ESSAYS IN GAME THEORY AND INSTITUTIONS

A Dissertation

by

BIRENDRA KUMAR RAI

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2006

Major Subject: Economics

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ABSTRACT

Essays in Game Theory and Institutions. (August 2006)

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This dissertation is a compilation of essays highlighting the usefulness of game theory in understanding socio-economic phenomena. The second chapter tries to provide a reason for the strict codes of conduct that have been imposed on unmarried girls in almost every society at some point of time in its history using tools from classical game theory. If men prefer to marry submissive women, then parents of girls will have an incentive to signal the submissiveness of their daughters in various ways in order to attract better matches. At the same time, parents will find it costlier to signal the submissiveness of girls who are not really submissive. This line of reasoning thus helps us interpret phenomena such as veiling, foot-binding, and sequestration of women in general as signals of submissiveness.

The third chapter attempts to rationalize some of the ad hoc rules proposed for dividing a bankrupt estate using tools from evolutionary game theory. The ad hoc rules differ from each other because of the axioms that are imposed *in addition to* efficiency and claims boundedness. Efficiency requires that the estate be completely divided between the claimants, and claims boundedness requires that no claimant be awarded more than her initial contribution. This dissertation tries to show that an ad hoc rule can be rationalized as the unique self-enforcing long run outcome of Young's [46] evolutionary bargaining model by using certain intuitive rules for the Nash demand game.

In the fourth chapter I present a simple model of conflict over inputs in an economy with ill-defined property rights. Agents produce output from the land they hold, which in turn can be allocated to consumption or the production of guns. There is no agency to enforce rights over the initial land holdings, and the future holdings of land are determined using a contest success function that depends on the guns produced by both agents. I characterize the equilibria in which only one, both, and none of the agents produce guns, as a function of the total land and the inequality of initial land holdings for general forms of utility, production, cost, and contest success functions.

To Amma and Babuji

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CHAPTER I

INTRODUCTION

The disappointments with the classical game theory literature that assumed perfect rationality on part of the agents has over the years given way to models that assume economic agents are boundedly rational. It is only in very simple strategic situations that we can hope the assumption of perfect rationality will give behaviorally sound predictions. The questions I raise in my dissertation and the tools I use to address them reflect this realization.

The second chapter tries to provide a reason for the strict codes of conduct that have been imposed on unmarried girls in almost every society at some point of time in its history using tools from classical game theory. If men prefer to marry submissive women then parents of girls will have an incentive to signal the submissiveness of their daughters in various ways in order to attract better matches. At the same time, parents will find it costlier to signal the submissiveness of girls that are not really submissive. This line of reasoning thus helps us interpret phenomena such as veiling, foot-binding, and sequestration of women in general, as signals of submissiveness. The undefeated equilibria of the underlying game lead to a unique separating outcome when the frequency of obedient and submissive girls is not quite high; and to a unique pooling equilibrium when this frequency is sufficiently high.

The third chapter attempts to rationalize the adhoc rules proposed in the literature for dividing a bankrupt estate using tools from evolutionary game theory. The adhoc rules differ from each other because of the axioms that are imposed

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This dissertation follows the style of Journal of Economic Theory.

in addition to efficiency and claims boundedness. Efficiency requires that the estate be completely divided between the claimants, and claims boundedness requires that no claimant be awarded more than her initial contribution. The paper tries to show that an adhoc rule can be rationalized as the unique self-enforcing long run outcome of Young's [46] evolutionary bargaining model by using certain intuitive rules for the Nash demand game. If the agents bargain in the framework of the usual demand game, the long run stochastically stable equilibrium turns out to be equal division of the estate. If, in addition to the usual rules, demanding more than one's initial claim leads to a zero payoff (claims boundedness) then the long run equilibrium corresponds to the constrained equal awards rule. If the rules of the demand game capture both efficiency and claims boundedness, then the long run equilibrium corresponds to the division proposed by the truncated claims proportional rule.

In the fourth chapter I present a simple model of conflict over inputs in an economy with ill-defined property rights. The economy consists of two agents initially holding unequal amounts of the total available land. The agents produce output from the land they hold which in turn can be allocated to consumption or the production of guns. There is no agency to enforce rights over the initial land holdings and the future holdings of land are determined using a contest success function which depends on the guns produced by both agents. Agents maximize the weighted sum of utility from current consumption and the utility from future land holding. I characterize equilibria in which only one, both, and none of the agents produces guns, as a function of the total land and the inequality of initial land holdings for general forms of utility, production, cost, and contest success functions. The final chapter provides concluding remarks and raises some questions for future research.

CHAPTER II

OF VEILS AND WEDDING RINGS

A. Introduction

Women have been viewed and valued very differently across cultures throughout our history. Even today these differentials are enormous within and across countries. It would be fair to say that parents in several cultures try hard to inculcate the traits of submissiveness and obedience in their unmarried daughters (Cheung [9], Ebrey and Watson [15], O'Faolain and Martines [37], Hill [23], Klapisch-Zuber [25], Stone [42]). Apart from the behavioral indoctrination, different cultures have come up with novel ways to alter the manner in which girls interact with the space and society around them. Some of the enduring features include the sequestration of girls, the veil in Islamic countries, the now extinct foot-binding in China, and genital mutilation in several parts of Africa.

Marriage is probably the most social event in the life of a girl in traditional societies. The importance of the measures parents take to prepare their daughters for marriage can be gauged by the beliefs regarding an ideal wife and the severity of punishment for transgressions after marriage. The prescriptions for an ideal spouse have hardly ever stressed a man who is obedient to the women, but in all cultures at some time in their history the ideal woman has been one who is obedient to her husband. Similarly, there is hardly any evidence of a society that imposes greater controls on the premarital sexual behavior of men. At all times in history adulterous women have been at least as strongly punished as adulterous men (Murstein [30]). The legal codes of present day Syria, Jordan, Morocco, and Haiti do not recognize a man killing his wife as a murderer if the wife is accused of adultery. In Brazil and Colombia similar laws have been struck down over the past two decades. It is estimated that nearly two thousand women become the victims of 'honor killings' by their close kin every year, sometimes for 'crimes' such as talking to men other than their relatives. It is difficult to deny that the breach of the marriage contract by women has been considered a bigger crime, than a breach by men.

Providing her husband with a child has been one of the main functions of a married girl in traditional societies. We assume that men will prefer those women as marriage partners who can establish the credentials for post marital fidelity. This is because men do not have absolute certainty regarding the paternity of a child, and they care more about their 'own' children. This argument has a long history starting from the ancient Greco-Roman philosophers, and carried on by medieval theologians and ideologues to the modern evolutionary biologists, anthropologists, and economists (Alexander and Tinkle [1981], Klapisch-Zuber [25], O'Faolain and Martines [37]).

Although girls in traditional societies are confined both before and after marriage, the main purpose of confinement seems to differ. Parents confine their daughters prior to marriage to signal her credentials for post marital fidelity. Husbands confine their wives after marriage to avoid moral hazard. It seems more reasonable to analyze premarital confinement as a signaling game, whereas a principal-agent model suits the analysis of post marital confinement.

The mechanics of the Gale-Shapely matching algorithm involve sequential proposals from one side and rejection/acceptance from the other side of the market given the rankings of all players (Roth and Sotomayor [39]). The primary focus of matching literature is to determine the set of stable matchings. We, instead, intend to elaborate on the process through which the players come up with their rankings and show that parents confine their daughters so that men would rank them higher. As a clarification, imagine the approach graduate schools might take to deal with a large number of applicants. In the first stage, the committee might come up with rules of thumb, like anyone with an undergraduate GPA of less than 2.0 would not be admitted. Once the list has been shortened by using one or several such arbitrary (but undoubtedly useful) criterion, the committee might then review each application and come up with the final admissions in the second stage of decision making. Our efforts could be thought of as explorations of only the first stage of admissions process and the main aim is to show that restrictions on women are equilibrium outcomes of a suitably defined game. The second stage which involves determining the exact pairings and their properties has been dealt with extensively by the matching literature and we do not pursue it here (Becker [6]).

It is widely accepted in the sociology and anthropology literature on cross-cultural studies that within communities that do confine females, rich families confine their girls and women more severely than the poor ones (Broude [8]). Purely economic motivations can explain why women of poor families are less confined; but they do not explain why women of rich families are more confined. If labor markets are not sufficiently developed, or the production technology is not favorable for women to earn high returns, women of rich families may not work. But this does not imply that rich men would 'choose' to confine the women in their families in various ways. Therefore, for the sake of clarity, we consider an economy in which women only have reproductive value and no productive value.

If we are willing to accept that women have a comparative advantage in rearing young children and household work we might be able to explain why married women spend more time indoors. But this does not explain why they would not be allowed to interact with other men. In any case this line of reasoning does not help us to understand why girls are confined prior to marriage. Therefore, we only try to explore premarital confinement of girls by their parents and argue that this obviates the reasons for post marital confinement. Parents of girls usually start placing restrictions on her around the onset of puberty and maintain it till her marriage (Broude [8]). Different girls might get married at different ages but we assume only qualitative differences in the extent of claustration and thus avoid the dynamic issues that might arise (Noldeke and van Damme [36]).

It is assumed that there exist temperamental differences among girls in the degree to which to they are submissive/obedient at the time parents start confining them. The extent of confinement by parents positively affects their degree of obedience. It is also assumed that the costs incurred by the parents of a girl in this process are psychic in nature with the cost of confinement to any extent being less for a more obedient girl than for a less obedient girl. There is asymmetric information in the sense that potential suitors can only (perfectly and costlessly) observe the extent of confinement but not the true nature of a girl.

We abstract from any transfers associated with marriage. Each man is assumed to care more about his 'own' children which in turn motivates him to consider the likelihood that a girl will remain fidel after marriage and sire 'his' children. This concern of men is formalized by assuming that the value of a girl to a potential suitor depends on her inherent character and extent of confinement. A suitor is valued by the parents of a girl because of the resources he can provide to their daughter and her children. The utility derived by the parents of a girl is thus assumed to be the wealth of the suitor net of their psychic costs incurred in raising her. Men decide whether to propose to a girl or not, and the girl's parents choose whether to accept the proposal or not. We employ sequential equilibrium as the solution concept for the game and discuss the nature and meaning of the equilibria that emerge.

B. The Model

The inherent nature of girls (their submissiveness/obedience) when parents start confining is assumed to be of two types $t \in (h, l)$, with a common prior, $P(h) = q$. The extent to which parents of a girl confine her is denoted by e. The confinement of a girl reflects not only the physical constraints on her movement, association, and interaction with other men but also the moral and behavioral education and indoctrination by family members. Confinement in this model plays a role similar to education in job signaling models. The cost of confinement to the extent e for a type-t girl is given by $c(t, e)$, with $c(t, 0) = 0$, $c_e(t, e) > 0$, $c_{ee}(t, e) > 0$, $c_t(t, e) < 0$, and $c_{et}(t, e) < 0$ for all $e \geq 0$. Thus, there is no cost to the parents if they do not confine their daughter, marginal costs of confinement are positive and increasing with increases in the extent of confinement for both types of girls, and the marginal cost of confining type-l girl is greater than that of type-h girl, at all levels of confinement. The net utility to the parents of a girl of a proposal from a suitor having wealth w is $[w - c(t, e)]$.

The value of the girl-(t, e) to potential suitors is $[\alpha v(t, e)]$ with the constant $\alpha \in (0, 1)$, and $v(h, e) > v(l, e)$ for all e, and $v_e(t, e) > 0$ for both t and all e. This value function of men defined over girls primarily reflects the likelihood of the child from the girl being his own. This should not be taken to imply that girls or their parents do not care about the nature and character of suitors. However, we presume it would add little in our efforts to explain why it is men that often set up extensive and sometimes excessive mechanisms to confine women. Also, the value function should be thought of as reflecting the 'perceived' value to men from marrying the girl-(t,e). The positive partial effect of confinement ignores the possibility that extreme confinement might lead a girl to become more rebellious and less obedient. The indifference curves of parents of girls are upward sloping and convex in the and the (e,w) space. The value functions of girls as perceived by men are linear in the (e,w) space.

It is assumed that every man will have to bear some unavoidable cost of providing for the girl he ends up marrying and the resulting children, irrespective of the type of the girl and her level of confinement. It might well be the case that the expenditure by a man on his wife is determined endogenously. However, we presume that men enter the marriage market with a rough idea of how much they would have to spend on their future wife and any adjustments to this expenditure can be done only after he actually gets married. Further assume that the absolute amount of this expenditure is increasing in the wealth of men. A convenient way to formalize this is to assume that the fixed cost equals a constant fraction of wealth. As a result the resources spent by different men will be different for the same girl. The utility function of men is thus assumed to be

$$u_m(t,e,w) = \begin{cases} \alpha \cdot v(t,e) + (1-\alpha).w & if\ married\ to\ girl-(t,e) \\ w & if\ unmarried \end{cases} \tag{2.1}$$

The timing of the game is as follows: nature determines the type of each girl with $P(h)$ being q, parents choose the extent of confinement of girls, men observe the extent of confinement and update their beliefs regarding the type of each girl, decide upon the girls they would be willing to propose to, and then parents of girls decide whether to accept an offer or not. We are assuming that it is the parents of the girl who ultimately decide her match.

The value of a girl known to be of type-t with certainty is $[\alpha v(t,e)]$. If type is private information then $\mu(h|e)$ is the common assessment by suitors that the girl is of type-h after having observed her extent of confinement e. The expected value of girl-(t,e) to suitors is $\alpha[\mu(h|e).v(h,e)+(1-\mu(h|e)).v(l,e)]$. I make the simplifying assumption that men would not mind proposing to those girls that leave them at least as well off as in the unmarried state. Thus a girl-(t,e) would receive proposals only from men having wealth w such that

$$\alpha.[\mu(h|e).v(h,e)+(1-\mu(h|e)).v(l,e)]+(1-\alpha).w \geq w \tag{2.2}$$

This helps us define the critical suitor wealth function, the wealthiest man willing to propose to a girl of unknown type, as a function of her observable level of confinement.We have

$$w_c(e)=\mu(h|e).v(h,e)+(1-\mu(h|e)).v(l,e) \tag{2.3}$$

where $w_c(e)$ will be linear in e with an intercept and slope between the intercepts and slopes of $v(l,e)$ and $v(h,e)$. Without loss of generality it can be assumed that for any girl-(t,e) there exist men with wealth $w \geq w_c(e)$. This ensures that we will always have equilibria in which each girl would get a proposal. Also, a continuous distribution of wealth actually simplifies the analysis because with a discrete number of wealth classes we will have to characterize the equilibria as a function of the 'levels' of wealth associated with those discrete classes.

$d(e|w)$ is an indicator function which equals 1 if the suitor having wealth w would be willing to propose to a girl who has been confined to the extent e, and equals 0 otherwise. Sequential Equilibrium (Kreps and Wilson [27]) will be used as the solution concept for this signaling game.

Since we have allowed girls to be of only two types there will be bunching of all girls at a single level of confinement in a pooling equilibrium, and at two levels of confinement in the separating equilibrium. Thus, each parent would optimally want to match their daughter to the same man in the pooling equilibrium. Similarly, in the separating equilibrium, the optimal suitor for daughters of one set of parents would be a particular man, and the remaining parents would prefer to match with another particular man. This feature of the equilibrium can not be eliminated even by allowing the type of girls to be a continuum unless we make additional assumptions on the measures of types of girls and wealth of men. We take this to imply that men use the type and confinement of women as a preliminary and crude sorting device to decompose the set of all women into two non-overlapping sets- one comprising of girls they would be willing to propose to, and the other consisting of girls they would not propose to. As mentioned earlier, the aim of this paper is to show that parents of girl would confine their daughters lest resourceful men put them in the set they will not propose to.

1. Complete Information About Type

We start our analysis with the complete information case where we assume that the type of a girl is common knowledge. Parents of girl-(t, e) will accept the proposal from the critical suitor in order to maximize their utility. Since the critical suitor's wealth is increasing in e parents of a girl choose the optimal extent of confining her by maximizing the net benefit, i. e.,

$$max_e \quad [v(t, e) - c(t, e)] \tag{2.4}$$

With complete information regarding types the optimal extent of confinement is denoted as $e^*(t)$, and the critical suitor's wealth $w^*(t) = v(t, e^*(t))$. The assumptions on the value function of girls and the cost function suggest that the net marginal benefits for type-h parents are greater than that for the type-l parents, for any given e. The optimal extent of confinement is reached when the net marginal benefits of confinement become zero. We thus expect type-h parents to optimally choose a greater level of confinement for their daughters. The girl-(t, e) offers the same confidence of paternity to every man. Since rich men spend greater amount of resources in absolute terms they will in turn demand a greater level of confidence, which will be reflected in their decision of proposing to relatively high (t, e)-girls.

2. Incomplete Information About Type

We now assume that the type of a girl is private information and only the extent of confinement is publicly observable. The parents of type-l girls can always ensure themselves of a proposal from the suitor having wealth $w^*(l)$. The worst that can happen to a parent of type-h girl is suitors believing that their daughter is instead type-l. In such a case the optimal level of confinement chosen by parents of this type-h girl will be denoted by $e^*(h \to l)$.

a. Separating Equilibria

The characterization of separating sequential equilibrium involves the consideration of both the no-envy and envy cases. In the no-envy case type-l parents do not find it profitable to mimic the confinement choice made by the type-h parents; in the envy case they do, unless the type-h parents choose a sufficiently high level of confinement. In the no-envy case the indifference curve of type-l parents passing through $(e^*(l), w^*(l))$ intersects the indifference curve of type-h parents passing through point $(e^*(l), w^*(l))$ intersects at an $e > e^*(h)$; in the envy case these indifference curves intersect at an $e < e^*(h)$. The strategy of parents of girls in the no-envy separating equilibrium, wherein the type-l girls can not profitably pretend to be type-h, thus involves

$$[e(l) = e^*(l), e(h) = e^*(h)] \tag{2.5}$$

The beliefs of men after observing these choices of e are

$$\mu(h|e^*(l)) = 0 \quad \text{and} \quad \mu(h|e^*(h)) = 1 \tag{2.6}$$

$$d(e|w) = \begin{cases} 1 & if\ e = e^*(l)\ \ and\ \ w \leq w^*(l) \\ 1 & if\ e = e^*(h)\ \ and\ \ w \leq w^*(h) \\ 0 & otherwise \end{cases} \tag{2.7}$$

The simple interpretation of these strategies is that if a girl is confined more, she is more desirable. In order to completely characterize this separating equilibrium we need to specify the beliefs of men for out of equilibrium choices of e, which will in turn determine the rest of their strategy. One set of beliefs that satisfies the required conditions is that the girl is of type-h only if she has been confined to an extent at least as high as $e^*(h)$. Formally

$$\mu(h|e) = \begin{cases} 1 & if\ e \geq e^*(h) \\ 0 & otherwise \end{cases} \tag{2.8}$$

The strategy of men in the no-envy case can now be fully specified as

$$d(e|w) = \begin{cases} 1 & if\ e \leq e^*(h)\ \ and\ \ w \leq w^*(l) \\ 1 & if\ e \geq e^*(h)\ \ and\ \ w \leq w^*(h) \\ 0 & otherwise \end{cases} \tag{2.9}$$

Intuitively this means that a girl who has been confined to a greater extent will receive proposals from all those who are willing to propose to the less confined girl, and in addition some more proposals from wealthier men. Since $e^*(h)$ is the best response by the type-h parents in response to the value placed on high type girls $(v(h, e))$, it is also the best response in this case. Similarly, $e^*(l)$ is the best response of the type-l parents in response to the value placed on their girls as their maximum payoff $[w^*(l) - c(l, e^*(l))]$ among all the choices of $e < e^*(h)$ is realized by choosing $e = e^*(l)$.

