algebraic properties to the unit interval in conventional topology. This similarity is evident in certain cohomology theories such as algebraic K-theory. One can perform another

such localization in the local flasque and local projective model structures, since these structures remain left proper, cellular, and in fact are also simplicial by Theorem II.31

Definition II.57. Define a motivic model structure to be the left Bousfield localization of a model structure with respect to the maps $X \to X \times \mathbb{A}^1$.

Remark II.58. Naturally this yields notions of motivic flasque and motivic Bousfield-Kan model structures on the category of simplicial presheaves. Again, one can show these are Quillen equivalent and both are Quillen equivalent to the $\mathbb{A}^1$-homotopy theory of Voevodsky and Morel ([13], [7]).

Theorem II.59. *The motivic flasque and motivic Bousfield-Kan structure are proper, cellular, and simplicial.*

Theorem II.31 shows that these are left proper, cellular, and simplicial whereas [19] Theorem 2.2.7 yields right properness (the class of fibrations in [19] is included in the flasque and Bousfield-Kan structures).

F.   Conclusion

Thus far we have defined the essential elements of homotopy theory necessary to work in the context simplicial presheaves over a small site. Although we are inevitably interested in applications to smooth schemes of finite type over a field k, many of the arguments proved in subsequent chapters work in more general contexts. The author has made no attempt to clarify or explain the categorical background for sheaves in the introduction, as the details of this discussion become relevant various definitions and concepts will be introduced.

Of the more specific model structures listed in the last section, we shall frequently pass between these to simplify subsequent arguments. We are at no loss in doing so

since each of the structures listed yield the same class of weak equivalences. In the case of the Bousfield-Kan model structure, one has a notion of fibration and weak equivalence in which the results of simplicial homotopy theory are immediately relevant. Because the fibrations are easy to understand, it is often simple to verify that a map is a weak equivalence. The Flasque structure is slightly less intuitive, but offers many technical advantages in applying general results about model categories to our specific context. Because the cofibrant objects are more abundant, results about conventional sphere objects in the context of schemes become easier to prove. In this sense presheaves with the flasque model structure are similar in flavor to spaces with non-degenerate basepoint. This will allow us to generalize some topological results beyond the simplicial sphere objects, which are inadequate, motivicly, to capture homotopy relevant to cohomology theories.

A topic that has not been discussed in this chapter in much detail is that of pointed model categories. Since most of our results will be concerned in the pointed context, it bears pointing out that homotopy theory in this context is understood by forgetting the basepoint. A pointed morphism of presheaves, therefore will be called a weak equivalence (cofibration, fibration) whenever the map is a weak equivalence (cofibration, fibration) in the unpointed homotopy theory. One thing that makes the pointed context less convenient to work in is the restriction on mapping spaces. For example, in the pointed context, basepoint preserving morphisms will be denoted $Hom_\bullet(A, B)$ and although we can give this the structure of a simplicial category, for example $Hom_\bullet(A \wedge \Delta[n, k]_+, B)$, the pleasant properties of Definition II.40 may not be guaranteed. It is true, however, that the pointed homotopy theory is cofibrantly generated with generating (acyclic) cofibrations given by attaching disjoint basepoints to the generating (acyclic) cofibrations in the unpointed context.

CHAPTER III

T-MODELS IN THE CONTEXT OF $SSet_{\bullet}^{\mathcal{C}^{op}}$

This chapter contains the basic ideas generalizing the topological notion of **T**-algebras to the context of $SSet_{\bullet}^{\mathcal{C}^{op}}$ where $\mathcal{C}$ is a small category closed under finite coproducts. As in the topological case, one can give the category of **T**-algebras the structure of a model category, in fact a proper cellular cofibrantly generated model category. Unfortunately, in practice the restrictions placed on $\mathcal{C}$ in this construction are superficially prohibitive and we also explore ways, particularly in the context of smooth schemes of finite type over a field k, to work around this restriction.

A. Introduction

In the topological context, a diverse collection of results exist toward identifying whether or not a space is a loop space. Recognition principles exhibit loop space classification based on a variety of different criteria. Classical examples typically frame the recognition principle in terms of the existence of an action by some nice collection of spaces. Generally, these theories require stringent topological considerations both about the space acting and on the underlying topological space being classified. These requirements make such models difficult to generalize to other areas of mathematics.

The relevance of motivic homotopy theory in cohomological considerations makes it desirable to develop a notion of loop recognition similar to the techniques available for working with topological spaces. Developing loop recognition over sheaves of a comma category of smooth schemes of finite type must, therefore, require models of loop recognition applicable to more general contexts but comparable to the classical topological machinery.

As an example of such a theory, we consider algebraic theories developed in [23],

[3], [4]. Here loop recognition is approached through comparing more topologically general loop spaces to spaces with an action of a larger collection of functions than in the classical examples, i.e. the little cubes operad. The advantage to this approach is that some of the stringent requirements of the underlying space are removed. Although [4] still makes use of many properties of the topological category, the generality of this approach suggests applications to more general contexts.

Section B defines algebraic theories and $\mathbf{T}$-algebras and gives an analogous definition for the category of presheaves over a finite complete category $\mathcal{C}$. Classically, an algebraic theory is a special type of functor from topological spaces into the category $SSet_{\bullet}$. More generally, when one considers other categories, such as $SSet_{\bullet}^{\mathcal{C}^{op}}$, these special types of functors are called $\mathbf{T}$-models. One may think of the category of $\mathbf{T}$-models as presheaves over an enlargement of the underlying category $\mathcal{C}$. Unfortunately, this classical description is in some ways too general to for general model structures when trying to mimic topology and properties of $\mathbf{T}$-algebras. We will discuss a specific class of $\mathbf{T}$-models which seem to be a more useful generalization of $\mathbf{T}$-algebras to the context of $SSet_{\bullet}^{\mathcal{C}^{op}}$. We will also develop develop a free $\mathbf{T}$-model functor, $\mathcal{F}_{\mathbf{T}}$, with adjoint $\mathcal{U}$.

Special attention will also be given to a pair of adjoint functors Ω^T and $B_{\mathbf{T}}$, Ω^T is a type of free loop space functor not unlike the classical topological loop functor. Together these functors are reminiscent of the loop and suspension functors of classical topology. They will be used extensively in understanding the homotopy theoretic properties of our $\mathbf{T}$-algebras.

Section C contains the development of the model category structure for $\mathbf{Alg_T}$. Weak equivalences and fibrations in $\mathbf{Alg_T}$ can be defined via a forgetful functor back to $SSet_{\bullet}^{\mathcal{C}^{op}}$. This is how the model structure is defined in the topological setting. We proceed by defining classes of generating cofibrations and generating fibrations and

use these to set up the structure of a cofibrantly generated model category. The main result of this section is:

Theorem III.1. *If fibrations and weak equivalences are defined via the forgetful functor $\mathcal{U}$, and cofibrations are those maps having the left lifting property with respect to acyclic fibrations, then $\mathbf{Alg_T}$ is a proper simplicial cellular model category.*

The proof of the result will utilize several convenient properties of the $\mathcal{F}_\mathbf{T}$ and $\mathcal{U}$ functors and the elements of the proof will be repeated in subsequent results. This is a somewhat different approach in the treatment of the model structure of $\mathbf{T}$-algebra in classical topology [23], [22], [3].

Section D contains ideas for applying this new $\mathbf{Alg_T}$ machinery in contexts where the underlying category may not be small or may not be closed under finite coproducts. Particular interest is given to the category of smooth schemes of finite type over a field k, Sm_k, but all of the results in this section apply to a slightly broader context. Given a Grothendieck topology (note not basis) on a category $\mathcal{C}$, the idea is to choose some small subcategory which contains a sufficient number of objects so that the isomorphism classes of these objects contain Grothendieck covers for the objects in the larger category. This small subcategory can then be co-completed so as to contain finite coproducts and endowed with a restriction of the topology from the larger category. Sheaves over this smaller category are in one-to-one correspondence with sheaves over the original.

This is done carefully in the case of Sm_k. In this approach, one can apply the ideas of $\mathcal{F}_\mathbf{T}$ and $\mathbf{T}$-models to the category of sheaves over $\mathcal{C}$ instead of $SSet_\bullet^{\mathcal{C}^{op}}$ and by working over this smaller category can escape the restrictions of smallness and closure under finite coproducts. Since the smaller category is likely not closed under pullbacks, the standard approach of Grothendieck bases must give way to the more

general strategy of sieves.

B.   Algebraic Theories and **T**-Models

Lawvere introduced the notion of algebraic theory, essentially a formalization of an algebraic object with n-ary operations and equational relations. Algebraic theories have proven useful in understanding loop spaces [4]. The purpose of this section is to adapt the idea of algebraic theories to the context of $SSet_\bullet^{\mathcal{C}^{op}}$ in such a way that they may be used to understand loop spaces in $SSet_\bullet^{\mathcal{C}^{op}}$ combinatorially. We start by introducing the definition of of a simplicial algebraic theory as given in [23]. Following this approach we first introduce Γ^{op}.

Definition III.2. Let Γ^{op} denote a skeletal category of finite sets with objects $n^+ = \{0, 1, \ldots, n\}$ for all non-negative integers n and morphisms maps of sets such that 0 is sent to 0. Γ^{op} has a symmetric monoidal smash product given by lexicographically ordering the elements of $n^+ \wedge m^+$.

Remark III.3. Segal's Γ is the opposite to our category Γ^{op}.

Definition III.4. [23] 2.1 A *simplicial theory* is a pointed simplicial category **T** together with a product preserving functor $\Gamma \to \mathbf{T}$. It is required that **T** have the same objects as Γ and that $\Gamma \to \mathbf{T}$ is the identity on objects. A morphism of simplicial theories is a product preserving simplicial functor commuting with the functor from Γ. A category with no simplicial structure yet having the same properties as above will simply be called an *algebraic theory*.

Remark III.5. A product in Γ is given by the coproduct in Γ^{op}. Hence $n^+ \times m^+ := n^+ \bigvee m^+$, the one point union.

Example III.6. Let S^1 denote the realization of the simplicial circle in $Top_\bullet$, the category of pointed topological spaces, pointed at some convenient point depending on

the model used. Let $S^1_\bullet$ denote the opposite of the full subcategory in $Top_\bullet$ generated by wedge products of the circle with itself, including of course the set consisting of single point for the empty wedge. The obvious functor given by $n^+ \to \bigvee_n S^1$ with the canonical identification of set theoretic maps defines an algebraic theory.

Example III.7. Let $\mathcal{C}$ be an arbitrary cocomplete pointed simplicial category with object A. Then denote by $A_\bullet$ the full subcategory consisting of pointed coproducts of A with the zero object for the empty coproduct. $A^{op}_\bullet$ is an algebraic theory. Since $\mathcal{C}$ is a simplicial category, we may let $A_\bullet$ inherit the simplicial structure of $\mathcal{C}$. In doing so $A^{op}_\bullet$ becomes a simplicial theory. Here, if we let $\mathbf{T} = A^{op}_\bullet$, then $Hom_{\mathbf{T}}(n^+, m^+)_i :=$ $Hom_{\mathcal{C}}(\vee_m A, \vee_n A)_i$.

Example III.8. In the example above let S^0 denote the disjoint union of the zero object with itself pointed by one copy of the zero object. Denote this algebraic theory by $\mathbf{T}^{S^0}$, this yields a full, faithful inclusion of Γ into $\mathcal{C}$.

Much attention shall be focused on the category of simplicial sheaves over a site $\mathcal{C}$. For the purposes of this section, we shall make the further assumption that $\mathcal{C}$ is a small site closed under finite coproducts; although, as mentioned in the introduction, this condition is somewhat artificial (in fact, one may get around it entirely). A full discussion in the context of smooth schemes of finite type over a field k is contained in section D.

Definition III.9. Let $\mathcal{M}$ be a pointed simplicial category. By a *Model of* $\mathbf{T}$ *in* $\mathcal{M}$ we shall mean a product preserving simplicial functor $\mathbf{T} \to \mathcal{M}$. A morphism of $\mathbf{T}$-models is a natural transformation of functors. If X is a $\mathbf{T}$-model we call $X(1^+)$ the underlying object of X. It is standard to denote the category of $\mathbf{T}$-models by $\mathbf{Alg_T}/\mathcal{M}$ or if $\mathcal{M}$ is understood, to simply use $\mathbf{Alg_T}$. If $\mathcal{M} = SSet_\bullet$, a $\mathbf{T} \to \mathcal{M}$ is typically called a $\mathbf{T}$-*algebra*.

Suppose one has a morphism of algebraic theories, for example there is a canonical morphism of $\phi : \mathbf{T}^{S^0} \to \mathbf{T}^A$ in any cocomplete pointed simplicial category $\mathcal{C}$, such a morphism induces, via composition, a canonical product preserving functor $\phi^* : \mathbf{Alg}_{\mathbf{T}^A} \to \mathbf{Alg}_{\mathbf{T}^{S^0}}$.

For simplicial presheaves over a small site $\mathcal{C}$, for example $\mathcal{M} = SSet_\bullet^{\mathcal{C}^{op}}$, we observe that a model for $\mathbf{T}$ is a covariant functor $\mathbf{T} \to SSet_\bullet^{\mathcal{C}^{op}}$ which is naturally equivalent to a functor $\mathbf{T} \times \mathcal{C}^{op} \to SSet_\bullet$. Hence, we may regard $\mathbf{T}$-models in the category of pointed simplicial presheaves over $\mathcal{C}$ as pointed simplicial presheaves over the product category $\mathbf{T}^{op} \times \mathcal{C}$. We consider an instructive definition.

Definition III.10. For $A \in SSet_\bullet^{\mathcal{C}^{op}}$ let $\mathbf{T} = \mathbf{T}^A$ with $\mathcal{M} = SSet_\bullet^{\mathcal{C}^{op}}$, define a functor $\Omega^{\mathbf{T}} : SSet_\bullet^{\mathcal{C}^{op}} \to \mathbf{Alg}_{\mathbf{T}}$ by

$$\Omega^{\mathbf{T}}(C)(n^+) := \underline{Hom}_\bullet(\vee_n A, C)$$

where $\underline{Hom}_\bullet$ denotes the internal hom functor of pointed simplicial presheaves.

The functor $\Omega^{\mathbf{T}}$ provides a rich source of $\mathbf{T}$-models in the category of pointed simplicial presheaves, but in fact $\mathbf{T}$-models of this sort enjoy additional properties; namely, given any natural transformation of pointed presheaves $\vee_n A \wedge U_+ \to \vee_m A \wedge V_+$, where U and V are representable presheaves in $SSet_\bullet^{\mathcal{C}^{op}}$, one has a natural map

$$\underline{Hom}_\bullet(\vee_m A \wedge V_+, C) \to \underline{Hom}_\bullet(\vee_n A \wedge U_+, C).$$

Hence, $\Omega^{\mathbf{T}}(C)$ is a presheaf over the full subcategory of pointed simplicial presheaves with objects of the form $\vee_n A \wedge U_+$.

Whereas loop spaces in $SSet_\bullet^{\mathcal{C}^{op}}$ are defined in terms of internal hom, it seems clear that if we wish to capture the homotopy theory of loop spaces with a notion of a $\mathbf{T}$-model, we should specialize to a definition that captures all the morphisms of

internal hom.

We now present our own definition of a simplicial algebraic theory for use with $SSet_\bullet^{\mathcal{C}^{op}}$.

Definition III.11. Let $\mathcal{C}$ be a small category closed under finite products and co-products. A simplicial algebraic theory of $SSet_\bullet^{\mathcal{C}^{op}}$ generated by a cofibrant object $A \in SSet_\bullet^{\mathcal{C}^{op}}$, denoted $\mathbf{T}^A$, will consist of a set of objects $(n+, U)$ with $U \in \mathcal{C}$ and morphisms $Hom_{\mathbf{T}^A}((n^+, U), (m^+, V)) := Hom_{SSet_\bullet^{\mathcal{C}^{op}}}(\vee_m A \wedge V_+, \vee_n A \wedge U_+)$.

Notation III.12. Whenever the generating object A is either understood or unimportant, the theory shall be denoted simply by $\mathbf{T}$.

We may now define $\mathbf{T}$-models in this context.

Definition III.13. We shall call a functor $F : \mathbf{T}^A \to SSet_\bullet$ a $\mathbf{T}^A$-model if

$$F(n^+ \vee m^+, U) \simeq F(n^+, U) \times F(m^+, U)$$

for any m and n in $\mathbb{Z}$, $U \in \mathcal{C}$, and where $\simeq$ denotes isomorphism. We shall shall also borrow from classical notation and denote the category of all such functors $\mathbf{Alg_T}$ with morphisms natural transformations preserving the products above.

Definition III.14. Define
$$\mathcal{U} : \mathbf{Alg_T} \to SSet_\bullet^{\mathcal{C}^{op}}$$

by

$$\mathcal{U}(F)(U) := F(1^+, U),$$

for $U \in \mathcal{C}$.

We now define free objects in $\mathbf{Alg_T}$ by constructing a left adjoint to $\mathcal{U}$. We start by noting that given a morphism $\vee_n S^0 \wedge U_+ \to \vee_m S^0 \wedge V_+$ and smashing

with the identity on A yields a morphism $\vee_n A \wedge U_+ \to \vee_m A \wedge V_+$ and so we have a natural inclusion $\mathbf{T}^{S^0} \hookrightarrow \mathbf{T}^A$. Using this inclusion we define a functor Ω_{T^A} by $\Omega_{T^A}(o)(f) := Hom_{SSet_\bullet^{\mathcal{C}^{op}}}(o, f)$ for $o \in \mathbf{T}^{S^0}$ and $f \in \mathbf{T}^A$.

Hence, given a choice of $o \in \mathbf{T}^{S^0}$ and $F \in SSet_\bullet^{\mathcal{C}^{op}}$ we create a dinatural transformation:

$$S^{(O,F)} : \left(\mathbf{T}^{S^0}\right)^{op} \times \mathbf{T}^{S^0} \to SSet_\bullet$$

given by $S^{(O,F)}(a,b) := \Omega_{T^A}(O)(a) \wedge \Omega^{T^{S^0}}(F)(b)$.

Definition III.15. [16] The coend of a functor $S : C^{op} \times C \to X$ is a pair, $(x, \zeta : S \dashrightarrow d)$, consisting of an object $x \in X$ and a dinatural transformation ζ, universal among dinatural transformations from S to a constant. We write:

$$d = \int^c S(c,c).$$

Definition III.16. Given $Y \in SSet_\bullet^{\mathcal{C}^{op}}$, define

$$\mathcal{F}_{\mathbf{T}}(Y)(n^+, V) := \int^{d \in \mathbf{T}^{S^0}} S^{((n^+,V),Y)}(d,d).$$

Clearly $\mathcal{F}_{\mathbf{T}}(Y) : \mathbf{T}^{S^0} \to SSet_\bullet$, and so we have a functor $\mathbf{T} \to SSet_\bullet^{\mathcal{C}^{op}}$. We will show this preserves products in the desired sense and so yields a functor $\mathcal{F}_{\mathbf{T}} : SSet_\bullet^{\mathcal{C}^{op}} \to \mathbf{Alg_T}$.

Remark III.17. The assumption that $\mathcal{C}$ is a small site becomes important here since the coend is only guaranteed to exist if it can be expressed as a colimit over a small diagram in $SSet_\bullet$. To apply this to categories that are not small, one may choose a small equivalent category and take the colimit over the smaller category since in some sense one is merely removing some isomorphisms from the colimit diagram.

