MIDDLE SCHOOL STUDENTS' REPRESENTATIONAL UNDERSTANDINGS AND JUSTIFICATION SCHEMES: GLEANINGS FROM COGNITIVE INTERVIEWS

A Dissertation

by

SHIRLEY MARIE MATTESON

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2007

Major Subject: Curriculum & Instruction

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Approved by:

Chair of Committee,	Robert M. Capraro
Committee Members,	Mary Margaret Capraro
	Dennie Smith
	Yvonna Lincoln
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ABSTRACT

Middle School Students' Representational Understandings and Justification Schemes:
Gleanings from Cognitive Interviews.

(August 2007)

Shirley Marie Matteson, B.M.E., Greenville College; M.M., Hardin-Simmons University

Chair of Advisory Committee: Dr. Robert M. Capraro

This dissertation investigated several aspects of middle graders' mathematical understanding based on representational models. Twenty (11 male, 9 female) sixth grade students were interviewed about their solution strategies and answer justifications when solving difficult mathematics problems. The interview participants represented a stratified demographic sampling of the student body of a culturally diverse middle school in a suburban school district in the southwestern United States.

Data from the interviews were analyzed qualitatively. This involved "chunking" cognitive interview transcripts into sections. Major themes were identified and manuscripts were developed around those themes. One theme examined the interviewers' ethic of care behaviors. Carol Gilligan noted differences in male and female ethic of care behaviors, but it was Nel Noddings who discussed the importance of such behaviors in the educational community. So what impact could the gender of the interviewer have on cognitive interviews? After considering ethic of care behaviors explicated by Hayes, Ryan and Zseller's (1994) study with middle grades students, the

interview transcripts were examined for specific positive and negative ethic of care behaviors.

The theme of students' justifications of mathematical solutions was also selected. The major undertaking involved developing a justification scheme applicable across mathematical strands and grade levels. The justification scheme that emerged was based on the work of Guershon Harel and Larry Sowder. The first-level schemes of Language, Mechanistic, Authoritarian, and Visual were used to classify and define the justifications. Several second-level schemes were also defined. The justification scheme framework was applied to students' cognitive interview responses on four difficult mathematics problems.

The third theme investigated the symbiosis of justification schemes with mathematical representations. This study examined possible links between representational formats and justification scheme categories. The premise of this study was that representations "trigger" students' choices of justification schemes. Student responses were analyzed as to which aspect of the mathematical representation received the students' initial attention. The students' understanding of the representation was pivotal to their solution, as well as the students' reasoning, or justification, of the answer. Students focused on key aspects of the problem and developed solutions based on that information.

DEDICATION

To my family, friends, colleagues, teachers, and mentors – THANK YOU for your influence, support, and guidance over the years.

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I would like first to express appreciation for the guidance of my committee chair, Dr. Robert M. Capraro, whose writing skills and frank comments were invaluable to me during my studies and subsequent dissertation. I would also like to thank my other committee members, especially Dr. Dennie Smith for encouraging the article format of this dissertation, and Dr. Mary Margaret Capraro and Dr. Yvonna Lincoln for their insights and feedback on this research and its related manuscripts.

I also acknowledge several fellow graduate students whose support helped me achieve my dream of obtaining my PhD. Heather Louder Joffrion played a pivotal role. Her friendship and steadfastness in prayer saw me through many stressful semesters and she well understood the pressure of being simultaneously a teacher and a student. I also acknowledge the insight of Dr. Tamara Carter in proposing I investigate the article format for my dissertation. In addition, I recognize the sacrifices of my colleagues and the students at Palo Alto Middle School in Killeen, Texas. They endured a great deal while I undertook my doctoral studies and I appreciate their flexibility and support.

Finally, I would like to acknowledge the prayer support and encouragement of my mother, as well as the members of my church, whom I consider my extended family. The members of my Sunday School class, my fellow Praise Team members, and the church staff have provided stability for me during my studies, numerous hours of commuting, and extended absences. I feel richly blessed by their commitment to continually uplift me in prayer.

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CHAPTER I

INTRODUCTION

The purpose of this dissertation is to present, in manuscript format, three articles that emerged as a result of interviewing sixth grade middle students while solving difficult mathematics problems. I present the following pieces: 1) a methodological discussion of how a researcher's ethic of care behaviors influence interactions with students in cognitive interview settings guided by theories of Gilligan (1982) and Noddings (1984), 2) an investigation of the reasoning and justification schemes middle school students use in mathematical problem solving, adapted from the works of Sowder and Harel (1998, 2003), and 3) an investigation of the representational "triggers" that influence students' justification schemes when solving *n*th term problems.

Impetus for the Manuscripts

The three manuscripts took root from a study, conducted for a graduate level mathematics education class, based on the mathematical representations used to present algebra problems on a state standardized assessment. What emerged from that study were two manuscripts, one published and one under consideration, concerning 1) representational parity, (Matteson, 2006a), and 2) the relationship between mathematical representations and mathematical literacy (Matteson, 2006b). Investigating the role mathematical representations play in mathematical literacy (Matteson, 2006b) was the precursor to the dissertation study. The results of the Matteson, 2006b study indicated that students were exposed to an increasing variety of representations on standardized

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This dissertation follows the style of *American Educational Research Journal.*

assessments as they progressed through more advanced grade levels. However, students without mathematical literacy skills, which include representational fluency, appeared to be limited in their ability to interpret and utilize multiple mathematical representations in the solution process. Although mathematics education researchers have focused on the students' creation of mathematical representations (Ainsworth, Bibby, & Wood, 2002; Hiebert, Carpenter, & Moser, 1982), limited research has been conducted that investigated students' interpretations of representations appearing on standardized assessment instruments (Bennett, Morley, Quardt, & Rock, 1999) as well as their representational fluency.

Data Limitations for the Original Study

The two precursor studies used extant state-level data, which limited the scope of further explorations given the direction of the findings. While the data were useful for the examination of representational parity, the data limited a more detailed investigation of the relationship between mathematical representations and mathematical literacy. In order to appropriately and thoroughly examine the relationship between mathematical representation and literacy, direct access to students was necessary. However, because of the constraints of the state standardized testing procedures, such access would never be made available. Instead a procedure was developed in which items from a released state mathematics assessment were administered to the same grade level of students as in the state's test administration. A sample of students was immediately interviewed after completing the test items. Thus data were obtained through cognitive interviews that had not been previously available through the state's assessment procedures.

Richer Data Through Cognitive Interviews

Cognitive interviews provide richer and deeper information concerning what students are thinking when solving mathematical problems (Willis, 1999). Using cognitive interview process provides the opportunity to elicit broader responses, as well as the potential to improve the quality of data. The cognitive interviews with sixth grade middle school students were conducted in order to gain insights into their reasoning and solution strategies with mathematical representations. Two Texas A&M University mathematics education faculty members interviewed 20 (11 male, 9 female) sixth graders in face-to-face semi-structured cognitive interviews at the students' middle school campus. The middle school campus was located in a suburban setting and was one of eleven middle schools in the culturally diverse school district (see Table I.1.)

During the cognitive interviews, students talked through their explanations of how they solved four mathematics word problems that originally appeared on the 2004 Texas Assessment of Knowledge and Skills (TAKS) Mathematics Test. The test items were selected because of their difficulty level. When initially administered to 289,449 sixth grade students, 50% or more of the students answered these items incorrectly. The items used various mathematical representations to present mathematical information. Students were asked to think-aloud, provide justifications for, and to explain their reasoning for their solutions of the four test items.

Table I.1.
Campus and District Demographics

Category	Campus %	District %
African American	55.5%	40.5%
Asian/Pacific Islander	3.8%	4.3%
Hispanic	16.8%	19.5%
Native American	0.7%	0.7%
White	23.2%	35.1%
Economically Disadvantaged	48.2%	48.7%
Limited English Proficiency	2.2%	7.1%
At-risk	57.6%	47.7%
Mobility	25.6%	32.3%

The cognitive interviews were transcribed and the transcripts were analyzed in order to ascertain how students used the mathematical representations presented in four problems to determine a solution. The two interviewers, mentor for qualitative research, and I divided the interview transcripts into subsections. After dividing the data, through a process called "chunking," major themes were identified. During the early discussions concerning themes, I mentioned noticing that the interviewers differentially interacted with the students. In struggling with trying to explain the differences in the interactions between the interviewers and students, my mentor for qualitative research and I discussed examining these differences from the perspective of an ethic of care which might impact subsequent research questions. The resulting study appears in Chapter II. Therefore, the analyses for Chapters III and IV have been formed by the findings from Chapter II.

Mathematical Representations and Mathematical Literacy

The cognitive interviews were analyzed for evidence of students' mathematical reasoning and justification schemes. The studies in Chapters III and IV examine how students translate or interpret the various mathematical representations as they justify their solutions. As a result of this process and iterative coding, several modifications were made to Sowder and Harel's (1998, 2003) justification scheme. The study in Chapter IV details that representations "trigger" students' solutions.

Overview of the Dissertation

This study was organized into five chapters and follows style guidelines of the *Publication Manual of the American Psychological Association, Fifth Edition*. This introduction is Chapter I of the study and outlines the content that follows (with Chapters II-IV fashioned as manuscripts for scholarly publication), Chapter V presenting an overall summary and conclusion, and five Appendixes.

Chapter II is a methodological piece that emerged while transcribing the cognitive interviews. Since the interview transcriptions provided the major portion of the data used for the studies presented in Chapters III and IV, an examination of the methodological concerns raised in Chapter II provides insights and background information concerning cognitive interview potentials and problems. The purpose of the study presented in Chapter II was to address these specific research questions:

1. What differences emerge within a cognitive interview format when male and female interviewers conduct student interviews under the influence of their personal behavior of care?
2. What are the methodological implications for using multiple interviewers in conducting cognitive interviews?

Implications for cognitive interviews in mathematics research are also discussed, because interviewers may exhibit differential ethic of care behaviors when conducting cognitive interviews, which may potentially influence interactions with students and data collected.

Chapter III contains an investigation of students' mathematical reasoning and justification schemes when solving challenging problems. This study introduces a justification framework that was adapted from the work of Guershon Harel and Larry Sowder. Harel and Sowder's framework, initially developed for analysis of geometric proofs, was adapted for use with other mathematical strands and their representational formats. The revision clarifies and defines first and second-level schemes and adds other schemes. During cognitive interview transcript analysis, a relationship appears to emerge between particular justification schemes and mathematical representation formats. The purpose of the study presented in Chapter III was to address these specific research questions related to the relationship of justification schemes and mathematical representation formats:

1. Do middle school students favor a particular justification scheme?

2. What is the relationship between the representational format and the justification schema?

Chapter IV presents a richer investigation into students' mathematical justification schemes and examines mathematical representations as "triggers" for students' solutions. Results from the study presented in Chapter III indicate a symbiotic relationship between students' interpretations of mathematical representations and use of justification schemes. The study in Chapter IV looked for evidence of this relationship in further detail. The purpose of the study presented in Chapter IV was to address the research question "What is the relationship between justification schemes and representational models as triggers for student mathematical understanding?"

Chapter V discusses the broader implications for the studies presented in Chapters II-IV. Methodological issues that emerged in the study presented in Chapter II are discussed, including the implications for using multiple interviewers in cognitive research studies and the possible influence on results. Although ethic of care behaviors for the interviewers were different, students were still encouraged to verbalize their solutions and mathematical understandings. The cognitive interviews were then examined for justification schemes present in student solutions, as presented in the study in Chapter III. The scheme has the potential to be applicable across mathematical strands and grade levels. The justification schemes used by the student may have a relationship to the mathematical representations used in the problem. The term chosen to describe this relationship is representational trigger. The concept of a "trigger" proposes the possibility of a symbiotic link between justification schemes and mathematical representations. Students may be influenced by representational characteristics when determining solutions to mathematics problems. This is the study presented in Chapter IV. Interviewing students' about their justifications of solutions to problems, and thereby obtaining insights into how they make use of mathematical representations, is an important educational tool. Ideas for future investigative studies with mathematical representations and student justification schemes are discussed in Chapter V.

Readers who desire more details as to the various types of mathematical representations used in standardized assessments are referred to an earlier article written by the author entitled *Mathematical Literacy and Standardized Mathematical Assessments* (Appendix A). This study explored the relationship between mathematical representations and mathematical literacy (Matteson, 2006b). This article is reprinted in Appendix A and provides a foundational understanding of mathematical representations. This background knowledge will assist the reader in understanding the studies presented in Chapters III and IV concerning students' justification schemes and their relationship to mathematical representational "triggers."

CHAPTER II

THE INFLUENCE OF ETHIC OF CARE BEHAVIORS IN RESEARCH INTERVIEW SETTINGS

Synopsis

This study considered the methodological implications of a qualitative study that involved two research practitioners as interviewers, one male and one female, who conducted semi-structured cognitive interviews with middle school students. During the reading and analysis of interview transcriptions, differences were noted between the interviewers' interactions with the students. Ethic of care behaviors noted by Hayes et al. (1994) was evident in the interviews. Data analysis found differences in the frequency of both positive and negative ethic of care behaviors exhibited by each interview and possibly influenced by gender. Implications for the methodological designs of studies involving semi-structured cognitive interviews conducted by multiple researchers are discussed.

Introduction

One unique aspect of qualitative research is the insertion of the researcher's perspective into the analysis of data. In essence the researcher holds a dual role as both researcher and a research subject, which adds a complex layer to the subsequent analysis of data. This complexity is magnified when one considers that asking questions is a natural process of human development, in that the interpretation of the "spoken or written word always has a residual of ambiguity, no matter how carefully we word the questions and how carefully we report or code the answers" (Fontana & Frey, 2005, p.

697). Researchers are drawn to interviews as a method of giving voice to various issues, and yet serve simultaneously as both recorder and interpreter of those interviews and, therefore, have the potential to let the researcher's voice drown out those of the participants.

This dual role of recorder and interpreter emerges particularly in qualitative studies utilizing semi-structured and unstructured interview settings. Within the structured interview setting provisions are made so the researcher maintains a neutral role, although not every contingency can be anticipated in interview settings. Researchers must consider a) gaining access to the setting and participants, b) understanding the language and culture of the respondents, c) deciding on how to present oneself, d) locating an informant, e) gaining trust, f) establishing rapport, and g) collecting empirical materials (Fontana & Frey, 2005). Qualitative studies using multiple interviewers have the potential for added layers of complexity to develop, especially since each interviewer much decide on how to present oneself, gain trust, and establish rapport with the participant within a limited time frame. In addition, there are ethical considerations concerning aspects of caring that have emerged from within the field of educational research (Noddings, 1986). Teachers, in the role of the *one-caring*, and students, in the role of the *cared-for*, have developed a unique relationship. In this relationship "caring is largely reactive and responsive. Perhaps it is even better characterized as receptive. The one-caring is sufficiently engrossed in the other to listen to him and to take pleasure or pain in what he recounts" (Noddings, 1986, p. 19). An educational researcher, when establishing a rapport with a school-aged participant, may unintentionally take on the role of a teacher, which then transports the researcher from an impartial observer role to that of a caring individual.

In a study of sixth graders' mathematical reasoning and justification schemes used in problems solving situations, two researchers, one male and one female, conducted student interviews in order to more efficiently collect data. It was surmised that subtle differences might emerge within the data gathered by more than one interviewer and that each interviewer's ethic of care would influence interactions with the students. This study investigates how an educational researchers' *ethic of care* (EOC) can influence the interviewers' interactions with students. Specifically this study focused on these questions:

What differences emerge within a semi-structured cognitive (Beatty, 2003; Willis, 1999) *interview format when male and female interviewers conduct student interviews under the influence of their personal behavior of care? What are the methodological implications for using multiple interviewers in conducting cognitive interviews?*

Literature Review

Ethic of Care

The concept of contrasting views concerning the ethic of care first emerged in Gilligan's (1982) *In a Different Voice* work in developmental psychology. Gilligan portrayed the EOC as a feminine perspective and contrasted it with the *ethic of justice* as a masculine perspective. According to this researcher, females were guided by "care and responsibility in relationships" (Gilligan, 1982, p. 73), whereas males were guided by “rights and rules” or fairness (p. 73). Although both genders may exhibit these characteristics, there is a predominance of a specific attitude within each gender, with females interpreting care as “an activity of relationship, of seeing and responding to need” (p. 62). Noddings vaulted the ethic of caring into the educational forefront by proposing that when one becomes a teacher, one enters a special caring relationship with one’s students in which “the student is infinitely more important than the subject matter” (Noddings, 1984, p. 176). It has been suggested that a teacher’s EOC is apparent in the activities of “modeling, dialogue, practice, and confirmation” (Owens & Ennis, 2005, p. 395). Researchers have attempted to identify traits of caring teachers from the perspectives of teachers (Collinson, Killeavy, & Stephenson, 1998; Rice, 2001) and students of various ages (Alder, 2002; Alder & Moulton, 1998; Bosworth, 1995; Hayes et al., 1994; Noddings, 1992; Teven & Hanson, 2004; Vogt, 2002). Such studies have indicated that there are numerous behaviors that educators exhibit when caring for their students and that what is considered a caring behavior is different for various age levels of students.

Studies have examined care as an ethic for those involved in educational research in which they have a “professional, functional or emotional bond” (Costley & Gibbs, 2006, p.89.) An EOC should be apparent in educational research so that the atmosphere of trust and professional respect is maintained within the complexities of methodological design and the analysis and reporting of results (Noddings, 1986). If this assumption were correct, it would be logical to conclude that a possibility exists that researchers’ behaviors of care influence how one interviews students. In other words, an interviewer's ethics of care would influence data collection.

Cognitive Interviews

There are two main cognitive interviewing methods: *think-alouds* and *verbal-probing* (Willis, 1999). In the think-aloud interview students verbalize what they are thinking in response to a stimulus such as a question. The interviewer rarely interjects comments, except to encourage the student to continue to express what they are thinking. There are several advantages of this method including its open-ended format, minimal training requirements, and freedom from interviewer imposed biases. However, there are also disadvantages, such as sometimes students are reluctant or resist responding, may pursue off-track lines of thought, or feel pressured to respond in a specific way. In using the verbal-probing method the interviewer asks for additional information in order to gain a deeper understanding of the student's thinking. "The strongest justification for the more probe-based paradigm is that it generates verbal material that questionnaire designers find useful, but that may not emerge unless a cognitive interviewer specifically asks for it" (Beatty, 2003, p. 13). There are disadvantages to verbal-probing, including the potential to introduce artificiality or bias because of the line of probing the interviewer chooses. However, this type of interview shifts from analyzing responses to questions to collecting rich data. Verbal-probing can be performed *concurrently* or *retrospectively* (Willis, 1999). Concurrent probing is characterized by an interchange or dialogue between the interviewer and the student. Retrospective probing is more of a debriefing, which would occur at the end the interview and requires the student to remember what they were thinking after the fact, which may not result in an accurate recollection. Furthermore, verbal probing can use specifically *scripted* probes, in which the questions are developed prior to the interview, as opposed to *spontaneous* probes (Willis, 1999). The spontaneous probe is sometimes misunderstood as being less scientific; however, the spontaneous format does provide for flexibility in pursuing important and interesting issues that emerge during the interview. Cognitive interviews may also incorporate a combination of scripted and unscripted probes.

Theoretical Framework

A semi-structured interview format that utilized both think-aloud and verbal-probing was employed in this study of students' reasoning and justification schemes used when solving difficult mathematical problems. This format was regarded as having the best potential for obtaining deep and valuable insights concerning students' justification schemes. The interviewers engaged in concurrent probing and used a combination of scripted and spontaneous questions. It was anticipated that subtle differences would emerge between our male and female interviewer. Differences were found in the interactions between the interviewers and students as the transcriptions were read and analyzed. These differences influenced the data collected and subsequent analysis and reporting of results, although they did not substantially introduce an element of bias in the data collected. For this study, it was theorized that an educational researcher's EOC would be evident in the interviewers' interactions with students.

It was important to identify ethics of care that would be applicable to middle school student interview settings. Several of the EOC behaviors that emerged from the analysis of middle-class, suburban sixth-grade student interviews by Hayes et al. (1994), and generally confirmed by Ferreira and Bosworth (2001), were used in analyzing the interview transcripts. Hayes et al. elaborated on 11 students' perceptions of care that included, from greatest to least in frequency of reference, 1) responding to the individual, 2) helping with academic work, 3) encouraging success and positive feelings, 4) providing fun and humor, 5) providing good subject content, 6) counseling the student, 7) interest in all students/fair, 8) avoiding harshness, 9) listening, 10) managing the class well, and 11) other. In our study the male and female interviewers had only a brief time to establish rapport with the middle school students. Since the interviewers' primary directive for this study was to *listen* and probe students' explanations of their mathematical problem solving strategies, several of Hayes et al.'s behaviors were not relevant to this study. Three of the first four student perceptions that emerged during the Hayes study became the focus of this study. They include: 1) responding to the individual, 2) encouraging success and positive feelings, and 3) providing fun and humor. Hayes et al.'s descriptions of each of these perceptions were then adapted for this study. The interviewers demonstrated *responding to an individual* through engaging in discussions about family or other volunteered personal information, which indicated an interest in the students' whole life. Exchanges in which students were encouraged, praised, or made to feel successful demonstrated *encouraging success and positive feelings.* Moments of laughter or humor demonstrated *providing fun and humor*. Two responses that emerged during initial readings of the transcriptions were included in the transcription analysis. Both responses demonstrate what was considered a negative EOC of the individual and were labeled 1) *avoiding assisting* and 2) *ignoring the individual.* Not pointing out an error or failing to guide the student to the correct item answer was labeled as avoiding assisting. When the interviewer did not follow up on students' personal comments, the behavior was labeled as ignoring the individual.

