KNOWING MATHEMATICS FOR TEACHING:

A CASE STUDY OF TEACHER RESPONSES TO STUDENTS' ERRORS AND DIFFICULTIES IN TEACHING EQUIVALENT FRACTIONS

A Dissertation

by

MEIXIA DING

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2007

Major Subject: Curriculum & Instruction

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Approved by:

Co-Chairs of Committee,	Gerald Kulm
	Yeping Li
Committee Members,	Donald Allen
	Mary Margaret Capraro
	Radhika Viruru
Head of Department,	Dennie Smith

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ABSTRACT

Knowing Mathematics for Teaching: A Case Study of Teacher Responses to Students' Errors and Difficulties in Teaching Equivalent Fractions. (August 2007)

Meixia Ding, M.S., Nanjing University

Co-Chairs of Advisory committee: Dr. Gerald Kulm
Dr. Yeping Li

The goal of this study is to align teachers' *Mathematical Knowledge for Teaching* (MKT) with their classroom instruction. To reduce the classroom complexity while keeping the connection between teaching and learning, I focused on *Teacher Responses to Student Errors and Difficulties* (TRED) in teaching equivalent fractions with an eye on students' cognitive gains as the assessment of teaching effects. This research used a qualitative paradigm. Classroom videos concerning equivalent fractions from six teachers were observed and triangulated with tests of teacher knowledge and personal interviews. The data collection and analysis went through a naturalistic inquiry process.

The results indicated that great differences about TRED existed in different classrooms around six themes: two learning difficulties regarding critical prior knowledge; two common errors related to the learning goal, and two emergent topics concerning basic mathematical ideas. Each of these themes affected students' cognitive gains. Teachers' knowledge as reflected by teacher interviews, however, was not necessarily consistent with their classroom instruction. Among these six teachers, other than one teacher whose knowledge obviously lagged behind, the other five teachers

demonstrated similar good understanding of equivalent fractions. With respect to the basic mathematical ideas, their knowledge and sensitivity showed differences. The teachers who understood equivalent fractions and also the basic mathematical ideas were able to teach for understanding. Based on these six teachers' practitioner knowledge, a *Mathematical Knowledge Package for Teaching* (MKPT) concerning equivalent fractions was provided as a professional knowledge base. In addition, this study argued that only when teachers had knowledge bases with strong connections to mathematical foundations could they flexibly activate and transfer their knowledge (CCK and PCK) to their use of knowledge (SCK) in the teaching contexts. Therefore, further attention is called for in collaboratively cultivating teachers' mathematical sensitivity.

DEDICATION

To my parents Banlian Ding and Zufeng Gu

For their care, support, and encouragement

To my husband Xiaobao Li and my son Muzi

For their love, patience, and understanding

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1. INTRODUCTION

As a member of the Middle School Mathematics Project (MSMP), one of my jobs is to observe, transcribe, and analyze the U.S. middle school teachers' classroom teaching videos. During the several-year classroom video observations, I was struck by two events. The first one was about teacher interventions in cooperative learning classrooms. Even though some U.S. teachers tended to use cooperative learning methods, they focused little on students' mathematical thinking (Ding, Li, Piccolo, & Kulm, 2007). For example, some teachers just walked around classrooms asking simple questions such as "do you agree or disagree". As a result, even though some classrooms looked busy, many students were actually off task. Through the examination of teacher interventions in cooperative learning classrooms, I found that teachers' intervention qualities showed great differences (Ding, Li, Piccolo, & Kulm, 2007). However, questions such as "why teachers do what they do" and "what enables them to do what they do" (Schoenfeld, 1998, p.7) in classrooms are yet to be answered.

The second striking event also occurred during my observations of MSMP videos. In the video taped lessons of equivalent fractions, some students could correctly explain why $\frac{3}{4}=\frac{6}{8}$ in oral language - multiplying numerator and denominator both by 2- while making mistakes in the written forms such as ¾ × 2 = 6/8. Some of them even made mistakes in their verbal representations such as "doubling 2". Teachers in these videos showed different responses to these errors. Some of them ignored it (maybe not realizing it) while others dealt with it but with different degrees of attention (Ding & Li, 2006).

The dissertation follows the style of *Journal for Research in Mathematics Education.*

This phenomenon again raised questions: Why did some teachers address the errors while the others did not? What were the underlying causes for teachers' different responses to student errors?

These two striking events about classroom teaching in the MSMP project - using cooperative learning without addressing students' mathematical thinking and different responses to student mathematical errors – made me think about this question – what really matters in teachers' instructional decisions in mathematics classroom teaching? With regard to the above examples, I wonder whether teachers have enough mathematical knowledge to address students' thinking and whether teachers have enough knowledge to discover, grasp, and deal with these types of errors. Put another way, how does teachers' mathematical knowledge make a difference in their classroom instruction and what is the useful mathematical knowledge that allows teachers to make proper decisions? These questions call for further research.

1.1 Rationale for This Study

Regarding mathematics teaching, there is a widespread agreement about the close relationship between teacher knowledge and classroom instruction (Ball, 2000; Clark & Peterson, 1986; Dewey, 1904/1964; Grossman, 1990; Ma, 1999; Shulman, 1986, 1987). The National Council of Teachers of Mathematics [NCTM] (2000) pointed out that "effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (p.11). As a result, what teachers are able to do during classroom instruction depends fundamentally on teacher knowledge: knowing subjects, knowing students, and knowing contexts (Shulman, 1986).

However, how these knowledge components work together in real classroom situations and specifically, what makes mathematical knowledge *usable* for teaching are still unsolved (Ball, Lubienske, & Mewborn, 2001). Recently, researchers at the University of Michigan (Ball, 2006; Hill, Rowan, & Ball, 2005; RAND Mathematics Study Panel, 2003) proposed a novel theory about teachers' *Mathematical Knowledge for Teaching* (MKT), with the aim to answer the above questions. Case studies concerning topics such as fractions, place value, and mathematical definitions have been researched well enough to illustrate what MKT looks like. Nevertheless, regarding particular topics, very few studies reached the same depth as that of Ma's (1999) *Profound Understanding of Fundamental Mathematics* (PUFM) where teachers' knowledge packages demonstrated connectedness, thoroughness, basic ideas, and multiple perspectives. However, Ma's (1999) knowledge packages were developed from teacher interviews rather than observing teaching practice. As a result, combining both ideas of MKT and PUFM and identifying specific *Mathematical Knowledge Packages for Teaching* (MKPT) from real teaching contexts seems much needed.

This study will investigate the issue in depth. Through the case study of *Teacher Responses to Students' Errors and Difficulties (TRED)* in teaching equivalent fractions, I attempt to align teachers' MKT with their teaching behaviors and also to identify a useful MKPT for teaching equivalent fractions. Two things are worthy of mention here. First, this study is not limited to cooperative learning mathematics classes (Ding, Li, Piccolo, & Kulm, 2007) because teacher's MKT is displayed in various classroom contexts. Second, this study is not limited to teacher responses to student errors (Ding & Li, 2006) because errors are usually associated with learning difficulties and “no errors” does not mean “no difficulties.”

There are two reasons to select TRED as a research focus. First, as Hiebert and Wearne (1993) pointed out, “instruction is too complex an activity to describe completely” (p.395). To reduce classroom complexity while keeping the connection between teaching and learning, Hiebert (1993) suggested researchers reduce two dimensions under investigation – scope and specificity. *Scope* includes the size of the sample and the length and breadth of classroom activities. To reduce size of sample, I focus on several teachers’ classroom teaching. To reduce length and breadth of classroom activities, I focus on TRED rather than covering every aspect of classroom teaching. *Specificity* means the “degree of precision or detail or grain-size of the analysis of relationships between learning and teaching” (Hiebert, 1993, p.223). To reduce specificity, I use a global assessment of student learning to measure teachers’ instructional effects.

The second reason to select TRED as a research focus is because TRED is an important indicator of teacher knowledge (Ball, 2006; Ma, 1999; Shulman, 1986). Ma’s (1999) study explored four topics to compare Chinese and U.S. teachers’ subject matter knowledge. Among these four topics, two of them were associated with how teachers respond to students’ invented or erroneous strategies. In addition, Ball (2006) also used teacher responses to students’ errors as illustrations of the MKT components.

My case study is situated in the context of teaching and learning equivalent fractions. Through the descriptions of six teachers’ responses to students’ errors and difficulties in learning equivalent fractions, I expect to analyze how teachers’ MKT relates to their instructional decisions which might influence students' understanding of equivalent fractions. Since fractions are difficult but significant topics in school mathematics (Ball, 1993; Lamon, 1999; Leinhardt & Smith, 1985; Ohlsson, 1988; Post, Cramer, Behr, Lesh, & Harel, 1993; NCTM, 2006), this topic provides me enough opportunities to see how students struggle in learning and how teachers respond to students' difficulties and errors. Moreover, since teaching and learning cannot be absolutely separated, research on effective teaching with an eye on students' learning as a measurement of teaching effects, will provide meaningful information for identifying a useful MKPT for teaching equivalent fractions.

1.2 Purpose of This Study

As previously mentioned, this study focuses on TRED. The case serves as a window for teachers and educators to examine the complexity of how teachers' MKT affects their classroom instruction which in turn influences students' mathematical learning. In addition, this study expects to identify a useful knowledge package (MKPT) for teaching equivalent fractions, which serves as a starting point for future research on other critical concepts. What follows are the research questions for this study:

1. How do teachers respond to students' errors and difficulties in learning equivalent fractions?
2. How does teachers' *Mathematics Knowledge for Teaching* (MKT) contribute to instructional decisions in dealing with students' errors and difficulties?
3. How do students' classroom responses correspond to teachers' different instructional decisions in dealing with students' errors and difficulties?

1.3 Method of This Study

To answer these research questions, this study uses a qualitative method. In comparison with the quantitative approach, this method "more easily allows for the discovery of new ideas and unanticipated occurrences" (Jacobs, Kawanaka, & Stigler, 1999, p. 718). Since classroom teaching is complex and it is hard to focus on both teaching and learning, previous research on teaching generally lacks the connection to learning (Hiebert, 1993). To address this issue, Hiebert and Wearne (1988) provided an approach to reduce the complexity while keeping the connection between teaching and learning. The approach includes four steps: (1) the selection of content, (2) the identification of key cognitive processes, (3) the design of instruction, and (4) the assessment of student changes. This method guided the procedures of this study. Briefly, , the content in this study focuses on equivalent fractions; the cognitive processes are reflected by student errors and difficulties; the instruction concerns TRED, and the assessment is students' corresponding responses to teachers' instructional decisions.

Specifically, six teachers in the *Middle School Mathematics Project* (MSMP) participated in this study. Four teachers were from Texas while the other two were from Delaware. For each teacher, two video-taped lessons concerning equivalent fractions in the school year 2002-03 were observed and analyzed in a naturalistic way (Lincoln & Guba, 1985). According to Ball and Bass (2000), this approach was a *job analysis*. That is, through close observation of several video tapes, theoretical ideas or hypotheses might occur which served as a lens for viewing other teachers' practice. As a result, the hypotheses might be reinforced, modified, or rejected to adapt to the new data. In addition, each teacher was interviewed through responding to (1) a short video clip of his/her own teaching, (2) a designed case concerning TRED, and (3) true or false questions about equivalent fractions. These qualitative interviews (Rubin & Rubin, 2005) were conducted for triangulation with the observed videos. These data went through an ongoing process of analysis and interpretation and finally were presented with qualitative research resources (Bogdan & Biklen, 2003; Denzin & Lincoln, 2000; Lincoln & Guba, 1985; Miles & Huberman, 1984).

1.4 Limitations of This Study

This study has at least two limitations. First, the video tapes were selected from the school year 2002-03. However, teachers' interviews were conducted four years later. It is possible that teachers could not accurately recall the reasons for their classroom actions. To decrease this limitation, I showed the teachers short video clips. In addition, teachers' current answers to the main interview questions might not truly be consistent with their MKT when they taught these lessons. Nevertheless, the interview data could at least be assumed as teachers' higher level of knowledge at that time. As a result, to capture teachers' MKT, I mainly rely on video data with the triangulation of teacher interviews.

Second, students' response data were used as a measure for assessing teachers' instructional effects. However, the MSMP project video camera generally followed teachers' actions rather than students' behaviors during video taping. Therefore, some evidences which might reflect students' cognitive changes were possibly missing.

1.5 Significance of This Study

This case study is significant in several ways. First, the case itself is important. *Teacher Response to Student Errors and Difficulties* (TRED) is a critical indicator of teacher knowledge and relates directly to students' mathematical understanding. Therefore, this case connects teachers' instructional behaviors and their cognitive processes. Meanwhile, it also connects teaching and learning in the real contexts. The detailed portraits allow teachers, researchers, and teacher educators, to see in depth the complexity of classroom instruction and to be aware of various embedded issues.

Second, this study contributes to teaching and learning equivalent fractions through the efforts of identifying a MKPT for this topic. As the RAND Mathematics Study Panel (2003) pointed out, in order to develop a better understanding of MKT, some important questions needed to be answered. For example, "what specific mathematical knowledge of mathematical topics and practices is needed for teaching particular areas of mathematics to particular students" and "what knowledge and expectations about students' mathematical thinking and capabilities are needed for teaching specific mathematics and mathematics practices to particular students" (p. 23). In this study, I raise common student errors which are likely to be ignored by most teachers when teaching equivalent fractions (Ding & Li, 2006). Typical types of errors serve as a window, connecting various mathematical knowledge pieces, such as fraction concepts, fraction operations, multiple representations in interpreting and reasoning equivalent fractions, and some other fundamental ideas concerning the equal sign, "0", and "1". Therefore, this study, through exploring curriculum, teacher classroom responses, and students' learning processes on the same topic, provides insights concerning critical components of teachers' mathematical knowledge for teaching equivalent fractions.

Third, this study is not limited to equivalent fractions. Through the portraits of how and why teachers responded to students' errors in certain ways, it is expected to show how teachers' MKT affects their classroom instruction, resulting in students' different understanding. Therefore, other mathematics teachers will have a greater sense of effective teaching such as how to anticipate students' possible difficulties and confusion, how to be aware of the accurateness of one's own representation, and how to capitalize on students' errors to deepen their understanding of critical knowledge pieces. In addition, this study closely examined the components of MKT, especially teachers' SCK – the use of mathematical knowledge in the teaching context. Through the discussion of the relationships between teachers' CCK, PCK, and SCK, this study argues that one of the critical factors that enables SCK is teachers' knowledge base which has strong connections to the mathematical foundations. This finding not only enriches the theory of MKT but also points out a direction for future researchers and mathematics educational reforms, that is, to collaboratively improve teachers' mathematical sensitivity through various opportunities and contexts.

Lastly, this study also serves as a starting point or a window for identifying useful MKPTs concerning various critical concepts. Substantive research on each of these topics as pointed out by NCTM (2006) *Curriculum Focal Points* provides a professional knowledge base for this field. When the components of MKPT are identified from real contexts as *practitioner knowledge*; they will offer pedagogically useful information for teaching practice and thereby to develop teachers' *professional knowledge* (Hiebert, Gallimore, & Stigler, 2002).

1.6 Outline of This Study

In the following sections, I first review prior research related to my study. I then introduce the methodology adapted from Hiebert and his colleagues' theory (Hiebert, 1993; Hiebert & Wearne, 1988) and some qualitative resources (e.g., Denzin & Lincoln, 2000; Lincoln & Guba, 1985). The results section includes student learning difficulties and common errors as identified from prior studies and curriculum analysis. In addition, two emergent themes are also reported. As a result, teachers' responses to these errors and difficulties, and students' corresponding responses to teachers' instruction are provided under each theme. Based on these findings, a MKPT for teaching equivalent fractions is provided. In addition, the relationships between MKT, TRED, and students' cognitive gains and the related issues about MKT and mathematical sensitivity are also discussed.

2. LITERATURE REVIEW

In this section, I review prior research related to my study from three aspects: (1) Teachers' *mathematical knowledge for teaching* (MKT), (2) *Teacher responses to student errors and difficulties* (TRED), and (3) Teaching and learning equivalent fractions. These three aspects are not absolutely separated because teachers' MKT can be reflected from TRED in the context of teaching and learning equivalent fractions. Therefore, these three aspects are somewhat intertwined in the following sections.

2.1 Teachers' Mathematical Knowledge for Teaching (MKT)

MKT is a new type of knowledge identified by Ball and others at the University of Michigan. According to Hill et al. (2005), MKT means "the mathematical knowledge used to carry out *the work of teaching mathematics*" (p. 373).

2.1.1 MKT and prior research

To understand MKT, I first situate it in the context of previous research on teacher knowledge. Since Ball and her colleagues (Ball, 2006; Ball & Bass, 2003a) claimed that MKT was developed from and complemented to Shulman's (1986) *Pedagogical Content Knowledge* (PCK), I then specifically compare MKT with PCK. These comparisons help to clarify where MKT comes from and why MKT is needed.

Ball and others started to conduct research on a *pedagogically useful mathematical knowledge* needed for teaching beginning in 1996 (Ball & Bass, 2000). This work was actually built on prior research in this field (e.g., Begle, 1979; Clark and Peterson, 1986; Dewey, 1904/1964; Grossman, 1990; Leinhart & Smith, 1985; Ma, 1999;

Shulman, 1986, 1987). According to Ball et al. (2001), there were two prior research approaches to teachers' mathematical knowledge. One emphasized characteristics of *teachers* while the other emphasized teachers' *knowledge*. As a result, MKT was "built on both lines of prior work – on *teachers* and on *knowledge* – but shifted to a greater focus on *teaching* and on *teachers' use of mathematical knowledge"* (p.441).

MKT and prior research on teachers. Studying characteristics of *teachers* such as their course taking as well as asking questions about how their qualifications related to student learning is unwarranted. There is little evidence to support the effects of mathematics course work on teaching. Begle (1979) conducted a meta-analysis of studies on teacher variables on students learning and found that course taking produced positive effects on student achievement in only about 10% of cases and negative effects in about 8% of cases. Begle, therefore, claimed that the belief that "the more a teacher knows about his subject matter, the more effective he will be as a teacher" demanded "dramatic modification" (p.51).

Regarding this interesting but strange finding that taking more advanced mathematics courses makes no difference for teaching, Ball and her colleagues provided several explanations. The first explanation is from the perspective of compression and unpacking of mathematics. According to Ball et al (2001), a powerful characteristic of mathematics is its compression. "When ideas are represented in compressed symbolic form, their structure becomes evident, and new ideas and actions are possible because of the simplification afforded by the compression and abstraction. Mathematicians rely on this compression in their work" (Ball & Bass, 2003a, p.11). However, teaching mathematics requires teachers a kind of decompression, or "unpacking", of ideas (Ball &

Bass, 2003a). As a result, the increasing mathematical work accompanied by increasing compression of knowledge, may interfere with the unpacking of content that teachers need to teach. The second explanation of the effect of advanced mathematics courses is that, when a teacher takes more course work in mathematics, he or she will experience more conventional approaches to teaching mathematics, which may “imbue teachers with pedagogical images and habits that do not contribute to their effectiveness with young students” (Ball, et al, 2001, p.442). The third explanation of the advanced mathematics courses problem is that success in traditional mathematics classes - often derived from memorizing formulas and performing procedure - does not necessarily provide teachers the kind of knowledge needed to teach mathematics for understanding (Ball, 1990). In summary, research on teachers’ characteristics or credentials contributes little to teacher effectiveness.

