PRESERVICE TEACHERS' KNOWLEDGE OF LINEAR FUNCTIONS WITHIN MULTIPLE REPRESENTATION MODES

A Dissertation

by

ZHIXIA YOU

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2006

Major Subject: Curriculum and Instruction

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ABSTRACT

Preservice Teachers' Knowledge of Linear Functions within

Multiple Representation Modes. (August 2006)

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Dr. Yeping Li

This study examines preservice teachers' knowledge in the case of linear functions. Teachers' knowledge in general consists of their subject matter knowledge and their pedagogical content knowledge. In this study, teachers' subject matter knowledge is examined by looking at their ability to adapt to different representation modes. The framework for subject matter content knowledge consists of five components: (1) flexibility across formal mathematical symbolisms; (2) flexibility between visual and algebraic representations; (3) flexibility within visual representations; (4) flexibility with real-life situations, and (5) procedural skills. In terms of pedagogical content knowledge, two aspects were examined across five corresponding components. These two aspects were knowledge of students' conceptions and misconceptions, and teachers' teaching strategies.

The primary source of data for the study was from two tests and six interviews. The results showed preservice teachers performed poorly in terms of representation flexibility. Furthermore, most of the preservice teachers had limited knowledge of the nature and sources of students' mistakes as well as effective teaching strategies to help students with their misconceptions. In terms of knowledge structure, representation flexibility was found to be significant in both CK and PCK compared to procedural skills. Moreover, the representational flexibility in terms of CK seemed to strongly predict the overall PCK performance. Representational flexibility seemed to be related to the use of instructional representations. Overall, there was a strong relationship between various components of CK and PCK.

DEDICATION

To my parents Juying Wang and Wenlong You

For your nurture, unconditional love and sacrifices

To my husband Christopher Fong

For your endless support and encouragement

To my brother Zhifeng You

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Thank you to all the people who have helped – 谢谢 !

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TABLE OF CONTENTS

Page

ABSTRACT iii

DEDICATION v

ACKNOWLEDGEMENTS vi

TABLE OF CONTENTS ix

LIST OF FIGURES xi

LIST OF TABLES xii

CHAPTER

I INTRODUCTION 1

Rationale 1
Statement of the Problem 8
Significance of the Study 9
Research Questions 9
Limitations 10
Delimitations 10
Definitions 11

II LITERATURE REVIEW 12

Teachers' Knowledge 12
Functions 21
Multiple Representations 23
Research on Teachers' Knowledge of Functions 33
Theoretical Framework 40

III METHODOLOGY 46

Pilot Study 46
Current Study 49

CHAPTER Page

IV RESULTS....67

Descriptive Information on Subjects....67
Quantitative and Qualitative Analyses....68
Results for Research Question 1....69
Results for Research Question 2....106
Results for Research Question 3....132

V DISCUSSION AND SUMMARY....142

Discussions of Main Findings....143
Discussions of Additional Findings....150
Contribution of Current Study....153
Limitations of the Current Study....156
Conclusions....157

REFERENCES....159

APPENDIX A....175

APPENDIX B....178

APPENDIX C....188

APPENDIX D....200

APPENDIX E....210

APPENDIX F....222

APPENDIX G....224

APPENDIX H....227

APPENDIX I....229

APPENDIX J....231

VITA....233

LIST OF FIGURES

FIGURE		Page
4.1	Measurement Model for Preservice Teachers' CK	77
4.2	Second Order Model for Preservice Teachers' CK	78
4.3	Measurement Model of Preservice Teachers' PCK.	111
4.4	Second Order Model of Preservice Teachers' PCK	112
4.5	Correlation Model of Preservice Teachers' CK and PCK	133
4.6	Prediction of Preservice Teachers CK on PCK	135

LIST OF TABLES

TABLE Page

2.1 Framework of Representation Flexibility 40

3.1 Spring 2006 Enrollment 50

4.1 Mathematics Courses, Grade Point Average, and Mathematics Grand Point Averages..68

4.2 Descriptive Statistics for Items of Testing Preservice Teachers' CK 71

4.3 Descriptive Statistics for Factors of Testing Preservice Teachers' CK 72

4.4 Frequency of Correctness of Items for Testing Preservices' CK (Elementary Teacher Education Level)...72

4.5 Frequency of Correctness of Items for Testing Preservices' CK (Middle School Teacher Education Level) 73

4.6 Regression Analysis for Preservice Teachers' CK 74

4.7 Multivariate Analysis of Preservice Teachers' CK between Different Levels of Teacher Education..75

4.8 Between Level of Teacher Education Effects of Preservice Teachers' PCK....75

4.9 Tests of Equality of Level of Teacher Education Mean Score of Preservice Teachers' PCK 75

4.10 Standardized Canonical Discriminant Function Coefficients 76

4.11 Classification Results for CK..76

4.12 Goodness of Fit Indices for Testing Preservice Teachers' CK 79

4.13 Standardized Path Loading of Latent Constructs on CK 79

4.14 Standardized Path Coefficients of Testing Preservice Teachers' CK 80

4.15 Correlation Matrix for Different Constructs in CK (Discriminant Validity)....80

4.16 Descriptive Statistics for Factors of Testing Preservice Teachers' PCK 107

TABLE Page

4.17 Correlation among Judgment, Misconception and Instruction 107

4.18 Regression Analysis of Preservice Teachers' PCK.. 108

4.19 Multivariate Analysis of Preservice Teachers' PCK between Different Levels of Teacher Education... 108

4.20 Between Level of Teacher Education Effects of Preservice Teachers' PCK.. 109

4.21 Tests of Equality of Different Level of Teacher Education Mean Score of Preservice Teachers' PCK.. 109

4.22 Standardized Canonical Discriminant Function Coefficients 109

4.23 Classification Results for PCK.. 110

4.24 Goodness of Fit Indices for Testing Preservice Teachers' PCK 112

4.25 Standardized Path Loading of Latent Subconstructs on PCK......................... 112

4.26 Standardized Path Coefficients of Testing Preservice Teachers' PCK........... 113

4.27 Correlation Matrix for Different Constructs in PCK (Discriminant Validity) 114

4.28 Goodness of Fit Indices for Relationship between Preservice Teachers' CK and PCK 133

4.29 Standardized Path Coefficients of Correlation between Preservice Teachers' CK and PCK 134

4.30 Goodness of Fit Indices for Prediction of Preservice Teachers' PCK by CK. 135

4.31 Standardized Path Loading of Latent Subconstructs Predicting Preservice Teachers' PCK 135

CHAPTER I

INTRODUCTION

Rationale

The advancement and prevalence of technology in the world necessitates a greater understanding of algebra and algebraic reasoning (Swafford & Langrall, 2000). Meanwhile, the National Council of Teachers of Mathematics (NCTM) has recommended that algebra be studied by all K-16 students, including those who are "low achieving or underserved studied" (Edwards, 1999). The development of algebra, according to historians, goes through several stages. Algebra in its earliest stage is called "rhetorical algebra," in which statements are expressed in natural language. When abbreviations and special signs are mixed with natural language, the term "syncopated algebra" is used. "Symbolic algebra" is obviously a movement toward simplicity and economy, but at the same time it is more detached from the concrete (Wheeler & Lee, 1986). Algebraic statements in natural language (for example, a description of a procedure for finding a root of an equation) are clearly meaningful. However, the same procedure in algebraic symbolic terms is silent about the nature of the task, as Wheeler and Lee (1986) stated the possibility for interpreting the procedure as applying to a totally different situation from the one that suggested the statement in the first place.

This dissertation follows the style of the *Journal for Research in Mathematics Education*.

This is the characteristic of modern algebra, but meanwhile it is also the source of pedagogical problems.

Students' Challenges in Learning Algebra

Algebra is a major hurdle for elementary students and beyond. A large number of students face challenges in the introduction of algebra (Kieran, 1992; Wheeler & Lee, 1986). They have difficulties transferring from arithmetic to algebra (Wheeler & Lee, 1986). For example, the *Concepts in Secondary Mathematics and Science* (CSMS) research project done by Hart (1981) concluded that most 13 to 15-year-old students tested failed to interpret letters as generalized numbers or even as specific unknowns. Moreover, many high school students, as well as college students and college graduates were found unable to deal with algebra as well (Davis, 1985).

Studies of students' failure in algebra courses are widespread (Arcavi, 1995; Knuth & Stephens, 2006; National Research Council [NRC], 1998). Literature has addressed some of the major difficulties students have in terms of algebra. As Booth (1988) stated, students' difficulties in learning algebra were mainly reflected in three areas: (1) way of thinking algebraically, (2) notation and convention, and (3) letters and variables. Regarding letters and variables, Kuchemann (1981) summarized two major difficulties that students have regarding the interpretation of algebraic letters: the letter is ignored, given an arbitrary value, or used as the name of an object, and the letter is treated as a specific unknown number or generalized number. In addition to the difficulties in understanding variables, learning to solve problems using algebra is difficult as well. Blanton and Kaput (2005) reported the similar difficulties in algebra. It is well known that students often use an arithmetic approach instead of algebraic equations to represent algebraic word problems (Stacey and MacGregor, 2000; Swafford & Langrall, 2000). Algebra has its own notation and convention. For example, functions can be expressed as different forms of representation, such as table, graph and algebraic expressions. Students often have a hard time mastering the multiple forms of functions and understanding the connections between different representations. Students are challenged in their efforts to link tabular and graphical forms of representations to algebraic forms of representations (Dunham & Osborne, 1991; Knuth, 2000; McCoy, 1994, Schoenfeld et al., 1993).

Instructional Factor

A question raised is: “what factors may contribute to students’ difficulties in learning algebra?” The efforts to answer this question have never been trivial. Educators and researchers have generated different interpretations of students’ difficulties in algebra learning. In addressing the possible sources of the students’ difficulties in learning algebra, MacGregor and Stacey (1997) identified several reasons: students have intuitive assumptions and pragmatic reasoning about an unfamiliar notation system, they often base their interpretation on analogies with symbol systems used in everyday life or in other area of mathematics, and their interpretation often get interference from new learning in mathematics or often their misinterpretation is based on a false foundation created by misleading teaching materials. Arcavi (1995) and Li (1999) argued that the challenges students face in algebra are derived from the curricular issues. Cognitive researchers traced students’ difficulties to their intelligence, and they argued cognitive reasons are the main source of the difficulties.

Nevertheless, one of the main possible reasons researchers have general consensus on is instructional factors (Cunningham, 2005; Dunham & Osborne 1991; Knuth, 2000). Dunham and Osborne (1991) have pointed out that the overemphasis of symbolic manipulation and algebraic representation suggest that students’ differences may result from instructional factors instead of cognitive factors. In traditional classes, instruction using mathematical representations is seldom carried out. Algebra is often taught with a focus mainly on symbol manipulation skills, such as how to solve equations. There is a lack of emphasis on problem representation skills, such as understanding what a word problem means and how different forms of representations link to each other (Brenner et al., 1997). In Katz’ (1997) historical survey of algebra and its teaching, he describes the current situation regarding algebra teaching: algebra has always been taught with problem solving and the problems have not changed much over decades, and the problems are not real-life problems—they are usually artificial. Instruction is too often focused on one single representation or use more than one representation but without making links between different representations.

Reform mathematics proposed new standards for learning and teaching algebra. In its *Principles and Standards for School Mathematics* (PSSM), the National Council of Teachers of Mathematics (NCTM, 2000) states the goal of instructional programs from pre-kindergarten through grade 12 in algebra should enable all students to:

- Understand patterns, relations, and functions,
- Represent and analyze mathematical situations and structures using algebraic symbols,

- Use mathematical models to represent and understand quantitative relationships, and
- Analyze change in various contexts. (p. 37)

The PSSM further states the detailed standard for various grades. In grades 3-5, students need to “begin to use variables and algebraic expressions as they describe and extend patterns.” By the middle grades, students should be able to “understand the relationships for particular purposed functions” (p. 38) and need to be able to “work with multiple representations of functions—including numeric, graphic and symbolic” (p. 38). For secondary school students, they need to “be comfortable using the notation of functions to describe relationships” (p. 38).

The reform standards call for new teaching approaches in algebra, as stated in the PSSM. The algebra teaching should focus on “understanding patterns, relations, and functions” (p. 38), “represent and analyze mathematical situations and structures using algebraic symbols” (p. 38) and “use mathematical models to represent and understand quantitative relationships” (p. 39). Swafford and Langrall (2000) concluded the emphasis in the curriculum should be more on developing and linking multiple representations than generalizing problem situations. Zehavi (2004) suggested the mathematics community should give more attention to developing students’ symbol sense in learning algebra. By symbol sense, he means the awareness of symbolic relationship and the ability to select a possible symbolic representation of the problem.

Teachers’ Knowledge

There are many factors influencing instructional approaches in the classrooms such as curriculum, teachers’ knowledge, etc. Teachers, regardless of curriculum, make the ultimate instructional decisions in classrooms. The results obtained from several studies have led to the general agreement that teachers' knowledge is a key factor influencing instruction methods and eventually students' achievement (Cunningham, 2005). However, teachers have been found to have limited knowledge to be able to perform what reform standards require. Over reliance on algebraic representation and symbol manipulation procedure may suggest teachers themselves do not have sufficient content knowledge. For example, Stein et al. (1990) reported that over-utilizing algebraic procedures to explain function concepts may indicate teachers' limited knowledge. Skemp (1987) argued teachers who teach with a focus on algebraic procedures may come up with correct answers, but they may not understand the function in a conceptual level.

Many teachers do not have a solid understanding of mathematics knowledge and serious misunderstandings were found at almost every level and every topic investigated (i.e., the concept of zero, division, proof, function) regardless of knowledge of rules, procedures and concepts (Even & Tirosh, 1995). Even and Tirosh (1995) further pointed out that insufficient subject matter knowledge on the part of teachers is a widespread and frequent phenomenon whose consequences for the actual teaching should be investigated.

In terms of teachers' knowledge in algebra, researchers identified teachers' difficulties as well. For example, teachers experience challenges in representing algebraic word problems and understanding algebraic equations (Van Dooren et al., 2003) and have difficulties in making connections among algebraic, tabular and graphical representations of functions (Even, 1990; Norman, 1992; Stein et al., 1990).

Van Dooren et al.’s study (2003) showed some middle school preservice teachers could only represent algebra word problems arithmetically instead of algebraically. Even’s study (1990) reported secondary preservice mathematics teachers were also unable to make connections between algebraic and graphical representations. In their case study, Stein et al. (1990) found one algebra teacher considered graphs only as checking devices and failed to see tables, graphs and algebraic expressions as multiple forms of representations of the same function.

The focus of this dissertation, therefore, is not on students and their algebra learning but on mathematics preservice teachers who will play an important role in stimulating and supporting algebra learning processes. The information with respect to the teachers is important because teachers, regardless of curriculum and textbook, make the ultimate instructional decisions and opportunities for students to learn (i.e., the process of learning is influenced by the teacher) (Even, 1993). In order for teachers to be maximally effective in teaching a concept, they must obtain an adequate level of mathematical knowledge of the concept to provide instruction (Simon, 1993).

Although there are many studies on teachers’ subject matter knowledge and pedagogical content knowledge (PCK) related to functions, few have assessed preservice teachers’ representation competency and how it influences knowledge of students and their teaching strategies. Since linear functions are primary to the study of algebra and play an important role in initiating the idea of transfer for students’ future study of functions (Cunningham, 2005), the purpose of the present research, therefore, is to provide a systematic study of teachers’ knowledge of linear functions. In particular, this study focuses on two aspects of preservice teachers’ mathematical knowledge that are central to understanding and teaching linear functions (y = mx + b)—the flexibility among and within modes of representations of functions and an understanding of students' learning and thinking.

Statement of the Problem

Multiple representations of functions are a major topic throughout mathematics teaching and learning. Although student understanding of the function concept is widely studied, few studies have been done involving preservice teachers' knowledge of function, which leaves teacher education programs lacking in an understanding of this domain. It remains true that the number of studies involving preservice teachers' knowledge with respect to representation flexibility in the domain of functions has been sparse, as well as systematic studies of teachers' representation flexibility and how it affects their knowledge of students' thinking and teaching strategies in linear functions.

The major goal of this study is, therefore, to uncover some insights of preservice teachers' knowledge in particular, representation flexibility and their understanding of students' thinking in linear functions. More specifically, this study seeks to examine preservice teachers' flexibility within a representation and across multiple representation modes in terms of linear function, investigate preservice teachers' knowledge of students' conceptions and misconceptions and their teaching strategies in linear functions, and explore the relationship between preservice teachers' representation flexibility and their understanding of students' thinking and misunderstandings.

Significance of the Study

Functions are regarded as one of the most important mathematics topics from elementary curriculum and beyond, and a flexible use of multiple representations indicates a deeper understanding of the function concept. However, not only are a large number of students struggling with the translations of multiple representations of functions, but also both preservice (Even, 1990; 1993) and in-service teachers (Lloyd & Wilson, 1998; Norman, 1992) have limited knowledge of translations between representations of functions. In addition, there has not been a systematic study of teachers' knowledge in terms of representation flexibility and its impact on teachers' understanding of students' cognition and teaching strategies. In this sense, the present study has some implications for preservice teachers' conceptual development in teacher education.

Research Questions

This study sought answers to the following questions:

1. How do preservice teachers demonstrate their content knowledge in terms of representation flexibility and procedural skills in linear functions? Does preservice teachers' subject matter knowledge vary for those in elementary and middle grades teacher education?
2. How do preservice teachers demonstrate their pedagogical content knowledge in terms of their knowledge of students' conceptions and misconceptions and their instructional strategies for addressing students'

misconceptions in linear functions? Does preservice teachers' pedagogical content knowledge vary for those in elementary and middle grades teacher education?

3. How does preservice teachers' subject matter knowledge influence their knowledge of students' conceptions and misconceptions of linear functions and their instructional approach for addressing students' misconceptions?

Limitations

There are three major limitations in this study. They are related to the number and the selection of participants. Only a total of 104 participants were involved with this study. The participants were limited to the preservice teachers in their late stage of teacher education from Texas A&M University. However, the number and the selection of the participants are based on the purpose of the study as well as the time and resources limit.

Delimitations

The selection of participants is delimited to the preservice teachers from Texas A&M University because the university consists of a wide range of student population from diverse background.

The examination of teachers' knowledge in this study is delimited to the specific topic – linear function because it is a primary topic in function for middle school students who start to learn algebra.

Definitions

Algebraic representation: A symbolically written equation (Moschkovich et al., 1993).

Graphic representation: A function appearing on a coordinate system (Moschkovich et al., 1993).

Pedagogical content knowledge: Knowledge of how to transfer subject matter knowledge into an understandable representational system, along with an understanding of what makes learning a specific topic easy or difficult (Shulman, 1987; Wu, 2004).

Representation flexibility: *Transformations* within a representation system, *translations* among different representational systems, and *real-life situations* that are needed to use mathematical ideas in everyday situation.

Subject matter content knowledge: knowledge shared by most educated adults, such as knowledge of mathematics.

Tabular (ordered pair) representation: A linear functional represented by a table or a set of ordered pairs (Moschkovich et al., 1993).

CHAPTER II

LITERATURE REVIEW

Teachers' Knowledge

The single factor which seems to have the greatest power to carry forward our understanding of the teachers' role is "the phenomenon of teachers' knowledge" (Elbaz, 1983, p. 45). There is a consensus that teachers' knowledge is one of the most important influences on what is done in classrooms and ultimately on what students learn (Fennema & Franke, 1992).

Fennema and Franke (1992) summarized several components of teachers' knowledge that received major attention from researchers: content knowledge, knowledge of learning, knowledge of mathematical representations, and pedagogical knowledge. Their research model examined the integration and relationship among knowledge of mathematics, knowledge of mathematical representations, knowledge of students and general knowledge of teaching and decision-making. Knowledge of mathematics is critical for teachers to help students, which is shown in Ball's (1988) statement: "knowledge of mathematics is obviously fundamental to being able to help someone else learn it" (p. 12). Post et al. (1988) further noted that, "A firm grasp of the underlying concepts is an important and necessary framework for the elementary teacher to possess…[when] teaching related concepts to children…" (pp. 120).

By content knowledge, Shulman (1986) refers to the amount and organization of knowledge in the mind of the teacher:

> We expect that the subject matter content understanding of the teacher be at least equal to that of his or her lay colleague, the mere subject matter major. The teacher need not only understand that something is so; the teacher must further understand why it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened and even denied. Moreover, we expect the teacher to understand why a given topic is particularly central to a discipline whereas another may be somewhat peripheral. This will be important in subsequent pedagogical judgments regarding relative curricular emphasis.

Teachers' subject matter content knowledge, after several decades of being neglected, is getting more attention from researchers (Stein et al., 1990). In her book, *Knowing and Teaching Elementary Mathematics*, Ma (1999) compared United States and Chinese elementary teachers' mathematical knowledge. Ma argued that profound understanding of fundamental mathematics is essential for effective teaching. The notion of profound understanding of fundamental mathematics comprises a thorough and well-connected understanding of arithmetic domain, and it also includes how each topic fits into the big picture—the overall conceptual structure of the discipline as well as the connection between mathematics topics and more conceptually advanced mathematics ideas (Irwin & Bana, 2001). Ma described a "knowledge package" of 72 Chinese elementary teachers that consisted of the organization and connectedness of ideas in an elementary mathematics domain. The study indicates teachers need not only to know content conceptually, that is, know the connections among ideas, but also must know the representations for teaching and common misconceptions that students have with specific content area (Ball et al., 2001).

Teachers' subject matter content knowledge can be organized into two categories, procedural and conceptual knowledge (Hiebert, 1986). Procedural knowledge is defined as the fact of mathematics symbols and procedures to solve a problem.

Conceptual knowledge is defined as “a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information” (pp. 3-4). Cognitive researchers attempted to study the organization of teacher knowledge by elucidating an individual’s mental presentations (Gardner, 1986). These researchers explored the way objects are represented and how they are related. Hiebert and Lefevre (1986) concluded that only with an understanding of mathematical processes or when learners are able to build relationships between existing and new knowledge can mathematics learning occur. The interrelationship of ideas is an important aspect of conceptual understanding.

Teachers’ subject knowledge impacts the decisions teachers make about classroom instruction. Furthermore, teachers’ knowledge of mathematics is directly related to student learning (Eisenberg, 1977; Erickson, 1986; Fennema & Franke, 1992). More specifically, it is teachers’ conceptual knowledge that is directly related to student learning.

Evidence shows that there is a direct relationship between teachers’ conceptual knowledge of subject matter and student learning as well as classroom instruction (Leinhardt & Smith, 1985; Steinberg et al., 1985; Brophy, 1991). Leinhardt and Smith (1985) conducted a longitudinal study comparing student teachers and expert teachers’ knowledge in fractions. Leinhardt and Smith interviewed both student and expert teachers and observed their classroom teaching and had them complete various card sorts. They made a conclusion that by comparing student and expert teachers’ knowledge, “the more experienced and competent teachers exhibited a more refined hierarchical structure of their knowledge” (p. 252), that is, those expert teachers demonstrated not only procedural understanding of fractions problems, but also obtained a conceptual understanding—the interrelationships of the procedures. Furthermore, Steinberg et al. (1985) explored the relationship between teachers' conceptual knowledge and their teaching. By examining the specific subset of mathematics knowledge they were teaching and their teaching approach, they found teachers who had connected and interrelated knowledge were able to teach more conceptually while those without this type of conceptual knowledge adopted a more procedural teaching approach.

The impact of teachers' conceptual knowledge on classroom instruction is confirmed in the statement by Brophy (1991):

> Where [teachers'] knowledge is more explicit, better connected, and more integrated, they will tend to teach the subject more dynamically, represent it in more varied ways, and encourage and respond fully to student comments and questions. Where their knowledge is limited, they will tend to depend on the text for content, de-emphasize interactive discourse in favor of seatwork assignments, and in general, portray the subject as a collection of static factual knowledge. (p. 352)

To teach mathematics effectively, teachers need to know not only what a function is but also the underlying "why." Researchers even suggest some teachers may need to re-learn the content of what they teach and to learn to use multiple representations and to increase their content knowledge of connections among different representations (Stein et al., 1990; Wilson, 1994).

In addition to knowing the subject conceptually, An et al. (2004) have argued effective teaching also requires profound pedagogical content knowledge. Knowledge of mathematical representations, knowledge of students from Fennema and Franke's (1992) model can be categorized into Shulman's notion of pedagogical content knowledge.

PCK “goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching.” (Shulman, 1986, p. 9). By PCK, Shulman included:

> For the most regularly taught topics in one’s subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others. Since there are no single most powerful forms of representation, the teacher must have at hand a veritable armamentarium of alternative forms of representation, some of which derive from research whereas other originate in the wisdom of practice. Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and basically that students of different ages and background bring with them to the learning of those most frequently taught topics and lessons. If those preconceptions are misconceptions, which they so often are, teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners… (pp. 9-10)

In summary, PCK is described by Shulman as knowing the ways of representing the subject matter to make it comprehensible to students, along with an understanding of what makes learning a specific topic easy or difficult (Ball, 1988; Even, 1993; Shulman, 1986). Focusing on teaching and learning mathematics, Hill et al. (2005) build on Shulman’s (1986) notion of pedagogical content knowledge, where they used the term “mathematics knowledge for teaching.” Mathematics knowledge for teaching is defined as:

> The mathematical knowledge used to carry out the work of teaching mathematics. Examples of this “work of teaching” includes explaining terms and concepts to students, interpreting students’ statements and solutions, judging and correcting textbook treatments of particular topics, using representations accurately in the classroom, and providing students with examples of concepts, algorithms, and proofs. (Hill et al., 2005, p. 373)

Teachers need not only know the common knowledge content (i.e., knowledge of the subject) a proficient student or mathematician would have but also “specialized” knowledge used in teaching student mathematics (Hill et al., 2005). For example, teachers must have adequate knowledge to compute 35*25. Certainly, many adults, and certainly all mathematicians, would know enough to answer this item correctly. Therefore, it is "common" content knowledge, not the knowledge for teaching. Teacher's specialized mathematics content knowledge requires them to know different approaches of solving a multidigit multiplication problem—35*25—and be able to evaluate whether these approaches would work in general. Teachers also need to be able to show or represent 35*25 using pictures or manipulatives to make the mathematics rules easy for students to understand.

The central idea of a new notion of teaching is that teachers need to be aware of the mental representation that a student is building in his or her head; the teacher needs to recognize the student's representations as accurately as possible and tries to provide for that student precisely those experiences that will be most useful for further development or revision of the mental structures that are being built (Davis & Maher, 1997). Reform mathematics requires teachers to teach for understanding. A powerful way to do this, according to *Principles and Standards for School Mathematics* (NCTM, 2000), lies in a teacher's ability to transfer his/her subject matter content knowledge into a representational system that is understandable to students. How abstract mathematical ideas are represented in instruction makes a significant difference in students' understanding (NCTM, 2000).

In connecting mathematical knowledge and the use of mathematical representations, Fennema and Franke (1992) argued that knowledge of representation is an integral part of teachers' knowledge. They argued how mathematics is presented in instruction is closely related to the PCK. Fennema and Franke further noted that the process of teaching is to take complex subject matter and to translate it into representations that can be understood by students. “The translation of mathematics into understandable representations is what distinguishes a mathematics teacher from a mathematician…mathematics is comprised of a large set of highly related abstractions, and if teachers do not know how to translate those abstractions into a form that enables learners to relate the mathematics to what they already know, they will not learn with understanding” (Fennema & Franke, 1992, p. 153). For example, teachers can use real-world situations to help students understand addition and subtraction. Teachers can also use concrete and pictorial representations to represent mathematical ideas. Evidence shows that use of both real-world situations and concrete or pictorial representations help students learn abstract mathematics ideas with understanding; therefore, for teachers to facilitate learning with understanding, they need to understand and know how to use representations to deliver their mathematical ideas (Fennema & Franke, 1992).

In one sense, knowledge of representation is teachers’ content knowledge (i.e., they themselves need to know understand multiple representations of mathematical ideas). The flexibility with multiple forms of representations reflects a deep conceptual understanding of a concept. In another sense, knowledge of representation is part of teachers’ pedagogical content knowledge, the way to represent their ideas to make it understandable to learners. Davis and Maher (1997) also stated “how a teacher introduces (or defines) a new idea can make a big difference to subsequent student success” (p. 98). In their study, Davis and Maher gave an example of how the representations a teacher uses can make a difference in students’ learning. When a teacher introduced the new idea of “odd” and “even” integers, she introduced an earlier idea-matching two-by-two with which the children were already familiar. She therefore gave children “tools to think with,” which is a powerful (internal) mental representation through external representation (Davis, 1992).

McDiarmid et al. (1989) stated that mathematics pedagogy can be seen as a repertoire of instructional representations. Instructional representations build the connection between the teacher and the mathematics, and construct the relationship between what the teacher knows and what the teacher does (Stump, 2001). Many teachers lack the knowledge of representation (Ball, 1990). Researchers have documented teachers having limited knowledge of instructional representations. Ball (1990) investigated 19 preservice teachers’ abilities to develop a representation of 13/4 divided by ½ using a story problem or other kind of model. Almost all of the teachers could obtain the correct answers to the problem. However, none of the teachers could develop an appropriate representation. Orton (1988) did a study on 20 in-service elementary teachers. In his study, he asked teachers to teach a fraction concept to a hypothetical student who had a specific misconception about fractions. The results showed that most of the teachers used procedural and symbolic representations rather than a representation that would promote the student’s conceptual understanding. Ball’s (1993) study identified elementary teachers having limitations in their own mathematics understanding and not being able to use instructional representations effectively. Even (1993), Norman (1992), and Wilson (1994) observed that preservice secondary teachers had limited repertoires of instructional representations regarding the concept of function.

Understanding of students’ conceptions and misconceptions are another aspect of PCK. Many studies have shown that students often have their own way of thinking which is not necessary parallel to the structure of the curriculum or instruction (e.g., Even, 1993; Hershkowitz et al., 1987; Kieran, 1992; Schoenfeld et al., 1993; Tirosh & Graeber, 1990). Accordingly, cognitive researchers proposed the importance of studying students' conceptions and misconceptions. Mayer (1987) stated that knowledge students' conceptions and misconceptions as well as their sources are necessary for teachers. He argued that teachers need to know not only what students are doing but also what the students are thinking while they are producing the answers. As von Glasersfeld (1987) says, "the teacher's role will no longer be to dispense 'truth' but rather to help and guide the student in the conceptual organization of certain areas of experience…[what teachers need to have is] an adequate idea of where the student is."

An understanding of student ways of thinking enables teachers to make appropriate decisions for helping and guiding students in their knowledge construction (Even & Tirosh, 1995). Even and Tirosh further stated a teacher who was aware of student conception could challenge and extend student thinking and modify or develop appropriate activities for students. Starting from students' limited conceptions, the teacher can come up with strategies to help students build more sophisticated ones. Teachers' strategies in response to students' misconceptions is an important component of PCK because a teacher's judgment on whether students' answers are correct is based on the teacher's content knowledge, but the content knowledge is not enough for developing a reaction that can help students construct their knowledge (Even & Tirosh, 1995). In addition to having knowledge of common students' conceptions and misconceptions related to specific topics, teachers should understand the reasoning behind students' conceptions and identify sources of misconceptions and strategies to help students correct their misconceptions.

In summary, teachers' knowledge plays an important role in classroom practice. Some of the main components that researchers agree upon teachers' knowledge are: subject matter content knowledge and PCK. Subject matter knowledge comprises procedural knowledge and conceptual knowledge. Existing researches have identified that conceptual knowledge is a key component that interrelates with teachers' PCK. Because the ability to flexibly transfer between representations is an indicator of a deep understanding of functions, representation flexibility is considered as conceptual knowledge. The second category of teachers' knowledge, PCK, includes knowledge of representation, knowledge of students' conceptions and misconceptions and teaching strategies. To teach is to transfer what is known to various representations that are understandable to students. A repertoire of instructional representations is necessary for teachers. Moreover, effective teaching requires teachers to be aware of students' misconceptions and their sources. Therefore, the main purpose of this study is to investigate preservice teachers' CK, PCK and their relationship, specifically on linear functions.

Functions

Functions are defined as a way to express the relationship or co-variation between two or more variables (Romberg et al., 1993) and are regarded as "a set of ordered pairs, a correspondence, a graph, a dependent variable, a formula, an action, a process or an object" (Selden & Selden, 1992, p. 4). Functions do not have any specific expression, follow some regularity or are described by a graph with any particular shape. However, there are explicit requirements: they should be defined on every element in the domain, and there should be only one element (image) in the range for each element in the domain.

There is a general consensus about the central role functions play in the learning and teaching of mathematics (Beckmann et al., 1999; Ferrini-Mundy & Lauten, 1993; Leinhardt et al., 1990; Schwartz, 1992; Selden & Selden, 1992). Functions, a central topic in both pre-algebra and algebra, are one of the most important mathematical topics, and they have been heavily recommended for inclusion through elementary curriculum and beyond (Kaput, 1989; NCTM, 1989, 2000; Romberg et al., 1993; Yerushalmy & Schwartz, 1993). For example, Romberg et al. (1993) state that, “Functions are one of the most powerful and useful notions in mathematics” (p. 1). Furthermore, functional relations are a common occurrence in real life situations (Hendrick, 1922). Breslich (1928) stated that “without functional thinking there can be no real understanding or appreciation of mathematics” (p. 28). In the past, research has been carried out on the difficulties students experience when learning functional relationships, as well as the sources of these difficulties and the ways in which they can be prevented and remedied through instruction (Brenner et al., 1997; Filloy & Sutherland, 1996).