Methodology

Participants and Setting

The student participants in the study include twenty (11 male, 9 female) sixth grade middle school students from a culturally diverse school district in the southwestern United States. Students selected for interviews were a stratified representation of the genders and ethnicities of the campuses' sixth grade student body of 223 students and were instructed in both regular and pre-advanced placement mathematics classes. Regular mathematics classes cover the sixth grade mathematics curriculum, but pre-advanced placement mathematics classes cover seventh grade objectives and prepare the students for a pre-algebra course the following year. Students may self-select the pre-advanced placement classes and do not have to meet specific criteria.

Two state university mathematics education faculty members, one male and one female, interviewed the students participating in the study. Both individuals were experienced in conducting student interviews with middle-grades students. Students were randomly placed with an interviewer. The female interviewer met individually with ten students, while the male interviewer met individually with nine students. Due to time constraints, the last student interview of the day was conducted with both interviewers present, but the female interviewer was the primary interviewer, although the male interviewer was present and participated infrequently in the interview. Interviews occurred during the school day and took place in offices located just off the school library. Interviews lasted between 8 and 44 minutes, depending on student responses.

Data Sources

Audio and videotapes were made of each student interview in which students explained and justified their solutions to four difficult mathematics problems (Matteson, Capraro, Lincoln, Capraro, 2007). The audiotape recorder was placed on the table between the interviewer and student and the video camera was focused on the student and any work examined or created during the interview. The primary function of the audiotape was to clarify verbal responses that were difficult to decipher from the videotapes due to background noise or softness of the response. Transcriptions of each interview were created from the videotape. These were then reread while listening to the audiotape, which provided a clearer and more accurate audio record. The written transcriptions were edited as necessary for accuracy.

Coding of the Transcriptions

Data analysis occurred in stages. First the interview transcriptions were read and marked for the positive EOC behaviors of 1) responding to the individual, 2) encouraging success and positive feelings, and 3) providing fun and humor. The interview transcriptions were also marked for negative EOC behaviors, as evidenced by 1) avoiding assisting and 2) ignoring the individual.

A member check was conducted to ensure validity and reliability. Four interviews (20%), two conducted by each interviewer, were read and marked for positive and negative ethic of care behaviors by the first author and each the two interviewers who conducted the interview. A lottery system was used to determine which interviews were read. The four interviews were subsequently discussed and consensus was obtained concerning the various EOC categories and frequencies of such occurrences. The focus of these discussions was to ensure an accurate count of the specific EOC behaviors for each interviewer by verifying the intention of the interviewers' interactions with the students.

One EOC, encouraging success and positive feelings, was determined to contain a subcategory in that both interviewers within that specific EOC behavior used two levels of response. The transcripts were subsequently recoded to reflect these subcategories. *Extended* responses included acknowledgement of specific behaviors, while *abbreviated* responses included one word or a short phrase. The following transcription excerpt[1] demonstrates the extended level of encouraging success and positive feelings. All students were given pseudonyms.

Sherri: What I did, I already multiplied a hundred by eighty because I just did the area. And then I multiplied thirteen by eight and then I got the area of that. So once I got my answer of a hundred feet by eighty feet, I just subtracted what thirteen feet by eight feet was. And then I got my answer.

Interviewer: Seems like you did that one without a problem. That one seems REALLY easy for you. You didn't even stop for a minute to think about what you did.[2]

An abbreviated response is demonstrated in the following exchange, which could have easily been extended to acknowledge the correctness of David's definitions of perimeter and area.

Interviewer: What's the difference between perimeter and area? What do you have to do differently?
David: Perimeter is just finding out what is AROUND and area is finding out like how much is inside it.
Interviewer: That's correct. So now you went back, you wanted to find out the area, so what's this? What's over here?

Data Analysis

The time codes from the videotapes were used to determine the amount of time each interviewer spent with a student. This data were used to calculate the average length of an interview conducted by the male and female interviewers. A frequency analysis and percentage of the positive and negative EOC behaviors was created for each interviewer by coding the video transcriptions for both behaviors of care. Frequency information was used to create graphs that compared the frequencies and percentages of the positive and negative behaviors of each interviewer by gender.

Results and Discussion

The verbal interactions between the interviewer and student were used to answer the question "*What differences emerge within a semi-structured cognitive interview format when male and female interviewers conduct student interviews under the influence of their personal behavior of care?"* Transcription analysis first concentrated on the length of the student interviews. Table II.1 shows the differences in the length of the interviews conducted by each researcher. The male interviewer talked with students 9 minutes and 37 seconds longer (56.6%) than the female interviewer. Lever (1976) noted when boys and girls were involved in activities such as games, that boys' activities lasted longer. It could be argued that the male predisposition of staying with something for a longer period of time accounts for the significant difference in the length of the interviews. However, a more logical explanation is that, compared to the female interviewer, the male interviewer worked with students with a larger number of wrong answers. The male interviewer's students averaged 1.56 (*SD* = 1.13) correct responses while the female interviewer's students average 2.30 (*SD* = 1.16) correct responses. This means the female interviewer's students outscored the male interviewer's students by 47.4%. Therefore, the discrepancy in the interview lengths may be attributed to the interviewer working with the student to understand problems they answered incorrectly.

The previous cited Hayes et al. (1994) study listed the EOC behaviors from greatest to least in frequency: 1) responding to the individual, 2) encouraging success and positive feelings, and 3) providing fun and humor. The male and female interviewers' frequencies slightly reorder these behaviors by moving the first behavior to last place, in other words behind the second and third behaviors. This change of order may be indicative of the lack of time available during the interviews in which one would gain a deeper understanding of the student as an individual.

Table II.1.
Length of Student Interviews by Interviewer by Interview Order

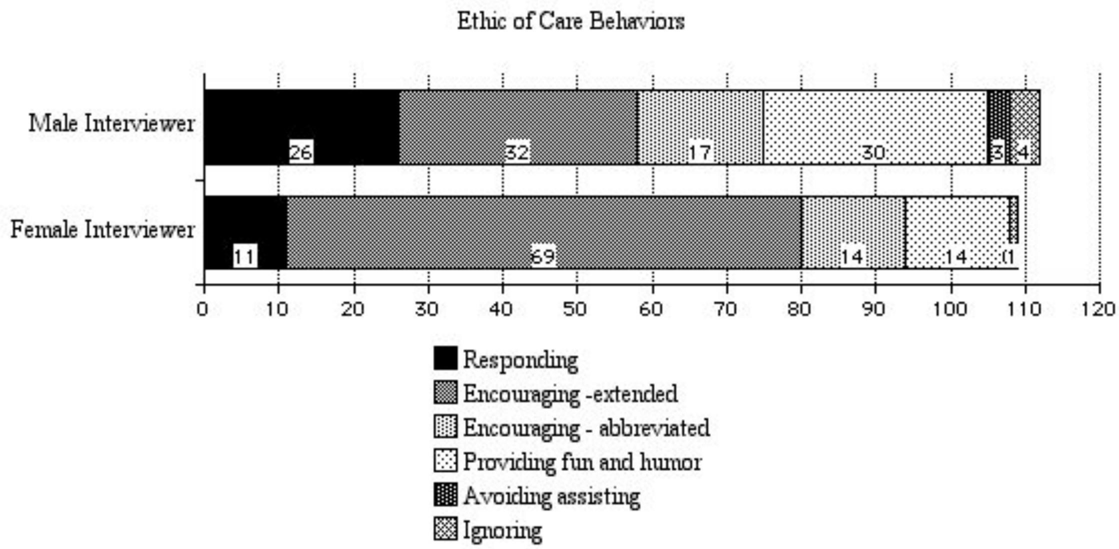
Male Interviewer		Female Interviewer	
*Student	Length	*Student	Length
Sylvia	43:55	Tanya	19:15
Maria	[1]36:26	Amber	25:58
Joey	25:22	Jenny	13:36
Marcus	22:02	Simon	13:08
Craig	33:54	Alejandro	29:55
Sherri	[2]18:10/17:18	Jacob	13:15
Angelina	18:58	Conrad	13:31
Dominic	27:19	David	12:40
Sharon	[3]17:09/14:07	Carl	19:59
		Sam	8:36
Mean	27:02/26:36	Mean	16:59
Both Interviewers			
Anne	15:56		

Notes: *All names are pseudonyms. [1]Videotape ran out, audiotape continued – added minutes from that recording to create total interview length. [2]Interviewer is seen turning off the videotape, but remote control did not work. [3]Interviewer engages student in personal dialogue, but cognitive interview is clearly complete.

The frequency and percent of EOC behaviors of the male and female interviewers were calculated (see Table II.2). The interviewer's frequencies for each positive behavior are reported in parenthesis after each behavior label. The male interviewer's frequencies were responding to the individual (26), extended encouraging success and positive feelings (32), abbreviated encouraging success and positive feelings (17), and providing fun and humor (30) for a total of 105 positive EOC responses. The female interviewer's frequencies were responding to the individual (11), extended encouraging success and positive feelings (69), abbreviated encouraging success and positive feelings (14), and providing fun and humor (14) for a total of 108 positive EOC responses. Although the male interviewer conducted interviews that were almost 50% longer than the female interviewer's, there was not a corresponding increase in the total number of EOC behaviors. Figure II.1 provides a visual comparison of the frequencies of the EOC behaviors by male and female interviewers. The male interviewer's frequencies are very similar or more balanced, for three of four positive behaviors, with the frequency of the behavior of abbreviated encouraging success and positive feelings lower than the other behaviors. The stacked bar graph shows the female interviewer's EOC is most frequently expressed through encouraging success and positive feelings in that she specifies or extends her responses during the interview setting.

Table II.2.
Frequencies and Percentages of Positive and Negative Ethic of Care Behaviors by Interviewer

Ethic of Care	Male Interviewer Frequency (%)	Female Interviewer Frequency (%)
Positive behaviors	105 (93.8%)	108 (99.1%)
Responding to the individual	26 (23.2%)	11 (10.1%)
Encouraging success and positive feelings		
extended	32 (28.6%)	69 (63.3%)
abbreviated	17 (15.2%)	14 (12.8%)
Providing fun and humor	30 (26.8%)	14 (12.8%)
Negative behaviors	7 (6.3%)	1 (0.9%)
Avoiding assisting	3 (2.7%)	
Ignoring the individual	4 (3.6%)	1 (0.9%)
Total behaviors	112	109



Note: Female interview had no avoiding assisting behaviors.

Figure II.1. *Comparisons of Frequencies of Ethic of Care Behaviors by Gender of Interviewer.*

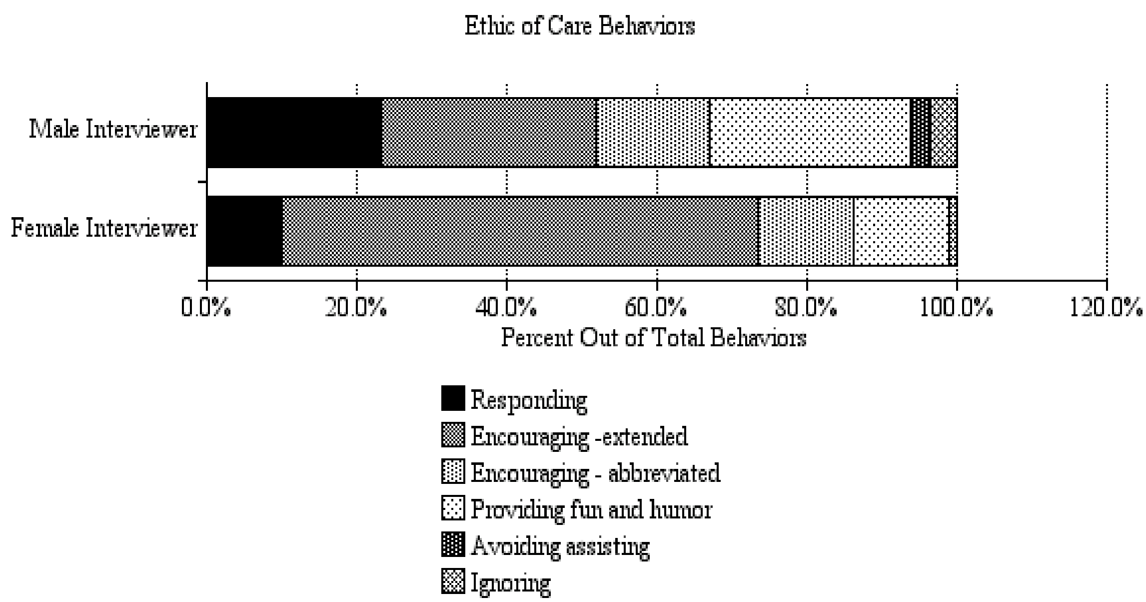


Figure II.2. *Comparisons of Percentages of Ethic of Care Behaviors by Gender of Interviewer.*

Figure II.2 provides a visual comparison of the percentages of EOC responses by male and female interviewer. This figure shows that the male interviewer's behaviors of responding to the individual and abbreviated states of encouraging success and positive feelings are given approximately 50% more often than the female interviewer. This aligns with both the increase in the length of time interviewed and the average correct score of the students interviewed by the male interviewer.

Only eight negative EOC behaviors were found during the transcription analysis, seven for the male interviewer and one for the female interviewer. Three times, with three different students, the male interviewer avoided assisting a student to obtain a correct answer. Four times, all with the same male student, the male interviewer ignored the individual by not following up on personal comments. The female interviewer exhibited only one of the negative behaviors, that of ignoring the individual, by not following up on personal comments.

These results indicate that the male and female interviewers, in general, exhibited the specific gendered attitudes of EOC behaviors as proposed by Gilligan (1982). The male interviewer's negative behavior of avoiding assisting students in finding the correct solution to a test item could be interpreted as an example of "rights and rules" by conducting interviews focused on student justifications. However, such a generalization is difficult to confirm with such a limited set of data, just three occurrences. It is somewhat easier to generalize about the gendered attitude of the female interviewer as approximately 60% of her EOC behaviors, exhibited "care and responsibility" as she saw and responded to her perceptions of student needs by making comments that encouraged the students' success and positive feelings.

In regards to the second question, *"What are the methodological implications for using multiple interviewers in conducting cognitive interviews?"* this study points to careful consideration of methodological issues especially those connected to cognitive interviews conducted with students. Large-scale studies incorporating student interviews would be difficult to manage due to the need to employ multiple interviewers in order that the study is conducted in a timely manner. When multiple interviewers are utilized, safeguards must be considered so that data collection is not suspect or compromised. This is especially true in educational research settings in which the interchanges between the student and interviewer provide important data and insights into critical educational issues. If students feel uncomfortable in the interview setting, which is always a potential problem, then the opportunity for the collection of rich data is also lost. It should be noted that the unintentional exhibition of several important and specific EOC behaviors by the two interviewers in this study do not appear to interfere with data collection, and may actually have facilitated a deeper exploration of the mathematical understandings held by the student. However, the results of the study do raise a concern of whether or not educational research should limit access to students to those who have significant experience with students in the specific field of which the research is conducted. Is there a quality aspect to what insights can be garnered from those who conduct research from outside the field, as opposed to those who are familiar with the field's specifics? Qualitative researchers might argue that this is indeed the case.

In the reporting of results, researchers in all fields should carefully consider the following questions when designing methodologies utilizing multiple interviewers: Are interviewers' EOC behaviors important to the topic or data being collected, that is, would EOC behaviors enhance the data collected or is there a potential negative impact on the data collected? Should interviewers be matched by specific EOC characteristics to the participants in studies involving specific and/or sensitive topics? Should methodologies include protocols that inform or caution interviewers about potential gender differences in EOC behaviors? Additionally, researchers need to recognize and account for nuance influences that specific EOC behaviors may provide in obtained results.

Endnotes

[1]Transcription symbols and abbreviations include:

. . . brief silences or pauses of 1 or 2 seconds
(6 sec) indicates silence of a specific number of seconds
// interruption or overlapping speech
[provides and indication of what is occurring visually on the tape]
ALL CAPS - words or phrases emphasized by the speaker

[2]Interview transcriptions were edited to remove nonessential verbalizations and utterances such as "umm," "uh," and others that do not substantially address mathematical content or the purpose of the interview. Researchers who engage in traditional discourse analysis deem all elements of the interview as important, but our focus is providing an example of an ethic of care.

CHAPTER III

EXTRICATING JUSTIFICATION SCHEME THEORY IN MIDDLE SCHOOL PROBLEM SOLVING

Synopsis

Twenty middle grades students' were interviewed to gain insights into their reasoning about problem solving strategies using Harel and Sowder's (1998) Justification Scheme as our theoretical lens and the basis for our analyses. This justification scheme was modified slightly making it more broadly applicable and thus accounting for recent research developments in the cognitive sciences. During cognitive interviews, students reasoned about their solutions to four contextualized problems. We found that students who used various combinations of justification schemes were more successful problem solvers as compared to students who focused on justification schemes suggested by the representations used to present the problems. Support was found for Sowder and Harel's (1998) assertion that Justification Schemes can be both beneficial and problematic when students lack the sophistication to combine various schemas in the solution process. Implications for pedagogy are included.

Critical Issues in Problem Solving

Mathematical problem solving has remained a focus of researchers for almost half a century. Polya's *How to Solve It* (Polya, 1957) became a framework for numerous researchers as they explored factors that make individuals successful problem solvers and how they explain or justify their solutions. Research to date has focused on each of the four steps of the problem solving process and the goal of this study is to examine the "look back" step at a granularity conducive to understanding student thinking about their solution and the process for arriving at the solution.

Problem solving skills are considered foundational to building mathematical knowledge because as one learns to apply and adapt strategies, it encourages monitoring and reflecting on one's own learning (National Council of Teachers of Mathematics, 2000). Various factors have been considered to facilitate students' development of a mathematical disposition and the necessary skills needed to persist in solving challenging problems. Emphasis has been placed on teaching of problem-solving strategies (Hohn & Frey, 2002; Polya, 1957) and numerous studies focus on students' understanding and use of those problem-solving strategies (Hegarty, Mayer, & Monk, 1995; Jiménez, & García, 2002; Mastromatteo, 1994; Pape & Wang, 2003). Unfortunately it has been shown that, even when students acquired computational skills and conceptual knowledge of problem-solving procedures and strategies, such knowledge was not a strong indicator in predicting which students would be successful mathematical problem solvers (Ballew & Cunningham, 1982).

Skill development has not dramatically affected students' problem solving success, consequently focus was placed on the teacher's role in the development of student' mathematical thinking skills (Fennema et al., 1996; Greenes, 1995; Henningsen & Stein, 1997). Along this line of research three separate but parallel paths have been explored: the teacher's role in (a) establishing a supportive learning environment, (Newman, 2002), (b) providing opportunities for mathematical discourse (McClain, Cobb, Gravemeijer, & Estes, 1999) and (c) responding to students' mathematical thinking (Davis, 1997; Doerr, 2006). Even with pedagogical, context, and content changes in place, student success in problem-solving situations has remained unpredictable and elusive.

Consequently, education researchers have turned to examining the characteristics of successful problem solving students. Researchers have investigated students' reading ability, linguistic competence, or computation skills (Ansley & Forsyth, 1990; Hashway & Hashway, 1990; Leong & Jerred, 2001; Sovik, Frostrad, & Heggeberget, 1999). The continued struggle with pinpointing factors that predict students' success in problem-solving situations has pointed to interactions between affective and cognitive variables (Boekaerts, Seegers, & Vermeer, 1995). Cognitive development theorists, such as Piaget and Vygotsky, have been linked to mathematical problem solving. Researchers from various constructivist perspectives have analyzed or categorized students' explanations of strategies and thinking in problem solving situations with students at all age levels (Carpenter, Frank, & Levi, 2003; Evens & Houssart, 2004; Greenes, 1995; Hiebert et al., 1982). The constructivist approach generally examines the organization and structure of processes involved in solving problems with regard to cognitive demand within the confines of prior knowledge. The assumption with this theory is that students gain fluency in expressing their mathematical thinking, and increase their skills of generalizing and justifying answers, when provided opportunities to investigate the "big ideas" (Greenes, 1995) that underlie various mathematical strands. Research along theses lines has identified a consistent, albeit small, relationship between solving arithmetic problems and particular cognitive abilities (Hiebert et al., 1982). These findings are aligned well with the notion that justification schemes are tied to algorithms (structured and unstructured), and procedures (formal and informal).

The nexus of reflecting on one's own knowledge and cognition also has been a line of inquiry. Researchers have examined this link between cognition and one's self-assessment and monitoring of learning, or metacognition (Lucangeli, Coi, & Bosco, 1997; Lucangeli, Tressoldi, & Cendron, 1998). Results showed that students benefit from opportunities to engage in metacognition since "One of the most valuable lifelong skills students can acquire is the ability to look back and reflect on what they have done and what they still need to do. Students who develop a habit of self-assessment will also develop their potential for continued learning" (Stenmark, 1991, p. 6). Reflecting on one's work, or Polya's fourth step of "looking back" (Polya, 1957), has garnered much attention in educational research and gave rise to understanding student strategies for justifying their solutions. This critical reflection process provides both evidence of students' reasoning or justification of their answer as well as insights into their mathematical sophistication.

In order to become successful mathematical problem solvers, students must develop reasoning skills and be able to articulate a justification as they make and evaluate conjectures and generalizations in all mathematical strands. The National Council of Teachers of Mathematics (NCTM) Standards state, "reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts" (NCTM, 2000, p. 56). NCTM's recent publication, *Curriculum Focal Points,* emphasized that K-8 grade level instructional focal points should incorporate “an application of reasoning to justify procedures and solutions” (NCTM, 2006, p.10). Learning to express one’s mathematical thoughts requires practice, as does evaluating one’s line of thinking. Reasoning skills must be cultivated and developed early in one’s mathematical studies and, as students progress through the grade levels, reasoning and proof skills should become more varied and sophisticated. Students should formulate plausible conjectures, test and revise conjectures as necessary, and use inductive reasoning to reach a generalization (NCTM, 2000). Even elementary children have been found to use external, empirical, and analytic proof schemes, or combinations of proof schemes (Flores, 2002). A natural progression of reasoning skills would include elementary students using concrete materials to test their conjectures, middle school students expressing reasoning through written and verbal mathematical language and appropriate symbols, and high school students incorporating more formal mathematical language and conventional proof formats with arguments clearly and logically written out.