In the envy case, by definition, the h-type parents will not be able to distinguish their daughters from the l-types by choosing $e^*(h)$. The maximum level of confinement which type-l parents will have an incentive to mimic leaves them exactly as well off as they are when their type is perfectly known to be l. Parents of type-h will be able to signal their high type only by choosing an $e \geq e_s > e^*(h)$. The strategies of parents of girls will thus be

$$[e(l) = e^*(l), e(h) = e_s] \tag{2.10}$$

The equilibrium beliefs of men would be

$$\mu(h|e^*(l)) = 0 \quad \text{and} \quad \mu(h|e_s) = 1 \tag{2.11}$$

An intuitive specification of out of equilibrium beliefs of men is

$$\mu(h|e) = \begin{cases} 1 & if\ e \geq e_s \\ \\ 0 & if\ e < e_s \end{cases} \tag{2.12}$$

This implies that men, while trying not to be deceived by type-l parents posing as type-h, will form beliefs such that only if the extent of confinement is greater than the minimum level of confinement that certainly differentiates the types, would they be willing to accept a girl as type-h. Finally, the complete strategy profile of men would be

$$d(e|w) = \begin{cases} 1 & if\ e < e_s\ \ and\ \ w \leq w^*(l) \\ \\ 1 & if\ e \geq e_s\ \ and\ \ w \leq v(h, e_s) \\ \\ 0 & otherwise \end{cases} \tag{2.13}$$

Men having $w \leq w^*(l)$ will propose to both types of girls. The type-h girls would be able to attract additional proposals from men having wealth $w \in [w^*(l), v(h, e_s)]$.

As discussed above, the l-type parents have two equally good options: choosing $e^*(l)$ and receiving a proposal from the suitor having wealth $w^*(l)$, or choosing e_s which raises the maximum offer to $v(h, e_s)$. We shall always assume in this paper that any indifference is resolved in favor of the lower extent of confinement. The choices of $e > e_s$ for h-type parents are clearly inferior to e_s as they lead them to lower indifference curves. Since the indifference curve of the l-type parents passing through $(e_s, v(h, e_s))$ is tangent to their value function $(v(l, e))$, hence the less steep indifference curves of type-h parents passing through $(e_s, v(h, e_s))$ lies above $v(l, e)$ for $e < e_s$. The choices of $e < e_s$ are thus inferior for type-h parents as any such choice of e will lead the men to believe that they are of type-l, and thus the best these choices can do is allow them to reach an indifference curve that is tangent to $v(l, e)$. Thus, the best response of h-type parents to the strategy of men is to choose $e = e_s$.

There exist other separating equilibria involving a different choice of e by the type-h parents. For e slightly greater than e_s such that $[v(h, e_{s+}) - c(h, e_{s+})] > [v(l, e) - c(h, e)]$, the type-$h$ parents prefer to signal the high quality of their daughters by choosing e_{s+}. Substituting e_{s+} for e_s in the beliefs and strategy of men in the previous separating equilibrium, along with the strategy $[e(l) = e^*(l), e(h) = e_{s+}]$ for parents of girls, completely specifies this equilibrium.

An example of the separating equilibrium that differs from the one in envy case off the equilibrium path involves modifying the beliefs of suitors such that it leaves the optimal strategies of parents $(e(l) = e^*(l), e(h) = e^*(h))$ unchanged. We need to ensure that both types of parents will end up on lower indifference curves if they deviate from their equilibrium strategies. This can be accomplished by allowing men to attach a strictly positive but sufficiently low probability (q') that the girl is type-h after observing an $e \in (e^*(h), e_s)$ such that for all $e \in (e^*(h), e_s)$ the critical suitor wealth function $w(e|q')$ lies strictly below the indifference curve of type-l parents passing through the point $(e^*(l), w^*(l))$.

b. Pooling Equilibria

Pooling equilibria emerge in both the envy and no-envy cases and the steps used in characterizing them are same. Hence, we only discuss the more interesting envy case. The worst that could happen to an h-type girl is that men believe her to be of type-l. The parents of type-l will have the incentive to pool only if pooling allows them to reach an indifference curve higher than the one they can attain when their type is perfectly known or believed by the suitors to be l. The utility that both types can ensure by responding optimally when they are believed to be type-l irrespective of their true type will be termed as their reservation utility (Banks 1991).

The minimum (maximum) e at which pooling is possible is the maximum (minimum) of the minimum (maximum) level at which the two types are willing to pool. Pooling equilibria are not possible below a certain value of the prior. All values of e in the set $[e_p^{min}, e_p^{max}]$ can support a pooling equilibrium similar to the one described below. We can characterize these equilibria completely by replacing e_p^* by any value of $e \in [e_p^{min}, e_p^{max}]$.

In the pooling equilibrium parents of both types of girls choose the same extent of confinement, say e_p^*. Therefore, the updated belief of suitors will be the same as their prior belief. This in turn implies that the expected benefit each girl, having been confined to the extent e_p^*, provides to any suitor is $\alpha[q.v(h, e_p^*) + (1 - q).v(l, e_p^*)]$. The point ($e_p^*$, w_p^*) lies on the line that gives critical suitor's wealth, $w_p(e) = [q.v(h, e) + (1 - q).v(l, e)]$, where the indifference curves of both types intersect. This point also determines the wealth level of the critical suitor in this case. All men with wealth $w \leq q.v(h, e_p^*) + (1 - q).v(l, e_p^*) = w_p^*$, will be better off proposing to any girl. We can specify the equilibrium strategy of men as follows.

$$d(e|w) = 1 \quad \text{if} \quad e = e_p^* \quad \text{and} \quad w \leq w_p^* \tag{2.14}$$

We still need to specify the beliefs of men, $(\mu(h|e))$ for out of equilibrium choices of e by parents of girls, which will in turn determine the remaining part of men's strategy. The simplest belief that men might have is that any extent of confinement other than e_p^* implies that the girl is of type-l. Thus, if the beliefs of men are

$$\mu(h|e) = \begin{cases} q & if\ e = e_p^* \\ 0 & if\ e \neq e_p^* \end{cases} \tag{2.15}$$

then their strategy would be

$$d(e|w) = \begin{cases} 1 & if\ e = e_p^*\ \ and\ \ w \leq w_p^* \\ 1 & if\ e \neq e_p^*\ \ and\ \ w \leq w^*(l) \\ 0 & otherwise \end{cases} \tag{2.16}$$

This implies that all men with $w \leq w_p^*$ will propose to any girl whose level of confinement is observed to be e_p^*. On the other hand, if the level of confinement is anything other than e_p^*, men take it as an unambiguous signal that the girl is of type-l, and only men with $w \leq w^*(l)$ offer a proposal. Now we need to argue that given the strategies and beliefs of men e_p^* is indeed the optimal choice by both types of parents.

The parents of girl-(t, e) always choose e to maximize $[w(e) - c(t, e)]$. In this case, the belief structure of men suggests that parents choose either e_p^* or the extent of confinement that maximizes $[v(l, e) - c(t, e)]$. Given the shape of indifference curves and value functions, type-h parents would choose e_p^* since their indifference curve passing through (e_p^*, w_p^*) lies above the value function for type-l girls. Similarly, type-l parents will also be better off choosing $e = e_p^*$ since the best alternate option $(e^*(l))$ leaves them on the lower indifference curve passing through $(e^*(l), w^*(l))$.

Yet another category of pooling equilibria can be generated by holding e at e_p^* but varying the beliefs and strategies of men for out of equilibrium choices of e as follows.

$$\mu(h|e) = \begin{cases} q & if\ e = e_p^* \\ q & if\ e \geq e_p^h \\ 0 & if\ e \leq e_p^h\ \ except\ for\ e = e_p^* \end{cases} \tag{2.17}$$

where e_p^h is the extent of confinement at which the indifference curve of the type-h parents through the point (e_p^*, w_p^*) crosses the wealth function given by $w = q.v(h, e) + (1 - q).v(l, e)$. The intuition for this specification comes from the observation that the parents of type-h are indifferent between (e_p^*, w_p^*) and (e_p^h, w_p^h) but the type-l parents strictly prefer (e_p^*, w_p^*) over (e_p^h, w_p^h). This structure of beliefs implies that if a girl is confined to an extent greater than a sufficiently high level (e_p^h) then men take it as a signal of her extreme desirability that forces her parents to 'protect' her from undesirable men. On the other hand, if she is confined any less than e_p^h but not at the socially prevalent extent (e_p^*) then it is believed to signal that she is not obedient enough or not good enough, thereby reflecting her l-type. The resulting strategy of suitors is

$$d(e|w) = \begin{cases} 1 & if\ e = e_p^*\ \ and\ \ w \leq w_p^* \\ 1 & if\ e \geq e_p^h\ \ and\ \ w \leq w_p^* \\ 1 & if\ e \leq e_p^h\ \ except\ for\ e = e_p^*\ \ and\ \ w \leq w^*(l) \\ 0 & otherwise \end{cases} \tag{2.18}$$

If a girl is confined to the socially prescribed level e_p^* then all men with $w \leq w_p^*$ propose to her. The same holds true if a girl is confined at a sufficiently higher level, i.e., if $e \geq e_p^h$. (Although men believe that if the girl is confined sufficiently more she must be of type-h, the strategy specifying that only men with $w \leq w_p^*$ propose to her constitutes an equilibrium, because the equilibrium strategy of all parents is to confine their daughters to the socially prescribed extent.) If she is confined less than the sufficiently high level but not at the socially prescribed one, she is considered to be of type-l and only those men with $w \leq w^*(l)$ propose to her.

The type-$h(l)$ parents are worse (better) off in the e_p^* pooling equilibrium as compared to the complete information equilibrium. However, only in the envy case there exists a critical value of the common prior, q, above which type-h parents are better off in pooling equilibria rather than the separating equilibrium.

C. Refinements of Equilibria

The previous sections describe the various equilibria that can possibly emerge. Several refinement criterion have been proposed for signaling games to isolate the plausible equilibria. Since we have considered only two types of girls (senders) the Intuitive Criterion (Banks and Sobel [5], Cho and Kreps [10]) will provide us with a unique prediction from all the possible sequential equilibria. It has been argued that apart from the discontinuity in the unique predicted outcome as the prior goes from a value very close to unity, to exactly unity, the logical foundations of forward induction as embodied in the Intuitive Criterion are myopic and thus inconsistent with perfect rationality. We will therefore compare the final prediction of the Intuitive Criterion with those obtained from the concept of Undefeated equilibrium (Mailath et al. [28]) that tries to overcome these shortcomings. We divide the following discussion into three parts differentiated by low values of the prior that allow no pooling equilibrium, medium values of the prior that provide the type-h parents with highest utility in the separating outcome, and finally high values of the prior that provide the type-h parents with highest utility in the pooling equilibrium.

1. Unique Equilibrium for Low Prior

If the maximum level of e at which type-l is willing to pool is lower than the minimum e at which type-h would be willing to pool, then no pooling equilibria are possible. Thus, in the envy case, we will have separating equilibria with $[e(l) = e^*(l), e(h) \geq e_s]$ and

$$\mu(h|e) = \begin{cases} 1 & if \ e \geq e(h) \\ 0 & otherwise \end{cases} \tag{2.19}$$

Consider any separating equilibrium in the envy case with $e(h) > e_s$. (The same arguments apply in the no-envy case with e_s replaced by $e^*(h)$.) This implies that e_s will be an off the equilibrium path message with $\mu(h|e_s) = 0$. But e_s is a dominated choice for the type-l parents, and so if e_s is observed then $\mu(h|e_s) = 1$. If so, type-h parents will deviate to e_s as it provides them with a higher utility, thereby upsetting the proposed equilibrium. This line of reasoning suggested by the Intuitive criterion will lead to the unique predicted outcome $[e(l) = e^*(l), e(h) = e_s]$ and

$$\mu(h|e) = \begin{cases} 1 & if \ e \geq e_s \\ 0 & otherwise \end{cases} \tag{2.20}$$

The Undefeated Equilibrium concept will also select this same outcome as its unique prediction in this case; but the logic is different. The intuitive criterion eliminated the separating equilibria having confinement of type-h girls greater than e_s on the basis of unreasonable out of equilibrium beliefs by arguing that the out of equilibrium message e_s is a dominated choice for the type-l parents but still men attach a positive probability to this message having been sent by them. The logic of undefeated equilibrium while eliminating the separating equilibria with $e(h) > e_s$, utilizes the fact that there exists another sequential equilibrium in which e_s is sent by at least one of the two types of senders. Moreover, it is that very type (the h-type) which prefers this alternative e_s equilibrium. Since the beliefs of men in the $e(h) > e_s$ equilibrium are not consistent with the beliefs in the $e(h) = e_s$ equilibrium, the e_s equilibrium is said to defeat the $e(h) > e_s$ equilibria. The only undefeated equilibrium this process of elimination gives is the e_s separating equilibrium, same as the one predicted by the Intuitive Criterion.

2. Unique Equilibrium for Medium Prior

We now consider the case where the prior is high enough to allow pooling but not high enough to make any pooling equilibria at any prior in this range more attractive to type-h parents, than the Riley separating equilibrium. For this range of the priors, we have argued earlier that both pooling and separating equilibria are possible. Separating equilibria with $e(h) > e_s$ fail the intuitive criterion by a reasoning similar to the one employed in the case of low priors. In order to understand why all the pooling equilibria also fail the intuitive criterion, consider a candidate pooling equilibrium in the envy case with $[e(l) = e(h) = e_p^*]$ and

$$\mu(h|e) = \begin{cases} q & if\ e = e_p^* \\ 0 & if\ e \neq e_p^* \end{cases} \tag{2.21}$$

This implies that e_s will be an out of equilibrium message. The specification of the e_p^* pooling equilibrium under study requires $\mu(l|e_s) > 0$. But e_s is a dominated choice for type-l parents, and so if e_s is observed the receivers must attribute it to the h-types. This in turn implies that $\mu(l|e_s) = 0$. If so, the h-types will have an incentive to deviate to e_s as it leads them to a higher indifference curve, thus upsetting the proposed equilibrium. Hence, the only equilibrium consistent with the intuitive criterion is the Riley separating outcome with $e(h) > e_s$.

The undefeated criterion will give the same result in this case as well. Let pooling at e_p^* be the equilibrium under consideration which involves $\mu(l|e_s) > 0$. The only sequential equilibrium in which e_s is sent as an equilibrium message has the h-types sending it. Also, it is only the h-types that prefer this Riley outcome over the pooling equilibrium with $e = e_p^*$, and it has $\mu(l|e_s) = 0$. All the pooling equilibria are thus defeated by the Riley separating equilibrium. It also defeats all the other separating equilibria with $e(h) > e_s$. Hence, in the case of medium priors also, both the intuitive and the undefeated criterion select the Riley separating equilibrium as the unique prediction.

3. Unique Equilibrium for High Prior

We now consider the case where the prior is high enough not only to allow pooling but also to make some pooling equilibria, at each value of the prior in this range, more attractive to type-h parents than the Riley separating equilibrium. This is the most interesting of the three cases as it clarifies the logical inconsistencies of forward induction. Consider the pooling equilibrium with $[e(l) = e(h) = e_p^*]$ and

$$\mu(h|e) = \begin{cases} q & if\ e = e_p^* \\ 0 & if\ e \neq e_p^* \end{cases} \tag{2.22}$$

Any level of confinement $e_{oeq} \in (e^{'}, e^{"})$ will be an out of equilibrium message. Regardless of the beliefs men will form after observing e_{oeq}, the l-types would be worse off by sending it compared to the e_p^* pooling equilibrium. If men instead believe that e_{oeq} was sent by the h-types, then their best response would be to offer $v(h, e_{oeq})$, and this would yield a higher utility to the h-type parents than they were getting in the e_p^* pooling equilibrium under study. Thus, h-types would deviate from e_p^* to e_{oeq}, thereby upsetting the pooling equilibrium. This is where the argument of Intuitive Criterion ends while eliminating the e_p^* pooling equilibrium because it involves unreasonable beliefs at out of equilibrium messages e_{oeq}.

Is the reasoning employed above sound? If men believe that it is the h-type parents sending e_{oeq}, then they will indeed be better off by sending it. But, if all players in the game are assumed to understand the underlying logic of intuitive criterion then the l-type parents should realize that after observing e_p^* men will conclude that it must have been sent by the l-types. The crux of undefeated criterion is that beliefs at off equilibrium path information sets can not be adjusted while keeping beliefs on the equilibrium path unchanged. Thus, the l-type parents cannot ensure that their daughters will receive proposals from men having wealth $w \in [w^*(l), w_p(e_p^*)]$. Moreover, if they were to deviate to e_{oeq} then these l-types would definitely get proposals from men having $w > w^*(l)$. And if they do choose e_{oeq}, then it no longer remains an unambiguous signal of an h-type girl.

What is the logical end of this thought process? More importantly, what would be 'reasonable' beliefs after a deviation from the equilibrium path is observed. The undefeated criterion suggests that the beliefs that should be formed after observing an out of equilibrium message should be consistent with the one that is formed when this message is sent as an equilibrium message in some other sequential equilibrium. The Riley separating equilibrium will defeat all the other separating equilibria as in the previous two cases. Now, let the Riley equilibrium be under consideration and the pooling at e_p^* be the alternate equilibrium. With respect to the Riley equilibrium, e_p^* is an out of equilibrium message with $\mu(h|e_p^*) = \mu(h|e_p^*) = q$. Since both types prefer the alternative pooling equilibrium to the separating equilibrium under study, the beliefs in the separating equilibrium after observing e_p^* should be consistent with the beliefs in the alternate pooling equilibrium. But, they are inconsistent; the separating equilibrium specifies $\mu(l|e_p^*) = 0$. Hence, the alternate equilibrium (pooling at e_p^*) defeats the equilibrium under study (the Riley separating equilibrium).

The only equilibrium that might be undefeated in this case can be a pooling equilibrium. For a given q in this range of priors, let the equilibrium under study be the pooling equilibrium at $e_p > e_p^*(h))$. Consider the pooling equilibrium that provides the h-types with highest utility $e_p^*(h)$ as the alternate sequential equilibrium. With respect to the e_p equilibrium, $e_p^*(h)$ is an out of equilibrium message which is preferred by both types. The $e_p^*(h)$ pooling equilibrium has $\mu(h|e_p^*(h)) = \mu(h|e_p^*(h)) = q$. But, if $e_p^*(h)$ is observed, the beliefs in the e_p pooling equilibrium assign zero probability to h-types. This inconsistency implies that the alternate pooling equilibrium defeats the e_p pooling equilibrium. Next, let the equilibrium under study be the pooling equilibrium at $e_p < e_p^*(h))$ with pooling at $e_p^*(h)$ again being the alternate equilibrium. It is only the h-types that prefer the alternate equilibrium over equilibrium under study. But, the e_p equilibrium will involve $\mu(h|e_p^*(h)) < 1$. Thus beliefs at an out of equilibrium message in the equilibrium under study are inconsistent with the beliefs at this message when it is an equilibrium message. This inconsistency again leads us to conclude that the pooling equilibria at $e_p < e_p^*(h)$ are defeated by the pooling equilibrium at $e_p^*(h)$.

Now, let the equilibrium under study be the $e_p^*(h)$ pooling equilibrium. Any pooling equilibrium towards the right of this point can not be a candidate alternate equilibrium as it is not preferred by either of the types. Pooling equilibria towards the left of $e_p^*(h)$ are only preferred by l-types. Moreover, if any $e < e_p^*(h)$ is observed, then the $e_p^*(h)$ pooling equilibrium will assign this deviation to the l-types (by construction, it assigns any deviation from the equilibrium only to l-types). Thus, the beliefs in the $e_p^*(h)$ pooling equilibrium are consistent according to the undefeated criterion, making it the unique prediction.

The intuitive criterion always selects the Riley separating equilibrium as its unique predicted outcome. The undefeated criterion selects that equilibrium which provides the highest payoff to the h-types; the Riley separating equilibrium for low and medium values of the prior, and the $e_p^*(h)$ pooling equilibrium for high values of the prior.

D. Conclusion

The refinement based on undefeated criterion suggets that when the frequency of h-type girls is very low, the parents of these girls will try to separate their daughters from the l-types. When the frequency of h-type girls becomes sufficiently high, h-type parents make no efforts to separate as the expected value of a girl to men is very close to the value of h-type girls. The implication of the model that pooling will take place at increasingly higher levels of confinement as the frequency of high type girls increases might seem counterintuitive. However, it might be the case that the marginal value of confinement as perceived by men is endogenous, and decreases with an increase in the frequency of high types. This modification will lead to pooling at lower levels of confinement. A crucial question is how to determine the prior in a society. Is it a statistic based upon past observations in the society that can only change with the behavior of the population? Or, can exogenous factors lead to changes in the prior without affecting the behavioral patterns? These questions have to be answered before we can see how our model performs because while comparing the confinement of women across societies we would have to categorize them according to low, medium, or high prior societies.