MKT and prior research on teacher knowledge. Concerning the prior research on teacher knowledge, researchers in different programs examined knowledge under different circumstances with different goals. Sherin, Sherin, and Madanes (2000) compared three research programs concerning teacher knowledge: (1) Shulman and others’ research on teachers’ knowledge growth at Stanford University; (2) Leinhardt and others’ studies on teacher expertise at the Learning Research and Development Center at the University of Pittsburgh, and (3) Schoenfeld and others’ work on developing a model of the teaching process at the University of California, Berkeley.

Shulman and his colleagues focused on how teachers learn to teach. They divided the knowledge required for effective teaching into seven separate categories that together made up the “knowledge base” for teaching. These categories are subject matter knowledge, pedagogical content knowledge, general pedagogical knowledge, curriculum knowledge, knowledge of learners, knowledge of school contexts, and knowledge of educational aims (Shulman, 1987; Wilson, Shulman, & Richert, 1987). Among these categories, the particularly important one is pedagogical content knowledge (PCK) – subject matter knowledge that is specialized for teaching. According to Shulman (1986), teaching requires “ways of representing and formulating the subject that make it comprehensible to others” (p.9). Shulman and his colleagues’ work has had a significant contribution to research on teaching. The introduction of PCK helped the research community to focus on the importance of intertwining the disciplined knowledge and teaching practice. However, this type of knowledge cannot equip teachers with the ability and flexibility to handle all practice, especially the complexity of real classroom teaching. This limitation will be elaborated later.

Leinhardt and other’s work has been to characterize teacher expertise, through comparing expert and novice teachers. Leinhardt and Smith (1985) examined expert teacher’s knowledge in light of eight teachers’ performances on interview tasks and three teachers’ real classroom teaching of reducing fractions. Through the use of semantic nets, these researchers argued for two core areas of knowledge for teacher expertise: lesson structure and subject matter knowledge. “The strategy - sorting teachers by their actual in-class mathematical performance - is rare in literature and, unfortunately, not well explicated in their published work” (Learning Mathematics for Teaching [LMT], 2006, p.4). As a result, even though Leinhardt’s work was based on classroom teaching, a promising method in exploring teacher knowledge for teaching mathematics, her contributions were relatively theoretically-based. Thus, the type of mathematical knowledge needed for teaching particular mathematics needs further research.

Schoenfeld and others' work was to explain teachers' moment to moment decision making during teaching contexts - why teachers do what they do and what enables them to do what they do in classrooms. As a result, Schoenfeld (1998, 2000) developed a model of the teaching process. Central to Schoenfeld's model was an examination of teacher knowledge. This model could be used to explain teachers' classroom behaviors and contributed to the field of research on teachers' cognitive processes, critical factors to effective teaching. However, a similar problem related to this model was that it was conducted at some distance from practice. It could not tell teachers what kind of mathematical knowledge they specifically needed to teach particular topics and how they could use this type of knowledge in order to teach well.

In addition to the above three research program about teacher knowledge, it is worth mentioning Ma's (1999) study. This study, exploring elementary mathematics teachers' subject matter knowledge, made progress toward *pedagogically useful mathematical understanding* (Ball & Bass, 2003a), thus, it drew both mathematicians' and math educators' attention. Through the comparison of 72 Chinese and 23 U.S. elementary teachers' mathematical knowledge for teaching, Ma (1999) pointed out that Chinese teachers had specific knowledge packages concerning particular teaching topics. These knowledge packages, as maps in skilled drivers' minds, consisted of (a) key ideas that "weigh more" than others in the package, (b) sequences for developing the ideas, and (c) "concept knots" that link related ideas. "Ma's notion of knowledge packages represents a particularly generative form of and structure for pedagogical content knowledge" (Ball et al. 2001, p.449). Based on these portraits of Chinese teachers' knowledge, Ma raised *Profound Understanding of Fundamental Mathematics* (PUFM) in terms of the (a) depth which referred to large and powerful basic ideas, (b) breadth, which had to do with multiple perspectives, (c) thoroughness, which was essential to weave ideas in a coherent whole, and (d) connectedness, which related to the above three.

Ma's (1999) approach takes an important step toward solving the problem of mathematical knowledge for teaching because it improves our understanding of the knowledge required for teaching mathematics. However, this approach, through quantitative interviews of what teachers know, even though it reveals the contents and structures of teachers' mathematical knowledge, also leaves gaps in efforts of solving the problem of mathematical knowledge for teaching (Ball et al. 2001). Since we do not know whether teachers would actually do what they said in interviews and whether what they said in interviews would be actually effective, Ma's portraits do not necessarily illuminate the knowledge that is critical to good practice. "To understanding the mathematical work of teaching would require a closer look at practice, with an eye to the mathematical understanding that is needed to carry out the work" (Ball et al., 2001, 449).

In summary, prior studies of mathematical knowledge still have limited use for determining what type of mathematical knowledge is really useful for teaching and how teachers should use this type of knowledge to effectively teach for understanding. An unsolved problem left by prior research is about teachers' mathematical knowledge for teaching, an ongoing effort made by researchers at the University of Michigan. Since Ball and others claim MKT was developed from PCK, I compare these two categories in the following sections.

MKT and PCK. One hundred years ago, Dewey (1904/1964) pointed out the gap between content knowledge and pedagogical knowledge. This gap was targeted by Shulman's (1986) pedagogical content knowledge (PCK), a special form of knowledge that bundles knowledge of subject, knowledge of context, and knowledge of students. This type of knowledge could help teachers anticipate students' learning difficulties and possible errors and have ready strategies to address those issues. However, a body of such bundled knowledge may not always equip teachers with flexibility to deal with the complexity of teaching practice. This is becauseteaching practice are embedded with both regularities and uncertainties (Ball & Bass, 2000). Even though PCK enables teachers to manage those regularities, it cannot prepare teachers for a significant proportion of uncertainties of teaching. For example, during the teaching context, teachers may need to decipher the mathematics in a student's idea or to consider the relative values of alternative representations in the face of a particular mathematical issue. To handle all the routine and nonroutine problems in teaching practice, *a "pedagogically useful mathematical understanding"* (Ball & Bass, 2000; Ball, et al, 2001) is needed.

MKT is exactly this type of knowledge pedagogically and mathematically useful for teaching practice. It was "built on pedagogical content knowledge by complementing what it offers for practice" (Ball & Bass, 2000, p.88). Ball (2006) compared the relationship between MKT and PCK in the following Figure (see Figure 1):

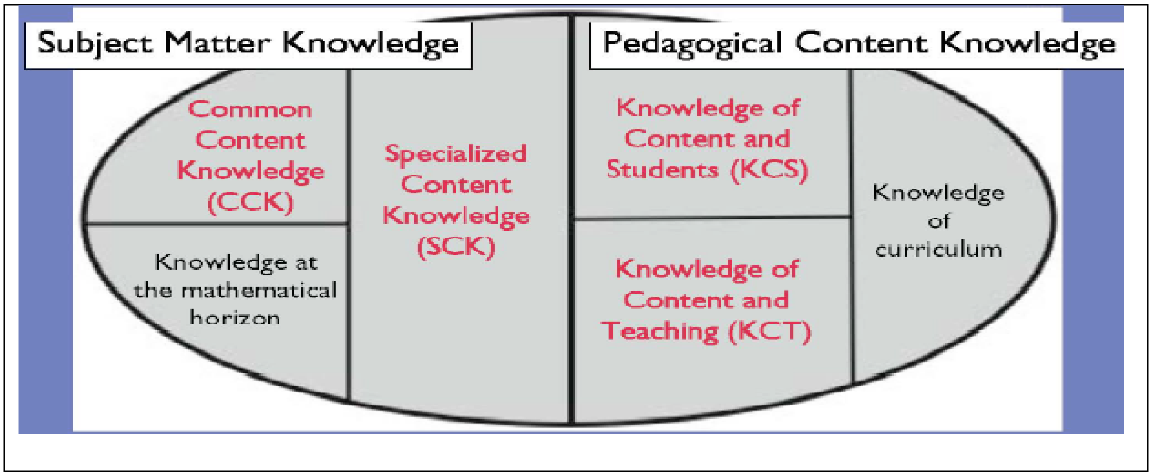


Figure 1. MKT and Shulman's PCK.

Figure 1 shows the relationship between MKT and PCK - MKT includes PCK but it has more components. As we see from the figure, PCK includes *Knowledge of Content and Students* (KCS) and *Knowledge of Content and Teaching* (KCT). In contrast, MKT includes two main additional categories: *Common Content Knowledge* (CCK), a knowledge base expected of any well-educated adult, and *Specialized Content Knowledge* (SCK), a knowledge base mainly needed by teachers in their work and beyond that expected of any well-educated adult.

To illustrate the differences among PCK and MKT, Ball (2006) provided examples to show how the different knowledge components played different roles concerning student errors (see Figure 2).

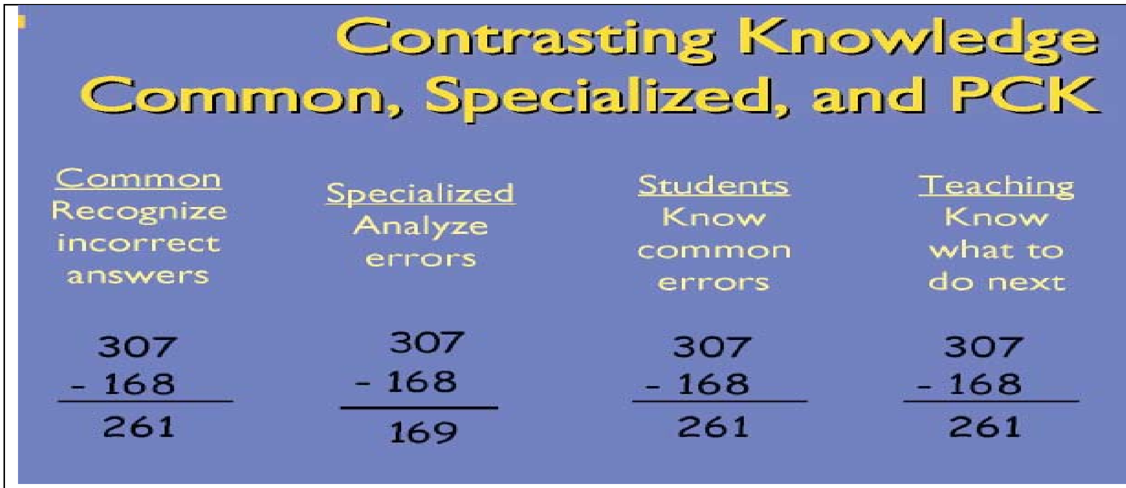


Figure 2. Example for the components of MKT.

Regarding to the same multiple digit number subtraction, 307-168, students might have various errors. People with PCK will have the ability to anticipate student common errors:

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 261 \end{array}$$

They could also find a bug related to student errors here –subtracting the small number from the big number. As a result, teachers will have ready approaches during their lesson plan so they know what to do next when they face this type of error. Briefly, PCK is a type of powerful knowledge outside the teaching context allowing teachers to sufficiently prepare for instruction in advance. As previously mentioned, people with MKT not only have PCK but also have CCK and SCK. CCK enables them to recognize errors whereas SCK enables them not only to recognize but also to analyze errors and evaluate alternative ideas. Regarding the more complex error:

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 169 \end{array}$$

A teacher with SCK may specifically analyze what happened here. For example, why this student subtracted correctly in the ones place but made a mistake in the tens place. Obviously, this student subtracted 0 from 6 and got the incorrect number “6” in the tens place. But did this mean the student also had a bug of “subtracting the small number from the big number”? If he had this bug, why didn’t this bug occur in the ones place? Also, if he had this bug, why did he not simply subtract 1 from 3 in hundreds place and then get the answer 269 instead of 169? Since the student got “1” in hundreds place, it showed that the value “3” in hundreds place was decomposed for the regrouping in tens place. Therefore, even though this student did not really subtract 6 from 10 in tens place, he seemed to not have that bug. This example demonstrates that teachers with SCK are able to mathematically analyze the source of student errors during teaching context. Of course, they can use various strategies such as scaffolding questions to encourage students to explain their ideas. Based on the analysis, teachers may further guide students to explore these errors. As a result, SCK equips teachers with the abilities to analyze students’ thinking and address students’ alternative representations mathematically and pedagogically during the teaching context.

2.1.2 What does MKT entail?

MKT is a type of pedagogically useful mathematical knowledge for the *work of teaching mathematics* (Hill et al., 2005). Examples of the *work of teaching* are interpreting and evaluating alternative solutions, and producing and evaluating mathematical explanations (Ball & Bass, 2006). The above example about dealing with student errors is only part of teachers’ MKT.

"Students not only make mistakes, they ask questions, use models, and think up their own non-standard methods to solve problems" (Ball, Hill & Bass., 2005, p.20). As a result, teaching also entails using representations. What is an effective way to represent the meaning of an idea? What are the connections between alternative representations? What are mathematical principles underlying these representations? Which representations will be easily accepted by students? All these considerations demand teachers have a sound mathematical understanding which is also usable for teaching students. Ball (1993) suggested teachers jointly construct fruitful representational contexts with students. According to Ball, a representational context has a broader meaning than a specific instructional representation. It encompasses the ways in which a teacher and students use a particular representation (it could be provided by a teacher or invented by students) as well as the meanings and discourses it makes possible during their work. As a result, a representational context offers thinking space and requires a teacher to (1) consider the mathematical content, (2) consider students and how they learn, and (3) put representational context into use. Ball illustrated this idea through an example of how she herself, as a third grade teacher, guided students to solve an open-ended problem: "You have a dozen cookies and you want to share them with the other people in your family. If you want to share them all equally, how many cookies will each person in your family get?" (Ball, 1993, p.176). Student Cassandra came up with 12 ÷ 5. She drew her chart as a tool to display her reasoning. After she distributed 10 of her cookies by making hash marks easily, she then drew two circles to represent the two leftover cookies. She cut them first in half and then in quarters and at last added another line to each cookie (see Figure 3).

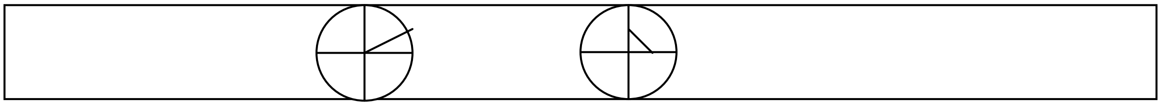


Figure 3. Dividing a circle shape cookie to five pieces.

Facing this type of representation, Ball, as a teacher asked herself:

> Did she mean them to be equal but just did not know how to draw fifths properly? Dividing a circle into five equal parts is no easy task. Or did Cassandra not recognize that equal size is a crucial aspect of dividing something like cookies equally? (p.179)

Ball also debated about how to respond to Cassandra: “Should I question her further about her solution? She was not at all dissatisfied with it and it made compelling sense in many ways (p.180).” At this time, Ball saw an opportunity to respect Cassandra’s genuine attempt – to distribute 12 cookies among five members of her family – thus, she decided to adjust the representational tool. Ball suggested Cassandra draw rectangular cookies because that would be easier to divide evenly so that everyone would get the same amount. As a result, Cassandra drew a new picture (see Figure 4).

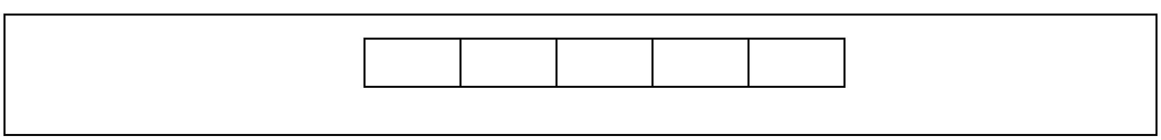


Figure 4. Dividing a rectangular-shaped cookie into five pieces.

Cassanda called these pieces “halves.” Ball told her they were not “halves” but “fifths” and asked her to think of a reason why that made sense. Cassanda quickly replied that it made sense because the cookie had been divided into five now-equal pieces. From this example, Ball (1993) pointed out that teachers need to be able to capture the fallacies embedded in students' representations and need to help them expand and deepen their understanding. When teachers help students expand their understanding, they can interweave mathematical ideas of fractions, geometry, and measurement with students' reasoning and notation such as what and how they learn, and what they might find exciting or interesting. Based on these processes, teachers can make justifiable decisions about students' representations including their construction, use, and adaptation. The jointly-constructed representational context forms one of the meanings entailed by MKT.

Teachers' MKT also demands a fluency and accuracy of mathematical language for their careful mathematical work. An emergent theme in Ball and her colleague's research is the precision of mathematical language and the need for a specialized fluency with mathematical terms such as what counts as a mathematical explanation and how to use symbols with care (Ball, Hill, et al., 2005). "Precision requires that language and ideas be meticulously specified so that mathematical problem solving is not unnecessarily impeded by ambiguities of meaning and interpretation. But precision is relative to context and use" (Ball & Bass, 2003a, p.8). An example about what counts as precise and usable mathematical language is the definition of even number used by third graders (Ball, 2006; Ball & Bass, 2000). According to Ball (2006), there could be various proposed definitions. For example, (a) an even number is a number that can be divided into two equal parts; (b) an even number is any multiple of 2; (c) an even number is any integer multiple of 2; (d) an even number is any number whose unit digit is 0, 2, 4, 6, or 8; and (e) a whole number is even if it is the sum of a particular whole number with itself. Ball

(2006) claimed that teachers should not only know the domain to which these definitions are conventionally applied but also compare and justify their usefulness and precision. In addition, teachers should be cognizant of possible consequences if students use these definitions to determine the status of specific numbers. Regarding the above definitions, both (c) and (e) are correct. However, (c) is not useable by third graders because they have no idea about integers, whereas (e) is both correct and useful because this definition is consistent with the general definition of integer and is also understandable by third graders. In contrast, all the other definitions are flawed. According to definitions (a) and (b), any number could be an even number. For example, “7” is an even number because it could be divided into two equal parts with each part as 3.5; or because it could be formed by 3.5 multiplying by 2. According to definition (d), the number “36.7” is an even number because its unit digit is “6”. In summary, teachers should have a good sense of when each definition might be useful. Knowing and being sensitive to these definitions and being able to use them in the context of student responses, can enable teachers to consider the possible reasons of particular solutions and to manage other kinds of situations that might arise. “Knowing what definitions are supposed to do, and how to make or select definitions that are appropriately and usefully precise for students at a certain point, demands a flexible and serious understanding of mathematical language and what it means for something to be precise” (Ball & Bass, 2003, p.8).

2.1.3 Strength and weakness of research on MKT

The research on MKT has at least three merits. First, it emphasizes “mathematics” entailed in the teaching practice. This could be seen from the above examples about representational contexts and definitions of even number. MKT requires sufficient understanding of mathematics and special ability to deconstruct/unpack one's own compressed mathematical knowledge into a less polished form, where basic components are accessible and visible (Ball & Bass, 2000, 2003a) .