Our study examined middle school students’ reasoning and justification skills when working with assessment problems that had already been identified as challenging. Our work focused on students’ justification schemes as we analyzed the interplay between understanding mathematical representations, use of problem-solving strategies, and students’ mathematical literacy skills in solving problems.

Analytic Justification Model

We acknowledge students need to be well versed in problem solving strategies and teachers’ perform a pivotal role in providing rich problem solving experiences for students. We concur that there are complex issues regarding mathematical literacy, computation skills, and affective considerations that need further examination as to their influence upon students' problem solving success. Having examined various theoretical models considered to facilitate students' development of mathematical reasoning and justification skills needed to persist in solving challenging problems we are most interested in students' reasoning and justification explanations as indicators of student problem solving success.

Harel and Sowder (1998) primarily used their justification scheme with university students studying geometry, but substantial evidence exists that their justification scheme may fit well across mathematical strands such as linear algebra (Harel, 1997) and number theory, (Martin & Harel, 1989). Therefore, an analytic model was derived from Harel and Sowder (1998) and Sowder and Harel (1998, 2003) to accommodate current thinking in regard to mathematical representations (e.g., Capraro & Capraro, 2006; Goldin, 2000) and cognitive models (Augustyniak, Murphy, & Phillips, 2005; Garrett, Mazzocco, & Baker, 2006; Steele & Johanning, 2004). The specific ways in which the Sowder and Harel (1998) model was adapted was used to examine justification schemes of participating middle grades students' as they solved four specific word problems.

The two frameworks have several key differences. Harel and Sowder focused on justifying results, whereas the framework used in this study focused more on how students arrived at the results. Like Sowder and Harel, the use of the terms *justification* and *proof* are used in the most general sense and ***not*** specifically related to geometry, but other terminology was seriously revised. For clarity, and because of findings since the Sowder and Harel study, their *ritual* category was replaced with *mechanistic* to reflect a simple application of some previously learned process or procedure, and *symbolic non-quantitative* was replaced with two categories, “language” and “visual,” to reflect the dichotomy, in Sowder and Harel’s original coding strategy, between explanations that were related to language or some visual form. The visual category includes all classes of representations (i.e., pictorial, numbers, symbols, and written/verbal).

Each of the first-level schemes, Mechanistic, Authoritarian, Language, and Visual, is comprised of multiple second-level, more refined classifications (see Figure III.1). For example, the Mechanistic category contains two second-level categories of algorithms and procedures. These terms are differentiated by defining the algorithm as a well-known, easily recognized process that is regularly generalizable, whereas, procedures are less formal, tend to be more idiosyncratic, and are less likely to be generalizable. If one were inclined to use the terms *reasoned* or *proof*, procedures are likely to contain attempts at *proving* their case by *reasoning* about problem context, a simpler problem, or a similar problem regardless of logical similarities. The Authoritarian category is comprised of second-level categories similar to those of Harel and Sowder (1998) and Sowder and Harel (2003), but differs in that it includes “self” as an authority. Harel and Sowder’s framework concentrated on external sources, such as textbooks or the teacher, and ignored recognizing the idea of the mathematical development of the individual as a justification.

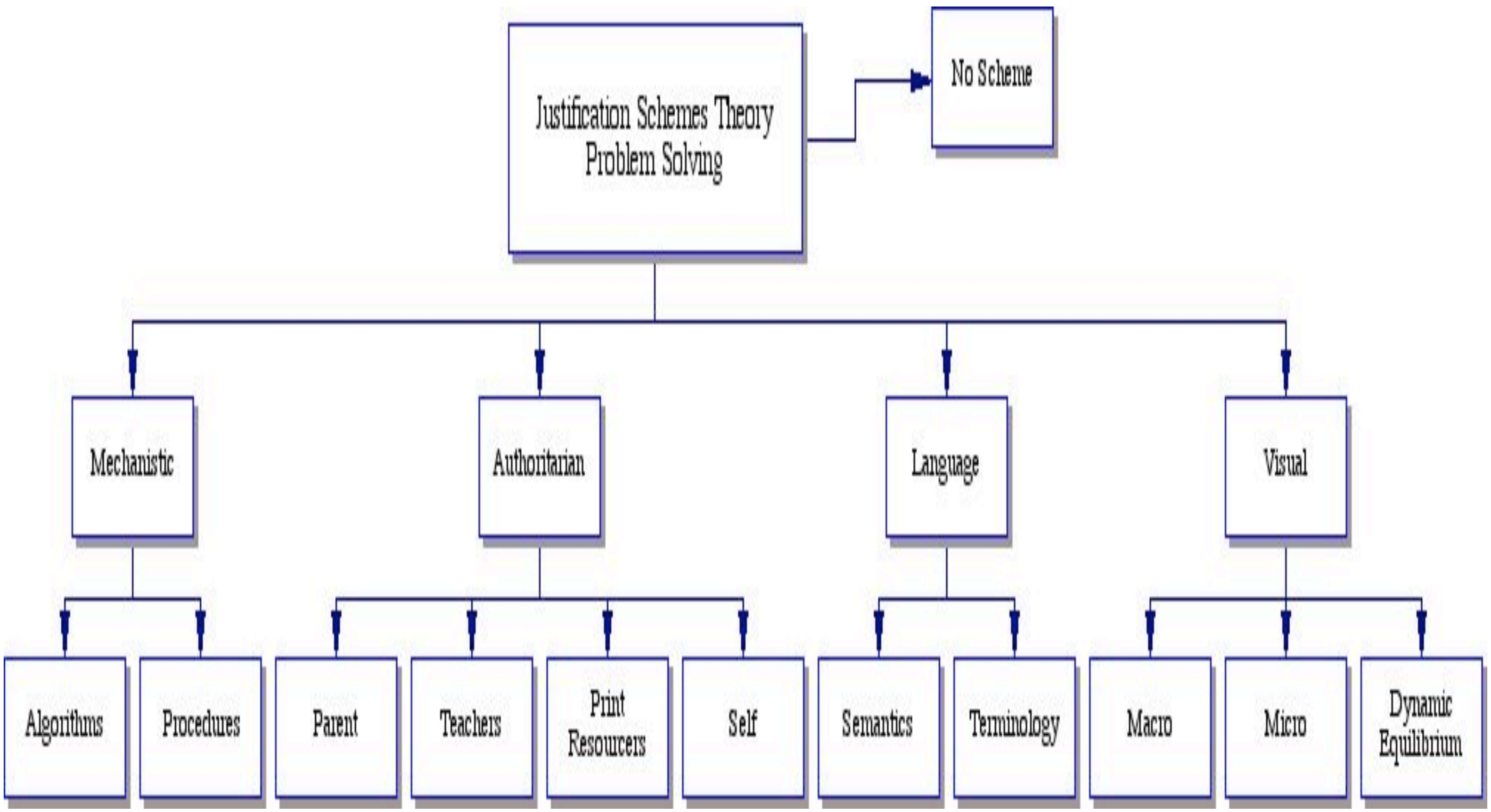


Figure III.1. *Student Justification Scheme*

Language is comprised of semantics and terminology. Semantics attenuates nuance differences indicating sophistication with understanding the interrelationships among words, i.e., the abililty to derive meaning from the terms. Hence, terminology is the simple understanding of specific words essential for solving the problem. Visual is composed of three categories: macro, micro, and dynamic equilibrium. The macro case occurs when students examine the entire problem and engage the presentation at the *whole* level without attempting to examine smaller units within the problem. Micro occurs when students focus on the details ignoring the relationship to the whole presentation. The dynamic equilibrium category indicates students who move between both the larger presentation and its smaller components when explaining their justification scheme. The "No scheme" category was added in anticipation that participants might not indicate any justification scheme and it would be important to understand how often a student did not fit the model.

Research Purpose

During an analysis of the 2003 and 2004 released versions of Texas Assessment of Knowledge and Skills (TAKS) mathematics test for grades 3 through 8, it was noted that each assessment contained several test items in which *less than 50%* of the students examined, over 260,000 per grade level, answered correctly. The number of problems in which less than 50% of the students answered correctly increased throughout the grade levels and by Grade 8, 24% of the 2004 TAKS mathematics test items were answered incorrectly by 50% of the students. The research team found that this problem solving context was appropriate to the theoretical model and important and timely given the weight given at state and federal levels to student performance on high stakes tests.

Although 50% or more of the states' sixth graders did not correctly answer the four test items used in this study, we avoided evaluating students' thought processes from the perspective of a simple error analysis. Instead we incorporated an *interpretive orientation* of the student responses as we evaluated "students' ideas with the aim of accessing their understandings, seeking information through more elaborated responses, and asking for demonstrations or explanations" (Doerr, 2006, p. 6). A *hermeneutic orientation* was subsequently used when the interviewers probed further after the students' provided initial comments and chose to "interact with their students, listening to their ideas and engaging with them in the negotiation of meaning and understanding" (Doerr, 2006, p. 6). This study focused on an interpretive orientation of the justification schemes used by middle graders on difficult mathematics word problems. Specifically we sought answers to the following:

Do middle school students favor a particular justification scheme? What is the relationship between the representational format and the justification schema?

Method

Participants and Setting

The participants in the study were twenty (11 male, 9 female) sixth grade middle school mathematics students from a culturally diverse school district in the southwestern United States. The middle school campus included students from grade six through eight. The participants were a representative sample of the ethnicities and genders in the sixth grade class of 223 students. Two female mathematics teachers instructed the sixth grade students. The teachers collaborated on lesson plans, activities, and assessments. The concepts used in the study were familiar to the students.

Two mathematics education faculty members conducted the individual student interviews. Both faculty members had prior experience with conducting student interviews and understood the semi-structured format of interviewing students. Student interviews were conducted during the school day. Researchers have noted that interviewing students concerning their problem solving strategies provides insights into students' mathematical thinking and aids the teacher in determining misconceptions (Crespo, 2000; Crespo & Nicol, 2003).

Instruments

Four assessment items, numbered 16, 24, 32 and 34 (see Figures III.2-III.5), previously identified as having been correctly solved by 50% or less of the state's sixth grade students during the prior year's test administration, served as the discussion focus of the interview. Students responded to four questions about the assessment items on which they were interviewed. They were asked to (a) rank the four test items from easiest to hardest, (b) indicate what made a mathematics problem easy, (c) indicate what made a mathematics problem hard, and (d) elaborate on what they did when they worked a math problem (see Appendix B).

Procedure

All sixth grade middle school students had been administered the previous year's released version of a state mathematics assessment one month before the interviews. The assessment booklets and student answers from that test administration were made available to the interviewers. Four of the assessment items were re-administered to the students immediately prior to the interview. Students were then interviewed individually with students' responses to the four questions concerning the test items incorporated into the interview format. Students started with the test item they considered the easiest to solve. The interviewer used a semi-structured protocol that allowed flexibility in asking for clarification or probing further into students' reasoning and justification. Students were allowed to see their answer from the previous test administration. The rationale for selecting the test item was never revealed to the student because research suggests that when students are faced with unfamiliar or difficult problem solving tasks they may become recalcitrant (Doerr, 2006; Newman, 2002).

Data Sources and Analysis

The data sources included test booklets containing the four assessment items from an earlier administration of the entire released test, students' written responses to the four questions concerning the assessment items, student work created when solving the four mathematics test items immediately prior to the interview, and student interviews. Appendix B contains the four questions asked the students concerning the test item. Appendix C summarizes the students' responses to questions B and C. The students were audio and videotaped during the individual interview sessions. The video camera was set up to focus on the student and any work created during the discussion of test item solutions. The audiotape assisted in decoding student responses that may have been obscured due to background noise or students speaking softly. Videotapes were transcribed[1], then compared to the audio transcriptions for further clarity. Student written work from a prior administration of the four test items, the written responses to four questions concerning the assessment items, and solutions discussed during the interview were used to triangulate as well as augment data sources.

Cognitive interview. The model for cognitive interview followed the caveats outlines by Beatty (2004) with special consideration to avoiding influencing responses by asking participants to verbalize any thoughts that came to mind. Analyses of the data took place in stages. After initial video transcription, they were reread while listening to the audiotapes of the interview sessions to ensure an accurate transcription of the interview. The second stage involved dividing the video transcripts into sections relating to a) students explanations, b) interviewers' follow-up questions, and c) students' responses summarizing problem difficulty. Four interviews were read and divided into meta-categories by all three investigators, and the section divisions of each investigator were compared for continuity and cohesion. Standardization of divisions were revised and refined during this process. Three researchers then iteratively paired to code the remaining 16 transcripts. This process was used to control rater drift and a maturation threat. Reliability was monitored at each scoring between each pair. Areas of discrepancy concerning the divisions of the transcriptions were discussed and a consensus was made before the data were placed onto note cards for categorization. A process we term reconciliation, where the two coders come together to discuss their categorizations and come to agreement on any discrepancies, was used to ensure that raters remained consistent to the framework. During reconciliation each categorization was justified and explained and discussion continued until agreement. Minor differences were accepted without reconciliation such as one coder beginning a section with the transition statement or ending the previous section with that same transition statement. After initial coding, this difference was minimized.

During the second stage we noticed that some students were reticent during the interviews, while others were open and spoke freely. We agreed to focus the initial data analysis on student responses for the test items that each individual student ranked as *very easy* or *easy* to solve. We believed that even the reticent students would be more likely to talk freely on an item they perceived as easy to solve.

The final phase of the analysis was to reread the portions of the interviews where students discussed their solutions to those test items they had ranked as very easy or easy. Each interview was independently reread and marked for justification schemes by two investigators. The investigators then met and shared their observations concerning students' use of justification schemes. Many times students concentrated on the representational format of the test item, which appeared to focus students on considering a specific justification scheme. In almost every instance, students often expanded on their original justification scheme by using additional justification schemes. During our conversations, we determined that students' justifications were not always blatantly obvious and sometimes we inferred a connection to a specific justification scheme. We are sensitive to the concept of extrapolating results beyond the actual data presented and our rationale for such inferences will be clearly identified in the discussion. We believe that our numerous combined years of experience with teaching middle grade mathematics students and understanding the context of the discussion that occurs before and after the portions presented in this manuscript allow us to make reasonable inferences of the students' justification schemes even when they were not explicitly stated by the students.

Discussion of Justification Schemes

Our discussion and findings is organized in numerical order by test item. We first present a summary of the ethnicities, genders, and success rate of our 20 students who labeled the item as very easy or easy then discuss the student justification schemes. Our examination of the transcripts found a variety of schemes were used for each test item. However, one important caveat is that students may have used a particular scheme other than what they verbalized during the interview. Therefore, inferences are based on the triangulation of available information from the previous student work and answers, with their current work and answers with their cognitive interview. We again note that the representation used to present the test item may have focused students to use a particular justification scheme or caused students to ignore or overlook specific information because of an overemphasis on item presentation.

Schemes and Combinations of Schemes of Justification

Of the two items students identified easiest only 55% answered them correctly indicating these problems were deceptively difficult for this sample of sixth graders to solve. Ironically, each of the four test items was ranked as very easy or easy to solve by one or more students. Eleven students ranked item 16 (patio problem) as very easy or easy with 8 of the 11 (72.7%) answering the item correctly. Nine students ranked item

24 (tax rate) as very easy or easy with 6 of the 9 (66.6%) answering the item correctly. Seven students ranked item 32 (perimeter) as very easy or easy with 3 of the 7 (42.9%) answering the item correctly. Thirteen students chose item 34 (*n*th term) as very easy or easy with 5 of the 13 (38.5%) answering the item correctly.

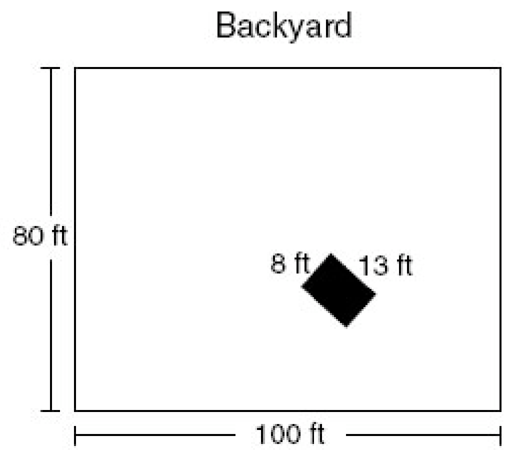
Table III.1.
Percentage Correct Comparison Between the Population and the Sample

		Sample	Population			
Test	State Mathematics	%	Answer Choices			
Item #	Objective	Correct	A/F	B/G	C/H	D/J
16	4 – Concepts & Uses of Measurement	55	15	47*	31	7
24	6 – Mathematical Processes & Tools	45	50*	41	5	4
32	4 – Concepts & Uses of Measurement	60	13	24	22	41*
34	2 – Patterns, Relationships, Algebraic Thinking	35	32	48*	16	3

Note. *Correct answer choice; Population N = 289,449; Sample n = 20

Table III.1 contains a comparison of the population (N=289,449) percentage rate answering correctly to the study sample. With the exception of items 32 and 34 the other two items were comparable. Students in the sample outperformed the population by 19% on item 32 but the sample underperformed by 13% on item 34.

16 A family put a rectangular patio in their backyard and planted grass in the rest of the yard. The rectangular backyard is 100 feet by 80 feet, and the patio is 13 feet by 8 feet. What is the area of the backyard that is planted with grass?



F 402 sq ft

G 7,896 sq ft

H 8,000 sq ft

J 8,104 sq ft

Released Test Item – Grade 6, 2004 TAKS Test

Figure III.2. *Test Item 16.* (Reprinted with permission from the Texas Education Agency. Copyright © 2004 by the Texas Education Agency. All rights reserved.)

Item 16

Test item 16 used a pictorial representation of a yard and patio to present the problem (see Figure III.2). Students were to find the area of the grass of the backyard, which meant students needed to understand that in order to solve the problem they must subtract the area of the patio from the area of the yard. Six male students (four Black and two Hispanic) and five females (three Black and two White), labeled this item as very easy or easy and 72.9% of these students correctly answered the item. Only one Back male, one Hispanic male, and one Black female incorrectly answered the item. Students who answered the item correctly understood the concept of area, performed the computations correctly, and understood that the patio needed to be subtracted from the area of the backyard. Generally students who did not correctly solve the problem started finding the perimeter, did not understand the term patio, or did not subtract the two areas. The one Hispanic female who incorrectly answered the item indicated that patio is the whole yard and that patio is a synonym for yard. In Spanish the word patio can mean yard.

Student justifications. The analysis of student responses to test item 16 indicates that some students missed important clues presented visually or verbally. Some students launched into a description of a Mechanistic procedural justification scheme by manipulating the numbers presented on the large rectangle and ignored the term area, which would appear in both Mechanistic algorithm and Language terminology justifications. Misunderstanding and mispronouncing the term patio as “pay-she-o” led to misapplication of the Language semantics justification, and ignoring or overlooking the small rectangle representing the patio lead to a Visual microview justification. Such errors would result in computing the perimeter of the shapes or not understanding that the presence of a patio would reduce the area of grass needed for the backyard.

Conrad's (each student was given a pseudonym) response is indicative of the combination of justification schemes that resulted in a correct response.

Interviewer:	So you said number 16 was the easiest one. Why was it the easiest?
Conrad:	We're studying about area right now.
Interviewer:	So how what did you do that gave you the right answer? I see the right answer there. How'd you get it?
Conrad:	I multiplied eighty by one hundred feet and eight by thirteen feet and when I got that solution I just subtracted the two sums.
Interviewer:	Why would you subtract?
Conrad:	Because they only wanted how much area of the was-how much grass it had.
Interviewer:	So why'd you subtract this thing? [points to the diagram]
Conrad:	Because it was covering the grass.
Interviewer:	What is that in there?
Conrad:	That's a patio.
Interviewer:	And does the patio have grass on it?
Conrad:	No.[2]

Conrad began his explanation with an Authoritarian self/teacher justification when he stated this problem is easy because he has been working with area and understands using the multiplication of the dimensions to solve the problem. We labeled his mathematical calculations Mechanistic procedural because he did not use the terms length or width (or units), just the numerical dimensions appearing in the diagram although it could be argued he is using a Mechanistic algorithm justification. His comment in response to why he subtracted the two sums indicated he understood the patio covered the grass and is an example of a Language semantics justification scheme. Conrad's comments implied that he saw all parts of the diagram, including the dimensions and the black rectangle that represented the patio. In doing so, Conrad understood the meaning of each individual part in relationship to solving the problem indicative of a Visual dynamic equilibrium scheme.

Simon ranked item 16 only as easy, but also did not have difficulty selecting the correct answer.

Interviewer: So how did you figure this one out?
Simon: Well first it says the regular back yard-the rectangular back yard is one hundred feet by eighty feet. And then the patio is thirteen by thirteen feet by eight feet. Then it asks, "What is the area of the back yard that is planted with grass." And I know that a patio wouldn't have any grass on it cause you know they use concrete and stuff. They cement over it. So first I multiplied this in order to get the area of the whole back yard. Then I multiplied thirteen times eight in order to get the area of the patio. Then I subtracted that then I got seven thousand eight hundred and ninety-six.

Simon's explanation clearly showed his understanding of the word patio. Simon used Language semantics and terminology, Authority self, and Mechanistic procedural justification schemes in his explanation of his solution. In subsequent analyses of other test items, we found successful students frequently employed similar justification schemes and always in combination with other schemes.

24 Felicia went shopping for clothes. She bought a pair of jeans priced at \$28.00, a sweater priced at \$32.50, and a belt priced at \$18.75. If there was an 8.75% tax on clothing items, which procedure could be used to find the amount of tax Felicia paid?