The equilibria suggest that under the assumptions of the model parents of girls can never confine their daughters enough to satisfy the concerns of richest of men. In our model these men will not even offer proposals to any girl. We interpret this as suggesting that if they get married they will have an incentive to guard their wives. This might be the reason for continued confinement of girls married to rich men when these men do have the option of not doing so.

CHAPTER III

EVOLUTION OF DIVISION RULES

A. Introduction

...the unjust is what violates the proportion;
for the proportional is intermediate,
and the just is proportional.

Nicomachean Ethics, Aristotle

The literature on non-cooperative bargaining has primarily focused on the question of how agents would divide a given amount of surplus. The simplest game theoretic representation of the problem is the Nash demand game involving two agents and certain rules to map demands of agents into payoffs. The multiplicity of Nash equilibria in the demand game led to the development of the axiomatic approach (Nash [32], [33]) and extensive form non-cooperative models (Rubinstein [40]) to select one out of the several Nash equilibria.

The surplus over which bargaining takes place is assumed to be exogenous in most of the studies. However, even in the most common examples alluded to in the bargaining literature(landlords and tenants, workers and management), the surplus is *created*, and the involved parties have an idea of their claims over the surplus. This paper deals with situations that could be termed as bargaining under the shadow of claims. It only deals with those bargaining problems where the initial claims of both parties are unambiguous and common knowledge, and the final surplus is not large enough to honor all the initial claims completely. The simplest example is bargaining among creditors to divide up a bankrupt estate.

The model considered in this paper has two populations of equal size, each characterized by an exogenous level of claim (c_l, c_h). Every period N pairs are formed with each pair comprising of one low and one high claimant. During each period, agents in every pair bargain over the same amount $e(\leq [c_l + c_h] = 1)$ within the framework of a *modified* Nash demand game. The agents are assumed to be myopic best responders, who sometimes exhibit inertia, and sometimes experiment with non-best response strategies. The paper tries to analyze the long run outcome of this dynamic process. Specifically, it aims to come up with a possible explanation for the ad hoc division rules (in particular, proportionality) from non-cooperative bargaining in this evolutionary setting (a framework similar to Young [46]).

Young [46] embeds the demand game in an evolutionary framework and utilizes the idea of stochastic stability to select the unique long run equilibrium. He specifies the evolutionary dynamic in a manner that makes the Nash equilibria of the demand game non-absorbing thereby allowing for transitions among the various equilibria; and then identifies that Nash equilibrium which, in some sense, is easy to get to but difficult to escape. Binmore, Samuelson and Young (BSY [7]) clarify the role of the various adaptive dynamics that have been employed in the literature to model the behavior of agents in evolutionary models of bargaining. They also provide an alternative way to identify the long run stochastically stable equilibria. However, Young [46] and BSY [7] assume the surplus to be exogenous and do not deal with the claims of the involved parties.

Moulin [29] provides an excellent survey of the ad hoc division rules that have been proposed in the literature to divide an amount of surplus that is no more than the total amount that went into creating it. The constrained equal awards rule divides the remaining estate equally subject to the constraint that no claimant gets more than her initial contribution. The most common way of dividing a surplus that is insufficient to completely satisfy all the (well defined) original contributions is to divide it proportionally to the initial contributions. The use of proportionality is widespread in both formal and informal (Ellickson [16]) environments. The *bankruptcy laws* of most countries dictate compensating creditors within a particular priority class proportionally to their initial contributions. The *York-Antwerp Rules* guiding maritime commerce have proportionality as the motivation behind the general average rule that is used to divide the cargo losses suffered during travel (Knight [26]). The truncated claims proportional rule is derived from the proportional rule through a simple modification. It first redefines the claim of an agent to be the minimum of her initial contribution and the size of the estate; and then divides the estate proportionally to the redefined (truncated) claims.

The existing literature (Aumann and Maschler [3], Thomson [43]) has looked extensively into the connection between the ad hoc division rules, the axiomatic and the cooperative bargaining solutions. There exist a few studies that try to come up with non-cooperative games that will have the division suggested by a particular ad hoc division rule as the equilibrium. For example, Dagan, Serrano, and Volij [14] assume that there exists a socially accepted rule to solve claims problems involving two agents. Under this assumption, they devise a non-cooperative game involving any finite number of agents which has the n-person generalization of this bilateral rule as its subgame perfect equilibrium. It is important to note that they do not answer how the society comes up with the particular bilateral rule.

Ellingsen and Robles [17] and Troeger [45] develop an evolutionary model in which two agents bargain (in the non-cooperative framework of the Nash demand game) over a surplus that is created by one agent's investment. However, they aim to establish that evolution eliminates the hold up problem. To understand what is unique about a division rule from a non-cooperative perspective, should we focus on the ex-ante incentives for *investors* that a particular rule for dividing the surplus would create? Or, should we focus on the *bargaining* after the surplus is realized; or, both? Ellingsen and Robles [17] show that efficient investment can be sustained in the long run even if the ultimatum game (with the investor being the responder) is used to model the bargaining interaction. Thus, it seems that several rules of the demand game can lead to efficient investment in the long run. At the same time, each set of rules for the demand game leads to a different division of the surplus. Hence, our analysis focuses on the bargaining interaction and assumes investments to be exogenously given. (It must be emphasized that both Ellingsen and Robles [17] and Troeger [45] allow only one agent to invest and the return function is riskless).

Gachter and Riedl [20] report that experimental subjects acting as a third party allocate the remaining estate (which is less than the sum of initial claims) in proportion to the initial claims. But, when two subjects having different initial claims engage in unstructured anonymous bargaining over the remaining estate, the results are statistically different from the division that is proportional to initial claims. This result motivates us to look deeper into the psychological differences that arise when a person is asked to act as a third party versus when he happens to be a bargainer himself.

Having considered the results of Troeger [45] and Gachter and Riedl [20], this paper takes off by first asking: what must be the considerations of a third party that suggests proportional division of a bankrupt estate; then tries to come up with the rules for the Nash demand game that reflect these considerations; and finally establishes the long run prediction of the evolutionary process using these rules for the demand game. It is shown that several of the ad hoc division rules can be obtained as the unique long run prediction of the evolutionary model by suitably changing the rules of the demand game. The reason behind the inability to obtain some of the ad hoc rules (that include the proportional rule) as the long run outcome will be discussed.

The structure of the paper is as follows. Section 2 describes the model in detail. A modified set of rules for the demand give is also given. Section 3 of the paper illustrates in detail the steps involved in finding out the long run stochastically stable equilibrium. Section 4 shows the importance of rules of the demand game in determining the long run outcome. If the agents bargain in the framework of the usual demand game, the long run stochastically stable equilibrium turns out to be equal division of the estate. If, in addition to the usual rules, demanding more than one's initial claim leads to a zero payoff (claims boundedness) then the long run equilibrium corresponds to the constrained equal awards rule. If the rules further specify that agents will obtain positive payoffs only if the sum of their demands equals the estate size (efficiency), then the long run divisions are those prescribed by the truncated claims proportional rule

To motivate the analysis in Sections 4.2 and 4.3, suppose we ask a person to act as a third party and divide an amount $e \leq 1$ between two agents who had initially contributed $c_l = 0.4$, and $c_h = 0.6$. Two things will most likely be observed. First, the division proposed by the third party will never give any agent more than her initial contribution for any $e \leq 1$. For example, if $e = 0.9$, no person acting as a neutral third party will violate *claims boundedness* (i.e., suggest giving the low (high) claimant more than $0.4(0.6)$). Second, the proposed divisions will be *efficient* (for example, the third party is unlikely to suggest the division $[0.4, 0.4]$ for $e = 0.9$, and let the remaining amount go waste). Instead of asking why third parties behave in this manner, the paper takes these two presumptions as helpful cues to come up with the rules that should be used to structure the bargaining between the two agents.

The rules that capture both efficiency and claims boundedness lead to the emergence of *truncated-proportional* division as the unique long run outcome. Consider the case with $(c_l, c_h, e) = (0.4, 0.6, 0.5)$. The maximum feasible payoff to the high claimant is $e = 0.5$, and she will end up losing atleast $[c_h - min(c_h, e)] = 0.1$. This is referred to as the *sunk* claim as it is beyond recovery. The equilibrium division turns out to be *as if* the claims of agents have been truncated from c_j to $min(c_j, e)$, and then the estate is being divided proportionally to these truncated claims.

The emergence of *truncated*-proportionality clarifies the reason behind the failure to obtain *exact*-proportionality as the long run outcome within this framework. This framework disregards claims that become *sunk*. On the other hand, proportionality requires dividing the leftover estate in proportion to the original contributions, even when a part of the contribution is beyond recovery for one or both the agents. It is only when the remaining estate is large enough to feasibly compensate even the high claimant, that exact-proportionality is the stochastically stable long run equilibrium.

B. The Model

The model considers a family of economies with each economy indexed by the tuple (c_l, c_h, e), where $e \leq (c_l + c_h) = 1$. Each economy in this family is assumed to consist of two distinct populations (low claimants and high claimants). The size of each population is N. Each bargaining pair consists of a L-claimant and a H-claimant. The decision of agents regarding whether to contribute, and if so how much, is not modeled explicitly. Only one level of investment for each population is considered. In every pair the low (high) claimant is assumed to have contributed

c_l (c_h). The agents come together in the form of a bargaining pair only after the realization of bankruptcy to decide upon the division of the remaining estate. It is assumed that for every pair during each time period in a particular economy, the size of the pie that remains after bankruptcy equals $e \in (0, 1]$. Thus, in a particular economy (belonging to the family) c_l, c_h, and e take the same numerical values across all pairs for all time periods. For example, there will be an economy in the family with $(c_l, c_h, e) = (0.4, 0.6, 0.5)$. So, in this economy during each time period N pairs are formed with the low (high) claimant in each pair having contributed 0.4 (0.6). Every such pair during each time period has to decide how to divide 0.5. The interest lies in figuring out the unique long run division that will emerge in each economy in the family, and then comparing them to the ad hoc division rules for claims problems.

A seemingly serious drawback of the set up is the assumption of fixed values of c_l, c_h, and e for a particular economy, irrespective of pairings and time. The first move towards greater realism might be to let e vary. However, if we allow e to take different values, then there will be no simple way of specifying the best response dynamic. For example, let $(c_l, c_h) = (0.4, 0.6)$ for all pairs at all times in an economy; but let e take two values 0.5 or 0.9. Should the two agents in a pair that are bargaining over $e = 0.5$ during the current period be allowed to draw inferences from the past play in cases with $e = 0.9$? If not, and players respond only according to the plays in the previous period that had $e = 0.5$, then allowing for two values of e does not help in any way. Troeger [45] takes this route when he assumes that agents in a pair state their current optimal demands by consulting the distribution of past demands in only those cases that had the same amount of surplus to be divided as this pair has in the current period. (This question does not even arise in Young [46] and BSY [7] as the bargaining always takes place over one unit of exogenously given surplus). Unfortunately, it is very difficult to model the other case where we wish to allow agents to best respond to all the observed cases. Thus, the assumption of fixed values of c_l, c_h, and e for a particular economy is partly for innocuous convenience and partly because of the lack of a simple theory to model learning across similar (but not the same) situations.

The bargaining interaction of a pair is modeled as a modified Nash demand game– the *blame game*. The demand game involves the two agents in a pair stating their demands simultaneously. The rules of the game specify what happens in case the sum of demands does or does not exceed the pie to be divided. The usual specification involves the agents being awarded their demands in case the sum of their demands does not exceed the pie, and the agents obtaining nothing if the sum of their demands exceeds the pie. Let d_l (d_h) be the demand of the low (high) claimant, and x_l (x_h) be her payoff. Demands greater than e are ruled out. The rules of the blame game try to capture the fact that the bargainers have well defined initial contributions. The rules are first described in words and then defined formally.

1) The agents are awarded their demands in case the sum of their demands does not exceed e.
2) If sum of the demands exceeds e, then two cases have to be considered.
    a) If both agents demand more or both demand less than their original contributions, then both get zero. This reflects the thought that we can not pin down the responsibility for the sum of demands exceeding the estate size on either agent.
    b) If one demands more than her claim and the other less than her claim, then the former receives zero and the latter receives her demand.

$$(x_l, x_h) = \begin{cases} (d_l, d_h) & if \;\; d_l + d_h \leq e \\ (0,0) & if \;\; d_l + d_h > e, d_l > c_l, d_h > c_h \\ (0,0) & if \;\; d_l + d_h > e, d_l \leq c_l, d_h \leq c_h \\ (d_l, 0) & if \;\; d_l + d_h > e, d_l \leq c_l, d_h > c_h \\ (0, d_h) & if \;\; d_l + d_h > e, d_l > c_l, d_h \leq c_h \end{cases} \tag{3.1}$$

Pairs of demands of the form $(d_l^*, d_h^*) = (d_l^*, e - d_l^*)$ that completely exhaust the pie constitute the Nash equilibria. Let D_l^* denote the range of equilibrium payoffs to the low-claimant. Formally,

$$d_l^* \in D_l^* = \begin{cases} [0, e] & if \;\; e \leq c_l \\ [0, c_l] & if \;\; c_l < e \leq c_h \\ [e - c_h, c_l] & if \;\; c_h < e \end{cases} \tag{3.2}$$

The minimum equilibrium payoff to the lower claimant is the remainder that will be left if the higher claimant is fully compensated, subject to the constraint that it should be positive. Similarly, the maximum payoff to the lower claimant equals her original contribution, subject to the constraint that $e > c_l$. This reveals that the equilibria of the modified demand game incorporate some elements of forward induction type of reasoning. The general expression is

$$d_l^* \in D_l^* = [(D_l^*)_{min}, \;\; (D_l^*)_{max}] = [max(0, e - c_h), \;\; min(e, c_l)] \tag{3.3}$$

and

$$d_h^* = (e - d_l^*) \tag{3.4}$$

In a one shot interaction the modified Nash demand game has multiple pure strategy Nash equilibria [1]with the equilibrium payoff to the low claimant varying from $(D_l^*)_{min} = max(0, e - c_h)$ to $(D_l^*)_{max} = min(e, c_l)$. The technique developed by Foster and Young [18] is utilized to find the stochastically stable equilibria. In order to do so several assumptions have to be made which are described next.

1. The Unperturbed Dynamic

The agents make their demands from the discrete and finite set $[\delta, 2\delta, \ldots, e-\delta]$ having cardinality K (assuming $e = (K+1)\delta$), where δ can be thought of as the least count of the monetary scale used in the economy. With a discrete strategy space, any efficient division of e will be of the form $((K+1-k)\delta, k\delta)$, where $k \in [1, K]$. An efficient division $((K+1-k)\delta, k\delta)$ that happens to be a Nash equilibrium of the blame game will be referred to as the k-equilibrium. The low (high) claimants will be relatively better off in an equilibrium with low (high) value of k. Evolution of the process occurs in discrete time. Assuming both populations are of equal size N, in each time period N random bargaining pairs consisting of a high and a low claimant will be formed. Each pair has complete knowledge of the original contributions, and faces the same problem of dividing e within the framework of the modified demand game. The state at the end of period t is $s_t = (n^l, n^h)_t$, where n^j is a K dimensional vector representing the number of agents in population $j \in (L, H)$ playing the pure strategy $k \in [1, K]$. The unperturbed adjustment dynamic is assumed to be the best response dynamic such that the current period demand of every agent maximizes her expected payoff given the previous period distribution of demands in the opponents' population. This dynamic specification

[1]It is only when $e = 1$ that there exists a unique Nash equilibrium for the one shot game with each agent demanding her original contribution, a result that motivated the choice of rules for the demand game.

can be concisely represented as a Markov chain $M_{(0,0)}$ on the finite state space S consisting of all pairs $s = (n^l, n^h) \in R^K \times R^K$, with $\sum n_k^l = \sum n_k^h = N$. The transition matrix for the process is denoted by $T_{(0,0)} = [p_{ij}]$, where p_{ij} is the probability that the process lands in state i at time $(t+1)$ given it was in state j at time t. The process is time homogeneous as the transition probabilities do not depend on time.

Restricting the demands of the agents to the finite set $[\delta, 2\delta, \ldots, e - \delta]$ allows us to proceed with the calculation of the stochastically stable equilibria by making the underlying Markov chain, $M_{(0,0)}$, finite. It also ensures that the best response of agents in the blame game will be a function and not a correspondence. Best response functions are more likely to lead to singleton absorbing sets for $M_{(0,0)}$ thereby resulting in the blame game being weakly acyclic.

The best response behavior of agents has at least two unpleasant implications regarding the evolution of the process, given the motivation of this study. First, suppose the process starts with all low claimants playing the same pure strategy k, and all high claimants playing the same pure strategy $k^{'}$, with $k \neq (K + 1 - k^{'})$. Given this initial state, under the best response dynamic, the process will keep cycling and the two populations will end up mis-coordinating for ever. Second, the pure strategy strict Nash equilibria will be recurrent states as the probability that the process returns to this state at some time in future, given that it is (or, was) in this state, is unity. Our interest lies in the long run behavior summarized by the stationary probability distribution over states. A probability distribution over the states is stationary if, once realized at some time, the probability distribution over states at all subsequent times remains the same. Since the recurrent class is not unique, the stationary distribution depends on the initial state. Let $v_{(0,0)}(s|s_0)$ be the relative frequency of the occurrence of state s till time t, given the initial state is s_0. Then,

$$\lim_{t\to\infty} v_{(0,0)}(s|s_0) = \mu_{(0,0)}(s|s_0) \tag{3.5}$$

In other words, the process is non-ergodic and can converge to any of the several pure strategy Nash equilibria [2] depending on the initial state, and once the process reaches any such state it gets stuck there. As a result, issues pertaining to equilibrium selection in the long run can not be addressed. Both of these problems (perpetual mis-coordination and initial state dependence) can be eliminated if deviations from best response behavior on part of the agents are introduced.

2. The Perturbed Dynamic

Following BSY [7], a perturbed best response dynamic is defined that first incorporates inertia to eliminate continual mis-coordination; and then allows the agents to play experimental non-best response strategies to overcome initial state dependence of the long run outcome. Let the probability that an agent states the same demand as in the previous period be $\lambda \in (0,1)$, and the probability with which she best responds be $(1-\lambda)$. We can now define the time homogeneous transition matrix $T_{(\lambda,0)}$. It can be proved that there always exists a $\lambda_{min} \in (0,1)$ that will get rid of the perpetual mis-coordination. When inertia is added to the best response dynamic the blame game becomes weakly acyclic. In other words, only the pure strategy Nash equilibria [47] will be the absorbing sets [3] of $M_{(\lambda,0)}$. Intuitively, even a small amount of inertia breaks the cycle of mis-coordination by moving the state

[2] If there exist absorbing sets that are not singletons, then it is also possible that the process reaches an absorbing set without actually converging to a pure strategy Nash equilibrium.

[3] Since the absorbing sets are singletons, we can now refer to them as the absorbing states

from the corners to the interior of the state space. The process will be aperiodic but not irreducible for all $\lambda \in (0,1)$. As every Nash equilibrium is still an absorbing state, the stationary distribution still depends on the initial state, and no meaningful discussion of equilibrium selection is as yet possible. Formally,

$$\lim_{t\to\infty} v_{(\lambda,0)}(s|s_0) = \mu_{(\lambda,0)}(s|s_0) \tag{3.6}$$

To make the Nash equilibria non-absorbing it is further assumed that when an agent goes on to state a demand that is different from the one in previous period, then she experiments with probability $\epsilon > 0$. While experimenting, the agent is equally likely to state any demand from the feasible set. Thus, in each period an agent responds inertially with probability λ, plays a best response with probability $(1-\lambda)(1-\epsilon)$, and engages in random experimentation with probability $(1-\lambda)\epsilon$. The time homogeneous transition matrix for the resulting process is denoted by $T_{(\lambda,\epsilon)}$.