Remark III.18. $\Omega^{\mathbf{T}^A}$ applied to a simplicial presheaf inherits simplicial structure ac-

cording to

$$\Omega^{\mathbf{T}}(X)(m^+)_n = \underline{Hom}(\vee_m A, X_n).$$

We observe that $\mathcal{F}_{\mathbf{T}}$ applied to a simplicial presheaf is equivalent to applying $\mathcal{F}_{\mathbf{T}}$ in each simplicial dimension and using the functorial maps induced by the simplicial maps on X to recover the simplicial structure of $\mathcal{F}_{\mathbf{T}}(X)$.

The coend guarantees the existence of the free **T**-model associated to a presheaf F, but if we wish to verify some elementary properties, it is handy to see this construction more directly. To this end we appeal to categorical tensor products. Recall that if X is a $\mathcal{C}$ diagram in $\mathcal{M}$, a simplicial model category (e.g. $SSet_\bullet^{\mathcal{C}^{op}}$) and K is a $\mathcal{C}^{op}$ diagram in $SSet_\bullet$, one defines $X \otimes_\mathcal{C} K$ as the co-equalizer to the diagram

$$\bigvee_{(\alpha,\alpha')\in\mathcal{C}\times\mathcal{C}} \vee_{\alpha\to\alpha'\in Mor_\mathcal{C}(\alpha,\alpha')} X_\alpha \wedge K_{\alpha'} \rightrightarrows \bigvee_{\alpha\in Ob(\mathcal{C})} X_\alpha \wedge K_\alpha$$

in the category $SSet_\bullet^{\mathcal{C}^{op}}$, where the two arrows are the evident evaluation maps.

Definition III.19. Let $\mathcal{C}$ in the tensor product be the category $\mathbf{T}^{S^0}$, then we have

$$\mathcal{F}_{\mathbf{T}}(Y)(n^+, U) = \Omega_{T^A}(n^+, U) \otimes_\mathcal{C} \Omega^{T^{S^0}}(Y).$$

Lemma III.20. $\forall V \in SSet_\bullet^{\mathcal{C}^{op}}, \mathcal{F}_{\mathbf{T}}(V) \in \boldsymbol{Alg_T}$.

Proof. We require that $\mathcal{F}_{\mathbf{T}}(V)$ preserve products in $\mathbf{T}^A$, this is a combinatorial fact following the definition and the requirement that $\mathcal{C}$ is closed under finite coproducts. Specifically we must show that for any $V \in \mathcal{C}$ that $\mathcal{F}_{\mathbf{T}}(V)(n^+, U)$ is isomorphic to $\prod_n \mathcal{F}_{\mathbf{T}}(V)(1^+, U)$. We construct the maps between the simplicial sets to sketch the proof; the details of which are consistent with material elsewhere in this section. It suffices to consider the case where $n = 2$.

For $\mathcal{F}_{\mathbf{T}}(V)(1^+, U) \times \mathcal{F}_{\mathbf{T}}(V)(1^+, U) \to \mathcal{F}_{\mathbf{T}}(V)(2^+, U)$ we construct the map on representatives of the form $g_i \wedge h_i$, for $i = 1, 2$. Let us assume that $g_i : A \wedge U_+ \to$

$\vee_{n_i} A \wedge (W_+)_i$, $h_i : \vee_{n_i} S^0 \wedge (W_+)_i \to V$. We note that we may construct $G_i :$
$A \wedge U_+ \to \vee_N A \wedge (W_+)_i$ and $H_i : \vee_N S^0 \wedge (W_+)_i \to V$ where $N = max\{n_1, n_2\}$ using
the maps induced by the inclusion of the smaller of the indices into the other. It is
immediate that $g_i \wedge h_i$ is equivalent to $G_i \wedge H_i$ under the identification induced from the
inclusion in the tensor product. We may then define $H : \vee_N A \wedge ((W_+)_1 \bigvee (W_+)_2) \to V$
in the obvious fashion and $\tilde{G}_i$ via composition of G_i with the evident inclusions
into the coproduct. Again, these yield equivalent pairs which now may be used
to canonically form the representative given by the maps $G_1 \amalg G_2 : \vee_2 A \wedge U_+ \to$
$\vee_N A \wedge ((W_+)_1 \bigvee (W_+)_2)$ and H.

The reverse map is trivial and that the compositions are the identity, and hence
induce isomorphisms, follows by inspection. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proposition III.21. $\mathcal{F}_{\mathbf{T}}$ *is the left adjoint to* $\mathcal{U}$.

Proof. Let $f \in Hom_{SSet_\bullet^{\mathcal{C}^{op}}}(X, \mathcal{U}(Y))$ where $X \in Spc$ and $Y \in \mathbf{Alg_T}$. We start
by constructing a $\mathbf{Alg_T}$-morphism $\tilde{f} : \mathcal{F}_{\mathbf{T}}^i(X) \to Y$. We first construct morphisms
from $E(\vee_n A \wedge U_+) := \coprod_{\mathbf{T}^{S^0}} \Omega_{\mathbf{T}^A}(n, U) \wedge \Omega^{\mathbf{T}^{S^0}}(X)$ to $Y(\vee_n A \wedge U_+)$ as follows. Given
$x \in E(\vee_j T \wedge U_+)$ we note that $x = g \wedge h$ where $g : \vee_j (A \wedge U_+) \to \vee_i A \wedge V_+$ and $h :$
$\vee_i S^0 \wedge V_+ \to X$. We observe, however that h corresponds to a natural transformation
$\vee_i V_+ \to X$, which by composition with f yields a map of presheaves $\vee_i V_+ \to U(Y)$,
which corresponds naturally to an i-tuple of elements in $Y(1^+, V)$. Mapping this
under $Y(g) : \prod_i (Y(1^+, V))) \to Y(J^+, U)$ yields the desired map. Suppose now
that $l \in Hom_{\mathbf{T}^A}((n, U'), (i, U))$. Then $\mathcal{F}_{\mathbf{T}}(l)(x)$ is represented by $lg \wedge h$ and so
$\tilde{f}(lx) = Y(lg)(f(h)) = Y(l)Y(g)(f(h))$ as required. Of course in order for this to
actually descend to a map from $\mathcal{F}_{\mathbf{T}}(X)$ to Y we must verify that the equivalence
relation induced by the co-equalizer is preserved. Suppose then that $g \wedge h \sim g' \wedge h'$,
for example there exists in $Hom_{SSet_\bullet^{\mathcal{C}^{op}}}(\vee_i S^0 \wedge U_+, \vee_j S^0 \wedge V_+)$ a map $k : \vee_i S^0 \wedge U_+ \to$

$\vee_j S^0 \wedge V_+$, which in a natural way induces a map $k : \vee_i A \wedge U_+ \to \vee_j A \wedge V_+$, such that $k \circ g = g'$ and $h' \circ k = h$. We must verify that our candidate for $\tilde{f}$ satisfies $\tilde{f}(g \wedge h) = \tilde{f}(g' \wedge h')$. Whereas we may regard Y as a presheaf over the category $\mathbf{T}^{S^0}$, we note that the Yoneda lemma guarantees that $Hom_{SSet_\bullet^{\mathcal{C}^{op}}}(\vee_i S^0 \wedge U_+, \mathcal{U}(Y)) = \prod_i Hom_{SSet_\bullet^{\mathcal{C}^{op}}}(U_+, \mathcal{U}(Y)) = \prod_i \mathcal{U}(Y)(U) = \prod_i Y(1^+, U) = Y(i^+, U)$ and so one has a commutative diagram:

$$
\begin{array}{ccc}
Hom_{SSet_\bullet^{\mathcal{C}^{op}}}(\vee_j S^0 \wedge V_+, \mathcal{U}(Y)) & \longrightarrow & Y(j^+, V) \\
\downarrow{\scriptstyle k^*} & & \downarrow{\scriptstyle Y(k)} \\
Hom_{SSet_\bullet^{\mathcal{C}^{op}}}(\vee_i S^0 \wedge U_+, \mathcal{U}(Y)) & \longrightarrow & Y(i^+, U).
\end{array}
$$

Moreover, we note that $f : X \to \mathcal{U}(Y)$ induces natural morphisms $Hom(\vee_i S^0 \wedge U_+, X) \to Hom(\vee_i S^0 \wedge U_+, \mathcal{U}(Y)) = \prod_i Hom(U_+, \mathcal{U}(Y)) = \prod_i Hom_{\mathbf{Alg_T}}(\Omega^{T^A}(A \wedge U_+), Y) = Hom_{\mathbf{Alg_T}}(\Omega^{T^A}(\vee_i A \wedge U_+), Y)$ and so we have $Y(g)\,(f(h)) = Y(g)\,(f(h' \circ k)) = Y(g)\,(k^* f(h')) = Y(g)\,(Y(k)\,(f(h'))) = Y(kg)\,(f(h')) = Y(g')\,(f(h'))$ as required.

This defines a map $\Phi : Hom(X, U(Y)) \to Hom(\mathcal{F}_\mathbf{T}(X), Y)$. To get the inverse map, we take a morphism in $\mathbf{Alg_T}$, $F : \mathcal{F}_\mathbf{T}(X) \to Y$, and define $\check{F} : X \to U(Y)$ by sending $x : U_+ \to X$ to $F(id_{A \wedge U_+} \wedge x)$. To see this is indeed a natural transformation, suppose $f : V \to U$ in $\mathcal{C}$. We must show that $\tilde{F}(f(x)) = U(Y)(f)(\tilde{F}(x))$. We already know that $F(\mathcal{F}_\mathbf{T}(f)(id \wedge x)) = Y(f)F(id \wedge x)$ and we know that $\mathcal{F}_\mathbf{T}(f)(id \wedge x) = h \wedge x$ where $h = f'(id_{A \wedge U})$ (this just involves following the definition of $\mathcal{F}_\mathbf{T}$) with f' the natural map induced by f on $A \wedge V \to A \wedge U$. h, however can also be written as $id_{A \wedge V}(f')$ and so we have that $h \wedge x \sim id_{A \wedge V} \wedge f(x)$. Combining the computations we have that $\tilde{F}(f(x)) = F(id_{A \wedge V} \wedge f(x)) = F(\mathcal{F}_\mathbf{T}(f)(id_{A \wedge U} \wedge x)) = Y(f)(F(id_{A \wedge U} \wedge x)) = U(Y)(f)\tilde{F}(x)$ as required.

So now we have a map back, call it Ψ. We see immediately that the composition $\Psi\Phi$ is the identity, and so it remains to show that $\Phi\Psi$ is also the identity.

Since maps of the form $\vee_i S^0 \wedge V_+ \to X$, where X is a presheaf over $\mathcal{C}$, are in one-to-one correspondence with products of maps $V_+ \to X$ which are in one-to-one correspondence with elements $X(V)$ by the Yoneda lemma, we shall represent an element of $X(V)$ by $x(v)$ and adopt the notation that $\prod_i x(v)$ represents a map $\vee_i S^0 \wedge V_+ \to X$. To see that $\Phi\Psi$ is the identity, we must show that a natural transformation from $\mathcal{F}_{\mathbf{T}}^i(X) \to Y$ is completely determined by where the representatives of the form $id_{A \wedge U_+} \wedge x(u)$ are sent. We note that if we have a representative of the form $(h : \vee_j A \wedge U_+ \to \vee_i A \wedge V_+) \wedge \prod_i X(v)$, that $F(h \wedge \prod_i X(v)) = F(\mathcal{F}_{\mathbf{T}}(X)(h)(id_i \wedge \prod_i X(v))) = Y(h)F(id_{\vee_i A \wedge V_+} \wedge \prod_i x(v))$. That is F on a representative $h \wedge \prod_i x(u)$ is determined by F on $id_{\vee_i A \wedge V_+} \wedge \prod_i x(u)$. It suffices then to see that F on $id_{\vee_i A \wedge V_+} \wedge \prod_i x(u)$ is determined by some collection $id_{A \wedge V_+} \wedge x(u)$. For this let $i : A \wedge U_+ \to \vee_i A \wedge U_+$ be the inclusion into one component of the coproduct. The equivalence on $\mathcal{F}_{\mathbf{T}}$ shows that $i \wedge \prod_i x(u) \sim id \wedge \rho(\prod_i x(u))$. Now since $F(id_{\vee_i A \wedge U_+} \wedge \prod_i x(u))$ is the product of $F(i \wedge \prod_i x(u)) = F(id_{A \wedge V_+} \wedge \rho \prod_i x(u))$ the proof is complete. $\qquad\square$

We already know by the universal properties of the coend that applying $\mathcal{F}_{\mathbf{T}}$ to an element $\vee_n S^0 \wedge U_+$ of $SSet_{\bullet}^{\mathcal{C}^{op}}$ yields the model represented by $\vee_n A \wedge U_+$ given by $\Omega^T(\vee_n A \wedge U_+)$ but one can see this directly from the definition of $\mathcal{F}_{\mathbf{T}}$ using tensor products.

Proposition III.22. *Let $V \in \mathcal{C}$. $\mathcal{U}\mathcal{F}_{\mathbf{T}}(V_+)$ is isomorphic, in the category of presheaves, to $\underline{Hom}_{SSet_{\bullet}^{\mathcal{C}^{op}}}(A, A \wedge V_+)$.*

Proof. This is actually a tautological statement involving the definitions. To illustrate the construction I explicitly demonstrate the isomorphism. To construct a morphism of presheaves from $\mathcal{U}(\mathcal{F}_{\mathbf{T}}(V))$, one proceeds much as above by constructing correspondence between elements of $\coprod_{\mathcal{D}_i} Hom(A \wedge U_+, \vee_i A \wedge W_+) \wedge Hom(\vee_j S^0 \wedge W_+, V)$ to $Hom(A \wedge U_+, A \wedge V_+)$. This is trivial since presheaf morphisms $Hom(A \wedge U_+, \vee_j A \wedge W_+)$

correspond under isomorphism to morphisms $Hom(A \wedge U_+, A \wedge \vee_j W_+)$, and composition with the induced map yields the desired morphism. This clearly descends to a map $\mathcal{U}(\mathcal{F}_\mathbf{T}(V))$ since the identification defining the co-equalizer is preserved under composition. This is a presheaf morphism since the restriction maps of $\mathcal{U}(\mathcal{F}_\mathbf{T}(V))$ are identical to those of $\underline{Hom}(A, A \wedge V_+)$. The map is actually a surjective map of presheaves since we note if one has a morphism $f \in Hom(A \wedge U_+, A \wedge V_+)$ then this is in the image of the pair $f \wedge id_V$. All that remains is to show injectivity. Suppose that $g \wedge h$ and $g' \wedge h'$ are both sent to the same morphism in $Hom(A \wedge U_+, A \wedge V_+)$. For convenience assume that $g : A \wedge U_+ \to \vee_i A \times W_+$, $h : \vee_i S^0 \wedge W_+ \to V_+$, $g' : A \wedge U_+ \to \vee_j A \wedge W'_+$, and $h' : \vee_j S^0 \wedge W'_+ \to V_+$. We must show that these maps are in the same equivalence class in $\mathcal{U}(\mathcal{F}_\mathbf{T}(V))$. To see this, however, it suffices to observe that both are in the equivalence class $gh \times id = g'h' \times id$. $\qquad \square$

Corollary III.23. *Let S be a simplicial set and U a representable sheaf, then*

$$\mathcal{F}_\mathbf{T}((S \times U)_+) \simeq \underline{Hom}_{SSet_\bullet^{\mathcal{C}^{op}}}(A, A \wedge (S \times U)_+),$$

if $\mathbf{A}$ is a pointed representable presheaf.

Proof. We start by assuming that S is a finite set, then $(S \times U)_+$ is just a pointed disjoint union of a finite number of copies of U. Isomorphism in this instance is trivial since $\mathcal{C}$ is assumed to be closed under finite coproducts. We generalize to the situation that S is an arbitrary set by exploiting the fact that $A \wedge U_+$ is a quotient of representables and so is compact and hence $Hom(A \wedge U_+, -)$ commutes with filtered colimits and then using the fact that S is the colimit over finite subsets. Now given a simplicial set S we note we have a levelwise isomorphism of simplicial sets and since both $\mathcal{F}_\mathbf{T}^i$ and $\underline{Hom}$ are functors the result follows. $\qquad \square$

We note that for an object in $SSet_\bullet^{\mathcal{C}^{op}}$, which is the quotient of representables,

namely $A \wedge U_+$; a map $f \in Hom(A \wedge U_+, \vee_i A \wedge X_i)$ induces a map $\tilde{f} : A \times U \to \vee_i A \wedge X_i$, which is a choice of element in $(\vee_i A \wedge X_i)(A \times U)$. This, however, corresponds by the definition of a colimit in the category of presheaves to an indexed choice of map $A \times U \to A \wedge X_i$ and so $\tilde{f}$ factors as $A \times U \to A \wedge X_i \to \vee_i A \wedge X_i$. We have then that $Hom(A \wedge U_+, \vee_i A \wedge X_i) \simeq \vee_i Hom(A \wedge C_+, A \wedge X_i)$ where X_i are representable. Using Proposition III.22 we have proved:

Corollary III.24. *Let Λ be an index set then,*

$$\mathcal{F}_{\mathbf{T}}(\vee_{\lambda \in \Lambda}(S_\lambda \times U_\lambda)_+) \simeq \underline{Hom}_{SSet_\bullet^{\mathcal{C}^{op}}}(A, A \wedge \vee_{\lambda \in \Lambda}(S_\lambda \times U_\lambda)_+),$$

where U_λ is representable and S_λ is a simplicial set, whenever A is a pointed representable.

To complete this section we introduce one more functor useful in future consideration. Recall Definition III.10. We may construct an adjoint of Ω^T by use of coends as above.

Definition III.25. Let $\mathbf{T}^A$ be as above. Define $B_T : Alg_{\mathbf{T}}^A \to SSet_\bullet^{\mathcal{C}^{op}}$ by

$$B_{\mathbf{T}}(F) := \int^{t \in \mathbf{T}^A} S_F(t, t)(U)$$

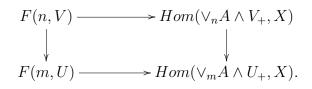where $S_F((n, U), (m, V))$ is the dinatural transformation given by $F(n^+, U) \wedge \vee_m A \wedge V_+)$.

Lemma III.26.

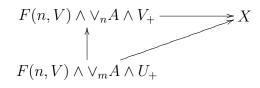$$B_{\mathbf{T}} : Alg_{\mathbf{T}} \leftrightarrow SSet_\bullet^{\mathcal{C}^{op}} : \Omega^T$$

form an adjoint pair of functors.

Proof. One may see this by observing what it means to have a morphism of $\mathbf{T}$-algebras
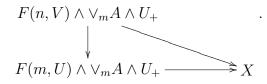
$F \to \Omega^T(X)$. In this case we have commutative diagrams:

$$
\begin{array}{ccc}
F(n,V) & \longrightarrow & Hom(\vee_n A \wedge V_+, X) \\
\downarrow & & \downarrow \\
F(m,U) & \longrightarrow & Hom(\vee_m A \wedge U_+, X).
\end{array}
$$

By adjunction, this diagram induces two commutative diagrams:

$$
\begin{array}{ccc}
F(n,V) \wedge \vee_n A \wedge V_+ & \longrightarrow & X \\
\uparrow & \nearrow & \\
F(n,V) \wedge \vee_m A \wedge U_+ & &
\end{array}
$$

and

$$
\begin{array}{ccc}
F(n,V) \wedge \vee_m A \wedge U_+ & & \\
\downarrow & \searrow & \\
F(m,U) \wedge \vee_m A \wedge U_+ & \longrightarrow & X
\end{array}
$$

The coherence relations implied by these commutative diagrams are precisely the relations in the quotient defining the tensor product. □

C.   Homotopy Theory of $\mathbf{Alg_T}$

In this section we discuss a model structure for $\mathbf{Alg_T}$ and describe some of its properties. Surprisingly, $\mathbf{Alg_T}$ has many nice properties which will allow for localization and provide a rich context for comparison with the homotopy theory on $SSet_\bullet^{\mathcal{C}^{op}}$. For this section we will assume that $SSet_\bullet^{\mathcal{C}^{op}}$ has the flasque model structure outlined in Chapter I. We will also assume that $\mathbf{T}$ is as described in example $III.7$, that is $\mathbf{T} := \mathbf{T}^A$. Since we are using the flasque model structure, any pointed representable is cofibrant.