F Multiply the tax rate by the sum of the prices of the clothing items

G Add the prices of the clothing items to the tax rate

H Add the prices of the clothing items

J Multiply the tax rate by the price of the most expensive clothing item

Released Test Item – Grade 6, 2004 TAKS Test

Figure III.3. *Test item 24.*(Reprinted with permission from the Texas Education Agency. Copyright © 2004 by the Texas Education Agency. All rights reserved.)

Item 24

Item 24 was presented in text (verbally) with no visual component (see Figure III.3). Students were asked to identify the procedure for calculating tax on several items of clothing. Four male students (two Black and two White) and five female students (four Black and one Hispanic) labeled this item as very easy or easy and 66.6% of these students correctly answered the item. One Black male, one White male, and one Black female incorrectly answered the item. Several students expressed a lack of confidence in their answer choice. There was confusion about the meaning of the term "tax" – one verbalized it as "eight dollars and seventy-five tax." Two male students believed they had nothing to work but rather only had to choose the correct strategy. Successful students knew that tax was figured by multiplying as other adults had explained this to them. Students knew that tax was *added* to the amount you paid, but did not always understand how it was determined and selected the answer choice "Add the prices of the clothing items to the tax rate."

Student justifications. The verbal representational format used to present item 24 may have guided students' choice of justifications. Students were asked to choose the "procedure" to calculate tax for a series of items and needed to understand the meaning of tax rate as opposed to the term tax. Reading errors indicated difficulties with Language semantics or terminology justifications and obscured the Mechanistic justification needed to solve the test item.

Craig: "Felicia went shopping for clothes. She bought a pair of jeans priced at twenty-eight dollars, a sweater priced at thirty-two fifty, and a belt priced at eighteen seventy-five. If there was an eight dol, eight seventy-five tax on the clothing items, which procedure should be, could be used to find the amount of tax Felicia paid."

Interviewer: What did you choose?
Craig: G.
Interviewer: Why'd you choose G?
Craig: Because if you . . . if you go through it has there's tax on each one of them . . . and . . . G was "add the price of the clothing items to the tax rate" which meant that - I THINK it meant what you put twenty-eight you have to add that to twenty-eight. Eighteen seventy-five and thirty-two fifty. . . cause. . .
Interviewer: I'm not following you. Let's try that again. You picked G, right?
Craig: Uh-huh.
Interviewer: And it says, "Add the prices of the clothing items to the tax rate."
Craig: Uh-huh.
Interviewer: Right. Explain that to me again then.
Craig: Well if like. . . eight seventy-five was that's the tax requires //
Interviewer: I have a question for you. Is eight point seventy-five percent the tax or is it the tax rate?
Craig: Oh:h... (4 sec) Oh, then that means I... (3 sec) Oo:h, whoops. (2 sec) Then that means I:I-uh (16 sec) So. Oh. (13 sec) I got lost there then.

Although Craig self-identified this item as his "very easy" problem, he is clearly struggling with the Language elements of the problem. Craig, and other students, had a misconception of the term *tax rate*. Students at this level have worked with percentages and rates, but are more familiar with tax as a specific amount, not the "tax rate" used to calculate the tax. Several students indicated familiarity with the concept of the tax through prior personal experience and incorporated an Authority justification in explaining their solution. The familiarity with tax, lack of understanding of the term "tax rate", and an overconfident use of the Authority self-justification scheme resulted in incorrect solutions.

Sherri: I chose G "Add the price of the clothing items to the tax rate."
Interviewer: Explain to me how that works.
Sherri: Well all I did was I did it mentally. I thought that, I mean I didn't thought, but I knew that you were suppose to add the jeans together

cause if there was a problem like that, but them when the tax price came up I read the um answer choices and then I thought that G would be a good one to add the tax. Like if you go to a store to buy a candy and the tax is ten cents, then you'll add that to the amount of money you're suppose to pay.

Interviewer: So if the tax is ten cents then you add that to the amount you have to pay.

Sherri: Cause usually when I go to the store I pay a lot of tax for my candy and stuff.

Sherri incorrectly answered this item because she relied on her Authoritarian self-justification to confuse the issue of what procedure is used to calculate tax on an item. Her personal experience of buying an item has focused her on the term “sum” as she combines the Language terminology and Mechanistic procedural schemes in determining her solution. Sherri had a conceptual understanding of tax as being added to the cost of the items, but did not know how to determine the amount of the tax. In her personal experiences a cash register automatically calculated the amount of tax.

David correctly answered item 24. He also used an Authoritarian justification scheme, crediting himself and his mother, who provided him with additional critical information.

Interviewer: So number 24 you said was VERY easy. Tell me about that cause some other kids didn't think that was very easy.

David: Because I really didn't have to do like a lot of MATH. All I had to do was figure out which one I-what I had to do.

Interviewer: And so how did you know what to do?

David: Because, when I usually go shopping, or something, they would add all of it up and then on the receipt it would have tax and the total at different //

Interviewer: So you knew that there was a way that they got from the amount of the clothes and then you saw the tax right there and then you saw the total and it was different from the amount of the clothes. So what did they DO?

David: At first I didn't know so I asked my mom and she said whatever the tax was there they timesed it by that and then they added it on.

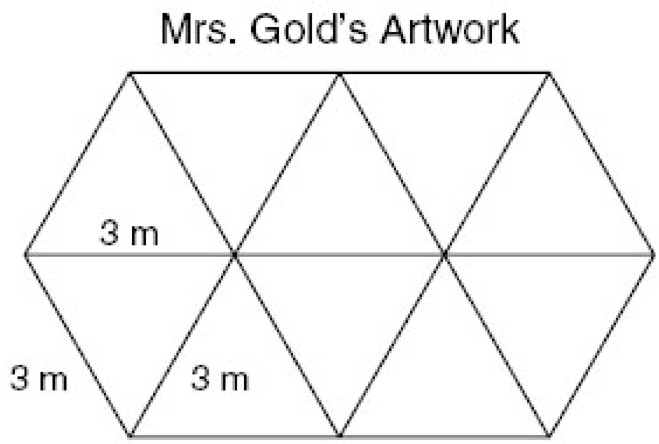
David understood that multiplication was important in determining the tax because he had a previous conversation with his mother concerning why there is a subtotal, amount of tax, then a total amount on a sales receipt. This was an important item of information for him to have and resulted in him correctly answering the question.

Another student, Sam, also incorporated Language semantics and Mechanistic procedure justifications to support using multiplication to compute the tax. Sam's teacher was mentioned in his Authoritarian justification, as she was the individual who provided him with an understanding of how to use a tax rate to "get the right tax." Both students chose this as very easy because they only had to *think* about, and did not have to actually calculate, the answer.

Sam: Cause you don't have to do that much then they thinking with this one. You just have to choose multiple choice and learned this when I was in class.
Interviewer: How'd you learn it? Cause you got it right so what'd ya do?
Sam: My teacher showed me how to do tax and she showed the different percents and stuff. And she showed us how to multiply to get the right tax and stuff like that.
Interviewer: So you knew you had to multiply.

These three interview excerpts indicate that Authoritarian justification can be both beneficial and harmful (see Sowder & Harel, 1998).

32 Mrs. Gold designed a piece of art by outlining equilateral triangles with wire.



How much wire did Mrs. Gold use to complete her piece of art?

F 9 m

G 33 m

H 90 m

J 57 m

Released Test Item – Grade 6, 2004 TAKS Test

Figure III.4. *Test Item 32.* (Reprinted with permission from the Texas Education Agency. Copyright © 2004 by the Texas Education Agency. All rights reserved.)

Item 32

Question 32 used a pictorial representational and involved finding the amount of wire to outline a group of equilateral triangles (see Figure III.4). Only seven students labeled this item as very easy or easy. Two male students (one Black and one Asian) and five females (three Black, one White, and one Hispanic) labeled this item as very easy or easy but 42.9% of these students correctly answered the item. The two males and one Black female were the only ones to successfully answer this item. Successful students used a strategy of writing the number 3 on *each side or line* and counting the number of threes. Student who incorrectly solved the problem labeled the *inside* of every triangle with a three or did not consider that some of the sides of the triangles were shared with another triangle.

Assessment item 32 was presented as a diagram. The phrasing of the question and the visual presentation guided students to use Language and Visual justification schemes. A common misconception was not recognizing that some of the side segments of the equilateral triangles were shared, which resulted in students counting segments more than once. The difficulty of keeping track of the number of line segments in the diagram, often appeared in explanations incorporating a Mechanistic scheme.

Student justifications. Anne is a student who used a Visual microview justification scheme, which meant she focused on the perimeter of one triangle in making her calculations.

> Interviewer: What were we suppose to do there?
> Anne: What I did was I added up the one triangle, which was three plus three, plus three which is the area of the one triangle.
> Interviewer: Area?
> Anne: What the triangle was and then I timesed nine which is what I got from the three numbers times ten because there's ten triangles and I got ninety.

In addition, Anne's use of the term *area* may indicate she misunderstood the concept in the problem, meaning that her Language terminology justification scheme was also incorrectly focused. Anne's Mechanistic procedural justification scheme could have produced the correct answer, but was influenced by her Visual microview justification scheme, which evolved into a Visual macroview as she saw all 10 triangles, however she did not see the overlapping lines on the drawing.

Tanya concentrated on the markings of the three meters, which led her to start her explanation with a Visual microview justification. Her use of Language terminology helped her understand outline as "around," which in turn expanded her Visual justification to a macroview while she marked each segment in the diagram.

Interviewer: Why did you think that one was the easiest?
Tanya: Because all I have to do on this problem is go like right here - OK they already gave me three . . . the three me //
Interviewer: meters //
Tanya: Yeah, meters, so all I have to do around this whole entire thing is write three meters on each side. So that would be easy and then I just have to add up all of the threes and then I get my answer.
Interviewer: OK, and what did you get?
Tanya: I got [remarks paper by adding little lines and 3s to the drawing, then recounts the line/numbers] I got fifty-seven.

Tanya completed her explanation of solving the problem by using a Mechanistic procedural scheme after writing a three on each line segment indicating that she understood that some were shared among triangles. She added up the number of 3s marked on the line segments and determined the correct answer to be 57 meters. Simple strategies such as marking which lines had already been counted were infrequently used by the students. According to Tanya's explanation, the computations occurred after other decisions, so in this instance her justification was influenced by her Visual microview. Several students struggled with keeping track of the numbers they obtained, even when, as in Tanya's case, they thought it would be easy to do so.

34 What is the rule to find the value of a term in the sequence below?

Sequence

Position, *n*	Value of Term
1	1
2	4
3	7
4	10
5	13
n	?

F $n + 3$

G $3n - 2$

H $3n$

J $n - 2$

Released Test Item – Grade 6, 2004 TAKS Test

Figure III.5. *Test Item 34.* (Reprinted with permission from the Texas Education Agency. Copyright © 2004 by the Texas Education Agency. All rights reserved.)

Item 34

This item was presented as a table of numbers with column headings (see Figure III.5). Labels such as Sequence, Position, *n,* and Value of the Term were used to identify various parts of the table. This item was the only one to use symbolic representations.

Thirteen students chose item 34 as very easy or easy. Ten male students (five Black, three White, one Hispanic, and one Asian) and three female students, (two Black and one White) labeled the *n*th term problem as very easy or easy. Only three Black males, one Asian male, and one Black female, or 38.5% of the students correctly answered this item. Students who were successful with the problem were able to verbalize why the other answer choices were incorrect and were able to demonstrate how $3n - 2$ worked in the problem. Unsuccessful students worked out a recursive relationship for only the right column, in other words they focused on $3 + 1 = 4$, $3 + 4 = 7$ or a variation of this. These students saw no link across the columns from left to right. This is indicative of a microview in the Visual scheme. These students were also confused as to how to compute the $3n$ portion of the expression.

Student justifications. This assessment item was presented as a chart. Students needed to have an understanding of various mathematical symbols and vocabulary, such as N and "term", in order to successfully solve the problem. More students selected this problem as very easy or easy than any other test item, however it had the lowest percentage of correct answers. We noted that students used an incorrect justification of Visual microview in justifying the answer choice as students often looked at the right and left columns of numbers independently, not keying in on the phrasing of the problem or the chart labels. Language terminology justifications provided examples of students misinterpreting *N* as "number" or a specific answer.

Dominic's response demonstrates that a combination of the Visual microview and Mechanistic procedural justifications could result in an incorrect answer.

Dominic: Well what I was suppose to do on this one is put like N plus three so the position which is N plus I'm suppose to add three to it for the value of the term.
Interviewer: And does that And does that work?
Dominic: Yes sir.

Interviewer: So show me, explain how that works cause I want to see how N plus three equals one.
Dominic: The value of the term is one and then the position equals N. And then number two is well it just goes one to five, but this one plus three equals four and four plus three equals seven and so basically //
Interviewer: So that's how you got N plus three.
Dominic: the position.

Dominic concentrated on the pattern of the numbers in the right column. His concluding statement showed that he did not understand the phrase "for the value of the term" and had interpreted *N* both as *number* and the *position* equaling *N*.

Joey also was confused as to the meaning of *N*. Joey initially justified his solution by using both Language semantics and terminology. He looked at key words and reflected on the meaning of "by the term". He immediately saw a number pattern that he used in a Mechanistic procedural scheme.

Interviewer: What did you do there?
Joey: What I did, I was trying to look, I underlined the key words. That's one of my strategies. And then, I cause when it says "by the term" I went to the final term place. And list what the difference is like, by what I do is four minus one is three. And then so I keep on doing JUST to make sure. And then four, seven minus four equals three, so I knew there was a pattern going on. So I looked, I looked down on my choices and then I knew it wasn't J. Which is nine minus two, cause it was three. And then so, I looked at my choices again. And then I chose F as N plus three.
Interviewer: N plus three. And what represents, what's N in there? In your problem.
Joey: The N is the, like the answer.
Interviewer: So point to the N's, where the N's are on your table.
Joey: Right here.
Interviewer: So is THAT N. Is that the only N in your table?
Joey: And right here.
Interviewer: That's a question mark. So is that an N?
Joey: Where? Here?
Interviewer: So if, then you think five could be an N? What else? Any, Could anything else be an N?

Joey: Umm. Four three two one.
Interviewer: Those could all be N's. So then you think its N, whatever represents N, plus 3 gives you the answer in the other column? Is that what you're telling me?
Joey: [student stares at the problem] I think I just made a mistake.

Joey's initial use of Visual microview was expanded to a Visual macroview due to the continued probing of the interviewer. Joey eventually acknowledged the meaning of the numbers on the right hand side of the chart and that lead him to realize that he misunderstood the problem.

Students who correctly solved item 34 incorporated combinations of Visual dynamic equilibrium, Language terminology, and Mechanistic algorithm justification schemes. They all understood the meaning of 3*n* as "three times six" and methodically went through the answer choices to confirm or reject the various expressions. Sharon's response demonstrates that a combination of justification schemes results in a correct solution.

Sharon: Well, first I looked at F which is N plus three. And since one plus three isn't one I decided that couldn't be the right answer. So I marked that off. Then I looked at H, cause I decided I'm gonna skip over some. I looked at H and it said three times N. So one times I mean three times one does not equal one. It equals three. So I skipped over that. Then I went down to J. And it said N minus two. Well one minus two will be negative one. So I decided that wasn't right and then I skipped back up to G. And I looked. Three times N minus two. Well three time one equals three and then when you subtract two from that equals one. So I just had to do it with number two just to make sure. And I did three times two equals six minus two which equals four. And I did it to three cause I didn't have to go too far. And it worked.
Interviewer: So what would N be after five? You got five, what would be there? What could be there?
Sharon: Six.
Interviewer: Write it in there for me. [student picks up pencil and starts writing] And what would come after thirteen?

Sharon: [student whispers to herself – Eighteen] Sixteen.

Summary of Findings

Students used a variety of justification schemes in solving the four test items. All four first-level categories appeared in student solutions. There was no predominant use of a specific justification scheme among the students interviewed, however, most problems had a group of students who used similar justification schemes. A more exhaustive study of all 20 students' solutions to a particular problem may provide a definitive preference for a particular justification scheme for that problem type. This is hinted at in solutions presented for item 34, the *n*th term problem, which 13 students labeled as easy or very easy. Further analysis is necessary before other conclusions can be made.

Reflecting on Student Responses

Do middle school students favor a particular justification scheme? Student responses seem to indicate no particular justification scheme was favored for correct answers, although many of the examples demonstrate that successful solutions were the product of similar patterns of justification schemes. However, all successful solutions made use of combinations of various schemes.

What is the relationship between the representational format and the justification schema? The type of representation used to present the test item was a factor in focusing student's attention and selection of justification schemes. Problems presented in a Language format required students to understand mathematical terminology. The presentation of only textual material generally limited students to a single scheme.

Items presented in a Visual format revealed student preferences on either a micro or macroview. Students focusing on either a micro or macroview were not successful when compared to students who demonstrated Visual dynamic equilibrium. Problems incorporating multiple representations generally resulted in more incorrect solutions, even though a larger numbers of students ranked such problems as very easy or easy. The visual component of the problem may be why many students regarded a problem as easy to solve. For example, the *n*th term item seemed particularly problematic because successful student solutions generally incorporated multiple justification schemes. Students often used a Visual microview justification scheme when examining the chart in the *n*th term item for patterns, and then oversimplified the pattern by using a Mechanistic procedural justification. They overlooked the importance of the Language terminology in understanding the column headings. Students' Mechanistic algorithm justifications were also important with the *n*th term problem as students interpreted the meaning of the various expressions.

Implications for Mathematics Educators

It is important that mathematics educators use a variety of representations with students and model strategies for understanding various representations, for example, using micro, macro, and dynamic equilibrium justification schemes. Perhaps students have been conditioned to look at diagrams, charts, pictures, and the like from a Visual microview perspective due to the emphasis teachers have placed on key words and phrases. This conditioning may explain why many of the participants relied on the Visual microview justification scheme.

In addition, students must be provided with opportunities to justify their mathematical solution strategies, both formally and informally. Carefully listening to students' explanations permits educators to identify flaws in student justification schemes, as well as to guide students to consider other means for solving problems.

More generally, this iteration of Harel and Sowder (1998) seems to be appropriate and shows the applicability of their Justification Scheme for use with middle grades students. Importantly, the modification from Harel and Sowder's original ritual scheme to mechanistic scheme allows for the examination of the set of procedures apart from what some might consider an affective entanglement involved in the ritual scheme. Therefore, the use of the mechanistic scheme differentiated between two forms of knowledge often examined in constructivist terms (Carpenter et al., 2003; Evens & Houssart, 2004; Greenes, 1995; Hiebert et al., 1982) and more recently in terms of situated cognition (Boaler, 2000; Cobb & Bowers, 1999).

From a teaching perspective, it is important for teachers to understand that modeling justification schemes is as important as modeling the solution strategy itself. However, this should occur after the teacher has ascertained what students think and understand about various problems and the representations in which they are presented. In complex problems, whether visual or verbal, various justification schemes should be taught and incorporated into the modeling process. While researchers are commonly aware that multiple representations facilitate conceptual development during the initial phase of teaching and learning, these same multiple representations may overly complicate problem solving when various justification schemes are not attenuated during the initial teaching processes. This study lends support for the stratification and fluid application of the original Justification Scheme (Harel & Sowder, 1998) and provides foundational work for expanding the usefulness of Justification Scheme Theory across mathematics content and within K-12 educational settings.

Endnotes

[1]The following symbols were used when transcribing the videotapes:

. . . short silences or pauses
(5 sec) indicates silence of a specific number of seconds
// overlapping speech or interruption
[gives some idea of what is happening on the tape]
ALL CAPS means that word or phrase was emphasized by the speaker
"quoted material" student was reading part of the problem

[2]This selection, and all subsequent interview transcriptions appearing in this section, was edited to remove nonessential utterances such as "umm," "uh," and other verbalizations that did not substantially address the mathematical content or purpose of the interview. We understand that traditional discourse analysis would include all elements of the interview as important, but our focus is the students' justification scheme.

CHAPTER IV

ARE MIDDLE GRADERS INFLUENCED BY REPRESENTATIONAL TRIGGERS IN NTH TERM TASKS?

Synopsis

This study specifically looked at sixth grade students' understandings of the algebraic concepts found in *n*th term problems in which relationships between numbers are explored. Cognitive interviews were conducted with 20 sixth grade middle school students. Students verbalized their solutions to challenging math problems. These responses were subsequently analyzed for evidence of the students' justification schemes (Matteson et al., 2007) and for algebraic conceptual understandings. The mathematical representation(s) students focused on when solving the problem appeared to "trigger" the students' justification schemes. Students' initial representational focus and understandings of the interconnectedness of the mathematical representations were found to be critical factors in arriving at a correct solution. Implications for instructional settings are discussed.

Importance of Study

Algebraic representations have been noted as particularly problematic because students frequently misunderstand graphical or symbolic representations of algebraic concepts (Koedinger & Nathan, 2004; Meira, 1995; Yerushalmy, 1997). A student's ability to correctly interpret mathematical representations is one of several critical skills involved in mathematical problem solving. Cognitive interviews of students have the potential to provide important information regarding students understandings of representations and the subsequent role such representations play in selecting solution strategies and determining the correctness of solutions.