The strategies an agent is allowed to play while experimenting might alter the equilibrium that will be selected as the (ultra) long run outcome. If the experimental strategies are chosen at random from the set of feasible strategies then every state is accessible from every other state in a finite number of periods. The process becomes irreducible, with the unique recurrent class being the whole state space. Also, the process is aperiodic because there does not exist any state to which the process will continually return with a fixed time period (greater than one). This helps us in two ways. Irreducibility implies that the process can potentially escape even a Nash equilibrium because in presence of experimentation Nash equilibria cease to be the absorbing states. Irreducibility, together with aperiodicity, implies that the stationary probability distribution over states will be unique and independent of the initial state.

It might be reasonable to assume that agents engage in state dependent experimentation. If agents experiment rationally then they will never play a strategy which (if established as the equilibrium) would give them a lower payoff than what they obtain in the currently established equilibrium. The time homogeneous transition matrix for this specification will be denoted by $T_{(\lambda,\epsilon^R)}$. This process is aperiodic but we do need to argue that it is irreducible. Recall that the low claimants receive their maximum payoff in the 1-equilibrium, and the high claimants receive their maximum payoff in the K-equilibrium. Suppose, the process is in the k-equilibrium at time t, with $1 < k < K$. There is a positive probability that all low (high) claimants rationally experiment in period $(t+1)$ by playing the strategy 1 (K) which (if established as the equilibrium) will provide them their maximum possible payoff. Now, in period $(t{+}2)$ there will be a positive probability that all agents in both populations best respond. This will lead each low claimant to play strategy K, and each high claimant to play strategy 1. At this point agents in both populations are playing their minimum payoff strategies. With rational experimentation, every state now becomes accessible. Thus, even with rational experimentation the process is irreducible for every $\epsilon > 0$ and will have a unique stationary distribution that is independent of the initial state. However, the two stationary distributions can be different. Formally,

$$\lim_{t\to\infty} v_{(\lambda,\epsilon)}(s|s_0) = \mu_{(\lambda,\epsilon)}(s) \tag{3.7}$$

$$\lim_{t\to\infty} v_{(\lambda,\epsilon^R)}(s|s_0) = \mu_{(\lambda,\epsilon^R)}(s) \tag{3.8}$$

3. Stochastic Stability

Stochastic stability relates to the limit of the stationary distribution as the probability of experimentation goes to zero. The state s is stochastically stable if

$$\lim_{\epsilon \to 0} \mu_{(\lambda,\epsilon)}(s) > 0 \tag{3.9}$$

The stochastically stable states are those that are most likely to be observed in the long run as experimental play by agents becomes exceedingly rare. The states that can be reached via experimental play by few agents, but escaped only if a large number of agents experiment, are the prime candidates for being the stochastically stable states. The predictive power of the analysis is greater the fewer is the number of states that receive positive probability weight under the above limit.

The algorithm for identifying the stochastically stable states first requires calculating the stationary distribution of the process for an arbitrary $\epsilon > 0$, and then finding the states that receive positive probability weight as ϵ approaches zero. The limiting operation is easy but the usual technique for calculating the stationary distribution is very cumbersome if the state space is large as it involves solving a huge system of equations.

The next section begins with some useful definitions from graph theory and describes how to identify the stochastically stable state(s) in a relatively straightforward manner using directed graphs (Friedlin and Wentzell [19], Young [46] and [47]).

4. The Minimal Tree

A graph comprises of two types of elements: nodes and edges. An edge connects a pair of nodes. A graph in which the edges have a sense of direction are called directed graphs. A path is a collection of alternate nodes and edges such that each node in this collection is incident to a minimum of zero and a maximum of two edges in this collection. The (in) out-degree of a node is the number of edges directed (in) out-ward at the node. A path in which all the interior nodes have in and out degree of one is called a directed path. A node is reachable from some other node in the graph if there is a directed path that starts at the latter and ends at the former. A cycle in a graph is a collection of alternate nodes and edges such that each node in this collection has in and out degree of one. A graph is acyclic if it contains no cycles, and unicyclic if it contains exactly one cycle. The graph is connected if it is possible to establish a path from any node to any other node in the graph. A tree is a connected acyclic graph. Rooted trees are directed acyclic graphs with $(|S| - 1)$ edges such that each edge is directed towards the root node, and from every node there is one and only one directed path to the root node. However, there can be several trees rooted at the same node. A weighted graph associates a real number with every edge in the graph. The weight of a path in a weighted graph is the sum of the weights of the edges in the path.

Let G_0 be the complete directed graph constructed by using each of the K absorbing states (i.e., each of the pure strategy Nash equilibria) of the time homogenous Markov chain $M_{(0,0)}$ as a node. The weight on the directed edge $(k \rightarrow k^{`})$ is taken to be the minimum number of experimenting agents required to move the process from the $k$-equilibrium to the $k^{`}$-equilibrium, often referred to as the resistance of this transition. Consider any one of the trees rooted at node $k \in [1, K]$. It will feature a directed path having $K - 1$ directed edges. The resistance of this rooted tree is defined as the sum of the resistances of the edges along its path. The resistance of each tree, rooted at each of the K nodes, can be calculated in a similar manner. Let Γ_0 represent the collection of all the trees in G_0. The stochastically stable state(s) is the one that serves as the root of the tree having minimal resistance (the minimal tree) among all the rooted trees in Γ_0 (Friedlin and Wentzell, [19]). However, if K is large then figuring out the minimal tree by explicitly calculating the resistance of each rooted tree becomes very tedious. The following section relies heavily on Young [47] and BSY [7] in establishing that the stochastically stable division of a bankrupt estate corresponds to the divisions proposed by the constrained equal awards rule, if the underlying interaction is assumed to be the blame game described in Section 2.

C. The Long Run Equilibrium

It has been argued in the previous section that the absorbing sets of $M_{(\lambda,0)}$ are singletons. This implies that $M_{(\lambda,0)}$ satisfies the no cycling condition (BSY [7]). Since every finite time-homogenous process reaches an absorbing set, $M_{(\lambda,0)}$ will eventually reach a Nash equilibrium, as only the Nash equilibria are the (singleton) absorbing sets of $M_{(\lambda,0)}$. Since we are only interested in identifying the stochastically stable equilibria we need to focus solely on $M_{(\lambda,\epsilon)}$ as ϵ tends to zero. This in turn implies that the minimal tree will be rooted at a Nash equilibrium. Hence, all we need to do is to find the tree rooted at a Nash equilibrium that has minimum total resistance. Those Nash equilibria that serve as the root of the trees having minimum total resistance will be the stochastically stable states of $M_{(\lambda,\epsilon)}$.

The equilibria of the blame game have been illustrated at the beginning of Section 2 for continuous strategy space. When the strategy space is discrete and demands are restricted to lie between δ and $(e-\delta)$, the Nash equilibria are

$$d_l^* \in D_l^* = \begin{cases} [\delta, e-\delta] & if \;\; e \leq c_l \\ [\delta, c_l] & if \;\; c_l < e \leq c_h \\ [e-c_h, c_l] & if \;\; c_h < e \end{cases} \tag{3.10}$$

Recall that d_l^* denotes an equilibrium payoff to the low claimant, and D_l^* denotes the set of equilibrium payoffs to the low claimant in the blame game.

In the following discussion only those strategies that are integral multiples of δ will be considered. Suppose, the process is currently in the equilibrium $(x, e-x)$. Let

$$X^+ = [d_l^* : d_l^* > x] \quad and \quad X^- = [d_l^* : d_l^* < x] \tag{3.11}$$

Thus, $x^+ \in X^+$ represents an equilibrium payoff to the lower claimant higher than x. The interest lies in figuring out the equilibrium that is most easily accessible from the current equilibrium at x. This most likely transition can either be on the left or on the right of x. We separately figure out the most easily accessible equilibrium to the right of x, and the most easily accessible equilibrium to the left of x. The easier of these two will in turn be termed as the most easily accessible equilibrium from the equilibrium at x. The relevant 2×2 games that need to be considered are shown in Figure 1. The game labeled $x^+ > x$ has two pure strategy Nash equilibria: $(x, e-x)$ and $(x^+, e-x^+)$. Suppose the economy is currently in the equilibrium $(x, e-x)$. The equilibrium $(x^+,\ e-x^+)$ can emerge if a sufficient number of L-claimants experiment with the higher demand of x^+. The 2×2 game helps us calculate the fraction of agents in the L-population that should randomly experiment with the higher demand of x^+ such that the best response for agents in the H-population is to demand $e - x^+$.

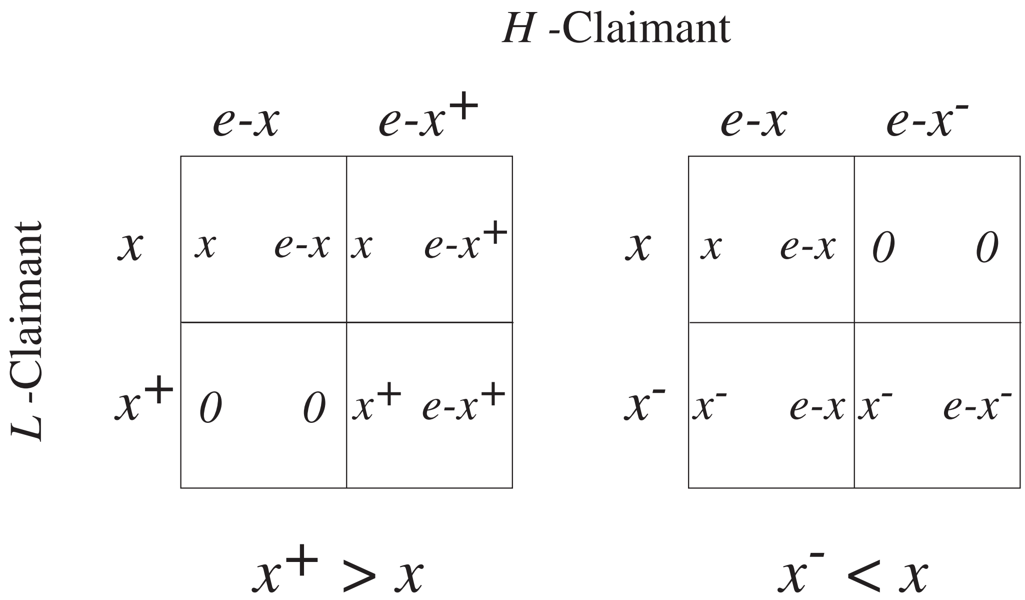

Fig. 1. The relevant games

Let $f_l[x \to x^+]$ denote the minimum fraction of L-agents that need to experiment and demand $x^+ > x$ such that the best response for H-agents is to demand $(e - x^+)$. Formally,

$$f_l.(e - x^+) + (1 - f_l).(e - x^+) = f_l.0 + (1 - f_l).(e - x) \tag{3.12}$$

$$\Rightarrow \quad f_l[x \to x^+] = \frac{x^+ - x}{e - x} \tag{3.13}$$

Similarly, let $f_h[(e - x) \to (e - x^+)]$ denote the minimum fraction of H-agents that will have to experiment with $(e - x^+)$ such that the best response for L-agents is to demand x^+. We have

$$f_h.x^- + (1 - f_h).x^- = f_h.0 + (1 - f_h).x \tag{3.14}$$

$$\Rightarrow \quad f_h[(e-x) \rightarrow (e-x^+)] = \frac{x}{x^+} \tag{3.15}$$

1. The Transition $(x \rightarrow x^+)$

The main proposition is arrived at through a sequence of simple results. Let $(x_l^+,\ e-x_l^+)$ be the equilibrium towards the right of x that is most easily accessible as a result of experiments initiated by agents in the L-population. Then

Result 1(a) *The least costly transition initiated by L-claimants towards $x^+ > x$ is the local transition*. This is because

$$x_l^+ = argmin_{x^+} \ \ f_l[x \rightarrow x^+] = argmin_{x^+} \ \ (\frac{x^+ - x}{e-x})$$

$$\Rightarrow \quad x_l^+ = (x^+)_{min} = (x+\delta) \tag{3.16}$$

Result 1(b) *The resistance to the most likely transition towards $x^+ > x$ initiated by experiments on part of L-claimants is*

$$f_l[x \rightarrow x_l^+] = \frac{\delta}{e-x} \tag{3.17}$$

Result 2(a) *The least costly transition initiated by H-claimants towards $x^+ > x$ is the extreme transition*. This is because

$$x_h^+ = argmin_{x^+} \ \ f_h[(e-x) \rightarrow (e-x^+)] = argmin_{x^+} \ \ (\frac{x}{x^+})$$

$$\Rightarrow \quad x_h^+ = (x^+)_{max} = \begin{cases} e-\delta & if \;\; e \leq c_l \\ \\ c_l & if \;\; e > c_l \end{cases} \tag{3.18}$$

Result 2(b) *The resistance to the most likely transition towards $x^+ > x$ initiated by experiments of H-claimants is*

$$f_h[(e-x) \rightarrow (e-x_h^+)] = \begin{cases} \frac{x}{e-\delta} & if \;\; e \leq c_l \\ \\ \frac{x}{c_l} & if \;\; e > c_l \end{cases} \tag{3.19}$$

Result $3(a)$ *The most likely transition from the equilibrium at x towards the right will be to the equilibrium at x^{++}, where*

$$x^{++} = \begin{cases} x+\delta & if \quad f_l[x \rightarrow x_l^+] \leq f_h[(e-x) \rightarrow (e-x_h^+)] \\ \\ e-\delta & if \quad f_l[x \rightarrow x_l^+] > f_h[(e-x) \rightarrow (e-x_h^+)] \;\&\; e \leq c_l \\ \\ c_l & if \quad f_l[x \rightarrow x_l^+] > f_h[(e-x) \rightarrow (e-x_h^+)] \;\&\; e > c_l \end{cases} \tag{3.20}$$

Result $3(b)$ *The resistance for the least costly transition for the equilibrium at x will be*

$$r^+(x) = \begin{cases} \frac{\delta}{e-x} & if \quad f_l[x \rightarrow x_l^+] \leq f_h[(e-x) \rightarrow (e-x_h^+)] \\ \\ \frac{x}{e-\delta} & if \quad f_l[x \rightarrow x_l^+] > f_h[(e-x) \rightarrow (e-x_h^+)] \;\&\; e \leq c_l \\ \\ \frac{x}{c_l} & if \quad f_l[x \rightarrow x_l^+] > f_h[(e-x) \rightarrow (e-x_h^+)] \;\&\; e > c_l \end{cases} \tag{3.21}$$

Result 4 *The least costly transition towards the right will be the local transition initiated by experiments of L-agents.*

For this result to be true it has to be proved that the most likely transition towards the right of any established equilibrium which is initiated by experiments of L-claimants requires less number of experimenting agents than the most likely transition initiated by H-claimants. Formally, we require

$$f_l[x \rightarrow x_l^+] \leq f_h[(e-x) \rightarrow (e-x_h^+)] \Rightarrow \begin{cases} x(e-x) \geq \delta(e-\delta) & if \;\; e \leq c_l \\ x(e-x) \geq \delta c_l & if \;\; e > c_l \end{cases} \tag{3.22}$$

It can be easily verified that both these inequalities hold true. The term $[x(e-x)]$ is the Nash product at the current equilibrium $(x, e-x)$. If $e \leq c_l$, the minimum value of Nash product $[\delta(e-\delta)]$ occurs when d_l^* is δ, or $e-\delta$. Hence, $x(e-x)$ will not be less than $\delta(e-\delta)$ if $e \leq c_l$. If $e > c_l$, then $x(e-x) \geq \delta(e-\delta) \geq \delta c_l$. The second part of this inequality holds because $e-\delta \geq c_l$. The first part holds because $\delta(e-\delta)$ is the minimum value of the Nash product. This completes the proof of Result 4.

2. The Transition $(x^- \leftarrow x)$

The procedure for calculating the equilibrium towards the left of the current equilibrium x that requires fewest experiments is the same. Calculations show that

$$f_l[x^- \leftarrow x] = (\frac{e-x}{e-x^-}) \tag{3.23}$$

$$f_h[(e-x^-) \leftarrow (e-x)] = (\frac{x-x^-}{x}) \tag{3.24}$$

Result 5 *The least costly transition initiated by L-claimants towards $x^- < x$ is the jump to the left most extreme*. This is because

$$x_l^- = argmin_{x^-} \;\; f_l[x^- \leftarrow x] = argmin_{x^-} \;\; (\frac{e-x}{e-x^-})$$

$$\Rightarrow \quad x_l^- = (x^-)_{min} = \begin{cases} \delta & if \;\; e \leq c_h \\ e - c_h & if \;\; e > c_h \end{cases} \tag{3.25}$$

Result 6 *The least costly transition initiated by H-claimants towards $x^- < x$ is the local transition to $(x - \delta)$.* This holds because

$$x_h^- = argmin_{x^-} \;\; f_h[(e - x^-) \leftarrow (e - x)] = argmin_{x^-} \;\; (\frac{x - x^-}{x})$$

$$\Rightarrow \quad x_h^- = (x^-)_{max} = (x - \delta) \tag{3.26}$$

Result 7 *The least costly transition from any equilibrium x towards the left will be the local transition initiated by the experiments of H-claimants.*

To establish this result it has to be proved that $f_h[(e - x^-) \leftarrow (e - x)] \leq f_l[x_l^- \leftarrow x]$. Note that

$$f_h[(e - x^-) \leftarrow (e - x)] \leq f_l[x_l^- \leftarrow x] \Rightarrow \begin{cases} x(e - x) \geq \delta(e - \delta) & if \;\; e \leq c_h \\ x(e - x) \geq \delta c_h & if \;\; e > c_h \end{cases} \tag{3.27}$$

The two inequalities indeed hold true as can be easily verified. The most likely transition towards left is again the local transition. However, it is initiated by the experiments of H-claimants. The resistance for moving from any equilibrium towards the most easily accessible equilibrium on left thus becomes

$$r^-(x) = \frac{\delta}{x} \tag{3.28}$$

Result 8 *It is easily verified that $r^+(x)$ is monotonically increasing in x, and $r^-(x)$ is* *monotonically decreasing in* x. $r^+(x) < r^-(x)$ *for all* x *that support a Nash equilibrium if* $e > 2c_l$. r^+ *intersects* $r^-(x)$ *at* $\frac{1}{2}e$ *if* $e \leq 2c_l$. *From an existing equilibrium* x *the least costly transition is to the equilibrium at* $(x + (-)\delta)$ *if* $r^+(x) < (>) r^-(x)$.

Proposition 1 *The stochastically stable equilibrium corresponds to the constrained equal awards rule if bargaining among the claimants takes place under the rules of the blame game .*

The minimal tree is given by the lower envelope of $[r^+(x), r^-(x)]$ when all least cost transitions are local (Young [47]). This minimal tree is rooted at $\frac{1}{2}e$ or c_l depending upon whether e is smaller or greater than $2c_l$. The constrained equal awards rule divides the estate equally subject to the constraint that no claimant gets more than her original contribution. Both agents get half of the estate if the estate is less than $2c_l$; the low claimant gets c_l if the estate is more than $2c_l$. Hence, the stochastically stable equilibrium exactly corresponds to the constrained equal awards rule. If we only allow for rational experimentation by agents then all transitions that involve a jump to an extreme will apriori be ruled out. However, the stochastically stable equilibrium remains unchanged as the least costly transitions for the demand game considered in this section always happen to be the local transitions initiated by rational experiments.

It is worth noting that Proposition 1 can also be obtained by a different choice of rules for the demand game. Consider the usual Nash demand game. Suppose, we add to it the rule that an agent demanding more than her initial contribution gets nothing, irrespective of the demand of her opponent. This can be interpreted as imposing claims boundedness on the final payoffs. The reader can verify that the Nash equilibria of the one shot demand game that imposes *claims boundedness* to the usual demand game are the same as those of the blame game. This gives us the following corollary to Proposition 1.

Corollary 1 *The stochastically stable equilibrium corresponds to the constrained equal awards rule if the requirement of claims boundedness is added to the rules of the usual Nash demand game*.