Second, the research on MKT directly targets "practice," aiming at solving the problems during teaching contexts. As Ball and Bass (2000) pointed out, teaching as a practice includes both regularities and uncertainties. The teaching regularities could be enabled by PCK while the uncertainties could not. MKT is exactly the pedagogically useful knowledge allowing teaching uncertainties. To identify what kinds of mathematical knowledge are needed for teaching, researchers in the *Learning Mathematics for Teaching* (LMT) project at the University of Michigan made great efforts to analyze detailed records of practice – videotapes and audiotapes of classroom lessons, copies of student classroom work, homework, as well as teacher's lesson plans and reflections. Therefore, MKT is quite practice-based.

Third, the research on MKT is believed as credible. Based on the detailed analysis of teaching records, the cross-disciplinary researchers in the LMT project including mathematicians, psychologists, math educators, and teachers, cooperatively designed an instrument of multiple-choice questions covering critical mathematical knowledge needed for teaching in many areas such as algebra, number, and geometry (Ball, Hill, et al., 2005; Hill & Ball, 2004; Hill et al., 2005; Hill, Schilling, & Ball, 2004). This instrument, as a knowledge base, was used to measure teachers' MKT. To verify the validity and reliability of this instrument, researchers in the project LMT additionally designed a coding rubric for measuring classroom teaching quality (LMT, 2006). As a result, they found there was a high correlation between teachers' scores in video analysis and teachers' pencil-and-paper performances on the multiple-choice questions ($r = 0.79$, $p < 0.01$). The videotaped ranking strengthened these researchers' view that "the pencil-and-paper assessments capture important knowledge for teaching mathematics, and helps illuminate the ways in which teachers' mathematical knowledge base can be put to use in classrooms" (LMT, 2006, p.33). In addition, the pencil-and-paper assessment was used in a survey concerning teachers' MKT with a nationally representative sample from 115 elementary schools during the 2000-2001 through 2003-2004 school years (Hill et al. 2005). About 700 first- and third- grade teachers (and almost 3000 students) participated in this study. Along with the survey, student achievement data were also collected. As a result, the researchers found teachers' performances on these questions - including questions for CCK and SCK – significantly predicted students' achievement. Through the efforts of identifying MKT from teaching practice, designing an instrument for measuring MKT, verifying the instrument by analyzing teaching tapes, using the instrument for a nation-wide survey, and connecting teachers' MKT with student achievement, the researchers in project LMT significantly contributes to the understanding of MKT, a critical factor for improving mathematical teaching and learning.

Still, concerning the research on MKT, there are at least two issues needed to be addressed. First, although the knowledge base of MKT was refined as a pencil-and-paper instrument of multiple-choice questions, it has limited use to improve real classroom teaching. As the researchers themselves pointed out, teachers do not improve student understanding simply by doing well on multiple–choice assessments (Hill et al., 2005). The paper-and-pencil instrument can serve as a measurement tool or a question pool for teachers' MKT. However, the measure cannot really serve as a base of *professional knowledge* (Hiebert et al., 2002). Of course, by knowing and understanding the answers of these multiple-choice questions, teachers will be possibly aware of some critical issues during their teaching practice. However, when MKT is transformed in the format of multiple choice question by question, what teachers learn from these questions will be piece by piece, and thus, fragmented. For example, even though there are several questions about teaching fractions in the instrument, it is difficult for teachers to see the connections between these questions or these knowledge pieces. As Ma (1999) pointed out, one of the critical characteristics of teachers' PUFM is connectedness, relating to breadth, depth, and thoroughness. If teachers' knowledge base lacks connectedness, it will not be mathematically and pedagogically useful for their teaching. Therefore, aiming at and beginning with teaching practice as researchers at the University of Michigan do, to find powerful knowledge packages regarding particular topics such as what Liping Ma provided, is currently a much needed work. It is true that much of Ball and her colleagues' work targeted classroom teaching in depth. However, most of their work focused on those "classical" individual cases such as fractions (Ball, 1993), definition of even number (Ball, 2006; Ball & Bass, 2000), multiplication and place value (Ball, Hill, et al., 2005) where these cases serve as illustrations of what MKT looks like and how MKT should be used. As Hill et al. (2005) claimed, what knowledgeable teachers do in classrooms and how knowing mathematics affects classroom instruction such as managing students' confusion and explaining concepts needs further research. Specifically, focusing on a particular topic, with the purpose of identifying a mathematical knowledge package that is pedagogically useful from the work of teaching is yet to be studied. As a result, to combine both ideas of MKT and PUFM to develop MKPTs concerning those curriculum focal points (NCTM, 2006) is a promising way for designing teacher professional development.

The second shortcoming of the research on MKT is that it lacks the connection with learning. In fact, this is a common shortcoming of research on teaching in this field (Hiebert, 1993; Hiebert & Wearne, 1988; Romberg & Carpenter, 1986). On the one hand, research on teaching provides informative descriptions of teacher thoughts and behaviors. But this work does not shed light on the way in which teaching influences learning. On the other hand, research on learning, even though it has a long tradition, has relatively little direct influence on teachers' instruction (Hiebert, 1993). As a result, the gap between teaching and learning should be bridged (Romberg & Carpenter, 1986) because the fundamental goal of effective teaching is to improve students' learning. As Hiebert (1993) argued,

> In order to improve students' learning, we must understand the way in which learning is connected with instructional activities. What kinds of classroom environments and activities are crucial for productive learning? Are these relationships dependent on particular teacher or student or subject matter variables or are there general principles that underlie the relationships? (p.221)

Therefore, Hiebert (1993) suggested research on teaching be related to students' cognitive processes or with student interactions with peers or the teacher. According to these considerations, research on MKT needs to connect more with students' learning process. Even though Ball (1993) paid attention to how teachers identify students' thinking (Carpenter, Fennema, & Romberg, 1993) and argued that knowing students' thinking was central to teachers' role in teaching for understanding, Ball did not record the evidence of the effects of teacher actions on students' cognitive performance. Even though researchers in the LMT project connected teachers' performance on paper-and-pencil test to student achievement, this is a remote connection. As these researchers themselves pointed out, one of their limitations was "a lack of alignment between our measure of teachers' mathematical knowledge and student achievement" (Hill et al., 2005, p. 399). Therefore, further research on MKT should have an eye on students' mathematical learning and use students' cognitive process as a measure of teaching effects

In summary, MKT as a sound theory of research on teaching, points out a promising direction of investigation. However, further efforts are needed to study in depth topic by topic and connect to students' learning for confirmation.

2.2 Teacher Responses to Students' Errors and Difficulties (TRED)

As previously mentioned, classroom teaching is complex. In order to research in depth, researchers usually select part of teaching activities (Hiebert & Wearne, 1993). Since TRED is a critical teaching behavior that relates to teacher knowledge and influences student understanding, it serves as a research focus in this study. In this section, I review prior studies concerning TRED.

2.2.1 TRED and teacher knowledge

TRED have been recognized as an important indicator for teacher knowledge (Hill et al., 2005; Leinhardt & Smith, 1985; Ma, 1999; Shulman, 1986). Shulman (1986) pointed out teachers' anticipation of students' errors and difficulties was part of teachers' pedagogical content knowledge:

> Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult….We are gathering an evergrowing body of knowledge about the misconceptions of students and about the instructional conditions necessary to overcome and transform those initial conceptions. Such research-based knowledge, an important component of the pedagogical understanding of subject matter, should be included at the heart of our definition of needed pedagogical knowledge. (p. 9-10)

Ma (1999) also connected TRED to teachers' subject matter knowledge. Ma conducted a study that compared mathematical understanding between U.S. and Chinese elementary teachers. Among her four interview questions, two of them related to teachers' responses to student errors or invented ideas. Ma's second interview question was about how teachers deal with students' mistakes in *Multidigit Number Multiplication:*

> Some sixth-grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate

```
  123
 ×645
 ----
```

> the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line. They were doing this:

```
  123
 ×645
 ----
  615
  492
  738
 ----
 1845
```

instead of this:

```
   123
 × 645
   615
  492
 738
 79335
```

While these teachers agreed that this was a problem, they did not agree on what to do about it. What would you do if you were teaching sixth grade and you noticed that several of your students were doing this? (p. 28-29)

All of the teachers in Ma's study considered this mistake as a problem of mathematical learning rather than carelessness. However, regarding identifying the reason for the problem and explaining how they would respond to student mistakes, U.S. teachers and Chinese counterparts demonstrated a big difference. Seventy percent of U.S. teachers viewed this mistake as a problem of carrying out the line-up problem whereas the majority of Chinese teachers interpreted the mistake either according to the distributive law, or conception of place value, or both. As a result, teachers' strategies to deal with this mistake turned out to be procedurally based versus conceptually focused. Based on this evidence, Ma (1999) pointed out that teachers' limited subject matter knowledge restricted their capacities to promote conceptual learning among students.

Even a strong belief of "teaching mathematics for understanding" cannot remedy or supplement a teacher's disadvantage in subject matter knowledge.…Ironically, with a limited knowledge of the topic, their perspectives in defining the students' mistake and their approach to dealing with the problem were both procedurally focused. (p.36)

Ma’s fourth interview question was about teacher responses to a student’s alternative idea /incomplete finding concerning *the Relationship between Perimeter and Area*. Through the comparison of teachers’ responses to student invented strategies, Ma classified teachers’ mathematical understanding into four levels. Based on these interviews, Ma identified Chinese teachers’ knowledge packages concerning each topic.

Ma’s (1999) study successfully uncovered teachers’ subject matter knowledge through the portraits of TRED. Based on this in-depth research, she was also able to provide Chinese teachers’ knowledge packages characterized by connectedness, depth, breadth, and thoroughness. However, her study only employed the interview method outside the classroom context. This method has its weakness because teacher knowledge is “characterized more by its concreteness and contextual richness than its generalizability and context independence” (Hiebert et al., 2002, p.3). It is necessary to see how teachers use their knowledge in real classroom contexts to deal with students’ errors and difficulties. This type of knowledge is called by Ball and her colleagues as Teachers’ MKT.

TRED as an important indicator of teacher knowledge was also reflected by the aforementioned study (Ball, 2006) where examples concerning student errors were used for differing MKT and PCK. In addition, Ball, Hill, et al. (2005) also explained the meaning of MKT by using the examples of dealing with students’ errors in the multiplication of whole numbers. They pointed out that effective teaching first requires teachers to be able to see students’ typical errors; then to analyze the source of errors which might need teachers’ mathematical consideration. Moreover, teachers need to explain the basis and principles of algorithms to students by using the words or representations that children understand. In a word, to address student errors and difficulties well demands teachers a deep and detailed understanding of mathematics that goes beyond carrying out algorithms.

2.2.2 TRED and cultural beliefs

As Stigler and Hiebert (1999) pointed out, even though U.S. mathematics teaching includes various approaches within the country, when it is compared with that of other countries, the internal differences are minimized and the culture-based patterns are more obvious. Ma's (1999) cross cultural study reflected the difference in teachers' knowledge. Meanwhile, it reflected a cultural difference concerning the views of student errors and difficulties. According to Bruner (1996), nothing is "culture free," but individuals are not simply a mirror of culture. Culture is the interaction between individuals that "both gives a communal cast to individual thought and imposes a certain unpredictable richness on any culture's way of life, thought or feeling" (Bruner, 1996, p.14). Education, specifically teaching, is then situated in a cultural setting and unitizes cultural resources. Teaching is a cultural activity because it is something one learns to do more by growing up in a culture than by studying it formally (Stigler & Hiebert, 1999). As a result, some teaching philosophy and characteristics are shared from person to person and inherited from generation to generation.

What is U.S teachers' cultural view of student errors and difficulties? Stigler and Hiebert (1999) clearly pointed out U.S. teachers' cultural view of student errors and difficulties in *The Teaching Gap,* the report of the TIMSS video survey (*N* = 238) among German, Japanese, and U.S. According to Stigler and Hiebert, many U.S. teachers believe that school mathematics is a set of procedures. Student learning, therefore, should be relatively error-free. The errors, confusion and frustration, in this view, should be minimized because they are signs that students have not mastered the earlier learning and teachers have not done their job well. In contrast, Japanese teachers view mathematics as a set of relationships between concepts and procedures. Students' learning, therefore, is a process of constructing connections. Allowing students to make mistakes and then reflect on their mistakes will provide opportunities for students to reach full understanding.

With different beliefs about errors, U.S. and Japanese teachers act in different ways during classroom instruction. U.S. teachers circle around the classrooms and sometimes reminded student if they see several individual students make the same mistakes: "Number twenty-three may be a little confusing. Remember to put all the x-terms on one side of the equation and all the y-terms on the other, and then solve for y. That should give the answer" (Stigler & Hiebert, 1999, p.92). In a word, U.S. teachers try hard to reduce errors and difficulties by presenting full information about how to solve problems. In contrast, Japanese teachers often begin their class with challenging problems and they help students understand and represent the problem in order that they can start working on a solution. Japanese teachers also encourage students to keep struggling in the face of difficulties rather than to decrease the level of difficulties or avoid difficulties. Sometimes they provide hints to support students' progress. Very few teachers would show students how to solve the problem midway through the lesson. As a result, "struggling and making mistakes and then seeing why they are mistakes are believed to be essential parts of the learning process in Japan" (Stigler & Hiebert, 1999, p.91).

Why do U.S. teachers have this cultural view? The cultural difference concerning teachers' views of student errors and difficulties is consistent with the difference in teachers' cultural beliefs on teaching and learning mathematics (Stigler & Hiebert, 1999). Stigler and Hiebert compared the U.S. and Japanese teachers' culture views from the following five aspects: (1) nature of mathematics, (2) nature of learning, (3) role of teacher, (4) individual differences, and (5) sanctity of lesson. According to Stigler and Hiebert, U.S. teachers believe that school mathematics is mostly a set of procedures and is learned by practicing materials incrementally, piece by piece. As a result, U.S. teachers take responsibility for "shaping the task into pieces that are manageable for most students, providing all the information needed to complete the task and assigning plenty of practice" (p.92). U.S. teachers also take responsibility for keeping students motivated, engaged, and attending. In contrast, Japanese teachers view mathematics as a set of relationships between concepts, facts, and procedures. The learning is a process for students struggling to construct connections between methods and problems. The teacher's role is then to help students construct connections during this process. To illustrate these cultural features in mathematics teaching and learning, Stigler and Hiebert provided two examples. First, U.S. teachers prefer to use the overhead projector because U.S. teachers believe that mathematics is set of procedures, mathematics learning therefore needs to gain students' moment to moment attention. By comparison, Japanese teachers do not use these tools because they want to show students the connection of concepts. Therefore, the blackboard is sufficient. Second, U.S. classes have frequent disturbances because although disturbances might be annoying, they will not affect mathematics teaching and learning because it is a set of procedures. However, these class disturbances seemed surprising and confusing to Japanese teachers and researchers. Considering these different beliefs of teaching and learning school mathematics, it is not surprising to see how U.S. teachers view student errors and difficulties substantially different from that of the other countries such as Japan and China.

Where is the source of this cultural view? U.S. teachers' common view of errors and difficulties is consistent with the cultural belief that mathematics learning is skill learning. This view has its tradition in American Education, originating in behaviorism, developed fully by E. L. Thorndike in the early 1900s and elaborated in different ways by B.F. Skinner and R. M. Gagne (Stigler & Hiebert, 1999). Thorndike was a behavioral psychologist and connectionist. His theory claimed that learning in one area could be automatically transferred to another area. Thorndike (1922) applied his stimulus-response learning theory to arithmetic teaching which influenced U.S. mathematics education for a long time (Bidwell & Clason, 2001). Skinner was the most famous follower of Thorndike. Based on Thorndike's stimulate-response theory, Skinner developed his learning theory of *operant conditioning.* According to this theory, a response to a stimulus can be made more frequent by reinforcement. "Adapting operant conditioning and programmed instruction to arithmetic and mathematics instruction requires dividing material into very small learning steps that can be individually reinforced" (Bidwell & Clason, 2001, p. 656). From this view, mathematics became a set of procedures which need to be learned by repeated practices, similar to the teaching strategies used in current classrooms. Gagne incorporated operant conditioning ideas into his learning theory of the hierarchical structure, which he used for describing learning concepts, principles, and structures.

Gagne (1965) advocated that knowledge must be organized and learned hierarchically with the later ones being slightly more difficult than the prior ones. “Gagne’s theory of structure led to greater care in sequencing instruction. Presenting mathematics in this way was sometimes called *guided learning,* a way of learning in opposition to the less-structured approach of *discovery learning*” *(*Bidwell & Clason, 2001, p.271*).* In current U.S. mathematics classrooms, many teachers use this approach to teach mathematics concepts – guiding students to practice series of exercises without considering the depth of their understanding (Bidwell & Clason, 2001). As a result, when teachers encounter student errors and difficulties, they do not tend to address them in depth.

2.2.3 TRED and motivation

As we see, how teachers respond to student errors and difficulties may relate to teachers’ knowledge. It could also be influenced by teachers’ cultural beliefs of teaching and learning school mathematics. Among these cultural beliefs, the concern of motivation is a noticeable one. It might hinder teachers to use their mathematical knowledge to address student errors or difficulties in depth.

The cultural belief of motivation and TRED. Motivation has been regarded as an important factor that influences students’ learning by most U.S. teachers. As previously mentioned, U.S. teachers take responsibility for keeping students attracted, engaged, and attending. The overhead projector is used for drawing students’ attention (Stigler & Hiebert, 1999). Many researchers have found that making mathematics fun was central to U.S. beginning and experienced teachers’ pedagogical reasoning (e.g., Ball, 1988; Ball, 1993; Eisenhart et al., 1993). Assuming that mathematics is inherently boring and hard to learn, these teachers thought their role was to find ways to motivate or engage students or to locate games to lighten the load for students (Ball, 1993). Methods such as praising students are used very often in classrooms. As a result, pointing out students' mathematical mistakes is naturally considered as negative. There is additional evidence showing that student motivation rather than mathematical content is a central belief among U.S. teachers. For example, cooperative learning methods are employed in classrooms with little focus on students' mathematical thinking (Ding, Li, Piccolo & Kulm, 2007); manipulatives are used by teachers without allowing students to see mathematical reasons (Li & Ding, 2006); and innovative activities designed for encouraging students especially girls tend to distort the mathematics in the process (Heaton, 1993). In summary, methods such as cooperative learning, manipulatives, or innovative activities are often used for motivating students or making it fun rather than teaching and learning mathematics. As a result, it is reasonable to assume that teachers' hesitation to explicitly point out students' errors or challenge students' thinking is due to their considerations of students' motivation. The concern that addressing students' errors will diminish students' motivation reflects the cultural views of teaching and learning mathematics. That is, mathematics is set of procedures that are not interesting. Errors are signs that the previous material was not mastered and the teachers' job has not been done (Stigler & Hiebert, 1999). Addressing students' errors or challenging students' thinking might embarrass students, resulting in low motivation which in turn, will negatively influence mathematical learning.

A constructivist's view of motivation and TRED. Ernst von Glasersfeld (1982), a leading figure of radical constructivism, developed Piaget's theory and also has had a great contribution to the field of mathematics education. This researcher (von Glasersfeld, 1996) claimed that teachers' responses to students' errors did matter in students' learning. He said, when a teacher boorishly dismissed a student's solutions as "wrong", it would demolish this student's motivation by degrading his/her efforts. In fact, whether a student would really work on an error or construct alternative understanding for the concept depends on whether this student would like to see the error as a "problem", which, in another word, depends on students' motivation (von Glasersfeld, 1989a).