Functional relationships are difficult for many students to understand, perhaps because it is predominantly taught with an emphasis on abstract algebraic forms rather than in a meaningful context (Demana et al., 1993; Dunham & Osborne, 1991; Karplus, 1979). In other words, the traditional teaching neglects the importance of multiple representations in the understanding of functions, which results in significant problems in students' understanding of the function concept. Various researchers have demonstrated that a large number of students encounter difficulties with transfer among representations to solve algebra problems (Dunham & Osborne, 1991; Eisenberg & Dreyfus, 1991; Goldenberg, 1988). For example, many junior high school students and even preservice teachers have difficulties in constructing graphs (Demana et al., 1993). In a study by Hart (1981), a survey of 3,000 middle-school students revealed students could not construct a functional relationship between data pairs and algebraic symbols.

A deep conceptual understanding of functions is demonstrated by an ability to effectively utilize multiple representations and to flexibly make translations among them. For students to develop a deep understanding of functions, they must have opportunities to solve problems that require them between algebraic, tabular and graphical representations (i.e., transfer problems) (Cunningham, 2005). It is argued that students' difficulties may stem from a teacher's lack of both subject matter knowledge and pedagogical content knowledge to conduct instruction that fosters representational skills of functions among students (Norman, 1992; Stein et al., 1990). Yerushalmy and Shterenberg (1994) stated that applying modeling processes besides teaching symbolic representations for the functions will help students better understand algebra functions.

Multiple Representations

There is some variation of how a representation is defined. Smith (2003) provided a summary of different definitions of representations. He organized the definition into two categories: object-oriented definition (Brinker, 1996; Goldin, 2003;

Pimm, 1995), and semiotic-oriented definition (Kaput, 1985; Lesh et al., 1987). Goldin (2003) defines a representation as:

A configuration of signs, characters, icons, or objects that can somehow stand for, or "represent" something else. According to the nature of the representing relationship, the term represented can be interpreted in many ways, including the following (the list is not exhaustive): correspond, denote, depict, embody, encode, evoke, label, mean, produce, refer to, suggest, or symbolize. (p. 276)

Kaput (1985) provided a semiotic-oriented definition:

> In its broadest sense, a representation is something that stands for something else, and so must inherently involve some kind of relationship between symbol and referent, although each may itself be a complex entity…A rigorous specification of a representation should include the following entities:
>
> - The represented world.
> - The representing world.
> - What aspects of the represented world are being represented?
> - What aspects of the representing world are doing the representing?
> - The correspondence between the two worlds. (pp. 383-384)

Although people gain meaning from representations through the "correspondence between the two worlds," multiple correspondences may result because what is represented can be differently interpreted (Smith, 2003).

Lesh et al. (1987) offered a definition that suggests a relationship between external representations and internal understanding: "The term representation here is interpreted in a naïve and restricted sense as external (and therefore observable) embodiments of students' internal conceptualizations—although this external/internal dichotomy is artificial" (p. 33). Lesh et al.'s (1987) statement indicated the possibility of examining a creator’s internal conceptualizations through external representations they create.

Internal representation is “networks of concepts and relationships” while external representations are used to describe “things that can be represented outside of the human mind” (p. 10). Internal representation refers to an individual’s internal psychological system of representation, including “…their visual imagery and spatial, tactile, and kinesthetic representation; their problem solving heuristics and strategies; their personal capabilities, including conceptions and misconceptions, in relation to conventional mathematical notations and configurations; their personal symbolization constructs and assignments of meaning to all these…” (p. 277).

Numerals, graphs and tables are examples of external representations (Goldin, 2003). According to Goldin (2003), external representation include “normative natural languages (e.g., ‘standard’ English); conventional graphical, diagrammatic, and formal notational systems of mathematics…” (p. 277). Mathematical power of the individual lies in the translation processes among representation modes, both external and internal (Goldin, 1987; Lesh, 1981; Lesh et al., 1983). Mathematics power requires competence in representations and its manipulation (Goldin, 2003). He states, “Mathematical concepts are learned powerfully when a variety of appropriate internal representations, with appropriate relationships among them, have been developed” (p. 278).

Davis (1992) argued internal representation is formed through various representations, such as the physical or pictorial models, symbolism, language (discussion of the ideas), or the concept placed in context. “A goal of education is to help individuals create internal representations that accurately mirror external representations" (Troutman & Lichtenberg, 2003, p. 10). The overarching goal of school mathematics is to develop students' powerful internal representations and learn how to infer the internal representations from their observable, external representations (Goldin, 2003). Internal representations are what build mathematics power and meaning which are the foundations of mathematics learning. A new emerging view of mathematics teaching is to emphasize on the meaning of the symbols rather than on symbols written on paper (Kaput, 1989).

Kaput (1989) argued the importance of external representations and the translations and transformations among and within external representations to the internal representation, ultimately, to mathematics meaning. For example, a Cartesian graph is a conventional representation that may correspond to an algebraic equation or function, or depict a set of data or express a qualitative relationship (Goldin, 2003).

Internal representation and external representation are closely related. Hiebert and Carpenter (1992) state the form of external representation that a student interacts influences how the student represents the quantity or relationship internally. On the other hand, how a student deals with or generates an external representation reveals how the student has represented the information internally (Hiebert & Carpenter, 1992).

Taken together, mathematics power is the foundation for mathematics learning. In turn, mathematics power requires a variety of internal representations. Many cognitive studies have come to a general consensus that the internal representations can be referred through external representations. Accordingly, the focus of the current study is on external representations. The following provides a summary of the importance of multiple external representations play in the teaching and learning of mathematics that is reflected in the current literature.

Smith (2003) stated representations which aid the solution of classes of problems are essential to mathematics and students must be able to use mathematical representations if they are to progress far within the mathematics discipline. The same idea is resonated in the PSSM (NCTM, 2000).

Instructional programs from pre-kindergarten through grade 12 should enable all students to:

- Create and use representations to organize, record, and communicate mathematical ideas;
- Select, apply, and translate among mathematical representations to solve problems; and
- Use representations to model and interpret physical, social, and mathematical phenomena (p. 67).

The PSSM (NCTM, 2000) stated representations “can play an important role in helping students and solve problems” (p. 68). In the problem-solving process, the representations provide “meaningful ways to record a solution method and to describe the method to others” (p. 68). Students need to “develop an understanding of the strengths and weaknesses of various representations” (p. 70). The teachers need to help students develop ways of interpreting and thinking about mathematics through representations (NCTM, 2000). In addition, in order to teach students to make and understand the standard representation, teachers need to engage students in purposeful shared activities using various representation forms (Monk, 2003). To do this, teachers need to be able to understand the appropriate use of representations (Smith, 2003). As NCTM (2000) notes, “When students gain access to mathematical representations and the ideas they represent, they have a set of tools that significantly expand their capacity to think mathematically" (p. 67).

With the increased awareness of the importance of representations, Goldin (2003) recommends that teachers should be able to do the following in their classrooms:

> (1) design activities to address well-formulated mathematical learning goals, posed in advance, that include internal and external representational capabilities; (2) choose tasks that the students can represent meaningfully but that offer challenges; (3) choose tasks that embody rich and varied representational structures, including contextual mathematics, abstract mathematics and visual imagery; (4) plan for major contingencies, making use of research that identifies cognitive obstacles or common representational difficulties; (5) encourage free problem solving by students, with guiding interventions; (6) maximize students' interaction with the learning environment, encouraging a variety of external representational modes, standard and nonstandard; (7) incorporate multiple, ongoing means of assessing students' learning through their representations; (8) develop a repertoire of proven, successful activities; (9) be alert to students' novel representations, strategies, and insights; and (10) balance these considerations with one another, compromising where appropriate and valuing highly personal teaching style. (p.281-282)

Lesh et al. (1987) argued external representations allow teachers and students to have a common language for communicating internal mathematics ideas. As Greeno and Hall (1997) noted "forms of representation can be considered as useful tools of constructing understanding and for communicating information and understanding" (p. 362). Students must be given opportunities to enhance their knowledge of mathematical representations and to solidify their mathematical understanding (Pape et al., 2001). In order to provide students with varied and accurate representations of mathematics ideas, teachers need to enhance their understanding and use of mathematical representations because a teacher's deeper understanding would enable them to encourage their students to use and transfer information among different representations (Fernandez & Anhalt, 2001). Furthermore, "to help students' progress valuing the representation, the teacher needs to understand how children view and relate to different mathematics representations" (Smith, 2003, p. 264).

Algebraic, graphical and tabular forms are documented as the three prominent representations of functions (Moschkovich et al., 1993). Cunningham (2005) presented these representations with a focus on linear functions of two variables. He defined algebraic representations as a symbolically written equation, tabular forms as a linear functional relationship represented by a table or a set of ordered pairs, and graphic representation as a function appearing on a coordinate system. Romberg et al. (1993) stated "a coherent body of knowledge about how the connections are developed among tables, graphs, and the algebraic expressions related to functions is desperately needed" (p. ix). Making connections among these three forms of representations of functions is highlighted as well (Romberg et al., 1993; Thompson, 1994).

Monk (2003) illustrated how graphical representations are important for mathematics power:

> (1) using graphs, students can explore aspects of a context that are not otherwise apparent, (2) the process of representing a context can lead to questions about the context itself, (3) using graphs to analyze a well-understood context can deepen a student's understanding of a graph and graphing, (4) students can construct new entities and concepts in a context beginning with important features of a graph, and (5) students can elaborate on their understanding of both a graph and its context through an iterative and interactive process of exploring both, and a group can build shared understanding through joint reference to the graph of phenomena in a context. (pp. 252-256)

Cognitively-oriented researchers have proposed models of mathematical competence, such as a problem-solving model in which good problem representational skill is a hallmark of mathematics understanding (Mayer, 1987). Problem representational skill refers to constructing and using mathematical representations in words, graphs, tables, and equations (Brenner et al., 1997). Being able to use multiple representations of a concept and to translate flexibly among them is regarded as a key component to understanding (Yerushalmy, 1997). In addition, supporting students with instruction on different forms of external representations enhances problem-solving skills (Brenner et al., 1997; Lewis, 1989). This perspective resonated in the PSSM (NCTM, 2000), which stressed the importance of multiple representations in the learning and understanding of mathematics. "Representation should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments and understandings to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling"(p. 67). Even for children in grades 3-5, the algebra standard of NCTM's *Principles and Standards for School Mathematics* requires students to be able to "represent and analyze patterns and functions, using word, tables and graphs" (p. 158). Standards in grades 6-8 requires students to be able to analyze and solve algebra function problems using graphs, tables and equations. In grades 9-12, students should be able to "interpret representations of functions of two variables" (p. 296) and use various representations and covert flexibly among them.

Researchers have documented a rich understanding of the function concept which requires one to be able to represent functions in multiple ways and to be able to move from one representation to another flexibly (Dubinsky & Harel, 1992; Eisenberg, 1992; Yerushalmy & Schwartz, 1993). Eisenberg (1992) stressed a key component of a robust understanding of the function concept is to be able to make connections between multiple representations. Yerushalmy and Schwartz (1993) confirmed a deeper and richer understanding of the function concept results from an ability to operate flexibly in symbols and graphs.

Janvier (1987) described translation as a psychological process involving moving from one mode of representation to another. Translation is not a mere switch from one mode to the other, however (Lesh, 1987). Instead, it enjoys the characteristics of moving back and forth between representations-dual directional and understanding the correspondence between the representations. Recognizing the same function in different forms of representations is also one of the key components in translation processes (Leinhardt et al., 1990). Furthermore, Lesh et al. (1987) illustrated that an ability to sort out common properties of different representations while recognizing the irrelevant characteristics of specific representation entails an understanding of a given mathematics idea. In the study, students having difficulty translating a concept from one representation to another are the same students who have difficulty solving problems and understanding computations. Accordingly, the authors concluded that strengthening the ability to make translations among representations enhances students' understanding.

Modern technology-rich systems encourage actions or interpretations within and across graphical and symbolic representations; however, technology cannot provide flexibility of use. The users need to learn to benefit from the new technology (Zehavi, 2004). Cuoco et al. (1996) contend that problem solving requires more than knowing multiple representations, and the role of the interplay between different representational structures and visualizing relationships among multiple representations are critical for developing symbol sense. Without stressing the connections among representations, the representations are only shown as separate procedures, and the understanding of function concept will not be increased (Thompson, 1994). The importance of connections among multiple representations calls for teachers to treat functions as relationships among quantities in their algebraic, symbolic and graphical forms.

In their study of grade 6 students' pre-instructional use of equations to describe and represent problem situations, Swafford and Langrall (2000) stated that there are potential benefits in examining the same problem through different representations such as diagrams, graphs, tables, verbal descriptions, and equations. They further stated that the use of multiple representations in and of themselves is not enough. For example, in the case of tables, students often fail to see a pattern that was consistent across the table, even though they could identify isolated patterns between pairs of dependent and independent variables. Whatever the representation, students need to make the connections between one representation and another (Swafford & Langrall, 2000).

Dreyfus (1990) has recommended that the learning process needs to proceed through four stages: (1) using a single representation, (2) using more than one presentation, (3) making links between parallel representations, and (4) integrating and flexibly switching among representations.

Although functions in their multiple representation modes are important concept of teaching and learning mathematics, only a limited amount of studies have been conducted to investigate teachers' knowledge of functions , especially with a focus on multiple representation modes.

Research on Teachers’ Knowledge of Functions

The result of prior studies on teachers’ knowledge of functions suggests that functions are a complex domain that even teachers demonstrate their lack of understanding on them. In his study of preservice secondary teachers’ knowledge of function concept, Even (1993) investigated 152 preservice secondary teachers. These teachers were asked to complete an open-ended questionnaire concerning their knowledge about functions. Meanwhile, 10 additional preservice teachers were interviewed. The results suggest that may of the participants do not have a modern function concept. By examining the responses of the questionnaire and analyzing the interviews, the study found that preservice teachers’ conception of function were lacking important characteristics of function (i.e., univalence and arbitraryness). Similar results are uncovered in Stein et al.’s (1990) study of in-service teachers. An experienced fifth grade teacher was videotaped as he taught a lesson on functions and graphing. Furthermore, an interview on the teacher’s subject matter knowledge and card sort task was conducted. The results showed that the teacher missed several key mathematical ideas of functions. He is found lacking an organized and representational understanding of the function concept.

Likewise, studies uncovered teachers’ insufficient PCK on functions. For example, Stump (2001) examined preservice teachers’ pedagogical content knowledge of slope, one of the important concepts in functions. Participants were three preservice teachers who were in a secondary mathematics methods course and then taught basic algebra. Their written assignments, interview transcripts, and transcripts of the basic algebra lessons were collected and examined. The development of their knowledge of students' difficulties with slope and their knowledge of representations for teaching slope were investigated. These preservice teachers were found to have sensitivity to both conceptual and procedural aspects of students' knowledge of slope. They should be able to expand their repertoires of representations for teaching the slope; however, they demonstrated limited knowledge for developing the concept of slope in real-world situations.

Taken together, both preservice and in-service teachers have limited subject matter knowledge of functions as well as their PCK, which led to an increased attention to the relationship between teachers' subject matter knowledge and their PCK in functions. How do teachers' limited subject matter knowledge on functions affect their teaching approaches? Existing results showed there was a relationship between teachers' subject matter knowledge and their performance in teaching in the case of functions.

On one hand, the results from some studies have established a general assertion that subject matter knowledge is directly related to the PCK, that is, the more the subject matter knowledge, the better performance in teaching. Limited subject matter knowledge results in lower PCK. For example, Sanchez and Llinares (2003) investigated four student teachers' pedagogical reasoning on functions. Interviews were conducted to explore four teachers' subject matter knowledge for teaching functions. Meanwhile, data relevant for processes of transformation of the subject matter described through critical interpretation, repertoire of representational modes and adaptation to pupils' mathematics thinking were collected. Four student teachers in the study viewed the concept of function as a correspondence between sets but differed in their subject-matter knowledge for teaching both in the different aspects of concepts. Furthermore, four other students differed in the use of a representation repertoire to structure learning activities. All of this affected the use of graphical and algebraic modes in their planning of subject matter to be presented to students.

Even’s (1993) study suggests preservice teachers lack understanding in the key components of functions, and their classroom approach reflected the lack of key components as well. To give an example, one of the participants in the study did not include key ideas in his definition of a function that each input number must be mapped to a unique output number, while in his classroom instruction, he did not include the univalence nature of function. Even (1993) concluded that a powerful PCK based on meaningful subject matter can produce effective teaching.

On the other hand, some studies suggest that the relationship between subject matter knowledge and the PCK is far more complex. For example, Wilson (1994) describes how a preservice secondary teacher’s understanding of functions developed as she participated in a mathematics education course integrating mathematical content and pedagogy. Before the course, the teacher viewed functions as computational activities (e.g., function machines, point plotting, vertical line test, etc.). After the course, her understanding function grew substantially. Thus, the course influenced her understanding of function (subject-matter knowledge) but her anticipated approach to teaching was less significantly affected by it.

In summary, these studies provide a lens for understanding the role of subject matter knowledge of functions played in teacher learning (Cooney & Wilson, 1993) and point out further research is needed to understand the relationship between the components of teacher knowledge (Wilson, 1994) and ultimately how teachers' knowledge relate to students' learning and achievement.

In connecting teachers' knowledge and student learning, Cunningham (2005) did a study on teachers' performance on problems requiring transfer between algebraic, numeric and graphical representations. The study involved 28 algebra teachers, and they were surveyed to determine the amount of class time they used for different types of linear function problems that require the transfer between algebraic, tabular and graphical representations. For example, a problem asking students to graph the linear equation $3x + 2y = 12$ using the slope and y-intercept is to test the transfer from algebraic representation to graphical representation (N to G). The survey also asked how many times these problems appear on their teacher-made assessments. The results showed that teachers spend less class time on graphical to tabular transfer problems, and these problems appear less frequently on assessments. These are exactly the type of questions that students have the most difficulty with. The study concluded the less instructional time teachers dedicated to the problems with which students have the most difficulty may indicate teachers themselves were not familiar with those graphical to tabular transfer problems.

When studying teachers' knowledge of linear functions, researchers have proposed several theoretical perspectives on functions. One strand is focused on the dual nature of the function concept, that is, function can be viewed as a process or an object. For example, Breidenbach et al. (1992) proposed process and object view of describing functions:

> A process conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done….A function is conceived of as an object if it is possible to perform actions on it, in general actions that transform it. (p. 263)

Moschkovich et al. (1993) developed a two-dimensional framework for interpretation of linear functions. One dimension refers to the means of representing linear functions with a focus on the most common symbolic representations: algebraic, tabular and graphical. The other dimension refers to the perspective from which a linear function is envisioned, which is process-oriented and object-oriented vision of a linear function.

Schwarz and Dreyfus (1995) developed another framework related to functions. They stated that acquisition of the function concept is closely connected to actions on objects and the conservation of invariants (i.e., properties of functions) under actions (O’Callaghan, 1998). Action on objects is to manipulate, compare and transform the objects in the graphical, tabular and algebraic settings. The objects change their essence and thus become objects of a new kind which is called representatives. Thus, new objects, new actions, and new links among the objects and actions of different settings could be created, but the function properties are invariant under actions.

In his study, O’Callaghan (1998) examined the impact of the Computer-Intensive Algebra (CIA) and traditional algebra curricula on students’ understanding of the function concept. CIA is a function-oriented curriculum that focuses on a problem-solving approach based on the modeling of realistic situations, an emphasis on conceptual knowledge, and the extensive use. This curriculum focused less on symbol-manipulation skills and more on the conceptual understanding of the functional concept. He proposed a function model, which consists of modeling, interpreting, translating and reifying. In his study, modeling refers to the ability to represent a problem situation using functions. It is a process entailing a transition from a problem situation to a mathematical representation of that situation. Interpreting, according to O'Callaghan, it is a reverse procedure of the modeling process. Systems, symbols, tables and graphs are functions that have three core representations (Kaput, 1989). O'Callaghan (1988) defined translating as the ability to transfer from one representation of a function to another of three core representation systems. Reifying is the process of creating a mental object from procedures.

Kaput (1989) described four sources of mathematical meaning which are organized into two complementary categories. The first category is named *referential* extension, which refers to translations between mathematical representation systems and translations between mathematical representations and non-mathematical systems (including physical systems, as well as natural language, pictures, etc.). The second category, *consolidation*, refers to pattern and syntax learning through transformations within and operations on the notations of a particular representation system and building through the reification of actions, procedures and concepts.

Lesh et al. (1987) made a similar argument to define mathematical meaning:

> Part of what we mean when we say that a student "understands an idea like "1/3" is that (1) he or she can understand the idea embedded in a variety of qualitatively different representational systems, (2) he or she can flexibly manipulate the idea within given representational systems (i.e., perform translations), and (3) he or she can accurately translate the idea from one system to another. (p. 36)

Summary

The review of research on preservice and in-service teachers' knowledge of functions shows teachers have a limitation in their subject matter knowledge. In general, the results indicated teachers' limited subject matter knowledge is reflected in their teaching approach (i.e., their pedagogical content knowledge). In the literature, subject matter knowledge is found to be a critical component for influencing the teacher's understanding of students' conceptions and misconceptions. However, some results show that even if a teacher's understanding of function concept improved, their teaching approach still did not change substantively. Therefore, the relationship between different components of subject matter knowledge and pedagogical knowledge is far more complex than it appears; further research is needed in this area.

The review of research on teachers' knowledge of functions provides a context for the current dissertation study. Subject matter knowledge, as related to the special topic that is addressed in the study-linear functions, is organized into representation flexibility and procedure knowledge, which corresponds to the proposed "know that" and "know why" in the literature. Regarding the PCK, the study focuses on two aspects: (1) knowledge of students' conceptions and misconceptions and (2) the teaching strategies used in addressing students' misconceptions. The complex relationship between different components of teachers' knowledge necessitates the theme of this study.

Different theoretical frameworks the researchers have developed in their studies provide information on how teachers' knowledge on linear function is structured. Three major types of framework have been proposed for knowledge on linear functions. One differentiates the interpretation of linear functions by objective-orientated and process-orientated dimensions, the second linked the acquisition of the function concepts with actions on objects and the conservation of invariants and the last divide the competence of problem-solving of linear functions into four components: modeling, interpreting, translating and reifying. The theoretical framework of current study will combine and revise the frameworks of these previous studies.

Theoretical Framework

By reformulating and synthesizing the above two arguments in terms of linear functions, a three-dimensional framework was developed for representation flexibility presented in this study (see Table 2.1) and defined *Representation flexibility* as *transformations* within a representation system, *translations* across different representational systems, and *real-life situations* that are needed to use mathematical ideas in everyday situations.

Table 2.1
Framework of Representation Flexibility

Flexibility across formal mathematical symbolisms (within algebraic representation)	Flexibility with visual (graph or tabular) representations		Flexibility with real-life situations
	Flexibility within visual representations	Flexibility among algebraic and visual representations	
• Transformations between standard form and others algebraic equations • Transformations between standard form and algebraic fractional forms • Different expressions for slopes	• Local property transformations • Global property transformations	• Cartesian connection • Entity-oriented connections	• Situation -> Equation • Equation -> Situation

Thompson (1985) argued that the goal of education is to develop intelligence through problem solving. He thought that to account for the building and expressing of mathematical meaning one has to learn how to create relationships through problem solving. Since functions can describe the relationships among differing quantities, learning and solving them are essential for constructing mathematics meaning. Therefore, the framework suggested here uses a problem solving environment that looks at how to solve functional problems in multiple representational contexts.

Flexibility across Formal Mathematical Symbolisms

There are several forms to express linear functions. The standard form ax + by + c = 0 is the most common one that is addressed in the mathematics curriculum. Other than the standard form, there are slope-intercept form y = mx + b or y = - (a/b)x - c/b, one-point form $(y - y_1) = m(x - x_1)$ and two-point form (y-y1)(x2-x1) = (x-x1)(y2-y1). Some forms are easier to use than others based on specific problem situations. For example, a problem asks which equation represents the line that is parallel to y = 2x - 5 and that goes through the point (1, 4). To solve this problem, one-point form would be a better choice since both the slope of the line and a point on the line are known. Students need to be able to understand the similarities and differences of these forms. As Lesh (1987) stated, “Good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representational systems that they instinctively switch to the most convenient representation to emphasize at any given point in the solution process” (p. 38). As mentioned above, two-point form can be expressed as (y-y1)(x2-x1) = (x-x1)(y2-y1). Besides this form, there are other ways to represent it, such as, $(y - y_1) / (y_2 -$

y1) = $(x - x_1) / (x_2 - x_1)$ and $(y - y_1) / (x - x_1) = (y_2 - y_1) / (x_2 - x_1)$. These were organized in a separate category named as Transformations of Standard form to Algebraic Fraction Form for the reason that they combine fraction form and algebraic form which could be more difficult for students to understand. Different expressions for slopes are also categorized in Flexibility across Formal Mathematics Symbolisms because slope is an important concept in function and it can be represented in many symbolic forms. The most common form is expressed as m in y = mx + b. Meanwhile, it can also be represented by any two points on the line, such as, $(y_2-y_1) / (x_2 - x_1)$ or $(y_1 - y_2)/(x_1 - x_2)$.

Flexibility with Visual Representations

The most commonly used forms, symbols, tables or ordered pairs and graphs, are called the three core representation systems in functions (Kaput, 1989). In this study, both tables and graphs are defined as visual representations. Accordingly, Flexibility with Visual Representations is organized into two categories: (1) flexibility within visual representations and (2) flexibility between algebraic and visual representations. Leinhardt et al. (1990) described two features when using a specific graph representing a function situation: a local process (e.g., one regarding point-by-point attention) and a more global one (e.g., trend direction). Based on these two features, Flexibility within Visual Representation was subcategorized into local property transformations, that is, ordered pairs and graph transformations, such as plotting a number of points and finding the properties of certain points (ordered pairs), and global property transformations, which is also about ordered pairs and graph transformations but more of trend direction, such as intervals of increase or decrease, intervals of extreme increase or decrease.

Flexibility between Algebraic and Visual Representations

Flexibility between Algebraic and Visual representations is based on a framework proposed by Moschkovich et al. (1993). Their framework has two dimensions: (1) the representational forms for functions (i.e., algebraic, graphical and tabular forms) and (2) the perspective from which a linear function is viewed and operated (i.e., process perspective and object perspective). “From the process perspective, a function is perceived of as linking x and y values: For each value of x, the function has a corresponding y value” (p. 71). The crucial information contained in the process perspective is in the statement of a *Cartesian connection* which is “a point is on the graph of the line L if and only if its coordinates satisfy the equation of L” (p. 73). In other words, when a graph goes through a point, then coordinates of that particular point satisfy the equation of that graph. In comparison, the object perspective refers to when “a function or relation and any of its representations are thought of as entities—for example, algebraically as members of parameterized classes, or in the plane as graphs that, in colloquial language, are thought of as being ‘picked up whole’ and rotated or translated” (p. 71). Based on Moschkovich et al.’s (1993) framework, two aspects of Flexibility between Algebraic and Visual representations are organized: (1) Cartesian connection and (2) Entity-oriented connections, which involve recognizing relevant properties of algebraic and visual representation, and making connections among them when treating functions as an entity. For example, the size of the slope m in the equation of $y = mx + b$ relates to the steepness of the graph, and whether the slope m is negative or positive decides the direction of the graph.

Flexibility with Real-Life Situations

As Selden and Selden (1992) noted, modeling real-world situation is one of the most common and important uses of functions. Sierpinska (1992) described modeling real life situations using functions as a sine qua non condition for understanding functions. Meanwhile, using real-life situations to interpret functions in its multiple representations is an essential factor to reach a conceptual understanding of functions (Monk, 2003). O’Callaghan (1998) in his function model describes modeling and interpreting as two components that relates functions to real life situations. He defines modeling as a transition from a problem situation to a mathematical functional representation of that situation, and interpreting as the reverse procedure of modeling (i.e., to interpret functions in their different representations [tabular, graphical, and algebraic forms]) in terms of real-life applications. Since both interpreting and modeling are to construct a connection between functions and real-life in the current study, this category is named Flexibility with real-life situations, and consists of two characteristics: (1) ability to interpret several different kinds of everyday problem situations that are similar to the function model and (2) ability to transfer from word problem situations to various forms of functions.

In summary, understanding connections between and among different representations of functions is a strong indicator for conceptual understanding. According to Simon (1993), a good teacher needs not only a concrete, contextualized and procedural knowledge, but also an abstract and conceptual understanding of mathematics. The conceptual framework, a synthesis of the researcher’s ideas with those of other mathematics educators discussed above, provides a basis for investigating not only teachers' procedural skills but also more importantly their conceptual understanding of linear function, that is, representation flexibility and how their understanding of mathematics content impacts their knowledge of students' thinking and misconceptions of linear functions.

Procedure Skills

Since procedural knowledge is one component of teachers' subject matter knowledge, it is also included as part of the theoretical framework in addition to representation flexibility in multiple forms of functions. The procedural knowledge consists of manipulating symbols and performing algebra procedures, for example, to solve $(4-x) = 1/2(3x-1)$.

CHAPTER III

METHODOLOGY

The study implemented both quantitative and qualitative measures and analyses to investigate preservice teachers' knowledge of linear functions within multiple representation modes. The first section of this chapter describes the pilot study, which was conducted to evaluate the test instrument. The details of methodology for the current study are then reported. The current study was expanded from the pilot study to provide a more comprehensive understanding of preservice teachers' knowledge of linear functions.

Pilot Study

In order to assess preservice teachers' representation flexibility and procedural skills in linear functions, a test instrument named Test A-Pilot (Appendix B) was developed containing 27 questions. Each set of three questions was designed to assess one of the components of the theoretical framework proposed in the Chapter II (see Table 2.1). Some of the questions in Test A-Pilot were adapted from Brenner et al. (1997), Conference Board of the Mathematical Sciences [CBMS] (2001); Knuth (2000), Leinhardt et al. (1990), and O' Callaghan (1998), and others were designed by the researcher. These problems were chosen because they contain different modes of representations of linear functions, and involve the transfer between varied representations. Another instrument Test B-Pilot (Appendix C) was designed to assess preservice teachers' pedagogical content knowledge, in particular knowledge of subject matter, knowledge of students' conceptions and misconceptions and instructional approaches to students' misconceptions. Test B-Pilot contained 11 items; each consisting of a set of three questions that corresponded to the three categories based on the theoretical framework from Chapter II (see Table 2.1).

The pilot study was conducted with students (n=38) from an elementary mathematics methods course. The pilot study served two purposes: (1) to determine if Test A-Pilot and Test B-Pilot were sufficiently valid and reliable to provide information to answer the theoretical research questions of the study and (2) to determine if the items were worded appropriately so that sensible and valid responses were obtained, and the length of the test was reasonable. For Test A-Pilot, the pilot study showed that except for the first question, the rest of the questions were feasible for capturing students' flexibility in mathematics symbolism, visual representation and real-life situations. Question 1 was intended to test the flexibility across mathematical symbolism, namely transformations between standard form and other algebraic equations, transformations between standard form algebraic fractional forms, as well as different expressions for slopes. However, the pilot study indicated that subjects transformed all forms of linear functions to slope-intercept form to get the answer. For example, the original intention of question 1 was to see if a student would be able to find any two distinct points from the problem given and write out a two-point form, while the pilot study uncovered that subjects computed two point form: $(y + 2)(x - 2) = (x + 2)(y - 6)$ to get the slope-intercept form: $y = 2x + 2$ to decide whether $(y + 2)(x - 2) = (x + 2)(y - 6)$ is a correct answer to the problem. Likewise, subjects who knew that m in the slope-intercept y =

mx + b stands for slope, computed (4-2) / (1-0) to see if it was equal to 2 to decide if it was correct. The original intention was to see if participants were able to see points (1, 4) and (0, 2) and use these two points to find the slope. Thus, question 1 in the pilot study was changed to ensure it assessed the preservice teachers' representation flexibility across mathematical symbolisms (Appendix B, Test A), by forcing students to use various mathematical forms to represent linear functions.

The pilot study results indicated that the items in Test B were suitable for capturing preservice teachers' pedagogical content knowledge in linear functions, and the sample size for quantitative data was sufficient to determine preservice teachers' pedagogical content knowledge, their use of instructional representation, and the relationship between subject matter knowledge and pedagogical content knowledge. The open-ended questions in Test B also offered substantive information that could be analyzed qualitatively in order to further investigate teachers' knowledge. At the same time, interviews were added to the current study.

With respect to the second purpose of the pilot study, the items in Test A-Pilot and Test B-Pilot were well understood by preservice teachers who took the test. The test items were valid in terms of language sensibility. There was sufficient time for everyone to complete the items appropriately.

Current Study

Participants

This study included preservice mathematics teachers who were in their last stages of study in teacher education programs. They had already taken all of the required mathematics courses and were completing courses related to methodology of teaching mathematics. A majority of the participants were seniors and a few were juniors. Specifically, a total of 104 preservice teachers from Texas A&M University enrolled in the elementary, and middle school degree programs in Fall 2006 participated in the study. These preservice teachers were enrolled in the following courses: ECFB 440, Mathematics Methods in the Elementary Grades, MASC351, Problem Solving in Mathematics, MASC450 Integrated Mathematics, and MEFB460 Mathematics Methods in the Middle Grades (Table 3.1). Catalogue descriptions of the courses are in the appendices (see Appendix G). ECFB 440 is a teaching methods course designed for elementary school preservice teachers. The other courses are designed for a middle school certification program. MASC351 is a problem solving course while MASC450 is an integrated mathematics course. MEFB 460 is a teaching methods course for teachers of elementary grades. Because MASC 351, MASC 450 and MEFB 460 are all targeted for middle school mathematics preservice teachers and serve a goal to equip preservice teachers with effective teaching methods that could help students learn, they were collapsed into one cohort while students from ECFB is another cohort.