Importance of Algebra and Algebraic Thinking for Learning

Mathematics educators have called algebra the "gatekeeper" to higher mathematics and numerous professions (e.g. Checkly, 2001; Edwards, 2000; Usiskin, 2004). The National Council of Teachers of Mathematics' (NCTM) noted in the *Principles and Standards for School Mathematics* "algebraic competence is important in adult life, both on the job and as preparation for postsecondary education" (NCTM, 2000, p. 37). In the past decade algebra instruction has undergone intensive scrutiny. Reforms have been suggested that would promote the development of algebraic thinking skills in early grade levels. The reform suggestions have included a) integrating the learning of algebra with the learning of other subject matter, b) encouraging different forms of algebraic thinking, c) building on students' natural linguistic and cognitive abilities, and d) engaging students in active learning (Kaput, 1998). While such propositions were initially met with skepticism, research focused on algebraic thinking of young children has found that, with proper instruction, young children were capable of understanding numerous algebraic concepts including: a) generalizing additive number properties (Carpenter & Franke, 2001; Vance, 1998; Yakel, 1997), b) thinking functionally (Blanton & Kaput, 2004; Schliemann et al., 2003), c) using algebraic expressions as rules for input-output situations (Goodrow & Schliemann, 2003; Schliemann et al., 2003), and d) assigning meaning to the variables used in algebraic expressions (Schliemann, Carraher, Brizuela, & Jones, 1998). It has also been demonstrated that as elementary students gained experience with algebraic thinking skills, they developed algebraic reasoning skills through which they justify fundamental mathematical properties and operations (Carpenter & Levi, 2000; Carpenter et al., 2003).

Transitioning to Algebraic Thinking from Arithmetic

Some have proposed that algebra gives meaning to arithmetic and students have struggled with learning to represent and manipulate variables only because "algebra enters the mathematics curriculum too late and at odds with students' knowledge and intuitions about arithmetic" (Carraher, Schliemann, & Brizuela, 2001, p. 131). Elementary students have demonstrated foundational algebraic thinking skills, such as making generalizations and conjectures, when arithmetic concepts were explicitly linked with basic algebra themes through real-life occurrences (Chappel, 1997). Research focused on middle grade students has concentrated on developing students' understanding of the "big ideas" of algebra in a logical fashion. Such ideas included algebraic notation, variable and function, properties of numbers, and language development (Edwards, 2000; Pegg & Redden, 1999).

Developing Facility with Representational Models

As students progress through the grade levels, they should become more fluent in using, understanding, and interpreting a variety of representations. Initially, the focus was placed upon the students' algebraic language development, as algebra has been described as a language having "five major aspects: (1) unknowns, (2) formulas, (3) generalized patterns, (4) placeholders, and (5) relationships" (Usiskin, 1999, p. 5). Researchers have proposed that the language of algebra should be introduced as early as kindergarten, because these students would benefit from being exposed to algebraic foundations through patterns, explorations concerning the relational aspect of equality, and problem solving strategies that include both inductive and deductive reasoning skills (Carraher, Schliemann, Brizuela, & Earnest, 2006; Kaput, 1998; Perry & Dockett, 2002; Usiskin, 1999). Waiting until the middle grades to introduce algebraic language has been shown to delay a student's development of the procedural or conceptual skills needed to translate written word problems into meaningful mathematical equations (Capraro & Joffrion, 2006).

Facility with Algebraic Representational Models

Researchers have focused not only on students' understandings of the verbal language present in algebra problems, but also on the graphical and symbolic representations of algebra concepts (e.g. Brenner et al., 1997; Koedinger & Nathan, 2004; Meira, 1995, Yerushalmy, 1997). Students' must become fluent in understanding numerous mathematical representations used in algebra problems (Matteson, 2006b). The study presented here specifically focused on one type of algebra problem presented to middle school students - an *n*th term problem. Previous investigations involving *n*th term problems have noted several common student errors including: misunderstanding patterns in the form of an + b (Stacey, 1989); treating each column as an independent number sequence; concentrating on the recursive relationship of a number pattern; and misinterpreting of the meaning of symbolic notations such as 5a or 7n, because of a lack of understanding of concatenation (Chalouh & Herscovics, 1999). These misunderstandings could also be classified as difficulties in assigning appropriate mathematical meanings to a) *symbolic* representations in the forms of *a*n + *b* and 5*a* or 7*n*, b) *numerical* representations in the form of number sequences and patterns, c) *graphical* representations in the form of vertical or horizontal tables of numbers or functions graphed on coordinate grids, and d) *verbal* representations such as creating verbal rules to describe each pattern. Students must "understand the relationships among tables, graphs, and symbols and to judge the advantages and disadvantages of each way of representing relationships for particular purposes" (NCTM, 2000, p. 37).

The Effects of Representational "Triggers"

This study considered whether the various types of representations appearing in *n*th term problems "trigger" students' interpretations of the problem and thereby influenced the selection of a correct solution to the problem. In order to ascertain if, in fact, the representational format of mathematics problems mediated students' understandings of problems, students' justification schemes and explanations of mathematical thinking when solving an *n*th term problems were examined. The *n*th term problem in this study incorporated several categories of mathematical representations. Therefore, a variety of justification schemes were expected to emerge as students explained their solution strategies depending upon which representation(s) the student focused.

Use of Justification Schemes for Understanding Student Thinking

Important insights as to what students' were thinking were gathered from the students' verbal and written explanations to the *n*th term problem. This study used the expanded justification scheme previously described in a study by Matteson et al. (2007).

The theoretical framework was based on the foundational works of Harel and Sowder (1998) and Sowder and Harel (1998, 2003), which investigated how students explain, justify, and support their answers to mathematical problems. The terminology of *justification* and *proof* is applicable to other mathematical strands and, in this study, will be used to convey students' justification and proof for their answers about algebraic concepts.

The theoretical justification model used in this study contained four first-level schemes: Mechanistic, Authoritarian, Language, and Visual (see Figure IV.1). Each of the first-level schemes contained second-level categories. The Mechanistic level was divided into algorithms and procedures. Algorithm was defined as an easily recognized process that was generalizable at least within the expected knowledge contained at any particular grade and the student demonstrated functionality with it. Procedure was defined as less formal, less generalizable, and more individualized. The Authoritarian category was divided into four second-level categories. The Language category was divided into semantics, those unique differences that indicate understanding of the interrelationships among words and phrases, and terminology, which addresses an understanding of specific mathematical terminology. The Visual category was composed of three categories. Students using a macro Visual justification scheme examined the entire problem and interacted with it as a *whole*, without considering smaller units. A student using a micro Visual justification scheme ignored the whole and concentrated on smaller details. Students using the dynamic equilibrium category were capable of

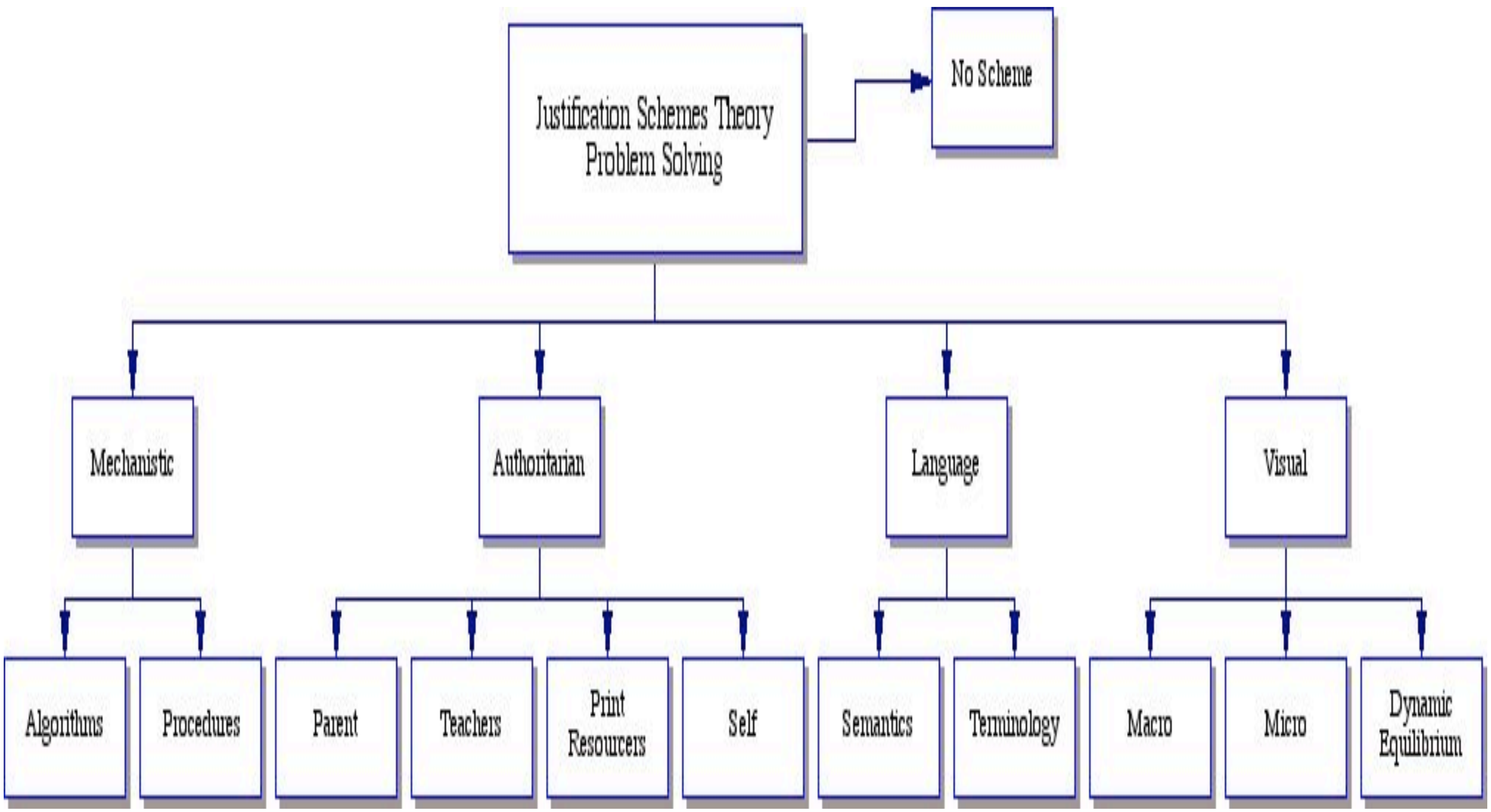


Figure IV.1. *Problem Solving Justification Scheme* balancing between both extremes. A final justification scheme, the No scheme, was used in those infrequent incidences in which the student did not indicate any particular justification model.

Research Question

Results from a study on students' choices of justification schemes used in solving difficult problems (Matteson et al., 2007) appeared to suggest the representational forms of mathematics problems "trigger" students to select and use specific justification schemes. The question that this study addresses is: *What is the relationship between justification schemes and representational models as triggers for student mathematical understanding?*

Method

Participants and Setting

A stratified process based on gender, ethnicity, and at-risk status was used to select 20 students from among 213 sixth-grade middle school students. The students were from a culturally diverse school district in the southwestern United States. Of the 20 students, 11 were male and 9 were female. Nine of the students were termed at-risk and were classified as such for various reasons. Four of these nine students had failed one or more portions of the state assessments in reading, mathematics, writing, or science in previous grade levels, two of them had been retained in an elementary grade level and failed one or more portions of the state assessments, one had been retained, one was designated as homeless, and one student who had been retained and failed one or more portions of the state assessment also received special education services. The 12

Black, 4 White, 3 Hispanic, and 1 Asian student were asked to discuss their mathematics solutions to a test item addressing the *n*th term concept, which was part of a set of four items administered to the students.

Cognitive Interview

The students participated in cognitive interviews (Beatty, 2004). Two mathematics education faculty members from a large public university conducted the interviews. Both interviewers were familiar with the cognitive interview format and had prior interview experience with middle grade students. The students were video and audiotaped explaining their solutions to the *n*th term problem. Listening to students' explanations of solutions requires "more time than simply asking for one-word answers and telling children whether or not they are correct" (Mewborn & Huberty, 1999, p. 245).

This study utilized two paradigms of cognitive interview methods, *think-alouds* and *verbal-probing* (Beatty, 2003; Willis, 1999) in order to facilitate the movement from an *interpretive* to *hermeneutic orientation* in understanding the students' mathematical thinking (Doerr, 2006). The interviewers encouraged students to think-aloud in providing details about how they had solved each of four mathematics test items. Only clarifying questions were asked of the students during this time. This portion of the interview provided the data for determining the justification schemes used by the student. After each student had explained their solution to a problem, the interviewers engaged in verbal-probing to elicit specific relevant information concerning the student's understanding of that problem. During verbal-probing, the interviewers used a mixture of *leading* and *teaching and telling* questions (Moyer & Milewicz, 2002). Questions concerning symbols, terminology, procedures, and the like were asked in order to gain a deeper insight into students' mathematical understandings. The focus of this portion of the interview was to uncover students' strengths and misconceptions by "listening to their ideas and engaging with them in the negotiation of meaning and understanding" (Doerr, 2006, p. 6). During this time the interviewers assisted students by addressing misconceptions that became apparent during the interview. This resulted in the collection of valuable information concerning students' cognitive understandings and interpretations of algebraic symbols and numerical relationships. Thereby, both interviewer and interviewee benefited from an extended discussion of the algebraic concepts imbedded within the test item.

Instruments

Data were collected from three written sources, which included a) a released version of the entire Grade 6 state mathematics assessment, b) four test items taken directly from the entire state mathematics assessment, and c) a brief survey exploring students' thoughts about four test items. The entire state mathematics assessment contained 46 test items, of which 7 of the items were correctly solved by 50% or less of the state's almost 290,000 sixth grade students. The cognitive interview focused on four of these seven items with students asked to provide explanations as to the solution strategies used in solving the problem.

The *n*th term item was one of the four items students discussed during the cognitive interview and was primarily selected for analysis because of the item's diversity in presentation and the accessibility of a large-scale administration for comparison. In addition, the *n*th term item used verbal, numerical, symbolic, and graphical representational elements while other test items used fewer representations. The *n*th term item was presented as a vertical chart entitled *Sequence* with column headings labeled with the terms *Position, n* and *Value of the Term* (see Figure IV.2).

34 What is the rule to find the value of a term in the sequence below?

Sequence

Position, *n*	Value of Term
1	1
2	4
3	7
4	10
5	13
n	?

F $n + 3$

G $3n - 2$

H $3n$

J $n - 2$

Released Test Item – Grade 6, 2004 TAKS Test

Figure IV.2. *Grade 6 Nth Term Test Item.* (Reprinted with permission from the Texas Education Agency. Copyright © 2004 by the Texas Education Agency. All rights reserved.)

Data Sources and Analysis

Analysis of the students' data sources included the written work generated by the students and the verbal responses obtained during individual student interviews. Students were both audio and videotaped during the interview sessions. The student and any work created during the interview were the focus of the videotape. The primary function of the audiotape was to aid in decoding obscured student responses, such as those resulting from students speaking softly or interference from background noises. The audiotape recordings were compared to the transcriptions made from the videotapes[1]. Data were triangulated from several written sources including (1) solutions in the student test booklet, (2) responses to the four questions about the difficulty levels of the items, and (3) work and verbal solutions generated during the cognitive interview.

Analysis of the videotaped student interviews involved several stages. In the first stage, video transcriptions were created and, in order to ensure accuracy, were compared to the audiotapes of the each interview session. The second stage involved segmenting the transcripts to form categories consisting of a) students explanations, b) follow-up interviewer questions, and c) students' responses in which they ranked the difficulty level of each test item from *very easy* to *very hard*. Initially, three researchers read four interviews and independently divided them into the major categories. The section divisions of each investigator were then compared and category standards were revised and refined for continuity and cohesion. These broad categories were then divided into subcategories for further analysis. The remaining 16 transcripts were coded by iteratively pairing researchers in order to control for maturation threat and rater drift.

Each pair's scoring was monitored for reliability with discrepancies discussed and a consensus made before final divisions were decided. A final step, termed *reconciliation,* was used to ensure consistency (Matteson et al., 2007). During this process both coders discussed the choice of categorizations and justified or explained their reasons, with discussion continuing until a consensus agreement was reached.

The final stage of the interview analysis involved rereading specific portions of the interviews in which students discussed their solutions to the *n*th term problem. Two investigators independently read and marked the transcripts for justification schemes. The investigators subsequently met and shared their observations and reconciled any discrepancies. In most instances the discrepancies occurred because students extended their responses, which resulted in additional, and frequently simultaneous, use of justification schemes. The investigators also noted that students appeared to initially use a justification scheme based upon the visual appearance of the test item, in this case a table, and ignored salient features such as column headings. A related study (Matteson et al., 2007) noted that student responses inferred the use of additional justification schemes, which also occurred in the responses to the *n*th test item. Data obtained by extrapolating results based upon inferences have been reported with an accompanying rationale for including such data. In order to validate interpretation, any such inferences have been based on the triangulation of various data sources - including a contextual analysis of the discussions before and after any excerpts presented in this manuscript, work appearing in the test booklet, student work created just prior to the cognitive interview, and work created during the cognitive interview.

Specific concepts that aligned to misunderstandings were marked as a common misconception(s). These misconceptions provided structure for discussion of the results and include, but were not limited to, finding and describing patterns, understandings concerning the relational aspect of patterns, and the meaning of the variable *n* and 3*n.*

Results and Discussion

This study investigated the following: *What is the relationship between justification schemes and representational models as triggers for student mathematical understanding?* The student interviews and written work provided insights as to the students' justification schemes and which mathematical representation(s) were focused upon when solving the problem. Students' conceptual understandings of various algebraic symbols and terminology were also identified during the transcription analysis.

The *n*th term problem (see Figure IV.2) was classified by 13 of the 20 students as very easy or easy. However, only 5 of the 13 students (38.5%) correctly answered this test item. Of the entire group of 20 students who were interviewed, only 7 of the 20 (35%) correctly answered the problem. The seven students who correctly solved this item included three Black males, one Asian male, two Black females, and one White female. One student, a Hispanic female, was not able to determine a solution.

Analysis of students' solutions revealed that students used a variety of justifications schemes when solving the *n*th term problem. Each of the categories of the justification schemes was mentioned, although students' clearly relied upon specific categories. It was noted that upon comparing student written work with the verbalized solutions, students did not always completely verbalize the justifications schemes they used when solving the test item. In those cases, inferences were made and verified within the context of additional data, as previously outlined.

Student justifications. All students appeared to simultaneously use Mechanistic and Visual justifications schemes when describing the pattern they used to determine the solution or demonstrating how to interpret the algebraic expressions that appeared as answer choices. However, there were two distinct approaches in using the categories of Mechanistic and Visual justification schemes and surprisingly this also delineated whether a student successfully answered the question or made an incorrect answer choice.

In regards to the 13 students who incorrectly solved the problem, seven of those students used a Visual micro view justification scheme in conjunction with a Mechanistic procedural justification. Five of the students who answered incorrectly used a Visual micro or macro view and combinations of Mechanistic procedural influenced by a Language justification. The one remaining student did not offer a solution and the response was coded as having No scheme. The twelve students who arrived at an incorrect solution focused initially on the numerical representation in the test item. In other words, these students were visually drawn to one of two number patterns - five used a recursive pattern and seven used an additive pattern. The recursive pattern was found in the right-hand column of numbers, such as 3 + 1 = 4, 3 + 4 = 7. The additive pattern of +0, +2, +4, + 6 emerged as students attempted to describe a relationship between the columns of numbers on the left and right hand sides. The use of these patterns was indicative of either a Visual micro or macro view that appeared to be triggered by the numerical representation of the test item. This group of students did not justify why the other solutions were incorrect, however some of the students did verbalize that they were not sure that their solution was correct or indicated they did not understanding the meaning of the various symbolic representations used in the *n*th term item. It was also noted that none of the students who incorrectly answered the *n*th term item used an Authoritarian justification scheme.

In contrast, the seven successful students relied upon a Visual dynamic equilibrium justification scheme to determine the relationship. The students did not visually concentrate on the number pattern in the vertical chart. Instead these students used the algebraic symbols appearing in the symbolic representations of the answer choices *and* number patterns of the numerical representation the in determining the solution. For this group, interpreting multiple representations triggered their use of the Visual dynamic equilibrium justification scheme. These students moved concurrently between several justification schemes and their explanations demonstrated flexibility in using aspects of the several mathematical representations, such as numerical, symbolic, and verbal. In addition, this group verbalized a Mechanistic justification scheme for explaining why other answer choices were incorrect, although not all answer choices were considered. Instead, students stopped when they encountered a solution they could not rule out. All seven students were able to verbally demonstrate how $3n - 2$ worked in the problem for other numbers and their success was facilitated by their interpretation of the meaning of symbolic representations such as $3n$. Four of the students correctly answering the *n*th term problem also used an Authoritarian justification scheme in that they acknowledged the role of a teacher or mathematics materials in acquiring an understanding of the meaning of 3*n*.

In summary, the representations used in the *n*th term item appeared to trigger the students' choices of particular justification schemes. For the students who incorrectly answered the item, this resulted in concentrating on specific representational aspects to the exclusion of other critical elements or over emphasizing the importance of a particular representation. In order to correctly solve *the n*th term problem the student must have acquired an understanding of numerous mathematical and algebraic concepts including: the meaning of "term", the variable *N,* and the notation of 3*n*; the ability to find and describe a relational mathematical pattern; and knowledge that a rule must apply to all situations.

Common Understandings

Ability to Find and Describe a Pattern

Finding and describing patterns was the primary focus for students who unsuccessfully solved the problem. These students responded to the trigger of the numerical representation of the number patterns appearing in the vertical chart. The students verbalized a simplistic additive or recursive pattern, and sought a solution based on this verbalization, which explains why a number of them chose the first answer. However, after verbalizing their solutions, several students realized their answer choice was incorrect. This provides evidence that a students' self-evaluation of the initial representational trigger is an important step in determining a correct solution. Jenny's (all students were given a pseudonym) response is typical of students who changed answers.