We endow $\mathbf{Alg_T}$ with a model structure by defining fibrations and weak equivalences via the forgetful functor U. Cofibrations are defined by the left lifting property

with respect to trivial fibrations. In fact, this yields a proper simplicial cellular model structure via the following definition.

Definition III.27. Let W consist of all maps $F : X \to Y$ with $X, Y \in \mathbf{Alg_T}$ such that $U(F) : U(X) \to U(Y)$ is a weak equivalence of simplicial presheaves. Let I consist of the images of the generating cofibrations of $SSet_\bullet^{\mathcal{C}^{op}}$ under the functor $\mathcal{F}_\mathbf{T}$ and J the images of generating acyclic cofibrations under $\mathcal{F}_\mathbf{T}$. Define the fibrations, F, to be all maps which have the right lifting property with respect to J (i.e. J-inj), and let the cofibrations, C, to be I-cof.

Theorem III.28. *The classes of weak equivalences, cofibrations, and fibrations in Definition III.27 give* $\boldsymbol{Alg_T}$ *the structure of a proper, simplicial, cellular model category.*

To prove the theorem we will need two preliminary results. The first is the following lemma.

Lemma III.29. *The images of the generating acyclic cofibrations under $\mathcal{F}_\mathbf{T}$ are weak equivalences in the model structure of* $\boldsymbol{Alg_T}$*, the image of a generating acyclic flasque cofibration in $SSet_\bullet^{\mathcal{C}^{op}}$ under the functor $\mathcal{U}\mathcal{F}_\mathbf{T}$ is an acyclic injective cofibration.*

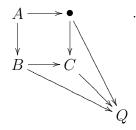The proof of this fact is made substantially simpler via the following lemma.

Lemma III.30. *If C is a quotient of representables, then the covariant functor on $SSet_\bullet^{\mathcal{C}^{op}}$, $Hom(C, \bullet)$, commutes with pushouts of diagrams of monics.*

Proof. We first show that the result holds for $Hom(B, -)$ when B is a representable object. In this case consider a pushout diagram

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
W & \longrightarrow & Z
\end{array}
$$

where $X \to Y$ and $X \to W$ are monic. On the level of presheaves, pushouts are taken objectwise and so we may write $Z(B) = W(B) \amalg_{X(B)} Y(B)$. We observe however that this is simply the statement $Hom(B, Z) = Hom(B, W) \amalg_{Hom(B,X)} Hom(B, Y)$ and the result follows.

Now suppose that $C = B/A$ where A and B are representables we have for an object Q the commutative diagram

$$
\begin{array}{ccc}
A & \longrightarrow & \bullet \\
\downarrow & & \downarrow \searrow \\
B & \longrightarrow & C \\
& & \searrow \\
& & Q
\end{array} \quad .
$$

The data in this diagram suggests that an element of $Q(C)$ corresponds to a choice of element in $Q(B)$ which is sent to the trivial element of $Q(A)$ under the map $Q(B) \to Q(A)$ where by trivial element we mean the image of the basepoint representative in $Q(\bullet)$ under the map $Q(\bullet) \to Q(A)$.

Assume now that Q is the pushout of some diagram as above(for notational simplicity assume $Q = Z$), then the the commutativity of the diagram corresponds to a choice of element in $Z(C)$ which is represented by a choice of element in $W(B) \amalg_{X(B)} Y(B)$ sent to a trivial representative in $W(A) \amalg_{X(A)} Y(A)$. Without loss of generality assume that the element of $W(B) \amalg_{X(B)} Y(B)$ is represented by the image of an element of $W(B)$. The corresponding element identified to the morphism $A \to Z$, as the element represented by the basepoint, corresponds to a unique element of $W(A)$ since for presheaves monics are injective objectwise. Since this element has to be the one given by the basepoint of $W(A)$ we see that our represenative in $W(B) \amalg_{X(B)} Y(B)$ yields a representative in $W(C)$. An analogous argument applies to the cases where the representative for $B \to Z$ is in $Y(B)$ and $X(B)$ and so we see that $Hom(C, Z)$ is

isomorphic as a set to $Hom(C, W) \amalg_{Hom(C,X)} Hom(C, Y)$, since the identification in this amalgamation is the one induced by the morphisms $Hom(C, X) \to Hom(C, Y)$ and $Hom(C, X) \to Hom(C, W)$ that are induced by the corresponding morphisms for A and B. $\square$

We now prove Lemma III.29.

Proof. Suppose $f : X \to Y$ is a generating acyclic cofibration in $SSet_\bullet^{\mathcal{C}^{op}}$. We must first show that $\mathcal{UF}_\mathbf{T}(f)$ is also a weak equivalence. Since f is a weak equivalence $X(C)$ is weak equivalent to $Y(C)$ for every $C \in \mathcal{C}$. We know from Proposition III.23, Lemma III.30, and the fact that $\mathcal{F}_\mathbf{T}$ commutes with pushouts that since the domains and codomains of generating acyclic cofibrations are pushouts of monics of representables tensored with simplicial sets, it suffices to show that $\underline{Hom}_{SSet_\bullet^{\mathcal{C}^{op}}}(A, A \wedge X) \to \underline{Hom}_{SSet_\bullet^{\mathcal{C}^{op}}}(A, A \wedge Y)$ is a weak equivalence. To this end we point out that $Hom(A \wedge U_+, A \wedge X)$ is the pullback of the diagram:

$$
\begin{array}{ccc}
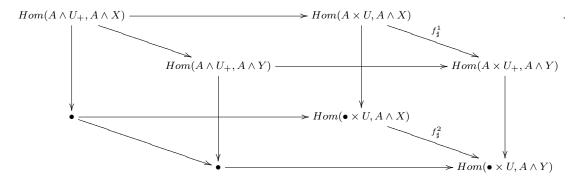& & Hom(A \times U, A \wedge X) \, . \\
& & \downarrow \\
\bullet & \longrightarrow & Hom(\bullet \times U, A \wedge X)
\end{array}
$$

Moreover, since A is pointed we have that $\bullet \times U \to A \times U$ is a retract diagram and the map $Hom(A \times U, A \wedge X) \to Hom(\bullet \times U, A \wedge X)$ is a projective fibration. To see this we first note that it is surjective. This fact follows from the retract statement since the canonical map $A \times U \to \bullet \times U$ induces the necessary surjection. It suffices then to show that given a commutative diagram:

$$
\begin{array}{ccc}
\Delta[n, k] & \longrightarrow & Hom(A \times U, A \wedge X) \\
\downarrow & & \downarrow \\
\Delta[n] & \longrightarrow & Hom(\bullet \times U, A \wedge X)
\end{array}
$$

with $n \geq 2$ that the diagram admits a lift. This is nearly trivial since A is a simplicially constant presheave and X is at worst a pushout, in the case of codomains of acyclic cofibrations, of products of representables with the constant simplicial objects $\Delta[m, r]$ and $\Delta[m]$ for $m, r \in \mathbb{N}$. Because of this fact, the bottom arrow represents a choice of map $f : \bullet \times U \to A \wedge X_n$ which is just a choice of representative in $A(\bullet \times U)$ and a choice in $X_n(\bullet \times U)$. To construct a lift it suffices to find a compatible choice in $X_n(A \times U)$ but elements of X_n are represented by pairs of elements in representables with elements of the constant simplicial presheaf, since the simplicial maps of X have no effect on the choice of element in the representable portion of the representative, and since the simplicial maps on the right most arrow of the diagram are the identity, the lift exists with respect to the representative, which then descends to the to the pushout, in the case of codomains, as required.

Now there is a commutative cube:

with $f_\sharp^i$ both weak equivalences and since the right vertical arrows are fibrations the map induced on $Hom(A \wedge U_+, A \wedge X) \to Hom(A \wedge U_+, A \wedge Y)$ is also a weak equivalence in the Bousfield-Kan structure, since it is right proper. In particular, the maps are flasque weak equivalences of presheaves as required since the weak equivalences are identical between the two categories.

To prove the second part of the lemma we observe that since the acyclic flasque

cofibrations are monics, which induce inclusions $Hom(A \wedge U_+, X) \to Hom(A \wedge U_+, Y)$, we have that the image of f under $\mathcal{U F_T}$ is actually an objectwise injection, which is an injective cofibration in $SSet_\bullet^{\mathcal{C}^{op}}$. $\square$

Proof. We first show that the structure on $\mathbf{Alg_T}$ outlined above is the structure of a cofibrantly generated model category using the cofibrantly generated model recognition theorem, Theorem II.24. Then, we verify the conditions of being a cellular model structure as per the requirements of Theorem II.30.
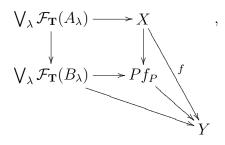
It is trivial to see that W, defined by the image under the forgetful functor, which in particular preserves retracts, is closed under retracts and the "2 of 3" composition rule.

Condition (1) of Theorem II.24 is a corollary to the stronger requirement of condition (3) of Theorem II.30 discussed below.
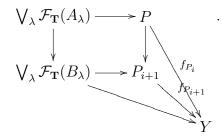
To prove condition (2) of Theorem II.24, suppose that $f : X \to Y$ is a J-cofibration, that is it satisfies the left lifting property with respect to J-fibrations. We shall see in the proof to condition (1) of Theorem II.30 that if h is I-fibration, then $U(h)$ is a acyclic fibration of simplicial presheaves and so has the right lifting property with respect to all generating acyclic cofibrations in $SSet_\bullet^{\mathcal{C}^{op}}$. By adjunction, h has the right lifting property with respect to all maps in J; hence, f has the left lifting property with respect to h.

To see f is a weak equivalence, we factor f as a J-cell complex as follows. First
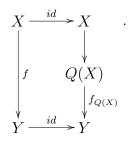
we construct the pushout diagram:

$$\begin{array}{ccc}
\bigvee_\lambda \mathcal{F}_{\mathbf{T}}(A_\lambda) & \longrightarrow & X \\
\downarrow & & \downarrow \\
\bigvee_\lambda \mathcal{F}_{\mathbf{T}}(B_\lambda) & \longrightarrow & Pf_P \\
& & \searrow f \\
& & Y
\end{array} \quad ,$$

where $A_\lambda \to B_\lambda$ is a generating acyclic cofibration in $SSet_\bullet^{\mathcal{C}^{op}}$ and the outer diagram commutes. Given such a pushout P_i we construct the next element in the J-cell by constructing the pushout diagram:

$$\begin{array}{ccc}
\bigvee_\lambda \mathcal{F}_{\mathbf{T}}(A_\lambda) & \longrightarrow & P \\
\downarrow & & \downarrow \\
\bigvee_\lambda \mathcal{F}_{\mathbf{T}}(B_\lambda) & \longrightarrow & P_{i+1} \\
& & \quad f_{P_i} \\
& & \quad f_{P_{i+1}} \\
& & Y
\end{array} \quad .$$

Taking the transfinite composition of all such pushouts and denoting the colimit over the pushouts by $Q(X)$ we have a factorization of f as in the diagram:

$$\begin{array}{ccc}
X & \xrightarrow{\ id\ } & X \\
\downarrow & & \downarrow \\
\Big\downarrow f & Q(X) & \\
& & \Big\downarrow f_{Q(X)} \\
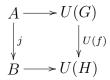Y & \xrightarrow{\ id\ } & Y
\end{array} \quad .$$

By the small object argument it follows that $f_{Q(X)}$ has the right lifting property with respect to elements of J and so the diagram admits a factorization of the identity map $Y \to Y$. This factorization makes f a retract of $X \to Q(X)$ which is weak equivalence by Lemma III.29, the fact that U preserves pushouts, and that W is closed under transfinite composition.

Condition (3) and (4b) are a corollaries to condition (1) of Theorem II.30 proved below.
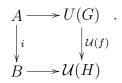
Now we proceed to the conditions of the cellular structure. Proof of condition (1): Suppose $f : G \to H$ is a trivial fibration. Then we know that $\mathcal{U}(f)$ is a weak equivalence and f has the left lifting property with respect to maps of the form $\mathcal{F}_{\mathbf{T}}(j)$ with j a generating acyclic cofibration in $SSet_{\bullet}^{\mathcal{C}^{op}}$ whenever the relevant commutative diagram exists. In particular, a commuting diagram

$$
\begin{array}{ccc}
\mathcal{F}_{\mathbf{T}}(A) & \longrightarrow & G \\
\downarrow{\scriptstyle \mathcal{F}_{\mathbf{T}}^{2}(j)} & & \downarrow{\scriptstyle f} \\
\mathcal{F}_{\mathbf{T}}(B) & \longrightarrow & H
\end{array} \quad,
$$

admits a lift $H : \mathcal{F}_{\mathbf{T}}(B) \to G$. Such a diagram, however is equivalent, via adjunction to a diagram:

$$
\begin{array}{ccc}
A & \longrightarrow & U(G) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle U(f)} \\
B & \longrightarrow & U(H)
\end{array}
$$

and a lift H is equivalent to one $h : B \to U(G)$. Hence, by adjunction f is a fibration in $\mathbf{Alg_T}$ if and only if $\mathcal{U}(f)$ is a fibration in $SSet_{\bullet}^{\mathcal{C}^{op}}$. Consequently, $\mathcal{U}(f)$ is an acyclic fibration and admits a lift to every commutative diagram of the form:

$$
\begin{array}{ccc}
A & \longrightarrow & U(G) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathcal{U}(f)} \\
B & \longrightarrow & \mathcal{U}(H)
\end{array} \quad.
$$

Here, $i : A \to B$ is a generating cofibration. Again, by adjunction, such a diagram

admits a lift if and only if every diagram of the form

$$
\begin{array}{ccc}
\mathcal{F}_{\mathbf{T}}(A) & \longrightarrow & G \\
\downarrow{\scriptstyle \mathcal{F}_{\mathbf{T}}(i)} & & \downarrow{\scriptstyle f} \\
\mathcal{F}_{\mathbf{T}}(B) & \longrightarrow & H
\end{array}
$$

admits a lift which consists of all maps in I as required.

Proof of condition (2): This is by definition.

Proof of condition (3) & (4): This follows by adjunction and the corresponding result from $SSet_{\bullet}^{\mathcal{C}^{op}}$ as in [13] Lemma 3.10.

Proof of condition (5): We note first that I consists of monomorphisms which are effective monomorphisms trivially. We also note that the pushout of a monomorphism in the category of sets, and so therefore in the category of presheaves of simplicial sets, is also a monomorphism and so an effective monomorphism. Transfinite composition monomorphisms induces a monomorphism as well as so we have that all I-cell complexes are effective monomorphisms.

To prove properness we start by observing that right properness follows immediately since as shown above $\mathcal{U}$ detects fibrations and weak equivalences, and as a right adjoint functor commutes with pullbacks.

To show left properness we note that cofibrations are retracts of I-cell complexes and so it suffices to show the result holds for I-cell complexes. Since I cell complexes are level-wise monomorphisms and therefore injective cofibrations under the image of $\mathcal{U}$, which also preserves pushouts, the result follows by left properness of the injective model structure on presheaves. $\qquad\square$

It follows trivially from the proof to the theorem that:

Proposition III.31. *With the above structure of a model category, one has a Quillen*

pair

$$\mathcal{U} : Alg_{\mathbf{T}} \leftrightarrow SSet_{\bullet}^{\mathcal{C}^{op}} : \mathcal{F}_{\mathbf{T}}.$$
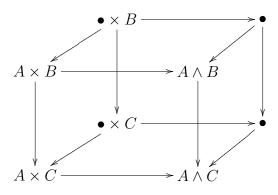
We also find we have a Quillen pair given by $B_{\mathbf{T}}$ and $\Omega^{\mathbf{T}}$.

Proposition III.32. $(B_{\mathbf{T}}, \Omega^{\mathbf{T}})$ *form a Quillen adjoint pair.*

To prove this we need a lemma:

Lemma III.33. *If $\bullet \to A$ is a pointed representable presheaf and $B \to C$ is an pointed (acyclic) cofibration, then so is $A \wedge B \to A \wedge C$.*

Proof. If we ignore the basepoint, then $A \times B \to A \times C$ is an (acyclic) cofibration. This follows since A is representable and I and J are closed under products with the identity on representables. We have the following commuting cube:

of pushout squares. Each vertical arrow between the top and bottoms a (acyclic) cofibration, and then, so is the map of pushouts as required. $\square$

We use this to prove Proposition III.32

Proof. This is standard since $\mathcal{U}(\Omega^{\mathbf{T}}(Y)) = \underline{Hom}(A, Y)$ which preserves fibrations and trivial fibrations by adjunction and Lemma III.33. $\square$

D. $\mathcal{F}_\mathbf{T}$ and Smooth Schemes over a Field k

The purpose of this section is to discuss how one can work when the underlying category is given by $\mathcal{C} = Sm_k$. The first problem we encounter is that Sm_k is not small, nor is it closed under finite coproducts. There are two potential approaches to this problem, the first is to construct a new category E which is equivalent to Sm_k and small. To this small category we can then add finite coproducts which by a simple argument is still small. The $\mathcal{F}_\mathbf{T}$ construction may be applied to an $F \in SSet^{Sm_k^{op}}$ on isomorphism classes of objects in Sm_k via this smaller category E with no ambiguity since presheaves preserve isomorphism.

Another approach to this issue is to work instead with a category of sheaves in some subcanonical topology on Sm_k. Sheaves of this type are more geometrically natural than presheaves and so are preferable for certain applications anyway. This section explores this idea in more detail and provides some discussion regarding the relationship between $\mathcal{F}_\mathbf{T}$ and sheaves.

We will assume that the topology being used for Sm_k is an enlargement of the Zariski Topology, actually we are interested in the completely decomposed topology of Nisnevich [20] but as we make no explicit use of the properties of this other than containment of the Zariski topology, we shall stick to this generality.

We start by choosing a small full subcategory of Sm_k which will contain enough geometric data to classify sheaves. Certainly, a sufficient choice would be one such that every scheme can be expressed as a colimit of representatives from the isomorphism classes of the objects of the subcategory, we shall call such a subcategory a *Zariski subcategory*. Take for this subcategory all objects of the form $Spec(k[\mathbf{X}]/\sim)$ where $\mathbf{X}_n$ denotes a finite collection $\{x_1, \ldots, x_n\}$ of indeterminants and $\sim$ denotes the equivalence relation generating the quotient by some ideal I. We note that the

set of objects is bounded by the cardinality of the product $\coprod_{\mathbb{N}} \mathcal{P}(k[\mathbf{X_n}])$ where $\mathcal{P}(X)$ denotes the powerset of X. Since the morphisms between objects correspond to morphisms of algebras and morphism induced by diagrams of algebras, we see this is indeed a small subcategory. Moreover, since every scheme can be covered by a finite collection of affine open subschemes, each of which is isomorphic to some $Spec(K[\mathbf{X}_n])$ we see that this is a Zariski subcategory of Sm_k. Call this full subcategory $\tilde{E}$.