Table 3.1
Spring 2006 Enrollment

K-12 Level	Courses and Instructors	Total Enrollment
Elementary Certification Program	**ECFB 440**: Elementary Mathematics Method Course Goldsby, D., Capraro M.M., Raulerson T. & Pavliska A.	150
Middle School Certification Program	**MASC 351**: Problem Solving in Mathematics Capraro, M.M. & Kulm, G.	28
	MASC 450: Integrated Mathematics Goldsby, D. & Capraro, R.	56
	MEFB 460: Methods of Teaching Middle Grades Mathematics Ezrailson, C. & Slough, S.	36

Most of the preservice teachers, both from elementary and middle school certification programs, have taken two mathematics prerequisites, MATH 141 and STAT 303, some mathematics specialized courses, MATH 365 and MATH 366, and some other mathematics courses such as MATH 142, MATH 367, MATH 368 and MATH 403. In MATH 141 and MATH142, students learn linear equations and applications, linear forms and systems of linear equation, matrix algebra and applications, linear programming, probability and applications, statistics, derivatives, curve sketching and optimization, techniques of derivatives, logarithms and exponential functions with applications, integrals, techniques and applications of integrals, multivariate calculus. MATH 365 covers a range of topics, such as sets, relations, functions, whole numbers, numeration systems etc. Chapter IV provides a summary of the mean number of mathematics courses completed and their reported grade point averaged for each of the cohorts.

The forms for the use of human subjects were filed and approved by the Texas A&M University Institutional Review Board (IRB). Since the study did not involve participants under the age of 18, the expedited procedure was used by the IRB. The consent form (Appendix A) included the purpose of the study, no negative or positive effect for participating in the study, the time devoted to the study confidentiality; contact information of the researcher, and contact information of the participant (Appendix C).

Instrumentation

Tests. Some items of Tests A and B are adopted from Brenner et al. (1997), CBMS, Knuth (2000), Leinhardt et al. (1990), and O'Callaghan (1998). The rest of the items were designed by the author. Test A was designed to access preservice teachers' subject matter knowledge with an emphasis on representation flexibility and procedural skills and consisted of four parts. Part I was to assess preservice teachers' flexibility across mathematical symbols. Part II focused on testing flexibility with visual representations. Part III was designed to test flexibility between visual representations. Part IV deals with preservice teachers' flexibility with real life situations. The inclusion of Part V was intended to access preservice teachers' procedural skills. In comparison, Test B consisted of 11 open-ended questions and each one accesses two components of preservice teachers' pedagogical content knowledge, i.e., their understanding of students' thinking and misconceptions in linear functions. Test B is also divided into five parts corresponding to the five parts of Test A but in terms of teaching, namely, flexibility across mathematical symbols, flexibility with visual representations, flexibility between visual representations, flexibility with modeling process and procedural skills in teaching.

Survey

The survey was designed to obtain background information of the participants. There were three parts of the survey: (1) the mathematics courses taken, (2) mathematics education courses taken and (3) the Overall GPA and mathematics GPA of the participants. The participants were asked to identify their gender as well.

Procedure

Data for this study were collected in two phases. In the first phase, professors in Texas A&M University teaching elementary and middle school mathematics education courses were contacted to discuss their willingness to devote one hour of their class to administering or distributing tests to their students enrolled in their courses. The packages contained an introduction, consent forms, Test A (Appendix B), Test B (Appendix C), and a background survey. The introduction provided details concerning the purpose of the study, time devoted to the study, confidentiality and contact information of the research. Test A and Test B were developed from the revision of Test A-Pilot and Test B-Pilot. Test A was designed to assess preservice teachers' content knowledge, in particular, representation flexibility and procedure skills. Test B was to access preservice teachers' understanding of students' thinking and misconceptions in linear functions. The background survey (Appendix F) queried participants about demographic and academic information regarding gender, grade point average, mathematics courses taken and mathematics education courses taken. The participants were notified that both tests and survey were for research purposes. After the tests, the preservice teachers were requested to fill out the background survey.

In the phase two, after completing the tests, participants were notified via email, phone or face-to-face about the interview. Six participants were selected based on their performance on the Test A and Test B. Approximately one week after the tests were collected; interviews were conducted in a designated room at Texas A&M University. Designation of times and meeting locations follow regulations of Texas A&M University. Each interview lasted about 45 minutes and was audio taped. The purpose of the interview was to determine preservice teachers' thinking behind their initial responses to Test A and Test B, clarify and examine their representation flexibility for some selected questions, and further explore the relationship between their subject matter knowledge of linear relationships and their pedagogical content knowledge.

In order to maintain consistency among interviewees and gain as much information as possible about the connections between the preservice teachers' subject matter knowledge and their pedagogical content knowledge in the case of linear functions, a semi-structured task-based interview process (Goldin, 2000) was implemented. An interview protocol (Appendix E) was designed to guide the interview. Meanwhile, the protocol allowed some deviation in response to some particular participant answers. Following the protocol, preservice teachers were asked to review their answers to some questions selected from Test A and Test B, describe their solutions to the selected questions, elaborate their thoughts corresponding to the responses, model their teaching approaches on certain problems, and solve related additional problems.

Data Collection

Data were collected from background surveys, Test A, Test B and interviews. The primary source was from Test A and Test B. A total of 104 elementary and middle school preservice teachers served as subjects for data collection and analysis. First, Test A and Test B were graded based on empirically and theoretically designed rubrics to provide numerical data for quantitative analysis. Secondly, answers to certain questions of Test A and the whole Test B were categorized into different groups for qualitative analysis. Three elementary preservice teachers and three middle school preservice teachers were interviewed in May, 2006. All six interviews were audio taped and transcribed to analyze teachers' knowledge of linear functions.

Data Analysis

Both quantitative and qualitative methodologies were employed to answer research questions proposed in Chapter I. Because Test A contained objective questions that had numerical results, quantitative methods were applied to analyze preservice teacher's knowledge structure and their performance in different components of representation flexibility as well as in procedural skills. Meanwhile, the problem solving process in Test A was also analyzed qualitatively in order to provide a more a complete picture of preservice teachers' representation flexibility and procedural skills. Test B consisted of 11 open-ended questions, and the responses were coded based on the rubrics; therefore, quantitative analysis was conducted to see how well preservice teachers perform in PCK items. Qualitative analysis was conducted to understand the nature and structure of preservice teachers' response to Test B. Data was described by preservice teachers' PCK in term of four categories, namely (1) explanations of students' misconceptions, (2) instructional strategies, (3) the use of instructional representations and (4) errors made. Furthermore, the relationship between teachers' subject matter knowledge and PCK was investigated both quantitatively and qualitatively.

Coding Scheme

Since the current study intended to investigate preservice teachers' knowledge of functions within multiple representation modes from both quantitative and qualitative data, it included two systems of coding scheme respectively for qualitative and quantitative analysis. For quantitative analysis, the coding of Test A was based on the correctness of answers. Test B was used to assess preservice teacher's PCK. The grading of Test B was based on a rubric, which had corresponded to the numeric score 0 to 2. Qualitative analysis started by reviewing answers of about 15 participants. This analysis resulted in preliminary categories of responses for each question and each subquestion in Test A and Test B. The opinions of experts in mathematics education were adopted in the process of revising the preliminary categories. The final categories were created by collapsing similar categories and adding important new categories based on preliminary categories.

Grading Tests for Quantitative Analysis

For Test A, participants received 1 point for each item with a correct answer and 0 for an incorrect answer. Since there were four parts in Test A, a subtotal score was given for each part of the test. These four scores corresponded to each preservice teacher's mastery of knowledge that was assessed in each part. The total score for Test A was used as an indication of the preservice teacher's overall mastery of linear functions.

In Test B, in order to evaluate preservice teachers' understanding of students' thinking and misconceptions, each open-ended question consisted of three sub-questions: (1) to judge the correctness of students' response (2) to identify students' thinking based on their response, and (3) to explain the strategies that can assist students. For the first sub-question, preservice teachers received 1 point for a correct judgment of correctness of a student's response and 0 point for an incorrect judgment. For the second and third sub-questions, the solutions were coded based on a rubric, and the numerical scale (0-2) was given. The rubric for each sub-question b is as follows:

0 No response, completely incorrect, irrelevant or incoherent.

1 The response provides a partial or complete understanding of students' conceptions and misconceptions and exhibits some understanding of the sources of students' misconceptions.

2 The response provides an accurate and complete description of students' conceptions and misconceptions. It demonstrates a deep and conceptual knowledge of the sources of students' misconceptions.

In sum, the score for the second sub-question may range from 0 to 2. Likewise, the grading was conducted in a similar manner for the third sub-question: "If you think the student has misconceptions with respect to the problem, how would you assist this student?" Similarly, the possible score for the third sub-question is from 0 to 2. Rubric for the third sub-question was as follows:

0 No response, completely incorrect, irrelevant or incoherent.

1 The response provides a partial or complete description of strategies for addressing students' misconceptions. However, the strategies reveal factual or procedural nature, and entail some conceptual nature.

2 The response provides a complete description of strategies for addressing students' misconceptions. Furthermore, the response entails accurate and complete conceptual strategies.

Taken together, a total possible score of each question in Test B can be from 0 to 5. For each teacher, the total score on Test B indicates his/her knowledge of subject matter, knowledge of students' conceptions and misconceptions, and their instructional approach in response to students' misconceptions in linear functions.

For the grading of Test B, one rater used rubrics (Appendix E and F) to score all the tests collected. Two weeks later, the rater randomly selected 10 tests, and rescored them again. The intra-rater agreement is 99% of all corresponding tests. A second rater also used the rubrics to grade twenty randomly selected tests independently. The inter-rater agreement was 92% of all corresponding tests. The differences were resolved after discussions.

Coding Structure for PCK

After determining the numeric scores for Test A and Test B, the responses in three sub-questions of each question in Test B were segmented and coded as a whole to view different categories of teachers' PCK, namely teachers' explanations of students' misconceptions, instructional strategies, and the use of instructional representations. The response to each question in Test B from all preservice teachers was classified into the following four categories.

First, preservice teachers' explanation for students' misconceptions. It refers to the verbal statement presented by preservice teachers in response to students' misconceptions, such as concepts, principles and procedures. The explanations in this category do not include those that state an instructional strategy, and it was assessed by subquestion b in Test B. It includes the judgment of students' misconceptions, identification of students' misconceptions and awareness of the sources of students' misconceptions. It was organized into five categories as follows:

I The response provides a partial description of students' conceptions and misconceptions but exhibit a lack of essential understanding of the sources of students' misconceptions.

II The response provides a complete description of students' conceptions and misconceptions but exhibit a lack of essential understanding of the sources of students' misconceptions.

III The response demonstrates partial understanding of students' conceptions and misconceptions and exhibits some understanding of the sources of students' misconceptions.

IV The response demonstrates partial complete understanding of students' conceptions and misconceptions and exhibits some understanding of the sources of students' misconceptions.

V The response provides an accurate and complete description of students' conceptions and misconceptions. It demonstrates a deep and conceptual knowledge of the sources of students' misconceptions.

Second, instructional strategy used. Instructional strategy in this study refers to the strategy that preservice teachers implemented to help students and it was assessed in subquestion c of Test B. The teaching strategies serve the instructional purposes. On one hand, teachers can use an instructional strategy that focused on conceptual understanding by emphasizing the concept and definitions of functions and slopes, integrating various representations of concepts, interrelating functions and equations to real life situations to represent abstract concepts, and addressing the relationship between different components of concepts. On the other hand, teachers could perform routine procedures, for example, perform routine computations and demonstrate the procedures of how to reach an answer. Thus, a scheme of instructional strategy used by preservice teachers in this study was developed: conceptual instructional strategy and procedural instructional strategy. Teaching strategies were codes in five categories as follows:

I The response provides a partial description of strategies for addressing students' misconceptions. However, the strategies provided reveal factual or procedural nature.

II The response provides a complete description of strategies for addressing students' misconceptions. However, the strategies provided reveal factual or procedural nature.

III The response provides a partial description of strategies for addressing students’ misconceptions. Furthermore, the strategies provided entail some conceptual nature.

IV The response provides a complete description of strategies for addressing students’ misconceptions. Furthermore, the strategies provided entail some conceptual nature.

V The response provides a complete description of strategies for addressing students’ misconceptions. Furthermore, the response entails accurate and complete conceptual strategies.

Third, use of instructional representations. Teaching strategies were further coded based on subjects’ different instructional representations they would use for teaching strategies. The instructional representations were organized into four categories: (1) visual-based instructional representation, (2) algebraic-based instructional representation, (3) verbal statement, and (4) activities. For example, for question 10 in Test B using two points to obtain slope, some participants stated in their teaching strategies that they would just tell students what the formula was. This could be categorized as verbal statement.

Two raters were involved in the categorization of participants’ PCK. Field experts were also consulted in the creation of categorization. One rater categorized each question in Test B. The intra-rater agreement coded for a period of two weeks was 99%. A second rater also coded independently ten tests. The inter-rater agreement of all correspondence code is 95%. The differences were resolved through discussions. Finally, the data were entered into statistical software SPSS, and the frequency of different mode of representations for each teacher across different items and each item across different teachers were calculated. Accuracy was ensured by checking a random sample of data.

Data Analysis Procedure

The following part of this chapter provides detailed information of the specific analyses employed to address each research question.

Research Question 1

How do preservice teachers demonstrate their subject matter content knowledge (CK) in terms of representation flexibility and procedural skills? Do preservice teachers' subject matter knowledge vary for those in elementary and middle grades teacher education?

Descriptive statistics were employed on data obtained from all four sections of Test A to show the preservice teachers' performance on three categories of representation flexibility, namely, (1) representation flexibility across mathematics symbols, (2) representation flexibility with visual representations and representation flexibility with real-life situations, and (3) procedure skills. The mean and standard deviation for each item across different teachers were reported, as well as for each teacher across different items. The frequencies of incorrect answers provided by preservice teacher on sub-items of each questions were also included. The frequencies were reported across different levels of teacher education as well as for different subcategories of the five constructs of Test A. Only the first four parts of Test A were

organized into subcategories. In Part I (flexibility across formal mathematical symbolisms), the percentages of incorrect answers were reported across three sub-areas:
(1) transformations between the standard form and others algebraic equations
(2) transformations between standard form and algebraic fractional forms, and
(3) different expressions for slopes. In Part II (flexibility with visual representations), statistical tests were examined to see how well preservice teachers performed in items that assessed flexibility within visual representations and flexibility among equations and visual representations. Part III contained subcategories of flexibility from problem situations to algebraic equation and flexibility from algebraic equations to problem situations. Part IV further categorized teachers' flexibilities to transform real life situations to mathematical representations and to transform mathematical representations to real life situations.

Multiple Analysis of Variance (MANOVA) and discriminant analysis were conducted to analyze preservice teachers' performance on their subject matter knowledge of linear functions across their different K-12 levels of teacher education. A predetermined design of contrast was used for the MANOVA analysis.

A confirmatory factor analysis applying structural equation modeling (SEM) technique was conducted to establish the structure of different components of preservice teachers' content knowledge in linear functions, as well as to verify the validity and reliability of Test A. Split-half reliability test using SPSS was used to check the reliability of Test A.

A regression analysis using preservice teachers' Grade Point Average on general college courses and mathematics courses to predict their CK was conducted using SPSS simple regression model.

To answer items in Test A, the participants provided not only the answers but also the problem-solving process. Therefore, similarities in responses were identified through constant comparison. The answers were categorized for possible patterns and themes.

Research Question 2

How do preservice teachers demonstrate their pedagogical content knowledge in terms of the use of instructional representations, their knowledge of students' conceptions and misconceptions and their cognitive behavior for approaching students' misconceptions? Does preservice teachers' pedagogical content knowledge vary for those in elementary and middle grades teacher education?

Descriptive statistics were reported to show the preservice teachers' performance in PCK, namely their knowledge of subject matter, their knowledge of students' conceptions and misconceptions, and their instructional approaches in response to students' misconceptions. The scores was analyzed across the three categories mentioned above. The percentage of incorrectness of preservice teachers' judgment on students' answers to the questions were reported, while the mean score and standard deviation for preservice teachers' understanding of students' conceptions and misconceptions, as well as their instructional approaches were included.

MANOVA and discriminant analysis were conducted to analyze preservice teachers' performance on PCK of linear functions in their different K-12 levels of teacher education.

Similar to Test A, a confirmatory factor analysis using structural equation modeling (SEM) technique was conducted to establish the structure of different components of preservice teachers' content knowledge in linear functions, as well as to verify the validity of Test B. Split-half reliability analysis using SPSS was used to test the reliability of Test A.

A regression analysis using preservice teachers' Grade Point Average on general college courses and mathematics courses to predict their PCK was conducted using SPSS simple regression model.

The open-ended questions in Test B were analyzed qualitatively as well. The solutions were outlined, summarized and evaluated in different categories to determine possible patterns and themes.

The interviews were transcribed, and the analysis began by listening to the taped interview and editing the transcriptions. The interviews were then analyzed by person, by grade level and by theme in the following manner. First, each person's answer to an interview question was summarized and important comments were recorded. Then, responses to each interview question were compared across the two grade levels, elementary preservice teachers and middle school preservice teachers. Additional attention was given to several PCK themes, such as teachers' explanations of students' misconceptions, the various instructional strategies, and the use of instructional representations.

Research Question 3

How does preservice teachers' subject matter knowledge influence their knowledge of students' conceptions and misconceptions of linear functions and their instructional approaches for addressing students' misconceptions?

Another SEM correlation model was used to test the relationship between preservice teachers' subject matter knowledge and pedagogical content knowledge in linear functions within multiple representation modes. The correlation of five sub-constructs in Test A with corresponding sub-constructs in Test B was also investigated. The objective is to find out how preservice teachers' subject matter knowledge and pedagogical content knowledge are related in terms of flexibility across mathematical symbols, flexibility within visual representations, flexibility between visual representations, flexibility with modeling process and procedural skills.

The interview transcriptions were analyzed by person together with their performance in both Test A and Test B. This analysis provided a summarized qualitative description of the nature of the relationship between the subject matter knowledge and the PCK being studied.

Validity and Reliability

The current dissertation study employed a mixed method approach to analyze data, i.e., both quantitative and qualitative approaches. For the quantitative approach, the validity of Test A and Test B were obtained from the analysis of SEM, and the split-half reliability of both tests was also computed and reported.

Qualitative methodology, emphasizes "…the study of people's understandings" (Bogdan & Biklen, 1992, p. ix). To ensure the validity (true value) of the naturalistic research design, triangulation (Rossman & Rallis, 1998) was applied in the study. To triangulate is to ensure that data are drawn from several sources to strengthen the robustness of the work. The current study includes background surveys, tests and transcriptions of audio recordings. In addition, triangulation was ensured by incorporating opinions by field experts, such as committee chairs, members and other researchers (see acknowledgements) to evaluate rubrics, grading processes, analysis framework, interpretations and implications for accuracy and completeness.

Reliability, in qualitative research refers to whether the same findings can be found when the study is replicated with similar participants and circumstances. To ensure the reliability, this study documented the process of collecting, analyzing and interpreting the data. Furthermore, the study used multiple approaches to gather data to ensure the completeness and complexity.

CHAPTER IV

RESULTS

The report of the results of the current study consists of two sections. In the first section of this chapter, descriptive statistics of preservice teachers' academic achievement distribution offer an overall introduction to the subjects of the study. Section two gives the results of quantitative and qualitative analysis of preservice teachers' performance on Test A and Test B. Quantitative analysis using multiple statistical techniques including descriptive statistics, MANOVA and Structural Equation Modeling were use to investigate preservice teachers' levels of Content Knowledge (CK) and Pedagogical Content Knowledge (PCK), the relationship between CK and PCK, as well as the reliability and validity of the tests. Further, qualitative analyses provide detailed information about preservice teachers' subject matter content knowledge and pedagogical content knowledge of linear functions within multiple representation modes as well as the relationship between these two types of knowledge.

Descriptive Information on Subjects

The survey completed by the preservice teachers provided basic information on their academic background. Two important factors of their academic experience, overall college Grade Point Averages and mathematics Grade Point Averages, were analyzed using descriptive analysis. As indicated in Table 4.1, the average college GPA of the subjects from elementary mathematics teacher certification programs was 3.31, ranging from 2.80 to 4.00. The mathematics GPA of this group of subjects was 2.89 with a range from 2.00 to 4.00. For those subjects from middle school certification program, the average college GPA is 3.38, from minimum 2.75 to maximum 3.98. The subjects from both programs had similar college GPA. However, the mathematics GPAs of subjects from middle school certification program was 3.31, higher than its counterpart group.

Table 4.1
Mathematics Courses, Grade Point Average, and Mathematics Grand Point Averages

Number of College Mathematics Courses	Overall College Courses GPA	College Mathematics Courses GPA
Elementary School Certification Program		
Mean	3.31	2.89
Range	2.80-4.00	2.00-4.00
Middle School Certification Program		
Mean	3.38	3.31
Range	2.75-3.98	2.75-4.00

Note. GPA = General Point Average in 4.0 scale.

Quantitative and Qualitative Analyses

The study focused on three aspects of preservice teachers’ subject matter knowledge: their representation flexibility and procedural knowledge, pedagogical content knowledge, that is, knowledge of students’ conceptions and misconceptions and their strategies for helping students with misconceptions, and the relationship between the aspects of subject matter knowledge and pedagogical content knowledge being studied. The report of both quantitative and qualitative analysis is organized based on the three research questions of this study. The report of qualitative analysis of preservice teachers’ CK follows the five knowledge structure of Test A, each representing a representation mode described in Chapters II and III. Detailed information about preservice teachers' PCK is organized according to the qualitative categories of the answers in order to examine the subjects' level of understanding of students' misconceptions and use of proper strategies for instruction.

Results for Research Question 1

Research Question 1

How do preservice teachers demonstrate their subject matter content knowledge (CK) in terms of representation flexibility and procedural skills? Does preservice teachers' subject matter knowledge vary for those in elementary and middle grades teacher education?

Preservice teachers' performance on Test A, i.e., their knowledge in subject matter content was analyzed using descriptive statistics and multivariate analysis. Data was analyzed across subjects' level of teacher education and different subconstructs and subitems of Test A. Confirmatory factory analyses using Structural Equation Modeling (SEM) technique were conducted determine the validity of the test and importance of each constructs in testing CK.

Quantitative Analysis

Overview of Preservice Teachers' CK

Descriptive statistics for each question and sub-items of Test A were generated to explore preservice teachers' level of subject matter content knowledge and their performance on each sub-item of the test. Table 4.2 shows descriptive statistics for each question in Test A. Except for AI1, middle school preservice teachers had higher mean score and lower standard deviation than elementary school preservice teachers. Descriptive statistics shown in Table 4.3 for each subconstruct of Test A also suggests that middle school preservice teachers had a better knowledge across all five parts of CK than elementary preservice teachers. Based on the information from descriptive statistics of Test A, a multivariate analysis along with a discriminant analysis was conducted to determine if preservice teachers from different levels had different levels of knowledge in the five aspects of CK of linear function.

The frequency of correct responses of preservice teachers from both levels of teacher education on each sub-item of Test A is reported in Tables 4.4 and 4.5. The results show that elementary school education preservice teachers performed poorly on the following 11 items, with more than 30% incorrect responses: 2), 3), 4), 5), 7), 8), 15), 16), 17), 21), 22), especially for item 21) on which only 10 of 58 preservice teacher gave the correct answer. Middle school education preservice teachers performed better than elementary school education preservice teachers, yet more than 30% of them made mistakes on items 7), 16), 17), 21), and 22). Their percentage of incorrect responses on item (21) was similar to elementary preservice teachers. When examining the items of Test A, it is obvious that the items of high frequency of correct responses belong to flexibility across mathematical symbolisms (FAMS), flexibility between visual representations and algebraic representations (FBVRAR) and flexibility with real-life situations (FWRS).

Table 4.2
Descriptive Statistics for Items of Testing Preservice Teachers' CK

		Mean	Minimum	Maximum	Std.
FAMS					
AI1	E	2.05	0	3	1.07
	M	2.80	1	3	.542
AI2	E	1.16	0	2	.951
	M	1.83	0	2	.569
AI3	E	2.02	0	3	1.12
	M	2.28	0	3	1.05
FWVR					
AII1	E	1.69	0	2	.537
	M	1.83	1	2	.383
AII2	E	3.71	2	4	.562
	M	3.91	3	4	.285
AII3	E	.345	0	1	.479
	M	.717	0	1	.455
FBVRAR					
AIII1	E	1.00	0	2	.637
	M	1.00	0	2	.758
AIII2	E	.603	0	1	.493
	M	.935	0	1	.249
AIII3	E	1.43	0	2	.728
	M	1.61	0	2	.493
FWRS					
AIV1	E	0.638	0	2	.765
	M	.743	0	2	.743
AIV2	E	1.138	0	2	.963
	M	.978	0	2	.977
AIV3	E	2.17	0	3	1.03
	M	2.33	0	3	.790
PS					
AV2	E	.845	0	1	.365
	M	.978	0	1	.147
AV4	E	.810	0	1	.395
	M	.957	0	1	.206
AV5	E	.707	0	1	.459
	M	.978	0	1	.147

(Note: AI1-AV5 represents questions in each part of test A; E standards for subjects in elementary certification program while M standards for those in middle school certification program)

Table 4.3
Descriptive Statistics for Factors of Testing Preservice Teachers' CK

		Mean	Minimum	Maximum	Std.
FAMS	E	5.22	0	8	2.70
	M	6.91	1	8	1.79
FWVR	E	5.74	3	7	.965
	M	6.46	5	7	.687
FBVRAR	E	2.77	0	5	1.23
	M	3.54	2	5	.912
FWRS	E	3.94	0	7	1.58
	M	4.04	1	7	1.69
PS	E	2.36	0	3	.742
	M	2.91	2	3	.284

Table 4.4
Frequency of Correctness of Items for Testing Preservices' CK (Elementary Teacher Education Level)

Items	Frequency (out of 58)	Percent (%)
(1)	51	87.9
(2)*	36	62.1
(3)*	32	55.2
(4)*	34	58.6
(5)*	33	56.9
(6)	41	70.7
(7)*	37	63.8
(8)*	39	67.2
(9)	54	93.1
(10)	44	75.9
(11)	58	100
(12)	47	81
(13)	55	94.8
(14)	55	94.8
(15)*	20	34.5
(16)*	31	53.4
(17)*	12	20.7
(18)	35	60.3
(19)	38	65.5
(20)	45	77.6
(21)*	10	17.2
(22)*	31	53.4
(23)	41	70.7
(24)	45	77.6
(25)	40	69
(26)	49	84.5
(27)	47	81
(28)	41	70.7

* indicates items with high frequency of incorrectness

Table 4.5

Frequency of Correctness of Items for Testing Preservices' CK (Middle School Teacher Education Level)

Items	Frequency (out of 46)	Percent (%)
(1)	46	100
(2)	42	91.3
(3)	41	89.1
(4)	42	91.3
(5)	42	91.3
(6)	42	91.3
(7)*	30	65.2
(8)	33	71.1
(9)	46	100
(10)	38	82.6
(11)	46	100
(12)	44	95.7
(13)	46	100
(14)	44	95.7
(15)	33	71.7
(16)*	31	67.4
(17)*	15	32.6
(18)	43	93.5
(19)	41	89.1
(20)	33	71.7
(21)*	8	17.4
(22)*	21	45.7
(23)	39	84.8
(24)	35	76.1
(25)	33	71.7
(26)	45	97.8
(27)	44	95.7
(28)	45	97.8

* indicates items with high frequency of incorrectness

A simple regression analysis was performed to determine how well GPAs predicted PCK. The analysis as shown in Table 4.6 revealed that Grade Point Averages on Mathematics (GPAM) is a good predictor of preservice teachers' pedagogical content knowledge. However, GPA on general college courses does not closely predict the preservice teachers' CK.

Table 4.6

Regression Analysis for Preservice Teachers' CK

Model	Standardized Coefficients	Sig	Adjusted R Square
GPAG	.136	.187	.008
GPAM	.385	.000	.137

Elementary and Middle School Preservice Teachers' CK

A MANOVA and discriminant analysis was performed to compare performance by levels and components of CD. As shown in Table 4.7, elementary and middle school preservice teachers scores were significantly different on CK. The Box's M is significant which means the covariance matrices differ between the two groups. However, since the sample size is relatively large, the log determinants of the group covariance matrices were checked, and the log determinant of the two groups did not differ greatly, which suggests the robustness of this discriminant analysis. Since there were two different major groups, only one discriminant function was generated. The Wilks' Lambda is statistically significant, justifying the discriminant function of the five CK constructs in classifying students into different major groups. Further, the univariate ANOVA information in Table 4.7 shows that elementary and middle school groups differed significantly on FAMS, FWVR, FBVR, and PS scores ($p<.01$), but not on Flexibility with Visual Representations ($p=.769$). Table 4.8 shows the results of tests of equality of group means, the smaller the Wilks' lambda, the more important the independent variable to the discriminant function. According to the results shown in Table 4.9, PS is the most important factor in discriminating the two groups. The standardized discriminant function coefficients shown in Table 4.10 serve the same purpose as beta weights in multiple regression, indicating the relative importance of the independent variables in predicting preservice teachers' levels of teacher education classification.

Table 4.7
Multivariate Analysis of Preservice Teachers' CK between Different Levels of Teacher Education

	Wilks' Lambda	Sig.	Box's M	Sig	Log Determinants	
					Group 0	Group 1
Level of Teacher Education	.015	.000	59.665	.000	2.122	-1.516

Table 4.8
Between Level of Teacher Education Effects of Preservice Teachers' PCK

Group	Test Factor	df	Mean Square	F	Sig.	Observed Power(a)
Level of Teacher Education	FWMS	1	73.175	13.335	.000	.161
	FWVR	1	13.120	38.18718.450	.000	1.000
	FBVRAR	1	15.116	11.67012.485	.001	.923
	FWRS	1	.233	11.073.087	.769	.909
	PS	1	7.788	15.10222.664	.000	.971

Table 4.9
Tests of Equality of Level of Teacher Education Mean Score of Preservice Teachers' PCK

	Wilks' Lambda	F	df1	df2	Sig.
FWMS	.884	13.335	1	102	.000
FWVR	.847	18.450	1	102	.000
FBVRAR	.891	12.485	1	102	.001
FWRS	.999	.087	1	102	.769
PS	.818	22.664	1	102	.000

Table 4.10
Standardized Canonical Discriminant Function Coefficients

	Function1
FWMS	.460
FWVR	.450
FBVRAR	.380
FWRS	-.259
PS	.450

Table 4.11 shows classification results that assessed how well the discriminant function works, and if it works equally well for each group of the dependent variables. Here it correctly classifies about 76.0% of the cases, though the proportion of mistakes for the two groups are not the same, which may be due to the different sample sizes of the groups and some extremely low scores in middle school education preservice teachers. Overall, there is a satisfactory discrimination for CK.

Table 4.11
Classification Results for CK

Level of Teacher Education			Predicted Group Membership 0	1	Total
Original	Count	0	39	19	58
		1	6	40	46
	%	0	67.2	32.8	100.00
		1	13.0	87. 0	100.00

76.0% of original grouped cases correctly classified.

Preservice Teacher's CK Measures and Structure

Structural Equation Modeling was performed to confirm the components of Test A. Figure 4.1 displays the Measurement Model I of Test A, which consists of 5 latent variables and 15 indicators. The five latent variables in ovals represent the five basic constructs of the test, Flexibility across Mathematical Symbols (FAMS), Flexibility within Visual Representation (FWVR), Flexibility between Visual

Representation and Algebraic Equations (FBVRAR), Flexibility with Real Life Situations (FWRS), and Procedural Skills (PS). Each latent variable was connected to three indicators in rectangles, which represented a set of three problems on Test A. In addition, each latent variable was allowed to covary with each other by default, which was indicated by a curved and two-headed arrow. Symbols e1 through e5 in circles of this figure represent residual error estimates associated with each indicator. A second factor model was also proposed to further confirm the design of Test A. The first part of this model is the same model as Measurement Model I. As shown in Figure 4.2, a second order factor, the general construct of Test A, was added to determine if the five sub-constructs of Test A covary and which constructs play the most important roles in testing preservice teachers' CK.