> Jenny: Well I did that's just You don't add anything. That you add two. That you add four. And then that you add six. So you would just um you keep going by twos.
> Interviewer: OK. You do add two each time, but is that one of the rules down here?
> Jenny: Uhuh. (Student shakes head back and forth.)[2]

Understanding the Meaning of the Variable N and the Expression 3n

Students who correctly answered the *n*th term problem understood the meaning of the symbolic representation of *N* and 3*n*. Six of the seven students who successfully answered the test item started their explanations by focusing on the four solutions that were given, which was their representational trigger. The seventh student said, "I usually don't look at my answer choices, but I had to look at the answer choices cause I was kind of stuck." As the numbers in the column titled "Position, *n*" were substituted in the expression 3*n*, the seven students demonstrated their understanding of the concept of concatenation, or the linking of the number to the variable by the operation of multiplication.

> Interviewer: Where did you learn that three next to a letter meant to multiply?
> Sharon: Um in math. It doesn't necessarily have to have a multiplication sign or a dot or something that separates it. It just has to have a number and either two numbers or a number and a letter or two letters. You've just gotta know -what the letter means. You have to be specific in that.

Common Misunderstandings

Some students had not acquired and understanding of the interconnectedness of mathematical representations. These students became fixated on certain representations and required assistance in interpreting the role other representations played in the problem. In the following excerpt, Joey initially focused on the verbal representation of the terminology in the problem and then turned his attention to finding the numerical representation of a number pattern. Joey appeared to understand the terminology because he went to the "final term place," but as he focused on the numbers in the chart, he struggled with the meaning of the variable in the problem. By indicating the question mark is *n,* he may have assumed a multiple interpretation of the meaning of *n* as both a number in the problem as well as the missing number in the right-hand column.

Interviewer: What did you do there?
Joey: What I did, I was trying to look, I underlined the key words. That's one of my strategies. And then, I cause when it says "by the term" I went to the final term place. And list what the difference is like, by what I do is four minus one is three. And then so I keep on doing JUST to make sure. And then four, seven minus four equals three, so I knew there was a pattern going on. So I looked down on my choices and then I knew it wasn't J. Which is nine minus two, cause it was three. And then so, I looked at my choices again. And then I chose F as N plus three.
Interviewer: N plus three. And what represents, what's N in there? In your problem.
Joey: The N is like the answer.
Interviewer: So point to the N's, where the N's are on your table.
Joey: Right here.
Interviewer: So is THAT n. Is that the only N in your table?
Joey: And right here.
Interviewer: That's a question mark. So is that an N?
Joey: Where? Here?
Interviewer: So if, then you think five could be an N? What else? Could anything else be an N?
Joey: Umm. Four three two one.
Interviewer: OK. Those could all be N's. So then you think its N, whatever represents N, plus 3 gives you the answer in the other column? Is that what you're telling me?

Students frequently regarded the variable *N* as a replacement for the *word* "number," such as found in formulas such as $A = lw$, where "A" stands for Area, "l" for length, and "w" for width. *N*th term problems appear to compound this misconception.

Implications

Students used a variety of justification schemes to determine the solution to the *n*th term problem and considered various components of mathematical problems. They were able to derive information form various mathematical representations. There appears to be a symbiotic relationship between being able to gain critical information from mathematical representations and the justification schemes students' use. Whichever representation the student focused on became a critical factor to successfully arriving at a correct solution. The data in this study appear to indicate that the concept of a representational trigger may be a significant mediating factor to student problem solving success, but additional studies are warranted before generalizations can be made.

The conceptual misunderstandings that were revealed during student interviews indicate the importance of providing numerous opportunities to interact with and evaluate multiple representational formats. Some students in this study had acquired a deeper understanding of how the various representations were linked to the conceptual understandings necessary to solve the problem. Students who did not understand the representations used in the problem, such as the symbolic representations of n and $3n$, became sidetracked by simpler representations, such as number patterns. They were focused on a less important representational aspect of the problem and chose incorrect solutions based on their misconceptions. Major misconceptions develop when formulas and symbols are introduced in elementary grades with little explanation about the meaning of included variables. Assisting students in understanding the symbolic representations became critical teaching points for the interviewers, as evidenced by the interviewers having to instruct the students in this study concerning the meanings of the representations used in the *n*th term problem.

The NCTM standards emphasize the teaching of multiple representations, (NCTM, 2000), but having students become familiar with numerous verbal, numerical, symbolic, and graphical representations is not enough. The results in this study seem to indicate that an overemphasis on the importance of terminology or symbols sometimes results in those aspects becoming a representational trigger for students, to the exclusion of other critical elements. Teachers must guide the students to understand the interconnectedness of these representational forms by creating deliberate connections to mathematical concepts. In addition, the teacher should instruct students to carefully examine all representational components for a given problem before deciding on a solution strategy.

Finally, teachers could incorporate both the “think-aloud” and “verbal-probing” interviewing methods in classroom discussions as students justify their mathematical reasoning. These interview techniques have the potential of eliciting important information that the teacher could use to inform subsequent lessons. It would be instructionally beneficial for teachers to gain critical insights into what representational aspects influence students’ mathematical thinking. Although conducting cognitive interviews are time consuming, the potential benefits to both students and teachers.

Endnotes

[1]Symbols appearing in the transcription

. . . silences or pauses of a brief duration, usually less than 3 seconds
(7 sec) silence for the indicated number of seconds
// interruption or overlapping speech
[insights as to what can be seen on the videotape]
ALL CAPS indicate word(s) emphasized by tone or volume

[2]Traditional discourse analysis includes all elements of the spoken word. We have chose to remove nonessential utterances in order to better address the purpose of the interview and to create a clearer understanding of the students' mathematical explanations.

CHAPTER V

SUMMARY AND CONCLUSIONS

In this dissertation I presented three integrated manuscripts that included: 1) a methodological discussion of how ethic of care behaviors influence interactions with students in cognitive interview settings guided by theories of Gilligan (1982) and Noddings (1984), 2) an investigation of the reasoning and justification schemes middle school students use in mathematical problem solving, adapted from the works of Sowder and Harel (1998, 2003), and 3) an investigation of the representational "triggers" that influence students' justification schemes when solving *n*th term problems. The manuscripts emerged during data analysis of cognitive interviews of sixth grade students. The first manuscript addresses a methodological issue, the second outlines a "new" theoretical framework, while the third postulates the theory of a representational "trigger" that influences how students arrive at a solution.

The findings from this dissertation further the mathematics teacher educators' and practitioners' understanding of students' mathematical thought processes by exploring the relationship of justification schemes and mathematical representations. In regards to justifications of mathematical solutions, teacher practitioners at various grade levels could readily adopt the revised justification framework presented in Chapter III and apply it to a variety of problem solving situations across mathematical strands. As students develop a deep understanding of the interconnectedness of various strands, this flexibility of the revised justification framework allows mathematics practitioners to better monitor students' development of mathematical reasoning skills.

Mathematics education researchers have examined the difficulty learners experience in translating and integrating information from different formats of representations (Ainsworth et al., 2002). However, other than studies conducted about the Graduate Record Examinations® (GRE) (Bennett et al., 1999; Katz, Lipps, &Trafton, 2002), investigations concerning students' interpretations of mathematical representations appearing on standardized assessments are limited. Testing constraints and limited classroom time hamper educators' efforts in obtaining important insights as to what students think and understand about various mathematical representations. Mathematics teacher practitioners understand that fluency in interpreting and using mathematical representations is a critical component to success in problems solving, but the influence of mathematical representations on students' solutions remains to be studied. Data analysis for the studies presented in Chapters III and IV revealed students' explanations of solutions including references to specific details that appeared in the mathematical representation of the problem, such as particular words or phrases, specific visual aspects of the problem, or specific symbols. Understanding which aspects students focus on in solving mathematical problems could inform pedagogical and curricular decisions.

Methodological Issues and Implications

This trilogy of articles resulted from an initial study of algebraic representations on standardized assessments undertaken during a graduate level mathematics education course. The data used for the *Reading Psychology* article (Matteson, 2006b), was extant data and had limitations. Therefore, a methodology was developed, which involved collecting data through cognitive interviews, so that richer insights into students' mathematical thinking could be obtained (Crespo, 2000; Crespo & Nicol, 2003). The specific cognitive interview techniques used were *think-alouds* and *verbal-probing,* (Beatty, 2003; Willis, 1999). The think-aloud format was used to ascertain which justification schemes were used by the student, while the verbal-probing format provided information concerning the students' representational understanding. This combination proved to be the most effective method of interviewing students.

Several qualitative educational research issues are discussed in Chapter II. These issues developed from noting differences between the male and female interviewer's interactions with the students during cognitive interviews. Numerous studies have investigated ethics of care behaviors of educators from various perspectives (Alder, 2002; Alder & Moulton, 1998; Bosworth, 1995; Collinson et al., 1998; Hayes et al., 1994; Noddings, 1992; Owens & Ennis, 2005; Rice, 2001; Teven & Hanson, 2004; Vogt, 2002). The Chapter II study used positive and negative ethics of care behaviors to categorize the differences in interactions of the interviewers and students. Many of what were termed as "positive" ethic of care behaviors as noted in Hayes et al., for example listening and helping with academic work, was applicable to the interviewers in the study. Negative ethic of care behaviors, such as avoiding assisting, have not been intentionally researched. Furthermore, such behaviors have not been addressed with educational researchers who use cognitive interviews as a means of data collection. Recent discussions have addressed care as an ethic within a research community, such as with work-based practitioner researchers (Costley & Gibbs, 2006), who asked "How should the researcher behave when the findings of the research might affect or even injure those to whom the research has a special professional, functional or emotional bond?" (p. 89). However, the implications of ethics of care behaviors have not been specifically examined for educational researchers, though the findings of this study indicate they do exist. Some of the ethic of care behaviors noted by Hayes et al., such as classroom management, would only be apparent if the interviewer spent an extended amount of time with the students, which would not be the case in a cognitive interview setting.

The influence ethic of care behaviors have on educational researchers and the research findings they report is also unknown and not well discussed in the literature and absent from mathematics education literature altogether. The findings presented in Chapter II points to a potential threat to study validity for mathematics educational researchers, indeed for the entire educational research community. Results for studies that utilize interviews as a means of data collection may be suspect because of being based on flawed interpretations when multiple interviewers are part of the study design. This is not to say that simply having multiple interviewers indicates fatal flaws, however, without careful examination of the ethic of care and possibly other considerations any results should be viewed with suspicion. Cognitive interviews are frequently used by those engaged in qualitative research because the interview format provides a setting in which deep and valuable data can be gathered. Cognitive interviews are not haphazard means of collecting data. The interview process must be carefully considered. Each of the cognitive interview formats has advantages and disadvantages

(Willis, 1999). Mathematics educational researchers frequently use interviews, but provide little detail as to the format of the interviews, much less the rationale for the use of a particular format. For example, ethic of care can have an effect on the depth, quantity, and quality of the data obtained and thereby influence the interpretation.

Most attention in qualitative studies is given to assuring the validity of the analysis of results, but those engaged in educational research are cautioned, “One of the issues around validity is the conflation between method and interpretation” (Guba & Lincoln, 2005, p. 205). Educational researchers must remain mindful that the data collection method must also address validity. This would include providing details concerning the cognitive interview format. Additionally, the methodological implications for studies using multiple interviewers have not been widely acknowledged or discussed within the applied research community. The validity of the data collected by multiple interviewers must be considered.

The reliability of results, another major concern of qualitative researchers in every discipline, could be greatly enhanced if the reader was aware of the qualifications of those conducting cognitive interviews. It is, therefore, suggested that methodologies include information regarding the background of those conducting the cognitive interviews. Such information would also lend to the credibility of the data collected as well as the resulting findings.

Evolving Theories

The articles in Chapters III and IV categorize students’ mathematical solutions using a revised student justification scheme based on the work of Harel and Sowder

(1998). The results presented in Chapters III and IV show students use a variety of justification scheme combinations when solving test items and that the schemes are not mutually exclusive in that students move easily among the categories. Mathematics students are often under the impression that specific problems require specific solution strategies or means of justification, which is an incorrect assumption according the results presented in Chapters III and IV. The revised justification framework adds credence to the idea that there are multiple ways of arriving at a particular solution even within specific mathematical strands. The concept of multiple solution strategies to mathematical problems is frequently at odds with curricular materials, pedagogical principles, and teacher beliefs.

The terminology of the revised justification scheme framework's first and second-level justification schemes seems easily understood. The categories of Mechanistic, Authoritarian, Language, and Visual, and the corresponding subcategories, are well-defined and appropriate for use with elementary and middle grades students, in addition to older students. Harel stated the changes made to the framework were "appropriate and wise" (Harel, personal communication). The inclusion of the *Authoritarian self* is different from Harel and Sowder's framework in which all Authoritarian justifications were limited to external sources (Sowder. personal communication). The scheme of Authoritarian self was added because several students cited themselves as an authority. These students reported having first-hand experience with the specific concept of tax.

The idea of a representational "trigger" emerged during the analysis of the students' justification schemes in Chapter III. The results of that study reveal that *groups* of students use similar justification schemes and these schemes appear to focus on a specific representational aspect of the problem. When analyzing students' explanations for each test item, it was noted that the students used several specific justification schemes more frequently than others. This finding was not anticipated. The idea of representations acting as an influential factor to "trigger" students' justifications was intriguing and was the major focus of the study appearing in Chapter IV. In this study the phrase *representational trigger* is used to denote what part of the problem receives the initial attention of the student. Exploring the concept of how or if a representation acted as a catalyst for how students solved problems seemed noteworthy. If evidence of a "trigger" were confirmed, this would be the next iteration on the role mathematical representational models play in mathematical success. The results of the study conducted in Chapter IV found that students' justification schemes are chosen based upon the representational trigger yielding the next iteration in the theoretical framework.

The manuscripts presented in Chapters III and IV build on the justification schemes of Harel and Sowder. Harel and Sowder's schemes were based on cognitive stages of students' development, and were initially focused on the understanding and application of geometric proofs. Harel and Sowder's external, empirical, and analytical justification schemes represented a "cognitive state, an intellectual ability, in student's mathematical development (Harel & Sowder, 1998, p. 244). The revised justification scheme improves upon Harel and Sowder's original work in that the revised framework is not developmentally based. Furthermore, the revised scheme is both flexible and applicable to a variety of mathematical problems, regardless of mathematical strand or representation, as demonstrated by the students' discussion of their solutions to the four test items.

Results from the studies presented in Chapters III and IV reveal the potential of the revised justification scheme framework as an additional means of describing students' problem solving abilities. By carefully analyzing students' responses, mathematics educators would be able to discover if students have strengths or deficiencies in such areas as interpreting visual representations, understanding formal and informal mathematical language, performing computational skills, and applying algorithms. The revised justification scheme improves upon Harel and Sowder's original work in that the revised framework is both flexible and applicable to a variety of mathematical problems, regardless of mathematical strand or representation, as demonstrated by the students' discussion of their solutions to the four test items.

Harel and Sowder (1998) proposed, "a person's proof scheme consists of what constitutes ascertaining and persuading for that person" (p. 244). The results of the study presented in Chapter IV indicate that specific features of mathematical representations "trigger" students' solutions, in other words the representations themselves are persuasive elements. Students often appear to have little or no rationale for determining solutions. Perhaps that is because mathematics educators are looking for key words, phrases, or strategies and not on the persuasive features of mathematical representations. The justification scheme framework allows educators to classify students' solutions based on a variety of data, not isolated phrases and words, and to investigate the influence of mathematical representations in relationship to students' justifications of solutions.

Looking Forward

The plan for my research agenda for the next three years is an outgrowth of the studies presented in this dissertation and a consequence of the rich data collected in connection with the dissertation process. Creating multiple manuscripts from the data collected in the cognitive interviews has allowed me to develop my writing skills and investigate several interesting themes that emerged during data analysis. One of the interesting findings that emerged from the study presented in Chapter IV was that members of two groups of students, those who successfully answered the test item and those who did not, used similar justification schemes that were based on a specific representational aspect of the problem. These findings suggest each group of students concentrates on a specific representational aspect and rejects the importance of other components. Unfortunately, there is no way to identify which representational aspect of a problem is most important because it differs from problem to problem. However, the instructional ramifications of this finding are enormous. Mathematics educators must create opportunities in which multiple representational formats are used to present concepts, and then evaluate students' understandings of various representational formats in order to be able to guide students into understanding and interpreting the information presented in those various formats.

Other themes are still being examined and further manuscripts are planned from the robust and descriptive dataset obtained during the cognitive interviews. For clarification and data augmentation another round of cognitive interviews were conducted to explicate and substantiate the key aspects of representational triggers, ethic of care, and cognitive interviewing strategies that incorporate a basal questioning rubric.

For the extant data the interview transcripts have been analyzed largely as an intact group. Preparations have been made for an in-depth examination of the data that concentrate on analyzing the responses of a small group of students or possibly an individual case study format will be used. The next anticipated manuscript follows the line of an individual case study of Maria, a second language learner, who struggles with mathematical content when confounded by verbal mathematical representations.

Several methodological changes have been planned for and, in some situations, implemented. These changes concern procedures, written artifacts, and the focus and structure of cognitive interviews of future studies. For example, future cognitive interviews should focus on investigating the idea of representational triggers by incorporating additional mathematical strands or using a mixture of representations to present the same mathematical concept. A mixture of representations, such as verbal, graphical, symbolic and numeric, would all be used within the same problem. Using this problem format, the data collected in the cognitive interview should indicate which representations "trigger" the student's solution. Finally, the information from this additional data set will then be analyzed for additional themes as well as for confirmation of already reported results. Doing so would address issues such as validity and reliability of results, and thereby lend credibility to the "trigger" theory.

Reflections

The format of developing individual manuscripts as chapters in a dissertation is gaining popularity. However, this format is not for every doctoral candidate, such as the novice writer or for one undecided about a dissertation topic, nor should it be undertaken unless his or her chair has published regularly, continues to pursue publication opportunities, and encourages students to seek publication opportunities. Outlining multiple articles can be intimidating, much less having the manuscripts in various stages of development, as there is a perceived lack of progress as one waits for comments from one's chair or the arrival of decision letters from editors. However, I believe the three-article dissertation format provides a unique forum in which to develop and refine one's skills as an educational researcher and value the practice I received in writing the various manuscript components.

Finally, I value the opportunities to work with other educational researchers in collecting and analyzing data, and collaborating on the preparation and development of manuscript themes. I am cognizant of the variety of skills, abilities, and unique perspectives that each individual brought to the collaboration. I am grateful for these experiences and look forward to future collaborations.

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APPENDIX A

MATHEMATICAL LITERACY AND STANDARDIZED MATHEMATICAL ASSESSMENTS*

Abstract

Mathematical literacy is an important skill that is gaining the attention of mathematics educators. Students are increasingly challenged on standardized assessments to read, create, use, and comprehend numerous mathematical representations as a way of demonstrating mathematical literacy. Test items assessing algebra concepts from the Texas Assessment of Knowledge and Skills (TAKS) test for Grades 3 through 8 were used for the study. The study examined the frequency and categories of external representations used to present and solve assessment items. The analysis showed a heavy emphasis on verbal representations even though algebra items were to use verbal, numerical, graphical, and symbolic representations. The variety of representations on assessments has implications for professional development opportunities for mathematics educators.

Reading Psychology: An International Quarterly, 26, 205-233. Copyright 2006 from Mathematical Literacy and Standardized Mathematical Assessments by Shirley M. Matteson. Reproduced by permission of Taylor & Francis Group, LLC., http://www.taylorandfrancis.com May vary in content slightly due to editorial changes.

The Link Between Mathematical Representations and Student Achievement on Standardized Assessments

Several decades ago mathematics teachers began to notice the shift to a greater emphasis on, and increase of, reading skills in solving mathematics problems (Culyer, 1988; Earp & Tanner, 1980; Thomas, 1988). The National Council of Teachers of Mathematics (NCTM) emphasized in its Communication Standard that, "Instructional programs from prekindergarten through grade 12 should enable all students to use the language of mathematics to express mathematical ideas precisely" (NCTM, 2000, p. 402). Mathematics educators continue to recognize the importance of reading comprehension skills to student success in mathematics (Adams, 2003; Barton & Heidema, 2002).

Educational assessment has become a national issue in the past decade. One of the provisions in the reauthorized Elementary and Secondary Education (ESEA) Act, better known as the *No Child Left Behind Act of 2001* (NCLB), stated, "By the 2005-2006 school year, states must, at a minimum, administer annual statewide assessments in math and reading (or language arts) to all students in grades three through eight, and at least once in grades nine through twelve" (Popham, 2002, p. 3). Each state has been required to establish learning standards, create a system of assessing students in mathematics and reading, determine a level of proficiency to be attained by the student, and report assessment results based on race, gender, socio-economic status, as well as other criteria.

Two seemingly disconnected events – the increased importance of reading skills in mathematics and an increased emphasis on mathematical assessments – have become the focal point for mathematics education reform in the United States. This researcher contends that the link between the two is much stronger than it appears on the surface. It is believed that an examination of the test questions on standardized assessments will confirm the connection, while possibly providing an insight into the reforms needed to once again have the United States at the forefront of mathematics education.