Now since Sm_k is not closed under finite coproducts neither is $\tilde{E}$ and so we must construct a new small category by adding them. For this we define E as the category consisting of all objects of $\tilde{E}$ together with finite coproducts of objects in $\tilde{E}$. Morphisms will consist of those from $\tilde{E}$ and those induced on the coproducts from component inclusion and the maps on $\tilde{E}$. It is clear that morphisms between coproduct objects, generated by finite inclusions and morphism in $\tilde{E}$, will still be set. Hence, E is small.

We now wish to understand sheaves on E and so must define a Grothendieck Topology. We recall the definition of a Grothendieck Topology here.

Definition III.34. [17] A Grothendieck Topology on a category $\mathcal{C}$ is a function J which assigns to each object $C \in \mathcal{C}$ a collection $J(C)$ of sieves on C, in such a way that

1. the maximal sieve $t_C = \{f \mid cod(f) = C\}$ is in J(C);

2. (stability axiom) if $S \in J(C)$ then $h^*(S) \in J(D)$ for any arrow $h : D \to C$;

3. (transitivity axiom) if $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all $h : D \to C$ in S, then $R \in J(C)$.

Remark III.35. Although the typical examples of Grothendieck topologies are given in terms of pretopologies [1] or Grothendieck bases, this definition will not suffice when

working with E which is not closed under pullbacks. Fortunately, it's well understood that a Grothendieck basis gives rise to a topology in the sieve sense [1],[17]

We now wish to give E a topology compatible with that of Sm_k in the sense that the sheaves on E will be comparable to those Sm_k. Suppose $x \in E$ such that $x \in Sm_k$. Let J_{Sm_k} denote to Grothendieck topology on Sm_k and let $S \in J_{Sm_k(x)}$ Define $\tilde{S}$ to be the sieve consisting of coproducts of maps in S with codomain in E. That this is a sieve is trivial. Let $G_E(x)$ consist of all such $\tilde{S}$ for every $S \in J_{Sm_k}(x)$. For $x \in E$ such that x is a coproduct of elements of Sm_k, we write $x = x_1 \amalg \cdots \amalg x_n$. Given a tuple $S_i \in J_{Sm_k}(x_i)$, define $\tilde{S}$ so the sieve consisting of all coproducts of tuples of coproducts from the S_i. That this is a sieve is also trivial and we let $G_E(x)$ consist of all such $\tilde{S}$.

Lemma III.36. G_E is a topology on E.

Proof. That the maximal sieve is contained is trivial. Suppose now that $x \in E$ such that $x \in Sm_k$, with $\tilde{S} \in G_E(x)$ and $h : y \to x$.

We first consider the case that $y \in Sm_k$ and so $h \in Mor_{Sm_k}(y, x)$. In this case we must confirm that $\{g : cod(g) = y$ and $hg \in S\}$ is a sieve in the sense of our definition. For this to be the case all maps in the sieve must result from induced maps from maps in some sieve in $J_{Sm_k}(y)$. Note that $\tilde{S} \in G_E(x)$ is generated by a sieve $S \in J_{Sm_k}(x)$ and this sieve pulls back via h to one in $J_{Sm_k}(y)$. We wish to show that $\widetilde{h^*S}$ is identical to $h^*\tilde{S}$. If $f \in \widetilde{h^*S}$ then f is induced by maps in h^*S, call these $\{f_i\}$. Now each f_i satisfies $hf_i \in S$ and so the coproduct satisfies $hf \in \tilde{S}$ as required. If $f \in h^*\tilde{S}$ then $hf \in \tilde{S}$ and so hf which is the pushout map of the compositions hf_i, each of which is in S, by definition is in $\widetilde{h^*S}$ as required. Hence $h^*\tilde{S}$ pulls back to a sieve in $G_E(y)$ as required.

Now if y is not in Sm_k then write $y = y_1 \amalg \cdots \amalg y_n$ and so h is the map induced

by a collection $h_i : y_i \to x$. It suffices to see that h^*S is the product of sieves in $G_E(y_i)$. To see this we must only confirm that h_i^*S is a sieve in $G_E(y_i)$ but this is the content of the previous argument.

Now suppose that x is a coproduct. A sieve in $S \in G_E(x)$ is given by the coproduct construction outlined above. In order for h^*S to be a sieve in our topology, we must confirm that the pullback of each of the component sieves giving S is still a sieve after pullback. This is the content of the case when $x \in Sm_k$ and the stability axiom is proved.

To show the transitivity axiom is a bit more technical. Suppose $\tilde{R}$ is a sieve on x, where $x \in Sm_k$ satisfying the hypothesis of the transitivity axiom. We must show the existence of a sieve $R \in J_{Sm_k}(x)$ generating $\tilde{R}$. Let $\tilde{S}$ be the hypothesized sieve in $G_E(x)$ and and for $h \in \tilde{S}$, let $\tilde{R}_h$ denote the covering sieve $h^*\tilde{R}$. Define $\check{R} = \{f \mid f \in \tilde{R} \text{ with } cod(f) \in Sm_k\}$, that is $\check{R}$ consists of all morphisms of $\tilde{R}$ which are morphisms in Sm_k. Define $R := \{f \mid f = gh \text{ where } g \in \check{R} \text{ and } cod(h) = dom(g)\}$. Confirming R is a sieve is trivial. We now must show R is a covering sieve.

We first show that for $h \in \tilde{S}$ such that $h \in S$, we have that $\widetilde{h^*R} = h^*\tilde{R}$. Let $f \in \widetilde{h^*R}$, then f is the coproduct of maps in h^*R with codomain in E. The coproduct of these maps composed h is then in $\tilde{R}$ which establishes the first inclusion. To show the other direction, assume $f \in h^*\tilde{R}$. Then f is the coproduct of maps in Sm_k and each of these composed with h is in $\tilde{R}$ which implies they are also in R. Taking the coproduct yields the inclusion in $\widetilde{h^*R}$.

Now we note that for each $h \in S$ such that $h \in \tilde{S}$, we have that h^*R is a covering sieve. We must show the same holds for each $h \in S$ such that $cod(h)$ is not in E. Let $y = cod(h)$. Then cover y with elements of E, the inclusions of which are contained in the Zariski topology which is contained in the topology for Sm_k and so it suffices to see that when we pullback h^*R along each of these inclusions we have a covering

sieve. This follows from the fact that for such an inclusion, call it i, $hi \in S$ and $hi \in \tilde{S}$ so the hypothesis yields the result. Hence, $\forall h \in S$, h^*R is a covering sieve and so R is a covering sieve.

The case when x is a coproduct of objects in Sm_k is similar to the approach for the stability axiom and will not be duplicated. It is clear that applying the construction of the definition of G_E to S yields our original $\tilde{S}$ which completes the proof. □

We may now compare sheaves over E to those over Sm_k. We describe this comparison in terms of a more general result. Essentially we've created a small category E such that every object in Sm_k can be covered by a finite collection of objects in E and given it the topology induced from these covers. We now show that sheaves over Sm_k are in bijective correspondence to sheaves over Sm_k. In fact, a more general statement is true.

Proposition III.37. *Let $\mathcal{C}$ be any site with small Zariski subcategory denoted by E. Then, up to isomorphism, the category of sheaves over $\mathcal{C}$, $Shv(\mathcal{C})$, is in bijective correspondence to the category of sheaves over E, $Shv(E)$, given by the restriction of the topology on $\mathcal{C}$ described above.*

To prove the lemma we require a few conventions regarding amalgamations and their properties. We follow [17].

Definition III.38. Suppose one has a category $\mathcal{C}$; let P be a presheaf and S a covering sieve of an object $x \in \mathcal{C}$. A matching family for S of elements of P is a function which assigns to every function $f \in S$ an element $P(cod(f))$, call it x_f, so that $P(g)(x_f) = x_{fg}$ for all composable morphisms g in $\mathcal{C}$.

Definition III.39. Using the conventions of the above definition, an amalgamation is a member $x \in P(C)$, such that for every $f \in S$ we have that $P(f)(x) = x_f$.

Using these we may define a sheaf as a presheaf such that every matching family has a unique amalgamation. The following proposition will be useful to us.

Proposition III.40. *[17] Let P be a presheaf on $\mathcal{C}$, with Grothendieck topology $\mathcal{G}$. P is a sheaf if and only if for any cover $\{f_i : x_i \to x \mid i \in I\}$ in a basis of $\mathcal{C}$, any matching family has a unique amalgamation.*

Remark III.41. This proposition only makes sense when $\mathcal{C}$ is closed under pullbacks. By cover we mean a cover in the usual sense of a Grothendieck basis. Fortunately, this is the case when $\mathcal{C} = Sm_k$.

We now prove Proposition III.37.

Proof. Suppose that F is a sheaf over $\mathcal{C}$, define $\tilde{F}(x) = F(x)$ whenever $x \in \mathcal{C}$ and $\tilde{F}(x) = \prod_i F(x_i)$ whenever $x = x_1 \amalg \cdots \amalg x_n$ for $x_i \in Sm_k$. We first show that $\tilde{F}$ is a sheaf.

Let $x \in E$ such that $x \in \mathcal{C}$. Then a matching family for $\tilde{F}$ for $S \in G_E(x)$ gives rise to a matching family for S and so has a unique amalgamation as required. When $x \in E$ is a coproduct of elements of $\mathcal{C}$, we note that each component has a unique amalgamation and the product of these gives the result.

Suppose now that two sheaves F_1 and F_2 yield the same $\tilde{F}$. Suppose $c \in \mathcal{C}$. Let $x \in F_1(c)$. Cover c by objects in E and use x to create a matching family over the cover. Since F_2 is a sheaf, there is a unique amalgamation $x' \in F_2(c)$. Map x to x'. In this way one creates an invertible natural transformation between F_1 and F_2.

We must show the correspondence is surjective. Given a sheaf, F, over E, create one over $\mathcal{C}$ by defining $F(c) := \lim F(x)$ over all arrows $x \to c$ where $x \in E$. $\qquad\square$

We now wish to define $\mathcal{F}_{\mathbf{T}}$ in the context of Sm_k. To do so we note that we may define $\mathcal{F}_{\mathbf{T}}$ as in the previous section but now over the small category $\mathcal{D} = \Gamma^{op} \times E$.

Here we adopt the same convention used in defining $\mathcal{F}_{\mathbf{T}}$ where $\mathcal{D}$ is regarded as a subcategory of $SSet_{\bullet}^{E^{op}}$.

This defines a functor over the category of $SSet_{*}^{E^{op}}$ to the category of $Alg_{\mathbf{T}}$ of models of $\mathbf{T}$ in $SSet_{*}^{E^{op}}$. Since there are canonical maps $Shv(Sm_k) \to Shv(E) \hookrightarrow SSet_{*}^{E^{op}}$ we have a functor from $Shv(Sm_k)$ into T-models in $SSet_{*}^{E^{op}}$.

We wish to see this as a functor to the category of $\mathbf{T}$-models in sheaves over E. To do so we continue to regard $\mathbf{T}$-models as presheaves over the category $\Gamma^{op} \times E$. There seem to be two reasonable appraoches. The first is to sheafify $\mathcal{F}_{\mathbf{T}}$ on each subcategory $n^+ \times E$. Since sheafification preserves finite products we still have a $\mathbf{T}$-model. Another approach is to give $\mathcal{D}$ the structure of a site in such a way that sheafification will automatically preserve product preserving properties without the individual application of sheafification to each subcategory. I briefly describe one way of doing this.

We give Γ^{op} a Grothendieck topology which includes as covering sieves for an object n^+, all sieves such that each element of n^+ is in the image of some map in the sieve. In particular this includes the sieves generated by the canonical covers given by $\{1^+ \to n^+\}$ consisting of the n inclusions of summands. That this is a topology is trivial to verify. We give $\mathcal{D}$ the product topology by defining a covering sieve for $n^+ \times e$ to be the sieve consisting of products (smash products since $\mathcal{D}$ is regarded as subcategory of $SSet_{\bullet}^{E^{op}}$) of maps in the covering sieve of n^+ with maps in the covering sieve of e. This clearly defines a sieve and it is easy to check this yields a Grothendieck topology after we include the global covering sieve of all morphisms (not all morphisms in $\mathcal{D}$ may result from smash products).

This gives $\Gamma^{op} \times E$ the structure of a site which allows us to define a sheafification functor, denoted $++$, for $SSet_{\bullet}^{\Gamma^{op} \times E^{op}}$, one should note that the choice of Grothendieck topology on Γ^{op} guarantees that the result of sheafification still respects the product

structure of Γ^{op}.

Since $\mathcal{F}_{\mathbf{T}}$ and $++$ are both left adjoints they commute with colimits. Since $\mathcal{F}_{\mathbf{T}}$ is a colimit over the same category that $++$ is defined over, and since the result of $\mathcal{F}_{\mathbf{T}}$ may still be regarded as a functor over $\Gamma^{op^{op}} \times E^{op}$; we note that these two functors commute, which yields the result:

Lemma III.42. *Let X be a Presheaf over E regarded as presheaf over $\Gamma^{op} \times E$ in the usual way, then*

$$\mathcal{F}_{\mathbf{T}}(++(X)) \simeq ++\mathcal{F}_{\mathbf{T}}(X).$$

In particular, $\mathcal{F}_{\mathbf{T}}$ takes sheaves to sheaves.

Again, everything here can be done in the same generality as our proposition. We now extend $\mathcal{F}_{\mathbf{T}}$ to schemes via the following result.

Proposition III.43. *$\mathcal{F}_{\mathbf{T}}$ may now be extended to functor,*

$$\mathcal{F}_{\mathbf{T}} : Shv(\mathrm{Sm}_k) \to \boldsymbol{Alg_T}/\mathrm{Sm}_k,$$

where $Alg_{\mathbf{T}}/\mathrm{Sm}_k$ denotes the category of $\mathbf{T}$-models in $Shv(\mathrm{Sm}_k)$.

Proof. The correspondence of $Shv(Sm_k) \to Shv(E)$ is clearly a functor and the previous lemma shows that $\mathcal{F}_{\mathbf{T}}$ defines a functor from $Shv(E)$ into $\mathbf{T}$-models in $Shv(E)$. The correspondence in Proposition III.37 shows that this is actually a $\mathbf{T}$-model in $Shv(Sm_k)$ and this correspondence is given by a functorial extension; hence, the composition of these functors yields the hypothesized functor $Shv(Sm_k) \to \mathbf{Alg_T}/Sm_k$. $\square$

Henceforth we will assume $\mathbf{Alg_T} = \mathbf{Alg_T}/Sm_k$ unless otherwise noted.

Proposition III.44.

$$\mathcal{F}_{\mathbf{T}} : Shv(\mathrm{Sm}_k) \leftrightarrow \boldsymbol{Alg_T} : \mathcal{U}$$

form an adjoint pair.

Proof. This follows from the fact that $\mathcal{U} : \mathbf{Alg_T} \rightarrow Shv(Sm_k)$ composed with the forgetful functor into $Pre(Sm_k)$ is left adjoint to $\mathcal{F_T}$ composed with $++$, but this composition is equivalent to restricting $\mathcal{F_T}$ to the category $Shv(Sm_k)$. $\qquad\square$

As in the previous section we may use $\mathcal{U}$ to endow $\mathbf{Alg_T}$ with the structure of a model category. Since we would like this to be compatible with the structure we gave for $\mathbf{T}$-models in presheaves, we implicitly make the assumption that the forgetful functor from $Shv(Sm_k) \rightarrow Pre(Sm_k)$ is used to endow $Shv(Sm_k)$ with a notion of a model category in the same way as $\mathbf{Alg_T}$. This yields immediately that:

Proposition III.45.

$$\mathcal{F_T} : Shv(\mathrm{Sm}_k) \leftrightarrow \boldsymbol{Alg_T} : \mathcal{U}$$

form a Quillen adjoint pair.

E.   Conclusion

We will now show that $\mathbf{Alg_T}$, the category of complete $\mathbf{T}$-models, captures much of the same combinatorial aspects of loop space recognition embodied in the topological variant of this construction [4].

Another observation worth pointing out that if a category $\mathcal{C}$ has a small subcategory including isomorphism classes for all the objects, then one can close this under small coproducts and develop a theory of $\mathbf{T}$-models. If one is willing to work with sheaves, then one can work over even smaller categories. In subsequent discussion regarding Sm_k we will assume that we are working in the realm of presheaves and assume that we have carried out such a construction. All the results involving presheaves will naturally hold in the context of sheaves.

CHAPTER IV

LOCAL MODEL STRUCTURES ON $\mathbf{Alg_T}$ AND A RECOGNITION PRINCIPLE

This chapter contains the main results concerning the approximation theorem and a recognition principle for motivic loop spaces similar in flavor to [4]. The Approximation theorem provides the basis for understanding how a motivic type localization of the model structure on $\mathbf{Alg_T}$ yields an equivalent model structure to the localization of $SSet_\bullet^{\mathcal{C}^{op}}$ with respect to maps inducing weak equivalence of loop spaces. This equivalence provides the foundation for comparing the homotopy type of a loop space with that of a $\mathbf{T}$-algebra.

A.   Introduction

Recall May's Approximation Theorem:

Theorem IV.1. *[18] Let X be a topological space, C_n the little n-cubes operad. Then there is a map*

$$\alpha_n : C_n X \to \Omega^n S_n X$$

where $\Omega^n S_n X = Hom(S^n, S^n \wedge X)$ which is a weak equivalence for connected X.

Here C_n is a tensor product between the little cubes operad and X in much the same way as the construction of $\mathcal{F}_\mathbf{T}(X)$ when $X \in alg$. In this sense our approximation theorem very much resembles May's topological example in the context of motivic loop spaces. Our approximation theorem is the following:

Theorem IV.2. *Let X be a cofibrant presheaf. Then for $\mathbf{T} = \mathbf{T}^A$, where A is cofibrant and the quotient of representables, and $\mathcal{C}$ is a site with an interval I, closed*

under finite coproducts, we have:

$$\mathcal{U}(\mathcal{F}_{\mathbf{T}}(X)) \simeq \underline{Hom}(A, A \wedge \tilde{X}),$$

where $\simeq$ denotes an I-local weak equivalence and $\tilde{X}$ denotes a fibrant replacement of X.

This is a more general statement than III.22 which is limited to representable presheaves.

Section C will exhibit a functor that yields a freely generated $\mathbf{T}$-model simplicially weak equivalent to the $\mathbf{T}$-model over which it is applied. This replacement functor will actually work in any context in which $\mathbf{Alg_T}$ has been defined, much unlike the results of the other sections which require a category with a notion of an interval. The results of this section will be crucial in later consideration of localization of $\mathbf{Alg_T}$, namely in Section D. Interesting in its own right, the material presented here provides some insight into the relationship between the homotopy type of $X \in \mathbf{Alg_T}$ and that of $\mathcal{U}(X)$ in the category $SSet_{\bullet}^{\mathcal{C}^{op}}$.

In order to make use of the $\mathbb{A}^1$-homotopy result of Section B, it will be necessary to understand potential localizations of $\mathbf{Alg_T}$. In particular we will try to develop a homotopy theory of $\mathbf{Alg_T}$ that is in some way compatible with $\mathbb{A}^1$ homotopy theory, in the sense that an $\mathbb{A}^1$-weak equivalence of $\mathcal{U}(X)$ and $\mathcal{U}(Y)$ for $X, Y \in \mathbf{Alg_T}$ should provide insight to the homotopy of X and Y. We will present a motivic localization of $\mathbf{Alg_T}$ to make this insight precise.