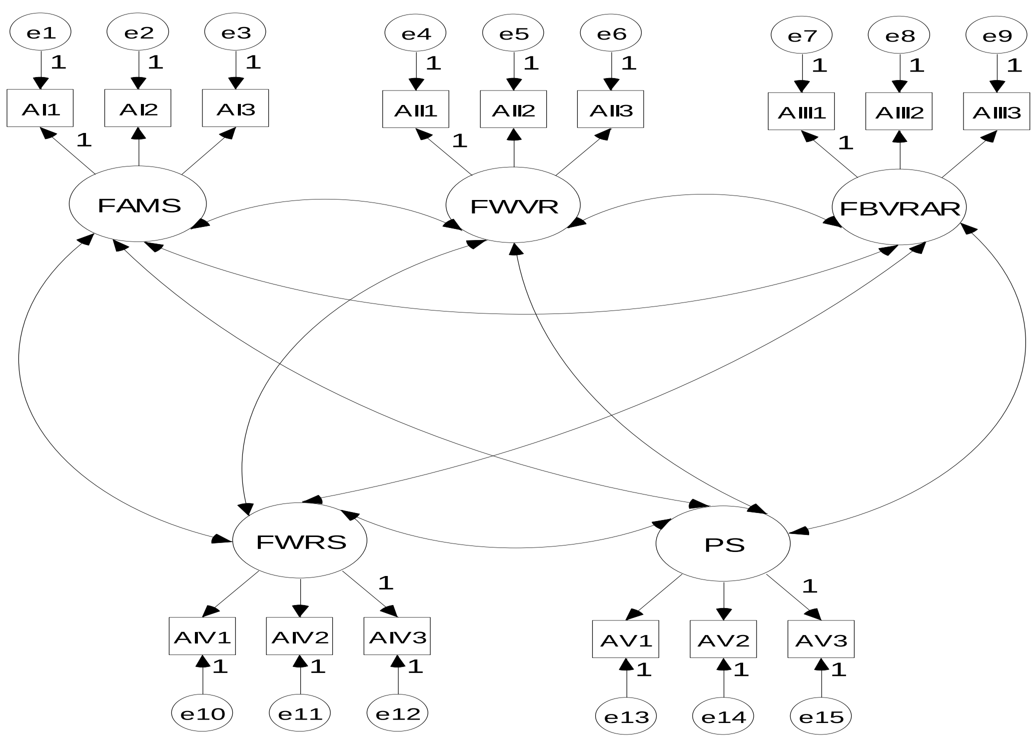


Figure 4.1. Measurement model for preservice teachers' CK

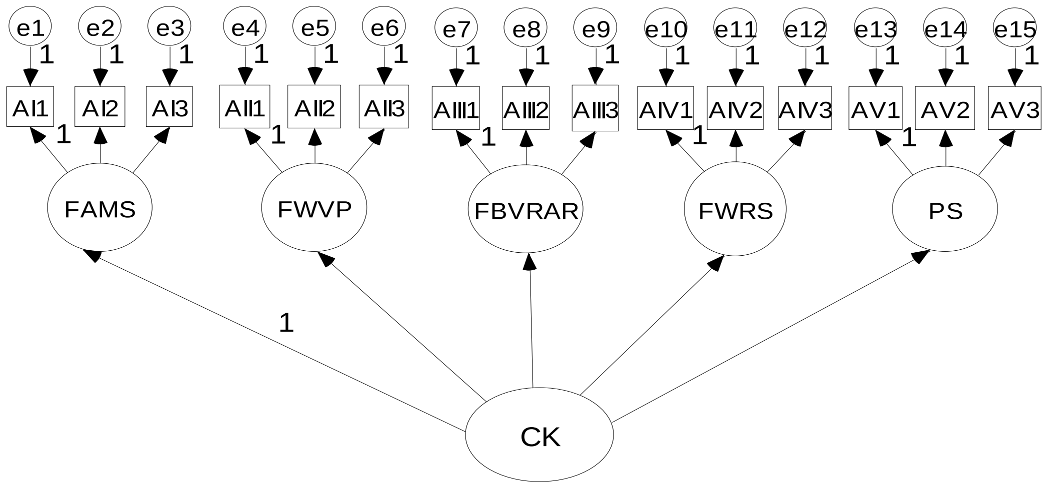


Figure 4.2. Second order model for preservice teachers' CK

Confirmatory Factor Analysis (CFA) was conducted to determine if the measurement model fit the present data as well as to examine the convergent and discriminant validity of Test A, The SEM confirmatory factor analysis was conducted through Mplus 4.0 (Mplus, 2006) using the maximum Likelihood (ML) method. The chi-square statistic, Comparative Fix Index (CFI), Root Mean Square Error of Approximation (RMSEA) are reported in Table 4.12 to demonstrate how the measurement models fit the data. Both of Measurement Model-CK and Second Order Model fit the data well according to the chi-square statistic with p-value .239 and .175 respectively. Further, CFI larger than .90 and RMSEA smaller than .05 also indicated a good fit of models. In addition, as shown in Table 4.13, the standardized path loadings of the five sub-constructs on overall latent constructs of CK in second order measurement model suggested that FWVR contributed the most to testing preservice teachers' CK, while FVRAR is not an important indicator of CK level. Further analyses of the qualitative data are needed to reveal the reason of this issue.

Table 4.12
Goodness of Fit Indices for Testing Preservice Teachers' CK

Model	Chi-square	df	P-Value	CFI	RMSEA (C.I.)
Measurement Model	90.726	82	.239	.957	.032 (.000-.065)
Second Order Model-CK	99.214	87	.175	.940	.037 (.000-.067)

Table 4.13
Standardized Path Loading of Latent Constructs on CK

	Standardized Path Loading	S.E.
FAMS	.570	.422
FWVR	.859	.087
FBVRAR	.095	.068
FWRS	.493	.368
PS	.558	.141

The standardized path coefficients of each indicator toward five latent sub-constructs are shown in Table 4.14. The majority of them were fairly high with $p<.01$ significance level, which suggests the convergent validity of the test items. Table 4.15 is the correlation matrix of the five latent sub-constructs, FAMS, FWVR, FBVR, FWMP and PS. It is obvious that there was no strong relationship among these latent sub-constructs indicating these sub-constructs were distinct in testing preservice teachers' CK. This also suggested the discriminate validity of Test A. Moreover, the Split-Half reliability test statistics, .742, shows that it is also a reliable test.

Table 4.14
Standardized Path Coefficients of Testing Preservice Teachers' CK

	Standardized Path Coefficients	S.E.
FAMS		
AI1	.785**	.085
AI2	.915**	.075
AI3	.623**	.102
FWVR		
AII1	.210*	.055
AII2	.204	.054
AII3	.860**	.110
FBVRAR		
AIII1	.286**	.071
AIII2	.987**	.030
AIII3	.150	.063
FWRS		
AIV1	.996**	.052
AIV2	.360**	.091
AIV3	.110	.091
PS		
AV2	.532**	.067
AV4	.205	.039
AV5	.565**	.058

Table 4.15
Correlation Matrix for Different Constructs in CK (Discriminant Validity)

	F1	F2	F3	F4	F5
FAMS	1.000				
FWVR	0.117	1.000			
FBVRAR	0.160	0.024	1.000		
FWRS	0.274	0.131	-0.067	1.000	
PS	0.233	0.104	0.111	0.147	1.000

In summary, from the results of CFA model fit, standardized path loading of indictors, correlation matrix of the five latent sub-constructs, and Split-Half reliability test, Test A is a valid and reliable measurement instrument of preservice teachers' CK. Subjects' scores of different test components can demonstrate their knowledge in different aspects of CK.

Qualitative Analysis

According to the descriptive analysis of each sub-item of Test A, one of preservice teachers' biggest problems on CK was the first part of the test, namely, the flexibility across mathematical symbolisms. Their test scores on the third part and fourth part, flexibility between visual representations and flexibility with real life situations, are also low. Therefore, the qualitative analysis will focus on these three modes of representations. Based on the standardized path loading of each sub-construct of preservice teachers' general CK, the second part of Test A, flexibility within visual representation is the most important component in testing preservice teachers' CK. Therefore, some details about this part of test will also be included. Since most students performed well on the last part of Test A, Procedural Skills, and it is not a significant indicator of preservice teachers' CK, further analysis is not regarded to be necessary in this study.

Flexibility across Formal Mathematical Symbolisms

Linear functions can be expressed in several forms of written symbolisms, such as standard form: $\mathrm{a}x + \mathrm{b}y + \mathrm{c} = 0$, slope intercept form: $y = \mathrm{m}x + \mathrm{b}$; and two other alternative forms: point-slope form: $y – y1 = m\ (x – x1)$ and two-point form. There are three formats for two-point form. $(y – y_1)(x_2 – x_1) = (x – x_1)(y_2 – y_1)$ is one of them. Another two formats of two-point form are $(y\text{-}y_1/x\text{-}x_1) = (y_2\text{-}y_1/x_2\text{-}x_1)$ and $(y\text{-}y_1/y_2\text{-}y_1) = (x\text{-}x_1/x_2\text{-}x_1)$. These mathematics symbolisms for linear functions have their own characteristics and serve different purposes. Slope-intercept, a main form addressed in the current curriculum, exposes students to the concept of slope and intercept. When a point and slope are known, then point-slope form can easily developed. Two-point form, in its format, looks complicated, but it is a form that could be related to the slope formula y_1-y_2/x_1-x_2.

Meanwhile, slope, a key component of symbolisms of linear function, by itself is also an important concept in teaching and learning functions. Slope for linear functions has the characteristics of being constant. Likewise, it can be expressed by several written symbolisms, such as m in $y = \mathrm{m}x + \mathrm{b}$, -a/b in $\mathrm{a}x + \mathrm{b}y + \mathrm{c} = 0$ and y_1-y_2/x_1-x_2.

Mastering the different mathematics symbolisms and the ability to use them flexibly is important for a complete understanding of linear functions. However, the descriptive analysis of the test scores shows that the subjects were challenged in their effort on this part of the test. Therefore, this section is devoted to a detailed description of preservice teachers' representation flexibility across formal mathematical symbolisms of linear functions covering three forms of mathematical symbolisms of linear function, slope-intercept form, point-slope form, two-point form and different expressions for slopes. Correspondingly, preservice teachers' answers to question 1, 2 and 3 in the first part of Test are analyzed qualitatively.

Slope-Intercept Form. Slope-intercept form, written as $y = \mathrm{m}x + \mathrm{b}$, is the most frequently introduced symbolisms of linear function in the middle school mathematics curriculum. Some sub-items of Questions 1 and 3 in Test A were related to the usage of this form. Almost all the preservice teachers' correctly answered these items, which indicated that they were able to apply slope-intercept form of symbolisms of linear function.

1. A line can be represented by the standard form: $ax + by + c = 0$ slope-intercept form, $y = mx + b$; point-slope form: $(y - y_1) = m(x - x_1)$; and two-point form: $(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$, where (x_1, y_1) and (x_2, y_2) are points on the line. Rewrite the line $2x - y + 5 = 0$ in slope-intercept form: ________________ point-slope form: __________________ two-point form:____________________

Being asked to transfer the standard form ax + by + c = 0 to the slope-intercept form in problem 1(a) in Test A, 97 out of 104 provided correct answer. The results suggested that the majority of first phase subjects had a good mastery of slope-intercept form. In addition, all the six second phase subjects successfully explained the process of how they obtained slope-form form. Meanwhile, they stated they preferred slope-intercept because that was the way that they were taught in the school. For example, April stated in her interview: “I definitely prefer slope-intercept form because it is simple.” Some further explained why they prefer the slope-intercept form by connecting it to the graph. For example Nicole said, “For slope-intercept [form], you can just look at it and you can independently see where it intersects, like where they the y intersection is and where the slope is.” At the same time, preservice teachers’ answers to Question 3 showed that they had a better understanding of slope as m in slope-intercept form y = mx + b than other expressions for the slope.

3. Slope can be represented as m in $y = mx + b$, $-\frac{a}{b}$ in $y = -\frac{a}{b}x - \frac{c}{b}$ $(b \neq 0)$ or $\frac{y_1 - y_2}{x_1 - x_2}$ when (x_1, y_1), (x_2, y_2) are points on the line. Write the slope of $\frac{1}{4}y - x - \frac{4}{5} = 0$ in the forms of

m: ______________________;

$-\frac{a}{b}$: ______________________;

$\frac{y_1 - y_2}{x_1 - x_2}$:__________________.

In problem 3, subjects were asked to write different forms of slopes from the standard form. Almost all of the subjects successfully transformed standard form to the slope-intercept form and wrote a correct m as slope.

¼ y – x – 4/5 = 0
¼ y = x + 4/5
y = 4x + 16/5
y = 4/1x + 16/5
so m = 4

However, the results showed that the subjects did not have a good mastery of two alternative forms. Many subjects were not able to figure out slope when they were asked to write slope in form –a/b or $\frac{y_1 - y_2}{x_1 - x_2}$. They further stated that they had never been introduced to those forms before.

Point-Slope Form. Given one point and the slope, a linear function can also be written in a point-slope form, $(y - y_1) = m(x - x_1)$. Preservice teachers' answers to the second item in Question 1 were analyzed to investigate their understanding of this form of symbolisms of linear function.

Three main problems were found from the answers as well as interviews of subjects. First, although about two thirds of the subjects did provide a correct point-slope form of linear function, many of them did not really understand the meaning of point-slope form. Many subjects converted the standard form to slope-intercept form and then manipulated the slope-intercept form to get the point-slope form.

The interviews revealed the process that these subjects used to obtain the point-slope form from slope-intercept form. They transformed $2x - y + 5 = 0$ to slope intercept form y = 2x + 5. They all seemed to recognize 2 as slope, and they tried to find a point that satisfied y = 2x + 5 instead of directly finding one that satisfied 2x – y + 5 = 0, such as (0, -5). Finally, they plugged the slope and points into the slope-point form $(y - y_1) = m\ (x - x_1)$ given by the question to get (y + 5) = 2(x - 0).

In addition, some of the subjects made mistakes in manipulating the slope-intercept form to get the point-slope form. They first got a slope-intercept form of the linear function, then mechanically plugged a point into the slope-intercept form without deleting the intercept. Consequently, many of the answers had the same pattern: (y-y1) = 2(x-x1)-5, indicating a lack of understanding of point-slope form and being confused by the slope-intercept form and point-slope form that was provided in the question.

Second, many subjects had misconceptions about finding points for the slope-point form. Some of them thought that only some specific points can satisfy the linear function but were not aware that infinite points on the line can satisfy the function, while some of them randomly chose the points but did not realize that the points should satisfy the linear functions of the line.

Some subjects were able to explain the reason they picked certain points. For example, Renee had no difficulty finding a point satisfying the equation of the line. In her case, she picked (1, 7). The interview showed that the reason for her to pick (1, 7) was because it satisfied slope-intercept form of the line (I=Interviewer; S=Subject).

> I: Ok, so you think (1, 7) is the only point that would satisfy the point-slope Form?
> S: No, it could be any point on the line because the line is never-ending so I was like, well any numbers I'm going to put in there.
> I: so you are saying any points that you can put in the equations of the line?
> S: Well that's what I was thinking at the time, now I'm getting confused but I'm…

At first, Renee was certain that any point that could be put in the slope-intercept form could work in the point-slope form. She came to this conclusion because the line was never ending. By saying this, she seemed to be aware of two facts: First, slope-intercept form and one-point form were just different forms of a straight line. Second, any point not just some specific points on the line could satisfy the point-slope form. However, she seemed not confident about the second fact when asked to confirm if she was saying any point that satisfy the equations of the line.

Anther problem with using point-slope form to express functions was that subjects made many mistakes in the computation of the points. Their process showed that they were on the right track of getting the points, however, apparent mistakes were made when they were computing the points. This suggests that their computation skills should be improved, since the questions only involve simple number processing.

Two-Point Form. Besides slope-intercept form y = mx + b and point-slope form $(y–y_1) = m(x–x_1)$, linear functions can be also expressed as two points in its multiple format, such as, $(y–y_1)(x_2–x_1) = (x–x_1)(y_2–y_1)$, $(y-y_1/y_2-y_1) = (x-x_1)/(x_2-x_1)$ and $(y-y_1)/(x-x_1) = (y_2-y_1)/(x_2-x_1)$.

The rationale behind two-point form of linear functions is that any two distinct points can determine a line. The two-point form utilizes the characteristic that linear functions have constant slopes. To write a two-point form, one needs to know two points on the line. Actually, this form is the most problematic part of preservice teachers' writing symbolisms of linear function. The first type $(y–y_1)(x_2–x_1) = (x–x_1)(y_2–y_1)$ was tested in the third item of Question 1 and the other two types, $(y–y_1)(x–x_1) = (y_2–y_1)(x_2-x_1)$, and $(y–y_1)(y_2–y_1) = (x–x_1)(x_2–x_1)$ were tested in the following Question 2.

2. A line can be represented by the standard form: $ax + by + c = 0$, two-point fractional form:

(1) $$\frac{(y-y_1)}{(x-x_1)} = \frac{(y_2-y_1)}{(x_2-x_1)}$$

OR (2) $$\frac{(y-y_1)}{(y_2-y_1)} = \frac{(x-x_1)}{(x_2-x_1)}$$

Rewrite the line $4x + y – 2 = 0$ using

form (1): ______________________

form (2): ______________________

Preservice teachers' answers to these test items showed that two-point form was the most challenging linear function form. Thirty-two of the 70 subjects were unable to write the two-point form correctly. The interviews subject showed that the difficulty may due to their unfamiliarity of two point form. For example, Renee, one of the second phase subjects, said:

> Well, I don't think I was ever taught the two point form, this was I think the first time I'd ever seen it.

Only 28 subjects (out of 104) were able to answer question 2 correctly. The following is an example of two correct points that a subject for two-point form.

--- (y – 2)/(x – 0) = (6 – 2)/ (1- 0)

--- (y – 2) / (6 – 2) = (x – 0)/ (1- 0)

Although a few of the subjects wrote out two-point form correctly based on standard form a*x* + b*y* + c = 0, it did not mean that the subjects had a thorough understanding of where two points were from or the rationale behind using two points to make a linear equation. From the interviews, it became apparent that some subjects just had a procedural understanding of finding and plugging points into forms given in the question. For example, even though Rachael gave correct answers to the questions, the interview revealed that she merely understood the procedure of plugging in points into mathematical forms but not the concept of two-point form.

> I: So what do you think of these three two-point forms? What is the relationship between these three?
> S: Right and I don't ever remember doing that either so whenever I was doing this, I just took my same numbers up here, no I didn't because it's a different equation, cause it was this equation but once again I just made up my own x1 and x2 and y1 and y2 by solving in here and then I just looked at the variables and every time it said x1 I just plugged in my x1 and every time it said x2…

Some of the subjects did not even know why two points can determine a line and make a function for a line. The following interview content suggested that Rachael did not understand why any two distinct points can make up a line, neither the relationship between the points and slope of a line. Further, she could not connect the two-point form of linear function with two-point form of slope, especially when it had generic x and y involved in it.

> I: …Ok. You did it right. Do you think any two points plugged into this equation would work?
> S: Yes
> I: Why?
> S: Well, any point where you choose like the x you plug it into the formula and you get that over…yes, any numbers would work as long as you make sure they plug into the formula.
> I: Why?
> R: Because…it's the same…they're the same distance apart.
> S: What do you mean by saying they are the same distance apart?
> I: It would give you the same numbers down here, I know as far as the slope formula it would… yeah we learned the slope formula for that.

Renee was the only student interviewed who understood the reason why two-point form worked. She explained that a line was formed by connecting two points. She also mentioned 2 points can determine a slope.

> I: So for the 2-point form, why do you think you plug in 2, then the form represents a line?
> S: Umm because there's on a slope in between those 2 points and if the points connecting those 2, like if there's a line connecting those 2 points then there's a line.

Although Renee said that it was her first time using two-point form for linear functions, she eventually realized the relationship between the two-point slope form and two-point linear function form. Especially, she understood the meaning of generic x and y.

> S: Well, I recognize this one [point to $(y – y_1) / (x – x_1)$] is a slope, then this one [point to (y2 – y1) / (x2 – x1)] is a slope as well.
> S: Right, and I recognize this one as slope
> I: And then if this is slope this has to be slope…

S: Because it's equal
I: Why do you think this is slope?
S: Because it's a generic y over a generic x compared to this one point right here so it's like any y that we have, any point on the line, so its just generic y, generic x so any point on the line can be found from our slope by just looking at this one point.

Recognizing the relationship between three two-point forms is not an easy step. Still, her recognition of $(y_2–y_1)/(x_2–x_1)$ as a slope does not necessarily mean that she understood the concept of slope. It may be because she remembered the formula of slope, i.e., $(y_2–y_1)/(x_2–x_1)$. But she was also able to point out $(y–y_1)/(x–x_1)$ which involved generic x and y as a slope. She said the right side of the equal sign which was $(y_2–y_1)/(x_2–x_1)$ was slope, then the left side which was $(y–y_1)/(x–x_1)$ had to be slope. She struggled and then managed to understand why $(y–y_1)/(x–x_1)$ was a slope as well, that is, generic x and y are any points on the line, therefore, when they were plugged into the formula, it still was a slope.

I: So what do you think of the relationship between these three [$(y–y_1)(x_2–x_1) = (x–x_1)(y_2–y_1)$, $(y-y_1)/(y_2-y_1) = (x-x_1)/(x2-x_1)$, $(y-y_1)/(x-x_1) = (y-y_1)/(x-x_1) = (y_2-y_1)/(x_2-x_1)$]?
S: It's just an expanded version instead of just saying like our slope is m, we're actually finding the slope when we're going through the equation so we have our two points and so when we put it into this form $(y-y_1)/(x-x_1) = (y_2-y_1)/(x_2-x_1)$ it's going to find the slope for us and then it's … it's just comparing any of our points on the line that we want to choose, choose any point here and point here and you can put it into this form so that you can find your line.

Renee related two-point form to the slope, viewing slope as the connection among different formats of two-point forms. She said any two points that satisfied the line employed to these two-point forms would come up with the same equation, and these forms could help to find the slope.

Summary

The qualitative analysis of first phase subjects' answers to the questions of Test A and the interview with second phase subjects, revealed that most preservice teachers were only familiar with the slope-intercept form of linear function y= mx + b and its corresponding slope form m. The reason was that these two forms were the most frequently introduced in current curriculum. Meanwhile, preservice teachers had a hard time applying point-slope form and two-point form of linear function and many of them claimed that they had never been taught the two-point forms. Therefore, it is not surprising that few of them understood the concept or rational of slope-point form and two-point form, let alone the relationship between slope form and function form. Moreover, some subjects did not even understand the point-line connections. Poor computation skill was another problem of some preservice teachers, considering it is a basic skill for a mathematic teacher.

Flexibility within Visual Representations

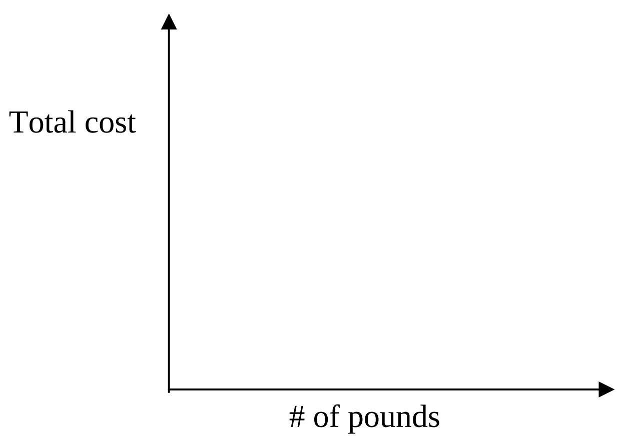
This part of Test A assessed preservice teachers' flexibility within visual representations. Both ordered pairs and graphs are visual representations for linear functions. To construct a line representing a linear function, some view it point-wise, while others recognize the global property of the line. By viewing a linear function point-wise, one can construct a line by plotting multiple points and connecting the points. When people use the global property of a line or linear function to form a line, they can either find the slope and intercept or two distinct points to define a line. Transferring between the ordered pairs and graphs reflects flexibility within visual representations. Three problems in Part II of Test A were designed to assess preservice teachers' flexibility within visual representations. Since very few subjects had difficulties with Question 2, only Question 1 and Question 3 are analyzed in the following.

1. At Speedy Delivery Service, the cost to deliver a package is $2.00 plus an additional $.50 per pound.

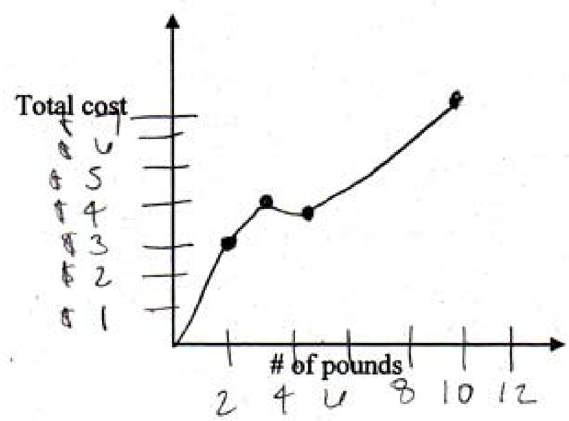
(a) Fill in the missing values in the table below and write an equation that represents the relationship between the number of pounds and the total cost.

Number of pounds	**Total Cost**
2	**?**
?	$3.50
4	**?**
?	$7.00

(c). Draw a graph that shows the relationship between the number of pounds and the total cost.



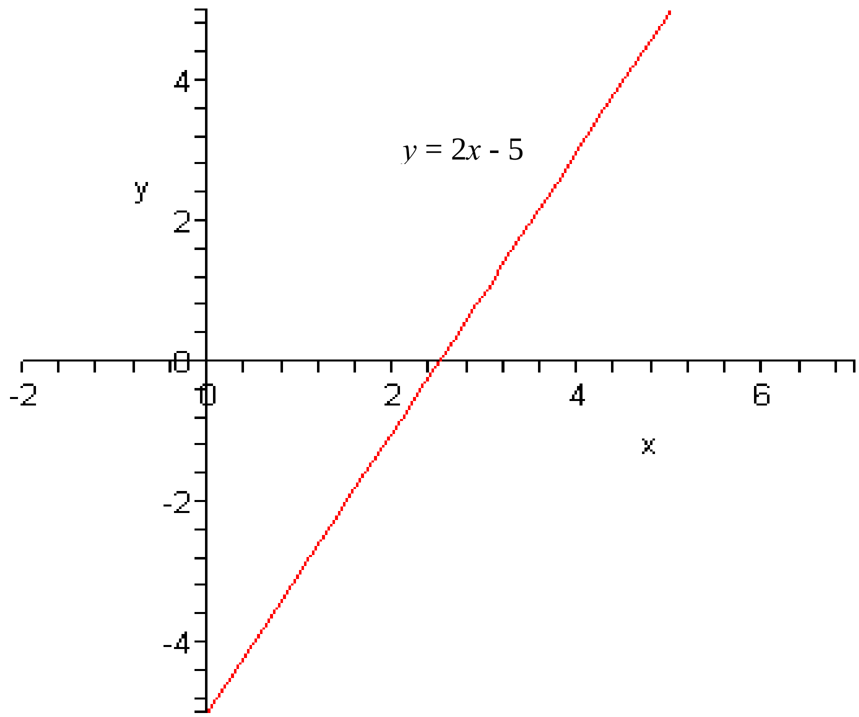
The second subitem of this question investigated preservice teachers' flexibility of visual representations in term of a graph and ordered pairs. Subjects' answers revealed that they lacked flexibility within visual representations. Although some subjects filled in the missing values and wrote out the equation correctly to represent the information given in the table, most of them plotted several points that satisfied the equation to form a line. Some of them did not have an accurate scale, so their graph turned out to be a curve instead of a straight line, such as the answer of this subject.



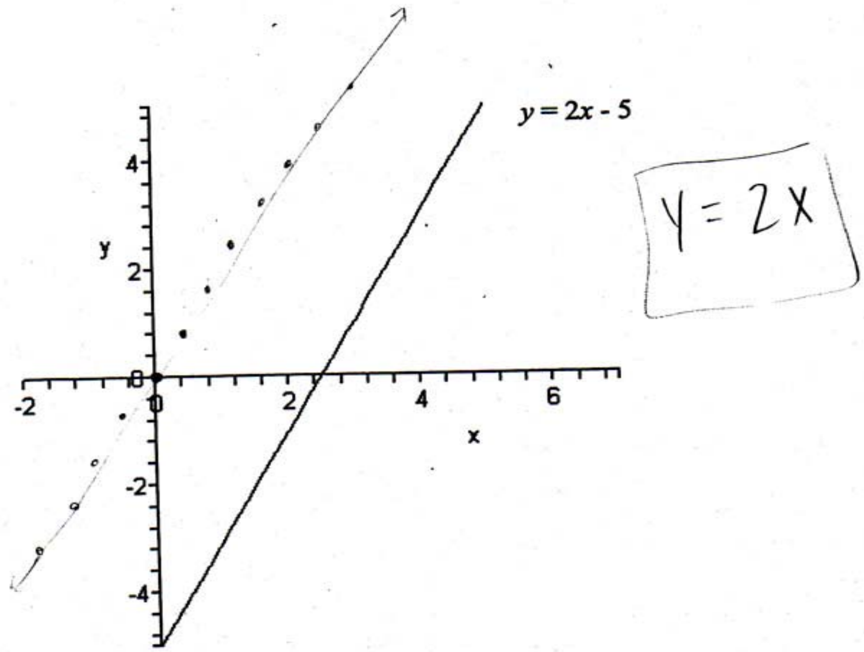
Fifty-two out of 104 did draw a straight line. However, a close investigation of the subject graphs revealed that the lines looked straight only because they had better scaling skills than those who got a curve. This indicated that these subjects only knew how to view a function point-wise. In order to investigate this further, the interviewed subjects were asked how they drew the line. The results showed that all of them (6) were unaware that to graph y = .5x + 2, one just needed two distinct points. Their approach showed that they dealt with functions point-wise and failed to see a linear function from a global perspective, that is, the graph of linear function was a straight line.

3. On the axis below, draw a line parallel to $y = 2x - 5$ that goes through the origin.

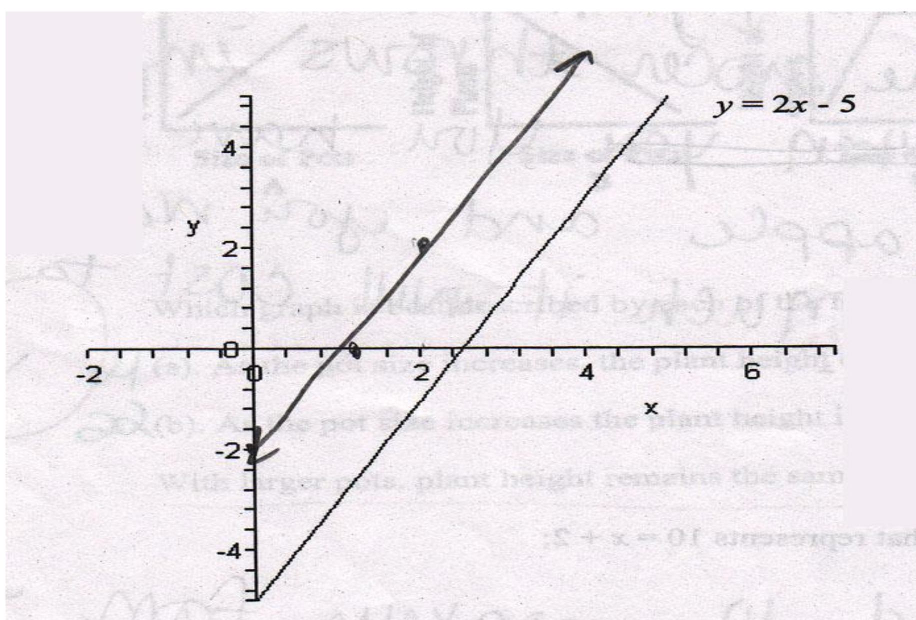
Write the equation.



Two major problems were found from subjects' answers to question 3. Although more than half of the students gave a correct equation and line, details of their graphs revealed that they did not have a global property view of graphing the equation and line. It seemed that they knew that the slope of the line should be 2 and it went though the origin, only a few of them draw a line directly from the fact that it was parallel to the given line and went through origin. Most of them again marked multiple points that satisfied the equation and connected them to form a line. For example, this subject used many points to determine a line that had already been known to be parallel to the given line.



Another problem that some subjects had on this question was that they did not know what the origin was, which was the main reason they could not give the right answer. For example,



Flexibility between Visual Representation and Algebraic Representations

Algebraic, tabular and graphical forms are the three most common representations of functions. The skills in translating between different representational systems, visual representations (tabular and graphical) and symbolic representations are important for understanding a concept (Lesh et al., 1987). In this section preservice teacher's representation flexibility between algebraic equation and visual representations of linear functions is examined. The relationship described by the Cartesian Connection, summarizes the main relationships between graph, ordered pairs, and algebraic equation. The Cartesian Connection states, according to Moschkovich et al. (1993) "a point is on the graph of the line L if and only if its coordinates satisfy the equation of L." In other words, it describes the relationship between the points on the graph of the line and the coordinates that satisfy the equation of the line. There are two components in this relationship. First, if a point is on the graph of the line L, then its coordinates satisfy the equation of L. Secondly, if coordinates of a point satisfy the equation of L, then the point is on the graph of L. Therefore, this section focuses on these two components of Cartesian Connection. The first two questions and the third question in Part III of Test A test preservice teachers' representation flexibility between visual representation and algebraic representation on the first and second components respectively.

Problem 1 and 2 focused on the first aspect of Cartesian Connection, i.e., when a point is on the graph of the line, then its coordinates satisfy the equation of L. The results showed that preservice teachers did not fully understand this aspect of the Cartesian Connection. Only 14 out of 104 successfully wrote an equation in problem 1, and 27 out of 104 could provide the equation correctly for problem 2.

1. Suppose that the following table gives the value (*V*), in dollars, of a car for different numbers of years (*t*) after it is purchased.

t	***V***
0	$16,800
2	$13,600
4	$10,400
6	$7,200
10	?

Write a symbolic rule expressing *V* as a function of *t*.