Even though von Glasersfeld (1996) reminded teachers not to be rude in correcting students' errors and difficulties, he did not suggest teachers ignore it. Instead, he suggested:

> A wiser teacher will ask the student how he or she came to the particular answer. In the majority of cases, the student, in reviewing the path (i.e., reflecting on the operations carried out), will either discover a hitch or give the teacher a clue to a conceptual connection that does not fit into the procedure that is to be learned. The first is an invaluable element of learning: it provides students with an opportunity to realize that they themselves can see what works and what does not. The second provides the teacher with an insight into the student's present way of operating and thus with a clearer idea of where a change might be attempted. (p.312)

The aforementioned example of Ball's (1993) study showed how to address students' errors with the consideration of their motivation. Facing a student's incorrect representation (circle), Ball debated about how to respond to the student Cassandra: "Should I question her further about her solutions? She was not at all dissatisfied with it and it made compelling sense in many ways (p.180)." At last, Ball saw the opportunity to respect this student's real attempt and encouraged her to use another representation (rectangle) because it would be easier to divide them evenly so that every one could get the same amount of the cookie. Obviously, the reason Ball did not directly say something was wrong with her solution was because of the concern for motivation.

Facing students' errors with the consideration of their motivation does not suggest ignoring errors or not allowing students to know them at all. Avoiding errors will not motivate students. In fact, von Glasersfeld (1996) argued that the only sources of real motivation were students' experiences of intellectual success and pleasure:

> More important still, we shall have to create at least some circumstances where the students have the possibility of experiencing the pleasure of finding that a conceptual model they have constructed is, in fact, an adequate and satisfying model in a new situation. Only the experience of such success and the pleasure they provide can motivate a learner intellectually for the task of constructing further conceptual models. (p.312)

Ignoring students' errors or not allowing students' to see their errors actually dismisses students' possible intellectual success and pleasure, the real source of interior motivation. von Glasersfeld (1989b) pointed out, a constructivism teacher was more likely to explore how and why a student solved the problem in his/her inventive ways. "This in turn makes it possible to build up a hypothetical model of the student's conceptual network and to adapt instructional activity so that it provides occasions for accommodations that are actually within the student's reach" (p. 12).

2.2.4 Positive views with alternative interpretations

U.S. teachers' common view of errors and difficulties, originating in behaviorism, has gradually changed in current classrooms due to the influence of constructivism. This could be seen from the above introduction of the constructivist's point of view concerning motivation and TRED. Hiebert et al. (1997) also argued that mathematics learning was a process of making sense and constructing understanding. Student mistakes should be viewed as a natural and important part of this process. This view is not to put a good face to a bad situation because if mistakes are treated appropriately, they can contribute to everyone's understanding. As a result, mistakes "should play a constructive role in classroom discussions" (Hiebert et al., 1997, p.48). To establish this constructive role of mistakes, teachers need to build a healthy classroom culture where students are free to take risks and to try ideas out without being ridiculed. "They need to feel that their thoughts can contribute to the classroom enterprise, even if they are not entirely correct" (p.49). According to Hiebert et al. (1997), to build such a classroom environment, teachers need to set a balanced tone. Correct or incorrect student suggestions should be both welcomed as potentially valuable opportunities for learning. Avoiding incorrect answers and applauding correct ones both convey something unnatural. A better approach is simply to discuss and analyze an answer with an explicit aim of learning something new. Teachers' summary and confirmation of what was learned from a particular error can also emphasize how one can learn from mistakes. Over time, a sound classroom culture could then be built. Ball (1993) also commented that after half of a year of hard work at creating a classroom in which it was safe to make mistakes or try things out, students in her class were able to respect another's thinking and were patient with faltering explanations. Moreover, these students were inclined to ask questions to ensure a classmate's thinking before suggesting revisions or disagreement. In summary, with a positive view and a healthy environment, teachers would not hesitate and students would not be embarrassed when they jointly dealt with errors and difficulties.

Capitalize on errors as springboards for inquiry. Researchers have advocated treating students' mistakes as opportunities for learning (Borasi, 1985, 1987, 1994, 1996; Ding, Li, Piccolo, & Kulm, 2007; Hiebert et al, 1997) rather than negative signs that need diagnosis and remediation (e.g., Confrey, 1990; Englehardt, 1982; O'Connell, 1999). A positive view of errors and a healthy classroom culture are prerequisites of addressing errors. How teachers use errors as opportunities for inquiry depends mainly on their *pedagogically useful mathematical understanding.*

Borasi (1987) advocated using errors as springboards for inquiry through analyzing the erroneous form a/b + c/d = (a + c)/(b + d). After pointing out two possible sources of the errors - (a) students could have confused the rules for adding fractions with the rules for multiplying fractions; and (b) students could try to operate with fractions as they did when adding whole numbers - Borasi argued for a third situation. That is, there were some real-life situations in which such a way of operating seemed indeed appropriate, which showed the spirit of capitalizing on errors as springboards for inquiry. She suggested that teachers, instead of looking for reasons why students made mistakes, could involve them in questioning whether there were particular contexts in which their wrong operations might be correct. The following real-life situations provided by Borasi (1987) was her evidence that a/b + c/d = (a + c)/(b + d) might be appropriate: "if you won 2 out of 3 games yesterday, and 5 out of 7 games today, altogether you have won 7 out of

10 games, and not 29/21" (p. 3). This game example illustrated the following misleading suggestion: if teachers capitalize on errors as springboards for inquiry, they may find that some erroneous formats turn out to be appropriate. The suggestion reflected this researcher's own confusion between fraction and ratio and a weak understanding of rational numbers.

The game example is actually a ratio problem. It is true that the ratio (a:b) could be written in the format as a/b. However, the rational number a/b has multiple meanings such as part of whole, measurement, quotient, operator, and ratio (Behr, Wachsmuth, Post, & Lesh, 1984; Lamon, 1999). Regarding the above example, if we use 2/3 and 5/7 to represent "won 2 out of 3 games yesterday" and "won 5 out of 7 games today" respectively, these two fractions "2/3" and "5/7" are actually ratios "2:3" and "5:7". As Lamon (1999) pointed out:

> The greatest difference between ratios and the other interpretations of rational numbers is in the way they combine through the arithmetic operations. The other interpretations of rational numbers are all different conceptually, but they are indistinguishable once they are written symbolically. They add, subtract, multiply, and divide according to the same rules. However, we do not operate on ratios in the same way that we do on fractions. (p.175)

According to Lamon (1999), the game example could be solved as: 2:3 + 5:7 = 7:10 or saying we won 7 out of 10 games. However, if we were adding fractions, we cannot write 2/3 + 5/7 = 7/10. Marshall (1993) also viewed the expression $\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$ as problematic and attributed this type of mistake to students' real-word experiences such as baseball games, cooking, or their grades in school. For example, in a multipart test, a student receives a/b grade for the first part with *a* indicating the number of items they answered and *b* representing the number of items in that part; and c/d being the result of a second part with similar definitions. The cumulative score, then, is (a + c)/(b + d). However, the aggregation of the two rations a/b and c/d, resulting in a weighted average, actually has a underlying process: $\frac{a}{b}(\frac{b}{b+d})+\frac{c}{d}(\frac{d}{b+d}) = \frac{a}{b+d}+\frac{d}{b+d}=\frac{a+c}{b+d}$, in which the situation is actually returning to part-whole, with the whole being *b* + *d*, the total items of two parts in that test. In fact, if we want to solve this problem by using the knowledge of "fraction", we should be able to see the real "whole" 10, the number of total games in two days. Therefore, the answer "7 out of 10 games" actually comes from "2/10 + 5/10" rather than "2/3 + 5/7". As a result, if one uses specific ratio problems to prove the erroneous fraction addition as correct, it will demonstrate a week understanding of not only ratio, but also fraction addition.

In California Assessment Program ([CAP], 1983), 42% of the students made the type of mistake in 1/5 + 3/4 = ☐ by selecting the first choice from: (a) 4/9, (b) 19/20, (c) 4/20, and (d) 3/20. With regard to this phenomenon, Marshall (1993) explained:

> The student may use his experiential and feature knowledge to build a "story" about this problem. In this case, the story may be about baseball, which suggests to the student to activate the ratio schema and to perform the incorrect algorithm of adding the numerators and then adding the denominators to form a new fraction. (p.282)

However, the problem 1/5 + 3/4 = ☐ demands a part-whole representation rather than a ratio. Obviously, when Borosi (1987) used the ratio example to prove a/b + c/d = (a + c)/(b + d), she made exactly the same mistakes as some students made when they add fractions. In fact, Borosi (1987) could attribute this type of error to students' real life experiences rather than to use the experience as a proof of its appropriateness. The written format 2/3 + 5/7 is a conventional mathematical expression for fraction addition which has a sole answer, 29/21. This is a mathematical fact, belonging to the *base of public knowledge*, one of the two bases (another base is mathematical language) for mathematical reasoning of justification (Ball & Bass, 2003b).

Although Borasi's insight of using errors as springboards for inquiry shows a positive view of students' errors, her way of exploring errors by finding the correctness for erroneous forms against mathematical facts is somewhat misleading. If teachers really agree with the written representation 2/3 + 5/7 = 7/10 as an appropriate one, students might lose their understanding of fractions, causing or entrenching the misconceptions regarding addition of fractions. Students could also be confused with fractions and ratios, resulting in their lower sensitivity of mathematics content and nature. Therefore, even though teachers should positively view errors as learning opportunities, their approaches to students' errors and difficulties greatly depend on teachers' own understanding of the domain and their mathematical knowledge for teaching. As a result, how to appropriately respond to student errors and difficulties deserves further study.

An alternative interpretation of "inquiry". Ding, Li, Piccolo, and Kulm (2007) used Borasi's (1994) suggestion of capitalizing on student errors for inquiry as one of the indicators to examine how teachers promote students' mathematical understanding in cooperative learning mathematics classes. In examining six teachers' interventions in teaching equivalent fraction lessons from two states, it was found in general, that these teachers were not skilled at promoting students' thinking by capitalizing on errors. The percentages of these six teachers' intervention sightings met the indicator "when students make mistakes, does the teacher use errors as springboards for inquiry" were 68%, 13%, 42%, 10%, 50%, and 0% respectively (Ding, Li, Piccolo & Kulm, 2007, p. 166). With regard to the common error "$3/4 \times 2 = 6/8$", very few teachers used it as an opportunity for inquiry. The authors, therefore, suggested that one of the ways to capitalize on this error for inquiry was to grasp the "=". Teachers could promote students' thinking by asking the following question: Since ¾ and 6/8 are equivalent fractions, if ¾ multiplies by 2, will the left side and the right side of the "=" still be equal? With this type of guidance, "students may not only have understood this concept but also have improved their understanding of the equal sign, a critical notion for later algebra study" (p.173). Since U.S. students have a weak understanding of the concept of equivalence (Capraro, Ding, Li, Matteson, & Capraro, 2007; Ding, Li, Capraro, & Capraro, 2007; McNeil et al., 2006) and understanding the "=" does matter in students' algebra learning (Knuth, Stephens, McNeil, & Alibali, 2006), Ding, Li, Piccolo, and Kulm (2007) suggested that teachers, especially elementary teachers, should have algebra eyes and ears even during their number lessons. In summary, these researchers modified Borasi's (1994) idea of using errors for inquiry from identifying appropriateness in erroneous forms to grasping errors as opportunities to both deepen students' understanding of current concepts (e.g., equivalent fractions) and lay foundations for their future study (e.g., algebra learning).

2.2.5 TRED and teaching context – when to respond and respond to what?

TRED relates to many factors. Teachers' knowledge affects the quality of their responses. Cultural beliefs including the concern for how motivation influences teachers' decisions on whether or not to respond and how to respond. Even for teachers who believe in using errors as learning opportunities, the interpretations of how to inquire could be different. Nonetheless, these aspects are quite general. Teachers' context-based concerns, views, and TRED might be more complex.

Student errors can occur on many occasions during the teaching context. The nature of errors varies from a slip of the tongue to conceptual mistakes. Errors could also be caused by certain bugs rooted in certain misconceptions reflecting students' learning difficulties (Li, 2006). As a result, should each of student errors be addressed at any time during classroom teaching? Or what are those appropriate occasions for teachers' TRED during context?

O' Connor (2001) examined the work of a fifth grade teacher, Mr. Anderson, in supporting student exploration of the relationships among fractions and decimals through the use of position-driven discussion, where "the teacher's role is not to provide validation of correct or incorrect hypothesis or evidence, but to support and clarify the contributions of students, often through revoicing moves" (p.150). In this case study, O'Connor pointed out a noticeable phenomenon, that is, the teacher sometimes ignored students' obvious mistakes. A typical example is the following discussion where student Clarence was trying to transforming 6/8 into a decimal:

> C: Because, um, six eighths can be reduced to three fourths and you can multiple the four to, by twenty, to get, um, to the power of ten which will be a hundred and then you multiply the three times twenty again will be sixty; that will be point sixty.
>
> T: What do you think about that? [addressing Sela]

S: I agree because, um, because you said – so three fourths and if you times a fourth times – if you times the four times twenty you get one hundred and if you do the three times twenty you get sixty and a hundred is [xx] with powers of ten.

T: Well that gets back to what Juana said. If you don't have a denominator that's a factor of a hundred, what should you look for, Juana?

J: You should look for an equivalent fraction to it that's less than that fraction.

In this example, the teacher did not correct Sela's and Clarence's incorrect multiplication "$4 \times 20 = 100$". Nor did she correct Juana's misuse of the terminology "an equivalent fraction that's less than that fraction" which actually meant an equivalent fraction in lower terms. Instead, she followed the conversation sequence concerning how to change the denominator into "powers of ten".

In terms of the phenomenon, O'Connor (2001), after discussing it with the teacher, argued that it actually reflected the teachers' skillfulness and success. She explained that the teacher's non-response to errors was "partially a matter of conscious judgment, and partially a matter of processing load" (p.175). On the one hand, during this exploratory talk, both students and teachers were under the greatest processing load. Students needed to figure out new ideas and present them in public while teachers needed to keep track of the sequence of student contributions, to monitor other students' understanding, and to prepare her own responses during two or three seconds. As a result, teachers are less likely to notice students' mistakes when their own processing load is heavy. On the other hand, students' excellent insights are possibly intertwined with computational mistakes. Thus, teachers need to judge what to focus on and when to focus on. This is because "stopping to focus on precision for precision's sake, when the idea can be understood without it, is a risk in terms of everyone's concentration" (O'Connor, 2001, p.175). Further, O'Connor provided a counterexample in which Mrs. Anderson did stop to correct student mistakes. This example was a summative talk - reviewing of a familiar idea. O'Connor pointed out, since everyone's processing load was lighter in this period, the teacher was able to pay closer attention to students' mistakes. More importantly, since that talk was summative, the teacher pursued the correctness and accuracy of the speech because "that is when it counts" (p.176). Based on these examples, O'Connor suggested, "when we are in the heavy lifting and framing stages of developing new ideas, stopping to correct every flaw is disruptive to the real work. When the ideas are ready for polishing, however, correctness in every respect must be the goal" (p.177).

As O'Connor (2001) argued, when many researchers and educators have noticed the importance of the precision of mathematical language, "they often fail to acknowledge that socialization into this practice must proceed in fits and starts" (p.177). Therefore, teacher actions such as not correcting student errors were often misconstrued. She further pointed out, in fact, not all activities in elementary and middle school classrooms would support an equally intensive focus on precise and correct language. This is because "active exploration of mathematical ideas is difficult, ideas do not emerge fully formed, refined and gleaming. They must be shaped, revised, scrutinized, reworked, and polished" (p.177).

O'Connor's study had significant contributions through raising practical questions and issues for debate; that is, she contrasted, "on the one hand, the concerns a mathematician might have for technical precision in the problem statement and, on the other hand, the efforts a teacher might make to state the problem in a way that is mathematically appropriate yet meaningful for fifth graders" (Thames & Ball, 2004, p. 429). O'Connor's case study also highlighted the significance of contexts for research on effective teaching. Mathematical understanding for teaching requires teachers not only knowing mathematics but also using it for the work of teaching in a pedagogically meaningful way.

Nevertheless, O'Connor's study left unsolved problems. This study suggested that, during explorative talk, teachers need not address those mistakes with little negative influence on local learning goals (e.g., $20 \times 4 = 100$ in that case) and leave these types of errors to the summative phase. However, another type of error which has no harm for the local learning goal but will possibly hinder students' future learning was not fully considered. Regarding the aforementioned error – an equivalent fraction which is less than that fraction (O'Connor, 2001) - was not addressed by the teacher since the central idea of that discourse was about "powers of 10" and this mistake was not an obstacle during that discussion. However, this mistake obviously reflects a common misconception held by many students, that is, reducing fractions makes smaller numbers (Ball et al., 2001; Leinhardt & Smith, 1985). This misconception is partially caused by students overgeneralization of whole numbers operations - multiplication makes bigger while division makes smaller- which reflected students' weak understanding of rational numbers. If teachers ignore this type of error, it could be entrenched over time, resulting in robust misconceptions (Li, 2006; Resnick, 1980). Since we are not sure whether such mistakes will occur in the summative phase to be addressed, it might be better to capitalize and correct them immediately for those obvious reasons. In addition, it seems hard to imagine how the discourse sequence would be disturbed by the teacher asking a simple question such as “less than that? You mean a lower term?” In a word, what is the appropriate time for teachers to address student errors may not absolutely follow O’Connor’s delineation of explorative and summative phases. Moreover, what types of errors are worthy of addressing needs teachers’ judgment that requires teachers’ understanding of mathematical content and student learning. As von Glasersfeld (1983) argued, teachers’ sound decision making demands they not only are able to have, or construct, models of students’ present conceptual structures but also the ones toward which instructional guidance might lead the children. During the lesson, teachers should not only have eyes on current knowledge but also be aware of students’ future learning. Only when teachers have this type of awareness, could they really treat student errors as learning opportunity or as opportunities for inquiry (Ding, Li, Piccolo, & Kulm, 2007), which, however, requests teachers’ MKT reaching a level with connectedness, breadth, depth, and thoroughness. Therefore, sound MKPT could contribute to TRED in real contexts.

2.2.6 Summary

Teacher responses to students’ errors and difficulties (TRED) is a critical but complex indicator of effective teaching. It is directly related to teacher knowledge but might also be shaped by cultural beliefs about mathematical teaching and learning, and influenced by considerations of student motivation. Even though there is an agreement on the positive view - using errors and difficulties for inquiry, what counts as “inquiry” and how to inquire in appropriate ways needs further discussion. In addition, in real teaching contexts, decisions about when to address student errors and what types of errors deserve addressing depend on teachers' MKT, specifically MKPT.

2.3 Teaching and Learning Equivalent Fractions

The purpose of this study is to align teacher knowledge with classroom instruction. The case TRED is situated in the context of teaching and learning equivalent fractions. As a result, this section reviews issues concerning teaching and learning equivalent fractions in an intensive way.

Fractions are a significant topic in school mathematics (Leinhardt & Smith, 1985). Both mathematicians and mathematics educators agree that understanding the meaning of fractions is critical because it lays a foundation for learning ratios, proportions, and percentages (Ball, Ferrini-Mundy, et al., 2005). Kieren (1993) pointed out understanding multiple meanings of fractions would significantly contribute to children's mathematical thinking. For example, deliberately using partitioning of continuous quantities can lead to intuitions of the infinitely small and a broadened conceptualization of numerical order. In addition, fractions are viewed as (a) measurement related to geometry and space, (b) ratios providing a window of probability, (c) operators highlighting certain algebraic properties and the notion of composite functions, and (d) decimals reflecting the base-10 numeration system.