D. Importance of Rules of the Demand Game

The rules of the demand game determine the payoffs resulting from the demands of agents and consequentially affect the analysis in two important ways. First, the rules determine the set of Nash equilibria of the one-shot demand game. Second, they bear upon the *criterion* that determines the most likely transition from an established equilibrium. This section analyzes the long run behavior of the process under some reasonable rules of the demand game. First the usual demand game is considered, and then the rules are modified to incorporate the idea of efficiency and claims boundedness as discussed in Section 1.

1. The Usual Demand Game

Let us consider the same basic set up as in the previous section with the only change being that the rules of the underlying game are those used in the usual Nash demand game.

1) Agents are awarded their demands if the sum of their demands does not exceed e.
2) They obtain nothing if the sum of their demands exceeds e.

For any $e \in (\delta, 1]$, the Nash equilibrium strategy vector will be of the form $(x, e-x) = (d_l^*, d_h^*) = (d_l^*, e - d_l^*)$, where $d_l^* \in [\delta, e-\delta]$. The search for the minimal tree will involve the consideration of exactly the same two games shown in Figure 1. The analysis is much easier because the equilibrium payoffs to the agents no longer vary with the value of e in relation to c_l and c_h. Note that the minimum value of the nash product will be $[\delta(e-\delta)]$, at $x = \delta$, and $(e-\delta)$. Thus, the nash product at any $x \in [2\delta, e-2\delta]$ will be greater than the Nash product at the extremes. This leads to the following simple result.

Result *Local transitions initiated by rational experiments are least costly*. Formally,

$$x(e-x) \geq \delta(e-\delta) \Rightarrow f_l[x \rightarrow x_l^+] \leq f_h[(e-x) \rightarrow (e-x_h^+)]$$

$$\Rightarrow \quad x^{++} = (x+\delta) \quad and \quad r^+(x) = \frac{\delta}{e-x} \tag{3.29}$$

Similarly,

$$x(e-x) \geq \delta(e-\delta) \Rightarrow f_h[(e-x) \leftarrow (e-x_h^-)] \leq f_l[x \leftarrow x_l^-]$$

$$\Rightarrow x^{--} = (x-\delta) \quad and \quad r^-(x) = \frac{\delta}{x} \tag{3.30}$$

Proposition 2 *Equal division is the stochastically stable equilibrium if we employ the rules of the usual demand game.*

The lower envelope of $[r^+(x), r^-(x)]$ gives the minimal tree, and the intersection of the two curves (if it exists) serves as root of the minimal tree since all least cost transitions are local. The minimal tree is rooted at $\frac{1}{2}e$, and thus equal division is the stochastically stable equilibrium. This is because

$$r^+(x) = r^-(x) \quad \Rightarrow \quad \frac{\delta}{e - x_{ss}} = \frac{\delta}{x_{ss}}$$

$$\Rightarrow x_{ss} = \frac{1}{2}e \tag{3.31}$$

2. Efficiency

Within the same dynamic framework, now consider the following rules for the demand game.

1) Agents are awarded their demands if the sum of demands *equals* e.
2) They obtain nothing if the sum of demands exceeds or falls short of e.

It is straightforward to see that the Nash equilibria of this demand game in a one-shot interaction will be of the form $(d_l^*, e - d_l^*)$. Since all the off-diagonal payoffs are zero, we will have to redo the analysis for identifying the stochastically stable equilibrium by considering the two games shown in Figure 2. All the notations carry the same meaning as in Section 3. We have

$$f_l[x \to x^+] = \frac{(e-x)}{(e-x)+(e-x^+)} \quad \Rightarrow \quad x_l^+ = (x^+)_{min} = (x+\delta) \tag{3.32}$$

$$f_h[x \to x^+] = \frac{x}{x+x^+} \quad \Rightarrow \quad x_h^+ = (x^+)_{max} = (e-\delta) \tag{3.33}$$

The least costly transition from any x towards the right will be the transition to the extreme right initiated by the H-claimants experimenting with their lowest payoff strategy. This is because $f_l[x \to x_l^+] \geq f_h[x \to x_h^+]$, for all $x \in [\delta, e-\delta]$.

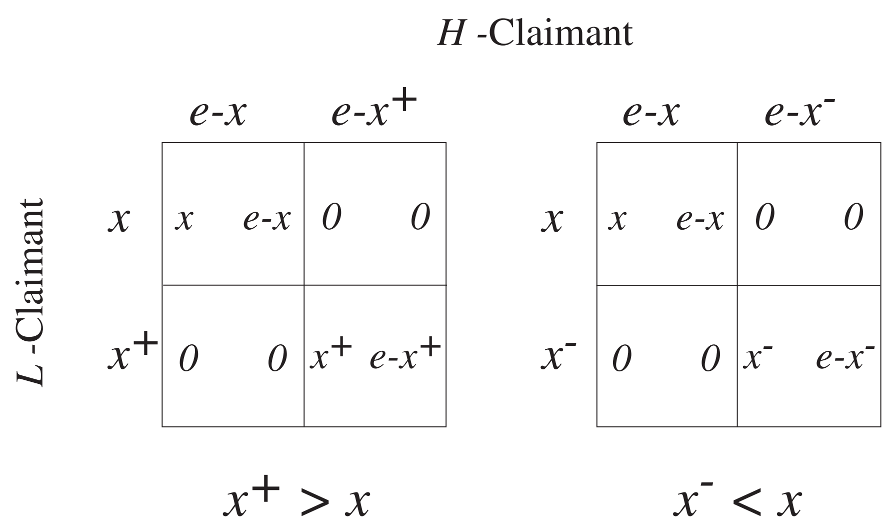

Fig. 2. The relevant games with efficiency

Similar calculations for the transitions towards left of x give

$$f_l[x^- \leftarrow x] = \frac{(e-x)}{(e-x)+(e-x^-)} \Rightarrow x_l^- = (x^-)_{min} = \delta \tag{3.34}$$

$$f_h[x^- \leftarrow x] = \frac{x}{x+x^-} \Rightarrow x_h^- = (x^-)_{max} = (x-\delta) \tag{3.35}$$

The least costly transition from any x towards the left will be the transition to the extreme left initiated by the L-claimants experimenting with their lowest payoff strategy. This is because $f_l[x \rightarrow x_l^-] < f_h[x \rightarrow x_h^-]$, for all $x \in [\delta, e-\delta]$. Thus, the resistance functions are

$$r^+(x) = \frac{x}{x+(x^+)_{max}} = \frac{x}{x+(e-\delta)} \tag{3.36}$$

$$r^-(x) = \frac{(e-x)}{(e-x)+(e-(x^-)_{min})} = \frac{(e-x)}{(e-x)+(e-\delta)} \tag{3.37}$$

Given an existing equilibrium at x, the least costly transition will be to the equilibrium on extreme right if

$$r^+(x) \leq r^-(x) \Rightarrow \frac{x}{(x^+)_{max}} \leq \frac{(e-x)}{(e-(x^-)_{min})} = \frac{(e-x)}{(e-x)_{max}} \tag{3.38}$$

The kalai-Smorodinsky solution for this bargaining problem would be the payoff vector $(x_{ks}, e - x_{ks})$, such that

$$\frac{x_{ks}}{x_{max}} = \frac{(e-x_{ks})}{(e-x)_{max}} \tag{3.39}$$

Thus, if the existing equilibrium provides the low claimant a payoff lower (higher) than what she would get in the Kalai-Smorodinsky solution, then the least costly transition will be to the equilibrium on extreme right (left). It is clear from the above calculations that $r^+(x)$ and $r^-(x)$ intersect at x_{ks}. We might be tempted to conclude that the lower envelope of $[r^+(x), r^-(x)]$ gives the minimal tree which in turn is rooted at x_{ks}, and thus the stochastically stable equilibrium should be $(x_{ks}, e - x_{ks}) = (\frac{1}{2}e, \frac{1}{2}e)$. It was mentioned earlier that this reasoning is applicable only when the least costly transitions are local. However, Proposition 10 of BSY [7] tells us that the stochastically stable equilibrium is indeed $(x_{ks}, e - x_{ks}) = (\frac{1}{2}e, \frac{1}{2}e)$. The result is summarized in the following proposition.

Proposition 3 *Equal division of the bankrupt estate is the stochastically stable equilibrium if the rules of the demand game ask for efficiency.*

3. Efficiency and Claims Boundedness

Efficiency alone is not enough for proportionality to emerge. Consider the following set of rules that are designed to capture both claims boundedness and efficiency.

$$(x_l, x_h) = \begin{cases} (d_l, d_h) & if \;\; d_l + d_h = e \\ (d_l, 0) & if \;\; d_l + d_h = e, d_l \le c_l, d_h > c_h \\ (0, d_h) & if \;\; d_l + d_h = e, d_l > c_l, d_h \le c_h \\ (0, 0) & if \;\; d_l + d_h \neq e \end{cases} \tag{3.40}$$

The set of Nash equilibria will consist of some allocations in which one agent gets zero payoff. Since such equilibria will be trivially easy to escape, stochastic stability calculations will be unaffected if these equilibria are ignored. The subset of Nash equilibria with strictly positive payoffs to the low claimants are given by

$$d_l^* \in \begin{cases} [\delta, e - \delta] & if \;\; e \le c_l \\ [\delta, c_l] & if \;\; c_l < e \le c_h \\ [e - c_h, c_l] & if \;\; c_h < e \end{cases} \tag{3.41}$$

The relevant games that need to be considered are again those illustrated in Figure 2. Following the same procedure and using the same notations we have

$$f_l[x \to x^+] = \frac{(e - x)}{(e - x) + (e - x^+)} \Rightarrow x_l^+ = (x^+)_{min} = (x + \delta) \tag{3.42}$$

$$f_h[x \to x^+] = \frac{x}{x + x^+} \Rightarrow x_h^+ = (x^+)_{max} = \begin{cases} e - \delta & if \;\; e \le c_l \\ c_l & if \;\; c_l < e \end{cases} \tag{3.43}$$

Similarly,

$$f_l[x^- \leftarrow x] = \frac{(e-x)}{(e-x)+(e-x^-)} \Rightarrow x_l^- = (x^-)_{min} = \begin{cases} \delta & if \;\; e \leq c_h \\ e - c_h & if \;\; c_h < e \end{cases} \tag{3.44}$$

$$f_h[x^- \leftarrow x] = \frac{x}{x+x^-} \Rightarrow x_h^- = (x^-)_{max} = (x-\delta) \tag{3.45}$$

It turns out that from any x the most likely transitions in either direction are the extreme transitions for all possible cases. Formally, $f_l[x \rightarrow x_l^+] \geq f_h[x \rightarrow x_h^+]$ and $f_h[x_h^- \leftarrow x] \geq f_l[x_l^- \leftarrow x]$

This further implies that the resistances take the same form as in the previous case. It was established in the previous case that from any established equilibrium at x the most likely transition will be to the extreme right (left) if the low claimants are receiving a lower (higher) payoff than suggested by the kalai-Smorodinsky solution. The Kalai-Smorodinsky division, which will also be the stochastically stable equilibrium, is

$$(x_l^{ss}, x_h^{ss}) = \begin{cases} (\frac{1}{2}e, \frac{1}{2}e) & if \;\; e \leq c_l \\ ([\frac{c_l}{c_l+e}]e, [\frac{e}{c_l+e}]e) & if \;\; c_l < e \leq c_l \\ (c_l e, c_h e) & if \;\; c_h < e \end{cases} \tag{3.46}$$

The stochastically stable equilibria exactly correspond to the truncated claims proportional rule. This rule first truncates the claim of each agent from c_j to $min(c_j, e)$; and then divides the estate proportionally to the truncated claims. This result in summarized in the following proposition.

Proposition 4 *The division suggested by the truncated claims proportional rule is the* *stochastically stable equilibrium if the rules of the demand game require both efficiency and claims boundedness*.

4. Why Not Proportional?

The results obtained in the previous section help us understand why the proportional division does not emerge as the stochastically stable equilibrium. For ease of exposition let us consider an example with $c_l = 0.4$, $c_h = 0.6$, and $e = 0.5$. Since $c_h > e$, the high claimant will definitely end up loosing the amount of $(c_h - e)$, henceforth referred to as the *sunk* claim. (It is easy to see that one, both, or none of the agents might have some sunk claims depending on the particular values of c_l, c_h, and e). If we consider this particular set of values then the proportional division would be $(2, 3)$. The usual demand game (Section 3.3), the modified demand game (Section 3.1-2), and the demand game requiring absolute efficiency (Section 4.1) will predict $(2.5, 2.5)$ as the stable division. The demand game requiring both absolute efficiency and claims boundedness (Section 4.2) will predict $(\frac{20}{9}, \frac{25}{9})$.

In all the formulations of the demand game the agents were restricted to demand no more than e, for obvious reasons. Irrespective of the rules of the demand game, that part of an agent's claim which is *sunk* will never figure in the calculations. However, as discussed earlier, when someone is asked to act as a third party his prescriptions will, in all likelihood, satisfy claims boundedness. More importantly, there is nothing to prevent this third party from giving serious thought to the original contribution of the high claimant (0.6), and not just the maximum feasible payoff satisfying claims boundedness $(min(0.6, 0.5) = 0.5)$. This suggests the following conjecture.

Conjecture 1 *An ad hoc division rule that uses the initial claims of agents while dividing* *the estate can not be obtained as the unique stochastically stable equilibrium by any choice of rules for the demand game.*

Most of the the division rules truncate the claims from c_j to $min(c_j, e)$, either directly or indirectly. It is transparent that even if some $c_j > e$, the proportional rule still uses the original contributions (c_h, c_l) to divide the remaining estate. This implies that the proportional rule does take into account the sunk claims which in turn makes it impossible to obtain it as the stochastically stable equilibrium in the evolutionary framework of this paper. It can not be denied that proportional division possesses certain properties that make it very attractive.For example, using proportional division ensures that there is no benefit to an agent from splitting her claim into several smaller claims, or merging her claim with the claims of other agents (Moulin [29]). However, this transfer-proofness of proportional division rule is vacuous when there are only two claimants.

E. Conclusion

The main question the paper tries to address is how to divide up scarce resources when the involved parties have claims over it. A simple example is the question of dividing up a bankrupt estate among the creditors. The existing literature has tried to come up with ad hoc rules from the perspective of a neutral third party. Proportional division is the most prominent of the several ad hoc rules. The ad hoc rules differ from each other because of the axioms that are imposed *in addition to* efficiency and claims boundedness. Efficiency requires that the estate be completely divided between the claimants, and claims boundedness requires that no claimant be awarded more than her initial contribution. This paper tries to explore if a rule will emerge in the long run if agents are asked to bargain amongst themselves. It thus deals with bargaining problems with verifiable initial claims of both parties. The surplus over which bargaining takes place is assumed to be insufficient to honor all the claims completely.

It is shown that an ad hoc rule can be rationalized as the unique self-enforcing long run outcome of Young's [46] evolutionary bargaining model by adding certain intuitive rules to the usual Nash demand game. If the agents bargain in the framework of the usual demand game, the long run stochastically stable equilibrium turns out to be equal division of the estate. If, in addition to the usual rules, demanding more than one's initial claim leads to a zero payoff (claims boundedness) then the long run equilibrium corresponds to the constrained equal awards rule. If the rules capture both claims boundedness and efficiency, then the long run divisions are those prescribed by the truncated claims proportional rule.

Proportional division of a bankrupt estate among the creditors seems so just and obvious that it is rarely debated. If we ask a person to act as an arbiter in such a case, the answer will most likely be to divide the estate proportionally to initial contributions. However, the inability of the framework to account for sunk claims stops us short of obtaining exact proportionality.

CHAPTER IV

INEQUALITY, INSECURE PROPERTY, AND CONFLICT

A. Introduction

Over the last two decades there has been an increasing interest in analyzing models of conflict in economies in which property rights are not well defined or difficult to enforce. The focus of this literature has been to characterize the equilibrium level of investments made by the agents in the economy to secure a fraction of the total output or the total inputs available in the economy using a non-cooperative game theoretic framework. Such investments are socially unproductive and lead to welfare losses which could be prevented if property rights were enforceable. These studies are related to the rent seeking literature wherein agents compete to win a prize (or a share of the prize) by making investments which are directly unproductive but determine the probability with which an agent obtains the prize.

Skaperdas [41] provides the canonical static model of conflict in an economy of two agents where the output depends on the simultaneous choice of productive inputs by both the agents out of their exogenously given equal endowments. The remaining endowment is invested in activities that are directly unproductive but influence the share of output an agent obtains. Neary [34] generalizes this model by incorporating inequalities in the initial endowments of the agents. Muthoo [31] analyzes a repeated game wherein agents produce their own output and have to decide whether to steal the other agent's output, given exogenous probabilities of successfully doing so. He concludes that if the agents are sufficiently patient then there exists a subgame perfect Nash equilibrium in which no agent tries to steal the other agents' output and thereby shows that respect for property rights can emerge even in the absence of an external enforcement agency.

Our interest lies in analyzing conflict over inputs in the production process and not the output. This is a crucial difference since a conflict over output, even in a repeated setting, does not affect the initial conditions in subsequent periods, but a conflict over inputs does. Grossman and Kim [21] consider a static model of conflict over inputs between two agents endowed with unequal amounts of the input that can be used to first produce defensive weapons, and then to produce output or offensive weapons. Their aim is to highlight that there exist equilibria in which both agents will invest in producing defensive weapons but not offensive weapons. Hirshleifer [24] considers a semi-dynamic model in which the fraction of the total input available in the economy an agent controls in the current period depends on his investment in weapons during the previous period. The focus of his paper is to analyze the conditions under which the economy converges to equilibrium allocations.

In this paper, we try to provide a static model of conflict in which agents care about current consumption and future shares of input. We believe this simple formulation better captures the realities of a situation of conflict over inputs. Specifically, we consider an economy of two agents, initially holding unequal amounts of the total available land. The agents produce output from the land they hold which in turn can be allocated to consumption or the production of guns. The future holdings of land are determined by the guns produced by both agents according to an exogenously assumed functional form. (This is referred to as the contest success function in the literature on conflict and rent seeking). Agents maximize the weighted sum of utility from current consumption and the utility from future land holding by simultaneously choosing how much to invest in guns. We characterize equilibria in which only one, both, and none of the agents produces guns, as a function of the total land and the inequality of initial land holdings.

Section 2 describes the set up of the model in detail and proves the existence and uniqueness of Nash equilibrium for general forms of utility function, production function, cost function for producing guns, and the contest success function. Given the total amount of land in the economy (L) and its initial distribution $(\theta, (1-\theta))$ between the two agents, four types of equilibria are possible- guns by none, guns by only one, and guns by both. In Section 3 we analyze the mapping between an (L, θ) pair and the type of equilibrium that arises when agents have the same utility, production, and cost functions. It also provides some comparative static results regarding the amount of guns being produced and the effect of higher valuations of future land holdings relative to current consumption. Section 4 considers the effect of simple forms of heterogeneities in the production and cost functions across the two agents. The discussion till Section 4 utilizes a contest success function that assumes agents share the land equally if none of them produces guns. Section 5 provides a condensed analysis of the problem using a contest success function that assumes that land holdings of agents do not change if none of them produces guns. Section 6 concludes and provides a brief discussion of the relation between the model used in this paper, the models employed in the literature on conflict, and the rent seeking literature.

B. The Model

$L(>0)$ is the total amount of land in the economy. $\theta \in (0, 0.5]$ is the fraction of the total land initially held by the poor agent. Thus the poor agent controls θL units and the rich agent controls $(1-\theta)L$ units of land initially. Land is the only input in production. Each agent decides how to allocate his output to consumption and production of guns. The cost of producing g units of guns is given by $c(g)$. Guns determine the future distribution of land between the two agents. The final fraction of land held by the poor agent is given by the contest success function $f_p(g_p, g_r)$, with $f_p(g_p, g_r) + f_r(g_p, g_r) = 1$. Each agent derives utility from current consumption and the amount of land he will hold in the future. $\beta(> 0)$ is the weight agents put on utility from future land holdings. $\lambda_y > 0$ and $\lambda_c > 0$ reflect the difference in the technology of producing output and guns between the two agents. The total utility of poor agent is denoted by V_p and that of rich agent by V_r, where

$$V_p[g_p, g_r] \quad = \quad u_p[\lambda_y y_p(\theta L) - \lambda_c c_p(g_p)] + \beta L f_p[g_p, g_r] \tag{4.1}$$

$$V_r[g_p, g_r] \quad = \quad u_r[y_r((1-\theta)L) - c_r(g_r)] + \beta L f_r[g_p, g_r] \tag{4.2}$$

The aim of the paper is to characterize the amount of guns being produced in equilibrium as it reflects the welfare loss in the economy due to the absence of well defined property rights. We are interested in obtaining results for general forms of utility function, cost function, production function and the contest success function.