Section E will explore how the motivic model structure on $\mathbf{Alg_T}$ introduced in Section D relates to the homotopy theory of $SSet_{\bullet}^{\mathcal{C}^{op}}$ with the motivic model structure. In the topological context, it has been shown that the right Bousfield localization of topological spaces with respect to maps inducing weak equivalence of loop spaces yields a model structure Quillen equivalent to the model structure on $\mathbf{Alg_T}$ [4]. In our

more general context it is necessary to find a homotopy structure with nice properties similar to the homotopy theory of topological spaces. It will be shown that the motivic localization of $\mathbf{Alg_T}$ provides exactly the right context for such considerations.

The following result is a natural consequence of the Quillen equivalence in Section 3.5.

Theorem IV.3. *A cofibrant object X in $SSet_{\bullet}^{\mathcal{C}^{op}}$ has the structure of an algebraic theory $Y \in \mathbf{Alg_{T^A}}$ if and only if X is weakly equivalent to the loop space $\underline{Hom}(A, B_{T^A}Y)$.*

This theorem and a comparison to the corresponding topological result is discussed in Section F.

B.   The Approximation Theorem for Free T-Models

In this chapter we demonstrate that free $\mathbf{T}$-models capture the homotopy theory of loop spaces of suspensions. This should be considered analogous to results similar to May's approximation theorem, mentioned above, which states:

Theorem IV.4. *[18] Let X be a topological space, C_n the little n-cubes operad. Then there is a map*

$$\alpha_n : C_n X \to \Omega^n S_n X$$

where $\Omega^n S_n X = Hom(S^n, S^n \wedge X)$ which is a weak equivalence for connected X.

The analogue of the approximation theorem in the context of $\mathbf{T}$-models will be stated with as great a generality as possible; and, partly due to this generality, one would not expect to have as cohomologically pleasant a result as May's. Most arguments here will assume an algebraic theory $\mathbf{T}$ given by the selection of a pointed cofibrant-representable object A in the category of pointed simplicial presheaves over a small index category $\mathcal{C}$, with finite coproducts as in example III.7. Since we have

endowed the category of **T**-algebras with a model structure derived from the flasque model structure on $SSet_\bullet^{\mathcal{C}^{op}}$, one notes that any representable object with a base point suffices for such an A. This applies, in particular, to representable models for loops in motivic homotopy theory over a field k.

Let $\tilde{X}$ denote a fibrant replacement for X via the techniques outlined in section C. We now proceed to prove Theorem IV.2, and a fundamental step in the proof is the following result.

Proposition IV.5. *If X is cofibrant-fibrant presheaf of sets, then one has*

$$\mathcal{U}(\mathcal{F}_{\mathbf{T}}(X)) \simeq \underline{Hom}(A, A \wedge X)$$

One has that $\mathcal{F}_{\mathbf{T}}$ preserves trivial cofibrations by Proposition III.31 and so with Lemma IV.5 we may prove IV.2 as follows.

Proof. Since $\tilde{X}$ is a fibrant replacement for X, one has that $X \to \tilde{X}$ is a trivial cofibration and hence $\mathcal{F}_{\mathbf{T}}(X) \to \mathcal{F}_{\mathbf{T}}(\tilde{X})$ is a weak equivalence since $\mathcal{F}_{\mathbf{T}}$ is Quillen adjoint to $\mathcal{U}$. Since $\tilde{X}$ is by construction a fibrant simplicial presheaf, which is cofibrant whenever X is, the result follows immediately from Proposition IV.5. $\qquad\square$

To prove Proposition IV.5 we begin with a definition.

Definition IV.6. Let $X \in SSet_\bullet^{\mathcal{C}^{op}}$ be a presheaf of sets. Denote by $\tilde{Q}(X)$ a functorial cofibrant replacement of X.

We recall that this cofibrant replacement is the transfinite composition of pushouts of generating cofibrations over X. If we have already assumed that X is a cofibrant object in the category of $SSet_\bullet^{\mathcal{C}^{op}}$ then the canonical map $\tilde{Q}(X) \to X$ is a weak equivalence of cofibrant objects. Since $\mathcal{F}_{\mathbf{T}}$ is a left Quillen adjoint it preserves weak equivalences between cofibrant objects and we have:

Lemma IV.7.

$$\mathcal{F}_{\mathbf{T}}(\tilde{Q}(X)) \to \mathcal{F}_{\mathbf{T}}(X)$$

is a weak equivalence.

We now show that:

Lemma IV.8.

$$\underline{Hom}_{SSet_{\bullet}^{\mathcal{C}^{op}}}(A, A \wedge \tilde{Q}(X)) \to \underline{Hom}_{SSet_{\bullet}^{\mathcal{C}^{op}}}(A, A \wedge X)$$

is a weak equivalence.

First, we naturally have that $\tilde{Q}(X)$ is fibrant whenever X is fibrant since $\tilde{Q}(X) \to X$ is an acyclic fibration. We also must show that whenever $X \to Y$ is a flasque fibration and A a cofibrant pointed representable object, then $A \wedge X \to A \wedge Y$ is still a flasque fibration. Although our proof applies to flasque fibrations, it should be noted that the same argument holds for Bousfield-Kan fibrations. Unfortunately, in this context we have much less flexibility when choosing an A to generate $\mathbf{T}$ as fewer representables are fibrant as pointed presheaves.

Lemma IV.9. *If $X \to Y$ is an acyclic flasque fibration between cofibrant objects, then $A \wedge X \to A \wedge Y$ is an acyclic flasque fibration whenever A is pointed representable and X is flasque fibrant.*

Proof. We start by noting that smashing with a pointed representable in the flasque model structure preserves cofibrations and acyclic cofibrations by Lemma III.33, and so preserves weak equivalences between cofibrant objects. It suffices then to show that the smash product preserves fibrations.

If we let $\Delta[n, k]$ denote the kth hat complex of $\Delta[n]$, the n-simplex represented

in $SSet_{\bullet}^{\mathcal{C}^{op}}$, then we must construct a lift to the diagram

$$U \otimes \Delta[n] \amalg_{U \otimes \Delta[n,k]} V \otimes \Delta[n,k] \longrightarrow A \wedge X$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$$
$$V \otimes \Delta[n] \longrightarrow A \wedge Y$$

where $U \to V$ is a monomorphism in $\mathcal{C}$. Since V is representable, a morphism $V \otimes \Delta[n]$ corresponds to n-simplex in the simplicial set $A \wedge Y(V)$, which can be represented by a pair $a(V) \times y(V)$, where $a(V) \in A(V)$ and $y(V) \in Y(V)$. We shall first assume that this is a non-trivial representative, i.e. neither $a(V)$ nor $y(V)$ are the basepoints of $A(V)$ or $Y(V)$. In this case to construct the lift we must find an element of $a(V) \in A(V)$ and $x(V) \in X(V)_n$ such that $a(V)$ is sent to $a(U)$ and such that $x(V)$ is sent to $y(V)$. That such an $x(V)$ exists follows from the fact that $X \to Y$ is a flasque fibration and the choice of $a(V)$ is of course the element from the representative of the bottom horizontal map in the diagram.

In the situation that $a(V)$ is the basepoint of $A(V)$ then we note the lift of the diagram is trivial and in the event that $y(V)$ is the basepoint of $Y(V)$ then the result follows from the fact that X is flasque fibrant. □

Lemma IV.8 now follows from the proof of Lemma III.29. To complete the proof of Proposition IV.5, it suffices to see:

Lemma IV.10.

$$\mathcal{U}(\mathcal{F}_{\mathbf{T}}(\tilde{Q}(X))) \to \underline{Hom}(A, A \wedge \tilde{Q}(X))$$
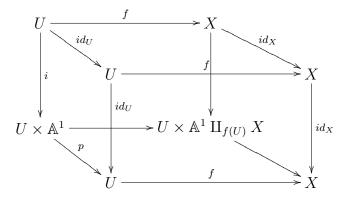
is an I local weak equivalence.

Remark IV.11. Since we are mostly interested in $\mathbb{A}^1$-homotopy theory, we write the proof only in this context. However, homotopy equivalence can be defined as in [19] in much the same way as below, and in this case the proof is identical for the more

general setting.

Proof. We start by noting that if X is a presheaf then there is a homotopy equivalence [19] between $U \times \mathbb{A}^1$ and U. Consider the map $H : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ induced by the ring homomorphism $k[x] \to k[x,y]$ sending $f(x)$ to $f(xy)$. Define maps $i_i : \mathbb{A}^1 \times spec(k) \to \mathbb{A}^1 \times \mathbb{A}^1$ as those induced by $f(x,y) \to f(x,0)$ and $f(x,y) \to f(x,1)$, respectively. It is clear now that $H \circ i_0$ is the identity on $\mathbb{A}^1$ and $H \circ i_1$ is the zero map. This creates a homotopy equivalence between $U \times \mathbb{A}^1$ and U.

Now consider the following commuting square.

Here the back and front squares are pushouts where i_U, i_V, and p are homotopy equivalences, hence the map on pushouts is a homotopy equivalence. We note these remain homotopy equivalences if we make f a pointed map by attaching a disjoint basepoint to each of the representables U and $U \times \mathbb{A}^1$ in the diagram. We now have an induced homotopy equivalence:

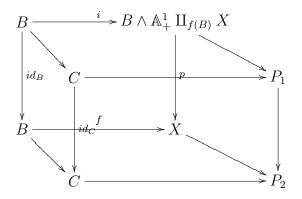$$U_+ \wedge \mathbb{A}^1_+ \amalg_{f(U)} X \to X \tag{B.1}$$

Since the maps are all $\mathbb{A}^1$-homotopy equivalences, we have that $\mathcal{U}\mathcal{F}_{\mathbf{T}}(U \times \mathbb{A}^1 \amalg_{f(U)} X) \to \mathcal{U}\mathcal{F}_{\mathbf{T}}(X)$ is a homotopy equivalence. Similarly, since smashing a homo-

topy equivalence preserves homotopy equivalence we have a homotopy equivalence $\underline{Hom}_{SSet_\bullet^{\mathcal{C}^{op}}}\left(A, A \wedge \left(U \times \mathbb{A}^1 \amalg_{f(U)} X\right)\right) \rightarrow \underline{Hom}_{SSet_\bullet^{\mathcal{C}^{op}}}(A, A \wedge X)$. To see the construction of both of these equivalences we note that a homotopy $H : B \times \mathbb{A}^1 \rightarrow C$ yields a homotopy equivalence on the level of $\underline{Hom}_{SSet_\bullet^{\mathcal{C}^{op}}}$ as follows.

First observe that $\underline{Hom}(A, B) \times \mathbb{A}^1 \cong \underline{Hom}(A, B) \times \underline{Hom}(\bullet, \mathbb{A}^1)$, where $\cong$ denotes isomorphism. Now $A \rightarrow \bullet$ induces a map $\underline{Hom}(\bullet, B) \rightarrow \underline{Hom}(A, B)$, and there is a natural map of presheaves $\underline{Hom}(A, B) \times \underline{Hom}(A, \mathbb{A}^1) \rightarrow \underline{Hom}(A, B \times \mathbb{A}^1)$. H now induces the morphism into the desired codomain $\underline{Hom}(A, C)$. Applying $\underline{Hom}(A, -)$ to H yields the desired $\mathbb{A}^1$ homotopy equivalence on the level of $\underline{Hom}$. This argument applies equally well to $\underline{Hom}_{SSet_\bullet^{\mathcal{C}^{op}}}$, since maps in this context are simply a subcollection of maps in $\underline{Hom}$ and inclusion of basepoint is preserved throughout the compositions.

To see the homotopy equivalence for $\mathcal{UF}_\mathbf{T}$, we use the the fact that $\underline{Hom}_{SSet_\bullet^{\mathcal{C}^{op}}}(A, -)$ preserves homotopy equivalences, which allows us to construct homotopy equivalence on the level of representatives from the terms of $Hom_{SSet_\bullet^{\mathcal{C}^{op}}}(A \wedge U_+, \vee_n A \wedge V_+) \wedge Hom_{SSet_\bullet^{\mathcal{C}^{op}}}(\vee_n S^0 \wedge V_+, X)$. It is trivial that these homotopy maps commute with the relations that define $\mathcal{UF}_\mathbf{T}$ (III.19) which completes the proof of the result.
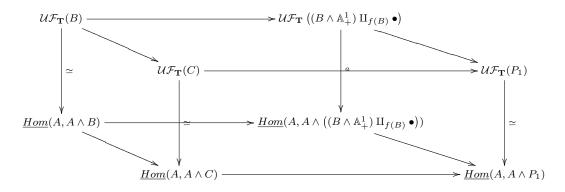
Using induced homotopy equivalences as in equation B.1, we consider pushout diagrams:

where $B := \left(\Delta[n,k] \times V \amalg_{\Delta[n,k] \times U} \Delta[n] \times U \right)_+$ and $C := (\Delta[n] \times V)_+$, with $B \to C$ is the map induced by an inclusion $U \hookrightarrow V$; $B \wedge \mathbb{A}^1_+$ naturally equivalent to $\left(\left(\Delta[n,k] \times V \amalg_{\Delta[n,k] \times U} \Delta[n] \times U \right) \times \mathbb{A}^1 \right)_+$; P_1 and P_2 are pushouts of the top and bottom squares respectively; i is the inclusion onto the image of i_0 in $\mathbb{A}^1$; and p is the map induced from the above pushout diagram.

As before, vertical maps are $\mathbb{A}^1$-homotopy equivalences and so $\mathbb{A}^1$-homotopy equivalences are induced between the images of P_1 and P_2 under $\mathcal{U}\mathcal{F}_{\mathbf{T}}$ and $\underline{Hom}(A, -)$. Moreover, the top square is a pushout of monics and Lemma III.30 implies that $\underline{Hom}_{SSet_\bullet^{\mathcal{C}^{op}}}(A, -)$ commutes with the pushout diagram.

Coproducts over the lower square are used to form the cofibrant replacement which is a transfinite composition of such pushouts starting out with $X = spec(k)$. We note that in this case we have the following commuting diagrams of pushout squares:

Now the left horizontal maps in the diagram are injective cofibrations in $SSet_\bullet^{\mathcal{C}^{op}}$, so by [11] 13.5.4, it suffices to see that the maps between the amalgamations, a, is a homotopy weak equivalence. This follows from the fact that $\mathcal{U}\mathcal{F}_{\mathbf{T}} \left((B \wedge \mathbb{A}^1_+) \amalg_{f(A)} \bullet \right) \to \mathcal{U}\mathcal{F}_{\mathbf{T}}(\bullet)$ is an $\mathbb{A}^1$ weak equivalence, $\mathcal{U}\mathcal{F}_{\mathbf{T}}(\bullet) \to \underline{Hom}(A, A \wedge \bullet)$ is an isomorphism by Proposition III.22, and $\underline{Hom}(A, A \wedge \left((B \wedge \mathbb{A}^1_+) \amalg_{f(A)} \bullet \right)) \to \underline{Hom}(A, A \wedge \bullet)$ is an $\mathbb{A}^1$ weak equivalence.

Since $\mathcal{U}\mathcal{F}_{\mathbf{T}}$ and $\underline{Hom}(A, \bullet)$ commute over coproducts we have now that $\mathcal{U}\mathcal{F}_{\mathbf{T}}(P_2) \to$ $\underline{Hom}(A, A \wedge P_2)$ is an $\mathbb{A}^1$ weak equivalence. Transfinite induction now implies $\underline{Hom}(A, A \wedge \tilde{Q}(X))$ is weak equivalent to $\mathcal{U}\mathcal{F}_{\mathbf{T}}(\tilde{Q}(X))$ as required. $\qquad\square$

C. Free Cofibrant Replacement in $\mathbf{Alg_T}$.

In this section we introduce a free cofibrant replacement functor that will be useful in questions concerning the localization of the model structure of $\mathbf{Alg_T}$. In particular, we shall show how to replace a cofibrant X in $\mathbf{Alg_T}$ with an object $(FU)_\bullet(X)$ which is cofibrant and flasque weak equivalent to X. This replacement will be a simplicial object in $\mathbf{Alg_T}$ such that each simplicial level is the image under $\mathcal{F}_{\mathbf{T}}$ of an object in $SSet_\bullet^{\mathcal{C}^{op}}$. In this way we will show how to replace cofibrant objects with free objects.

The results of this section are fairly general in contrast to the results of the chapter which often apply only to the case where $\mathcal{C}$ is a small site with an interval.

First, let $\epsilon : \mathcal{F}_{\mathbf{T}}\mathcal{U}(X) \to X$ be the counit of the adjunction $(\mathcal{F}_{\mathbf{T}}, \mathcal{U})$ and let $\eta : X \to \mathcal{U}\mathcal{F}_{\mathbf{T}}(X)$ be the unit. Define $\delta := \mathcal{F}_{\mathbf{T}}\eta\mathcal{U}$.

Definition IV.12. Let $\tilde{X} \in \mathbf{Alg_T}$, we define $(FU)_\bullet(\tilde{X}) \in \mathbf{Alg_T}$ by $(FU)_k(\tilde{X}) := (\mathcal{F}_{\mathbf{T}}\mathcal{U})^{k+1}(\tilde{X}_k)$, that is the diagonal of the bisimplicial T-model given by applying $\mathcal{F}_{\mathbf{T}}\mathcal{U}$ to $\tilde{X}$ $k + 1$ times for each $k \geq 0$. Face and degeneracy maps are given by

$$(FU_k\tilde{X} \xrightarrow{d_i} FU_{k-1}\tilde{X}) := (\mathcal{F}_{\mathbf{T}}\mathcal{U})^i\epsilon(\mathcal{F}_{\mathbf{T}}\mathcal{U})^{k-1}d_i : (\mathcal{F}_{\mathbf{T}}\mathcal{U})^{k+1}(\tilde{X}_k) \longrightarrow (\mathcal{F}_{\mathbf{T}}\mathcal{U})^k(\tilde{X}_{k-1})$$

and

$$(FU_k\tilde{X} \xrightarrow{s_i} FU_{k+1}\tilde{X}) := (\mathcal{F}_{\mathbf{T}}\mathcal{U})^i\eta(\mathcal{F}_{\mathbf{T}}\mathcal{U})^{k-1}s_i : (\mathcal{F}_{\mathbf{T}}\mathcal{U})^{k+1}(\tilde{X}_k) \longrightarrow (\mathcal{F}_{\mathbf{T}}\mathcal{U})^{k+2}(\tilde{X}_{k+1}).$$

This definition is modeled after a cofibrant type replacement given in [3]. Note that $(FU)_\bullet(X)$ comes equipped with a canonical simplicial map $\varphi : (FU)_\bullet(X) \to X$.

Define $\varphi : (FU)_\bullet(X) \to X$ by the maps $\epsilon^{k+1} : (FU)_k(X) \to X_k$.

We now consider some of the properties of the $(FU)_\bullet$ construction. The following two lemmas will depend on a convenient fact about free objects in $\mathbf{Alg_T}$ and so we will state it here.
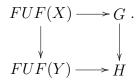
Claim IV.13. If $A \in SSet_\bullet^{\mathcal{C}^{op}}$ and $G \in \mathbf{Alg_T}$ then the morphism $\mathcal{F_T}(A) \to G$ is completely determined by the image of A in $\mathcal{U}(G)$ under the composition $A \to \mathcal{UF_T}(A) \to \mathcal{U}(G)$.