However, fractions are "exceedingly difficult for children to master" (National Assessment of Education Progress ([NAEP], 2001). Many studies have provided evidence that children learn fractions with great difficulty (e.g., Behr, Lesh, Post, & Silver, 1983; Hiebert, 1988; Tatsouka, 1984). Meanwhile, it is also a difficult topic for many teachers (Ball, 1993; Ball et al., 2001; Ma, 1999; Post, et al, 1993). Consequently, students are struggling with learning fractions while teachers are struggling with teaching it.

Among the topic of fractions, fractional equivalence is a fundamentally important concept (Post et al., 1993). It forms the basis to meaningfully operate on fractions and decimals. In addition, it helps children judge the reasonableness of their answers (Post, et al, 1993). As a result, NCTM (2006) *Curriculum Focal Points* lists understanding equivalent fractions as one focus in third grade based on the sense of “part-whole” relationship. Therefore, it is necessary and meaningful to review prior research on teaching and learning equivalent fractions. Four aspects are addressed in this section: (1) intrinsic difficulties for learning equivalent fractions, (2) teacher mathematical knowledge for teaching equivalent fractions, (3) children’s thinking characteristics in learning equivalent fractions; and (4) a semantic net of equivalent fractions provided by Leinhardt and Smith (1985). All these aspects together provide explanations of what makes equivalent fractions difficult to teach and learn and what are research-based strategies to deal with this issue. As a result, students learning difficulties and common errors in teaching and learning were interwoven in these reviews. Again, since teaching and learning cannot be absolutely divided, the above four aspects are possibly overlapped.

2.3.1 Intrinsic difficulties for learning equivalent fractions

To understand what makes equivalent fractions difficult to teach and learn, we need first to know what makes fractions difficult to teach and learn. This is because the concept *fraction* is the base for the “core of knowledge” in learning equivalent fractions

(Leinhart & Smith, 1985; NCTM, 2006). The following four critical features of fractions cause the intrinsic difficulties in learning equivalent fractions.

Multiple meanings. Kieren (1980) identified five interpretations of fractions, namely part-whole, quotient, measure, ratio, and operator. Behr et al. (1983) provided seven ideas – which they called *subconstructs* – of fractions: fractional measure, ratio, rate, quotient, linear coordinate, decimal, and operator. Lamen (1999) even provided 12 interpretations by using the example of ¾. These varied interpretations of fractions typically demonstrate the complexity of the concept of fraction. To truly understand the meanings of fractions, students should develop fluency among these alternative interpretations. Even though the definition of equivalent fractions is based on the "part-whole" relationship in elementary mathematics, if the "part-whole" interpretation turns out to be the only focus as is the case in traditional instruction, it will leave students "with a deficient understanding of the part-whole fractions themselves, and an impoverished foundation for the rational number system" (Lamen, 1999, p.30), resulting in learning difficulties concerning equivalent fractions over time. For example, to solve problems such as "3/4=?/5" also requires the knowledge of ratio and proportion (Post et al., 1993).

Abstraction of the concept. Compared with whole numbers, the concept fraction is more abstract, going beyond children's common sense and intuition. For example, with regard to "part-whole", "the 'whole' may be a continuous or a discrete quantity; the continuous quantity may be a linear, area, or volume model" (Saxe, Taylor, McIntosh, Gearhart, 2005, p.154). Regarding equivalent factions, teachers and textbooks, however, often provide incomplete information such as only use continuous models to illustrate the idea of equivalent fractions (Leinhardt & Smith, 1985). As a result, some students could use a common factor strategy to complete a given equivalent expression such as ½ = □/4 only based on a continuous quantity object rather than discrete elements, which shows students' understanding of equivalent fractions was still brittle and what they had mastered was actually numerical algorithms (Hunting, 1984). Therefore, the abstraction of the concept fraction brings obstacles for students' understanding of fractional equivalence. To truly master equivalent fractions requires a deep understanding of the concept fraction.

Complicated semantics. Ohlsson (1988) pointed out another difficulty of fractions is their complex semantics. "How is the meaning of 2 combined with the meaning of 3 to generate a meaning of 2/3" (p. 53)? In other words, because of the special format – a numerator and a denominator – fractions are not easily internalized as a workable concept by children. Students often see little relationship between numerators and denominators and often handle them as separated entities (e.g., 2/3 + 3/4 = 5/7) (Behr et al., 1984; Carpenter et al, 1993). Some students regard fractions as merely the result of two counting experiences – counting the number of shaded parts of a figure and the total number of parts - reported in a special format (D' Ambrosio & Mewborn, 1994; Moss & Case, 1999). Some students even think fractions are not numbers at all (Kerslake, 1986). The above semantic difficulties could be explained in term of Sfard's (1991) point of view *object* and *process* (or *structurally - operationally*). According to Sfard, most mathematical concepts can be conceived in two fundamentally different ways: as object or as process. Students have ontological difficulties in transition from process to object. When students treat the numerator and denominator of a fraction as separate numbers, their understanding are still at the process level rather than object level (Li, 2006), which leads to various errors such as thinking of 1/4 is greater than 1/3 because 4 is greater than 3 (Carpenter et al., 1993; Leinhart & Smith, 1985). This type of misconception can further hinder students' understanding of equivalence (Leinhardt & Smith, 1985), resulting in beliefs such as no number could be found to satisfy the equity for problems like 8/15=?/5 (Post, Harel, Behr, & Lesh, 1988).

Relation to whole number. Fractions are developed from whole numbers. Therefore, there is a consistency between these two number systems. For example, division of fractions is not different conceptually from division of whole numbers (Ball, 1990). However, as Kieren (1993) pointed out:

> Although intertwined with, sharing language with, and using concepts from whole numbers, rational number knowing is not a simple extension of whole number knowing. There are fundamental new axioms and properties; there are fundamentally distinct actions for the knower. (p.56)

Students often have difficulty overcoming their whole number thinking while they work with fractions or decimals (Behr et al., 1984; Hiebert & Wearne, 1985, 1986). Mack (1995) pointed out two challenges concerning written fractional notations: (1) students tended to interpret written notations in whole number terms. For example, some children regarded that the notation for five eighths of a figure could be written as 5 and 5/8; (2) students tended to interpret whole number notations of mixed number expressions in terms of fractions. For example, they interpret "2" in the subtraction expression "2 – 3/8" as "2/8" (also cited by Saxe et al., 2005). In addition, the aforementioned error $\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$ might also be influenced by students' whole number ideas (Behr et al., 1984; Borasi, 1987). This type of whole number thinking causes difficulties for learning equivalent fractions. In Post Behr, and Lesh's (1984) study, one student successfully answered ¾ = ?/8 and explained her process by finding a common factor. That is, she first identified a factor that changed 4 to 8, and then she used that factor "2" to change 3 to 6. However, when this student was asked to do the next task ¾ = ?/5, she said she would added 1 to the numerator because she first tried to find a factor to multiply, but she could not find a whole number, so she changed to add a whole number. This statement clearly demonstrated the dominance of whole numbers on children's thinking strategies. Another example of evidence that whole number knowing in learning equivalent fractions is children's beliefs that multiplication makes bigger and division makes smaller (Post et al., 1984). In Hart's (1981) study, even though some children knew that 2/3 and 10/15 were equivalent, they still thought 10/15 was bigger than 2/3. They explained, 3 go to 15 five and 2 goes to 10 five times but 10/15 was still bigger because 10 and 15 were bigger than 2 and 3.

In summary, these four features of fractions - multiple meanings, abstraction, special representation, and its close relationship with whole number – together cause intrinsic difficulties for teaching and learning equivalent fractions.

2.3.2 Teachers' mathematical knowledge for teaching equivalent fractions

The difficulties concerning teaching and learning equivalent fractions could also be due to teachers' own knowledge. Teachers' knowledge of rational numbers, a component in teacher decision making (Kieren, 1993; Schoenfeld, 1998, 2000) determines whether they have the ability to provide cognitively guided instruction in this topic for their students (Carpenter & Fennema, 1988; Kieren, 1993). As previously mentioned, von Glasersfeld (1983) argued that in making sound decisions in teaching rational numbers, teachers must not only be able to have models of students' current conceptual structures but also the ones toward which instructional guidance might lead the children. As a result, to teach equivalent fractions, teachers need to have a knowledge package (Ma, 1999) where the concept of equivalent fractions weighs the most but also closely ties with other related concepts, both the learned and the ones to be learned. Lacking this type of knowledge will make teaching and learning equivalent fractions more difficult.

Regarding teaching equivalent fractions, there is evidence of teachers' weak knowledge. For example, some preservice teachers thought questions such as ¾ = ?/9 could never be solved (Cramer & Lesh, 1988; Post et al., 1988), which might be due to their whole number thinking, or because of a weak understanding of multiple fractional meanings (e.g., ratio and proportion), or because they thought a fraction is not a number (Post et al., 1993). In an example with inservice teachers, Leinhardt and Smith (1985) found that all participants - 4 expert teachers and 4 novice teachers - correctly defined *equivalent* by emphasizing the regional equality. However, when they were asked "Is 3/7 = 243/567", five of these eight teachers were confused. Even though they tended to get 81 as a factor but they did not know what to do with it and eventually said these fractions were not equivalent or they did not know. A striking example of a response provided by Leinhardt and Smith is that:

> No, wait, let's see. Well, I'm saying that because you can divide 3 into 253 and 7 into 567, ahm, huh, not necessarily because you cannot, as they are, 3 and 7 don't both go into these numbers evenly (that is that 7 doesn't go into 243)…Okay, no.

> I'm just figuring, it's 81 over 81, if you divide it out that way… Isn't that funny, I teach, I'm teaching that and understanding it when I'm teaching it. Yeah it does, no, they aren't equivalent because you can't divide them by the same number. Like, I would have to divide 567 in order to see if they were equivalent and it isn't. Let's see 21…no, they aren't. Can you tell me? (p.253)

Moreover, when discussing equivalent fractions, six out of these eight teachers did not mention that raising or lowing terms was actually multiplying or dividing by a "1" (in this case 81/81). Even during the interview, these less knowledgeable teachers did not realize it was true.

Teachers' weak mathematical knowledge for teaching equivalent fractions partially relates to their weak understanding of some basic ideas. Ball et al. (2001) summarized research on various fundamental topics including factions and pointed out: "An overview of those studies of teachers' mathematical knowledge – elementary and secondary, preservice and experienced – reveals pervasive weaknesses in U.S. teachers' understanding of fundamental mathematical ideas and relationships" (p.444). One of the examples is "dividing by 0". In Ball's (1990) study, some teachers said 7 divided by 0 was zero; some teachers said "0" could not be a divisor because it was undefined. However, they did not know why it was undefined. The other teachers could not remember a rule at all and were unable to provide an answer. Lacking knowledge for these types of fundamental ideas will inhibit teachers' sensitivity to school mathematics. For example, many current curricula provide incomplete knowledge to students (Leinhardt & Smith, 1985). Regarding finding equivalent fractions, textbooks usually state the rule in the following ways: "multiplying the numerator and denominator by the same number" (Leinhardt & Smith, 1985). However, could a fraction be multiplied by 0/0? When a teacher lacks knowledge concerning "dividing by 0", it is hard for them to be sensitive to this type of incomplete textbook knowledge or imprecise mathematical language. As a result, teachers' own understanding of equivalent fractions is not in depth and their classroom instruction will be more procedural-focused.

Teachers' weak knowledge for teaching equivalent fractions is consistent with their general weak understanding of fractional concepts, computations, and relationships. For example, in Ball's (1990) study of 252 U.S. prospective teachers, including elementary and secondary candidates, only about 31% of them could appropriately represent 1¾ ÷ 1/2 , as a result, Ball concluded that teacher candidates' mathematical understanding tended to be rule-bound and thin. These "inadequately trained" (Stevenson & Stigler,1992, p.157) preservice teachers, when they become inservice teachers, will encounter difficulties in their teaching. Ma's (1999) comparison study, by using the same problem – representation for 1¾ ÷ ½ - as one of four interview questions, showed U.S. teachers subject matter knowledge lagged far behind their counterparts.

As Ball (1993) pointed out, teachers' own understanding matters in regard to their ability to teach and their sensitivity to school mathematics, which determines students learning opportunities and makes differences in their learning outcomes (Ball & Bass, 2003a). Even when teachers claim to teach for understanding but their own understanding of mathematics is grounded in rules and algorithms, their teaching will tend to focus on procedures rather than on creating a context for unpacking mathematical meanings (Ball, 1993; Leinhardt & Smith, 1985; Ma, 1999). As a result, when teachers' mathematical knowledge for teaching equivalent factions is weak, both teaching and learning equivalent fractions become difficult.

2.3.3 Students' thinking characteristics in learning equivalent fractions

To help students construct their understanding of equivalent fractions, teachers need to understand not only the content, but also how students learn this content. "This bifocal perspective – perceiving the mathematics through the mind of the learner while perceiving the mind of the learner though the mathematics – is central to the teacher's role in helping students' learn with understanding" (Ball, 1993, p.159).

Researchers in the Rational Number Project (RNP) (Post, Wachsmuth, Lesh, & Behr, 1985) based on the observation of a teaching experiment, (Behr et al., 1984), identified three thinking characteristics of successfully learning equivalent fractions: (1) thought flexibility in coordinating between-mode translations; (2) thought flexibility for transformations within a mode of representation; and (3) reasoning that becomes increasingly independent of specific concrete embodiments. Concerning the first thinking characteristic, students need to know simultaneously: (a) how to represent symbol m/n by the fraction embodiments such as pictures and manipulative objects - denoted as (m)/n; and (b) how to derive meaning of symbol m/n from the embodiments of fractions. Taken together, students need to make translations in either direction between the fraction symbols and their embodiments (see Figure 5):

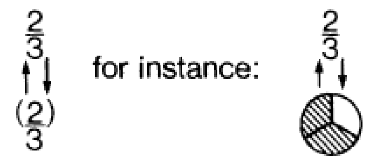


Figure 5. Translation between mathematical symbol and embodiment.

"This coordination of information between the mathematical symbol representation and fraction embodiment representation is called *coordinating translations*. The process often involves relating the physical transformation among, and within, fraction embodiments to a corresponding symbolic arithmetic operation" (Post et al., 1985, p.20). The second thinking characteristic is to make transformations within either symbolic or embodiment modes. For example, to solve 4/6 = □/3, students can do it by means of a transformation algorithm within symbolic mode, such as dividing 4 and 6 each by 2. It could also be solved within the embodiment system which requires students' ability to transform (4)/6 to (2)/3, a process involving either a physical or a mental (i.e., imagined) repartitioning (see Figure 6). Students' abilities to make these transformations within representational modes were referred to as thought flexibility for transformations.

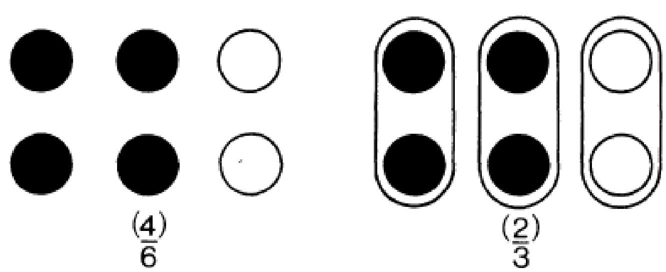


Figure 6. Transformation within embodiments to solve 4/6 = □/3.

As a result, regarding 4/6 = 2/3, students' first and second thought flexibility provide them three sources of information - (a) the observation that (4)/6 and (2)/3 have equal amounts covered, (b) the translation of (4)/6 to 4/6, and (c) the translation of (2)/3 to 2/3 - for the ability to make subsequent inference that 4/6 = 2/3 at the symbolic level. Limited thought flexibility in accomplishing any of (a), (b), or (c) above inhibits students' successive abstraction, the ultimate aim of instruction, which requires students' third thinking characteristic - independent reasoning and thought from embodiments

Post et al.'s (1985) report of students' thinking characteristics in learning equivalent fractions provided teachers useful information about teaching this topic. As a result, teachers should first develop students understanding of the concept *fraction*, the base of the core knowledge in equivalent fractions (Leinhardt & Smith, 1985), through transforming alternative representations. If students cannot understand the concept of fraction such as notations and part-whole references, it is impossible for them to understand the concept of equivalent fractions. However, the concept fraction is not easy for students to master. Saxe et al. (2005) explored the developmental relationship between students' use of fraction notations and their understanding of part-whole references. They found that the students' knowledge of notations and part-whole references somewhat developed independently. Students' use of notations does not lead to their development of part-whole relationships. As a result, teachers need to be aware of all these learning difficulties rooted in the concept fraction in order to construct students' understanding of equivalent fractions.

Second, Post et al.'s study pointed out the importance of representation, including concrete and symbolic, between and within these representational models. As previously mentioned, Post et al.'s (1985) identification of students' thinking characteristics is based on the observation of students' successful performance in one of RNP's teaching experiments with 4th, 5th, and 7th graders (Behr et al, 1984) in which Lesh's (1979) representational model was used for instruction. This model was an extension of Bruner's three representational models –enactive, iconic, and symbolic - to include verbal and real-word problem settings and situations. It emphasized nonlinear translations within and between models of representation (see Figure 7).

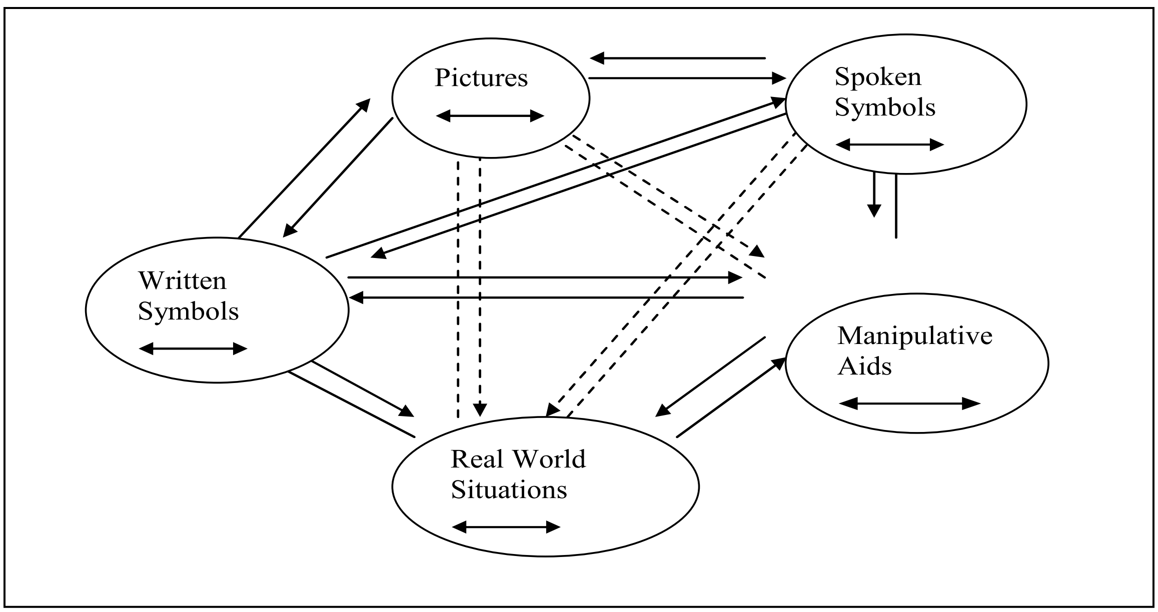


Figure 7. Lesh's (1979) translation model.