Assumption 1: $u' \geq 0$, $u'(0) = \infty$, $u'' < 0$, $u''' > 0$, $y' > 0$, $y'' \leq 0$, $c' > 0$, and $c'' \geq 0$. The marginal utility of consumption is positive and decreasing for all levels of consumption. The marginal utility of consumption is assumed to be infinite at $c = 0$ to ensure that both agents devote a strictly positive amount of their output to consumption for all values of L and θ under consideration. The production function is weakly concave, and the cost of producing guns is weakly convex.

Assumption 2(a): $0 < f_i < 1$, $\frac{\partial f_i}{\partial g_i} > 0$, $\frac{\partial^2 f_i}{\partial g_i^2} < 0$, $\frac{\partial f_i}{\partial g_j} < 0$, $\frac{\partial^2 f_i}{\partial g_j \partial g_i} \geq 0$ if $g_i \geq g_j$, $f(g,g) = 0.5$, $\frac{\partial f_i}{\partial g_i} = \frac{\partial f_j}{\partial g_j}$ at(g,g), $\frac{\partial}{\partial g_i}[\frac{\frac{\partial f_i}{\partial g_i}}{\frac{\partial f_j}{\partial g_j}}] \leq 0$.

Guns affect the future distribution of land. The future land holding of both agents is positive because we assume that infinite investment in guns is required by an agent to capture all the land, for any given finite investment in guns by the other agent. The marginal effect of guns is assumed to be positive but decreasing in one's own investment in guns for all levels of guns of the other agent. For any given amount of one's own guns, one's future land holding decreases as guns of the other agent increase. Agents share the land equally if they have the same amount of guns. An example of a contest success function that satisfies these assumptions is

$$f_p(g_p, g_r; \alpha) = \frac{0.5\alpha + h(g_p)}{\alpha + h(g_p) + h(g_r)}$$

where $\alpha > 0$ is a constant and $h(\cdot)$ is a increasing and concave function. The higher is α, the lower will be the effectiveness of guns in determining the final land holdings. Most of the contest success functions used in the conflict and rent seeking literatures are special cases of this specification (Amegashie [2]).

It might be argued that if both agents do not produce any guns then the final fractions held by the two agents should equal the initial fractions. This leads us to consider an alternative specification of the contest success function whose properties are summarized below.

Assumption 2(b): $0 < f_i^\theta < 1$, $\frac{\partial f_i^\theta}{\partial g_i} > 0$, $\frac{\partial^2 f_i^\theta}{\partial g_i^2} < 0$, $\frac{\partial f_i^\theta}{\partial g_j} < 0$, $f_p(0,0) = \theta$, $\frac{\partial f_p^\theta}{\partial \theta} > 0$, $\frac{\partial}{\partial \theta}[\frac{\partial f_p^\theta}{\partial g_p}] < 0$, $\frac{\partial}{\partial \theta}[\frac{\partial f_r^\theta}{\partial g_r}] > 0$, $\frac{\partial}{\partial g_i}[\frac{\frac{\partial f_i^\theta}{\partial g_i}}{\frac{\partial f_j^\theta}{\partial g_j}}] < 0$, $\frac{\partial^2 f_p^\theta}{\partial g_r \partial g_p} < 0$ if $g_p < g_r$, $\frac{\partial f_r^\theta}{\partial g_r} \rightarrow 0, if \theta \rightarrow 0$.

An example of a contest success function that satisfies Assumption $2(b)$ is

$$f_p^{\theta}(g_p, g_r; \theta, \alpha) = \frac{\theta\alpha + h(g_p)}{\alpha + h(g_p) + h(g_r)}$$

This contest success function embodies history dependence in the final shares and has not been studied in the literature on conflict to the best of our knowledge. Corchon [11] uses a function based on the same idea of history dependence to analyze rent seeking expenditures in a model where agents have unequal prior probabilities of obtaining the prize.

1. Existence and Uniqueness of Equilibrium

The poor agent chooses $g_p \in [0, c_p^{-1}[\frac{1}{\lambda_y} y_p(\theta L)]]$ to maximize V_p taking g_r as given. Similarly, the rich agent chooses $g_r \in [0, c_r^{-1}[y_r((1-\theta)L)]]$ to maximize V_r taking g_p as given. The optimal values (g_p^*, g_r^*) constitute the Nash equilibrium. Note that

$$\frac{\partial V_p}{\partial g_p} = -\lambda_c u_p'[\lambda_y y_p(\theta L) - \lambda_c c_p(g_p)].c_p'(g_p) + \beta L \frac{\partial f_p}{\partial g_p} \quad \forall\, g_p, g_r \tag{4.3}$$

The term $\lambda_c u_p'[y_p(\theta L) - c_p(g_p)].c_p'(g_p)$ gives the marginal cost of allocating output to guns. $\beta L \frac{\partial f_p}{\partial g_p}$ gives the corresponding marginal benefit. (For the sake of brevity we will not write the arguments whenever unnecessary. We will however keep the subscripts to distinguish the agents.) The marginal total utility from allocating output to guns is decreasing with respect to one's own investment in guns, as

$$\frac{\partial^2 V_p}{\partial g_p^2} = -\lambda_c[u_p'.c_p'' - \lambda_c u_p''.(c_p')^2] + \beta L \frac{\partial^2 f_p}{\partial g_p^2} < 0 \quad \forall\, g_p, g_r \tag{4.4}$$

Also,

$$\frac{\partial^2 V_p}{\partial g_r \partial g_p} = \beta L \frac{\partial^2 f_p}{\partial g_r \partial g_p} \quad \forall\, g_p, g_r \tag{4.5}$$

Similarly,

$$\frac{\partial V_r}{\partial g_r} = -u_r'.c_r' + \beta L\frac{\partial f_r}{\partial g_r} \quad \forall\, g_p, g_r \tag{4.6}$$

$$\frac{\partial^2 V_r}{\partial g_r^2} = -[u_r'.c_r'' - u_r''.(c_r')^2] + \beta L\frac{\partial^2 f_r}{\partial g_r^2} < 0 \quad \forall\, g_p, g_r \tag{4.7}$$

$$\frac{\partial^2 V_r}{\partial g_p \partial g_r} = \beta L\frac{\partial^2 f_r}{\partial g_p \partial g_r} \quad \forall\, g_p, g_r \tag{4.8}$$

Since, $f_p + f_r \;=\; 1$

$$\frac{\partial f_p}{\partial g_p} + \frac{\partial f_r}{\partial g_p} = \frac{\partial f_p}{\partial g_r} + \frac{\partial f_r}{\partial g_r} = \frac{\partial^2 f_p}{\partial g_r \partial g_p} + \frac{\partial^2 f_r}{\partial g_p \partial g_r} = 0 \tag{4.9}$$

Proposition 1: There exists a unique Nash equilibrium (g_p^*, g_r^*) for any given (L, θ) pair.

Proof: The strategy set of both agents is compact and convex. The payoff function is continuous and concave in a player's own strategy ($\frac{\partial^2 V_i}{\partial g_i^2} < 0$). Moreover, the Jacobian of the implicit form of the best response functions given by

$$J(g_p, g_r) = \begin{bmatrix} -[u_p'.c_p'' - u_p''.(c_p')^2] + \beta L\frac{\partial^2 f_p}{\partial g_p^2} & \beta L\frac{\partial^2 f_p}{\partial g_r \partial g_p} \\ \beta L\frac{\partial^2 f_r}{\partial g_p \partial g_r} & -[u_r'.c_r'' - u_r''.(c_r')^2] + \beta L\frac{\partial^2 f_r}{\partial g_r^2} \end{bmatrix}$$

is clearly negative definite as both the diagonal elements are negative, and the determinant of $J(g_p, g_r)$ is positive. Hence, there exists a unique Nash equilibrium for any given (L, θ) pair (Rosen, [38]). The proposition also holds for the contest success functions of the type $f_p^\theta(\cdot)$. We can characterize the equilibria that emerge for the various feasible (L, θ) pairs based upon whether only one, both, or none of the agents invest in guns. The model can thus lead to the following four types of equilibria.

No guns $(g_p^* = 0, g_r^* = 0)$: For a given (L, θ) pair, the no guns equilibrium will arise if even after consuming all the output the marginal cost of guns is no less than the marginal benefit of guns for both agents. Or,

$$\frac{\partial V_p}{\partial g_p}|_{(g_p=0,g_r=0)} \ \leq \ 0 \qquad and \qquad \frac{\partial V_r}{\partial g_r}|_{(g_p=0,g_r=0)} \ \leq \ 0 \tag{4.10}$$

Guns by rich only $(g_p^* = 0, g_r^* > 0)$: For some (L, θ) pairs it is possible that the rich agent finds it worthwhile to forego consumption and allocate some of his output to guns, but the poor agent does not. The (L, θ) pairs that can sustain this type of equilibrium can be fully described by

$$\frac{\partial V_p}{\partial g_p}|_{(g_p=0,g_r>0)} \ \leq \ 0 \qquad and \qquad \frac{\partial V_r}{\partial g_r}|_{(g_p=0,g_r>0)} \ = \ 0 \tag{4.11}$$

Guns by poor only $(g_p^* > 0, g_r^* = 0)$: This equilibrium will characterized by

$$\frac{\partial V_p}{\partial g_p}|_{(g_p>0,g_r=0)} \ = \ 0 \qquad and \qquad \frac{\partial V_r}{\partial g_r}|_{(g_p>0,g_r=0)} \ \leq \ 0 \tag{4.12}$$

Guns by both $(g_p^* > 0, g_r^* > 0)$: The equilibrium in which both agents produce guns is given by

$$\frac{\partial V_p}{\partial g_p}|_{(g_p>0,g_r>0)} \ = \ 0 \qquad and \qquad \frac{\partial V_r}{\partial g_r}|_{(g_p>0,g_r>0)} \ = \ 0 \tag{4.13}$$

C. Equilibria with Same Utility, Production, and Cost Functions

We first present the results for the baseline model in which both agents have the same utility, production, and cost functions, i.e., $\lambda_y = \lambda_c = 1$.

Lemma 1: There does not exist an equilibrium in which only the poor agent produces guns.

Proof: The first order conditions that characterize the equilibrium in which only the poor agent produces guns are given in equation (4.12). These conditions reduce to the following two equations.

$$u_r'[y_r((1-\theta)L_{pg})]c_r'(0) \quad \geq \quad \beta L_{pg}\frac{\partial f_r}{\partial g_r} \tag{4.14}$$

$$u_p'[y_p(\theta L_{pg}) - c_p(g_p)]c_p'(g_p) \quad = \quad \beta L_{pg}\frac{\partial f_p}{\partial g_p} \tag{4.15}$$

Note that, $g_p^* > g_r^*$ implies $\frac{\partial f_r}{\partial g_r} > \frac{\partial f_p}{\partial g_p}$. Similarly, $[y_r((1-\theta)L_{pg})] > [y_p((\theta L_{pg}) - c_p(g_p))]$ and $c_p'(g_p) > c_r'(0)$ imply $u_r'[y_r((1-\theta)L_{pg})]c_r'(0) < u_p'[y_p(\theta L_{pg}) - c_p(g_p)]c_p'(g_p)$. Hence, the two first order conditions can not hold simultaneously. This in turn proves that there can not exist an equilibrium in which only the poor agent produces guns.

Let $L_{ng}(\theta)$ be the maximum value of total land in the economy till which no agent produces guns for a given θ. The collection of $L_{ng}(\theta)$ values for all $\theta \in (0, 0.5]$ constitutes the upper boundary of the no-guns region in the (L, θ) space. One implication of $Lemma$ (1) is that at a given value of θ the rich agent will produce strictly positive amount of guns for all $L > L_{ng}(\theta)$. The poor agent will also start producing guns for this value of θ once the total land in the economy reaches a critical value denoted by $L_{bg}(\theta) \geq L_{ng}(\theta)$. The collection of $L_bg(\theta)$ values for all

$\theta \in (0, 0.5]$ constitutes the lower boundary of the region in the (L, θ) space in which both agents produce guns. We characterize these boundaries in the following two lemmas.

Lemma 2: The upper boundary of the no-guns region in (L, θ) space is upward sloping and convex.

Proof: The boundary of the no guns region in the (L, θ) space is defined implicitly by

$$\frac{\partial V_r}{\partial g_r} = 0 \quad \Rightarrow \quad u_r'[y_r((1-\theta)L)].c_r'(0) = k\beta L \tag{4.16}$$

where, $k = \frac{\partial f_p}{\partial g_p}|_{(0,0)} = \frac{\partial f_r}{\partial g_r}|_{(0,0)} = constant$. Total differentiation of equation (4.15) with respect to L, and θ gives

$$[(1-\theta)u_r''.y_r'.c_r'(0) - k\beta] \cdot dL_{ng} + [-L_{ng}u_r''.y_r'.c_r'(0)] \cdot d\theta = 0 \tag{4.17}$$

$$\Rightarrow \quad \frac{\partial L_{ng}(\theta)}{\partial \theta} = \frac{L_{ng}u_r''.y_r'.c_r'(0)}{[(1-\theta)u_r''.y_r'.c_r'(0) - k\beta]} > 0 \quad \forall\, \theta \in (0, 0.5] \tag{4.18}$$

Thus, the boundary of the no-guns region is upward sloping. This boundary is convex, as

$$\frac{\partial^2 L_{ng}(\theta)}{\partial \theta^2} = \frac{k\beta L_{ng}^2(u_r'''.(y_r')^2 + u_r''.y_r'') + L_{ng}(u_r''.y_r'.c_r(0))^2}{[(1-\theta)u_r''.y_r'.c_r'(0) - k\beta]^2} > 0 \tag{4.19}$$

Lemma 3: The lower boundary of the region in which both agents produce guns is downward sloping. The amount of guns along this boundary decreases as inequality decreases.

Proof: For each $\theta \in (0, 0.5]$, the maximum amount of land that can sustain a Nash equilibrium in which only the rich agent produces guns is implicitly defined by

$$\frac{\partial V_r}{\partial g_r}|_{g_p=0, g_r>0} = 0 \qquad and \qquad \frac{\partial V_p}{\partial g_p}|_{(g_p=0, g_r>0)} = 0 \tag{4.20}$$

It has been proved in $Lemma$ (1) that for any given $\theta \in (0, 0.5]$ if $L \leq L_{ng}(\theta)$, then no agent produces guns. If $L > L_{ng}$, the rich agent allocates a strictly positive amount to guns but the poor agent may still not find it worthwhile to invest in guns as it is possible that $\frac{\partial V_p}{\partial g_p}|_{(g_p=0, g_r>0)} < 0$. As we keep increasing L, we reach a critical value of L equal to $L_{bg}(\theta)$ such that in equilibrium $\frac{\partial V_p}{\partial g_p}|_{(g_p=0, g_r>0)}$ becomes zero and $\frac{\partial V_r}{\partial g_r}|_{(g_p=0, g_r>0)}$ is also zero. At this point the poor agent will also start investing in guns. The pairs $[L_{bg}(\theta), \theta]$ define the boundary of the region in (L, θ) space above which the equilibrium involves both agents investing in guns.

The $[L_{ng}(\theta), \theta]$ boundary was completely determined by only one equation (4.15). g_p^* and g_r^* were both zero along the upper boundary of the no-guns region. The first order condition of the rich agent ($\frac{\partial V_r}{\partial g_r}|_{(g_p=0, g_r=0)} = 0$) was sufficient to determine L_{ng} as a function of θ, because

$$\frac{\partial V_r}{\partial g_r}|_{(g_p=0, g_r=0)} = 0 \qquad \Rightarrow \qquad \frac{\partial V_p}{\partial g_p}|_{(g_p=0, g_r=0)} < 0$$

It is important to note that $[L_{bg}(\theta), \theta]$ boundary above which the poor agent also invests in guns can not be analogously characterized by using only the first order condition of the poor agent ($\frac{\partial V_p}{\partial g_p}|_{(g_p=0, g_r>0)} = 0$). This is because the equilibrium investment in guns by the rich agent (g_r^*) can vary along this boundary, unlike the $[L_{ng}(\theta), \theta]$ boundary along which g_r^* is always zero. Hence, we need the first order condition of both agents. Equation (19) defines this boundary. It should be interpreted as a system of two equations in three unknowns (L, θ, g_r). The two equalities in equation (4.19) can be rewritten as

$$u_r'[y_r((1-\theta)L_{pg}) - c_r(g_r)]c_r'(g_r) - \beta L_{pg}\frac{\partial f_r}{\partial g_r} \quad = \quad 0 \tag{4.21}$$

and

$$u_p'[y_p(\theta L_{pg})]c_p'(0) - \beta L_{pg}\frac{\partial f_p}{\partial g_p} \quad = \quad 0 \tag{4.22}$$

The collection of (L, θ, g_r) values that satisfy the above two equations define the boundary in (L, θ) space above which the poor agent also invests in guns. Suppose, (L^1, θ^1, g_r^1) is one such point. If we change θ slightly, both L and g_r will (potentially) vary to ensure that the two equalities still hold. This operation can be summarized by the total differentiation of each equation with respect to L, θ, and g_r as given below.

$$[(1-\theta)u_r''.c_r'.y_r' - \beta\frac{\partial f_r}{\partial g_r}] \cdot dL_{pg} + [u_r'.c_r'' - u_r''.(c_r')^2 - \beta L_{pg}\frac{\partial^2 f_r}{\partial g_r^2}] \cdot dg_r \quad = \quad [L_{pg}u_r''.c_r'.y_r'] \cdot d\theta \tag{4.23}$$

$$[\theta u_p''.y_p'.c_p' - \beta\frac{\partial f_p}{\partial g_p}] \cdot dL_{pg} + [-\beta L_{pg}\frac{\partial^2 f_p}{\partial g_r \partial g_p}] \cdot dg_r \quad = \quad [-L_{pg}u_p''.y_p'.c_p'] \cdot d\theta \tag{4.24}$$

Define

$$\Delta_1 \quad = \quad \begin{bmatrix} (1-\theta)u_r''.c_r'.y_r' - \beta\frac{\partial f_r}{\partial g_r} & u_r'.c_r'' - u_r''.(c_r')^2 - \beta L_{pg}\frac{\partial^2 f_r}{\partial g_r^2} \\ \theta u_p''.y_p'.c_p' - \beta\frac{\partial f_p}{\partial g_p} & -\beta L_{pg}\frac{\partial^2 f_p}{\partial g_r \partial g_p} \end{bmatrix}$$

$$\Delta_1^\theta(L_{pg}) = \begin{bmatrix} L_{pg}u_r''.c_r'.y_r' & u_r'.c_r'' - u_r''.(c_r')^2 - \beta L_{pg}\frac{\partial^2 f_r}{\partial g_r^2} \\ -L_{pg}u_p''.y_p'.c_p' & -\beta L_{pg}\frac{\partial^2 f_p}{\partial g_r \partial g_p} \end{bmatrix}$$

$$\Delta_1^\theta(g_r) = \begin{bmatrix} (1-\theta)u_r''.c_r'.y_r' - \beta\frac{\partial f_r}{\partial g_r} & L_{pg}u_r''.c_r'.y_r' \\ \theta u_p''.y_p'.c_p' - \beta\frac{\partial f_p}{\partial g_p} & -L_{pg}u_p''.y_p'.c_p' \end{bmatrix}$$

The determinant of $\Delta_1^\theta(L_{pg})$, and $\Delta_1^\theta(g_r)$ is unambiguously negative. The determinant of Δ_1 is positive under a weak assumption. Using Cramer's rule we obtain

$$\frac{\partial L_{pg}}{\partial \theta} = \frac{|\Delta_1^\theta(L_{pg})|}{|\Delta_1|} < 0 \;\; \& \;\; \frac{\partial g_r}{\partial \theta} = \frac{|\Delta_1^\theta(g_r)|}{|\Delta_1|} < 0 \;\; \textit{along the } (L_{pg}, \theta) \textit{ boundary.} \tag{4.25}$$

1. Guns and Welfare

We now give comparative static results on the amount of guns that are produced as the total land or the inequality in initial land holdings changes.