Proof. This is a Yoneda type argument. The image of A in $\mathcal{UF_T}(A)$ can be thought of as sending $x \in A(U)$ to $id_{A \wedge U_+} \wedge x$. It is clear now that the functorial properties of G determine the image of $f \wedge x$ for $f : A \wedge V_+ \to A \wedge U_+$. Similarly, the image of elements of the form $id_{A \wedge U_+} \wedge x$ determine the image of elements with representatives $id_{\vee_n A \wedge U_+} \wedge \prod_n x_i$ which completes the argument for $\mathcal{UF_T}(A)$. The product properties of algebras allow one to show that this in fact determines the entire morphism $\mathcal{F_T}(A) \to G$. $\square$

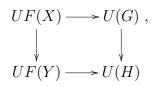**Lemma IV.14.** *Let X be a cofibrant object in $\mathbf{Alg_T}$, then $(FU)_\bullet(X)$ is Reedy cofibrant.*

Proof. By Theorem III.28 we have that $\mathbf{Alg_T}$ is a cofibrantly generated model category and so every cofibrant object is the retract of a relative I-cell complex, with I the set of generating I cofibrations. To prove that $(FU)_\bullet$ takes cofibrant objects to cofibrant objects, we start with a preliminary result about cofibrations. It will be convenient to adopt a simpler notation here, namely $F := \mathcal{F_T}$ and $U := \mathcal{U}$, for our following arguments.

Suppose that $X \to Y$ is a cofibration in $SSet_\bullet^{\mathcal{C}^{op}}$. Since F preserves cofibrations, we have that $F(X) \to F(Y)$ is a cofibration in the category of $\mathbf{Alg_T}$. We wish to show that in fact, $FUF(X) \to FUF(Y)$ is a cofibration as well. Suppose that $G \to H$ is

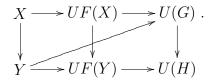an acyclic fibration in $\mathbf{Alg_T}$ and we have a commutative diagram:

$$
\begin{array}{ccc}
FUF(X) & \longrightarrow & G \\
\downarrow & & \downarrow \\
FUF(Y) & \longrightarrow & H
\end{array}
\quad .
$$

We know by adjunction that is is equivalent to a diagram in $SSet_{\bullet}^{\mathcal{C}^{op}}$

$$
\begin{array}{ccc}
UF(X) & \longrightarrow & U(G) \\
\downarrow & & \downarrow \\
UF(Y) & \longrightarrow & U(H)
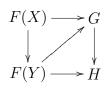\end{array}
\quad ,
$$

and by applying the counit we have a commutative diagram

$$
\begin{array}{ccccc}
X & \longrightarrow & UF(X) & \longrightarrow & U(G) \\
\downarrow & & \downarrow & & \downarrow \\
Y & \longrightarrow & UF(Y) & \longrightarrow & U(H)
\end{array}
\quad .
$$

By definition we know that the arrow $U(G) \to U(H)$ is an acyclic fibration and since $X \to Y$ is a cofibration we have a lift $Y \to U(G)$ such that the following diagram commutes:

$$
\begin{array}{ccccc}
X & \longrightarrow & UF(X) & \longrightarrow & U(G) \\
\downarrow & & \downarrow & \nearrow & \downarrow \\
Y & \longrightarrow & UF(Y) & \longrightarrow & U(H)
\end{array}
\quad .
$$

Once again, by adjunction we have a commutative diagram given by

$$
\begin{array}{ccc}
F(X) & \longrightarrow & G \\
\downarrow & \nearrow & \downarrow \\
F(Y) & \longrightarrow & H
\end{array}
$$

which induces a commutative diagram

$$\begin{array}{ccc} UF(X) & \longrightarrow & U(G) \\ \downarrow & \nearrow & \downarrow \\ UF(Y) & \longrightarrow & U(H) \end{array} \ .$$

It follows from the definition of the isomorphism of the adjunction that we may combine this data with our previous diagrams to obtain a commutative diagram:

$$\begin{array}{ccccc} X & \longrightarrow & UF(X) & \longrightarrow & U(G) \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ Y & \longrightarrow & UF(Y) & \longrightarrow & U(H) \end{array}$$

which one final application of adjunction yields:

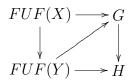$$\begin{array}{ccc} FUF(X) & \longrightarrow & G \\ \downarrow & \nearrow & \downarrow \\ FUF(Y) & \longrightarrow & H \end{array}$$

as required by the claim. We may iterate this argument to show that $(FU)^k F(X) \to (FU)^k F(Y)$ is also a cofibration whenever $X \to Y$ is.

Similar to the above argument, given a commutative diagram

$$\begin{array}{ccc} (FU)^k F(X) & \longrightarrow & G \\ \downarrow & & \downarrow \\ (FU)^k F(Y) & \longrightarrow & H \end{array}$$

one has

$$\begin{array}{ccc} U(FU)^{k-1} F(X) & \longrightarrow & U(G) \\ \downarrow & & \downarrow \\ U(FU)^{k-1} F(Y) & \longrightarrow & U(H) \end{array}$$

which by iterative application of the counit yields

$$
\begin{array}{ccccc}
X & \longrightarrow & U(FU)^{k-1}F(X) & \longrightarrow & U(G) \\
\downarrow & & \downarrow & & \downarrow \\
Y & \longrightarrow & U(FU)^{k-1}F(Y) & \longrightarrow & U(H)
\end{array}
.
$$

Again the outer square admits a lift which induces a map from $F(Y)$, which induces a map from $UF(Y)$ which induces a map from $FUF(Y)$, and so on; until one eventually has a map $U(FU)^{k-1}F(Y)$ and then the remainder of the argument is identical.

Since the generating cofibrations of $\mathbf{Alg_T}$ are given by the images of the generating cofibrations in $SSet_\bullet^{\mathcal{C}^{op}}$; we have that $(FU)^k$ sends generating cofibrations to cofibrations, and, consequently, pushouts of generating cofibrations to cofibrations as both F and U commute with colimits. Since $(FU)^k$ are functors they preserve retracts and since cofibrations are closed under retracts we have that $(FU)^k(X)$ is cofibrant for $k \geq 0$. $\qquad \square$

Lemma IV.15. *If $X \in \mathbf{Alg_T}$ is cofibrant, then so is $(FU)_\bullet(X)$.*

Proof. To show that $(FU)_\bullet(X)$ is cofibrant for cofibrant X we must demonstrate that the structural maps of the simplicial set $(FU)_\bullet(X)$ respect the maps induced by the functorial lifts for the images of cofibrant maps under $(FU)^k, \ \forall k \geq 0$. We proceed as above. Consider a commutative diagram of the form:

$$
\begin{array}{ccc}
(FU)_\bullet(\mathcal{F}_\mathbf{T}(A)) & \longrightarrow & G \\
\downarrow & & \downarrow \\
(FU)_\bullet(\mathcal{F}_\mathbf{T}(B)) & \longrightarrow & H
\end{array}
,
$$

where $A \to B$ is a cofibration in $SSet_\bullet^{\mathcal{C}^{op}}$ and $G \to H$ is an acyclic fibration in $\mathbf{Alg_T}$. Just as we can construct a simplicial map $(FU)_\bullet(\mathcal{F}_\mathbf{T}(A)) \to \mathcal{F}_\mathbf{T}(A)$, as in the iterative argument above we may construct maps $\mathcal{F}_\mathbf{T}(A_k) \to (FU)^{k+1}(\mathcal{F}_\mathbf{T}(A_k))$, and

so combining these maps we have a map $\mathcal{F}_{\mathbf{T}}(A) \to (FU)_{\bullet}(\mathcal{F}_{\mathbf{T}}(A))$. We must check that this indeed a simplicial map. This follows trivially from the functorial nature of the simplicial relationships between each $(FU)^{k+1}$, for example $(FU)^i \epsilon (FU)^{k-i}$, are the identity on the image of $\mathcal{F}_{\mathbf{T}}(A_k)$ in $(FU)_k(\mathcal{F}_{\mathbf{T}}(A))$. Since $\mathcal{F}_{\mathbf{T}}(A) \to \mathcal{F}_{\mathbf{T}}(B)$ is a cofibration and therefore admits a lift as in the diagram:

$$
\begin{array}{ccccc}
\mathcal{F}_{\mathbf{T}}(A) & \longrightarrow & (FU)_{\bullet}(\mathcal{F}_{\mathbf{T}}(A)) & \longrightarrow & G \ . \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{F}_{\mathbf{T}}(B) & \longrightarrow & (FU)_{\bullet}(\mathcal{F}_{\mathbf{T}}(B)) & \longrightarrow & H
\end{array}
$$

Since the map from $\mathcal{F}_{\mathbf{T}}(B) \to G$ induces a map on each simplicial level of $(FU)_{\bullet}(\mathcal{F}_{\mathbf{T}}(B))$, we need only determine that the lift respects the simplicial maps in $(FU)_{\bullet}(\mathcal{F}_{\mathbf{T}}(B))$. This is guaranteed by the fact that the simplicial maps of $(FU)_{\bullet}(X)$ are compositions of the functorial maps induced by the units and counits with the original simplicial maps of $\mathcal{F}_{\mathbf{T}}(B)$. We already know that the original simplicial maps of $\mathcal{F}_{\mathbf{T}}(B)$ commute with those of G by considering the lift to the outer rectangle, and the additional simplicial maps in $(FU)_{\bullet}(\mathcal{F}_{\mathbf{T}}(B))$ are identical to those of $\mathcal{F}_{\mathbf{T}}(A)$, which already commute with those of G and act as the identity on those of $\mathcal{F}_{\mathbf{T}}(B)$.

Again since $(FU)_{\bullet}$ commutes with pushouts and retracts, the result follows for all cofibrant X. $\qquad \square$

Theorem IV.16. *φ is a flasque weak equivalence.*

Proof. Flasque weak equivalences in $\mathbf{Alg_T}$ are determined by the functor $\mathcal{U}$. It is therefore necessary to show that $\mathcal{U}(\varphi)$ is a weak equivalence in $SSet_{\bullet}^{\mathcal{C}^{op}}$. To this end we note that a weak equivalence in $SSet_{\bullet}^{\mathcal{C}^{op}}$ with the flasque model structure is an objectwise weak equivalence, so we construct a homotopy inverse to $\mathcal{U}(\varphi)$, called ϕ, objectwise.

First we consider $\eta : Y \to \mathcal{U}\mathcal{F}_{\mathbf{T}}(Y)$. We note that η induces maps $\eta\mathcal{U} : \mathcal{U}(X) \to$

$\mathcal{U}\mathcal{F}_{\mathbf{T}}\mathcal{U}(X)$. We will often abuse notation and write η when $\eta\mathcal{U}$ is meant. In this way we construct a map $\mathcal{U}(X) \to \mathcal{U}((FU)_\bullet(X))$ given by $\eta^{k+1} : \mathcal{U}(X)_k = \mathcal{U}(X_k) \to \mathcal{U}(FU)_{k+1}(X_k)$. We assemble these maps to obtain ϕ.

Before proceeding it will be helpful to understand the maps η, ϵ, d_i, and s_i on representatives of elements in $(FU)_k(X)$. We consider representatives of the form $(f_1, \ldots, f_{q+1}, x_q)$, with each f_i such that $dom(f_{i+1}) = cod(f_i)$, by which we mean to represent $f_1 \wedge \cdots \wedge f_{q+1} \wedge x_k \in (FU)_{q+1}(X_q)$. Note since each X_q is an object of $\mathbf{Alg_T}$ (ignoring simplicial structure), it is sensible to regard $x_q \in \mathcal{U}(X_q)(c)$ as a map $A \wedge c_+ \to \mathcal{U}(X_q)$ for $c \in \mathcal{C}$. We shall exploit this fact in making sense of the following formulae:

$$d_i(f_1, \ldots, f_{q+1}, x_q) = (f_1, \ldots, f_{i+2}f_{i+1}, \ldots, f_{q+1}, d_i(x_q)) \qquad \text{(C.1)}$$

$$s_i(f_1, \ldots, f_{q+1}, x_q) = (f_1, \ldots, f_{i+1}, id_{dom(f_{i+2})}, f_{i+2}, \ldots, f_{q+1}, s_i(x_q)) \qquad \text{(C.2)}$$

$$\epsilon(f_1, x) = xf_1 \qquad \text{(C.3)}$$

$$\eta(f_1, \ldots, f_{q+1}, x_q) = (id, f_1, \ldots, f_{q+1}, x_q). \qquad \text{(C.4)}$$

Here we abuse notation by using d_i and s_i to refer to both the simplicial maps of the structure $(FU)_\bullet$ and of the simplicial $\mathbf{T}$-algebra X. We also introduce a new function η^0 which sends $(f_1, \ldots f_{q+1}, x_q)$ to $(id, f_1, \ldots, f_{q+1}, s_0(x_q))$.

From these equations it is clear that $\phi\varphi$ is the identity on X. We wish to show that $\varphi\phi$ is homotopy equivalent to the identity on $(FU)_\bullet(X)$. To see this we construct a simplicial homotopy (Definition 9.1 [18]) between the two morphisms.

Define $h_i : (FU)_k(X) \to (FU)_{k+1}(X)$ by the composition $s_0^i \eta^0 d_0^i$.

We first must see that $d_0 h_0$ is the identity on $(FU)_\bullet(X)$. Note that h_0 is just η^0 which sends $(f_1, \ldots f_{q+1}, x_q)$ to $(id, f_1, \ldots, f_{q+1}, s_0(x_q))$, but d_0 sends this back to

$(f_1 id, \ldots f_{q+1}, d_0 s_0(x_q)) = (f_1, \ldots f_{q+1}, x_q)$ as required.

We must also see that $d_{q+1} h_q$ yields $\varphi \phi$. Under h_q $(f_1, \ldots f_{q+1}, x_q)$ is sent to $(id_1, \ldots, id_{q+1}, f_{q+1} \cdots f_1, s_0^q s_0 d_0^q(x_q))$ where id_i denotes the ith copy of the identity map on the appropriate domain and $f_{q+1} \cdots f_1$ denotes composition. This representative is sent, under d_{q+1} to $(id_1, \ldots, id_{q+1}, d_{q+1} s_0^q s_0 d_0^q(x_q) f_{q+1} \cdots f_1)$, but since $d_{q+1} s_0^q s_0 d_0^q = d_{q+1} s_q s_0^q d_0^q = id$ the result follows.

Suppose now that $i < j$, then we must show that $d_i h_j = h_{j-1} d_i$. To see this we note that a representative $(f_1, \ldots, f_{q+1}, x_q)$ is sent to $(id_1, \ldots id_{j+1}, f_{j+1} \cdots f_1, f_{j+2}, \ldots, f_{q+1}, s_0^j s_0 \ d_0^j(x_q))$, under h_j, and then to $(id_1, \ldots, id_j, f_{q+1} \cdots f_1, \ldots, d_i s_0^j s_0 d_0^j(x_q))$ under d_i. On the other hand, it is first sent to $(f_1, \ldots, f_{i+2} f_{i+1}, f_{i+3}, \ldots, f_{q+1}, d_i(x_q))$ under d_i and then to $(id_1, \ldots id_j, f_{q+1} \cdots f_1, \ldots f_q, s_0^{j-1} s_0 d_0^{j-1} d_i(x_q))$ under h_{j-1}. Since $s_0^{j-1} s_0 d_0^{j-1} d_i = s_{j-1} d_i = d_i s_j$ the result follows.

If $i = j$ we must show that $d_i h_j = d_j h_{j-1}$. For the left hand side we have that $(f_1, \ldots, f_{q+1}, x_q)$ first goes to $(id_1, \ldots, id_{j+1}, f_{j+1} \cdots f_1, \ldots, s_0^j s_0 d_0^j(x_q))$ and then to $(id_1, \ldots, id_q, f_{j+1} \cdots f_1, \ldots d_j s_0^j s_0 d_0^j(x_q))$. On the right hand side it is first sent to $(id_1, \ldots, id_j, f_j \cdots f_1, f_{j+1}, \ldots, s_0^{j-1} s_0 d_0^{j-1}(x_q))$ and then to $(id_1, \ldots, id_j, f_{j+1} \cdots f_1, \ldots, d_j s_0^{j-1} s_0 d_0^{j-1}(x_q))$. As before the conclusion follows from the appropriate simplicial identities.

If $i > j + 1$ then we must show that $d_i h_j = h_j d_{i-1}$. This follows much as the above argument noting that since $i > j + 1$ we have that $i - 1 \geq j + 1$, which implies that function composition in the first $q+1$ terms of the representative will not overlap between h_j and d_i or d_{i-1}. The result then follows from an argument similar to the above and since $d_i s_j = s_j d_{i-1}$.

The last two simplicial relationships exist between s_i and h_j. If $i \leq j$ then we want $s_i h_j = h_{j+1} s_i$. On the righthand side, the composition sequence is given by $(id_1, \ldots, id_{j+1}, f_{j+1} \cdots f_1, \ldots, s_0^j s_0 d_0^j(x_q))$ and then $(id_1, \ldots id_{j+2}, f_{j+1} \cdots f_1, \ldots, s_i s_0^j)$

$s_0 d_0^j(x_q))$. On the left hand side the result is similar but with last term $s_0^{j+1} s_0 d_0^{j+1} s_i(x_q)$. The result follows from $s_i s_j = s_{j+1} s_i$ when $i \leq j$.

The last simplicial identity is now for $i > j$ and then $s_i h_j = h_j s_{i-1}$. The function composition does not overlap and the simplicial identity is the same as above.

This verifies that h is a simplicial homotopy on the level of $\mathcal{U}((FU)_\bullet(X))$ and since restriction to $c \in SSet_\bullet^{\mathcal{C}^{op}}$ merely identifies a choice of domain for the first element of the representative, which is preserved throughout all calculations, the theorem is proved. $\qquad\square$

D.   The Motivic Local Model Structure on $\mathbf{Alg_T}$

In this section when $\mathcal{C} = Sm_k$. Here we show how to localize $\mathbf{Alg_T}$ so that the homotopy theory corresponds to that of Morel and Voevodsky's $\mathbb{A}^1$-homotopy theory on Sm_k. Specifically, we wish to show that $\mathcal{U}$ induces a faithful embedding from the homotopy category of the localization of $\mathbf{Alg_T}$ to the $\mathbb{A}^1$-homotopy category. More precisely:

Theorem IV.17. *There exists a model structure on $\mathbf{Alg_T}$ such that if X, $Y \in \mathbf{Alg_T}$ then $\mathcal{U}(X) \simeq \mathcal{U}(Y)$, where $\simeq$ denotes $\mathbb{A}^1$-weak equivalence, implies that X is weak equivalent to Y in the homotopy theory on $\mathbf{Alg_T}$. Moreover, if f is a morphism in $\mathbf{Alg_T}$, then f is fibrant in $\mathbf{Alg_T}$ if and only if $\mathcal{U}(f)$ is fibrant in the motivic model structure of $SSet_\bullet^{\mathcal{C}^{op}}$.*

Whereas we have built the homotopy theory on $\mathbf{Alg_T}$ using the flasque model structure on $SSet_\bullet^{\mathcal{C}^{op}}$, we proceed as in [13] to localize $\mathbf{Alg_T}$ in a manner compatible with the localization of $SSet_\bullet^{\mathcal{C}^{op}}$ which yields a model structure Quillen equivalent to $\mathbb{A}^1$-homotopy theory.
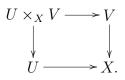
We first recall some definitions and results about flasque model localization.

Definition IV.18. [13] The *Local projective* (resp., *flasque, injective*) *model struc-ture* on $SSet_\bullet^{\mathcal{C}^{op}}$ is the left Bousfield localization (Definition II.27) of the objectwise projective (resp., flasque, injective) model structure at the class of all hypercovers.