According to this model, the RNP teaching experiment made heavy use of manipulative materials such as fraction circles, Cuisenaire rods, chips, paper folding, and number line. "The materials themselves were consistent with a cognitive perspective of the teaching-learning process" (Post et al., 1993, p.347). This teaching experiment also involved verbal, pictorial, symbolic, and real-word modes of representation. Researchers in this project believed that it was the translations within and between the various modes that made ideas meaningful for children. As a result, students acquired series thinking abilities needed by successful learning of equivalent fractions (Post et al., 1985)

Even though Post et al. (1985) highlighted the importance of "progressive independence of thought from embodiments" (p. 21) as the third thinking characteristic, the connection between concrete to abstract was not clear. According to Post et al. (1985), when students see the "equivalence" through the embodiments such as (4)/6 = (2)/3, (above condition a) and then to transform each embodiment to the symbol such as (4)/6 = 4/6 and (2)/3 = 2/3 (above condition b and c), they would be able to draw a correct conclusion 2/3 = 4/6. However, it is doubtful whether students can really see the connection between the embodiment representation (4)/6 = (2)/3 and symbolic representation 4÷2/6÷2 = 2/3. As Leinhart and Smith (1985) pointed out, mapping between the numerical representation and regional representation of fractional equivalence was not at all straightforward for students. From Leinhardt and Smith's (p.1985) point of view, this aforementioned example (see Figure 6) should be analyzed in the following way: the fraction 4/6 is represented by a set of six circles with four of them shaded. In order to show how 4/6 is equivalent to 2/3 with numbers, 4/6 is divided by 2/2. The comparable action in these circles was to repartition these circles to create three groups, two of them were shaded. In order to understand this comparison, one of the intuitive points needs to be considered, that is, division by 2/2 is equivalent to repartitioning. This type of mapping emphasized by Leinhardt and Smith (1985) clearly shows the transition from embodiment representation to symbolic representation concerning equivalent fractions. It, however, was uncovered in the study by Post et al.

(1985). Without this type of mapping, it is hard for students to really develop "progressive independence of thought from embodiments" (Post et al.1985, p. 21).

Regarding learning fractions, other researchers pointed out the importance of connecting students' informal knowledge (Ball, 1993) or the intuitive knowledge (Kieren, 1993) with their formal knowledge. According to Kieren (1993), intuitive knowledge means the confluent use of imagery, thought tools, and informal use of language. Imagery is the awareness of physical, geometric, visual, or numerical patterns. Informal use of language includes the use of informal language and emphasizes oral language. These two types of intuitive knowledge are somewhat related to representational system. In contrast, the "thought tool" is much different. With regard to fractions, the intuitive thinking tool is the act of partitioning. According to Piaget, Inhelder, and Szeminska (1957), partitioning - the division of the unit and its reconstruction from 1/n size parts – was a central feature of fraction knowing. Partitioning contains both multiplicative and additive natures (Kieren, 1993). However, both operations are independent in the rational number field. For example, to find an equivalent fraction for 2/3, we could bisect its embodiment such as an existing figure by partitioning (drawing a single line). This partition equals a multiplicative operation: $2/3 \times 2/2$ or $(2\times2)/(3\times2)$ or an additive operation: $(2 + 2)/(3 + 3)$. However, the multiplication here cannot be seen as repeated addition which could "make bigger". It is different from replicative thinking as with whole numbers (Kieren, 1993). For example, with the numerator 2 and denominator 3 respectively, both operations multiplication and addition make them bigger: $2\times2 = 2+2 = 4$, and $3\times2 =3+3=6$. However, with the whole fraction, "now multiplication does not make bigger. It is almost ironic that this nature of multiplication, related to the action of partitioning, yields such a simple computational procedure" (Kieren, 1993, p.55). This nature of partitioning raises potential confusion in children's conception of fractions. First, drawing a line is actually a division process, and yet it equals a multiplication by 2/2. Mapping these two representations are not at all straightforward. It turns out to be a learning difficulty for students and is often ignored by teachers and even researchers (Leinhardt & Smith, 1985). Second, multiplication by 2/2 is different from multiplication by 2, in which the later one is a replicative thinking that makes bigger and is also very familiar to children. As a result, additional difficulties and errors concerning equivalent fractions arise for students: (1) an equivalent fraction with higher or lower terms is often called as bigger or smaller numbers (Kieren, 1993; Leinhardt & Smith, 1985; O'Connor, 2001); (2) one of the dual roles of "1" – as multiplicative identity element or identity operator (not as a single replicate) (2/2 = 1) is easily overlooked (Kieren, 1993); and (3) multiplying by 2/2 is viewed as the same thing as multiplying by 2 (Ding & Li, 2006). All these partially rooted from students' intuitive thinking tool "partitioning" which is naturally confusing for children.

2.3.4 A semantic net of equivalent fractions – Leinhardt & Smith (1985) study

Leinhardt and Smith's (1985) study examined teachers' subject matter knowledge through the topic of reducing fractions. The semantic net developed by these researchers clearly shows the "core knowledge" (p.256) involved in reducing fractions. Since reducing fractions is one of the ways to find equivalent fractions, the core knowledge is the same as what the teachers in my study needed to teach equivalent fractions. Therefore, this section specifically introduces this semantic net including the strength and weakness it brings to Leinhardt and Smith's study.

The semantic net. According to Leinhardt and Smith (1985), the core knowledge involved in finding equivalent fractions includes two aspects. One is mathematics concepts and relationships involved in reducing fractions and the other is fraction procedures built around these concepts and relationships (see Figure 8).

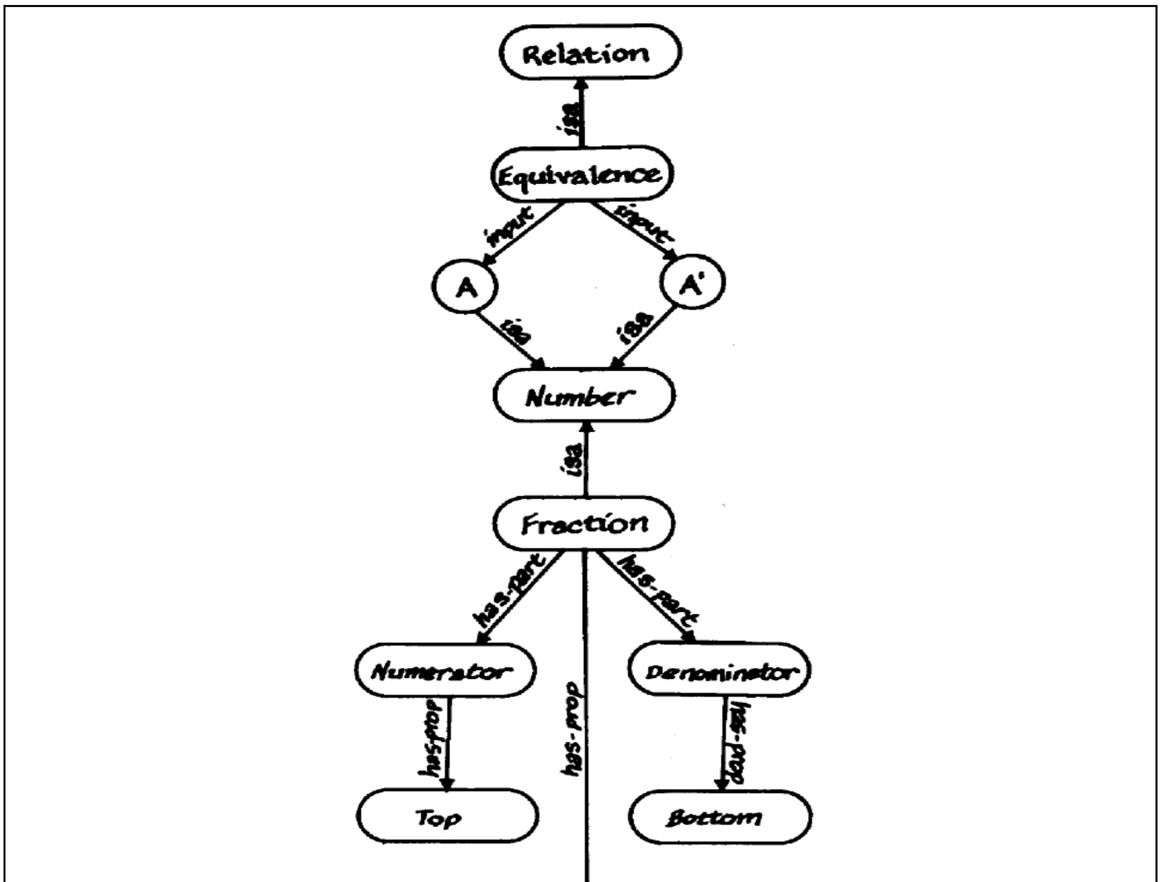


Figure 8. Leinhardt and Smith's (1985) semantic net for core knowledge in reducing fractions.

In general, this core is based on the concept *fraction.* A fraction is a number and has two parts: numerator on the top and denominator at the bottom. A fraction can also be modeled by different nonnumeric representations. Equivalence, as a critical concept in this core, is a relationship between two fractions. When both fractions are input to that relationship, a judgment of equivalence can then be made. As Leinhardt and Smith (1985) further pointed out, concepts in this core can be connected to other concepts which distinguish teachers from each other as well as from the textbook.

When this core of knowledge was connected to the other concepts in finding equivalent fractions, the relationship becomes complex. In the Leinhardt and Smith study, the textbook provided three different types of representations concerning the concept *fraction*: (1) region, (2) number line, and (3) discrete object. However, the concept of equivalent fractions was only built on region and number line presentations but not discrete object (see Figure 9).

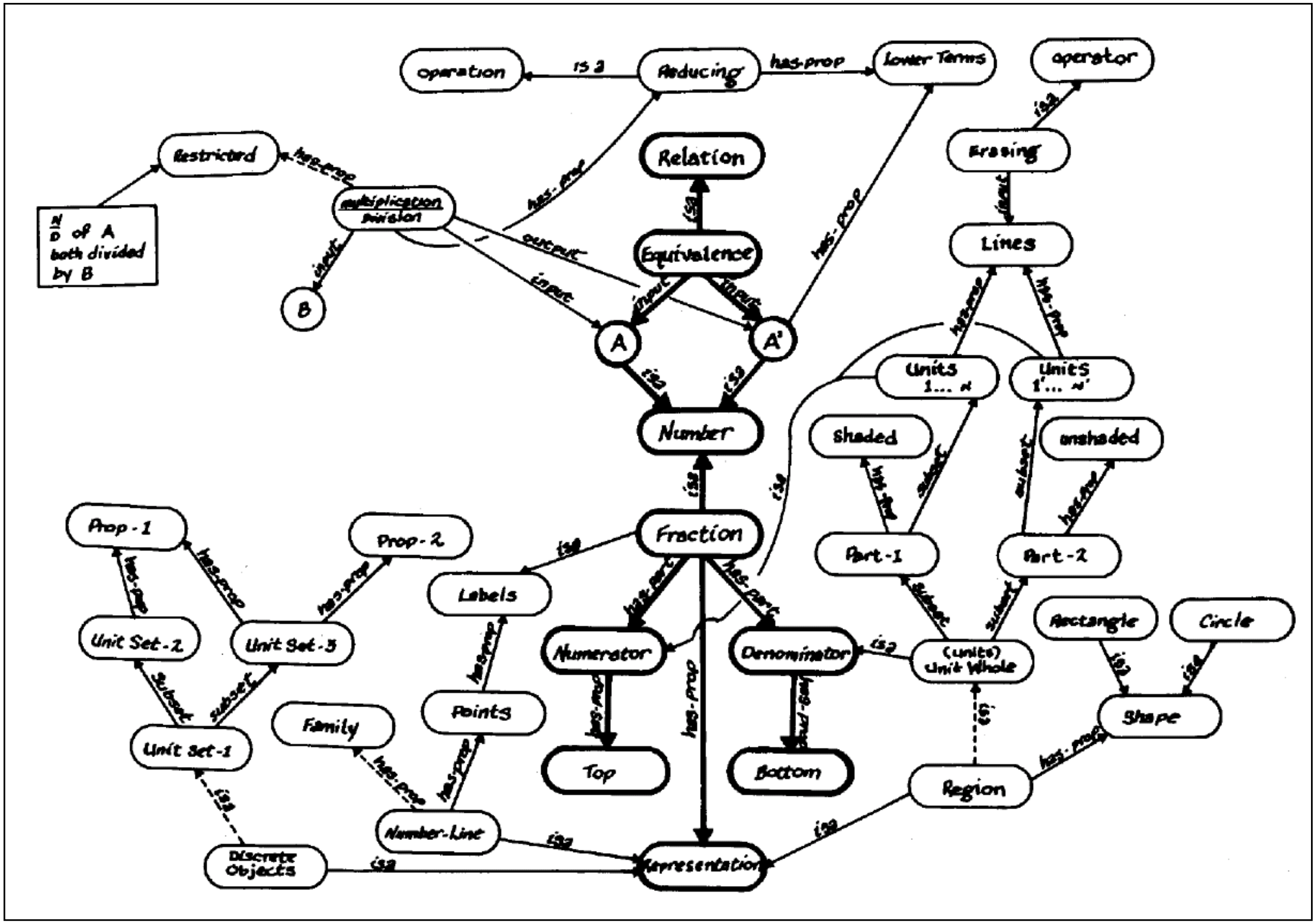


Figure 9. Leinhardt & Smith's (1985) semantic net for reducing equivalent fractions.

As seen from Figure 9, the textbook explicitly stated that a region representation had the property of shape. A rectangle and a circle were the two main shapes. A region was a unit whole (not stated clearly in the textbook) that could be divided into two parts, one shaded and the other unshaded. The shaded part corresponded to the numerator while the unit whole to the denominator. To find equivalent fractions, one could draw lines (add equal parts) or erase lines (remove equal parts) and the fractions represented by the shaded part to the unit whole would remain equivalent. Regarding number line representation, the textbook pointed out that a number line had the property of points that were labeled by fractions. There could be several different number lines, each of which had the property of a family because of the same denominator. The expression of equivalence was accomplished by lining up a group of number lines with equal-sized units in a vertical array. Therefore, it was possible to show how ½ on the twos number line was equivalent to (lines up with) 2/4 on the fours number line. Finally, unlike its discussion of region and number line representations, the textbook did not provide any description of how discrete objects could be used to show equivalent fractions. This partially provided a hint into why students in Hunting's (1984) study, could show their understanding of equivalent fraction with continuous quantity objects rather than discrete elements.

In addition to the above three representations, the textbook also noticed that multiplication and division could be used to generate equivalent fractions through the process of input-output and the relation of equivalence (see Figure 9). That is, multiplying or dividing the numerator and denominator of a fraction A by the same number B, one can get an equivalent fraction A'. However, even though equivalence could be generated by either multiplication or division, the textbook and the teachers mainly focused on multiplication - raising a fraction in higher terms - as a particular way to obtain “equivalence”. In contrast, division was explicitly mentioned to have the property of reducing and, therefore, creating a fraction in lower terms. As a result, these teachers failed to note the symmetry of multiplication and division. For example, one teacher mentioned equivalence and reducing fractions were opposite. As criticized by Leinhardt and Smith (1985), the textbook and teachers provided students incomplete information which may cause students’ incorrect inferences, reinforce their misconceptions, and produce inaccurate problem solutions (Resnick, 1980).

Strength and weakness of the semantic net. Leinhardt and Smith’s (1985) semantic net of reducing fractions provided critical core and connections between the core knowledge and the related concepts. This semantic net was similar to Ma’s (1999) knowledge package but the knowledge package was constructed from teacher interviews whereas the semantic net was developed from a textbook. If the teaching contents are similar, this semantic net could be used as a model or content base to examine teaching practice concerning equivalent fractions.

However, the semantic net in Leinhart and Smith’s (1985) study had at least two limitations. First, as these researchers themselves pointed out, “the two systems of representation are not yet well integrated” (p.270). One of the representations here refers to the region while the other number line. This is because Leinhardt and Smith clearly stated that “representation” in the semantic net meant “nonnumeric system” (p.255), in which “discrete objects” was not used to show the idea of fractional equivalence. Obviously, there are very few connections between region and number line systems in the semantic net (see Figure 9) where, however, student difficulties and potential errors were possibly embedded. Chazan and Ball (1999) pointed out that mapping representations between region and number line was difficult. For example, when working on number lines, some students were confused by the number of lines and pieces. In fact, “the number of pieces was what mattered here making the correspondence between the regional and linear models of fractions” (Chazan & Ball, 1999, P.6). Therefore, when Leinhardt and Smith’s semantic net is used for the research on teaching equivalent fractions, an unclear connections between these two representational systems should be noticed.

The second limitation of the semantic analysis in Leinhardt and Smith’s (1985) study is due to the weak connection to students’ learning. Brown (1993), based on the review of research on teaching rational numbers, pointed out “the extent to which the Leinhardt and Smith (1985) study probed the teachers’ understanding of rational number seemed quite limited” (p.201). One of reasons is that the semantic net was only about teacher’s knowledge rather than that of students. Therefore, this study “does not report the extent to which students participated in the classroom activities or what students actually learned from the lessons taught” (Brown, 1993, p.202). As a result, a measure of the effects of instruction was missing, resulting in the limited use in improving the understanding of effective teaching. Based on this review, Brown suggested relating teachers’ understanding of rational numbers, teachers’ decision making, classroom instruction, and students’ learning in a systematic way. As Carpenter et al. (1993) pointed out, “the semantic analysis of rational number concepts is clearly an important component of research on teaching, learning, curriculum and assessment. But the analysis cannot move too far ahead of the related research programs" (p. 5). In summary, the semantic analysis of effective teaching could not be isolated from student learning.

2.3.5 Summary

Due to the intrinsic features of the concept *fraction*, teachers' weak knowledge of equivalent fractions, students' learning characteristics, and the complex representational systems, equivalent fractions turn out to be very difficult in school mathematics. The possible difficulties and errors concerning equivalent fractions as identified from prior studies include: (1) Difficulty in the transition from region to number line representations (Leinhardt & Smith, 1985) which causes "pieces or lines" confusion (Chazan & Ball, 1999); (2) Difficulty in the transition from concrete (e.g. drawing a line) to symbolic representation (e.g., $\times 2/2$) (Leinhardt & Smith, 1985); (3) Difficulty in the transition between additive and multiplicative thinking in the rational number system (Kieren, 1993); (4) Errors such as $3/4\times2/2$ is the same as $3/4\times2$ (Ding & Li, 2006); (5) Errors such as bigger/smaller equivalent fractions (Leinhardt & Smith, 1985; O'Connor, 2001); (6)Failure to recognize the identity element such as "$a/a =1$" ($a \neq 0$) (Kieren, 1993; Leinhardt & Smith, 1985); and (7) Failure to recognize that "0" cannot be a divisor (Ball, 1990). In addition, some teachers/researchers also have these errors and difficulties or fail to recognize these errors. These findings from the literature provide the basis and lens for further study.