Lemma 4: For $L \in (L_{ng}, L_{bg})$, the rich agent produces more guns with an increase in L holding θ constant, but less guns with an increase in θ holding constant.

Proof: Consider an economy characterized by an (L, θ) pair such that $L_{ng} < L < L_{bg}$. In this economy only the rich agent would be producing guns. Moreover, for small changes in L, or θ the poor agent will still not produce guns. The effect of an increase in land while holding inequality constant (or, vice versa) can be obtained by the appropriate total differentiation of the first order condition of the rich agent. The first order condition of the rich agent for all θ and $L \in (L_{ng}, L_{bg})$ is

$$u_r'[y_r((1-\theta)L) - c_r(g_r)]c_r'(g_r) - \beta L\frac{\partial f_r}{\partial g_r} \;=\; 0 \tag{4.26}$$

Total differentiation of this equation with respect to L and g_r, and θ and g_r gives

$$\frac{\partial g_r}{\partial L} \;=\; \frac{[\beta\frac{\partial f_r}{\partial g_r} - (1-\theta)u_r''.c_r'.y_r']}{[u_r'.c_r'' - u_r''.(c_r')^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2}]} \;>\; 0 \tag{4.27}$$

And,

$$\frac{\partial g_r}{\partial \theta} \;=\; \frac{[Lu_r''.c_r'.y_r']}{[u_r'.c_r'' - u_r''.(c_r')^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2}]} \;<\; 0 \tag{4.28}$$

Lemma 5: For $L > L_{bg}$, an increase in land for a given level of inequality leads to higher production of guns by the rich agent. The effect on the poor agents' production of guns is not necessarily monotonic.

Proof: The first order conditions that characterize an equilibrium involving production of guns by both agents are given by equation (13), and can be rewritten as

$$u_r'[y_r((1-\theta)L) - c_r(g_r)]c_r'(g_r) - \beta L\frac{\partial f_r}{\partial g_r} \;=\; 0 \;\;\&\;\; u_p'[y_p(\theta L) - c_p(g_p)]c_p'(g_p) - \beta L\frac{\partial f_p}{\partial g_p} \;=\; 0 \tag{4.29}$$

Total differentiation of this system with respect to g_r, g_p, and L gives

$$[u_r'.c_r'' - u_r''.(c_r')^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2}]\cdot dg_r + [-\beta L\frac{\partial^2 f_r}{\partial g_p \partial g_r}]\cdot dg_p \;=\; [-(1-\theta)u_r''.c_r'.y_r' + \beta\frac{\partial f_r}{\partial g_r}]\cdot dL \tag{4.30}$$

$$[-\beta L\frac{\partial^2 f_p}{\partial g_r \partial g_p}]\cdot dg_r + [u_p'.c_p'' - u_p''.(c_p')^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2}]\cdot dg_p \;=\; [-\theta u_p''.y_p'.c_p' + \beta\frac{\partial f_p}{\partial g_p}]\cdot dL \tag{4.31}$$

We can now define

$$\Delta_2 = \begin{bmatrix} u'_r.c''_r - u''_r.(c'_r)^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2} & -\beta L\frac{\partial^2 f_r}{\partial g_p \partial g_r} \\ -\beta L\frac{\partial^2 f_p}{\partial g_r \partial g_p} & u'_p.c''_p - u''_p.(c'_p)^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2} \end{bmatrix}$$

$$\Delta_2^L(g_r) = \begin{bmatrix} -(1-\theta)u''_r.c'_r.y'_r + \beta\frac{\partial f_r}{\partial g_r} & -\beta L\frac{\partial^2 f_r}{\partial g_p \partial g_r} \\ -\theta u''_p.y'_p.c'_p + \beta\frac{\partial f_p}{\partial g_p} & u'_p.c''_p - u''_p.(c'_p)^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2} \end{bmatrix}$$

$$\Delta_2^L(g_p) = \begin{bmatrix} u'_r.c''_r - u''_r.(c'_r)^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2} & -(1-\theta)u''_r.c'_r.y'_r + \beta\frac{\partial f_r}{\partial g_r} \\ -\beta L\frac{\partial^2 f_p}{\partial g_r \partial g_p} & -\theta u''_p.y'_p.c'_p + \beta\frac{\partial f_p}{\partial g_p} \end{bmatrix}$$

The determinant of Δ_2, and $\Delta_2^L(g_r)$ is always positive, but that of $\Delta_2^L(g_p)$ is difficult to establish. We have

$$\frac{\partial g_r}{\partial L} = \frac{|\Delta_2^L(g_r)|}{|\Delta_2|} > 0 \quad and \quad \frac{\partial g_p}{\partial L} = \frac{|\Delta_2^L(g_p)|}{|\Delta_2|} > or < 0 \tag{4.32}$$

Lemma 6: For all $L > L_{bg}$, an increase in θ (equality) for a given L leads to higher production of guns by the poor agent.

Proof: The first order conditions that characterize an equilibrium involving production of guns by both agents are given by equation (26). Total differentiation of this system with respect to g_r, g_p, and θ gives

$$[u'_r.c''_r - u''_r.(c'_r)^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2}] \cdot dg_r + [-\beta L\frac{\partial^2 f_r}{\partial g_p \partial g_r}] \cdot dg_p = [Lu''_r.y'_r.c'_r] \cdot d\theta \tag{4.33}$$

$$[-\beta L\frac{\partial^2 f_p}{\partial g_r \partial g_p}] \cdot dg_r + [u'_p.c''_p - u''_p.(c'_p)^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2}] \cdot dg_p \quad = \quad [-Lu''_p.y'_p.c'_p] \cdot d\theta \quad (4.34)$$

Let us define

$$\Delta_2^\theta(g_r) \quad = \quad \begin{bmatrix} Lu''_r.y'_r.c'_r & -\beta L\frac{\partial^2 f_r}{\partial g_p \partial g_r} \\ -Lu''_p.y'_p.c'_p & u'_p.c''_p - u''_p.(c'_p)^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2} \end{bmatrix}$$

$$\Delta_2^\theta(g_p) \quad = \quad \begin{bmatrix} u'_r.c''_r - u''_r.(c'_r)^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2} & Lu''_r.y'_r.c'_r \\ -\beta L\frac{\partial^2 f_p}{\partial g_r \partial g_p} & -Lu''_p.y'_p.c'_p \end{bmatrix}$$

The determinant of $\Delta_2^\theta(g_p)$ is positive but that of $\Delta_2^\theta(g_r)$ does not have an unambiguous sign. Thus

$$\frac{\partial g_r}{\partial \theta} \quad = \quad \frac{8}{g}|\Delta_2^\theta(g_r)||\Delta_2| \quad > \; or \; < \quad 0 \quad and \quad \frac{\partial g_p}{\partial \theta} \quad = \quad \frac{|\Delta_2^\theta(g_p)|}{|\Delta_2|} \quad > \quad 0 \qquad (4.35)$$

It would be interesting to know the degree of welfare loss in the economy as measured by the fraction of total output spent on the production of guns. Unfortunately, the results in this section show that it is difficult to determine whether the production of guns is increasing or decreasing with changes in total and inequality for all the possible values. This is the drawback of using general forms of utility, production, cost, and contest success functions.

2. Effect of β

Lemma 7: Higher weight on future land holdings lowers the upper boundary of the no-guns region.

Proof: This follows from implicitly differentiating equation $()$ with respect to β and L, holding θ constant.

$$\frac{\partial L_{ng}(\theta)}{\partial \beta} = \frac{kL_{ng}}{[(1-\theta)u_r''.y_r'.c_r'(0) - k\beta]} < 0 \quad \forall\, \theta \in (0, 0.5] \tag{4.36}$$

Lemma 8: At a given (L, θ) such that $L_{ng} < L < L_{bg}$, the amount of guns produced by the rich agent will increase if β increases.

Proof: This follows immediately from the total differentiation of the first order condition of the rich agent (equation 21) with respect to g_r and β, while holding L and θ constant.

$$\frac{\partial g_r}{\partial \beta} = \frac{[L\frac{\partial f_r}{\partial g_r}]}{[u_r'.c_r'' - u_r''.(c_r')^2 - \beta L_{pg}\frac{\partial^2 f_r}{\partial g_r^2}]} > 0 \tag{4.37}$$

Lemma 9: An increase in β lowers the lower boundary of the region in which both agents produce guns.

Proof: We need to show that if β increases then the value of L that satisfies equalities in equations $(20) and (21)$ decreases, for any given θ. Total differential of this system with respect to L, g_r, and β gives

$$[(1-\theta)u_r''.c_r'.y_r' - \beta\frac{\partial f_r}{\partial g_r}]\cdot dL_{pg} + [u_r'.c_r'' - u_r''.(c_r')^2 - \beta L_{pg}\frac{\partial^2 f_r}{\partial g_r^2}]\cdot dg_r = [L_{pg}\frac{\partial f_r}{\partial g_r}]\cdot d\beta \tag{4.38}$$

$$[\theta u_p''.y_p'.c_p' - \beta\frac{\partial f_p}{\partial g_p}]\cdot dL_{pg} + [-\beta L_{pg}\frac{\partial^2 f_p}{\partial g_r \partial g_p}]\cdot dg_r = [L_{pg}\frac{\partial f_p}{\partial g_p}]\cdot d\beta \tag{4.39}$$

Define,

$$\Delta_1^{\beta}(L_{pg}) = \begin{bmatrix} L_{pg}\frac{\partial f_r}{\partial g_r} & u_r'.c_r'' - u_r''.(c_r')^2 - \beta L_{pg}\frac{\partial^2 f_r}{\partial g_r^2} \\ L_{pg}\frac{\partial f_p}{\partial g_p} & -\beta L_{pg}\frac{\partial^2 f_p}{\partial g_r \partial g_p} \end{bmatrix}$$

The determinant of $\Delta_1^{\beta}(L_{pg})$ is always negative, and thus

$$\frac{\partial L_{pg}}{\partial \beta} = \frac{|\Delta_1^{\beta}(L_{pg})|}{|\Delta_1|} < 0 \tag{4.40}$$

Lemma 10: For $L > L_{bg}$, an increase in β holding L and θ constant leads to higher production of guns by both agents.

Proof: The first order conditions that characterize an equilibrium in which both agents produce guns are given in equation (26). Total differentiation of this system with respect to g_r, g_p, and β gives

$$[u_r'.c_r'' - u_r''.(c_r')^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2}] \cdot dg_r + [-\beta L\frac{\partial^2 f_r}{\partial g_p \partial g_r}] \cdot dg_p = [L\frac{\partial f_r}{\partial g_r}] \cdot d\beta \tag{4.41}$$

$$[-\beta L\frac{\partial^2 f_p}{\partial g_r \partial g_p}] \cdot dg_r + [u_p'.c_p'' - u_p''.(c_p')^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2}] \cdot dg_p = [L\frac{\partial f_p}{\partial g_p}] \cdot d\beta \tag{4.42}$$

Define

$$\Delta_2^{\beta}(g_r) = \begin{bmatrix} L\frac{\partial f_r}{\partial g_r} & -\beta L\frac{\partial^2 f_r}{\partial g_p \partial g_r} \\ L\frac{\partial f_p}{\partial g_p} & u_p'.c_p'' - u_p''.(c_p')^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2} \end{bmatrix}$$

$$\Delta_2^{\beta}(g_p) = \begin{bmatrix} u_r'.c_r'' - u_r''.(c_r')^2 - \beta L\frac{\partial^2 f_r}{\partial g_r^2} & L\frac{\partial f_r}{\partial g_r} \\ -\beta L\frac{\partial^2 f_p}{\partial g_r \partial g_p} & L\frac{\partial f_p}{\partial g_p} \end{bmatrix}$$

The determinant of $\Delta_2^{\beta}(g_r)$, and $\Delta_2^{\beta}(g_p)$ is positive. Thus,

$$\frac{\partial g_r}{\partial \beta} = \frac{|\Delta_2^{\beta}(g_r)|}{|\Delta_2|} > 0 \quad and \quad \frac{\partial g_p}{\partial \beta} = \frac{|\Delta_2^{\beta}(g_p)|}{|\Delta_2|} > 0 \tag{4.43}$$

D. Effect of Heterogeneity

We now try to analyze the effect of simple forms of heterogeneity in production function and cost function across the agents. In order to simplify the analysis we will consider heterogeneity in only one of the functions at a time. The major aim is once again to come up with the characterization of the equilibrium.

1. Heterogeneous Production Functions

Let the production function of the poorly endowed agent be $\lambda_y y_p(\cdot)$, and that of the rich agent be $y_r(\cdot)$, with $y_p(\cdot) = y_r(\cdot)$ as before. We interpret λ_y as a parameter that reflects the technological difference between the two agents. By Proposition 1, there exists a unique Nash equilibrium for a given (L, θ) pair, and any given $\lambda_y > 0$. We make the additional assumption that $y_p(\cdot)(= y_r(\cdot))$ is a linear function of its argument. However, all the results in this section hold for any concave production function. It is easy to show that for a given $\lambda > 1$, there exists $\theta_y = \frac{1}{1+\lambda_y} \in (0, 0.5)$, such that for all $\theta > \theta_y$ the poorly endowed agent has a greater output than the richly endowed agent. When $\lambda_y \leq 1$, the poorly endowed agent never produces more output for all $\theta \in (0, 0.5)$, and all $L > 0$. The immediate implication is that for $\theta > \theta_y$ as L increases the poorly endowed agent will start producing guns first. The boundary of the no guns region now consists of two distinct curves as summarized in the following lemma.

Lemma 11: For $\lambda_y > 1$, the upper boundary of the no guns region is upward sloping and convex for $0 < \theta < \theta_y$, and downward sloping and convex for $\theta_y < \theta < 0.5$.

Proof: From the discussion above, the upper boundary of the no-guns region is defined by

$$u_r'[((1-\theta)L_{ng}^r)].c_r'(0) \;=\; k\beta L_{ng}^r \quad if \;\; \theta \leq \theta_y \tag{4.44}$$

and

$$u_p'[\lambda_y(\theta L_{ng}^p)].c_p'(0) \;=\; k\beta L_{ng}^p \quad if \;\; \theta \geq \theta_y \tag{4.45}$$

This boundary is convex and increasing in θ for $\theta < \theta_y$ (see $Lemma\ (2)$). Although convex, it is decreasing in θ for $\theta > \theta_y$ since

$$\frac{\partial L_{ng}^p(\theta)}{\partial \theta} \;=\; \frac{-\lambda_y L_{ng}^p u_p''.c_p'(0)}{[\lambda_y(\theta L_{ng}^p)u_p''.c_p'(0) - k\beta]} \;<\; 0 \quad \forall\, \theta \in (0, 0.5] \tag{4.46}$$

and

$$\frac{\partial^2 L_{ng}^p(\theta)}{\partial \theta^2} \;=\; \frac{\lambda_y^2 L_{ng}^p[k\beta L_{ng}^p(u_p'''.c_p'(0)) + (u_p''c_p'(0))^2]}{[\lambda_y(\theta L_{ng}^p)u_p''.c_p'(0) - k\beta]^2} \;>\; 0 \quad \forall\, \theta \in (0, 0.5] \tag{4.47}$$

From the analysis of the previous section we can conclude that for any $\theta < \theta_y$ there exists an $L_{bg}^p > L_{ng}^r$ such that the poor agent also produces guns for all $L > L_{bg}^p$. Similarly, for any $\theta > \theta_y$ there exists an $L_{bg}^r > L_{ng}^p$ such that the rich agent also produces guns for all $L > L_r$. The combination of $L_{bg}^p(\theta|\theta < \theta_y)$ and $L_{bg}^r(\theta|\theta > \theta_y)$ defines the boundary above which both agents produce guns. The following lemma characterizes this boundary.

Lemma 12: The boundary above which both agents produce guns is downward sloping for $\theta < \theta_y$, and upward sloping for $\theta > \theta_y$, for any given $\lambda_y > 1$.

Proof: For a given $\theta \in (0, 0.5)$ and $\lambda_y > 1$, the minimum amount of land above which both agents produce guns is implicitly defined by

$$\frac{\partial V_r}{\partial g_r}|_{(g_p=0,g_r>0)} = 0 \;\; \& \;\; \frac{\partial V_p}{\partial g_p}|_{(g_p=0,g_r>0)} = 0 \;\; for\; \theta < \theta_y \tag{4.48}$$

and

$$\frac{\partial V_r}{\partial g_r}|_{g_p>0,g_r=0} = 0 \;\; \& \;\; \frac{\partial V_p}{\partial g_p}|_{(g_p>0,g_r=0)} = 0 \;\; for\; \theta > \theta_y \tag{4.49}$$

The first set of conditions determine the boundary for the case where the poor agent is the last to produce guns. The first set of conditions defined for $\theta < \theta_y$ are exactly similar to the set of equalities in equation (20). Hence, the required boundary will be downward sloping for $\theta < \theta_y$, and the amount of guns being produced by the rich agent will be decreasing along the boundary (see $Lemma$ (3)). The second set of equalities defined for $\theta > \theta_y$ can be rewritten as

$$u'_r[(1-\theta)L^r_{bg}]c'_r(0)-\beta L^r_{bg}\frac{\partial f_r}{\partial g_r} = 0 \;\; \& \;\; u'_p[\lambda_y\theta L^r_{bg}-c_p(g_p)]c'_p(g_p)-\beta L^r_{bg}\frac{\partial f_p}{\partial g_p} = 0 \tag{4.50}$$

Total differential of this system with respect to L^r_{bg}, g_p, and θ gives

$$[(1-\theta)u''_r.c'_r(0)-\beta\frac{\partial f_r}{\partial g_r}]\cdot dL^r_{bg}+[-\beta L^r_{bg}\frac{\partial^2 f_r}{\partial g_p \partial g_r}]\cdot dg_p = [L^r_{bg}u''_r.c'_r(0)]\cdot d\theta \tag{4.51}$$

$$[\lambda_y\theta u''_p.c'_p-\beta\frac{\partial f_p}{\partial g_p}]\cdot dL^r_{bg}+[u'_p.c''_p-u''_p.(c'_p)^2-\beta L^r_{bg}\frac{\partial^2 f_p}{\partial g_p^2}]\cdot dg_r = [-\lambda_y L^r_{bg}u''_p.c'_p]\cdot d\theta \tag{4.52}$$

Using Cramer's rule to solve the above two equations gives

$$\frac{\partial L^r_{bg}}{\partial \theta} > 0 \;\; \& \;\; \frac{\partial g_p}{\partial \theta} > 0 \;\; \textit{along the } (L^r_{bg}(\theta > \theta_y), \theta) \textit{ boundary.} \tag{4.53}$$

2. Heterogeneous Cost Functions

Let the cost function of the poorly endowed agent be $\lambda_c c_p(\cdot)$, and that of the rich agent be $c_r(\cdot)$, with $c_p(\cdot) = c_r(\cdot)$ as before. We interpret λ_c as a parameter that reflects the difference in the technology of producing guns. By Proposition 1, there exists a unique Nash equilibrium for a given (L, θ) pair, and any given $\lambda_c > 0$. If $\lambda_c > 1$, then the poorly endowed agent has a cost disadvantage as well and the nature of results obtained in Section (3) does not change. However,

Lemma 13: For a given $\lambda_c < 1$, there exists a $\theta_c \in (0, 0.5)$ such that for all $\theta > \theta_c$ the poor agent starts producing guns before the rich agent as L increases.