For a precise definition of hypercovers see [1], [2], [8]. Restricting to the case of Sm_k, we need not consider all hypercovers for localization but instead may specialize to localizing the maps

$$U \amalg_{U \times_X V} V \to X$$

for all elementary Nisnevich squares ([19], Def 3.1.3)

$$
\begin{array}{ccc}
U \times_X V & \longrightarrow & V \\
\downarrow & & \downarrow \\
U & \longrightarrow & X.
\end{array}
$$

This yields:

Theorem IV.19. *[13] The local flasque (resp. injective) model structure on $SSet_\bullet^{\mathcal{C}^{op}}$ is the left Bousfield localization of the objectwise flasque (resp. injective) model struc-ture at the set consisting of maps*

$$U \amalg_{U \times_X V} V \to X$$

for all elementary Nisnevich squares.

That the local flasque model structure on $SSet_\bullet^{\mathcal{C}^{op}}$ is left proper, cellular, and simplicial is a natural consequence of left Bousfield localization[11]. It follows from [19] that the structures are also right proper.

We now wish to localize $\mathbf{Alg_T}$ in a compatible way. Namely, we will localize $\mathbf{Alg_T}$ at the class of all maps

$$U \amalg_{U \times_X V} V \to X$$

under the functor $\mathcal{F}_{\mathbf{T}}$ in the context of Sm_k. Since $\mathcal{F}_{\mathbf{T}}$ is a left adjoint the categorical properties of these maps remain intact and we may construct the left Bousfield localization of $\mathbf{Alg_T}$ with respect to these maps, denoted by $\mathbf{C}$.

Theorem IV.20. *The local flasque structure on $\mathbf{Alg_T}$ is the left Bousfield localization of the flasque structure on $\mathbf{Alg_T}$ at (the maps in) $\mathbf{C}$. This yields a left proper, cellular, simplicial model structure on $\mathbf{Alg_T}$.*
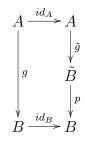
Proof. That the model structure is left proper, cellular, and simplicial as above follows from Theorem 4.1.1, [11]. □

It would be useful if fibrations were still detected by $\mathcal{U}$. Indeed, this is the case.

Proposition IV.21. *Let X and Y be in $\mathbf{Alg_T}$. $f : X \to Y$ is a fibration in the left Bousfield localization of $\mathbf{Alg_T}$ if and only if $U(f) : U(X) \to U(Y)$ is a fibration in the left Bousfield localization of $SSet_{\bullet}^{\mathcal{C}^{op}}$.*

Proof. We recall from Chapter 4 of [11] that the local acyclic cofibrations are given by representatives of the isomorphism classes of inclusions of subcomplexes that are S-local equivalences of complexes, where S is the class of maps defining the localization. It is clear that if $J^S_{SSet_{\bullet}^{\mathcal{C}^{op}}}$ denotes the acyclic cofibrations in the localization of $SSet_{\bullet}^{\mathcal{C}^{op}}$ that $\mathcal{F}_{\mathbf{T}}(J^S_{SSet_{\bullet}^{\mathcal{C}^{op}}})$ can be taken as a subset (or rather the representatives thereof) of $J^S_{\mathbf{Alg_T}}$. If $U(f)$ is a fibration then f has the right lifting property with respect to all maps in $\mathcal{F}_{\mathbf{T}}(J^S_{SSet_{\bullet}^{\mathcal{C}^{op}}})$ and we wish to show this implies f has the right lifting property with respect to maps in $J^S_{\mathbf{Alg_T}}$.

To see this consider a map $g \in J^S_{\mathbf{Alg_T}}$. We construct a factorization diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{\ id_A\ } & A \\
 & & \downarrow{\scriptstyle \tilde{g}} \\
\downarrow{\scriptstyle g} & & \tilde{B} \\
 & & \downarrow{\scriptstyle p} \\
B & \xrightarrow{\ id_B\ } & B
\end{array}
$$

where $\tilde{g}$ is the transfinite compositions of commutative diagrams over the maps $\mathcal{F}_{\mathbf{T}}(J^S_{SSet^{\mathcal{C}^{op}}_\bullet})$. We note that $J^S_{SSet^{\mathcal{C}^{op}}_\bullet}$ contains isomorphism classes of the generating cofibrations of the original model structure (before localization) of $SSet^{\mathcal{C}^{op}}_\bullet$ and so $\mathcal{F}_{\mathbf{T}}(J^S_{SSet^{\mathcal{C}^{op}}_\bullet})$ contains isomorphism classes of the generating cofibrations of $\mathbf{Alg_T}$. By the small object argument it follows that p has the right lifting property with respect to all maps in $\mathcal{F}_{\mathbf{T}}(J^S_{SSet^{\mathcal{C}^{op}}_\bullet})$ and so is an fibration in the original model structure, it also happens to be a local weak equivalence since $\tilde{g}$ is the pushout of local acyclic cofibrations. Since local acyclic fibrations are equal to the original class of fibrations, we have that $\tilde{B}$ is a retract of B. Hence it suffices to show that a local fibration has the right lifting property with respect to $\tilde{g}$. Since $\tilde{g}$ is the transfinite composition of pushouts of objects in $\mathcal{F}_{\mathbf{T}}(J^S_{SSet^{\mathcal{C}^{op}}_\bullet})$, this follows immediately. $\qquad\square$

Remark IV.22. This proof also contains the further result that maps in $J^S_{\mathbf{Alg_T}}$ are retracts of maps in $\mathcal{F}_{\mathbf{T}}(J^S_{SSet^{\mathcal{C}^{op}}_\bullet})$.

Theorem IV.23. *If f is a morphism in $\boldsymbol{Alg_T}$ such that $\mathcal{U}(f)$ is a local weak equivalence in $SSet^{\mathcal{C}^{op}}_\bullet$, then f is a local weak equivalence in $\boldsymbol{Alg_T}$.*

Proof. Let Z be $\mathbf{C}$-local in $\mathbf{Alg_T}$. A morphism $f : X \to Y$, such that $\mathcal{U}(f)$ is a local weak equivalence, induces a map $(FU)_\bullet(X) \to (FU)_\bullet(Y)$. Define $U_\bullet(X)$ as the simplicial object in $SSet^{\mathcal{C}^{op}}_\bullet$ with k^{th} level given by $\mathcal{U}(\mathcal{F}_{\mathbf{T}}\mathcal{U})^k(X_k)$. Then we note $(FU)_\bullet(X) = \mathcal{F}_{\mathbf{T}}(U_\bullet(X))$. We also note that $\mathcal{U}(FU)_\bullet(f)$ is a local weak equivalence

by the commutativity of the diagram:

$$\begin{array}{ccc} \mathcal{U}(FU)_{\bullet}(X) & \longrightarrow & \mathcal{U}(X) \\ \downarrow & & \downarrow \\ \mathcal{U}(FU)_{\bullet}(Y) & \longrightarrow & \mathcal{U}(Y). \end{array}$$

and Theorem IV.16. We now have weak equivalences of homotopy function complexes

$$map(X, Z) \simeq map(\mathcal{F}_{\mathbf{T}}(U(X)), Z)$$

and

$$map(Y, Z) \simeq map(\mathcal{F}_{\mathbf{T}}(U(Y)), Z).$$

We also have isomorphisms

$$map(\mathcal{F}_{\mathbf{T}}(U_{\bullet}(X)), Z) \simeq map(U_{\bullet}(X), \mathcal{U}(Z)) \simeq map(\mathcal{U}\mathcal{F}_{\mathbf{T}}(U_{\bullet}(X)), \mathcal{U}(Z))$$

and

$$map(\mathcal{F}_{\mathbf{T}}(U_{\bullet}(X)), Z) \simeq map(U_{\bullet}(X), \mathcal{U}(Z)) \simeq map(\mathcal{U}\mathcal{F}_{\mathbf{T}}(U_{\bullet}(X)), \mathcal{U}(Z)),$$

by Claim C. The result follows from the fact that $\mathcal{U}(FU)_{\bullet}(f)$ is still a local weak equivalence and the fact that $\mathcal{U}(Z)$ is local in $SSet_{\bullet}^{\mathcal{C}^{op}}$. $\qquad\square$

This yields a local flasque structure on $\mathbf{Alg_T}$, compatible with the local flasque structure on $SSet_{\bullet}^{\mathcal{C}^{op}}$ in precisely the way we want: Homotopy on $\mathbf{Alg_T}$ is determined by the homotopy type of the underling $SSet_{\bullet}^{\mathcal{C}^{op}}$. It will be important to subsequent discussion to see that Ω^T and B_T are still Quillen adjoint.

Proposition IV.24. *With the local flasque model structure on $\mathbf{Alg_T}$,*

$$B_T : \mathbf{Alg_T} \leftrightarrow SSet_{\bullet}^{\mathcal{C}^{op}} : \Omega^T$$

is a Quillen adjoint pair.

Proof. Since cofibrations are identical for left Bousfield localizations to those of the original model structure Theorem 3.3.1, [11], we note B_T preserves cofibrations.

To show that B_T preserves acyclic cofibrations we start with the preliminary claim that given a local object $Z \in SSet_\bullet^{\mathcal{C}^{op}}$, that $\underline{Hom}(A, Z)$ remains local. From [11] Proposition 4.2.5 we have that local objects are characterized by a lifting property with respect to the union of generating trivial cofibrations and a collection

$$\Lambda(\mathbf{C}) := \{\tilde{A} \otimes \Delta[n] \amalg_{\tilde{A} \otimes \partial\Delta[n]} \tilde{B} \otimes \partial\Delta[n] \to \tilde{B} \otimes \Delta[n] | (A \to B) \in \mathbf{C}, n \geq 0\},$$

where $\mathbf{C}$ is the class of maps over which the left Bousfield localization is taken and $\tilde{A} \to \tilde{B}$ denotes a cosimplicial resolution of f (see [11]). In our case $\mathbf{C}$ is the image under $\mathcal{F}_\mathbf{T}$ of the hypercovers and this is closed under taking smash products since distinguished squares are. Cofibrant resolutions remain cofibrant resolutions under smash products with cofibrant objects, as do generating flasque acyclic cofibrations in $SSet_\bullet^{\mathcal{C}^{op}}$ which proves the claim.

Whereas, by Remark IV.22, and the fact that B_T preserves pushouts, transfinite compositions, and retracts as a left adjoint functor it suffices to consider only the images of elements of $\mathcal{F}_\mathbf{T}(J^S_{SSet_\bullet^{\mathcal{C}^{op}}})$. The composition $B_T\mathcal{F}_\mathbf{T}$, however, is equivalent to taking the smash product with A and so we see that B_T sends $\mathcal{F}_\mathbf{T}(J^S_{SSet_\bullet^{\mathcal{C}^{op}}})$ to local weak equivalences, as above, and the result follows. $\square$

We now introduce the motivic model structure on $SSet_\bullet^{\mathcal{C}^{op}}$ which is simply another left Bousfield localization.

Definition IV.25. [13] Let $spec(k) \to \mathbb{A}^1$ be the morphism induced by $k[x] \to k$, given by evaluation at 0. The motivic projective (resp., flasque, injective) model structure on $SSet_\bullet^{\mathcal{C}^{op}}$ is the left Bousfield localization of the local projective (resp.,

flasque, injective) model structure at the set of maps $\mathbf{S}$ given by $X \to X \times \mathbb{A}^1$ induced from the inclusion $spec(k) \to \mathbb{A}^1$ for all $X \in SSet_{\bullet}^{\mathcal{C}^{op}}$.

In this way we recover a homotopy theory equivalent to the Morel-Voevodsky $\mathbb{A}^1$-homotopy theory.

As before we may localize $\mathbf{Alg_T}$ with respect to the maps $\mathcal{F}_{\mathbf{T}}(\mathbf{S})$. Since the proof of Theorem IV.20 does not depend on the maps being localized nor the specific properties of the model structure being localized the result still holds.

Theorem IV.26. *Let $\mathbf{Alg_T}$ have the motivic model structure described above, then $\mathbf{Alg_T}$ is a left proper, cellular, simplicial model category.*

Similarly, we also have Proposition IV.24, since the maps localized in $SSet_{\bullet}^{\mathcal{C}^{op}}$ are closed under products with A. Proposition IV.21 and Theorem IV.23 also are independent of the maps being localized. We shall call this the *Motivic Model Structure* on $\mathbf{Alg_T}$.

We state each of these results for referential purposes.

Proposition IV.27. *Let $\mathbf{Alg_T}$ and $SSet_{\bullet}^{\mathcal{C}^{op}}$ have the motivic local model structure described above. Then*

$$B_T : \mathbf{Alg_T} \leftrightarrow SSet_{\bullet}^{\mathcal{C}^{op}} : \Omega^T$$

are still and adjoint pair.

Proposition IV.28. *Let $\mathbf{Alg_T}$ and $SSet_{\bullet}^{\mathcal{C}^{op}}$ have the motivic local model structure, then a map $f : X \to Y$ in $\mathbf{Alg_T}$ is a fibration if and only if $\mathcal{U}(f) : \mathcal{U}(X) \to \mathcal{U}(Y)$ is a fibration in $SSet_{\bullet}^{\mathcal{C}^{op}}$.*

Proposition IV.29. *Let f be a morphism in $\mathbf{Alg_T}$ such that $\mathcal{U}(f)$ is a motivic weak equivalence, then f is a motivic weak equivalence in $\mathbf{Alg_T}$.*

Combining these results, Theorem IV.2 implies the following:

Theorem IV.30. *For $X \in SSet_{\bullet}^{\mathcal{C}^{op}}$ there is a motivic weak equivalence of $\mathbf{T}-models$*

$$\mathcal{F}_{\mathbf{T}}(X) \simeq \Omega^T(A, A \wedge \tilde{X}),$$

where $\tilde{X}$ is a fibrant replacment of X whenever X is cofibrant in $SSet_{\bullet}^{\mathcal{C}^{op}}$.

E. $\mathbf{Alg_T}$ and a Right Localization of $SSet_{\bullet}^{\mathcal{C}^{op}}$

The purpose of this section is to outline the construction of another localization of the motivic flasque $SSet_{\bullet}^{\mathcal{C}^{op}}$ using the adjoint functors Ω^T and B_T and to compare this model structure with the motivic model structure on $\mathbf{Alg_T}$. Naturally, we will be specifically be working in the context of simplicial presheaves over Sm_k and will choose $A := \mathbb{P}^1$ pointed at infinity.

We start by localizing $SSet_{\bullet}^{\mathcal{C}^{op}}$. Let $\mathbf{R} := \{f : X \to Y | \Omega(f) : \Omega(A, A \wedge \tilde{X}) \to \Omega(A, A \wedge \tilde{Y})$ is a motivic homotopy equivalence and $\tilde{X}$ and $\tilde{Y}$ are motivic fibrant replacements of X and Y respectively$\}$.

Definition IV.31. The right Bousfield localization of $SSet_{\bullet}^{\mathcal{C}^{op}}$ with respect to $\mathbf{R}$ will be denoted $\mathcal{LC}$.

Right Bousfield Localization assumes the use of a homotopy function complex as in Definition II.25. As a matter of convention we will assume that when localizing a model structure $\mathcal{M}$ for a category $\mathcal{C}$ that the homotopy function complex $map(W, A)$ will be giveN by a cofibrant replacement of W, say $\tilde{W}$ and a fibrant replacement of A, denoted $Ex^{\mathcal{M}}(A)$ as the simplicial mapping space:

$$map(W, A) := Hom_{\mathcal{C}}(\tilde{W} \otimes \Delta_{\bullet}, Ex^{\mathcal{M}}(A)),$$

where as a matter of convention if either W or A is already cofibrant or fibrant

respectively, then we may avoid using the replacement functors, specifically we make the following remark.

Remark IV.32. If X is cofibrant already, then a sufficient cosimplicial resolution is to take $\tilde{X} = X \times \Delta_{\bullet}$ [11], 16.1.3.

Also, if we already have a simplicial model category we may exploit the simplicial structure in the following way:

Remark IV.33. In the case of a simplicial model category, such as the motivic structure on $SSet_{\bullet}^{\mathcal{C}^{op}}$, we may take the function complex to simply be the internal simplicial mapping space.

Since we are frequently concerned with smash products in $SSet_{\bullet}^{\mathcal{C}^{op}}$ and $\mathbf{Alg_T}$, the following lemma will be useful.

Lemma IV.34. *$A \wedge U_+$ is a colocal object in $\mathcal{LC}$ for $U \in \mathcal{C}$.*

Proof. This follows from the fact that weak equivalences between fibrant objects can be factored as a trivial cofibration, which has a trivial fibration as a homotopy inverse, and a trivial fibration; the fact that motivic trivial fibrations are simplicial trivial fibrations objectwise; and the definition of $\mathbf{R}$. $\qquad\square$

As per the results of Section C, $\mathcal{LC}$ is a rightproper model structure for $SSet_{\bullet}^{\mathcal{C}^{op}}$. As we will be working with several model structures concurrently in the remaining arguments, we adopt the following convention.

Notation IV.35. The motivic model structure on $SSet_{\bullet}^{\mathcal{C}^{op}}$ will be denoted $\mathcal{MC}$. The localization described above will, as already remarked, be denoted $\mathcal{LC}$. It shall seldom be necessary to make such distinctions for $\mathbf{Alg_T}$, but unless noted otherwise we will assume it has the motivic local structure described above. Unless otherwise noted

fibrant replacement will always refer to fibrant replacement in the motivic model structure on either $SSet_\bullet^{\mathcal{C}^{op}}$ or $\mathbf{Alg_T}$.

Theorem IV.36. *Let $\mathbf{Alg_T}$ have the homotopy theory induced by the forgetful functor $\mathcal{U}$, then*

$$B_T : \mathbf{Alg_T} \leftrightarrow \mathcal{LC} : \Omega^T$$

form a Quillen adjoint pair.

To prove this we need a lemma.

Lemma IV.37. $B_T \mathcal{F_T}$ *sends generating projective cofibrations to cofibrations in $\mathcal{LC}$.*

Proof. This follows from the fact that $B_T \mathcal{F_T}$ of a generating projective cofibration $U_+ \wedge \partial\Delta[n] \to U_+ \wedge \Delta[n]$ is simply the smash product with A, the fact that $A \wedge U_+$ is colocal, Definition 5.2.1, and Theorem 5.1.1 in [11]. □

Proof. Since the fibrations of $\mathcal{LC}$ are identical to the fibrations of $\mathcal{MC}$, we note that Proposition IV.27 implies that Ω^T still preserves fibrations. It will be sufficient to show that B_T of a cofibration is still a cofibration. This reduces to showing that for a cofibration $f : B \to C$ in $\mathbf{Alg_T}$ that $B_T(f) : B_T(B) \to B_T(C)$ is a cofibration in $\mathcal{LC}$. This is equivalent to showing that $B_T(f)$ has the left lifting property with respect to acyclic fibration in $\mathcal{LC}$. Let $g : X \to Y$ be such an acyclic fibration.

By the definition of the model structure on $\mathcal{LC}$ we note that if $\tilde{X}$ and $\tilde{Y}$ are fibrant replacements for X and Y respectively in $\mathcal{MC}$, then $\tilde{g} : \underline{Hom}(A, \tilde{X}) \to \underline{Hom}(A, \tilde{Y})$ is an acyclic fibration by the proof to Theorem IV.26 and Propositions IV.28 and IV.27, in $\mathcal{MC}$. Again by the proof to Theorem IV.26 and Proposition IV.28, $\Omega^T(\tilde{g})$ is a trivial fibration in $\mathbf{Alg_T}$.