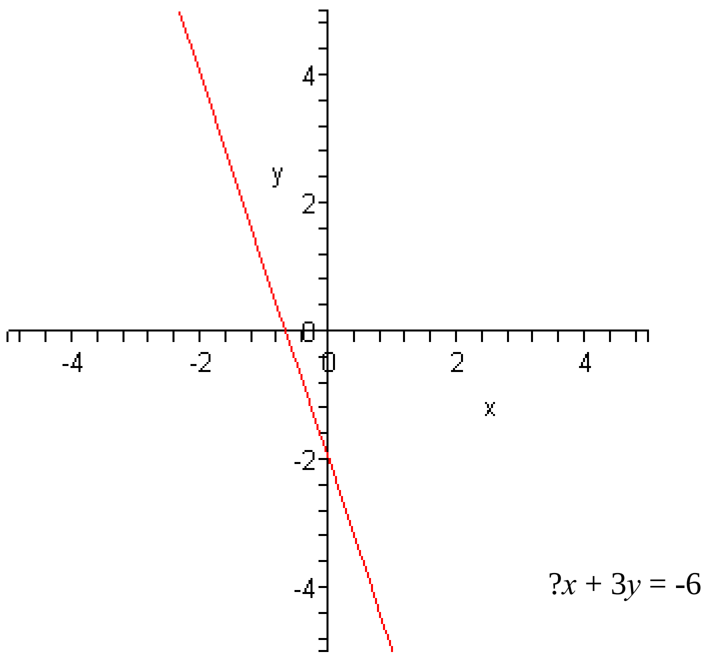
The results revealed that a large number of subjects had difficulties finding the equation from the pattern of the ordered pairs given a real life situation. Ninety-eight out of 104 were not able to provide a correct equation to describe the relationship between the number of years (t) after it was purchased and the value of the car. When looking at tables, one group of subjects only found the pattern for V. Some of them described verbally that the value of the car was decreasing with the number of years increasing and many of them even found that there was a constant decrease of the value and filled in a correct number, 800 in the question mark cell. However, they failed to come up with a linear function to express the relationship between the ordered pairs t and V.

Another group of preservice teachers remembered how to write an equation to represent the ordered pairs with linear relationship between the coordinates. Nevertheless, they wrote v = -3200t + 16800 (32 out of 104). By providing this answer, they assumed the rate of change is -3200, which shows they made a procedural error. They failed to notice that the amount of decrease value 3,200 was for every two years, i.e., 2t. So the rate of change should be -3200/2 since the rate of change for 2t was -3200 while for t it was -1600.

2. The graph below represents the equation ?x + 3y = -6

(We do not know the value of the coefficient of x).

Is it possible to find the missing value? If yes, what is the missing value?



There are three approaches to solve this problem. First, to solve this problem, one can start with computing and manipulating the equation, and then, find a point on the graph to plug in the equation to solve the problem, which according to Knuth (2000) is called an algebraic solution approach because the points on the graph are only used a means of supporting the algebraic approach. Twenty out of 27 correct answers revealed that subjects tended to compute the algebraic equation. To them, it seemed finding a point meant to compute an equation. Knuth (2000) reported similar findings in his study of college prep students' understanding of Cartesian Connection, with more than three fourths of the students employed algebraic approaches as primary solution source, even when graphical approaches may be easier and more efficient.

For those subjects who used algebraic approach, some of them (12 out of 20) could apply algebraic approach, but could not use graph as a means to find the solution. Angel wrote

> 3y/3 = (-?x – 6)/3
> y = -?x/3 -2

The response revealed that this subject tried to manipulate the algebraic equation. However, she did not connect her approach to graphical representation of the equation. Therefore, she failed to find the missing value in this case. Some of the subjects used the algebraic approach, and could manage to use the graph to find the missing value. For example, one subject answered:

> From the graph, (-2, 4)
> (-2)? + 3*4 = -6
> ? = 9

Few students applied the second approach, finding a point from graph to plug in the equation without manipulating the equation. The third approach was to find a slope from the graph, which required students' conceptual understanding of slope, rate of measured change of the line. However, it seemed that some of the subjects attempted to use this approach to find the slope but did not exactly know how to get the rate of change from the graph. Some subjects utilized the graph to find the slope of the line by finding two distinct points on the line. When the slope was known and equation was changed to slope-intercept form, then the missing value could be found. 29 out of 104 used this approach. Two correct answers of the subjects are in the following:

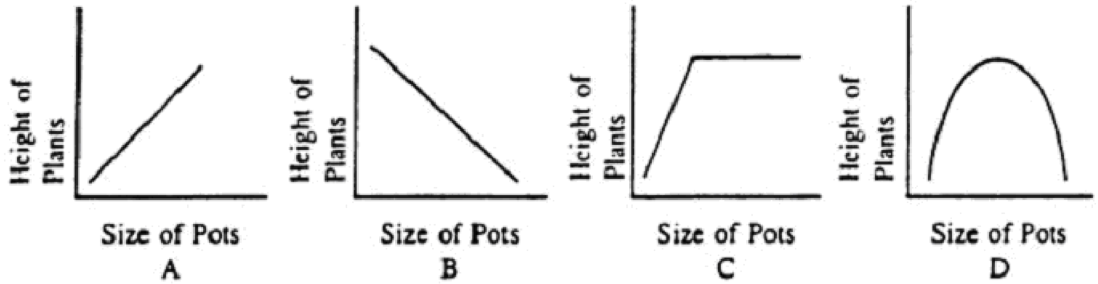
--- (0, -2) (-0.4, -0.8)
m = 1.2/ (-0.4) = -3
y = -3x – 2
?x + 3y = -6
3y = -?x -6
y = - ?x/3 – 2
so ? = 9
-- (0, -2)
(-2, 4)
(y2 – y1)/ (x2 – x1) = (4 – (-2)) / (-2 – 0) = -3

Among this group of the subjects, some thought that "?" in the equation represented the slope, which was the reason why they gave the answer -3 instead of 9 for the missing value. It again indicated that some students were only used to the slope in y = mx + b but not in other linear function forms which was consistent with the findings in Part I of the test.

Question 3 assessed the second part of the Cartesian Connection that states: If coordinates of a point satisfy the equation of L, then the point is on the graph of L.

Although there was no exact equation listed in the question, information about an important component of linear function, slope, was given in detail. The subjects were supposed to pick a graph to match the slope information in the question. The first sub-item was not problematic, except for those who were not careful enough and picked A by mistake. For the second part, about a quarter of subjects chose D instead of C, indicating that they did not have a clear idea how slopes can be graphed. These subjects might not be able to use the change between coordinates x and y to find the corresponding graph.

3



Which graph is best described by each of the following statements?

(a) As the pot size increases, the plant height decreases.

(b) As the pot size increases the plant height increases up to a certain pot size, and with larger pots, plant height remains the same.

Summary. Quantitative results of preservice teachers' scores on Part III of Test A suggested that they did not have good flexibility between visual representation and algebraic representation on the Cartesian Connection. Qualitative analysis of subjects' answers to the questions found that many of them were not able to use equations to express the information given by ordered pairs, even though they had already realized the relationship given by the ordered pairs. Most subjects did not know how to use graphic approaches to form an equation. There were also some subjects who had difficulty connection the graph with a linear function.

Flexibility with Real-life Situations

Problem representation skills in mathematics problem solving are an important part of mathematics learning. Actually there are two directions in the problem representation skills. One skill is to use algebraic representations to express word problems, while the other skill is to create a real life situation based on the expression of linear function. Both skills require conceptual understanding of linear function. Nevertheless, only the first skill is emphasized in the current curriculum. The results of Part IV of Test A reflected the status of teaching problem representation skills at school. A majority of preservice teachers (97 out of 104) were able to solve a word problem symbolically. However, when they were asked to describe a real life situation for equations, few preservice teachers was able to provide a correct situation. The following part of this section will use the first two questions of Part IV to discuss subjects' second skill of problem representation and the third question to review subjects' first skill of problem representation.

1.	Describe a real life situation that represents the equation $y = 6x+2$.
2.	Describe a real life situation that represents $10 = x + 2$.

Only 18 of 104 subjects could a correct real life situation for y = 6x +2. Among these preservice teachers, some gave specific explanations of what x and y represented and how their values were related. For example, three subjects' answers are shown below.

> Zach's allowance is $2 a week plus $6 for every chore he completes. x = the number of chores completed; y = total allowance. (Shelley)
>
> A store says that there is a $12.00 entry fee, but every item you buy after that is 6 dollars. x = number of items, y = total spent. (Hillary)
>
> Robert is training for a marathon, so, in order to stay out of the way of a real marathon that is taking place, he goes to the course where the marathon is taking place and starts at the 2 mile marker. He runs consistently at a rate of 6mph.

> Here, in $y = 6x + 2$, x would be his time and y would be his distance traveled. (Tracy)

The last example described a situation that showed the subject's deep understanding of linear function of $y = 6x + 2$. She described a situation that not only contained the information for the slope 6, which was the rate of change in this case, but also pointed out the intercept, which was the starting point. More importantly, she was the only one that mentioned the concept of constant that is, "runs consistently at a rate of a 6 mph."

Some subjects described a situation that could be developed into the relationship of $y = 6x + 2$, but they used specific numbers instead of a linear function to describe the relationship, for example, one subject said:

> Ashley wants to buy some bedding at pottery barn. There is a $2 online transaction fee that is required for each purchase. It also costs $6 per kg to ship an item. If the bedding weighs 3 kg, how much does she have to pay?
>
> So $y = 6(3) +2 = 18+2 =20$

X	Y
0	2
½	5
1	8

Some answers reflected that the subject did not understand the relationship expressed by the linear function. Therefore, their answers did not match the linear function at all. For example:

> Kim has 6 times as many soccer balls as Bob and then Kim got two more. (Kristen)

Her answer showed that she did not understand that x is a generic variable in the function and there is linear relationship of x change with y change.

Question 2 also required subjects to write a real life situation; however, it differed from Question 1 in two aspects. The first aspect was that x in Question 2 had a definite value (8), while x in y = 6x + 2, x could be any value. 10 = x + 2 could describe either an arithmetic situation or by an algebraic situation. Y = 6x +2 could mainly be described by an algebraic situation. The second aspect was that 10 = x + 2 only had one variable, while y = 6x + 2 involved variables x and y to describes the relationship between x and y. The quantitative results showed that subjects performed a bit better on Question 2 than on Question 1. About half of them provided a correct situation for the equation 10 = x + 2. Among these subjects, 28 of 104 who provided a correct situation for 10 = x + 2 used arithmetic word situations and only 10 gave an algebraic situation. The arithmetic approach generally starts the narration with the knowns, then expresses the unknowns after information on all the known information is provided. The Algebraic approach would first propose an unknown and continue the description of the problem with a focus on the unknown variables. In the examples below, the first subject described a situation for equation 10 = x + 2 that was primarily algebraic, and the second subject used an arithmetic word problem to express a real life situation.

> Stephanie had a certain amount of money. Stephanie’s mom gave her $ 2 so she could buy a $10 pair of shoes. How much money did Stephanie start with? (Kathy)
>
> Elisa has 10 tickets to the opera. She has already decided that she and her best friend Sue are going to be using tickets (2). How many more people could Elisa invite and take to the opera with those tickets remaining. (Kelly)

3. A truck is loaded with boxes, each of which weighs 20 pounds. If the empty truck weighs 4500 pounds, find the following:

   a. The total weight of the truck if the number of box is 75.______________

   b. The number of boxes if the total weight of the truck is 6740 pounds.

      ____

   c. Using W for the total weight of the truck and x for the number of boxes, write a symbolic rule (or equation) that expresses the weight as a function of the number of boxes.

Question 3 in Part IV explored preservice teachers' flexibility in terms of using algebraic representations to express real life word problems. The subjects' frequency of correctness on this question was much higher than on Question 1 and 2. Although some of the subjects had difficulty finding an equation for the word problem, most of them demonstrated good skills at applying algebraic equations to solve real life problems.

Summary. It seemed easier for subjects to write an algebraic equation for a real life word problem than to make up a situation. They were able to use different approaches to solve the word problems, such as slope-intercept approach, point-wise approach etc. It was more difficult for the subjects to describe a situation for an equation, especially when the linear equation involved two variables, x and y, and when the variables did not have definite values. For an easy equation such as $10 = x + 2$, students tended to use more of arithmetic word problems where an arithmetic approach was explicitly stated in the word problem. Not as many people were found using more algebraic focused situation to describe the equation $10 = x + 2$.

Results for Research Question 2

Research Question 2

How do preservice teachers demonstrate their pedagogical content knowledge in terms of the use of instructional representations, their knowledge of students' conceptions and misconceptions and their cognitive behavior for approaching students' misconceptions? Does preservice teachers' pedagogical content knowledge vary for those in elementary and middle grades teacher education?

Quantitative Analysis

To answer the second research question, similar statistical procedures were used to test preservice teachers' level of pedagogical knowledge and the validity and reliability of Test B.

Overview of Preservice Teachers' PCK

Table 4.16 shows descriptive statistics of each latent construct in the measurement models of Test B. Except for FWMS, high school preservice teachers have higher mean score and lower standard deviation than elementary school preservice teachers, which suggest that high school preservice teachers generally had a higher level PCK than elementary preservice teachers. Based on the information from descriptive statistics of Test B, a multivariate analysis and a discriminant analysis were also conducted to determine whether preservice teachers from different levels of teacher education had different levels of knowledge in the PCK of linear function.

Table 4.16
Descriptive Statistics for Factors of Testing Preservice Teachers' PCK

		Mean	Minimum	Maximum	Std.
FAMS	0	12.24	0	20	4.45
	1	11.37	0	21	4.68
FWVR	0	7.93	0	26	7.25
	1	17.37	1	43	8.31
FBARBR	0	4.56	0	11	2.90
	1	6.36	1	11	2.34
FWRS	0	5.36	0	17	4.66
	1	8.46	2	20	4.77
PS	0	3.82	0	17	4.53
	1	7.61	0	21	5.39

(Note: E standards for subjects in elementary certification program while M stands for those in middle certification program.)

The correlation analysis of the three components of each question in Test B shown in Table 4.17 revealed that there was strong relationship among the three sub-items of each question. In other words, preservice teachers' judgments of students' answers, understanding of students' conceptions and misconceptions, and instructional levels were highly correlated with each other.

Table 4.17
Correlation among Judgment, Misconception and Instruction

	Ba	Bb	Bc
Ba	1.000		
Bb	.693	1.000	
Bc	.723	.804	1.000

Note. Ba, Bb and Bc are three sub-questions of each question in Test B.

The simple regression analysis results shown in Table 4.18 revealed that Grade Point Averages in Mathematics were also a good predictor of preservice teachers' pedagogical content knowledge. However, regression coefficient of GPA on general college courses on PCK is not significant.

Table 4.18
Regression Analysis of Preservice Teachers' PCK

Model	Standardized Coefficients	Sig	Adjusted R Square
GPAG	.255	.052	.025
GPAM	.449	.000	.191

Elementary and Middle School Preservice Teachers' PCK

MANOVA and discriminant analyses shown in Table 4.19 suggested that preservice teachers of different major groups scored differently on PCK. The Box's M is not statistically significant (.219), suggesting that the assumption of homogeneity of covariance matrices is met. The similar log determinants of the two groups deliver the same information. Similar to Test A, only one discriminant function was obtained from SPSS. The Wilks' Lambda is statistically significant, which means that five PCK constructs can also be used to classify preservice teachers into different major groups. Moreover, the univariate ANOVA in Table 4.20 show that the two major groups differ significantly on FWVR-T, FBVRAR-T, FWRS-T, and PS-T ($p<.01$), but not on FWMS-T ($p=.769$). The results of tests of equality of group means in Table 4.21 suggest that FWVR-T is the most important variable of Test B in discriminating the two groups. The standardized discriminant function coefficients in Table 4.22 show the same indication about the importance of different variables on discrimination function.

Table 4.19
Multivariate Analysis of Preservice Teachers' PCK between Different Levels of Teacher Education

	Wilks' Lambda	Sig.	Box's M	Sig	Log Determinant	
					Group 0	Group 1
Level of Teacher Education	.660	.000	19.963	.219	14.322	14.836

Table 4.20
Between Level of Teacher Education Effects of Preservice Teachers' PCK

Group	Test Factor	df	Mean Square	F	Sig.	Observed Power(a)
Level of Teacher Education	FWMS-T	1	19.498	.942	.334	.161
	FWVR-T	1	2285.395	38.187	.000	1.000
	FBVRAR-T	1	83.174	11.670	.001	.923
	FWRS-T	1	245.652	11.073	.001	.909
	PS	1	366.768	15.102	.000	.971

a. Computed using alpha = .05

Table 4.21
Tests of Equality of Different Level of Teacher Education Mean Score of Preservice Teachers' PCK

	Wilks' Lambda	F	df1	df2	Sig.
FWMS-T	.991	.942	1	102	.334
FWVR-T	.728	38.187	1	102	.000
FBVRAR-T	.897	11.670	1	102	.001
FWRS-T	.902	11.073	1	102	.001
PS	.871	15.102	1	102	.000

Table 4.22
Standardized Canonical Discriminant Function Coefficients

	Function1
FWMS-T	-.450
FWVR-T	.788
FBVRAR-T	.101
FWRS-T	.104
PS	.322

Classification results shown in Table 4.23 indicate that this function correctly classifies about 76.0% of the cases with similar proportion of mistakes for the two groups. This would generally be considered a satisfactory level of discrimination.

Table 4.23
Classification Results for PCK

Level of Teacher Education			Predicted Group Membership 0	1	Total
Original	Count	0	43	15	58
		1	10	36	46
	%	0	7401	25.9	100
		1	21.7	78.7	100

The misclassified cases for discriminant function of preservice teachers' CK and PCK on levels of teacher education were compared to explore the relationships between them. According to the case-wise statistics of CK and PCK, there were only 6 subjects misclassified in both CK and PCK discriminant analysis. The other 38 subjects were only misclassified in CK or PCK. The qualitative analysis of data in this chapter provides more information about this issue.

Preservice Teacher's PCK Measures and Structure

Confirmatory Factor Analysis (CFA) using M-plus was also conducted to check the model fit of Test B, in testing preservice teachers' PCK level. There are also five sub-constructs for Test B corresponding to the five sub-constructs in Test A, namely, Flexibility across Mathematical Symbols in Teaching (FAMS-T), Flexibility within Visual Representation in Teaching in Teaching (FWVR-T), Flexibility between Visual Representation in Teaching (FBVR-T), Flexibility with Modeling Process in Teaching (FWMP-T), and Procedural Skills in Teaching (PS-T). Similar to Test A, two measurement models were used to decide the model fit of the data and to find out which construct contributed the most to the test. Measurement Model - PCK presented in Figure 4.3 show a five latent factor with 2 indicators for each latent variable (except for

FWMP-T which is connected to 3 indicators). The Second Order Model-PCK for Test B shown in Figure 4.4 is a second order CFA, which intends to determine how the sub-constructs covary to test preservice teachers' PCK.

The results are reported in Table 4.24, which show that both of the measurement models fit the data based on information from Chi-square statistics as well as other model fit indices, such as CFI and RMSEA. Table 4.25 provides information about standardized path loading of each sub-construct on preservice teachers' overall PCK. According to the path loadings, F2-T and F3-T contributed the most to testing preservice teachers' PCK, while F1-T is not an important part of Test B. Further information about this issue will be discussed in qualitative analysis.

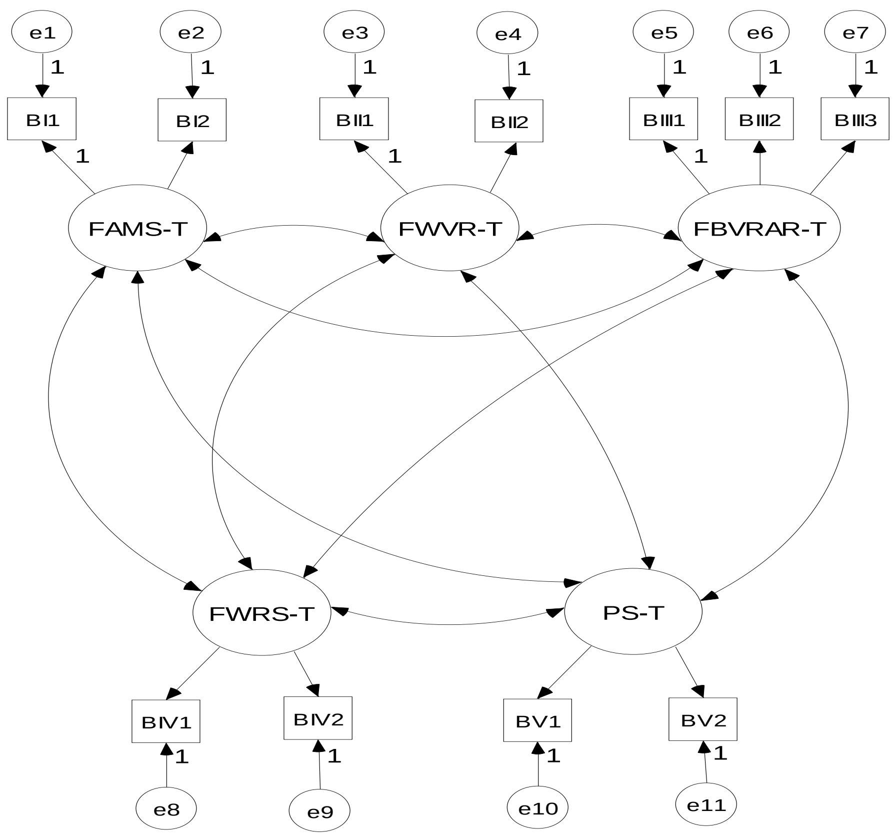


Figure 4.3. Measurement model of preservice teachers' PCK

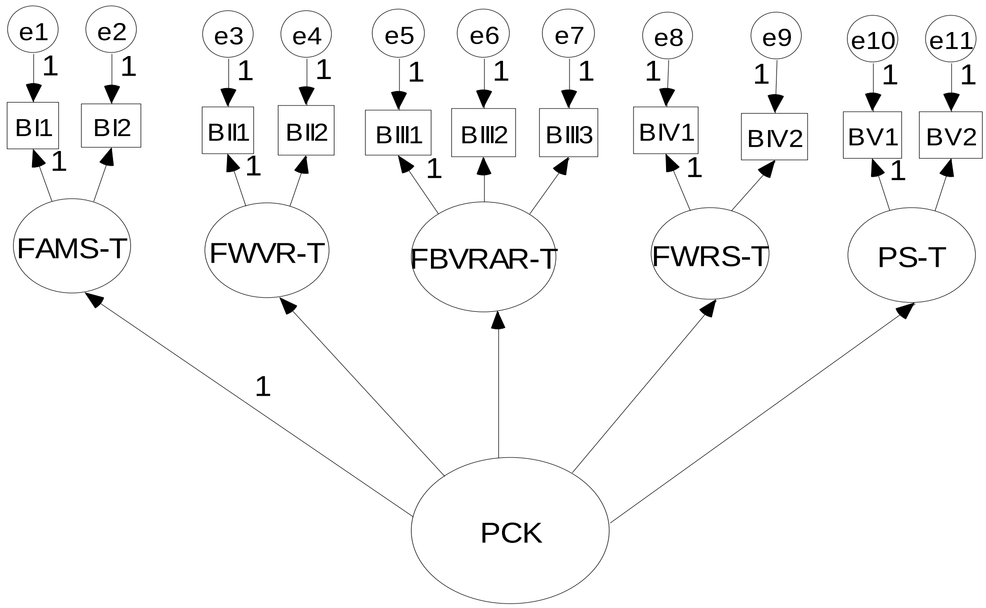


Figure 4.4. Second order model of preservice teachers' PCK

Table 4.24
Goodness of Fit Indices for Testing Preservice Teachers' PCK

Model	Chi-square	Df	P-Value	CFI	RMSEA (C.I.)
Measurement Model I -T	41.387	35	.2118	.968	.042 (.000-.085)
Second Order Model -T	51.979	40	.0972	.940	.054 (.000-.0091)

Table 4.25
Standardized Path Loading of Latent Subconstructs on PCK

	Standardized Path Loading	S.E.
FWMS-T	.232	.104
FWVR-T	0.958	.314
FBVRAR-T	.884	.292
FWRS-T	.757	.352
PS-T	776	.405

Further, the standardized path coefficients of each indicator toward five latent sub-constructs in Table 4.26 explain how well a set of indicators covary to test a single construct.

Most of the path coefficients are fairly high with $p<.01$ significance level, which suggests the convergent validity of the test items. The correlation matrix of the five latent sub-constructs, FAMS, FWVR, FBVR, FWMP and PS is shown in Table 4.27. The results suggest the existence of strong relationship among these latent sub-constructs indicating these constructs are not distinct in testing preservice teachers' PCK, which implies that the discriminate validity of Test B is not satisfactory. The Split-Half reliability test statistics, .742, shows that it is also a moderately reliable test.

Table 4.26
Standardized Path Coefficients of Testing Preservice Teachers' PCK

	Standardized Path Coefficients	S.E.
FWMS-T		
BI1	.324	.242
BI2	1.00**	.209
FWVR-T		
BII1	.565	.360
BII2	.550	.302
FBVRAR-T		
BIII1	.628*	.292
BIII2	.631*	.279
BIII3	.678*	.320
FWRS-T		
BIV1	.560	.384
BIV2	.618*	.303
PS-T		
BV1	.579	.472
BV2	.568	.293

Table 4.27

Correlation Matrix for Different Constructs in PCK (Discriminant Validity)

	F1	F2	F3	F4	F5
FWMS-T	1.000				
FWVR-T	.379	1.000			
FBVRAR-T	.160	1.053	1.000		
FWRS-T	.124	.923	.696	1.000	
PS-T	.263	.898	.724	.606	1.000

In brief, the results of CFA model fit, standardized path loading of indictors , correlation matrix of the five latent sub-constructs, and Split-Half reliability test justified Test B to be a valid and reliable measurement instrument of testing preservice teachers' PCK. From the test scores of the preservice teachers, we can find out how well they are prepared in their knowledge in terms of the five aspects of linear function.

Qualitative Analysis

Pedagogical content knowledge is one of the most important components of teachers' knowledge, which plays a significant role in teaching. This study investigated two aspects of preservice teachers' pedagogical content knowledge, the understanding of students' conceptions and misconceptions and their repertoire of teaching strategies. The former component was assessed through the first two sub-questions under each question of Test B, while the third sub-question assessed the latter component. The results of this test revealed that, for all five aspects of representational flexibility for teaching, he preservice teachers did not have sufficient knowledge or teaching experience of linear functions to assess students' conceptions and misconceptions when solving the problems. Further, the strategies that they used to teach the related subject matter mostly focused on the procedures of solving the problem rather than the conceptual knowledge of the problem. The following sections will provide some detailed information of subjects' understanding of students' conceptions and misconceptions and teaching strategies.

Preservice Teachers' Understanding of Students' Cognition

It is common that students make mistakes, from which teachers have the opportunity to diagnose students' conceptions or misconceptions of specific subject matter knowledge. It is only by understanding students' conceptions and misconceptions that teachers can correct students' mistakes effectively and help students understand the content better. Therefore, teachers' ability to assess students' conceptions and misconceptions is important for high quality teaching.

According to the descriptive statistics of Test B, the majority of subjects' answers to the understanding of students' conceptions and misconceptions fell into one of three categories. The answers in the first category failed to point out the mistakes made by the students or agreed with the student's incorrect solutions. The answers in the second and third categories recognized part or all of the procedural mistakes of students. However, few answers provided conceptual sources of the students' mistakes. Only a small percentage of answers were in the fourth and fifth categories that require part or full explanation of the sources of the students' mistakes.

Based on descriptive statistics of Test B, Question 2 and Question 6 in Test B were the most difficult questions for subjects to determine the conceptions and misconceptions underlying students' answers. About 80% of the subjects failed to point out students' misconceptions or agreed with the students' solutions.

Question 2 assessed subjects' flexibility with modeling process for teaching. It states: "Mary Wong just got a job as a clerk in a candy store. She already had $42. She will earn $7 per hour. How many hours will she have to work to have a total of $126? Write an equation to represent the problem." One student's response was as follows:

> 126 – 42 = 87/7 = 12 hours.

A majority of the subjects (90 out of 104) thought the answer provided by the student was correct and students did not have misconceptions at all in solving this problem. They said:

> yes, the answer is correct. She has a good understanding
> yes. She is thinking correctly!

Some subjects stated the students' conception of solving the problem correctly, but few of them realized that the students' were required to use a linear function instead of arithmetic method to get the answer. For instance, one subject (Jennifer) explained what the student was thinking as the following:

> Subtract amount has already known then divide by $7/hr to see how many hours needed to make $84. No misconceptions!

Interestingly, many subjects wrote an equation to solve the problem on the margin of the test paper during the test, but failed to point out that students should also use an equation. One subject wrote the equation:

$$\begin{aligned} y &= 7x + 42 \\ 126 &= 7x + 42 \\ 84 &= 7x \\ x &= 12 \text{ hours} \end{aligned}$$

However, she said that students' answer was correct. Some subjects could provide both arithmetic and algebraic approaches correctly, but still failed to see that the students were supposed to use an equation to solve the problem. For example, one subject (Parris) wrote:

> yes, the answer is correct.
>
> $$\begin{aligned} 7x + 42 &= 126 \\ 126 - 42 &= 7x \\ 84 &= 7x \\ 84/7 &= x \\ x &= 12 \end{aligned}$$

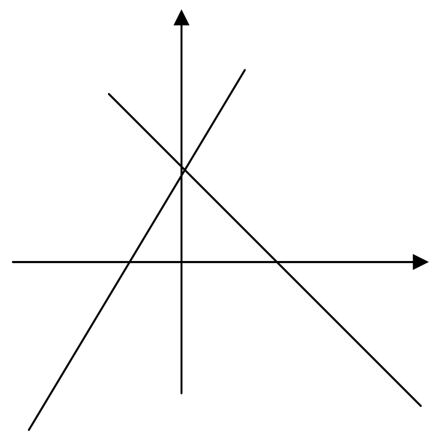
She used the algebraic approach to check if student's answer of 12 hours was correct or not. Then she provided an explanation for students' mistakes.

> She [Mary Wong] needs $126, she has $42 already, therefore, she needs to earn $126 – 42, which is $84. She will make $7 an hours, so she will have to work 12 hours to make $84. (84/7 =12)

These subjects did have knowledge of using a linear function to solve real life problem, however, they did not suggest students should use an algebraic equation to get the answer as requested by the problem. It seemed that they did not pay much attention to using a linear function to solve the problem but only focused on the final answer the student got. They overlooked the fact that this question was testing student's ability to use an algebraic method instead of arithmetic method to solve real life problem. The inability to use the algebraic method of solving problems can result in students' limited success in applying linear functions in real life situation. When mathematics problems get more complicated, the algebraic method is more effective to use to solve problems.

Another mistake of the student's solution is that he/she used the equal sign improperly. Only one out of 104 subjects were able to point out the "=" sign problem. The equal sign is a common error made by students. Teachers often treat the equal sign as the “it is.” The misunderstanding of equal sign could cause problems later on in their study.

Question 6 asked students to solve $2x + 5 = -x + 5$ using a graph, one student’s response was as follows:

Andy’s solution: $2x + 5 = -x + 5$ graph $y = 2x + 5$ graph $y = -x + 5$ answer (0, 5)	

The problem resembled a common mistake that students have when they use a graph to solve the equation, that is, the approach the student use was correct, but he did not realize that the x value was asked for in the problem, not an ordered pair.

The preservice teachers were asked to judge if the student’s answer was correct and also to explain what the student was thinking, which can reflect their flexibility between visual representation and algebraic representation for teaching. About three quarters of the subjects did not point out the student’s mistake or misconception. Some of the subjects did not know how to use the graph to solve the problem. For example, one Subject (Emily) said “I don’t know how they did it or how would solve it. I don’t understand.” Even though some subjects thought the student’s answer to be wrong, according to their description of the student’s misconception, they did not know how to solve this problem. One subject (Shelley) thought the students’ answer was wrong, but reasoned not that the student forgot that x is the only answer but the student used an incorrect procedure to solve the problem. Actually the students’ procedure of solving the problem is correct. The subject said:

> The student thought (2x+5) and (-x+5) were 2 separate lines when in fact it is just looking for x…two graphs cannot result from one value of x.

Another group of subjects approved the students’ procedure of solving the problem, but didn’t realize that only x value was needed, either. Renee wrote:

> Student saw that they had 2 graphs so they put them in y = form and graphed them where the lines intersect, is the solution.

For Question 10, 85% of the subjects pointed out that the students made a mistake. However, about half of them only provided a procedural explanation of student’s misconceptions without stating why the student made the mistake. The question is provided in the following:

> Coordinates (2, 5) and (3, 7) are two points on the line. What is the slope of the line?
>
> One student’s response was as follows:
>
> I cannot decide the slope of this line. Because slope is m in y = mx + b, for example, only if we are told that the line such as y = 2x + 5, then we know the slope is 2. Otherwise we don’t know.

Although it was not difficult for the subjects to judge that the student made a mistake, many of them did not think conceptually why the student made such a mistake or try to find the source of the mistake. Therefore, many of the subjects' answers were like this one "He doesn't know two point slope form."

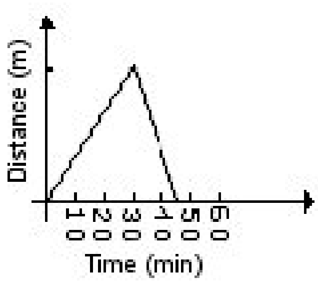
Some subjects gave a conceptual reason why the student made the mistake. For example, one subject (Ashley) provided correct, complete and conceptual reasons why the student's answer is not right.