The Language of Mathematics

Mathematics educators know that the language of mathematics differs from other subjects. The complex nature of mathematical concepts is further complicated by compactness of presentation, such as understanding the meaning of a formula or theorem. Relationships are implied or assumed and are hidden in technical terminology. Learning mathematics is like learning a foreign language with its own definitions of words and symbols. “Every word and abstract symbol must be read (or written) and understood with precision” (Fuentes, 1998, p. 82). Vocabulary issues are complex as

> Some words are special to mathematics, some are borrowed from ordinary usage, and some are familiar words with new and different meanings (e.g., factor, product, rational, origin, mean, real). Further, vocabulary and symbolic notation may carry equivalent meaning; for example, "+" is equivalent to sum, increase, positive, more, and, add, group, combine. (Fuentes, p. 82)

The Programme of International Student Assessment (PISA) developed an international definition of mathematical literacy. In doing so, PISA noted that mathematical literacy involves, but is not limited to, knowledge concerning such elements as procedures and facts, operational skills, and mathematical terminology (Organisation for Economic Co-operation and Development, 2003). An individual who failed to develop appropriate mathematical skills and knowledge would be presumed to be mathematically illiterate. Therefore, it is important that discussion and research continues on such topics as: effectively developing mathematical reading comprehension skills (Adams, 2003; Bratina & Lipkin, 2003), the role of reading in developing problem-solving strategies and skills (Pryer, n.d.), and training in appropriate strategies that assist students in developing technical reading skills applicable to the mathematics classroom (Barton & Heidema, 2002; Reehm & Long, 1996). Mathematics educators need to be aware of the role of reading as applied to standardized assessments, especially those of the multiple-choice format. Researchers need to concentrate on determining what connections exist between mathematical literacy and achievement on mathematical assessments as there have been few research studies acknowledging the growing importance of reading comprehensions skills as a predictor of success on mathematics standardized assessments (McGhan, 1995; Stober, 2003) or that address reading strategy accommodations for mathematics assessments (Tindal & Ketterlin-Geller, 2004).

One problematic issue of mathematics education is that students must read and comprehend a variety of mathematical *representations*—critical elements which support students' mathematical understanding, aid the student in communicating mathematical knowledge, create connections among mathematical concepts, and can be used in applying mathematical concepts in the real world (NCTM, 2000). This is an important, but complex and multi-dimensional topic considering there is an enormous degree of variety in portraying mathematical concepts and information. Unfortunately within the mathematical research community the term “representation” does not share a consensus in meaning (Luitel, 2002).

Mathematics researchers have explored the connections between representations and conceptual understanding and have written extensively about the external and internal representations that are essential to cognitive mathematical development (Duval, 1999; Goldin & Kaput, 1996; Holmes, 2004, Ozgun-Koca, 1998; Pape & Tchoshanov, 2001). Some have concentrated their efforts on identifying, examining, and defining various specific external representational forms (Brenner et al., 1997; Diezmann & English, 2001; Greeno & Hall, 1997; and Swafford and Langrall, 2000), while other researchers have concentrated on the larger schema of classifying or categorizing representations (Boerst, n.d.; Foley, 1996; Hughes-Hallett, et al. 1994; Lesh, 1979; Simundza, 2002).

It has been found that some students may not be able to read, create, or reason with various representational types (Diezmann & English, 2001) and therefore, may not find them useful in solving problems. The ability to interpret and analyze representations is predominate in standardized mathematical assessments. Effective educators may use a variety of assessment techniques in the classroom, but standardized assessments concentrate exclusively on what Luitel (2002) terms *external* representations, which require fluency with representations as “students connect verbal, numeric, graphic, and symbolic representations of relationships” (Texas Education Agency, 1998, §111.23.

Mathematics, Grade 6, (a) Introduction (2)). Such representations are visual constructs and include words, numerals, equations, symbols and signs, graphs, diagrams, charts, and tables.

Concerning representations of algebraic concepts, Brenner et al. (1997) found that prealgebra courses focused on symbol manipulation, but such courses "do not emphasize the underlying problem representational skills, such as understanding what a word problem means" (p. 664). This lack of experience with *verbal* representations is further compounded when students do not understand graphical or symbolic representations of algebra concepts (Koedinger & Nathan, 2004; Meira, 1995; Yerushalmy, 1997).

Rationale

Successfully arriving at solutions to mathematical problems is a combination of *problem representation skills* and *symbol manipulation skills* (Brenner et al., 1997). The former involves skills that "include construction and using mathematical representations in words, graphs, tables, and equations" (p. 666). Jitendra (2002) used the term *problem representation* as referring "to translating a problem from words into meaningful graphic representation" (p. 34). It has been documented that students pass through stages in developing representational fluency (Pape & Tchoshanov, 2001). Mathematics educators recognize the importance of representations to the development of mathematical literacy and that students must be provided with opportunities to gain expertise with a variety of representations in order to experience success on standardized assessments.

The Commission on Instructionally Supportive Assessment (CISA), a group supported by five national education associations, convened a panel to address issues raised in the NCLB Act. CISA's members were experts in the areas of instruction and assessment. The committee advanced nine requirements. The sixth of which, in part, stated, "A state . . . must provide well-designed assessment appropriate for a broad range of students" (Popham, 2002, p. 17) and that

> A state should secure evidence that supports the ongoing improvement of its states assessments to ensure those assessments are (a) appropriate for the accountability purposes for which they are used, (b) appropriate for determining whether students have attained state standards, (c) appropriate for enhancing instruction, and (d) not the cause of negative consequences. (p. 19)

CISA also elaborated upon the NCLB's professional development provisions for educators (NCLB, 2002, P.L.107-110, Part A, Subpart 1, Sec. 111 (c) (4)). The committee's eighth requirement specified, "A state must ensure that educators receive professional development focused on how to optimize children's learning based on the results of instructionally supportive assessments" (Popham, p. 19). Determining the focus of those professional developments, in light of assessment results, will require more than an analysis of teacher and student behaviors. Educators must be afforded the opportunity to examine the actual assessments themselves, not just examine the results of the test, as frequently happens in many states (EDinformatics, 2005).

Purpose

In this study, algebra problems on the 2003 and 2004 Texas Assessment of Knowledge and Skills (TAKS) tests for Grades 3 through 8 were analyzed by the representations used to present the problem, and the representations used to answer the question. Two effects were anticipated: (a) that students in earlier grades would be presented more pictorial and numerical representations and (b) that the shift to algebraic representations would gradually increase as students progressed through Grade 8. Following this premise, it was assumed that students in older grade levels would be presented more questions in verbal format and would respond to those questions using algebraic representations, or vice versa.

This study examined: (a) the categories of representations that were used in the questions and choices for the algebra items on the TAKS test, and (b) the implications of such representations as applicable to professional development opportunities for mathematics educators.

Rationale for Analyzing the Texas TAKS Test

Lack of National Standards

While NCLB mandated assessing students in mathematics and reading, it did not mandate the format of the assessment, nor did it establish a standard of achievement. These areas were left to individual states. Academic standards vary from state to state, which has resulted in difficulty in comparing the progress of students from state to state. Since it is left up to the individual states to “ensure that all students meet or exceed a ‘proficient’ level of academic achievement (as defined by each state) on required state assessments" (Popham, 2002, p. 4), no national standard exists concerning Adequate Yearly Progress (AYP) as mentioned in NCLB. The availability of released versions of the state assessments also varies from state to state but released tests are readily available in Texas (EDinformatics, 2005).

Variety and Availability of Assessments

The State of Texas administers yearly assessments in mathematics and reading for students in Grades 3 through 12. Other content areas, such as writing, social studies, and science, are assessed at specific grade levels. Most states do not currently have such an extensive assessment program nor do states release the entire test for each grade level (EDinformatics, 2005).

State Curriculum Aligned with NCTM Standards

The State of Texas TAKS assessment standards correlate directly to the curriculum statements found in the Texas Essential Knowledge and Skills (TEKS) documents. The State Board of Education adopted the TEKS in 1997 (TEA, 1998). Texas's mathematics TEKS are based on the National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (NCTM, 2000). The mathematics TEKS mandate that students use language skills to demonstrate conceptual understanding. For example, the following terms and phrases are used to encourage mathematical discourse: use appropriate language, select and use formal language, describe, analyze, communicate by generalizing, make summary statements, and evaluate. Furthermore, there is a student expectation concerning communication of mathematical concepts at every grade level from kindergarten through high school. As early as kindergarten, students are expected to (a) use "objects, words, pictures, numbers, and technology to explain and record mathematical observation; and (b) relate mathematics language and requisite skills to everyday language" (TEA, 1998, TEC. 111.12, Mathematics, Kindergarten, TEK K.14).

Research Using Previous Texas Assessments

Texas has a long history of using standardized tests to assess the academic progress of its students and openly provides released copies of the exams and reports of student results. Texas's student assessment program has been the focus of numerous studies concerning issues of student equity, ethnicity, and socio-economic status (Klein, Hamilton, McCaffrey, & Stecher, 2000; Linton & Kester, 2003; McNeil, 2000; Valenzuela, 2000; Viadero, 2000), most of which have involved various analyses of the Texas Assessment of Academic Skills test, the precursor to the TAKS test. Curriculum and assessment standards have been in place in Texas long before the passage of NCLB. In some ways, this makes the State of Texas both an expert and a target in the area of standardized assessments.

Algebra and Representations

The statewide TAKS mathematics assessments results are reported by objective, therefore test questions could be specifically isolated for a specific objective. There were two reasons why the study examined questions from Objective 2: Algebra, Patterns, and Functions: (1) "Algebraic competence is important in adult life, both on the job and as preparations for postsecondary education. All students should learn algebra." (NCTM, 2000, p. 37), and (2) algebra questions require students to understand a variety of representations.

Method

Subjects and Data Sources

The Texas Education Agency provided the four major sources of data for each of the grade levels analyzed in this study. An Item Analysis Summary Report (IA) of All Students was obtained for each grade level for the tests administered in the spring of 2003 and the spring of 2004. The IA report provided summary data on the percent of students responding by each multiple-choice answer on the test item. Correct answers were coded on the IA with an asterisk. Reports were also available for special populations of students, such as All Students Not in Special Education, but this study used the IA for All Students tested.

The TAKS Answer Key (AK) for each grade level was also obtained. The answer key provided much of the same information as the IA, but it also identified specific student expectations, or TEKS, which were tested under TAKS Objective 2: Algebra, Patterns, and Functions.

The Summary Report – Test Performance for All Students (SRTP) provided grade level data detailing how many public school students in the State of Texas were tested during the 2003 and 2004 test administrations. In 2003 the number of students tested at each grade level was Grade 3 (N = 266,983), Grade 4 (N = 273,229), Grade 5 (N = 280,047), Grade 6 (N = 283,564), Grade 7 (N = 283,305), and Grade 8 (N = 275,739). For the 2004 test, the number of students tested at each grade level was Grade

3 (*N* = 271,275), Grade 4 (*N* = 275,081), Grade 5 (*N* = 282,250), Grade 6 (*N* = 289,449), Grade 7 (*N* = 290,955), and Grade 8 (*N* = 286,223). Data used was for the entire public school population tested at that grade level regardless of gender, ethnicity, or socio-economic status. The SRTP also provided an average of items correct for each mathematics objective for each grade level.

The primary data source for the study was the released TAKS tests for Grades 3 through 8 administered in the spring of 2003 and the spring of 2004. The released tests provided the test items for the TAKS Objective 2: Algebra, Patterns, and Functions. Copies were made of each test item. The test items were carefully examined and coded by the both the category of representations used to present each question and the representational format the student was required to use in his/her answer.

Table A.1 presents a grade level analysis of the number of algebra items and corresponding percentage of algebra questions on the test. The percentage of test items assessing Algebra, Patterns, and Functions varied from 15% in Grade 3 to 20.8% in Grade 7. Six mathematics objectives were assessed in Grades 3 through 8, but Algebra, Patterns, and Functions generally received more than one-sixth of the emphasis (16.7%) on the test, as shown in the percentages for Grades 4, 6, 7, and 8. Each grade level assessment used varying numbers of questions for each mathematics objective, which was dependent upon the number of student expectations (TEKS) tested at that grade level for that objective.

Table A.1.

Number of Algebra Questions per Grade Level and as Percent of the Total Test Items

Grade Level	Number of Algebra Questions per Test Year	Total Test Items per Year	Algebra Questions as % of Test	% of Algebra Items Correct		69 % or Less[a]	%	70% or More[a]	%
	2003 & 2004			2003	2004				
Grade 3	6	40	15.0%	76	85	2	16.7%	10	83.3%
Grade 4	7	42	16.7%	77	81	3	21.4%	11	78.6%
Grade 5	7	44	15.9%	68	72	7	50.0%	7	50.0%
Grade 6	9	46	19.6%	61	64	13	72.2%	5	27.8%
Grade 7	10	48	20.8%	54	58	14	70.0%	6	30.0%
Grade 8	10	50	20.0%	60	67	15	65.0%	7	35.0%

Note: [a] 69% or Less and 70% or More reflects the number of questions that were answered correctly in those percentile ranges, which is also reported as a percent of the algebra items asked on the assessment.

Coding of the Data

Analysis of the algebra items on each of the tests was conducted after the test questions were identified by the TEKS student expectation. Only the TAKS Objective 2 items, those specifically identified as Patterns, Relationships, and Algebraic Reasoning, were coded for the type of representation used to present the question to the student and for the type of representation the student was required to use to answer the question. TAKS mathematic assessment item answers were primarily multiple-choice in format (95.9%). Four of the 98 algebraic questions (4.1%) in Grades 3 through 8 required students to calculate an answer and bubbled in the number(s) on a number grid: 2003 Grade 3 question 21, 2003 Grade 4 question 21, 2004 Grade 4 question 21, and 2004 Grade 8 question 21. Each algebra question received two representational category codes – one for the representational category used to present the question and one for the representational category students were required to use in responding to the question. The analysis of the 98 algebraic questions in this study produced 196 categorical codings.

The Introduction sections for the TEKS in Grades 6 through 8 state, "Students use algebraic thinking to describe how a change in one quantity in a relationship results in a change in the other; and they connect verbal, numeric, graphic, and symbolic representations of relationships" (TEA, 1998, §111.23. Mathematics, Grade 6. Introduction. (a) (2)). The terms of verbal, numerical, graphical, and symbolic, plus an additional term, *dual representation,* were used in the study. The coding structure and schema for the representations used in the algebra questions and answers for the grades 3 through 8 TAKS Mathematics assessment appears in Appendix A. Dual representations are shown as linking the four representational categories listed in the TEKS.

Algebraic questions and responses that were coded *numerical representations* revealed two different uses of numbers on the assessments. These uses were found to be either *numerical*, showing specific numerical formats for numbers - such as a decimal, fraction, or percent; or a *numerical list*, such as a list of numbers appearing as outcomes of probability or in completing a pattern/sequence of numbers.

The graphical category contained six distinct visual representations - *pictorial, model, horizontal charts, vertical charts, graphs, and coordinate graphs.* Pictorial representations used real-world objects such as toys. Models refer to an object used to represent something else, such as cups and counters representing an algebra expression.

Verbal representations require the use of written language to understand, describe, analyze, explain, or reflect upon numerical, algebraic, or graphic representations. In the mathematical research community, verbal representation refers to the use of either written or oral language (Brenner et al., 1997; Diezmann & English, 2001; and Swafford and Langrall, 2000). Unless the student has been identified as receiving an oral testing accommodation during a special education Admission, Review, and Dismissal committee meeting, TAKS Mathematics tests cannot be administered orally. The verbal category was not used to categorize test items that included brief phrases such as directions for solving the problem or explaining the meaning of a symbol in an equation or expression, nor were verbal descriptions used to describe in detail any visual representations.

Symbolic representations focused on symbolic notation and included the use of variables and formulas. Five symbolic representations were found - *equation, expression, algebraic equation, algebraic expression,* and *formula.* The difference between equations and expressions were determined by the presence or absence of a variable.

The dual representation category was created *a posteriori* when some test items utilized two representational categories. Analysis of the test items revealed that most test items included a verbal component as part of the directions, but an item designated as dual representation indicates that a second representational category provides essential information (see Appendix B). Neither representation presented sufficient details to "stand alone" and both representations must be understood in order to successfully understand and answer the problem. Seven different combinations of representations were noted – *numerical list/verbal, model/algebraic equation, horizontal chart/verbal, vertical chart/verbal, coordinate graph/algebraic equation, verbal/formula,* and *verbal/pictorial.*

Procedure

The 2003 and 2004 TAKS released mathematics tests for Grades 3 through 8 were downloaded from the TEA website. Using the IA and SRTP, test items for the algebra objective were located and removed from the main body of the test for analysis. This resulted in 98 test items from the 12 tests, two tests per grade levels 3 through 8 that addressed the Objective 2 - Patterns, Relationships, and Algebraic Reasoning. The test item number, grade level, year of test, percent responding to each multiple choice answer, and percent of students responding with the correct answer was input into an SPSS data file.

The test items were then coded for the category of representation used to present the question and the category of representation used in the correct response. The researcher coded all grade 3 through 8 algebra questions in three separate sessions, with sufficient time allocated to examine all 98 items in each session. During the first examination only four representational categories were initially used – verbal, numeric, graphic, and symbolic. However, it became apparent that a fifth category, the dual representation, would be needed in order to accurately reflect the representations that were used on the assessments.

The researcher performed a second coding two weeks later. Intra-rater dependability was based on comparisons of the coding results between the two sessions (Willson, 1980). Each test item was recoded for the representations for both the question and response. There was agreement on 93% of the items, with most disagreements occurring due to the introduction of the dual representations category.

Due to the creation of the category of dual representation, all test items were coded a third time. This resulted in a 98% agreement between the coding of the representations in the second and third session. No solutions were determined to include a dual representation, in other words all solutions were of one representational category.

After coding the questions and solutions to the TAKS algebra items, the representation codes were translated to a numerical format and input in the SPSS document in the appropriate category of either question or answer. The representations were input as to both representational type and representational category, although this study will focus only on representational categories used to present and answer the algebra questions.

The secondary focus of the study was determining trends regarding the success of students in using various representation categories. The percent of students correctly responding to each algebra item was coded as falling within a range of percents. For example, students responded correctly to test items at a 43%, 47%, and 44% level were grouped in the range of 40% to 49%. This resulted in ten ranges of numbers.

Data Analysis

To determine the categories of representations used in the question and answer choices for the algebra items on the TAKS test, a simple frequency distribution was generated. Several groupings of this information were conducted: the entire set of test items from Grades 3 through 8 (Gr3-8); representations in Grades 3 through 5 (Gr3-5); representations in Grades 6 through 8 (Gr6-8); and representations by individual grade levels. The frequency distribution was reported by both count and percent.

Results

The analysis of the frequency of representational categories reflected in the questions and solutions on the algebra items on the TAKS found a greater variety of representations used for the question than for the solution. Table A.2 shows that all five representational categories were used to present algebraic questions, but no dual representations were used in responses to the question.

Table A.2.
Frequencies and Percentages of Single and Dual Representational Categories used in the Questions and Solutions for Algebra Items

	Grades 3-5		Grades 6-8		Grades 3-8		Grades 3-5		Grades 6-8		Grades 3-8	
	n = 40	%	*n* = 58	%	*N* = 98	%	*n* = 40	%	*n* = 58	%	*N* = 98	%
	Questions						Solutions					
Numerical	4	10.0%	1	1.7%	5	5.1%	21	52.5%	26	44.8%	47	48.0%
Graphical	6	15.0%	14	24.1%	20	20.4%	None		5	8.6%	5	5.1%
Verbal	16	40.0%	34	58.6%	50	51.0%	7	17.5%	10	17.2%	17	17.3%
Symbolic	4	10.0%	3	5.2%	7	7.1%	12	30.0%	17	29.3%	29	29.6%
Dual Representation	10	25.0%	6	10.3%	16	16.3%	No Dual Representations Used as Solutions					

Note: Representations used on the test items for the 2003 and 2004 TAKS released tests have also been grouped by Grades 3-5, Grades 6-8, and Grades 3-8

The data revealed that algebra questions on the TAKS test for Gr3-5 students incorporated more dual representations than those in Gr6-8. Algebra questions for Gr3-5 students were represented most often in verbal (40.0%) or dual (25.0%) formats, while the questions for Gr6-8 students were represented most often verbally (58.6%) or graphically (24.1%).

In regards to responding to algebra questions, students in Gr3-5 and Gr6-8 usually responded with numerical and symbolic representations. Gr3-5 algebra questions responses appeared most often numerically (52.5%) or symbolically (30.0%), with similar percentages for Gr6-8 students of numerical (44.8%) or symbolic (29.3%). Table 3.A shows there was a great deal of similarity within the category percentages of the solutions to algebra questions for Gr3-8, Gr3-5, and Gr6-8, with the exception of no graphical representations as solutions for Gr3-5. Verbal representational percentages are within 0.3% and symbolic percentages are within 0.7% for the three groupings of students.

Ranking the representational categories used in the questions, from greatest to least, for Gr3-8 reveals resulted in the order of verbal (51.0%), graphical (20.4%), dual representation (16.3%), symbolic, (7.1%), and numerical (5.1%). The ranking of the representational categories for solutions, from greatest to least, for Gr3-8 resulted in the order numerical (48.0%), symbolic (29.6%), and verbal (17.3%), and graphical (5.1%). These two lists revealed a reverse in the listing of most categories. Therefore students most often were presented a question using one representational category and had to respond in a different representational category.

Table A.3.
Grade Level Frequencies and Percentages of Representational Categories used in the Questions and Solutions for Algebra Items on the 2003/2004 TAKS Released Tests

	Grade 3		Grade 4		Grade 5		Grade 6		Grade 7		Grade 8	
	n = 12	%	*n* = 14	%	*n* = 14	%	*n* = 18	%	*n* = 20	%	*n* = 20	%
Questions												
Numerical	3	25.0%	None		1	7.1%	None		None		1	5.0%
Graphical	5	41.7%	1	7.1%	None		4	22.2%	6	30.0%	4	20.0%
Verbal	3	25.0%	5	35.7%	8	57.1%	14	77.8%	9	45.0%	11	55.0%
Symbolic	1	8.3%	3	21.4%	None		None		2	10.0%	1	5.0%
Dual Representation	None		5	35.7%	5	35.7%	None		3	15.0%	3	15.0%
Solutions												
Numerical	8	66.7%	6	42.9%	7	50.0%	9	50.0%	7	35.0%	10	50.0%
Graphical	None		None		None		1	5.5%	2	10.0%	2	10.0%
Verbal	None		5	35.7%	2	14.3%	None		7	35.0%	3	15.0%
Symbolic	4	33.3%	3	21.4%	5	35.7%	8	44.4%	4	20.0%	5	25.0%
Dual Representation	No Dual Representations were used in the solutions for any grade level											

A cursory analysis of representational categories used to present algebra questions found that Grade 3 used numerical, graphical, verbal and symbolic representations, but no dual representations. Grade 4 used graphical, verbal, symbolic, and dual representations, but no numerical representations. Three categories: numerical, verbal, and dual representations were used on Grade 5 items. Grade 6 used only two categories of representations: graphical and verbal. Grade 7 used graphical, verbal, symbolic, and dual representations, but no numerical representations. Grade 8 was the only grade level to use all five categories: numerical, graphical, verbal, symbolic, and dual representations.