Proof: Suppose the rich agent starts producing guns before the poor agent for all $\theta \in (0, 0.5)$. The value $L^r_{ng}(\theta)$ at which this happens for a given θ will then be implicitly given by

$$u'_r[y_r((1-\theta)L^r_{ng}]c'_r(0) - k\beta L^r_{ng} = 0 \tag{4.54}$$

If the poor agent were to start producing guns before the rich agent for all $\theta \in (0, 0.5)$ then the L^p_{ng} at which this for a given θ will be implicitly given by

$$\lambda_c u'_p[y_p(\theta L^p_{ng})]c'_p(0) - k\beta L^p_{ng} = 0 \tag{4.55}$$

It can be verified that $\frac{\partial L^r_{ng}}{\partial \theta} > 0$, whereas $\frac{\partial L^p_{ng}}{\partial \theta} < 0$. Moreover, $L^r_{ng}(\theta \to 0) < L^p_{ng}(\theta \to 0)$, but $L^p_{ng}(\theta = 0.5) < L^r_{ng}(\theta = 0.5)$. Hence, we conclude that for any $\lambda_c < 1$ there exists a $\theta_c \in (0, 0.5)$ such that $L^p_{ng}(\theta_c) = L^r_{ng}(\theta_c)$, $L^r_{ng}(\theta | \theta < \theta_c) < L^p_{ng}(\theta | \theta < \theta_c)$, and

$L^p_{ng}(\theta|\theta > \theta_c) < L^r_{ng}(\theta|\theta > \theta_c)$.

Lemma 14: The lower boundary of the region in which both agents produce guns is downward sloping for $\theta < \theta_c$, and upward sloping for $\theta > \theta_c$, for any given $\lambda_c < 1$. (The proof is similar to that of $Lemma$ (12)).

E. History Dependence

The analysis till now has assumed that if none of the agents produces guns then they share the land equally in future irrespective of the heterogeneity in initial land holdings. In this section we characterize the equilibria for various (L, θ) pairs under the assumption that the final land holdings are the same as the initial land holdings if no agent produces guns. Specifically, we make the following assumptions.

Assumption 2(b): $0 < f_i^\theta < 1$, $\frac{\partial f_i^\theta}{\partial g_i} > 0$, $\frac{\partial^2 f_i^\theta}{\partial g_i^2} < 0$, $\frac{\partial f_i^\theta}{\partial g_j} < 0$, $f_p^\theta(0,0) = \theta$, $\frac{\partial f_p^\theta}{\partial \theta} > 0$, $\frac{\partial}{\partial \theta}[\frac{\partial f_p^\theta}{\partial g_p}] < 0$, $\frac{\partial}{\partial \theta}[\frac{\partial f_r^\theta}{\partial g_r}] > 0$, $\frac{\partial}{\partial g_i}[\frac{\frac{\partial f_i^\theta}{\partial g_i}}{\frac{\partial f_j^\theta}{\partial g_j}}] < 0$, $\frac{\partial^2 f_p^\theta}{\partial g_r \partial g_p} < 0$ if $g_p < g_r$, $\frac{\partial f_r^\theta}{\partial g_r} \to 0$, if $\theta \to 0$.

We again assume that $u_p(\cdot) = u_r(\cdot)$, $y_p(\cdot) = y_r(\cdot)$, and $c_p(\cdot) = c_r(\cdot)$. λ_y represents the difference in production technology, and λ_c the difference in the technology of producing guns.

1. Heterogeneous Production Functions

Let the production function of the richly endowed agent be $y_r(\cdot)$, and that of the poorly endowed agent be $\lambda_y y_p(\cdot)$. Proposition 1 still holds and thus there exists a unique Nash equilibrium for any feasible (L, θ) pair. It is difficult to separate the

(L, θ) space into regions that support one of the four possible types of equilibria. We will focus our attention on determining the effect of λ_y.

Conjecture 1: For θ sufficiently close to zero, the poorly endowed agent starts producing guns at a lower value of L than the richly endowed agent, for all $\lambda_y > 0$.
Proof: Suppose the richly endowed agent starts producing guns first, for all $\theta \in (0, 0.5]$. If so, the upper boundary of the no-guns region will be given by

$$\frac{\partial V_r}{\partial g_r}|_{(g_p=0,g_r>0)} = 0 \tag{4.56}$$

$$\Rightarrow \quad u_r'[y_r((1-\theta)L_{ng}^r]c_r'(0) - \beta L_{ng}^r \frac{\partial f_r^\theta}{\partial g_r} \quad = \quad 0 \tag{4.57}$$

Since, $\frac{\partial f_r^\theta}{\partial g_r} \rightarrow 0$ as $\theta \rightarrow 0$ (by Assumption 2(b)), $L_{ng}^r \rightarrow \infty$ as $\theta \rightarrow 0$.

Next, suppose the poorly endowed agent starts producing guns first, for all $\theta \in (0, 0.5]$. The upper boundary of the no-guns region will then be given by

$$\frac{\partial V_p}{\partial g_p}|_{(g_p>0,g_r=0)} = 0 \tag{4.58}$$

$$\Rightarrow \quad u_p'[\lambda_y y_p(\theta L_{ng}^p)]c_r'(0) - \beta L_{ng}^p \frac{\partial f_p^\theta}{\partial g_p} \quad = \quad 0 \tag{4.59}$$

Since, $\frac{\partial f_p^\theta}{\partial g_p} \rightarrow 1$ as $\theta \rightarrow 0$ (by Assumption 2(b)), L_{ng}^p will be a finite number as $\theta \rightarrow 0$. This proves the conjecture.

Conjecture 2: Higher values of $\lambda_y > 0$ make it more likely that the poorly endowed agent starts producing guns at a lower value of L than the richly endowed agent, for all $\theta \in (0, 0.5]$.

Proof: The lower L^p_{ng} for a given θ, the more likely it will be that the poorly endowed agent is the first to produce guns. We begin by showing that an increase in λ_y lowers the (L^p_{ng}, θ) boundary. Equation (46) gives

$$\frac{\partial L^p_{ng}}{\partial \lambda_y} = \frac{-u''_p.y_p.c'_p(0)}{[\lambda_y \theta u''_p.y'_p.c'_p(0) - \beta \frac{\partial f^\theta_p}{\partial g_p}]} \;<\; 0 \tag{4.60}$$

For $\lambda_y = 1$, $L^p_{ng}(\theta = 0.5) = L^p_{ng}(\theta = 0.5)$. The (L^r_{ng}, θ) boundary does not get affected by changes in λ_y. Thus, if $\lambda_y > 1$ then the poorly endowed agent will be the first to produce guns for θ sufficiently close to 0.5.

We have characterized the (L^r_{ng}, θ), and the (L^p_{ng}, θ) boundaries in the previous conjecture. Now we will show that both these boundaries are downward sloping when θ is close to zero, and convex for all $\theta \in (0, 0.5]$ and all $\lambda_y > 0$. Equation (54) gives

$$\frac{\partial L^r_{ng}}{\partial \theta} \;=\; \frac{L^r_{ng} u''_r.y'_r.c'_r(0) + \beta L^r_{ng} \frac{\partial}{\partial \theta}[\frac{\partial f^\theta_r}{\partial g_r}]}{[(1-\theta) u''_r.y'_r.c'_r(0) - \beta \frac{\partial f^\theta_r}{\partial g_r}]} \tag{4.61}$$

The sign of $\frac{\partial L^r_{ng}}{\partial \theta}$ can not be determined without assuming particular functional forms. However, the slope is negative when θ is close to zero. Moreover, the (L^r_{ng}, θ) boundary is convex since $\frac{\partial^2 L^r_{ng}}{\partial \theta^2}$ can be shown to be strictly positive.

Similarly, equation (55) gives

$$\frac{\partial L^p_{ng}}{\partial \theta} \;=\; \frac{-\lambda_y L^p_{ng} u''_p.y'_p.c'_p(0) + \beta L^p_{ng} \frac{\partial}{\partial \theta}[\frac{\partial f^\theta_p}{\partial g_p}]}{[\lambda_y \theta u''_p.y'_p.c'_p(0) - \beta \frac{\partial f^\theta_p}{\partial g_p}]} \tag{4.62}$$

The sign of $\frac{\partial L^p_{ng}}{\partial \theta}$ is difficult to determine. The (L^p_{ng}, θ) boundary is also convex as $\frac{\partial^2 L^r_{ng}}{\partial \theta^2}$ given by

$$\frac{\lambda_y\beta(L_{ng}^p)^2c_p'(0)[\lambda_y u_p'''.(y_p')^2+u_p''.y_p''][\frac{\partial f_p^\theta}{\partial g_p}-\theta\frac{\partial}{\partial\theta}[\frac{\partial f_p^\theta}{\partial g_p}]]+L[\lambda_y u_p''.y_p'.c_p'(0)-\beta\frac{\partial}{\partial\theta}[\frac{\partial f_p^\theta}{\partial g_p}]]^2}{[\lambda_y\theta u_p''.y_p'.c_p'(0)-\beta\frac{\partial f_p^\theta}{\partial g_p}]^2} \tag{4.63}$$

is strictly positive.

2. Heterogeneous Cost Functions

Let the cost function for producing guns be $c_r(\cdot)$ and $\lambda_c c_p(\cdot)$ for the richly and the poorly endowed agent, respectively. As before, $c_r(\cdot) = c_p(\cdot)$, and $\lambda_c > 0$. We will argue that a lower λ_c makes it more likely that the poorly endowed agent will produce guns first. The (L_{ng}^r, θ) boundary and its properties will be the same as in the previous case since while allowing for heterogeneity we are only altering the poorly endowed agent's specification. Now, suppose the poorly endowed agent starts producing guns first, for all $\theta \in (0, 0.5]$. The upper boundary of the no-guns region will then be given by

$$\frac{\partial V_p}{\partial g_p}|_{g_p>0,g_r=0} = 0$$

$$\Rightarrow \quad \lambda_c u_p'[y_p(\theta L_{ng}^p)]c_r'(0) - \beta L_{ng}^p\frac{\partial f_p^\theta}{\partial g_p} \quad = \quad 0 \tag{4.64}$$

Since, $\frac{\partial f_p^\theta}{\partial g_p} \rightarrow 1$ as $\theta \rightarrow 0$ (by Assumption 2(b)), L_{ng}^p will be a finite number as $\theta \rightarrow 0$. This helps us conclude that when θ is close to zero the poorly endowed agent will start producing guns at a lower value of L as compared to the richly endowed agent. The following derivative summarizes the effect of λ_c on the (L_{ng}^p, θ) boundary.

$$\frac{\partial L^p_{ng}}{\partial \lambda_c} = \frac{-u'_p.c'_p(0)}{[\lambda_c \theta u''_p.y'_p.c'_p(0) - \beta \frac{\partial f^\theta_p}{\partial g_p}]} \quad > \quad 0 \tag{4.65}$$

It is easy to verify that $L^p_{ng}(\theta = 0.5) = L^r_{ng}(\theta = 0.5)$ if $\lambda_c = 1$. The above derivative thus implies that for $\lambda_c < 1$ the poorly endowed agent will be the first to produce guns for values of θ close to 0.5.

3. An Example

It is difficult to characterize the equilibria for various values of θ away from 0 and 0.5 under the assumption that the final land holdings are the same as the initial land holdings if no agent produces guns, for all concave utility functions, concave production functions, and convex cost functions. We now try to do so for a particular case by assuming $u(x) = x^\gamma$, $y(x) = x$, where $0 < \alpha < 1$. The results in this section hold for all concave production functions of the form $y(x) = x^\delta$.

a. Heterogeneous Production Functions

Proposition 1: (i) The poorly endowed agent starts producing guns at a lower value of L than the richly endowed agent for all $\theta \in (0, 0.5)$ if $\lambda_y \geq 1$. (ii) For every given $\lambda_y < 1$, there exists a $\theta_y \in (0, 0.5)$ such that the rich agent is the first to produce guns for all $\theta \in (0, \theta_y)$; the rich agent produces guns first for $\theta \in (\theta_y, 0.5]$.

Proof: Suppose the richly endowed agent starts producing guns first, for all $\theta \in (0, 0.5]$.

$$-\alpha[(1-\theta)L^r_{ng}]^{\alpha-1}c'_r(0) + \beta L^r_{ng}\theta = 0 \quad \Rightarrow \quad L^r_{ng}(\theta) = [\frac{\alpha(1-\theta)^{\alpha-1}c'_r(0)}{\beta\theta}]^{\frac{1}{2-\alpha}} \tag{4.66}$$

Next, suppose the poorly endowed agent starts producing guns first, for all $\theta \in (0, 0.5]$. The upper boundary of the no-guns region will then be given by

$$-\alpha[\lambda_y \theta L^p_{ng}]^{\alpha-1} c'_p(0) + \beta L^p_{ng}(1-\theta) = 0 \quad \Rightarrow \quad L^p_{ng}(\theta) = [\frac{\alpha(\lambda_y \theta)^{\alpha-1} c'_p(0)}{\beta(1-\theta)}]^{\frac{1}{2-\alpha}} \quad (4.67)$$

Note that, $L^p_{ng}(\theta) \leq L^r_{ng}(\theta)$ for all $\theta \in (0, 0.5]$ if

$$\lambda_y^{\alpha-1}\theta \leq (1-\theta) \quad \Rightarrow \quad \lambda_y \geq 1 \quad (4.68)$$

Let us now consider the case of $\lambda_y < 1$. For any given $\lambda_y < 1$, $L^p_{ng}(\theta) \leq L^r_{ng}(\theta)$ if

$$\lambda_y^{\alpha-1}\theta \leq (1-\theta) \quad \Rightarrow \quad \theta \leq \frac{1}{1+\lambda_y^{\alpha-1}} = \theta_y \quad (4.69)$$

Moreover, since $\lambda_y < 1$, $\theta_y \in (0, 0.5)$. We can characterize the lower boundary of the region in (L, θ) space in which both agents produce guns in the manner described in $Lemma$ (12).

b. Heterogeneous Cost Functions

Proposition 2: (i) The poorly endowed agent starts producing guns at a lower value of L than the richly endowed agent for all $\theta \in (0, 0.5)$ if $\lambda_c \leq 1$. (ii) For every given $\lambda_c > 1$, there exists a $\theta_c \in (0, 0.5)$ such that the rich agent is the first to produce guns for all $\theta \in (0, \theta_c)$; the rich agent produces guns first for $\theta \in (\theta_c, 0.5]$.

Proof: Suppose the richly endowed agent starts producing guns first, for all $\theta \in (0, 0.5]$. The boundary of the no guns region will be given by

$$-\alpha[(1-\theta)L^r_{ng}]^{\alpha-1} c'_r(0) + \beta L^r_{ng}\theta = 0 \quad \Rightarrow \quad L^r_{ng}(\theta) = [\frac{\alpha(1-\theta)^{\alpha-1} c'_r(0)}{\beta\theta c'_r(0)}]^{\frac{1}{2-\alpha}} \quad (4.70)$$

Next, suppose the poorly endowed agent starts producing guns first, for all $\theta \in (0, 0.5]$. The upper boundary of the no-guns region will then be given by

$$-\lambda_c\alpha[\theta L_{ng}^p]^{\alpha-1}c_p'(0) + \beta L_{ng}^p(1-\theta) = 0 \quad \Rightarrow \quad L_{ng}^p(\theta) = [\frac{\lambda_c\alpha\theta^{\alpha-1}c_p'(0)}{\beta(1-\theta)}]^{\frac{1}{2-\alpha}} \tag{4.71}$$

Note that, $L_{ng}^p(\theta) \leq L_{ng}^r(\theta)$ for all $\theta \in (0, 0.5]$ if

$$\lambda_c^{\frac{1}{\alpha}}\theta \leq (1-\theta) \quad \Rightarrow \quad \lambda_c \leq 1 \tag{4.72}$$

Let us now consider the case of $\lambda_c > 1$. For any given $\lambda_c > 1$, $L_{ng}^p(\theta) \leq L_{ng}^r(\theta)$ if

$$\lambda_c^{\frac{1}{\alpha}}\theta \leq (1-\theta) \quad \Rightarrow \quad \theta \leq \frac{1}{1+\lambda_c^{\frac{1}{\alpha}}} = \theta_c \tag{4.73}$$

Since $\lambda_c > 1$, $\theta_c \in (0, 0.5)$. Once again, we can characterize the lower boundary of the region in (L, θ) space in which both agents produce guns in the manner described in $Lemma$ (12).

F. Conclusion

The literature on conflict can be classified along several dimensions- whether the conflict is over output or inputs, whether the output is produced jointly or separately by the agents, whether the framework is static, repeated or dynamic, whether the model is one of complete or incomplete information, whether the agents make simultaneous or sequential choices, and so on. In this paper we have analyzed a static model of conflict over an inexhaustible input between two agents where agents make simultaneous choices. Our model distinguishes itself from those analyzed in the literature till date on account of its generality and the analysis of history dependent contest success function.

The rent seeking literature can be thought of as the precursor to the literature on conflict. The standard rent seeking model involves several agents expending resources to win a pie of fixed size. The objective of this literature has been to characterize the existence of equilibrium levels of expenditures by the agents. The expected utility of an agent in a general model of rent seeking is given by

$$EU_i(g_i, G_{-i}) = p_i(g_i, G_{-i}).u_i(e_i + L - g_i) + (1 - p_i(g_i, G_{-i})).u_i(e_i - g_i)$$

where e_i is the endowment of agent i, L is the pie agents are competing over, g_i is the expenditure by agent i to increase his chances of obtaining the pie, and p_i is the probability with which agent i obtains the pie, i.e., the contest success function. The various studies can be characterized depending upon whether agents are risk neutral or not, whether the agents are homogeneous or heterogeneous, whether the endowment constraint is binding or not, whether the number of agents in the rent seeking context is fixed or there is free entry, whether agents make sequential or simultaneous expenditures, what is the form of the contest success function, and so on. The interested reader can refer to Nitzan [35], Tollison [44], and Cornes and Hartley [12], [13] for details. For a comparison of rent seeking and conflict models please refer to Hausken [22]. The rent seeking models with homogeneous risk neutral agents are special cases of the model given in the present paper because the two will be strategically similar as

$$EU_i(g_i, G_{-i}) = (e_i - g_i) + p_i(g_i, G_{-i}).L$$

This formulation is similar to the one studied in the present paper with the additional assumption that utility from current consumption is linear. However, the analysis of such a model will be slightly different from the one presented in this paper as there will exist additional equilibria involving no consumption by either one or both the agents. The unwelcome tradeoff we have to face while characterizing the equilibria using general forms of utility production, cost, and contest success functions is that we are unable to provide clear cut welfare calculations which is central to the rent seeking literature.

CHAPTER V

CONCLUSION

Game theoretic analysis of strategic interactions has become an integral part of the economics literature. The essays in this dissertation illustrate the versatility of economic reasoning. The second chapter provides a reason for the strict codes of conduct that have been imposed on unmarried girls in almost every society at some point of time in its history using tools from classical game theory. The third chapter rationalizes some of the adhoc rules proposed for dividing a bankrupt estate from an evolutionary perspective. The fourth chapter presents a simple model of conflict over inputs in an economy with ill-defined property rights. These studies leave some related issues unanswered, and also give rise to several methodological questions.

The signaling game presented in Chapter II assumes that all the men attach the same prior probability to a girl being submissive. The common prior assumption is frequently used in signaling games but there does not exist a strong conceptual foundation for it. Another damaging criticism is the use of refinements to eliminate some of the Nash equilibria. Cognitive limitations of individuals do not justify the use of high rationality solution concepts in one shot games and there is overwhelming laboratory evidence to support this. However, it can be expected that if individuals face similar situations over time then they might be able to approach an equilibrium through a process of trial and error. The third chapter on evolution of division rules takes this approach. The assumption that agents repeatedly face exactly the same situation is undeniably an oversimplification. We need models of learning across situations that are similar but not exactly the same. The literature on stochastic stability is built around the assumption that it is the errors or experiments by agents that help select the long run equilibria from the set of possible stage game equilibria. The fact that different specifications of the error process often lead to the selection of different equilibria as the long run outcomes casts doubt on the generality of the results. It would be desirable to supplement the theoretical results of this chapter with experimental evidence regarding what division people consider as fair in a situation like bankruptcy. The model of conflict presented in chapter IV questions the accepted wisdom underlying folk-theorem type of arguments by showing that an increase in valuation of future leads to increased conflict. The reason is that most the models used to elucidate the folk theorem employ repetitions of the same stage game where the initial conditions in successive periods are independent of the outcomes in preceding periods. I hope to address these questions in my future work.

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