> They are thinking that each side of the equals sign is a line, and they found the point of intersection, but the answer they are looking for is the x value in the point (0, 5), so x=0.

Question 1, assessing preservice teachers' flexibility between visual representation and algebraic representation in terms of concept of change, was an example of their subject lack of correct conceptual understanding of the student's mistake. The concept of change was fundamental to understanding functions and many real life situations (NCTM, 2000). In the case of linear functions, slope represents constant rate of change. A solid understanding of constant rate of change is a basis for more complex non-constant change. Understanding change in various contexts can start with qualitative change. Distance and displacement is an important topic when students learning algebra. Although about 80% of the subjects realized that the student made a mistake in graphing the relationship between time and Rebecca's walking distance, 68% of them only listed a procedural description of the student's mistake.

1. Rebecca took a walk to Hensel Park from home. She walked in a constant pace. It took her 20 minutes, and she rested there for 10 minutes. Then she walked back home, and it took her 15 minutes. Draw a graph that best represents the relation between time and the distance Rebecca walked.

   One student's response was as follows:

   (a) Is the answer correct?

   (b) What do you think the student is thinking about the problem and graph?

   (c) If you think the student has misconceptions with respect to the problem and graph, how would you assist this student?

This question provided a real life context that dealt with time and distance relationship. There were three types of relationships involved in this situation. First, Rebecca walked constantly from home to the park for 20 minutes. It was a straight line going upward and to the right as time goes by. Second, when Rebecca rested for 10 minutes, the distance did not change however the time elapsed. Third, Rebecca went back home from the park, and it only took her 15 minutes. The speed walking home compared to the speed walking to the park was faster. The "walking back home" part also involved change of directions.

The student's response in the problem revealed several common misconceptions. For the first 20 minutes, the student drew a line to represent 20 minutes and the distance walked, and at the same time the student realized it was a constant rate, so the graph was straight. When Rebecca rested for 10 minutes, the student did not realize that the distance remained the same while time elapsed, so the graph of "resting in the park" still resembled that of "walking to the park." For walking back home, the student confused the concept of displacement and distance.

"Walking to the park" and the graph. Almost all of the subjects (100 out of 104 subjects) were able to judge the student's response regarding walking to the park correctly, and be able to provide a correct graph to correspond to the first 20 minutes.

"Resting in the park" and the graph. More that half of the subjects realized that the students made a mistake in this part of the graph, but did point out the correct source of students' misconception that the student was not aware that the distance remained the same when time elapsed. Most of these subjects only described step by step the procedure of the student constructing the line. For example, one subject (Emily) said:

> I think they are think about walking somewhere /, stopping to rest and then going back to where they started.

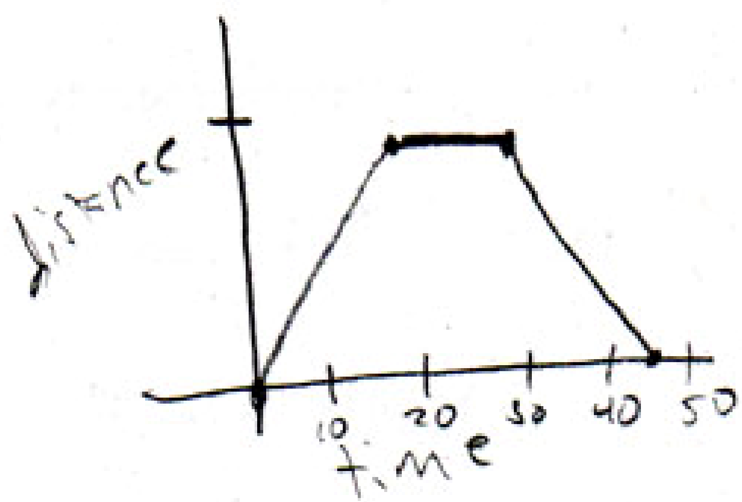
Some of the subjects only realized that the student ignored the resting time of the walk. Unawareness of resting time was then considered by some of the subjects as the source for student's mistake with the graph for "resting in the park." They assumed student did not realize Rebecca was resting and thought only about her walking time

> She does not realize that in her graph Rebecca is resting. She just sees that after 30 minutes Rebecca starts returning home. (Abby)

I think that the student was only thinking about her walking time. The student added in the resting time to the time it took her to get to the park. (Jennifer)

The second phase subject Rachel also gave a similar answer in the test. The following transcript of the interview revealed how she was thinking about the student's misconceptions.

I: Do you think the student had misconception based on the graph he provided?
S: Yes, here…10…20…30…40…50… I think that's enough. Ok it said he walked for 20 minutes it's a steady rate unless it says otherwise but then he rests for 20 minutes? So he's not going to go anywhere for 20 minutes. And then, he's going to walk for 15 more minutes and he's going to stop. And the student did not take into consideration. Well actually, hold on… he walked back home. So he's going to go, all the way back home in 15 minutes. But he's not going to go anywhere for those 10 minutes, so he's not going to move anywhere, so he's not going to go any distance. And this student was like, ok, well I am just going to walk all the way, he said that he's going to walk for 20 minutes, but for those 10 minutes that he rests, she still keep putting a distance. And then to get all the way back down in 15 minutes, she should end up all the way back down, so she's right about that, but she should of leveled it out for those 10 minutes.



Still some students successfully pointed out the conceptual source of student's mistake. For example, some of the answers were stated as the following:

They must think you need to add distance when you add time. (Britney)

> "resting" takes place at one point. Consider more of the locations, instead of distance. (Diana)

> The student is thinking that resting takes place at one point. (Kathy)

By providing this explanation, Britney correctly identified student' thinking in related to "resting in the park" and the graph. She thought that student graphed the "resting in the park" incorrectly may due to his confusion with the concept of location and distance, i.e., not aware that distance is different from location.

"Walking back home" and the graph. The majority of subjects (91 out of 104) failed to find out that student made a mistake when drawing the "Walking back home" graph, indicating their limited knowledge on the differences between displacement and distance as well as on time and distance relationship. Accordingly, they could not be able to explain student's error in this part because they themselves make the same error.

Only 14 subjects recognized student's mistake for graphing "walking back home." Some of the subjects attributed student's difficulties with graph of "walking back home" to their lack of confusion with slope and distance. They assumed student's downward graph for "walking back home" was because they thought walking to the park meant slope was increasing while walking back home slope should be decreasing.

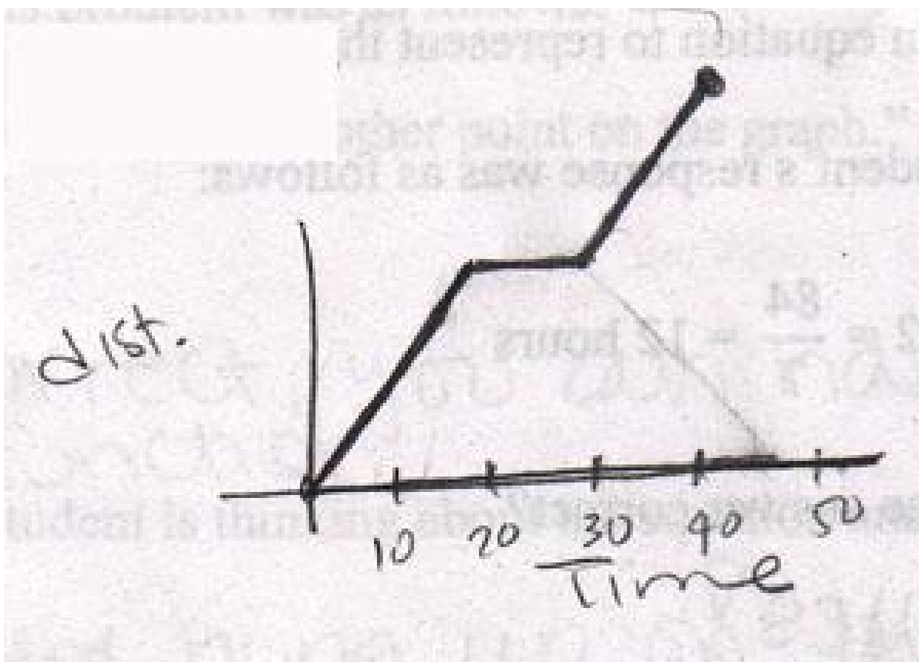
> They are thinking that you draw the slope increasing when walking to the park and decreasing when walking home. (Shelley)

In fact, relating a student's mistake on the graph with his understanding of slopes may sound reasonable. But by saying slope is increasing while walking to the park and decreasing when walking home, the subject herself did not seem to understand in either case slope was constant because it was a linear relationship.

Some other subjects also related student's error in relation to the slope. These subjects tried to explain that student's mistake was due to their knowledge of slope being positive or negative. One subject provided an explanation on student's error in relation to the sign of the slope.

> They must think … return time gets a negative slope while walking there time is positive.

Other subjects provided different but correct explanations for the student's error on the graph of "walking back home" and some of them even drew a correct graph for the problem. For example:



One of the explanations was that the student's graph of "walking back home" resulted from their confusion with the path and the distance. One subject answered the question as following.

> I think they are a picturing more of how her path will look, since she will be going to the park then coming back.

Others attributed the student's mistake to their misunderstanding of distance.

> I think the student assumes that walking back home subtract from the distance. The student is thinking that the time back (\) took 15. The student is not counting the distance.

> No [student's answer is incorrect]. They are not considering the distance correctly. They are thinking that since Rebecca is walking back home, the line on the graph goes back down.

A few of subjects of this group pointed out that student was confused by displacement and distance. For example:

> I don't think the student knows how to represent time passing but no distance and they seem to be representing displacement instead of total distance.

Renee, one of the interviewed subjects, illustrated how she almost made the same error as the student did. At first, she was thinking student only made mistake for the "resting in the park." Later, she realized that the question was about distance not displacement. She, therefore, provided the sources of the student's misconceptions. She explained:

> S: Well it's the whole displacement, distance thing, like you know, displacement, or your total distance is how far you walk but the displacement is how far you are from your original starting place so when I first drew it I went back down thinking cause I looked at what the student did and I was like, oh that's good, you know you turn around you go the other direction, so I was like ok, they just missed a little plateau in there, but then once I started writing about it I was like, wait, what am I doing, they still walked the whole distance,

Summary. Taken together, there are multiple reasons why a large number of preservice teachers were not able to correctly and conceptually point out students' conceptions and misconceptions about linear function problems. First, they might lack knowledge to judge the students solution. Second, some of them had the same misconception as the students. Third, many subjects only recognized the procedural mistake of the students but did not have a conceptual understanding of the problems.

Additionally, some preservice teachers could evaluate students' answers to be incorrect but their explanation contained their own misconceptions of the problems The few preservice teachers who did provide correct explanations of students' misconceptions set up a model for other preservice teachers how to translate students' mistakes into sources of students' misconceptions.

Preservice Teachers' Instructional Strategies

Using proper strategies to help students learn from misconceptions is an important part of teaching. Teachers need not only have knowledge of nature and sources of students' mistakes, but also a repertoire of instructional strategies to help students with their misconceptions. Knowledgeable decisions with instructional strategies should be an aim for teachers to obtain. It is only through effective instructional strategies that teachers can guide students on the right track of understanding and solving linear function problems. Some of the instructional strategies by the preservice teachers only provided procedural instruction to assist students to solve the problems, while others combined the procedural instruction with conceptual explanations of rationales for the problems. The latter category of instructional strategies is more effective in teaching students linear function, since they help students build up a conceptual understanding of linear function and its relationship with multiple representation modes. The results of the quantitative analysis of preservice teachers' answers to Test B indicated that the majority of them tended to offer procedural instruction and ignore the conceptual explanation. The following section will use

Question 1 and 10 to explore the details of how preservice teachers attempts to provide instruction to clear students' misconceptions.

Subjects' choice of instructional strategies for Question 1 varied from pure procedural instruction to fully conceptual explanation. As discussed in the earlier section, the major misconceptions of students' graph of time and Rebecca's walking distance are the relationship between distance and time as well as difference between the concepts of displacement and distance.

For the first misconception of the student, some subjects only provided procedural instruction to help students understand that there should be a straight line to represent the distance they walked during the resting time. Some examples are the following statements:

> Show them what is really going on. (Britney)

> Help the student realize that what the student draw show that she didn't stay at the park, that she went to the park and back. (Margaret)

The interview with one of the subjects, Rachael, revealed that she focused more procedural instruction.

> R: ok umm, I would, if a student was having a problem with it, I would get her to walk me through it, like talk me through it, and say, you know, ok you're walking for 20 minutes, how would you think the graph's going to go if you're walking 15 minutes and get them to show me and ask me why, and get them to explain it to me, and then I'd say ok, so you're going to rest for 10 minutes, now are you going anywhere during that time or are you going to … are you moving or are you not moving? So should you show you are moving? Or not? But you're still using time and then I'd ask, you know… when you came back home, how long did it take? I would talk it out with the student.

A small portion of the subjects were able to use more conceptual instructional strategies. For example, one of the interviewed subjects, Renee explained how she would assist students with the problem as the following:

> so what I said I would do for this to help the misconceptions is that I said that depending on how much time and how many students had the problem you could make up a little grid in your classroom or you could say like here's the trashcan and here's the chalkboard and say, I want to walk to the trashcan and y'all time me, ok this took me 10 seconds and so we can have little kids on the board writing and graphing it took me 10 seconds to walk all the way to the trashcan and if we want we can have places in the middle to keep the data going and then I'm going to stand there at the chalkboard and just erase something on the chalkboard and say where am I going, like how do you represent this on a graph, am I moving back and forth, am I going a distance, am I staying still, is time elapsing or is time not elapsing and just like introduce them to everything and then I'll say ok, I'm going to walk back to the trashcan and if the student on the board draws it back down like this then I'll say, are you telling me I didn't go anywhere? And they'll be like well, on your graph you said I didn't walk anywhere because the distance…wait this is right how I did it? I'll be like according to your graph, my total distance I walked nowhere and they'll be like, no but you just came back. And I'll be like but you still have to add up if this is 5 feet all the way here and this is 5 more feet, I want those 5 feet in my graph and so it will show them that they'll have to go up and then elapsed time will show them they have to… they have to use the elapsed time even if you're not going up but time is still ticking like on a stopwatch.

Her explanation showed that she had a full conceptual understanding of the relationship between time and distance. Moreover, she guided the students to understand this relationship by modeling real-life situation and question inquiring. She asked students to time her while she walked at a constant pace to the trashcan. It took her 10 seconds and then she had students to come up the board to graph it. What she did next was standing there at the chalkboard and guided students to think about both time and distance by asking whether the time elapsed or not and whether she was going a distance. Then the students were asked to make a graph that represented her standing at the chalkboard. This was a good example of using multiple strategies to help students conceptually understand the problem.

For the second misconception, a larger percentage of subjects who recognized the difference between displacement and distance used more conceptual instructional strategies to assist the students.

> I would explain that when she is walking home she is still adding to her total distanced traveled and the line would continue to go up. (Jenney)

> The distance Rebecca walked to return home is a positive addition to the total distance she walked. (Barbara)

Another example is an interview with subject May, whose explanation suggested that she fully understood the related concept and would guide the students to find the discrepancy between displacement and distance. Additionally, she used the real-life situation again to facilitate teaching concepts to students.

> That’s kind of tricky because that’s all in how you read the question and you’ll have to explain to the kids that’s all about how you read the question and if they want to know how far she walked total or how far from the house she walked it’s a discrepancy in there…. I think when they get involved and some of them come up to the chalkboard and write on the chalkboard and if you get a real life example of me walking somewhere, taking a break, maybe to get a drink of water and then walking back, they see you did travel the 10 feet it took and so the graph, you wouldn’t want to show you traveled 0 feet, but then we could do it again and we’d say if I started at the trash can, how far from the trash can am I when I end, they can see how the graph should be, umm like reading the graph would be different based on the question that was asked.

Question 10 dealt with the students’ misconceptions about slope and the fact that two points determine a line. Most subjects used procedural instructional strategies to help students’ with their misconceptions. About one third of the subjects merely gave a two-point slope formula to explain how to find the slope of the line. For example:

> To use the formula $m = y_2\text{-}y_1/ x_2\text{-}x_1$ and find that it is 7-5/3-2 or 2/1 m=2. (Kate)

> I would give them the formula $y_2\text{-}y_1/x_2\text{-}x_1$ and tell them to solve for the slope. (Helen)

> Remind them of $m = y_2\text{-}y_1/x_2\text{-}x_1$ (Shelley)

Another 10% of the subjects added a common informal explanation of the slope, rise over run, instead of the rate of change of a line. For example: Angelina gave the following answer:

> Remind them that (2, 5) and (3, 7) are points of a line of a graph, and that slope means rise over run. So I would have them plot the points on a graph and determine the slope. So counting the rise first you go up 2 and over 1, so the slope is 2/1=2. Then I would tell them the formula $y_2\text{-}y_1/x_2\text{-}x_1$ = 7-5/3-2 = 2/1 = 2.

Only 18 out of 104 subjects combined procedural instruction with conceptual explanation of slope, such as “tell him the slope formula and teach him what a slope actually is.” Another good example is the answer:

> I would first see if he understand what slope means when he looks at a graph on line (then with the equation)… then I would review or introduce the concept that slope can be found with $m = y_2\text{-}y_1/x_2\text{-}x_1$ or $m = y_1\text{-}y_2/x_1\text{-}x_2$ and explain what these x’s and y’s are with given points and follow with an example.

Summary. Quantitative results of preservice teachers’ scores on the third sub-question of Test B suggested that their capability of applying both procedural and conceptual strategies in teaching is not well developed. The qualitative analysis confirmed the quantitative results. Although some of the subjects were able to provide conceptual instructional strategies to help students with their misconceptions, most of them failed to give instruction or directly use procedural strategies without further explaining concepts of the problems. Too much attention was put on process and formula for solving the problem, while the concept or rationale of why students should use certain process and formula to solve the problem was often neglected. Many preservice teachers took problem solving of linear function as memorizing the formula instead of understanding the mathematical concepts.

Results for Research Question 3

Research Question 3

How does preservice teachers' subject matter knowledge influence their knowledge of students' conceptions and misconceptions of linear functions and their instructional approaches for addressing students' misconceptions?

Quantitative Analysis

To investigate the relationship between preservice teachers' CK and PCK and how the five sub-constructs Test A correspond to the five sub-constructs of Test B, a correlational model of confirmatory factor analysis combining two second order analysis was conducted. Figure 4.5 shows the structure of this model. The two second order latent variable CK and PCK were connected by a curve with two arrows to indicate their correlation with each other. So were the corresponding constructs of each test.

The model fit indices in Table 4.28 show that the correlation model fit the data well with chi-square statistics 0.0715 (>0.05), CFI larger than 0.9 and RMESEA smaller than 0.05. The standardized path coefficients of correlations between each pair of construct as well as between preservice teachers' CK and PCK are reported in Table 4.29. They suggest strong relationship between preservice teachers' CK and PCK, however, the correlations between each pair of constructs were not high.

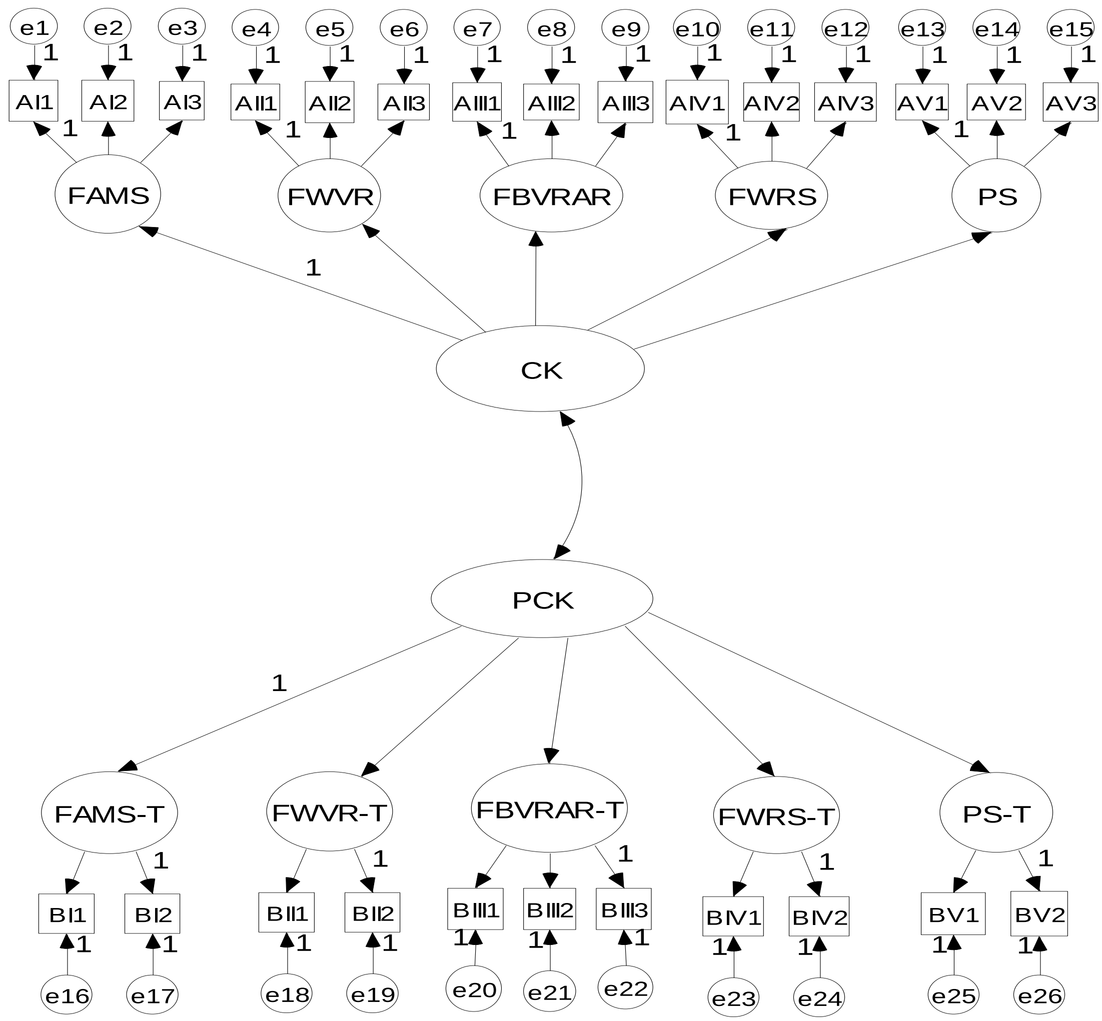


Figure 4.5. Correlation model of preservice teachers' CK and PCK

Table 4.28

Goodness of Fit Indices for Relationship between Preservice Teachers' CK and PCK

Model	Chi-square	Df	P-Value	CFI	RMSEA (C.I.)
Correlation Model	321.769	286	.0715	.924	.035 (.000-.053)

Table 4.29

Standardized Path Coefficients of Correlation between Preservice Teachers' CK and PCK

	Standardized Path Coefficients	S.E.
FWMS with FWMS-T	0.133	0.112
FWVR with FWVR-T	0.871	-0.031
FBVRAR with FBVRAR-T	0.381	0.136
FWRS with FWRS-T	0.261	0.032
PS with PS-T	0.591	-0.012
CK with PCK	0.890	0.091

The model fit indices in Table 4.28 show that the correlation model fit the data well with chi-square statistics 0.0715 (>0.05), CFI larger than 0.9 and RMESEA smaller than 0.05. The standardized path coefficients of correlations between each pair of construct as well as between preservice teachers' CK and PCK are reported in Table 4.29. They suggest strong relationship between preservice teachers' CK and PCK, however, the correlations between each pair of constructs were not high.

Further, a structural equation model with five latent variable predicting preservice teachers' CK was also analyzed using M-plus, as shown in Figure 4.6. The results in Table 4.29 show that the five constructs of Test A are able to predict preservice teachers' PCK. Information about each sub-construct as a predictor of preservice teachers' PCK is given in Table 4.30. It is obvious that preservice teachers' FWVR and PS are the most important variables in the prediction (Table 4.31).

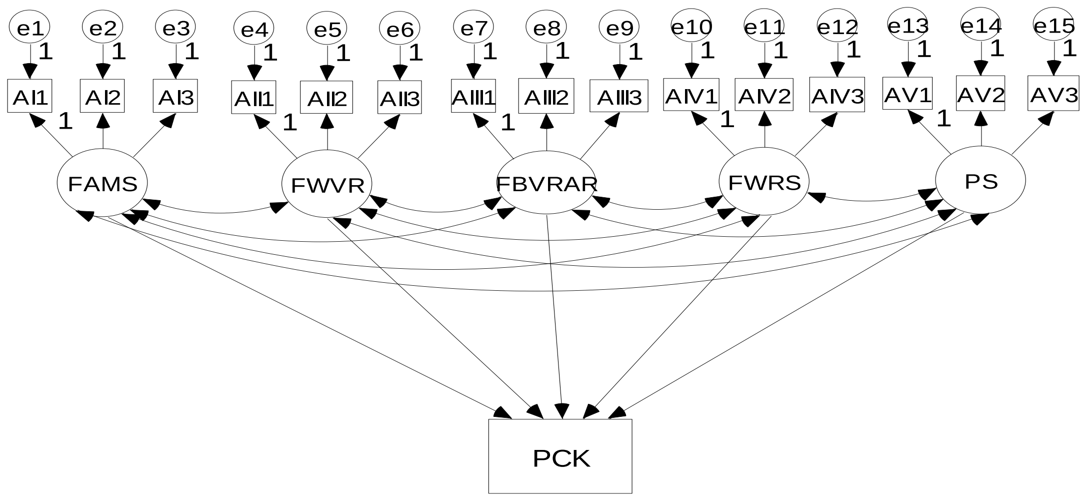


Figure 4.6. Prediction of preservice teachers CK on PCK

Table 4.30
Goodness of Fit Indices for Prediction of Preservice Teachers' PCK by CK

Model	Chi-square	df	P-Value	CFI	TLI	RMSEA (C.I.)
PCK Prediction Model	99.516	91	.254	.968	.957	.030 (.000-.062)

Table 4.31
Standardized Path Loading of Latent Subconstructs Predicting Preservice Teachers' PCK

	Standardized Path Loading	S.E.
FWMS	.376	.004
FWVR	.813	.002
FBVRAR	.402	.006
FWRS	.230	.009
PS	.824	.002

Qualitative Analysis

In this section the relationship between preservice teachers' subject matter knowledge and pedagogical content knowledge in the case of linear functions is investigated. Does higher subject matter knowledge influence pedagogical content knowledge? Is there some sort of relationship between these two aspects of teachers' knowledge? The results of the SEM correlation model of preservice teachers' CK and PCK shown in the first part of this chapter indicated a strong relationship between them. Further analyses about how they are related will be presented in this section of the paper.

Question 6 in Test A and Question 7 in Test B used the same problem to test preservice teachers' subject matter knowledge and pedagogical content knowledge of flexibility between visual representation and algebraic representation, specifically in using linear equations to express ordered pairs. In Question 6, the subjects were asked to solve the problem, while Problem 7 provided a hypothetical situation in which the student's solution to the problem was provided so that preservice teachers acted as teachers to evaluate students' answers. They were further asked to express students' thinking and their teaching strategies. The interviews were used to investigate how subjects' answers to Question 6 in Test A were connected to their responses to Question 7 in Test B.

Aston in her answer to problem 6 in Test A, wrote an incorrect equation: V = 16800 – 3200t. In order to further investigate what she was thinking while she provided the answer, she was asked to describe the reasoning for solving this problem.

I: How did you do this problem [point to question 6 in Test A]?
S: I subtracted that. The difference between them is 3200 divided by …and this is a time of two years. The difference, and there's a 2 years, ok but…umm, the value could go down, assuming this trend keeps up which I don't know is true in cars but if you're facing your information only on what you have here then you would have to say for every two years, the car will go down this amount of money. That's the table indicates.

It is obvious that she had no difficulties finding the pattern that the car value went down. She illustrated that for every two years the car's value went down the amount of 3200. Based on her understanding of the pattern in the problem, she was asked to explain how to express the pattern. She illustrated:

> Aston: ok, umm…as time…whatever your value is…your value goes down…twice…for each year…every two years…I am sorry I just totally fried my brain…ok… multiply it by years, and then subtract it from your value, your original value.

Aston seemed not sure of the way to express the functional relationship. When she was asked to describe the functional relationship, she started to express her doubts of how realistic the problem itself was. She finally accepted the assumption that the problem was realistic and tried to explain the functional relationship. Her mumbling of "twice", "every two year" and "for each year" showed signs of trying to figure out the rate of change. Unfortunately, she finally decided to write an equation on the paper (a blank paper that was presented to the subjects during the interview):

V = x (original value) – 3200* year

She then explained: "so if you take the original value minus 3200 times the number of years. That would give you your value."

Taken together, although Aston was not able to write a correct equation for the problem, she did have some understanding of the pattern regarding the value of the car and its year. The knowledge of pattern enabled her to evaluate the student's response in Problem 7 as incorrect. In response to the sub-question (b), she pointed out that the fact of student's mistake was that he focused on the missing value. Furthermore, she attributed the student's mistake to his failure to realize a function relationship by examining the paired orders in the table.

As expected, in the interview, she showed a general idea of the nature and sources of the student's mistake, which was consistent as her answer to sub-question (b) of Problem 7. However, the interview also revealed that her insufficient knowledge of the functional relationship limited her from going further to see student's mistake from a more thorough and specific perspective. She basically repeated her answer to sub-question b of problem 7:

> Aston: Ok if you look at the chart and he just doesn't understand the question at all because it's stated so symbolically. A rule expressing the inflection of time you can clearly see a function here from the other answers in the table. The student looked like he was trying to fill in the missing blank but not really answer the question which is just come up with an expression for how the car is devaluing.

When explaining the strategies to help students, she said: "I will tell him the value goes down 3200 every year." It is apparent she could not provide a further help to the students because of her limited content knowledge.

Another example to explore the relationship between preservice teachers' subject matter knowledge and pedagogical content knowledge is the comparison of subjects' answers to Question 2 and 3 of Part V in Test A and Question 5 in Test B.

A common mistake that students have when they are asked to simply an algebraic equation is to treat it as equations. Both Question 2 and 3 from Part V of A involved simplifying the algebraic expressions while question 3 also required solving the equation. Almost all the preservice teachers (91 out of 104) could solve these two problems correctly. This result showed that they have little difficulty with simplifying algebraic expressions and solving equations.

1. Simplify $5x - 2 - 6x + 7$

3. Solve $\frac{(x-2)}{2} = \frac{(x-3)}{4}$

Question 5 from test B offered a hypothetical situation where a student treated simplifying algebraic equations as solving equation, that is, to add "= 0" to the algebraic equation 2x + 7 + 3x - 9. Adding "= 0" is considered as a common error made among students.

5. Simplify 2x + 7 + 3x - 9.

One student's response was as follows:

$2x + 7 + 3x - 9 = 0$

$5x - 2 = 0$

$5x = 2$

$x = 2/5$

(a) Is the answer correct?

(b) How do you think the student is thinking about the equation?

(c) If you think the student has misconceptions with respect to the equation, how would you assist this student?

Interestingly, while few preservice teachers had difficulties when they were asked to simplify expressions, they did not make a correct judgment when they were presented with the student's error. Fifty-two of 104 subjects thought the student's solution is correct. Twenty-eight of 52 people who simplified problem 15 correctly could not detect student's mistake. Some of them even started to make the same error as students did. It seems their thinking was influenced by the student's mistakes.

> For problem 15 in test A, their answer was
> x+5
> Or
> 5 – x
>
> For problem 5, their answers were:
>
> ---yes [the student's answer is correct].
> 2x + 7 + 3x – 9 = 0
> 5x – 2 = 0
> 5x = 2
> x = 5/2
>
> ---Putting like variables together then solving for x.
>
> ---yes [the student's answer is correct]. The student went through all the same steps that I would go through and teach the students.
>
> ---yes [the student's answer is correct]. Combines like terms using basic additional subtraction set x by itself to solve.

Moreover, only 20 subjects who successfully simplified algebraic expressions were able to recognize student's answer as incorrect.

> No [the student's answer is incorrect]. She is thinking about solving equation for x, not just simplifying. (Julia)
>
> No [the student's answer is incorrect]. They are probably assuming that this expression is equal to zero, allowing you to solve for x. (Tracy)

These subjects had a clear understanding of student's mistakes, that is, recognizing the confusion between simplifying algebraic expressions and solving equations was the source of the student's mistake.

Summary

Generally speaking, preservice teachers' CK and PCK are strongly correlated, which was confirmed by both quantitative and qualitative analysis of data. However, as suggested by the literature (Wilson, 1994), this relationship between CK and PCK is indeed complicated. Question 6 is a good example that subjects' inadequacy of content knowledge in linear function will limit their PCK, namely, understanding of students' misconceptions and usage of instructional strategies. However, even though most subjects did well in solving the problems of Question 2 and 3 of Part V in Test A, few of them realized the student's mistake on Question 5 in Test B. Apparently subjects were easily confused by the student's mistake that most of them would not have when they themselves were asked to solving the same problem. It might be the consequence of carelessness of subjects when answering the questions or their lack of experience in identifying students' mistakes.