An analysis of the four categories of representations used in the solutions of the algebra items was also performed. Grade 3 used numerical and symbolic representations. Grades 4 and 5 used numerical, verbal, and symbolic representations. Grade 6 used numerical, graphical, and symbolic representations. Grades 7 and 8 used all four representational categories: numerical, graphical, verbal, and symbolic.

The frequency and percentages of the categories of representations by grade level revealed that at some grade levels over 50% of the questions were presented or solved in one particular representation category. Specifically noted were: Grade 3 – numerical representations of solutions (66.7%), Grade 5 – verbal representations of questions (57.1%) and numerical representations of solutions (50.0%), Grade 6 – verbal representations of questions (77.8%) and numerical representations of solutions (50.0%), and Grade 8 – verbal representations of questions (55.0%) and numerical representations of solutions (50.0%). There appears to be a theme of using numerical or verbal representations in algebraic assessment items.

The TAKS item analysis (IA) provided information concerning the percentage of students who correctly answering each algebra item on the test. As previously noted, the percentages of students responding correctly to an algebra item were grouped in a ten-point range for analysis. Tables A.4 and A.5 show the percentage of students correctly responding to a test item, as disaggregated into representational categories used in the questions and answers for Gr3-8. Graphical and verbal representations were used in 32 out of 46 (69.6%) of the questions that students successfully answered in the 70% to 99% ranges. The predominant categories of representations used as solutions were symbolic and numerical representations with 35 out of 46 (76.1%) of the answers in the 70% to 99% ranges.

Interestingly, 21 out of 50 (42.0%) verbal representations were correctly answered by 70% or more of the students, which means that 29 out of 50 (58.0%) test items were correctly answered by 69% or few of the students. The only other representational category to have less than 50% of the questions answered correctly by 70% or more of the students was the dual representation category, which may use a verbal representation as part of the problem.

Table A.4.
Percentage of Students, Grouped by Ten Point Range, Responding Correctly to Algebra Items by Representational Categories Used to Present Algebra Items for 2003/2004 TAKS Released Tests

Representational Category	Numerical		Graphical		Verbal		Symbolic		Dual Representations		Total	
Percent Answering Correctly	No.	%	No.	%	No.	%	No.	%	No.	%	*N* = 98	%
20 to 29			1	1.0%	1	1.0%					2	2.0%
30 to 39			1	1.0%	1	1.0%			1	1.0%	3	3.1%
40 to 49	2	2.0%	2	2.0%	5	5.1%	1	1.0%	1	1.0%	11	11.2%
50 to 59			3	3.1%	11	11.2%	1	1.0%	2	2.0%	17	17.3%
60 to 69			2	2.0%	11	11.2%	1	1.0%	5	5.1%	19	19.4%
70 to 79	1	1.0%	5	5.1%	6	6.1%	2	2.0%	4	4.1%	18	18.4%
80 to 89			4	4.1%	10	10.2%	1	1.0%	3	3.1%	18	18.4%
90 to 99	2	2.0%	2	2.0%	5	5.1%	1	1.0%			10	10.2%
Total	5	5.1%	20	20.4%	50	51.0%	7	7.1%	16	16.3%	98	100.0%

Table A.5.

Percentage of Students, Grouped by Ten Point Range, Responding Correctly to Algebra Items by Representational Categories Used to Solve Algebra Items for 2003/2004 TAKS Released Tests

Representational Category	Numerical		Graphical		Verbal		Symbolic		Total	
Percent Answering Correctly	No.	%	No.	%	No.	%	No.	%	*N* = 98	%
20 to 29	1	1.0%	none		1	1.0%	none		2	2.0%
30 to 39	1	1.0%	none		1	1.0%	1	1.0%	3	3.1%
40 to 49	7	7.1%	none		2	2.0%	2	2.0%	11	11.2%
50 to 59	9	9.2%	none		4	4.1%	4	4.1%	17	17.3%
60 to 69	9	9.2%	1	1.0%	2	2.0%	7	7.1%	19	19.4%
70 to 79	10	10.2%	2	2.0%	4	4.1%	2	2.0%	18	18.4%
80 to 89	5	5.1%	1	1.0%	3	3.1%	9	9.2%	18	18.4%
90 to 99	5	5.1%	1	1.0%	none		4	4.1%	10	10.2%
Total	47	48.0%	5	5.1%	17	17.3%	29	29.6%	98	100.0%

The ranges of scores were regrouped into two categories: 69% or fewer of the students correctly answering the questions and 70% or more of the students correctly answering the questions. The separation point of 70% between the two groups was chosen as it was deemed a compromise between using the upper quartile designation and a clear majority of two-thirds of the students tested. The entire group, Gr3-8, had 46 out of 98 (46.9%) of the students correctly answering at 70% or above on the algebra items. For Gr3-5 and Gr6-8 it was 28 out of 40 (70.0%) and 18 out of 58 (31.0%) respectively. The percentage of students responding correctly on algebra items was consistently higher for Gr3-5, whereas Gr6-8 consistently scored consistently lower. This information is presented in Table A.6.

The results for Gr3-8 were almost equal in the percent and number of students correctly responding to algebra items between the two designations of 69% or fewer and 70% or more. Seven of the nine representational categories used to present and solve the algebra items differed by less than 5%, with the other two under 9%. Several representational categories had opposite outcomes between the Gr3-5 and Gr6-8 grade levels, such as the verbal representations used to present the algebra problem (27.5% and 17.2%.) The numerical (37.5% and 8.6%) and symbolic (25.0% and 8.6%) representations used in solving the algebra question also varied greatly. Only the percentages of the verbal representations used in solving the algebra questions were comparable between Gr3-5 and Gr6-8 (7.5% and 6.9%.)

Table A.6

Students Correctly Answering Algebra Items on the 2003/2004 TAKS Released Tests at 69% or Less or 70% or More by Gr3-5, Gr6-8 and Gr3-8

	Grades 3 through 5 $n = 40$				Grades 6 through 8 $n = 58$				Grades 3 through 8 $N = 9$			
Representational Category	69% or Lower		70% or Higher		69% or Lower		70% or Higher		69% or Lower		70% or Higher	
	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%
Question												
Numerical	1	2.5%	3	7.5%	1	1.7%	none		2	2.0%	3	3.1%
Graphical	1	2.5%	5	12.5%	8	13.8%	6	10.3%	9	9.2%	11	11.2%
Verbal	5	12.5%	11	27.5%	24	41.3%	10	17.2%	29	29.6%	21	21.4%
Symbolic	1	2.5%	3	7.5%	2	3.4%	1	1.7%	3	3.1%	4	4.1%
Dual Representation	4	10.0%	6	15.0%	5	8.6%	1	1.7%	9	9.2%	7	7.1%
Solution												
Numerical	6	15.0%	15	37.5%	21	36.2%	5	8.6%	27	27.6%	20	20.4%
Graphical	none		none		1	1.7%	4	6.9%	1	1.0%	4	4.1%
Verbal	4	10.0%	3	7.5%	6	10.3%	4	6.9%	10	10.2%	7	7.1%
Symbolic	2	5.0%	10	25.0%	12	20.7%	5	8.6%	14	14.2%	15	15.3%

Discussion

Since the NCTM Representation standard and grade level TEKS indicate that "students' use of representations to model physical, social, and mathematical phenomena should grow through the years" (NCTM, 2000, p. 71) it was surmised that students in the upper grade levels would use a wider variety of representations. This philosophy is also reflected in the State of Texas mathematics TEKS for grade 3 through 8. While instructional programs may emphasize using "the language of mathematics to express mathematical ideas precisely" (NCTM, 2000, p. 402), standardized tests often revert to using a limited type and number of representations. Standardized tests frequently restrict the types of student responses, such as in multiple-choice assessments, and therefore students may not be given the flexibility or opportunity to create representational constructs or to explain their mathematical thinking, thus making it difficult to accurately assess the student's level of mathematical literacy.

Students in Gr3-5 were presented more questions in dual representation format. Ainsworth, Bibby, and Wood (2002) and Pape and Tchoshanov (2001) found learners experienced a greater understanding of concepts when combinations of representations were used. By using dual representations in the elementary grades, students are able to transition from concrete representations to abstract representations. In that sense, it is appropriate that the analysis of TAKS algebra assessment items reflects transitioning students through these stages of development.

Verbal representations were heavily emphasized throughout the algebraic questions and solution in the grades 3 through 8 TAKS assessments. Verbal and graphical representations were used to present 82.7% of the algebra questions for Gr6-8 students. It was assumed that a greater variety of representations would be used with older students. They should have been exposed to more symbolic representations as they progressed through the mathematics, prealgebra, and algebra curriculum. Of particular concern is the heavy emphasis of verbal representations (77.8%) in asking algebra questions to grade 6 students. Students need "opportunities to consider the advantages and limitations of the various representations they use" (NCTM, 2000, p. 208) and "need to work with each representation extensively in many contexts as well as move between representations in order to understand how they can use a representation to model mathematical ideas and relationships" (NCTM, 2000, p. 209).

The findings seems to contradict the NCTM *Principles and Standards for School Mathematics*, which states, "students' repertoires of representations should expand to include more-complex pictures, tables, graphs, and words to model problems and situations" (NCTM, 2000, p. 69). Table 2 revealed that for Grades 3 through 8 only 27.5% of the algebra questions were presented symbolically or graphically, and grade 3 through 8 students used graphical or symbolic representations for 34.7% of the responses. It should be noted that only a small section of each grade level's state assessment was analyzed by representational category. If the entire test had been analyzed in this manner, the results could have been quite different, but it is a reasonable conclusion that students should have a greater exposure to symbolic and graphical representational formats when solving algebra problems.

Symbolic representations appeared most frequently as solutions to the algebra questions in all six of the grade levels. Numerical representations were rarely used as a question, appeared primarily as a solution, and were used throughout the six grade levels. Graphical representations were used to present algebra question, but rarely used in the solutions. Mathematics educators need to provide students with opportunities to create graphical representations when solving algebra problems as the creation of and reasoning with charts and graphs will in all likelihood increase at the high school level. No connection concerning representations could be made for the decrease in the scores within the algebra strand from grade 3 through grade 7 (see Table 1).

Mathematics educators have acknowledged the following:

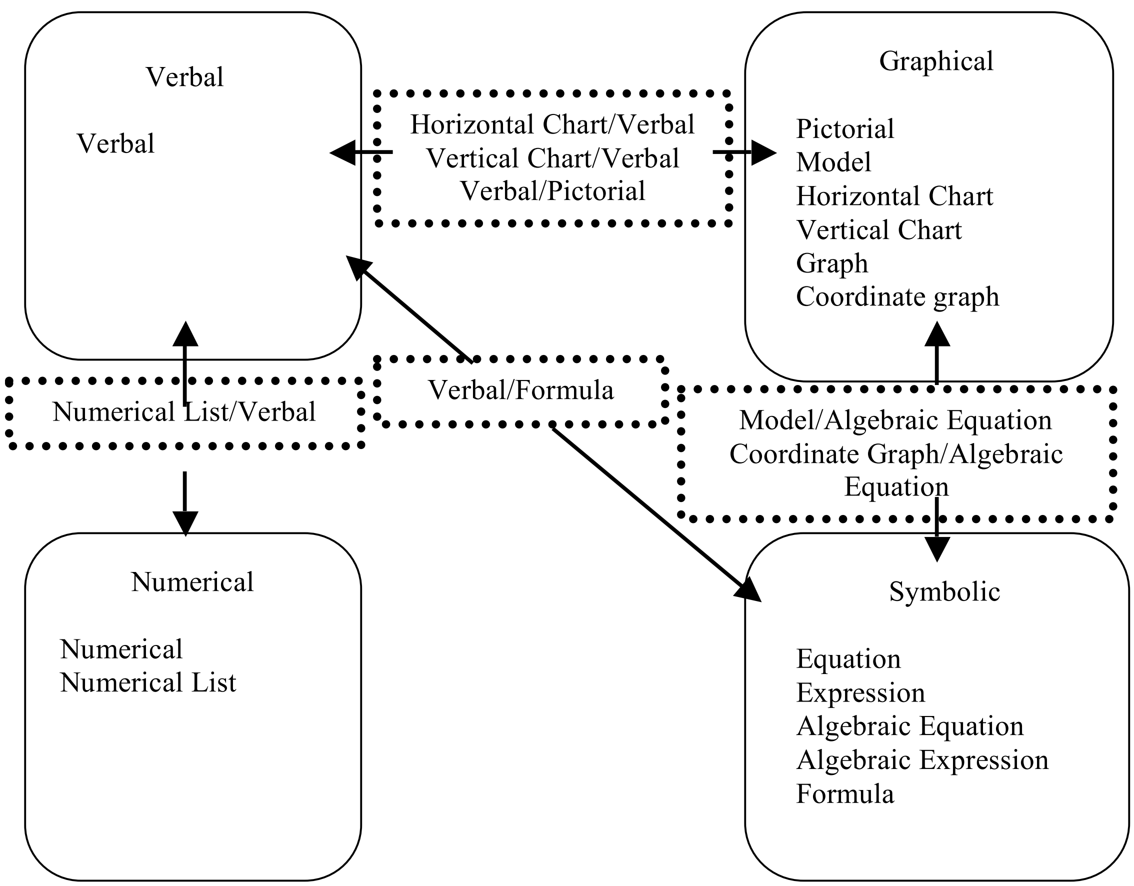
> Beyond the challenge of learning to process language in mathematics, there is the added issue of the language of mathematics. To help these students, teachers need to give them experiences in dealing with the same math topics in more than one form. . . . Students must become adept at translating from words to symbols, from symbol to word, as they search for meaning and solution. (Fuentes, 1998, p. 82)

CISA expressed hope that tests would be aligned with the academic content standards and be "deliberately designed to foster improved instruction" (Popham, 2002, p.19). Educators must become informed about state assessments and "given opportunities to either use state-provided classroom assessments or to learn how to develop their own classroom assessments" (Popham, p.19). Representations play key roles in determining students' understandings of concepts, and an examination of representations on standardized tests may help determine the validity of assessment results as well as provide direction for professional development, such as opportunities to explore mathematical literacy concerns. Research must continue to address a wide range of issues raised under NCLB: test design, assuring tests address the needs of a broad range of students, investigating the link between assessment and improvements in instruction, and examining the nature and scope of professional development opportunities focused on the results of analyzing instructionally supportive assessments.

Mathematics educators must remain mindful that the ultimate goal of developing mathematical literacy is larger than raising mathematical assessment scores. “In mathematics, students must communicate their mathematical ideas as part of understanding them . . . Without communication, students will not be able to reason, defend, or understand the conceptual basis of mathematics” (Center for the Education and Study of Diverse Populations, n.d., Communication: Math).

Representation Descriptions and Schema of Representational Types by Category

Numerical: Representation focuses on specific numerical values in a variety of formats such as decimal, fraction, and percent. This includes a list of numbers for outcomes of probability or completing a pattern/sequence of numbers.
Graphical: Representation focuses on pictorial, geometric, coordinate graphs, and other visual displays such as horizontal and vertical charts.
Verbal: Written language used to understand, describe, analyze, explain, or reflect upon numerical, algebraic, or graphic representations. Does not include brief phrases such as directions for solving the problem.
Symbolic: Representation focuses on symbolic notation and may include the use of variables and formulas.
Dual Representation: Contain two of the above listed representations categories. Neither representation presents sufficient details to “stand alone.” Both representations must be used in order to successfully understand and answer the problem.



Note: Dual representations appear inside dashed boxes. Arrows indicate representational categories that are linked by the dual representation.

TAKS Test Item Requiring Dual Coding for the Representation - Vertical Chart/Verbal

18. Each number in Set P is paired with a number in Set Q. The relationships for each pair of numbers is the same.

Set P	**Set Q**
1	7
4	10
8	14

If the number in Set P is 11, how will you find its paired number in Set Q?

F. Add 6 to 11

G. Multiply 11 by 6

H. Add 6 to 14

J. Multiply 11 by 3

Released Test Item – Grade 4, 2003 TAKS Test

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APPENDIX B

Student Questionnaire

Look at the four test problems in front of you on the table. Rank them in order from easiest to hardest by placing the test item number in the blank above the correct term.

__________	__________	__________	__________
very easy	easy	somewhat hard	hardest

1. Why did you rank them in this order?

2. What makes a problem easy for you to solve?

3. What makes a problem hard for you to solve?

4. What do you do when you have a hard test problem?

APPENDIX C

Written Responses from Student Questionnaire to Question 2 "What makes a problem easy for you to solve?" and Question 3 "What make a problem hard for you to solve?" by Frequency of Response

Easy	Hard
Problem attributes	
4 – Basic operations	2 – Numbers/information not given
3 – Clues or obvious answers	1 – Many procedures or steps
2 – Visual components	1 – Lots of numbers
1 – Bold print, key information	
1 – Short length	
1 – Almost all information given	
Concepts or content	
4 – Have done in class before (area, fractions, probability)	2 – Not covered or familiar with
	1 – Covered, but not proficient
	1 – Probability concepts
	1 – *Pi* concepts
Learner centered issues	
1 – Not tricky	6 – Confusing, did not understand, tricky, does not make sense
1 – Easily and quickly answered	2 – Figure out an answer that does not appear in the answer choices
1 – Work it mentally	2 – Think hard to work out
1 – Time needed to work it out	1 – Could not figure out ordering of problems from very easy to hard
	1 – Lots of time to solve

Note: Students may have made more than one classification of a comment.

APPENDIX D

Cover Letter Template

Date

Editor
Address
City, State, Zip

Dear Editor,

Enclosed is a manuscript entitled Title of manuscript________________________________
This manuscript is intended for publication in the name of journal. This manuscript has not been submitted to any other publication source.

This study specifically brief overview of the article and findings______________________

____

____

All original research procedures were consistent with the principles of the research ethics published by the American Psychological Association. The manuscript will be included as a chapter in Shirley M. Matteson's Ph.D. dissertation for Texas A&M University.

We hope that you and the reviewers find this manuscript interesting and that it merits acceptance for publication. Please refer correspondence to Dr. Robert M. Capraro, Texas A&M University 4232 TLAC; College Station, TX 77843-4232, telephone 979-845-8007 fax 979-845-9663 or rcapraro@coe.tamu.edu. We look forward to further communication from you.

Respectfully yours,

Shirley M. Matteson, Graduate Student
Teaching, Learning and Culture
Texas A&M University

VITA

Shirley Marie Matteson, 1602 Michele Drive, Killeen, TX 76542

EDUCATIONAL EXPERIENCE

Ph.D., **Texas A&M University,** Curriculum and Instruction with emphases in Mathematics Education and Educational Research, 2005.

M.M., **Hardin-Simmons University**, Instrumental Music, 1984.

B.M.E., **Greenville College,** Music Education, 1979.

National Board Certified Teacher - 2001 Early Adolescence/Mathematics.

Texas State Teacher Certificates - Provisional certification held in Elementary General (Grades 1-8), Secondary English (Grades 6-12), Secondary English Language Arts Composite (Gr. 6-12), and All-level Music (Grades PK-12).

Illinois State Teacher Certificate - All-Level Music.

PROFESSIONAL EXPERIENCE

1995 – present 6th grade mathematics teacher, **Palo Alto Middle School**, Killeen, TX.

2000 – 2001 Adjunct clarinet instructor, **Central Texas College**, Killeen, TX.

1984 –1995 Elementary music; Secondary language arts, band, mathematics teacher, **Killeen Independent School District**, Killeen, TX.

1983 – 1984 Graduate Assistant, **Hardin-Simmons University**, Abilene, TX.

1980 – 1983 Band director, **Ramsey Community Schools**, Ramsey, IL.

1979 – 1980 Junior high and elementary school band director, **Normal School District**, Normal, IL.

SELECTED PUBLICATIONS

Matteson, S. M. & Lincoln, Y. S. (in press). The Influence of Ethic of Care Behaviors in Research Interview Settings. *Qualitative Inquiry.*

Capraro, M. M., Ding, M., Matteson, S., Li, X., & Capraro, R. M. (2007). Representational implications for understanding equivalence. *School Science and Mathematics, 107,* 86-88.

Matteson, S. M. (2006). Mathematical literacy and standardized mathematical assessments. *Reading Psychology: An International Quarterly, 26,* 1-29.

Matteson, S. M. (2005). At-risk students and their performance on graphing assessments. *The Lamar University Electronic Journal of Student Research, 2*, http://dept.lamar.edu/lustudentjnl/archived_files_VOL2_summer05.htm

SELECTED PRESENTATIONS

Matteson, S. M. (2007, March). Textbooks' influence on the development of the concept of equality. Session in S. Matteson (Chair). ≥ *to 30 Years of Research on the Equals Sign.* With C. Lubinski, A. Otto (Illinois State University), E. Knuth (University of Wisconsin-Madison), R.M. Capraro, M. M. Capraro. Symposium presented at the Research Presession of the 85^{th} annual meeting of the National Council of Teachers of Mathematics, Atlanta, GA.

Matteson, S. M. (2006, Feb.). *Mathematical literacy and standardized mathematical assessments.* Session in R. Capraro (Chair) With Littlefield-Cook, J., Carter, T., Capraro, M.M., & Lager, C. Paper presented at the annual meeting of the Research Conference of Mathematical Leaders, Las Vegas, NV.