2.4 Summary of Literature Review

Teachers' MKT entails not only knowing mathematics but also using this knowledge for the work of teaching in real contexts. Various cases illustrated what MKT entails and how it can be obtained. Critical knowledge pieces identified from teaching practices strengthened the knowledge base (e.g., LMT project's multiple choices questionnaire) which, however, still lacks connectedness. Therefore, studies of MKT related to particular topics needs in-depth research (Ball & Bass, 2003a) to reach the level of PUFM where teachers' knowledge package demonstrates depth, breadth, and thoroughness (Ma, 1999). Since Ma's knowledge package is interview-based rather than practice-based, it is necessary to absorb both ideas from MKT and PUFM, to develop pedagogically useful MKPTs for particular topics.

Classroom teaching is complex and teaching and learning cannot be separated. To decrease the complexity but keep the connection between teaching and learning, researchers could reduce the dimensions under investigation (Hiebert, 1993; Hiebert & Wearne, 1988). As a result, typical teaching behaviors such as TRED could serve as a window for investigating how teachers' MKT influences their classroom instruction, which in turn affects students' cognitive performance. However, teacher responses to student errors could also be influenced by other factors such as cultural views, the consideration of student motivations, and other local classroom situations. As a result, research on TRED in rich and complete classroom context is important for improving trustworthiness of this type of study.

For research in depth, it is important to narrow the scope by focusing on certain topics. Since equivalent fractions is a significant topic as pointed out by NCTM (2006) *Curriculum Focal Points*, and it is also difficult because of the nature of fractions, student learning characteristics, and teacher knowledge, this topic deserves great research effort. Hiebert and Wearne's method (1988) of selecting equivalent fractions as a teaching content, student errors and difficulties as key cognitive processes, TRED as teachers' instructional focus, and student cognitive changes as measurement for teaching effects will provide tools to identify pedagogically useful information for teaching equivalent fractions. Starting from this topic, researchers could gradually develop various MKPTs other critical concepts and build a sound professional knowledge base.

As Hiebert et al. (2002) pointed out, the gap between research knowledge and teaching practice, resulting in teachers' rare employment of the research archives to help them analyze students' learning such as misconceptions, needs to be bridged. As a result, research in context and in-depth concerning how teachers' MKT contributes to their instructional decisions such as TRED in teaching equivalent fractions is much needed.

3. METHOD

To investigate the research questions, this study employs a qualitative method because this method allows for in-depth inquiry and discovery of new ideas. Through naturalistic inquiry of both classroom video and teacher interview data, I obtained valuable information concerning teachers' MKT. Since classroom teaching could not be separated from student learning, I also connected teachers' instruction with student cognitive gains during the inquiry process.

3.1 Research Paradigm and Its Legitimacy

The research paradigm of this study is qualitative inquiry, specifically, naturalistic inquiry (Lincoln & Guba, 1985). This paradigm matches well with my research purpose - to explore how teachers' MKT influences their classroom practices which in turn affects student learning. Since this exploration aims at developing teachers' professional knowledge – to identify a MKPT for teaching equivalent fractions - based on their practitioner knowledge in real classroom teaching (Hiebert et al., 2002), a natural inquiry process is needed and appropriate.

3.1.1 A naturalistic paradigm in this study

Lincoln and Guba (1985) pointed out 14 characteristics for naturalistic inquiry, which guided my research style during the inquiry process. What follows are the explanations of how my study includes these features, which also provides a general sense of the methods used in this study.

1) Natural setting. In this study, I mainly used classroom video and teacher interview data (elaborated upon later). The videos that I observed were taped in real mathematics classrooms. The video observation took place with the "the entity-in-context for fullest understanding" (Lincoln & Guba, 1985, p.39). Context information such as teacher interventions, student corresponding responses, time allotment, teaching methods (cooperative learning or directive teaching) were recorded in that context information was crucial in deciding the transferability of the findings in this study to other contexts (Licoln & Guba, 1985).

2) Human instrument. In this study, even though I had a simple coding scheme and main interview questions, I mainly depended on myself – a human as an instrument - to collect *teacher responses to students' errors and difficulties* (TRED). This means, during the video observation, the coding scheme was revised according to the ongoing analyses and reflections; during the teacher interview, the main questions were not asked in order. I adapted to teacher responses and asked follow-up questions in terms of teachers' answers.

3) Utilization of tacit knowledge. During the video observation and teacher interview, I paid attention to non-verbal clues or silent sightings because such information can more fairly and accurately mirrors the value patterns that I was looking for (Lincoln & Guba, 1985).

4) Qualitative method. In this study, I mainly used qualitative methods such as video observation and teacher interviews because these methods are sensitive to and adaptable to mutually shaping influences and value patterns (Lincoln & Guba, 1985).

5) Purposive sampling. "Purposive sampling can be pursued in ways that will maximize the investigator's ability to devise ground theory that takes adequate account of local conditions, local mutual shapings, and local values (for possible transferability) (Lincoln & Guba, 1985, p.40)." In this study, I purposely selected those teachers who used the CMP curriculum and agreed to be interviewed (elaborate upon in section 3.4).

6) Inductive data analysis. The data analysis in this study went through the process from practice-based to theory-oriented. I analyzed the classroom videos and teacher interviews. Based on practitioner knowledge, I aimed at developing some professional knowledge. This analysis process is likely to describe fully the classroom settings and to make decisions about transferability to other classroom settings easier.

7) Grounded theory. No priori theory concerning how teachers' MKT influences TRED in teaching equivalent fractions existed. As a result, I expected to find a MKPT concerning equivalent fractions based on my data.

8) Emergent design. The complexity of classroom instruction in addition to several waves of video observations and ongoing reflections brought me increasing information and new themes. As a result, the research questions were even modified during the inquiry process. Meanwhile, the coding scheme was also changed as an emergent design during the observation process. A similar situation happened with the teacher interviews.

9) Negotiated outcomes. After the video observation, I discussed selected teaching sightings with teachers through the interviews. One of the negotiation purposes was to check whether my interpretations based on the video observations were credible "because respondents are in a better position to interpret the complex mutual interaction- shapings-that enter into what is observed; and because respondents can best understand and interpret the influence of local value patterns" (Licoln & Guba, 1985, p.41). In addition, the transcriptions of teacher interviews also went though the process of member-checking for the negotiation of meanings with the participants.

10) Case study reporting mode. In this study, the case was how teachers respond to students' errors and difficulties during the same lessons. The case study mode was more adapted to a description of the multiple context because "it provides the basis for both individual 'naturalistic generalizations' (Stake, 1980) and transferability to other sites (thick description)" (Lincoln & Guba, 1985, p.41-42).

11) Idiographic interpretation. In this study, I interpreted the relationship between teacher knowledge, teacher instruction, and student learning in terms of examining teacher responses to students' errors and difficulties in particular classrooms. The meaningfulness and validity of the interpretations depended heavily on local classroom contexts.

12) Tentative application. My main purpose for this study was to understand teacher instruction in depth rather than to make broad applications of my findings. However, the findings could be transferred to the classroom settings which have similar situations.

13) Focus-determined boundaries. Even though this study employed a naturalistic inquiry with the human instrument and emergent design, I had focus-determined boundaries during both video observation and teacher interview, that is, my research focus. In addition, the coding scheme and main interview questions also provided me the foci during my investigation processes.

14) Special criteria for trustworthiness. The trustworthiness in a qualitative study includes credibility, transferability, dependability, and conformability (Lincoln & Guba, 1985). In this study, I provided a complete classroom context including two continuous lessons for each teacher. I interviewed teachers with the purpose of triangulation and obtaining more valuable information. I also reported my findings in a case study mode where I provided rich and thick descriptions. All these ensured the trustworthiness for this study.

In summary, even though I did not go through the field-based data collection process or interact with teachers over time, my study basically meets these 14 characteristics for naturalistic inquiry as claimed by Guba and Lincoln (1985). As a result, the research paradigm of this study is a naturalistic inquiry, one format of a qualitative study. The detailed inquiry process such as data collection and analysis are explained in later sections.

3.1.2 The legitimacy of this paradigm in this study

In this study, I mainly used qualitative methods – video observation and teacher interview - to collect and analyze teachers' MKT. This method contradicted with that of the study by Hill et al. (2005) where two main instruments were used to collect the data concerning teachers' MKT. One instrument was a highly structured self-report log in which teachers recorded the amount of time they devoted to mathematics instruction, the mathematics content covered on that day, and so on. The other instrument was a questionnaire concerning various items for measuring teachers' content knowledge for teaching mathematics. In other words, in Hill et al.'s study, teachers' MKT was measured by using many quantitative variables. As a result, a question naturally occurs in my study: Can I really get trustworthy information about teachers' MKT by using such kinds of naturalistic investigation approach? Does the paradigm for catching teachers' MKT have legitimacy if I mainly depend on my interpretations for classroom teaching by integrating with teacher interviews?

According to Kanberelis and Dimitriadis's (2006) "*Chronotopes of human science inquiry*", my study can be categorized as "reading and interpretation". My inquiry mode is hermeneutics which refers to the general process of coming to understand or constructing an interpretation of the classroom phenomenon – why teachers respond to students' errors and difficulties in certain ways. As Kanberelis and Dimitriadis (2006) pointed out, qualitative inquiry conducted within the chronotope of reading and interpretation, influenced by hermeneutics, "does not aim to generate foundational knowledge claims. Instead, it aims to refine and deepen our sense of what it means to understand other people and their social practices" (p. 11), which is exactly what I aimed to do in this study. As a result, the qualitative inquiry mode serves my research purpose - to understand classroom teaching in a deeper way - very well.

> Such modes of inquiry draw on the notion of the "hermeneutic circle" as a unique and powerful strategy for understanding "part" (a text, an act, a person) always involves also understanding the whole (the context, the activity setting, the life history) and vice versa. (p.10)

In this study, the rich contextual information, the expanded classroom settings, the detailed observations of teacher instruction, and the qualitative teacher interviews together form a powerful hermeneutic tool to understand how teachers' MKT affects their teaching behaviors. Based on the postpositive perspective, there is no absolute truth for teachers' MKT. The best approach to explore how teacher MKT influences their classroom teaching requires real contexts. As a qualitative researcher, I should "struggle with our daily work, especially respect to locating ourselves strategically within and across chronotpes and creating epistemology-theory-approach-strategy assemblages that are both principled and pragmatic" (Kanberelis & Dimitriadis, 2006, p.27).

3.2 An Approach Bridging Teaching and Learning

In this study, under the naturalistic inquiry paradigm, I combined teaching and learning during the investigation even though the focus of this study was effective teaching. As previously mentioned, classroom teaching is complex. To reduce the complexity, prior researchers either focused on only teaching or learning. As a result, a common shortcoming concerning research approaches on teaching was the lack of connection with learning. To solve this problem – reducing complexity while keeping the connection between teaching and learning - Hiebert and Wearne (1988) provided a method which had four steps. (1) the selection of content, (2) the identification of key cognitive processes, (3) the design of instruction, and (4) the assessment of student changes. This method, as a theoretical model, was used to guide the design of my study – considering both teaching behavior and corresponding students' cognitive gains.

According to Hiebert and Wearne's (1988) method, the teaching content in this study was about equivalent fractions. Teachers taught two continuous lessons in the *Connected Mathematics: Bits and Pieces I sixth-grade text* (simply called CMP curriculum). One lesson (Lesson 2.1) was *Compare Notes* while the other (Lesson 2.2) was *Finding Equivalent Fractions.* The centralized content was equivalent fractions in these two lessons.

Regarding key cognitive processes, I identified them as students' learning difficulties and errors through the sources of Leinhardt and Smith's (1985) semantic net, CMP textbook and teacher guide book, and prior researches related to this topic. A task analysis covering the above steps (1) and (2) is elaborated in section 3.3.

Concerning teacher instruction, I focused on TRED. Based on the errors and difficulties as identified from task analysis, I stated a naturalistic inquiry process including data collection and data analysis on video tapes triangulated with teacher interviews. The detailed procedure is explained in section 3.5.

Finally, I employed a global assessment of students' cognitive gains as a measure of teaching effects. For example, students' reported strategies in the summary part provided a general sense of their learning. I compared students' performances across classes under certain topics and I also followed individual students' cognitive changes whenever it was possible. The detailed procedure can also be found in Section 3.5.

3.3 Task Analysis

3.3.1 Task selection

Why CMP curriculum? The MSMP project employed four sets of middle school mathematics textbooks including CMP. In this study, the fraction lessons were selected from the CMP textbook. There are two reasons for only selecting CMP. First, CMP is a highly-ranked textbook (American Association for the Advancement of Science [AAAS], 1999). This textbook is inquiry-based and generally characterized as a student-centered learning curriculum (Rivette, Grant, Ludema, & Rickard, 2003). The textbook suggests that teachers employ the classroom instructional sequence - launch, explore, and summary – for each lesson. This teaching sequence enables teachers to create a supportive environment for students to grapple, analyze and solve interesting problems through a real-world context (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998). Since students' problem solving is a process of encountering difficulties, making mistakes, and struggling for reaching conceptual understanding, the classrooms using CMP will, therefore, provide me more opportunities to observe TRED. The second reason for selecting the same textbook is to reduce the curriculum factor which might affect teachers' instructional quality.

Why fraction lessons? This study used two lessons both concerning fractions, specially equivalent fractions. Since students have difficulties in flexible use and understanding written notation with fractions (Hiebert, 1988) and fractions are "exceedingly difficult for children to master" (NAEP, 2001), these two fraction lessons are applicable for my study purpose of examining TRED and the underlying reasons for teachers' TRED.

Why two continuous lessons? The two fraction lessons are both from Investigation 2: *Comparing Fractions*. These two lessons (2.1 and 2.2) are also continuous. Lesson 2.1: *Comparing Notes* asks students to compare three fractions 2/3, ¾, and 6/8 and identify the correct ones that represent $270 out of $360. Since both ¾ and 6/8 are correct answers, students may come up with the idea of equivalent fractions. Lesson 2.2: *Finding Equivalent Fractions* asks students to find equivalent fractions for 2/3 and ¾ (Detailed content is provided in Section 3.3.2).

The purpose of selecting two continuous lessons is to investigate teachers' actions in a real, rich, and completed context, and to reduce possible biases in the interpretation of TRED. For example, if a teacher in the first lesson did not address the error $3/4 \times 2 = 6/8$ at the end of this class, I should not simply conclude that this teacher was lacking MKT for teaching equivalent fractions. There could be multiple reasons: (a) the teaching time was running out (b) the teacher might have thought this error should be addressed in the follow-up lesson because the focus of that lesson was "finding equivalent fractions", or (c) this teacher might really lack MKT, resulting in his/her inability to address this type of error. As a result, it is necessary to observe this teacher's follow-up actions in the next lesson to capture additional information such as whether the same error occurred in the second lesson and whether the teacher addressed it this time. Therefore, combining information from two lessons to interpret teachers' actions should help to improve the trustworthiness of this study (Lincoln & Guba, 1985).

3.3.2 Teaching content

The CMP context for Lessons 2.1 and 2.2. As previously mentioned, Lesson 2.1 and Lesson 2.2 are from Investigation 2 in the sixth grade CMP textbook. Before this investigation, students were supposed to have developed the part-whole interpretation of fractions. According to the CMP teacher guide book (Lappan et al., 1998), students in Investigation 1: *Fund-raising Fractions* made fraction strips to explore the fund-raising process. Phrases such as "two thirds of the goal has been reached" were particular focused. The three components of fraction representations - the visual model (fraction strips), word names, and symbols– were explored.

Investigation 2: *Comparing Fractions* provides students a context to investigate equivalence through comparing fraction strips. The Lessons 2.1 and 2.2 reflect this idea. Another aim of this investigation is the creation of a number line and the use of benchmarks to compare fractions. As a result, from Lesson 2.3: *Making a Number Line*, CMP changes the representations from fraction strip to number line.

Lessons 2.1 and 2.1 both contain equivalent fractions. As the CMP teacher guide book points out, "The most important concept in understanding and using rational numbers is equivalence of fractions. This concept underlies operations with fractions, changing representations of fractions, and reasoning proportionally" (Lappan et al., 1998, p. 1f). These two lessons not only construct students' knowledge of fraction equivalence but also develop their prior knowledge in Investigation 1 as well as lay foundation for later learning of number-line representations.

Lesson 2.1 in the CMP textbook. In Lesson 2.1, students are asked to compare fractions: 2/3, 3/4, and 6/8, and to decide which fractions represent the meaning of the phrase "had raised $270 of the $360 they needed to reach their goal" through measuring a thermometer provided by the textbook. Since ¾ and 6/8 are both correct answers, the first lesson naturally raises the concept of "equivalent fractions". What follows are the fund-raising story and the two questions as provided by the textbook in problem 2.1 (Lappan et al., 1998, p.19):

> At the end of the fourth day of their fund-raising campaign, the teachers at Thurgood Marshall School had raised $270 of the $360 they needed to reach their goal. Three of the teachers got into a debate about how they would report their progress.

- Ms. Mendoza wanted to announce that the teachers had reached three fourths of the way to their goal.
- Mr. Park said that six eighths was a better description.
- Ms. Christos suggested that two thirds was really the simplest way to describe the teachers' progress.

Problem 2.1:

A. Which of the three teachers do you agree with? Why?

B. How could the teacher you agreed with in part A prove his or her case?

In addition to the written information, the textbook provides a thermometer that shows the fund-raising process (see Figure 10)

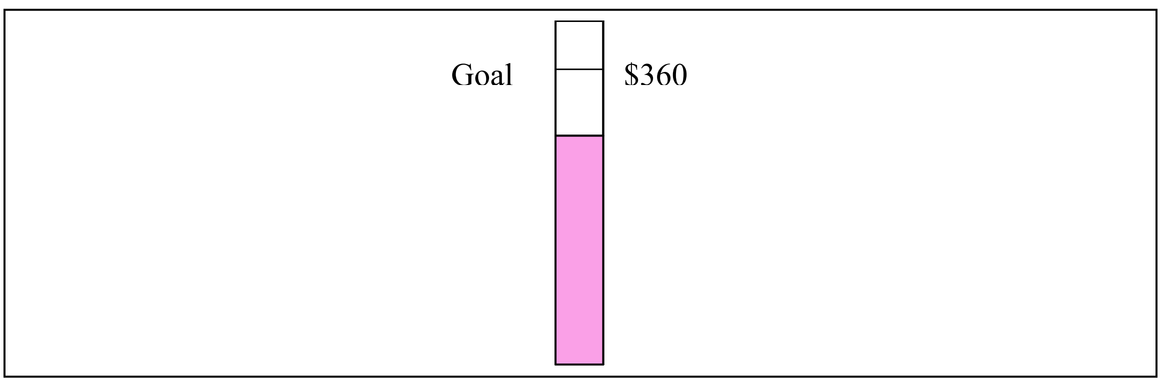
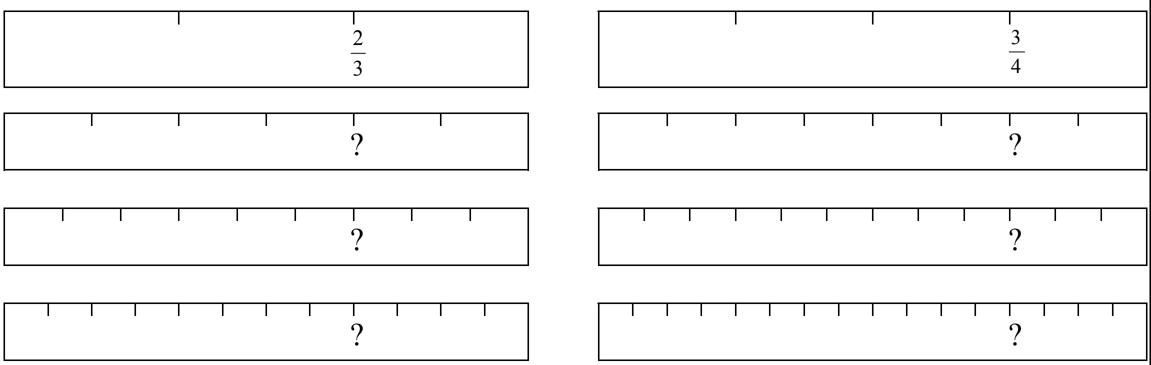


Figure 10. The thermometer showing fund-raising process in problem 2.1.