CHAPTER V

DISCUSSION AND SUMMARY

The first goal of the study was to examine preservice teachers' subject matter knowledge. Two aspects of teachers' subject matter knowledge were investigated, i.e. representational flexibility and procedural skills. Representation flexibility was investigated in four aspects, i.e., flexibility among mathematical symbolisms, flexibility within visual representations, flexibility between visual representations and algebraic representations, and flexibility with real-life situations.

The second goal of the current study was to investigate two aspects of preservice teachers' PCK, i.e., knowledge of students' conceptions and misconceptions and teaching strategies for helping students' misconceptions. Knowledge of the nature and sources of students' misconceptions helps teachers to make appropriate instructional decisions and enables them to teach meaningfully. In addition, effective teaching requires teachers to have a repertoire of teaching strategies to help students with their misconceptions.

The third goal of the study was to investigate the interrelationship between the two components of preservice teachers' content knowledge and two aspects of PCK. Previous studies have not reached a unanimous conclusion about the relationship between preservice teachers' CK and PCK. The current study applies both quantitative and qualitative methods to examine how preservice teachers' CK can affect their understanding of students' conceptions and misconceptions as well as the instructional strategies.

Discussions of Main Findings

1. Elementary and middle school preservice teachers' subject matter content knowledge in terms of representation flexibility and procedural skills.

The current study revealed that preservice teachers did not have the flexibility across mathematical symbolisms for linear relationships. The participants were best at the slope-intercept form of linear functions. Many of them failed to transfer from the standard form to one-point form and to multiple format of two-point form. Some mentioned that they had never been introduced two-point form before. Even for some participants who could be able to get the right one-point and two-point form, they still transferred all the alternative forms to the slope-intercept form to check if their answers were correct. Still others admitted that they just guessed one point or two points to plug in the one-point form and two-point form formula.

This finding is not surprising since the majority of the k -12 curriculum is focused only on slope-intercept form. Linear functions are generally introduced in slope intercept form. Students mainly learn $y = mx + b$, and they tend to remember m is slope and y is intercept without really understanding the concept of slope and intercept. So their knowledge of linear function is only limited to the mechanical memorizing of slope-intercept form.

More importantly, for some participants who obtained the right point for one-point form failed to understand that any point on the line would satisfy one-point form, thinking it has to be a certain point. They did not realize that slope and any one point could decide a line. The same misunderstanding holds for the two-point form. They were not aware that any two distinct points would make a two-point form. These misconceptions indicate that they did not understand the basic property of linear function, i.e., any two distinct points could decide a line.

The background survey revealed that almost all the preservice teachers took the required mathematics courses. But they lacked of knowledge of some basic properties of linear equations such as two points determine a line, and when the slope and one point on the line are known, one is able to write the equation for the line. Furthermore, the participants in the study did not have a deep understanding of linear functions and they could not connect different forms of linear functions. They attributed the reason to their teachers who did not introduce the other forms of linear functions other than slope-intercept form. It appears that these preservice teachers' teachers may not be able to teach other forms of linear functions other than slope to their students, which may result in the preservice teachers' vicious circle effect on their students.

Most of the participants knew how to write slope as m when they transferred the standard form to $y = mx + b$. Many of them failed to understand slope as $-a/b$ and $(y1-y2)/(x1 - x2)$. Stump (2001) reported similar findings in their study that preservice teachers knowledge of slope was only limited to m in $y = mx + b$.

Preservice teachers were found to be better at symbolic manipulations, such as solving equations, than other types of representations. In particular, they scored low on items on flexibility between visual representations and algebraic representations and flexibility with real life situations. First, preservice teachers seemed not able to transfer from symbolic representation to graphical representation. Similar findings were reported in Even's (1989) study of preservice teachers' knowledge about mathematical functions. Her study revealed that the participants had difficulties transferring from symbolic representation to graphical representations and vice versa when dealing with quadratic and sine functions as well as inverse functions. Many participants of the current study were not able to write an equation for the given ordered pairs even if they had already figured out the relationship or pattern of the ordered pairs. Some participants were not able to find a correct graph based on the given slope information of a linear function. These findings suggested that they did not have a conceptual understanding of what linear function or graph represented and the relationship between algebraic and visual representation of linear functions.

A trend was found in the study when participants dealt with real life situations. A majority of the participants were able to write an algebraic equation for real life word problems. However, when they were asked to provide a real life situation for an algebraic equation, only few of them could successfully describe a suitable situation. The participants were asked to real life situation to two algebraic equations, $y = 6x + 2$ and $x + 2 = 10$. Participants were found to perform better in describing the second equation than the first one. The possible reason could be that in the second equation, the unknown variable has a set value eight, however, the second equation $y = 6x + 2$, deals with two variables x, y had infinite values.

Although preservice teachers scored better on flexibility within visual representations, further examination of their answers revealed their lack of a global property view of graph and linear functions. Being asked to provide a graph based on the algebraic equations, they could only construct the graph by plotting out multiple points and then connecting them. By doing this, the participants showed they understood linear function point-wise instead of paying attention to the global properties, such as, a slope and a point can decide a line. Even though some participants wrote a linear equation to solve the problem successfully, their graph of the equation turned out to be a curve due to their poor scaling skill, which showed that they were not aware that the graph for a linear equation should be a straight line.

It is not surprising that participants did better at the symbolic manipulation skills than at other types of representations. Algebra is often taught with a focus on symbolic manipulation which results in a lack of understanding of different representations in learning functions. There have been consistent calls for developing student algebraic thinking beyond symbol manipulations (e.g., Silver, 1995; Steen, 1992).

The poor performance of participants in representational flexibility provided implications on what should be stressed in teacher education. First, besides symbolic manipulations, multiple representations and their connections should be more focused. Second, algebra should be taught more meaningfully related to the real life. Finally, new technology could be applied in the teaching to improve representation skills.

2. Elementary and middle school preservice teachers' PCK in terms of knowledge of students' misconceptions and their teaching strategies

In this study, strong connections were found between the ability of judging students' answers and their knowledge of students' thinking and teaching strategies. Further, knowledge of students' misconceptions strongly predicted the effectiveness of instructional strategies. The results showed that participant were not able to provide effective strategies if they did not know the nature and sources of students' mistakes. With limited knowledge of students' misconceptions, their strategies tended to be general and not mathematics content specific. It is apparent that teachers first need to have knowledge of students' misconceptions to provide effective instructional strategies. However, most preservice teachers did not perform well on either understanding students' misconceptions or applying proper instructional strategies.

The study found preservice teachers were not competent at figuring out the source of students' mistake even after they determined the students' answer to be incorrect. Often participants pointed out the students' procedural mistakes in solving the problem but not the conceptual misunderstanding of the problem. The lack of conceptual understanding of students' misconceptions may due to several reasons. First, since these teachers were preservice teachers, they had limited experience with students. The more experienced the teachers are, the more knowledge that they may have regarding students' common mistakes. Second, they themselves may have limited understanding or some misconceptions with the topic. It is hard to imagine teachers could help students' misconceptions if they themselves had trouble with the topic. For example, in this study, participants had difficulties attending to the entire graph, especially when the graph expressed the relationship between two simultaneously changing variables. In one question, preservice teachers were asked to evaluate students' graph on time and distance relationship. The results indicated that teachers themselves had misconceptions with graphs. Therefore, they would not be able to help students. This result seemed to differ from the findings of the existing literature (Janvier, 1987; Krabbendam, 1982) which showed students were more competent with graphs of functions in which one of the variables is time and time-dependent because familiarity of time and its unidirectional nature (time only increases). Preservice teachers as future teachers should be expected to perform better than students. One of the reasons that could explain why preservice teachers in the study did not do well in the time-distance graph is that the problem involved the concept of distance and displacement. Distance and displacement turned out to be a confusing topic for subjects.

A majority of participants were only able to provide strategies that focused on procedural knowledge, which is in accordance to the findings of their understanding of students' misconceptions. Most of the preservice teachers ignored the concepts or rationales of the problems on linear functions and took problem solving of linear function as memorizing the formula. Further, they did not know how to relate to the real life situations, and to use multiple representations to help students with their misconceptions.

3. Relationship between preservice teachers' subject matter content knowledge and PCK.

Generally strong positive relationships were found between preservice teachers' CK and PCK through quantitative analysis of data. Specifically, flexibility within visual representations and procedural skills were the two most important predictors for preservice teachers' PCK. However, qualitative analysis of preservice teachers' answers suggested that this relationship is not as straightforward but indeed very complex and intricate. This finding has also been suggested in many other literatures (Wilson, 1994).

Participants' inadequacy of content knowledge in linear functions will limit their PCK, namely, understanding of students' misconceptions and usage of instructional strategies. Wilson (1994) described how a preservice secondary teacher's understanding of function developed as she participated in a mathematics education course integrating mathematical content and pedagogy. The course influenced her understanding of function (subject-matter knowledge) but her anticipated approach to teaching was less significantly affected by it. This study helped to understand the role of the subject matter played in teacher learning and teaching. There is a necessity for deeper research into the relations of subject matter knowledge and pedagogical content knowledge.

Preservice teachers who perform better at the representation flexibility tend to understand the nature and sources of students' misconceptions better. Furthermore, when they use the strategies to help students' misconceptions, their strategies tend to be more conceptual. These findings correspond to the literature that teachers' conceptual knowledge has an influence on their pedagogical choices. Meanwhile, if the preservice teachers were lack of conceptual understanding of linear functions, their explanations of problems tended to be more procedural and non-mathematical.

High levels of CK cannot guarantee the same level of PCK. Sometimes participants were confused by the student's mistake, causing them to make errors that most of them would not have when they themselves were asked to solving the same problem. This implied that other factors exist that may affect teachers' PCK, such as teaching experience and pedagogical training in teaching mathematics. These factors are not considerations of the current study, since the participants are preservice teachers most of whom do not have any teaching and training experience before taking the tests. Future studies can made effort to involve multiple factors in prediction of teachers' PCK.

Discussions of Additional Findings

Other than the three research questions, the current study also developed some other findings about various factors affecting preservice teachers' CK and PCK.

Differences between Preservice Teachers from Different Levels of Teacher Education

Significant differences in CK and PCK between elementary education and middle school preservice teachers were found from multivariate analysis of the data. Specifically, for preservice teachers' CK, middle school education preservice teachers performed better in flexibility across mathematical symbolisms, flexibility within visual representations, flexibility between visual representations and algebraic representations, and procedural skills. However, no difference was found on elementary and middle school education preservice teachers' flexibility with real-life situations. This result may due to the fact that the questions testing flexibility with real-life situations were too difficult and beyond the knowledge of both groups of participants, since both groups performed poorly on this part of Test A. For preservice teachers' PCK, middle school education preservice teachers also scored higher than elementary education preservice teachers, especially on flexibility within visual representations, flexibility between visual representations and algebraic representations, flexibility with real-life situations and procedure skills. No significant difference was found on flexibility across mathematical symbolisms.

GPA on Mathematical Courses' Effect on CK and PCK

The current study also found that preservice teachers' Grade Point Average on mathematics course is a good predictor of their CK and PCK. Regression analysis using preservice teachers' Grade Point Average on general college courses and mathematics courses to predict their scores in Test A and Test B indicates that there is strong relationship between GPA on mathematic courses and preservice teachers' content knowledge and pedagogical content knowledge. However, GPA in general college courses is not a good predictor for preservice teachers' CK or PCK. This is a reasonable result since preservice teachers' GPA on mathematics courses is directly related to their performance on CK and in return has connection with PCK.

Preservice Teachers' Knowledge Structure

The second order confirmatory factor analysis provides some information about the structure of preservice teachers' subject matter content knowledge and pedagogical content knowledge. To be specific, the standardized path loading of each subconstruct and on general CK and PCK point to the importance of the sub-constructs in CK and PCK. For preservice teachers' subject matter content knowledge, flexibility within visual representations is the most important factor followed by flexibility across mathematical symbolisms and procedural skills. For preservice teachers' pedagogical content knowledge, flexibility within visual representation for teaching is again the most important component followed by flexibility between visual representations and algebraic representations and procedural skills. This result suggests that future measures on teacher’s knowledge about linear function should pay more attention on these components of multiple representation modes.

Instructional Representations Used by Preservice Teachers

From a close analysis of preservice teachers’ answers to the question how they would help students with the misconceptions, a pattern of their choice of instructional representations was found. Preservice teachers’ use of representations mainly focused on verbal representations, such as “I would explain to students” and “I would describe to students.” The second popular instructional representation the participants used was algebraic representation, which could easily be found in their answers, such as “I would remind them the formula.” Few real life situations and visual representations were used in preservice teachers’ teaching strategies, which have been suggested to be the most important components of teachers’ knowledge on linear functions with multiple representation modes in above section of discussion. Teachers’ choice of instructional representations will affect students’ exposure to various representations of linear functions (Monk, 2003). This has been verified from the current study through the fact that many participants claimed that they did not know certain representations because they had never been taught or shown. At the same time, studies have shown that students who have opportunities to use multiple representations will learn mathematics better, which requires a change in teachers’ instructional representations (Smith, 2003). This result also provides an important implication for improvement of mathematics teacher education.

Teachers' Knowledge and Students' Achievement

Although the current study does not include an analysis of relationship of teachers' knowledge and students' achievement, the findings of the study suggests the reason why students have high frequency of making certain mistakes in liner functions. Preservice teachers' errors found in this study are consistent with the students' common mistakes stated in the literature. For example, Cunningham's (2005) study found that teachers had difficulties transferring from graphical to tabular representations. The current study also demonstrated that preservice teachers had difficulty in transferring within visual representations. Thus, teachers' knowledge could be an important factor that affects students' achievement in mathematics.

Detailed information about how teacher knowledge will affect students' achievement is beyond the scope of the current study. However, the result of the study may provide some implications and theoretical support for future research in this field of study.

Contribution of Current Study

This study successfully extended prior study on teachers' knowledge. The results of the study suggest both the complexity of teachers' knowledge and the importance and feasibility of conducting such a study from multiple perspectives. Hopefully, this study not only contributes to our understanding of teachers' CK, PCK and the interrelationship, but also suggests a framework for research development in the future.

In particular, an integration of teachers’ knowledge and representation perspective provide a new dimension to investigate teachers’ knowledge

One of the main contributions of this study is its examination of the interrelationship between different components of teachers’ subject matter knowledge and PCK. Examining teachers’ knowledge structure in functions allows a different perspective to investigate teachers’ knowledge. Specifically, teachers’ representation flexibility was found to be significantly related to PCK. Representation flexibility is identified as a very important component in teachers’ subject matter knowledge. The aspects are included in the integrated knowledge.

Another contribution of this study is to examine teachers’ instructional representation to representation flexibility and to PCK. Teaching for understanding calls for a change in the way teachers teach. Students who are exposed to multiple representations tend to have a better achievement in mathematics. This study reflected preservice teachers’ limited knowledge in the case of functions. This study informed teacher educators of the important components of teacher knowledge in teaching linear functions. Teacher education should prepare preservice teachers to understand the different representations of linear functions, and conceptually understand linear functions in the school curriculum.

Previous studies on teachers’ knowledge have been primarily qualitative, such as studies by Ball, 1990, Ma, 1999, Leinhardt and Smith, 1985, and Even, 1989. The current study used both quantitative and qualitative methods to analyze preservice teachers’ subject matter content knowledge and pedagogical content knowledge.

One of the main features of this study is the combination of teachers’ knowledge and multiple representation modes of linear functions. Besides the investigation of teachers’ subject matter content knowledge and pedagogical content knowledge, the study also focused on process perspective of teaching and learning linear functions, i.e., representations of linear functions. Combining these two perspectives allows a more comprehensive examination of teachers’ CK and PCK of linear functions within different representation modes. This also helps to explore the structure of teachers’ knowledge in terms of multiple representation modes.

Flexibility with visual representations was strongly correlated with the corresponding components in PCK. This supports Shulman’s (1986), Hill et al., (2003), and others’ claims that CK and PCK are important components in teaching.

Implications for Mathematics Teacher Education

The study also provides some implications for the curriculum of preservice teachers in mathematics teacher education. Findings from the study support a curriculum that goes into depth and focuses more on conceptual understanding of linear functions. The mastery of subject matter content knowledge is a presupposition of teachers to understand students’ misconceptions and providing proper instructional strategies. Therefore, the reinforcement of subject matter content knowledge in mathematics teacher education is necessary. Further, the study suggests that representation flexibility is necessary and needed in preparing preservice teachers in the case of linear functions. Therefore, preservice teachers need to know how to use multiple representations to represent mathematical ideas accurately, and being able to transfer flexibly among different representation modes. This is also the responsibility of teacher education program to increase emphasis on this aspect.

Preservice teachers should have knowledge of students' cognitions and mathematics learning theories that can help them determine students' conceptions and misconceptions in solving mathematical problems. It is also preservice teachers' task to learn how to utilize effective instructional strategies and apply multiple instructional representations to help students with linear functions, which can be achieved by attending more instruction courses in teacher education. Additionally, field based courses and work with elementary and middle school students should be available for preservice teachers to learn from experienced teachers, build up their own repertoire of teaching strategies, and earn experience through classroom teaching practice.

Limitations of the Current Study

This study examined preservice teachers' knowledge structure in terms of CK and PCK using Structural Equation Modeling techniques. Due to the sample size requirement of using the SEM technique, the study combined both elementary and middle school preservice teachers and examined their knowledge structure as a whole, left unexamined measurement models and knowledge structures separately for elementary preservice teachers and middle school preservice teachers. It would be interesting to see if there are any differences between their knowledge structure and if there is any pattern regarding the differences. It certainly will provide some implications on specific content that needed be addressed in both elementary teacher education and middle school teacher education.

Furthermore, this study only contributes to the field for the understanding preservice teachers' knowledge of linear functions. However, there is much more to be investigated in the filed of teaching mathematics related to preservice teachers' knowledge. The study is only limited to preservice teachers in mathematics teacher education program from Texas A&M University. There is no way to investigate how different teacher education could contribute to teachers' knowledge of linear functions from the results of current study.

Conclusions

The current study of preservice teachers' knowledge in linear functions with multiple representation modes discovered considerable information about the three proposed research questions.

Preservice teachers have limited subject matter content knowledge in linear functions, especially the conceptual understanding of linear function and its multiple representation modes. This limitation has impeded their pedagogical content knowledge, namely, understanding of students' misconception and application of effective instructional strategies.

Preservice teachers' knowledge of students' cognition is far from adequate, and their instructional strategies turned out not to be effective, and unitary in terms of representations of linear functions. Verbal and algebraic representations occupied the most instruction the preservice teachers offered, while graphical and real-life situation strategies were rarely used.

Strong correlations between subject matter content knowledge and pedagogical content knowledge were found in the study, which suggest the needs to strengthen both perspectives of teachers' knowledge in mathematics teacher education programs.

The results of the study showed that the conceptual framework provided in the study for describing components of preservice teachers' CK and PCK served well for the purpose of the study. It provided a well-organized and clearly defined structure to build the research questions in the study, to guide the design of the test instruments to provide meaningful answers to the research questions, as well as to quantitatively and qualitatively analyze the data obtained from the tests.

The ultimate goal of this study was to investigate teachers' knowledge of linear functions and the effect of this knowledge on teaching and learning linear functions. It is also conducted with a clear intention to provide theoretical and methodological implications for improvement of teaching and learning in other mathematics content area. Hopefully, the research on teachers' knowledge in mathematics education can be extended, continued and improved by including other mathematics content area with the suggestions and implications of the current study.

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APPENDIX A

CONSENT FORM

Preservice Teachers' Knowledge of Linear Functions

within Multiple Representation Modes

I am conducting research for my dissertation during the 2006 spring semester to examine preservice teachers' knowledge of linear functions within multiple representation modes. This study will involve a maximum of 200 participants and there is no risk in participating. This research will not affect the results in your courses. Confidentiality on all documents will be strictly adhered to and kept in a secured desk drawer for up to 12 months and then destroyed. All information will be collected at Texas A&M University.

Time devoted to participating in this project would include:

1. Completing a background information survey.
2. Completing a one-hour test.
3. One possible 45 minute interview after the test.

This research study has been reviewed by the Institutional Review Board—Human Subjects in Research, Texas A&M University. For research-related problems or questions regarding subjects' rights, you may contact the Institutional Review Board through Ms. Angelia M. Raines, Director of Research Compliance, Office of the Vice President for Research at (979)458-4067, araines@vprmail.tamu.edu.

By responding to this survey, you acknowledge that you understand the following: your participation is voluntary; you can elect to withdraw at any time; there are no negative or positive benefits from participating in this research projects; the project will be used for student research; and the researcher has your consent to publish results obtained from the research.

If you have further questions, please contact me, Zhixia You, or my advisor, Dr. Gerald Kulm at 979-862-4407.

Please indicate if you would be interested in participating in this research project and fill out this form.

Thank you,

Zhixia You
Mathematics Education Doctoral Student
College of Education
Department of Teaching, Learning and Culture
4232 TAMU
College Station, TX77843-4232
zhixiayou@tamu.edu

****

I have read the above information. By signing this document, I consent to participate in this study. I understand that I can withdraw at any time without my relations to the University or courses being affected. I have been given a copy of this consent form.

____ Yes, I would be interested in participating in the linear functions project, "Preservice teachers' knowledge of linear functions within multiple representation modes" being conducted in the spring 2006 semester.

Name:____

The course that I am in:______________________________

Phone number:____________________________________

E-mail address:____________________________________

****

APPENDIX B

TEST A - PILOT

Name: ___________________

Test A - Pilot

1. Please decide if each of the following statements is a right or wrong representation of a line that is parallel to y = 2x - 5 and goes through the point (1, 4).

(a) y – 4 = 2(x - 1)

(b) y = 2x + 2

(c) $\frac{(y-2)}{x} = \frac{(y-4)}{x-1}$

(d) (y + 2)(x - 2) = (x + 2)(y - 6)

(e) $\frac{(y+2)}{(y-6)} = \frac{(x+2)}{(x-2)}$

(f) slope = 2

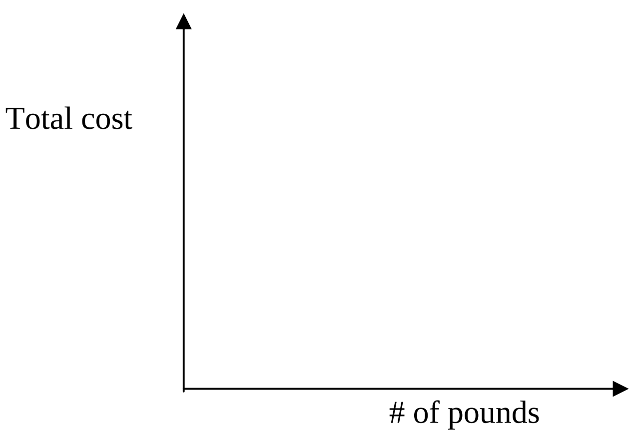
(g) slope = $\frac{(4-2)}{(1-0)}$

(h) slope = $\frac{(4-2)}{(0-1)}$

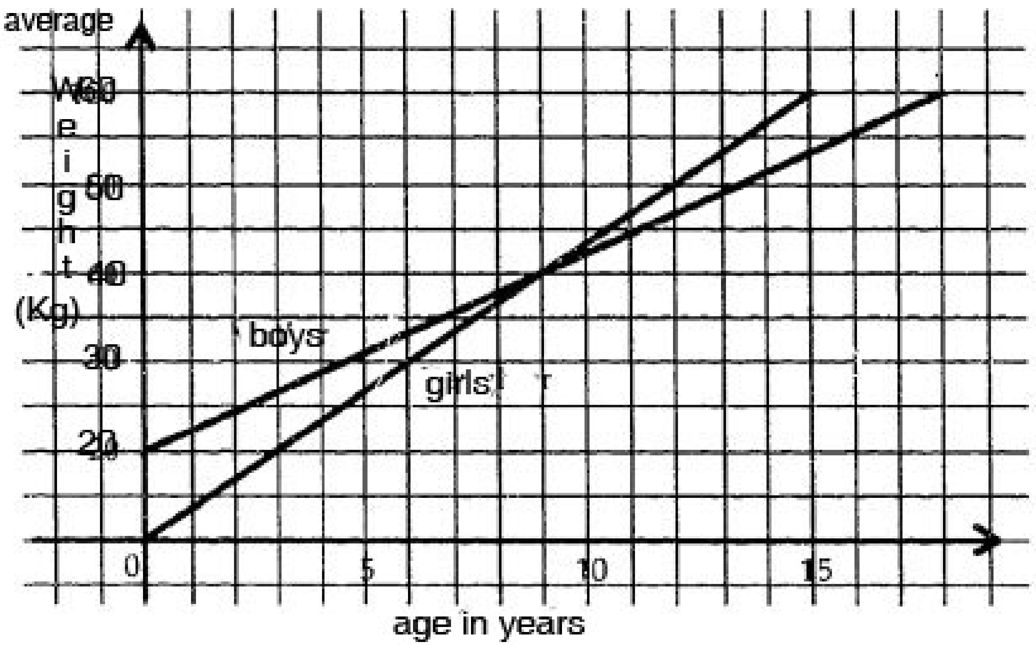
2. At Speedy Delivery Service, the cost to deliver a package is $2.00 plus an additional $.50 per pound. Fill in the missing values in the table below.

Number of pounds	Total Cost
2	**?**
?	$3.50
4	**?**
?	$7.00

Draw a graph that shows the relationship between the number of pounds and the total cost.



3. Using the following graph, answer the questions below.



(1) The average weight of boys at age 6 is ________________

(2) The average weight of girls at age 12 is _______________

(3) From what age do boys on average weigh more than 5 pounds?

____

(4) From what age do girls on average weigh more than 30 pounds?

____

(5) When (at what age) do the girls weight more than the boys?

____

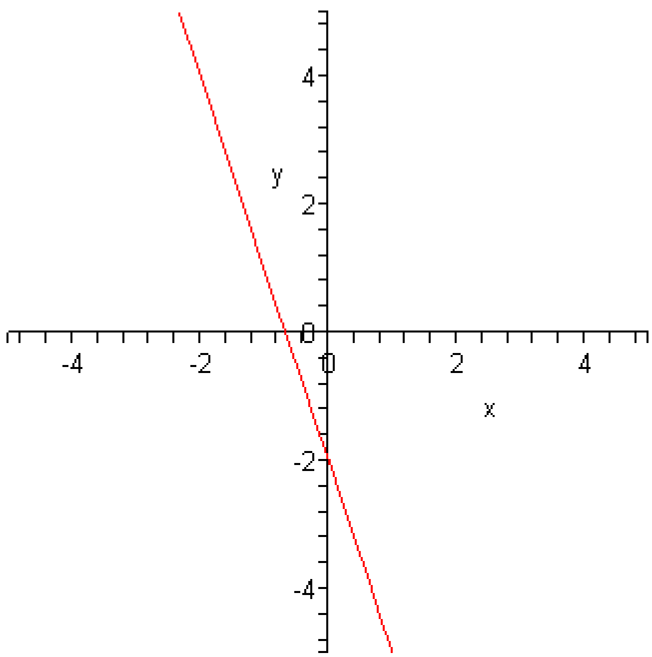
(6) By how many kilograms does the average weight of the girls increase between age 3 and age 8? _____________________

4. Suppose that the following table gives the value (V), in dollars, of a car for different numbers of years (t) after it is purchased.

t	V
0	$16,800
2	$13,600
4	$10,400
6	$7,200
10	?

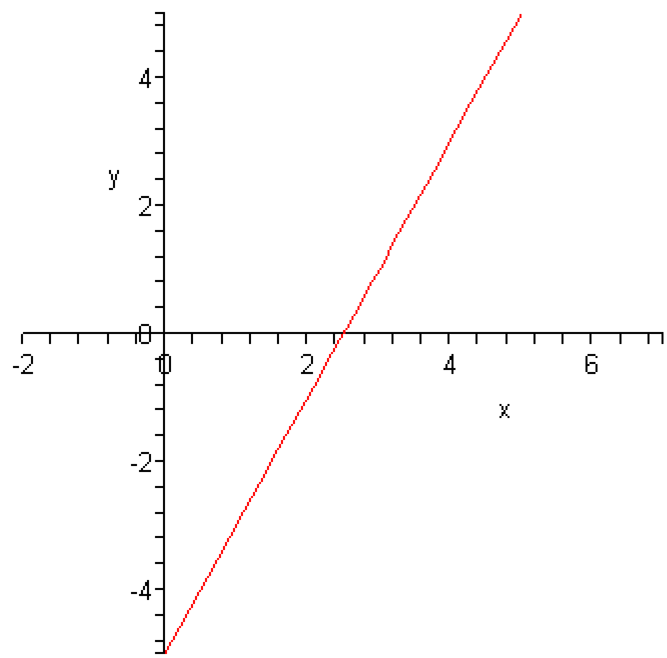
Write a symbolic rule expressing V as a function of t.

5. The graph below represents the equation ?x + 3y = -6 (We do not know the value of the coefficient of x). Find a solution to the equation without the missing value.



6. On the axes below, draw a line parallel to y = 2x - 5 that goes through the origin.

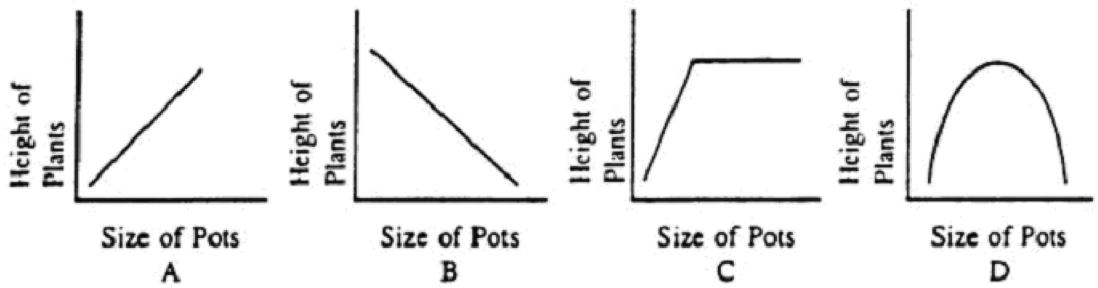
   Write the equation.



7. Describe a situation that represents equation $y = 6x + 2$;

8. Describe a situation that represents $10 = x + 2$;

9. Which graph is best described by each of the following statements?



11) As the pot size increases, the plant height decreases.

12) As the pot size increases the plant height increases up to a certain pot size. With larger pots, plant height remains the same.

Part IV

1. Describe a real life situation that represents the equation $y = 6x + 2$;

13) The situation can be:

10. A truck is loaded with boxes, each of which weighs 20 pounds. If the empty truck weighs 4500 pounds, find the following.

    a. The total weight of the truck if the number of boxes is 75. _____________

    b. The number of boxes if the total weight of the truck is 6740 pounds. ________

    c. Using W for the total weight of the truck and x for number of boxes, write a symbolic rule (or equation) that expresses the weight as a function of the number of boxes.

11. Solve $4 – x = \frac{1}{2}(3x - 1)$

12. y = 5(x - 5), if x = 163, find y.

13. Solve 4d – 18 = 58

14. Simplify 5x – 2 - 6x + 7

15. Solve $\frac{(x-2)}{2} = \frac{(y-3)}{4}$

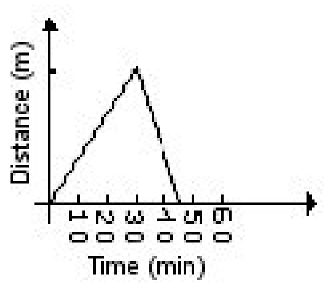
APPENDIX C

TEST B - PILOT

Test B- Pilot

1. Rebecca took a walk to Hensel Park from home. It took her 20 minutes, and she rested there for 10 minutes. Then she walked back home, and it took her 15 minutes. Draw a graph that best represents the relation between time and the distance Joan walked.

One student's response was as follows:



Is the answer correct? What do you think the student is thinking about the problem and graph? If you think the student has misconceptions with respect to the problem and graph, how would you assist this student?

2. Mary Wong just got a job as a clerk in a candy store. She already had \$42. She will earn \$7 per hour. How many hours will she have to work to have a total of \$126?

Write an equation to represent the problem.

One student's response was as follows:

$126 – 42 = \frac{84}{7}$ = 12 hours or \$126 - \$42 = a, $\frac{a}{7}$ = hrs need to work

Is the answer correct? What do you think the student is thinking about the problem and equation? If you think the student has misconceptions with respect to the problem and equation, how would you assist this student?

3. Could you provide a function so that its graph goes through (0, 2) and the value of y increases as x increases. For example: ___________.

One student's response to this problem was as follows:

"No. To do so you would need to know another point on the graph."

Is the answer correct? What do you think the student is thinking about the equation and graph? If you think the student has misconceptions with respect to the equation and graph, how would you assist this student?

4. If a line $y = kx + b$ does not go through the second quadrant, then what should k and b be?

One student’s response was as follows:

$k < 0, b \geq 0$

Is the answer correct? What do you think the student is thinking about the equation? If you think the student has misconceptions with respect to the equation, how would you assist this student?