We construct the projective cofibrant replacement of f in $\mathbf{Alg_T}$ in the usual way:

$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\ i\ } & B \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \tilde{C} & \xrightarrow{\ p\ } & C. \end{array}$$

Here we find $\tilde{B}$ by taking the projective cofibrant replacement of B with $i : \tilde{B} \to B$ a projective trivial fibration, which is still a flasque weak equivalence in $\mathbf{Alg_T}$. $p\tilde{f}$ is the functorial factorization of fi. Since B_T commutes with pushouts, transfinite composition, and retracts Lemma $IV.37$ implies that $\tilde{f}$ as a projective cofibration is sent to a cofibration in $\mathcal{LC}$ under B_T. Now $\tilde{g}$ is a fibrant replacement of the fibration g and so by Theorem 13.2.1(2) [11], it follows that $B_T(\tilde{f})$ has the right lifting property to g if and only if it has the right lifting property to $\tilde{g}$. $B_T(\tilde{f})$, however, has the right lifting property if and only if $\tilde{f}$ has the right lifting property with respect to $\Omega^T(\tilde{g})$, which follows since $\tilde{f}$ is a cofibration in $\mathbf{Alg_T}$. On the other hand, by the same argument, f has the right lifting property with respect to Ωg if and only if $\tilde{f}$ has the property.

Hence, f has the right lifting property with respect to Ωg and so $B_T(f)$ has the right lifting property with respect to g as required. $\qquad\square$

Our central result is that this Quillen adjunction is actually a Quillen equivalence. Consequently, the homotopy category of $\mathbf{Alg_T}$ is homotopy equivalent to the homotopy category of the right Bousfield localization of $SSet_{\bullet}^{\mathcal{C}^{op}}$, which is the proper context for considering loop spaces.

Notation IV.38. Up until this point we have been happily abusing notation using $\tilde{X}$ to denote either the cofibrant or fibrant approximation to X depending on context. Now let $Ex^{\infty}(X)$ denote the flasque fibrant replacement of X and let $\tilde{X}$ denote the flasque cofibrant (or motivic cofibrant) replacement of X.

Theorem IV.39. *The functors $B_T : \mathbf{Alg_T} \leftrightarrow \mathcal{LC} : \Omega$ form a Quillen equivalence.*

Proof. We start by proving that if $X \in \mathbf{Alg_T}$ is cofibrant, then $X \overset{\eta}{\to} \Omega^T B_T(X)$ is an $\mathbb{A}^1$ weak equivalence in $\mathbf{Alg_T}$. To this end we invoke the cofibrant replacement described in Definition V.1. In this replacement recall that we may write $(FU)_\bullet(X) = \mathcal{F_T}(U_\bullet(X))$, where we define $U_\bullet(X)_k := \mathcal{U}(\mathcal{F_T U})^k(X_k)$ and have the following diagram with commuting square:

$$\mathcal{F_T}(U_\bullet(X)) \overset{\tilde{\eta}}{\longrightarrow} \Omega^T B_T \mathcal{F_T}(U_\bullet(X)) \overset{p}{\longrightarrow} \Omega^T B_T \mathcal{F_T}(Ex^{\mathcal{M}}(U_\bullet(X)))$$

with a the arc over the top, and

$$X \overset{\eta}{\longrightarrow} \Omega^T B_T(X)$$

Here a is the $\mathbb{A}^1$ weak equivalence of Theorem IV.30. Since $U_\bullet(X) \to Ex^{\mathcal{M}}(U_\bullet(X))$ is a trivial cofibration in the flasque model structure on $SSet_\bullet^{\mathcal{C}^{op}}$, we see $\mathcal{F_T}(U_\bullet(X)) \to \mathcal{F_T}(Ex^{\mathcal{M}}(U_\bullet(X)))$ is a cofibrant weak equivalence in the flasque model structure on $\mathbf{Alg_T}$. Applying B_T to this yields yet another flasque cofibrant weak equivalence in $SSet_\bullet^{\mathcal{C}^{op}}$.

Now it follows from Lemma III.29 that for simplicially constant presheaves if $B \to C$ is a weak equivalence then $\underline{Hom}(A, B) \to \underline{Hom}(A, C)$ is a flasque weak equivalence, and so p is an $\mathbb{A}^1$-weak equivalence, by Theorem 2.14, [19]. By the 2 of 3 property of weak equivalences we have that $\tilde{\eta}$ is an $\mathbb{A}^1$ weak equivalence.

The leftmost vertical arrow is a flasque weak equivalence by Theorem IV.16. The rightmost arrow follows again from the fact that B_T preserves flasque weak equivalences between cofibrant objects as a left adjoint. Whereas flasque weak equivalences are object-wise simplicial weak equivalences, applying Ω^T is still a flasque weak equivalence (see proof of Lemma 2.3.2) and so an $\mathbb{A}^1$-weak equivalence. The 2 of 3 principle now implies η is a weak equivalence.

We can now prove the theorem. Let X be cofibrant in $\mathbf{Alg_T}$ and Y be fibrant in $\mathcal{LC}$. We wish to show $f : B_T(X) \to Y$ is a weak equivalence in $\mathcal{LC}$ if and only if adjoint $f^\sharp : X \to \Omega^T Y$ is a weak equivalence in $\mathbf{Alg_T}$.

Noting that $\Omega^T(f)\eta = f^\sharp$, we see it suffices to show that $\Omega^T(f)$ is a weak equivalence if and only if f is. $\Omega^T(f)$ being a weak equivalence implies that f is a weak equivalence in $\mathcal{LC}$ by the definition of our right Bousfield localization; so it suffices to see that when f is a local weak equivalence, $\Omega^T(f)$ is a weak equivalence in $\mathbf{Alg_T}$. Since X is cofibrant, we may take a free replacement of X as in Definition V.1. Since this has an associated a homotopy equivalence, we see the homotopy weak equivalence is preserved under Ω^T. Finally, whereas the composition of B_T and $\mathcal{F_T}$ is just the smash product with A, it suffices to prove the result for the case in which $A \wedge x \to Y$ is a right local weak equivalence.

We make the following two claims: First, if $\tilde{x}$ is the fibrant replacement of cofibrant x, then $A \wedge \tilde{x}$ is fibrant; second, $\Omega^T(A \wedge x) \to \Omega^T(A \wedge \tilde{x})$ is a weak equivalence. If these claims are true, then $h : A \wedge x \to Y$ being a weak equivalence in $\mathcal{LC}$ implies that $\Omega^T(h)$ is a weak equivalence in $\mathbf{Alg_T}$ by the definition of the right Bousfield localization.

We prove the second claim first. If $\tilde{x}$ is the fibrant replacement of x, then it is the transfinite composition of pushouts of acyclic cofibrations in $\mathcal{MC}$. Since $\mathcal{F_T}$ preserves these and Theorem IV.36 yields that B_T sends motivic acyclic cofibrations to acyclic cofibrations in the localization, we have that $B_T\mathcal{F_T}$ applied to the replacement is still an acyclic cofibration. Since these are also both left adjoints we note that the pushouts in $SSet_\bullet^{\mathcal{C}^{op}}$ over the transfinite composition defining $x \to \tilde{x}$ are still pushouts over acyclic cofibrations in a transfinite composition for $B_T\mathcal{F_T}(x) \to B_T\mathcal{F_T}(\tilde{x})$. Since weak equivalences are closed under transfinite composition, and since Ω^T commutes with transfinite compositions since $\mathbb{P}^1$ is representable, it suffices to show that Ω^T

applied to a pushout map preserves weak equivalences.

Let $x_i \to x_{i+1}$ be a pushout in a presentation of $x \to \tilde{x}$. Then we have a pushout diagram:

$$
\begin{array}{ccc}
B_T \mathcal{F}_{\mathbf{T}}(b) & \longrightarrow & B_T \mathcal{F}_{\mathbf{T}}(x_i) \\
\downarrow & & \downarrow \\
B_T \mathcal{F}_{\mathbf{T}}(c) & \longrightarrow & B_T \mathcal{F}_{\mathbf{T}}(x_{i+1})
\end{array}
$$

for $b \to c$ some coproduct of generating acyclic cofibrations in $\mathcal{MC}$. We note that since x is cofibrant then so is $\tilde{x}$ and, incidentally, so are all x_i in the presentation. We have that applying Ω^T to $B_T(\mathcal{F}_{\mathbf{T}}(x)) \to B_T(\mathcal{F}_{\mathbf{T}}(\tilde{x}))$ yields, via adjunction, a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{F}_{\mathbf{T}}(x_i) & \longrightarrow & \Omega^T B_T \mathcal{F}_{\mathbf{T}}(x_i) \\
\downarrow & & \downarrow \\
\mathcal{F}_{\mathbf{T}}(x_{i+1}) & \longrightarrow & \Omega^T B_T \mathcal{F}_{\mathbf{T}}(x_{i+1})
\end{array}
$$

where the horizontal arrows are weak equivalence by the first part to this proof and the left most vertical arrow is a weak equivalence as the image of an acyclic cofibration under $\mathcal{F}_{\mathbf{T}}$. The completes the proof of the second claim.

To see $\tilde{x}$ is motivic fibrant implies that $A \wedge \tilde{x}$ is as well, we start by noting that a presheaf is motivic fibrant if and only if it is flasque fibrant, $F(X) \to F(U) \times_{F(U \times_X V)} F(V)$ is an acyclic fibration for every distinguished square, and $F(X \times \mathbb{A}^1) \to F(X)$ is an acyclic fibration for every X.

The proof of Lemma IV.9 shows that $A \wedge \tilde{x}$ satisfies the first property. The second condition follows from the fact that when $F = \mathbb{P}^1$ the said map is a bijection since distinguished squares are cocartesian in Sm_k and so as a simplicially constant map is a simplicial acyclic fibration. Since the smash product with this map is a pushout over a diagram of monics it remains a weak equivalence and is still a fibration since projective fibrations, and so simplicial fibrations are preserved by smashing with a

simplicially constant bijection, similar to the proof of Lemma IV.9.

To show the final property it suffices to argue that $\mathbb{P}^1(X \times \mathbb{A}^1) \to \mathbb{P}^1$ is an acyclic fibration. Again since this is simplicially constant it suffices to see that the map is surjective. That is, if one has a map $X \to \mathbb{P}^1$, does it extend to a map $X \times \mathbb{A}^1 \to \mathbb{P}^1$. This follows from the fact that X is a retraction of $\mathbb{X} \times \mathbb{A}^1$ via the inclusion $0 \to \mathbb{A}^1$. $\square$

$\mathbf{Alg_T}$ therefore provides a context for understanding the homotopy structure of the right Bousfield localization of $SSet_\bullet^{\mathcal{C}^{op}}$ with respect to morphisms inducing weak equivalence of the obvious loop spaces. This is a surprising combinatorial encoding of the Bousfield localization and is not dissimilar from the results of [4] which apply in the case of pointed topological spaces.

It should be noted that the results here naturally imply the results of [4] since our construction applies to the category of presheaves over a category with only one object, $SSet_\bullet$.

We now consider some consequences of this homotopy categorical equivalence.

F.   Conclusion: Towards a Combinatorial Recognition Principle

This section explores the results of this chapter towards identifying whether or not a presheaf is weak equivalent to a loop space. The generality of the result is naturally tied to the generality of proof to Theorem IV.39, which relies on a nice choice of loop object, in that discussion $\mathbb{P}^1$; an interval type homotopy theory of a small site, for example $\mathbb{A}^1$; and that essentially all of this is being done over a nice model structure on a category of presheaves over a small site.

We begin with a brief corollary to Theorem IV.39.

Corollary IV.40. *Let $X \in SSet_\bullet^{\mathcal{C}^{op}}$, then X is weakly equivalent to an A loop space*

if and only if there exists $\tilde{X} \in \boldsymbol{Alg_T}$ such that $\mathcal{U}(\tilde{X})$ is A-local weakly equivalent to X.

Proof. X is weakly equivalent to a loop space if there exists a space Y such that $X \simeq \underline{Hom}(A, Y)$. This is equivalent to $X \simeq \mathcal{U}\Omega^T(Y)$ proving the first direction.

If there exists a $\tilde{X} \in \mathbf{Alg_T}$ such that $\mathcal{U}(\tilde{X}) \simeq X$ then applying a cofibrant replacement and the first part of the proof to Theorem IV.39 we have $\tilde{X} \simeq \Omega^T B_T(\tilde{X})$ and so $X \simeq \mathcal{U}(\tilde{X}) \simeq \underline{Hom}(A, B_T(\tilde{X}))$ as required. $\square$

Here the functor B_T provides the delooping of X when applied to the relevant algebraic theory. Regarding this another way, we may regard X to consist of a space coupled with an action from a category $A_\bullet \times \mathcal{C}$ as in Chapter II. This action is such that if $f : \vee_n A \wedge U_+ \to \vee_m A \wedge W_+$ then f induces a map $X^m(W_+) \to X^n(U_+)$.

This is not unlike the existence of an action by the an operad of May's recognition principle, and in the case that $\mathcal{C} = \{pt\}$ we recover precisely the statement of [4]. Like in May's construction, we have reduced understanding loop spaces to understanding the existence of an action by a combinatorial structure. Unfortunately, our structure is a good deal more complicated than the little cubes operad.

In future work the author wishes to understand, among other questions, whether the action of such an object is in any way a reasonable condition for recognizing loop spaces. Do, for example, certain spaces suggest the possibility of such an action more evidently than suggesting the structure of a loop space?

Regardless of the outcome of such investigation, this work does lay a common foundation in very wide collection of existing (and potential) homotopy categories for understanding the combinatorial structure encoding loop spaces.

CHAPTER V

SUMMARY

It has now been demonstrated that the topological notion of algebraic theories can be generalized to categories of presheaves over small sites. This generalization is distinct from Lawvere's original construction of **T**-models described in [23] in that we require functorality over a larger collection of morphisms. This requirement, however, still yields a definition compatible with Lawvere's formulation of **T**-algebras if we regard simplicial sets as simplicial presheaves over a category with one object, and so is not an unreasonable generalization. Surprisingly, the category formed by our **T**-algebras has some very nice properties, including admitting a model structure which is compatible with left Bousfield localization.

Several comments are worth noting here. First, the cumbersome requirement that $\mathcal{C}$ is closed under finite coproducts be may be omitted if the proof of Lemma III.20 is made more functorial. In particular, given a pair of elements in $\mathcal{F}_{\mathbf{T}}(V)$ represented by $h_1 \wedge g_1$ and $h_2 \wedge g_2$, if we note that the definition of the tensor product yields that $h_i \wedge g_i \simeq g_i h_i \wedge id$ for $i = 1$ or $i = 2$, then it becomes unnecessary to resort to the coproduct argument used in the proof. This simplifies the exposition of section D in chapter III as well, since now we may ignore the various manipulations to get around the presence of co-products.

Regardless of these modifications, chapter III describes a model structure for our **T**-algebras which in shares many of the nicer properties of the flasque model structure on presheaves. The rigid combinatorial structure of the category lends itself easily to the developments of chapter IV, and also seems useful for further homotopical arguments in the context of sheaves without losing the pleasant homotopy properties over presheaves.

In chapter IV we also see the development of the a most useful free replacement functor:

Definition V.1. Let $\tilde{X} \in \mathbf{Alg_T}$, we define $(FU)_\bullet(\tilde{X}) \in \mathbf{Alg_T}$ by $(FU)_k(\tilde{X}) := (\mathcal{F_T U})^{k+1}(\tilde{X}_k)$, that is the diagonal of the bisimplicial T-model given by applying $\mathcal{F_T U}$ to $\tilde{X}$ $k+1$ times for each $k \geq 0$. Face and degeneracy maps are given by

$$(FU_k\tilde{X} \xrightarrow{d_i} FU_{k-1}\tilde{X}) := (\mathcal{F_T U})^i \epsilon (\mathcal{F_T U})^{k-1} d_i : (\mathcal{F_T U})^{k+1}(\tilde{X}_k) \longrightarrow (\mathcal{F_T U})^k(\tilde{X}_{k-1})$$

and

$$(FU_k\tilde{X} \xrightarrow{s_i} FU_{k+1}\tilde{X}) := (\mathcal{F_T U})^i \eta (\mathcal{F_T U})^{k-1} s_i : (\mathcal{F_T U})^{k+1}(\tilde{X}_k) \longrightarrow (\mathcal{F_T U})^{k+2}(\tilde{X}_{k+1}).$$

Interesting in its own right, this construction is sufficient to capture the homotopy theory of the category of cofibrant $\mathbf{T}$-algebras. Every cofibrant $\mathbf{T}$-algebra may be replaced, up to homotopy, by a free $\mathbf{T}$-algebra. This is a tremendous insight into the homotopy structure of $\mathbf{Alg_T}$ and allows us to prove that the pleasant properties of the flasque model structure on $\mathbf{Alg_T}$ induced from $SSet_\bullet^{\mathcal{C}^{op}}$ are not only preserved under left Bousfield localization, but more importantly that homotopy is still recognized by the forgetful functor $\mathcal{U}$.

This fact makes the subsequent localizations to obtain the motivic homotopy structure on $\mathbf{Alg_T}$ compatible with $SSet_\bullet^{\mathcal{C}^{op}}$ in the following sense. Topological loop spaces have the algebraic structure of a $\mathbf{T}$-algebra, and the full subcategory of loop spaces is Quillen equivalent to the motivic model structure on $\mathbf{Alg_T}$. In the case of presheaves, this is the exactly content of Theorem IV.39.

There do remain some important questions regarding the relationships between the homotopy structure on $\mathbf{Alg_T}$ and $SSet_\bullet^{\mathcal{C}^{op}}$. For example, it is not yet known whether $\mathcal{U}(f)$ is a motivic fibration in the $SSet_\bullet^{\mathcal{C}^{op}}$ if and only if f is a motivic

fibration in $\mathbf{Alg_T}$. The author is fairly certain that this is true, but whereas the conclusion was not necessary for the localization results of section E in chapter IV, where we see that a presheaf has the structure of a $\mathbb{P}^1$ loop space if and only if it has the structure of a $\mathbb{P}^1$-algebra, the question was not investigated exhaustively.

Another observation is that correspondence between $\mathbf{T}$-algebra structure and loop space structure suggests a sort of recognition theorem; although, as of yet it appears to be no simplification of the classification problem since $\mathbf{T}$-algebra structure is difficult to detect in any context. As above the real insight concerns the homotopy structure of a category of motivic loop spaces. Theorem IV.39 has as a corollary that the full subcategory of motivic loop spaces admits a left proper, simplicial, cellular model structure as the motivic localization of the flasque structure for $\mathbf{Alg_T}$. This result comes complete with a set of generating cofibrations that are in every way compatible to the homotopy type of loop spaces in the category of presheaves. Homotopically, loop spaces are presheaves with additional composition operations suggested by the algebra structure. This fact may prove to be a rich source of information useful in the study of motivic loop spaces in a variety of contexts.

Finally, this work may provide a basis for eventually understanding infinite loop spaces in terms of another type of $\mathbf{T}$-algebra; potentially over colimits of choices of $\mathbf{T}$. An obvious first step in this direction is to iteratively consider whether an object and its corresponding de-looping have the action of a $\mathbf{T}$-algebra. If this is the case, can the existence of such a sequence of structures be phrased in terms of a single choice of object for $\mathbf{T}$. Intuitively, this seems plausible, since when $\mathbf{T} = \mathbb{P}^{\wedge 2}$ we see that at $\mathbb{P}^1$-algebra is certainly a $\mathbf{T}$-algebra. The precise relationship between $\mathbb{P}^1$-algebras and $\mathbf{T}$-algebras remains a topic of further investigation, which the author hopes to address in the near future.

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VITA

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