5. Simplify $2x + 7 + 3x - 9$.

   One student's response was as follows:

   $2x + 7 + 3x - 9 = 0$

   $5x - 2 = 0$

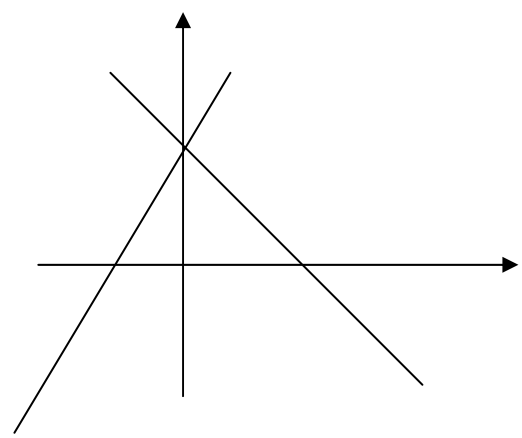
   $5x = 2$

   $x = 2/5$

Is the answer correct? How do you think the student is thinking about the equation? If you think the student has misconceptions with respect to the equation, how would you assist this student?

6. Solve 2x + 5= -x + 4 using graph.

One student's response was as follows:

Andy's solution: 2x+5= -x + 4 graph y = 2x + 5 graph y = -x + 5 answer (0, 5)	

Is the answer correct? What do you think the student is thinking about the equation and graph? If you think the student has misconceptions with respect to the equation and graph, how would you assist this student?

7. Suppose that the following table gives the value (V), in dollars, of a car for different numbers of years (t) after it is purchased.

t	V
0	$16,800
2	$13,600
4	$10,400
6	$7,200
10	?

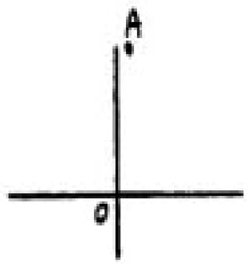
Write a symbolic rule expressing V as a function of t.

One student's response was as follows:

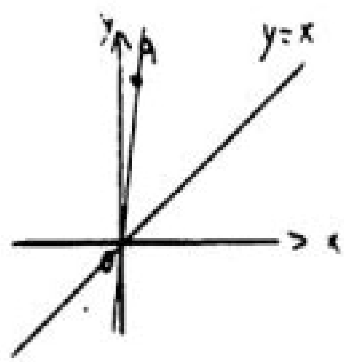
"No, because you don't know the value of car at the 10^{th} year."

Is the answer correct? What do you think the student is thinking about the equation and table? If you think the student has misconceptions with respect to the equation and table, how would you assist this student?

8. Find the equation of a line that goes through A and the origin O.



One student's response is as follows:

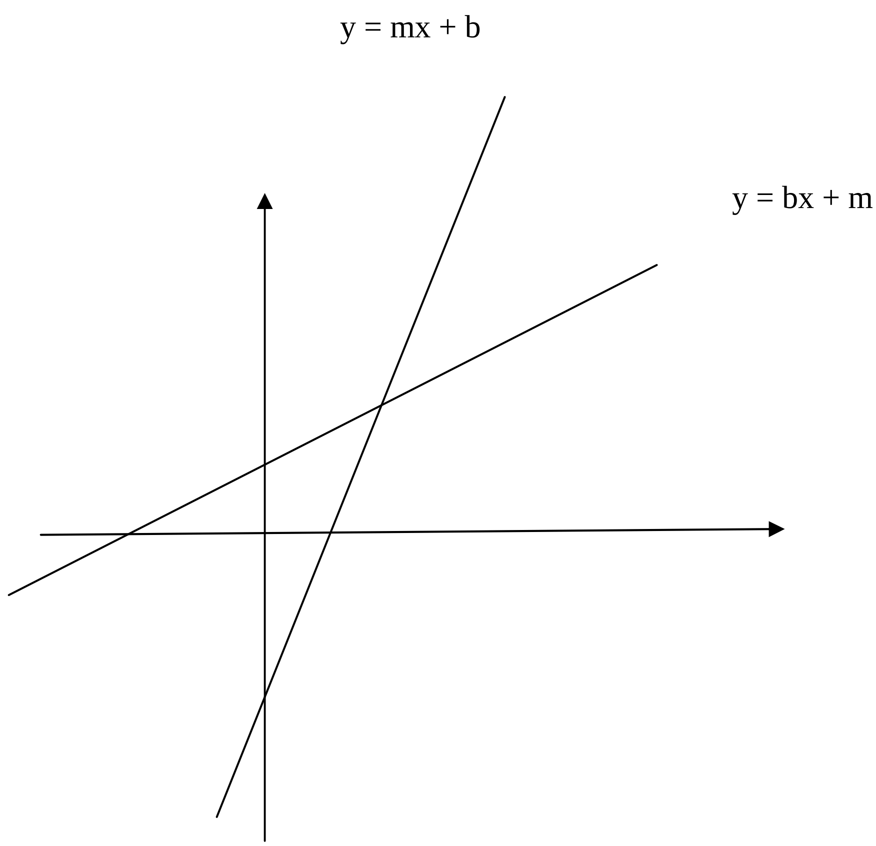


"Well, I can use the line y = x as a reference line. The slope of line AO should be about twice the slope of the line y = x, which is 1. So the slope of line AO is about 2, and the equation is about y = 2x, let's say y = 1.9x."

Is the answer correct? What do you think the student is thinking about the equation and graph? If you think the student has misconceptions with respect to the equation and graph, how would you assist this student?

9. The graph of the equation y = mx + b is shown in figure below. Draw a graph that represents y = bx + m.

    One student's response was as follows:



Is the answer correct? What do you think the student is thinking about the equation and graph? If you think the student has misconceptions with respect to the equation and graph, how would you assist this student?

10. Coordinates (2, 5) and (3, 7) are two points on the line. What is the slope of the line.

    One student's response was as follows:

    I cannot decide the slope of this line. Because slope is *m* in $y = mx + b$, for example, only if we are told that the line such as y = 2x + 5, then we know the slope is 2. Otherwise we don't know.

    (a) Is the answer correct?

    (b) What do you think the student is thinking about the problem?

    (c) If you think the student has misconceptions with respect to the problem, how would you assist this student?

11. Please decide which form is the easiest way to represent the equation of the line that is parallel to y = 2x - 5 and goes through the point (1, 4).

(Note: A line can be represented by the standard form: $ax + by + c = 0$
slope-intercept form, $y = mx + b$;
point-slope form: $(y - y_1) = m(x - x_1)$;
and two-point form: $(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$, where (x_1, y_1) and (x_2, y_2) are points on the line).

One student's response was as follows:

I would use $y = mx + b$ because it is the most common form.

(a) Is the answer correct?

(b) What do you think the student is thinking about the problem?

(c) If you think the student has misconceptions with respect to the problem, how would you assist this student?

APPENDIX D

TEST A

Name:________________________

Test A

Part I

1. A line can be represented by the standard form: $ax + by + c = 0$
slope-intercept form, $y = mx + b$;
point-slope form: $(y - y_1) = m(x - x_1)$;
and two-point form: $(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$, where (x_1, y_1) and (x_2, y_2) are points on the line.

   Rewrite the line $2x - y + 5 = 0$ in

   1) slope-intercept form: ________________

   2) point-slope form: __________________

   3) two-point form:____________________

2. A line can be represented by the standard form: $ax + by + c = 0$, two-point fractional form:

   (1) $$\frac{(y - y_1)}{(x - x_1)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

   OR (2) $$\frac{(y - y_1)}{(y_2 - y_1)} = \frac{(x - x_1)}{(x_2 - x_1)}$$

   Rewrite the line $4x + y - 2 = 0$ using

   4) form (1):_____

   5) form (2):_____

3. Slope can be represented as m in $y = mx + b$, $-\frac{a}{b}$ in $y = -\frac{a}{b}x - \frac{c}{b}$ $(b \neq 0)$ or $\frac{y_1 - y_2}{x_1 - x_2}$ when (x_1, y_1), (x_2, y_2) are points on the line. Write the slope of $\frac{1}{4}y - x - \frac{4}{5} = 0$ in the forms of

6) m=

7) $-\frac{a}{b} =$

8) $\frac{y_1 - y_2}{x_1 - x_2} =$

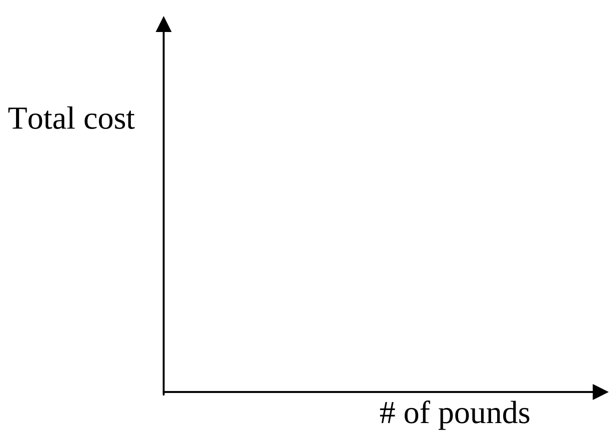
Part II

1. At Speedy Delivery Service, the cost to deliver a package is \$2.00 plus an additional \$.50 per pound.

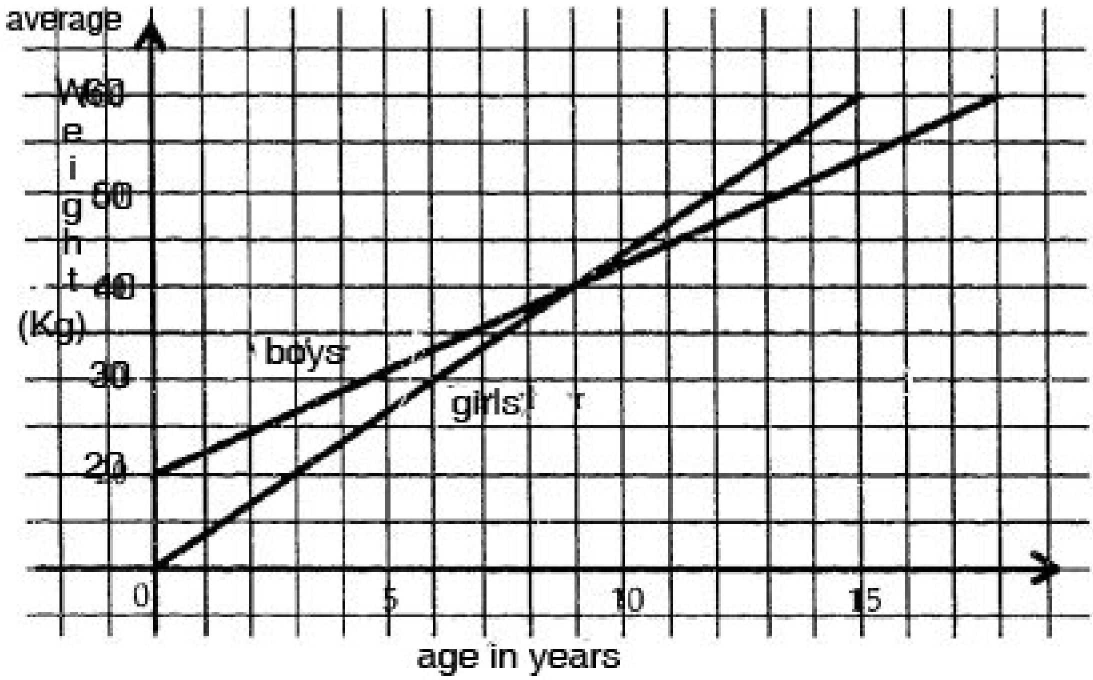
9) Fill in the missing values in the table below and write an equation that represents the relationship between the number of pounds and the total cost.

Number of pounds	**Total Cost**
2	**?**
?	\$3.50
4	**?**
?	\$7.00

10) Draw a graph that shows the relationship between the number of pounds and the total cost.



2. Using the following graph, answer the questions below.



14) The average weight of boys at age 6 is ___

15) From what age do girls on average weigh more than 30 pounds? _

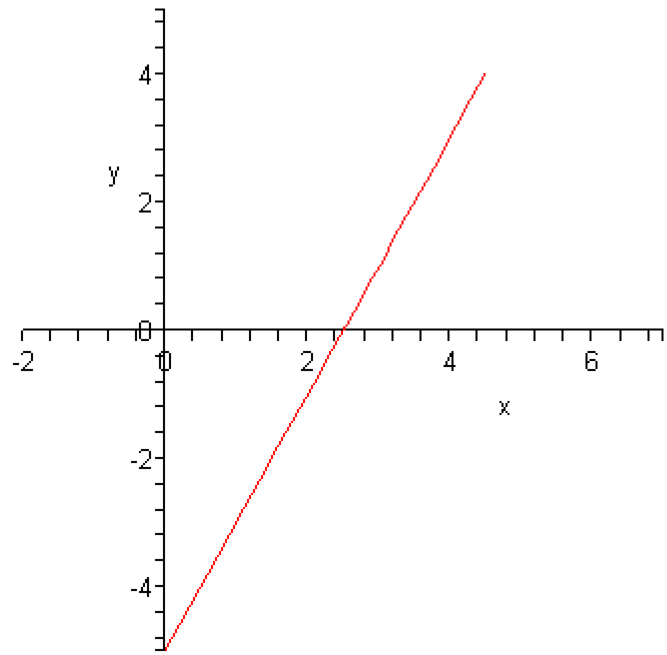
16) When (at what age) do the girls weigh more than the boys ______

17) By how many kilograms does the average weight of the girls increase between age 3 and age 8? _

3. On the axis below, draw a line parallel to $y = 2x - 5$ that goes through the origin.

18) Write the equation.

$y = 2x - 5$



Part III

1. Suppose that the following table gives the value (*V*), in dollars, of a car for different numbers of years (*t*) after it is purchased.

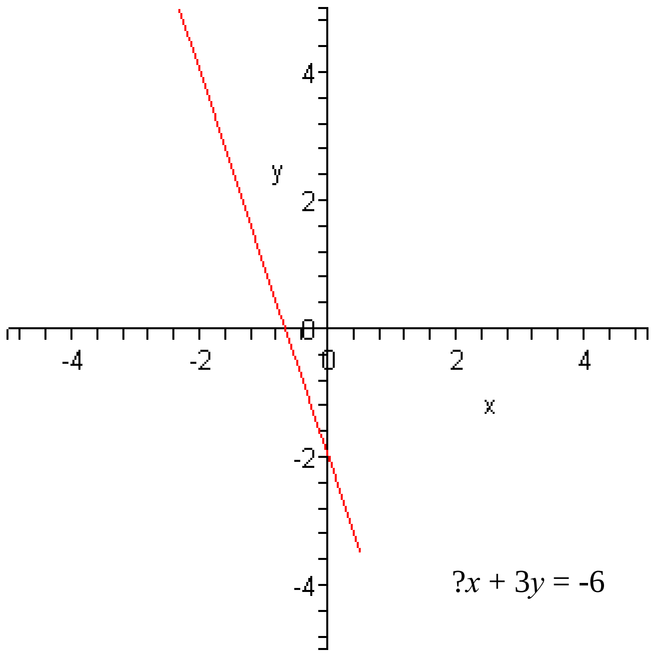
t	***V***
0	$16,800
2	$13,600
4	$10,400
6	$7,200
10	?

19) Write a symbolic rule expressing *V* as a function of *t*.

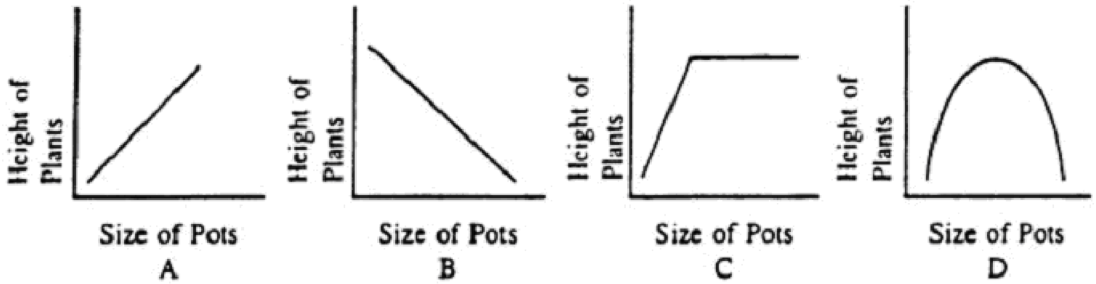
2. The graph below represents the equation ?x + 3y = -6
   (We do not know the value of the coefficient of x).

   20) Is it possible to find the missing value?

   21) If yes, what is the missing value?



3. Which graph is best described by each of the following statements?



22) As the pot size increases, the plant height decreases.

23) As the pot size increases the plant height increases up to a certain pot size. With larger pots, plant height remains the same.

Part IV

1. Describe a real life situation that represents the equation $y = 6x + 2$;

24) The situation can be :

2. Describe a situation that represents $10 = x + 2$;

    25) The situation can be:

3. A truck is loaded with boxes, each of which weighs 20 pounds. If the empty truck weighs 4500 pounds, find the following:

    23) The total weight of the truck if the number of box is 75.______________

    24) The number of boxes if the total weight of the truck is 6740 pounds.

    ____

    25) Using W for the total weight of the truck and x for the number of boxes, write a symbolic rule (or equation) that expresses the weight as a function of the number of boxes.

Part V

1. $y = 5(x - 5)$, if $x = 163$, find y

    26) The answer is:

2. Simplify $5x - 2 - 6x + 7$

    27) The answer is:

3. Solve $\frac{(x-2)}{2} = \frac{(x-3)}{4}$

    28) The answer is:

APPENDIX E

TEST B

Test B

Part I

1. Coordinates (2, 5) and (3, 7) are two points on the line. What is the slope of the line.

   One student's response was as follows:

   I cannot decide the slope of this line. Because slope is m in $y = mx + b$, for example, only if we are told that the line such as y = 2x + 5, then we know the slope is 2. Otherwise we don't know.

   (a) Is the answer correct?

   (b) What do you think the student is thinking about the problem?

   (c) If you think the student has misconceptions with respect to the problem, how would you assist this student?

2. Please decide which form is the easiest way to represent the equation of the line that is parallel to y = 2x - 5 and goes through the point (1, 4).

   (Note: A line can be represented by the standard form: $ax + by + c = 0$
   slope-intercept form, $y = mx + b$;
   point-slope form: $(y - y_1) = m(x - x_1)$;
   and two-point form: $(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$, where (x_1, y_1) and (x_2, y_2) are points on the line).

   One student's response was as follows:

   I would use $y = mx + b$ because it is the most common form.

   (a) Is the answer correct?

   (b) What do you think the student is thinking about the problem?

   (c) If you think the student has misconceptions with respect to the problem, how would you assist this student?

Part II

1. If a line $y = kx + b$ does not go through the second quadrant, then what should k and b be?

   One student's response was as follows:

   $k < 0, b \geq 0$

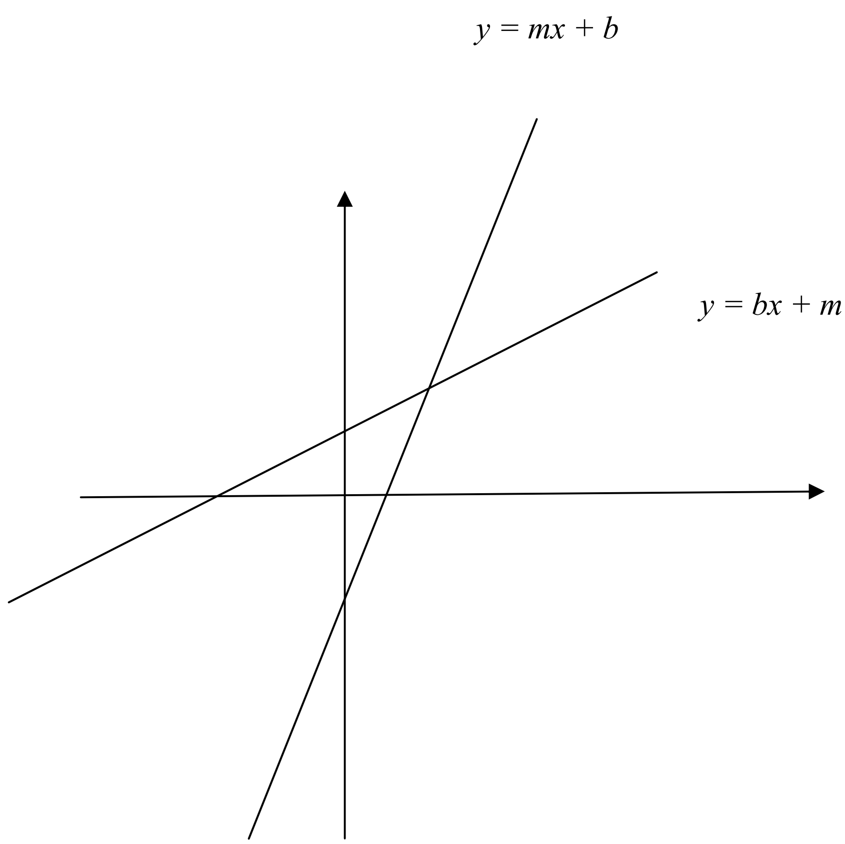
   (a) Is the answer correct?

   (b) What do you think the student is thinking about the problem?

   (c) If you think the student has misconceptions with respect to the problem, how would you assist this student?

2. The graph of the equation $y = mx + b$ is shown in figure below. Draw a graph that represents $y = bx + m$.

   One student's response was as follows:



(a) Is the answer correct?

(b) What do you think the student is thinking about the problem?

(c) If you think the student has misconceptions with respect to the problem, how would you assist this student?

Part III

1. Could you provide a function so that its graph goes through (0, 2) and the value of y increases as x increases? For example:_______________.

   One student's response to this problem was as follows:

   "No. To do so you would need to know another point on the graph."

   (a) Is the answer correct?

   (b) What do you think the student is thinking about the problem?

   (c). If you think the student has misconceptions with respect to the problem, how would you assist this student?

2. Suppose that the following table gives the value (*V*), in dollars, of a car for different numbers of years (*t*) after it is purchased.

t	***V***
0	$16,800
2	$13,600
4	$10,400
6	$7,200
10	?

Write a symbolic rule expressing *V* as a function of *t*.

One student's response was as follows:

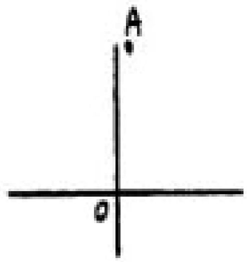
"No, because you don't know the value of car at the 10th year."

(a) Is the answer correct?

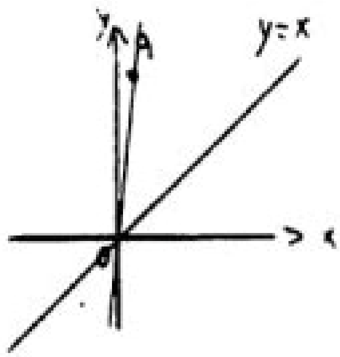
(b) What do you think the student is thinking about the problem?

(c) If you think the student has misconceptions with respect to the problem, how would you assist this student?

3. Find the equation of a line that goes through A and the origin O.



One student's response is as follows:



"Well, I can use the line y = x as a reference line. The slope of line AO should be about twice the slope of the line y = x, which is 1. So the slope of line AO is about 2, and the equation is about y = 2x, let's say y = 1.9x."

(a) Is the answer correct?

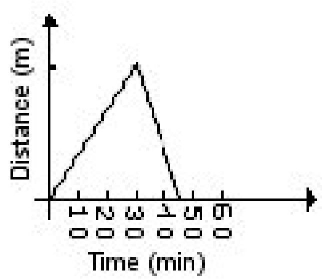
(b) What do you think the student is thinking about the problem?

(c) If you think the student has misconceptions with respect to the problem, how would you assist this student?

Part IV

1. Rebecca took a walk to Hensel Park from home. It took her 20 minutes, and she rested there for 10 minutes. Then she walked back home, and it took her 15 minutes. Draw a graph that best represents the relation between time and the distance Rebecca walked.

   One student's response was as follows:



(a) Is the answer correct?

(b) What do you think the student is thinking about the problem?

(c) If you think the student has misconceptions with respect to the problem, how would you assist this student?

2. Mary Wong just got a job as a clerk in a candy store. She already had \$42. She will earn \$7 per hour. How many hours will she have to work to have a total of \$126?

   Write an equation to represent the problem.

   One student's response was as follows:

   $$126 - 42 = \frac{84}{7} = 12 \text{ hours}$$

   (a) Is the answer correct?

   (b) What do you think the student is thinking about the problem?

   (c) If you think the student has misconceptions with respect to the problem, how would you assist this student?

Part V

1. Simplify $2x + 7 + 3x - 9$.

   One student's response was as follows:
   $2x + 7 + 3x - 9 = 0$
   $5x - 2 = 0$
   $5x = 2$
   $x = 2/5$

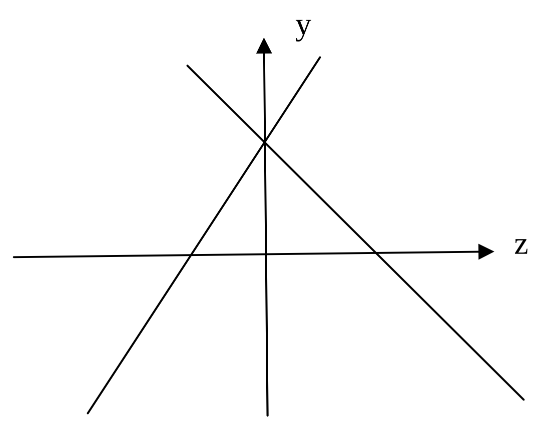
   (a) Is the answer correct?

   (b) What do you think the student is thinking about the problem?

   (c) If you think the student has misconceptions with respect to the problem, how would you assist this student?

2. Solve 2x + 5= -x + 5 using graph.

One student's response was as follows:

Andy's solution: 2x+5= -x + 5 graph y = 2x + 5 graph y = -x + 5 answer (0, 5)	

(a) Is the answer correct?

(b) What do you think the student is thinking about the problem?

(c) If you think the student has misconceptions with respect to the problem, how would you assist this student?

APPENDIX F

SURVEY

1. Name: ____________________________

2. Male ____ Female ____

3. Overall college grade point average: ____

4. College grade point average in mathematics: ____

5. Please check the mathematics courses you have taken at the college or university level”
   Differential calculus # credits ____
   Calculus # credits ____
   Differential equation # credits ____
   Advanced calculus # credits ____
   Topology # credits ____
   Euclidean geometry # credits ____
   Number theory # credits ____
   Linear algebra # credits ____
   Probability # credits ____
   Combinatorics # credits ____
   History of mathematics # credits ____
   Number analysis # credits ____
   Others:

6. Please list the education courses you have taken at the college or university level and the number of credit hours.
   __________________________________ ____
   __________________________________ ____
   __________________________________ ____
   __________________________________ ____
   __________________________________ ____
   __________________________________ ____
   __________________________________ ____

APPENDIX G

COURSE CATALOG

Course No.	Course Description
MATH 141	Business Mathematics I. Linear equations and applications, linear forms and systems of linear equation, matrix algebra and applications, linear programming, probability and applications, statistics.
MATH 142	Business Mathematics II. Derivatives, curve sketching and optimization, techniques of derivatives, logarithms and exponential functions with applications, integrals, techniques and applications of integrals, multivariate calculus.
MATH 403	Mathematics and Technology. Mathematical problem-solving and communication through the use of various theologies.
MATH 365	Structure of Mathematics I. Informal logic, sets, relations, functions, whole numbers, numeration systems, binary operations, integers, elementary number theory, modular systems, rational numbers and the system of real numbers.
MATH 366	Structure of Mathematics II. Geometry, measurement, and coordinate geometry.
MATH 367	Basic Concepts of Geometry. Formal development of geometry, finite, non-Euclidean and Euclidean.
MATH 368	Introduction to Abstract Mathematical Structures. Mathematical proofs, sets, relations, functions, infinite cardinal numbers, algebraic structures, structure of the real line.
STAT 303	Statistical Methods. Introduction to concepts of random sampling and statistical inference, estimation and testing hypotheses of means and variances, analysis of variance, regression analysis, chi-square tests
TEFB 273	Field-based course for introduction to schooling and classroom.
MEFB 352	Curriculum and Instruction for Middle Grades Curriculum. study of educational theory and instructional strategies appropriate to middle grades education including planning and development of interdisciplinary and multidisciplinary curricula, student centered learning and methodologies.

MEFB 460 Math Methods in Middle Grades. Examines theories, provides practice in teaching methods essential to successful mathematics learning; focus on content and criteria central to teaching mathematics for understanding, skill development, and problem solving; readings, discussions, analyzing, modeling and practicing mathematics teaching and learning.

MASC 351 Problem Solving in Mathematics. Problem solving strategies in math and science; evaluate conjectures and arguments; writing and collaborating on problem solutions; posing problems and conjectures; constructing knowledge from data; developing relationships from empirical evidence; connecting mathematics concepts; readings, discussions, and analysis will model and illustrate mathematics problems solving and proofs.

MASC 450 Integrated Mathematics. Integration and connections among topics and ideas in mathematics and other disciplines; connections between algebra and geometry and statistics and probability; focus for integration with authentic problems requiring various branches of mathematics.

ECFB 440 Mathematics Methods in Early Childhood Education. Analyzes contemporary curricula; implementation of methods relevant for active, authentic learning and age appropriate teaching of mathematics to young learners; considers state and national standards related to teaching and learning mathematics.

ECHE 332 Planning and Curriculum Development for Early Childhood Education. Field-based course that addresses curriculum development, planning and delivery strategies; examines curriculum from a variety of cultural and philosophical perspectives; explores a range of instructional strategies for enhancing, guiding and stimulating learning, and creating effective learning environments in EC-4 context.

APPENDIX H

RUBRICS FOR SUBQUESTION B IN TEST B

Rubrics for Sub-question b in Test B

0	No response, completely incorrect, irrelevant or incoherent
1	The response provides a partial or complete understanding of students' conceptions and misconceptions and exhibits some understanding of the sources of students' misconceptions.
2	The response provides an accurate and complete description of students' conceptions and misconceptions. It demonstrates a deep and conceptual knowledge of the sources of students' misconceptions.

APPENDIX I

RUBRICS FOR SUBQUESTION C IN TEST B

Rubrics for the Sub-question c in Test B

0	No response, completely incorrect, irrelevant or incoherent
1	The response provides a partial or complete description of strategies for addressing students' misconceptions. However, the strategies reveal factual or procedural nature, and entail some conceptual nature.
2	The response provides a complete description of strategies for addressing students' misconceptions. Furthermore, the response entails accurate and complete conceptual strategies

APPENDIX J

INTERVIEW PROTOCOL

Purpose: To examine the sources of preservice teachers' mistakes on subject matter content knowledge and deficiency on pedagogical content knowledge in linear functions within multiple representation modes. To provide detailed information of preservice teachers' procedure of solving linear function problems, drawing graphs, understanding students' misconceptions and using different strategies to teach students linear functions. To investigate how preservice teachers' subject matter content knowledge can affect their pedagogical content knowledge specifically on different problems of linear functions.

Interview Protocol: "My name is Zhixia You, and I am doing my dissertation study to examine preservice teachers' knowledge in linear functions. I think that you have signed the consent form and understand that this interview will be audio taped and I will make a transcript, however, your identity will be kept confidential.

You have completed a survey and a test titled 'mathematical knowledge for teaching. Today what I want to do is to ask some of the questions from the test and get a better idea of your thinking behind each question. I am going to ask you how you solve the problem and why you solve the problem that way. I am also going to ask you questions to clarify things that I may not understand. Do you have any questions?

Questions for Test A:

How did you do this problem?

Why did you do it this way?

What misconceptions the students might have when solving this kind of problem? How would you help them?

Questions for Test B:

Do you think the student's answer correct?

If yes, why do you think that it is a correct answer?

If no, where do you think the student made a mistake?

What misconceptions do you think the student has?

How would you help the students with the misconceptions?

VITA

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EDUCATIONAL BACKGROUND

2006. PhD, Curriculum and Instruction – Mathematics Education, Texas A&M University, College Station, TX.

2002. M.Ed., Curriculum and Instruction, Texas A&M University, College Station, TX.

PROFESSIONAL EXPERIENCE

2002 – 2005. Research Assistant for Middle School Mathematics Project (MSMP), Texas A&M University, College Station, TX.
Performed video analysis on MSMP teacher videos; Coordinated professional development summer workshops; Design and maintain the MSMP website (http://msmp.tamu.edu).

2002 – 2005. Assistant to Dr. Gerald Kulm, Curtis D. Robert Professor.
Designed online surveys and questionnaires for ITS project and performed data analysis and generate reports using SPSS; Designed and maintained MathEd website (http://mathed.tamu.edu).

2001 – 2002. Graduate Assistant, Department of Teaching, Learning and Culture, Texas A&M University. Worked with faculty involved with graduate studies, and communicated with graduate students.

PUBLICATIONS AND PRESENTATIONS

You, Z & Tong, F. (2006, March). *An Analysis of Teachers' Communication Strategies in Reducing Students' Misconceptions: The Case of Comparing Fractions.* Paper to be presented at the annual meeting of the Research Council in Mathematics Learning, Las Vegas, NV.

You, Z. (2006, Febuary). *Communication Practice in Middle School Mathematics Classroom: A Vygotskian Perspective.* Paper to be presented at the 29th annual meeting of the Southwest Educational Research Association, Austin, TX.

You, Z & Tong, F. (2006, January). *An Analysis of Teacher's Use of Representation in One Middle School Mathematics Classroom.* Poster session to be presented at the annual meeting of the Educational Research Exchange, College Station, TX.

Carter, T., Li, X., Sahin, A. You., Z., Zientek, L., English, S. & Jones, C. (2005). *How Do Students in Middle Grades Represent Data?* Paper presented at the 84th annual meeting of the National Council of Teachers of Mathematics, Anaheim, CA.