Lesson 2.2 in the CMP textbook. Lesson 2.2 further explores equivalent fractions. It first introduces the idea that "any quantity can be described by an infinite number of different fractions" (Lappan et al., 1998, p. 20), which leads to the definition of equivalent fractions. This definition is illustrated by three lined up fraction strips that represent the three equivalent factions ½, 2/4, and 3/6. Based on the above information, the textbook provides Problem 2.2 with two sets of ruler-shape fraction strips. These ruler-like fraction strips are not pure regional models like the thermometer in Lesson 2.1 but actually number line–oriented (Leinhart & Smith, 1985). Therefore, Lesson 2.2 is a critical transition from regional to number line representations. Based on these two sets of fraction strips, students are asked to label equivalent fractions (represented by "?") and find patterns that help them obtain equivalent fractions. Figure 11 shows Problem 2.2 (Lappan et al., 1998, p.21).

Problem 2.2

The fraction strips on the left below show $\frac{2}{3}$ and three fractions equivalent to $\frac{2}{3}$. The strips on the right show $\frac{3}{4}$ and three fractions equivalent to $\frac{3}{4}$. Study the two sets of strips. Look for patterns that will help you find other equivalent fractions



A. What are the three fractions shown that are equivalent to $\frac{2}{3}$? Name three more fractions that are equivalent to $\frac{2}{3}$.

B. What are the three fractions shown that are equivalent to $\frac{3}{4}$? Name three more fractions that are equivalent to $\frac{3}{4}$.

C. What pattern do you see that can help you find equivalent fractions?

Figure 11. Problem 2.2 in the CMP textbook.

3.3.3 Core cognitive processes – learning difficulties and common errors

According to Hiebert and Wearne's (1988) methods, one of important steps to research on classroom teaching is the identification of key cognitive processes. These key cognitive processes determine students' cognitive gains. As a result, they should be the teachers' instructional focus and also researchers' spotlights. In this study, I located students' core cognitive processes through the process of identifying their learning difficulties and common errors. Through the examination of the CMP textbook and teacher guide book, the literature review, and my personal teaching and research experiences, I identified the following critical points in learning equivalent fractions as reflected in Lessons 2.1 and 2.2.

Learning difficulty 1: Identify the "goal"/whole. In Lesson 2.1: Comparing notes, students are expected to use fraction strips to measure the fund-raising process as shown by the thermometer. Since the length of the thermometer is greater than the fund-raising goal $360, it causes an obstacle for students' investigation when they are suggested to use fraction strips. Students need to be aware that (1) their fraction strip should be lined up with the goal mark rather than the top of the thermometer, and (2) if their fraction strips are longer than the goal mark, they should figure out a way to use it appropriately. Even though these knowledge pieces are not the target of this lesson, they are related to students' understanding the concept *fraction*– what is the whole and what is the part - which forms the core knowledge (Leinhardt & Smith, 1985) of students' learning of equivalent fractions. If teachers do not realize this critical knowledge piece and do not make sure students understand this point, students tend to be stuck at the stage, resulting in lack of time to compare notes or identify equivalent fractions. In fact, if teachers allow students to become aware of this learning difficulty, it could be easily solved and students will therefore be able to move onto the learning goal in this lesson.

Learning difficulty 2: Counting lines or pieces. In Lesson 2.2, students are expected to find equivalent fractions through a set of fraction strips. These fraction strips look like a series of rulers with the same size but different marks. Compared with the thermometer in Lesson 2.1 and the number lines in Lesson 2.3 in the CMP textbook, the set of fraction strips in Lesson 2.2 is the transition between alternative representations. According to Leinhardt and Smith's (1985) semantic net, equivalent fractions could be represented through three representational systems: region, number line, and discrete objects. The regional representation is about a unit whole divided into shaded and unshaded parts. For example, the thermometer (a fraction bar) in Lesson 2.1 is a regional model. The number line representation of equivalent fractions is accomplished through lining up a group of number lines with equal-sized unit in a vertical array. As a result, the set of ruler-like fraction strips in Lesson 2.2, has both properties of regional and number line presentations. It is the transition from regional to number line representations. In fact, in the CMP textbook problem 2.3, the number lines are exactly made from fraction strips.

Mapping representations between region and number line is very difficult for students. One example of evidence is that, when working on number lines, many students are unclear about the differences between number of lines and pieces (Chazan &Ball, 1999). According to my personal working experiences in the MSMP project, I did observe students in some classes encountered this type of difficulty. The two sets of fraction strips in Lesson 2.2 require students transferring their regional model thinking

(Lesson 2.1) to number line model thinking. As a result, students may have the learning difficulties related to counting lines or pieces when they label the fraction strips.

The difficulty concerning "lines or pieces" is also not directly related to the learning goal - finding equivalent fractions. However, students need to name the fraction strips first. Only when they finish these labelings, can they move ahead to find patterns and to name more equivalent fractions. Therefore, how teachers are aware of and address this difficulty – lines or pieces – when students name the question marks on these fraction strips affects how far these classes reach.

Common Error 1: "Doubling Error" ($\frac{3}{4} \times 2 = \frac{6}{8}$). In a prior study concerning equivalent fractions, I identified a student mistake $\frac{3}{4} \times 2 = \frac{6}{8}$ that commonly existed in MSMP video tapes of Lesson 2.1: *Compare Notes* (Ding & Li, 2006). In Lesson 2.1, when students correctly solved the problem 2.1, they found ¾ and 6/8 were both correct answers for "$270 out of $360". Therefore, they said that ¾ and 6/8 were equivalent fractions: ¾ = 6/8. Some teachers asked students to explain why ¾ = 6/8. As a result, some students could verbally represent their understanding - multiplying the numerator and denominator both by 2 – they, however, made mistakes in the written format $\frac{3}{4} \times 2 = \frac{6}{8}$. In addition, there were also some students making verbal mistakes such as "multiplying by 2" or "doubling", which was actually the same mistake as "× 2" (simply called all these as "Doubling Error"). Since Lesson 2.2 is also about equivalent fractions, similar errors may occur again during students' explanations.

Clearly, this type of error is not a slip of the tongue or pen. It reflects students' learning difficulties and understanding of this topic. Since the learning goal in this study is to find patterns for obtaining equivalent fractions and the whys behind those patterns, this type of mistake is critical because it inhibits students' abilities to achieve the learning goal. When students wrote $\frac{3}{4}\times 2=\frac{6}{8}$ instead of $\frac{3}{4}\times\frac{2}{2}=\frac{6}{8}$, they confused "×2" and "$\frac{\times 2}{\times 2}$". One of the possible reasons is due to student difficulty in the transition from verbal to symbolic representations. When they said "multiply the numerator and denominator both by 2", they wrote down "×2" as a correspondence to "multiply by 2". Another possible reason is due to their familiarity with whole number operations ("× 2" rather than "× 2/2") and the negative influence of whole number thinking. Whatever the reason, this type of mistake indicates that students do not really understand the underlying principles of finding equivalent fractions. If students can understand "2/2=1 and "every number times 1 will not change the value", or if students can really visualize "$\frac{\times 2}{\times 2}$" and understand that process, they may overcome the errors and obtain conceptual understanding of equivalent fractions. As a result, how teachers view and respond to this type of error and difficulty not only relates to teachers' MKT but also affects students' cognitive gains. Even though the "Doubling Error" might not occur in some classes in this study, it is necessary to examine why this type of error did not occur because "no error" does not mean "no difficulty".

Common Error 2: Adding a fraction ($\frac{3}{4}+\frac{3}{4}=\frac{6}{8}$). Another error that occurred in my prior study of Lesson 2.1 (Ding & Li, 2006) was $\frac{3}{4}+\frac{3}{4}=\frac{6}{8}$. To explain why 3/4 and 6/8 were equivalent, some students explained their thinking in the above erroneous way.

This error can possibly occur in Lesson 2.2 where two sets of fraction strips are provided for students to obtain equivalent fractions for 2/3 and ¾ respectively. For example, as to 3/4, they may find 6/8, 9/12, and 12/16 by naming these strips. Based on these findings, students need to find patterns. If students correctly compare the change of numerators and denominators, they will find they actually keep adding 3 to the numerators while they keep adding 4 to the denominators. Symbolically, students are doing: $\frac{3+3}{4+4}=\frac{6}{8}$, $\frac{6+3}{8+4}=\frac{9}{12}$, and $\frac{9+3}{12+4}=\frac{12}{16}$. However, it is not easy for them to really understand this procedure. As a result, when students add 3 to the numerator and 4 to the denominator, they may make errors by saying "I keep adding 3/4" or by writing down incorrect formats like $\frac{3}{4}+\frac{3}{4}=\frac{6}{8}$.

Since the addition pattern is associated with the multiplication one - $\frac{3+3}{4+4}=\frac{6}{8}$ is the same thing as $\frac{3\times 2}{4\times 2}=\frac{6}{8}$ - teachers, therefore, should ensure students a correct understanding of this type of pattern. In addition, the erroneous format $\frac{3}{4}+\frac{3}{4}=\frac{6}{8}$ also reflects students' weak understanding of the concept fraction (the core knowledge for equivalent fractions) and will absolutely affect their later learning of fraction addition As a result, it is necessary for teachers to address this type of error and difficulty.

Why these difficulties and errors are critical cognitive processes? The above four difficulties or errors relate to either the necessary prior knowledge for learning equivalent fractions or parts of the learning goal. If these types of errors and difficulties are not addressed appropriately, they will turn out to be obstacles for students' exploration or misconceptions inhibiting students' further learning. However, if teachers do address these types of errors and difficulties, they will become learning opportunities, resulting in students positive cognitive gains. As a result, these four identified learning difficulties or errors serve as critical focal points for my further examination of teachers' classroom instruction and the relationship between teachers' instruction and MKT. Meanwhile, since my study is a naturalistic inquiry, the emergent theme might occur during my exploration process.

3.4 Participants and Data Sources

3.4.1 Participants

Six teachers from the Middle School Mathematics Project (MSMP) participated in this study. The MSMP project is a 5-year longitudinal study examining how the use of specific research-based instructional strategies in classrooms relates to lasting improvements in student learning.

To select the participants, I complied with the reliable and convenient principle (Rubin & Rubin, 2005) and gradually narrowed my focus. At first, all sixth-grade teachers ($N = 14$) who used CMP textbook during the school year 2002-03 were selected. Concerning fraction lessons, each teacher taught three lessons: Lessons 1.5, 2.1, and 2.2. The Lesson 1.5 was about folding fraction strips while Lessons 2.1 and 2.2 related to equivalent fractions. Even though Lesson 1.5 was not used in this study, I also observed this lesson for obtaining more information and to decrease bias for each teacher. Therefore, all these video tapes ($N = 14 \times 3$) were formally observed. During this process, I transcribed the critical sightings and wrote down my thoughts. After the first formal observation, I got a general impression about these teachers' classroom instruction. As a result, I narrowed my focus by selecting 8 teachers who demonstrated instructional differences. Among these 8 teachers, 4 were from Texas while the other 4 from Delaware. At this time, I started the second observation and also started contacting teachers for interviews. At last, four Texas teachers and two Delaware teachers agreed to be interviewed. Therefore, I eventually narrowed my focus to these six teachers and all of whom were female. One teacher left teaching for family reasons; one teacher left teaching for school administration; the other four teachers are still mathematics teachers. In this study, these teachers' pseudonyms are Kathleen, Jennifer, Lisa, Mary, Barbara, and Rose.

3.4.2 Data sources

Video data. My primary data source in this study is video data. Jacobs et al. (1999) stated three advantages of this type of data: (a) "relatively unfiltered through the eyes of researchers" and "arguably more 'raw' than other forms of data" (p. 720); (b) "more versatile than other forms of data" (p. 720) and can be viewed by researchers from diverse cultural and linguistic backgrounds who might bring fresh perspectives to video analysis and examine many facets of the data; and (c) permanent data source that can be watched, coded, and analyzed repeatedly from different dimensions.

In this study, even though I first observed all video tapes about fraction lessons (Lessons 1.2, 2.1, and 2.2) of all teachers who used the CMP textbook, I only reported teachers' instruction concerning Lessons 2.1 and 2.2 of those six teachers who were interviewed. Four of these teachers employed cooperative learning while two of them used directive teaching. The classes typically had 18-30 students and each lesson was approximately 24-58 minutes long. Table 1 shows the detailed information:

Table 1

Teacher(pseudonyms) and class information

Name	State	Teaching Method	Class Time (minutes)	
			Lesson 1	Lesson 2
Kathleen	TX	Directive teaching	30’38	30’27’’
Jennifer	TX	Cooperative learning	33’46’’	36’17’’
Lisa	TX	Cooperative learning	31’12’’	41’43’’
Mary	TX	Directive teaching	52’04’’	24’12’’
Barbara	DE	Cooperative learning	35’11’’	57’52’’
Rose	DE	Cooperative learning	41’15’’	46’56’’

Interview data. To better understand these teachers’ instructional decisions, I employed qualitative interviewing with these teachers (Rubin & Rubin, 2005). During the interview, I discussed three parts with teachers: (a) video clips of their own teaching, (b) a designed case concerning student errors in equivalent fractions, and (c) eight “True or False” questions (simply called T or F questions) (see Appendix 1 for interview material).

The purpose of sending teachers video clips was to help them recall those students and to provide necessary context information. Since these videos were 4 years old, teachers may have forgotten what they did and why they did it at that time. However, this limitation does not threaten the trustworthiness (Lincoln & Guba, 1985) of this study for the following reasons: (a) teachers could say something different even in an interview conducted right after a class, (b) my interview includes three parts rather than just video tapes, and (c) even if a teacher later was involved in professional development, my interview could at least reflect the higher level of her knowledge, which also provides valuable information for this study.

The designed case was to ask teachers' responses to students' invented strategies $3/4 \times 2 = 6/8$ and $3/4 + 3/4 = 6/8$. The purpose of the designed case was to triangulate what teachers did in classrooms and also collect more information about teacher knowledge. For those teachers who really addressed students' errors, I wanted to know why they responded in particular ways. For those teachers who did not encounter these types of errors, I wanted to know how they would address them if they attempted to do.

Concerning the eight T or F questions, I designed and revised them to develop increased levels of information. These questions were used for the examination of four themes: two common errors as previously mentioned and two critical issues concerning basic mathematics ideas emerged in the process of data collection: 0/0 and equivalence. These questions were not arranged in order (see Appendix 1). Question (2) and (6) were designed for error $3/4 \times 2 = 6/8$; Question (4) and (7) were used for $3/4 + 3/4 = 6/8$; Question (1), (5), and (8) for "0/0", and Question (3) and (4) for equivalence. In fact, question (7) was also related to teachers' sensitivity to mathematics notation. The detailed analyses will be provided when these questions are used in the results section.

In general, the interview data provided valuable information concerning teachers' MKT and can be used to triangulate with video sources, improving the credibility of the data interpretation. The procedure of the teacher interviews is elaborated in next section.

3.5 Data Collection and Data Analysis

Qualitative inquiry requires the simultaneous procedures of data collection and data analysis (Denzin & Lincoln, 2000; Lincoln & Guba, 1985). In this study, the data collection was also a process with ongoing data analysis. Based on the prior data analysis, I modified the direction of data collection.

3.5.1 Job Analysis on video data

A method of job analysis. The main method of my dissertation is actually "job analysis" (Ball & Bass, 2000, p. 89) which is based on observation of classroom videos. According to Lincoln and Guba (1985), a major advantage of direct observation is that it provides here-and-now experience in depth:

> Observation … maximizes the inquirer's ability to grasp motives, beliefs, concerns, interests, unconscious behaviors, customs, and the like; observation… allows the inquirer to see the world as his subjects see it, to live in their time frames, to capture the phenomenon in and on its own terms, and to grasp the culture in its own natural, ongoing environment; observation… provides the inquirer with access to the emotional reactions of the groups introspectively – that is, in a real sense it permits the observer to use *himself* as a data source; and observation… allows the observer to build on tacit knowledge, both his own and that of members of the group. (p.273)

Briefly, the observation allows researchers to capture information of the real context from his or her subject's view. Since my research purpose is to investigate what kinds of MKT are really useful for teaching equivalent fractions, I explored this topic through a detailed observation of those classroom data. Rather than identifying teachers' MKT only from curriculum or teacher interviews, I started with examining the teaching practice – the real classroom teaching tapes. According to Ball and Bass (2000),

> Examining the curriculum, although useful, is incomplete for it fails to anticipate the mathematical demands of its enactment in classrooms. Interviewing teachers, though also valuable, is incomplete because it infers teaching's mathematical demands from teachers' accounts of what they think or would do. Without knowing whether the teachers interviewed are actually able to help all students learn mathematics well, what they report remains in some significant ways unwarranted. In any case, neither of these approaches bridges the gap between knowledge and practice, except indirectly through inference or report. (p.89)

In contrast, the job analysis of classroom teaching focused on the actual work that teachers do and "discover what teachers need to know and what they need to be sensitive to regarding content to teach well" (Ball, 2000, p. 244). At the same time, this kind of analysis may bring some surprises or unanticipated insights (Ball, 2000).

Four waves of video observations. According to Lincoln and Guba (1985), my observation of the teaching tapes was a process of natural inquiry. As these researchers pointed out, "Early on, the observation may be very unstructured, a stage of defocusing or immersion (Douglas, 1976) … Later, the observations may become more focused as insights and information grow" (Lincoln & Guba, 1985, p.275). In general, this study involves 4 waves of video observation.

The first wave of observation was basically a random one. All three fraction lessons – Lesson1.5, 2.1, and 2.2 – of all teachers who used CMP curriculum ($N = 14$) were observed. Since I formerly transcribed some teachers' videos-tapes of Lesson 2.1 for my prior studies, during this process, I mainly observed the other teachers' classroom videos that I was not familiar with and all video-tapes about Lesson 1.5 and 2.2. As previously mentioned, even though I will not use Lesson 1.5 for my final report, I still observed it for more information and less bias. The purpose of this process was to obtain a general impression about teachers' instruction of fractions.

The second wave of observation was relatively formal. I still observed all 14 teacher's videos but only for Lessons 2.1 and 2.2. During this process, I transcribed all the interesting sightings and all the ones where students' errors or difficulties occurred. I took field-based notes and I wrote down my thoughts immediately after observing certain sightings. These thoughts were highlighted. I processed my data when I finished the observation for each teacher. I also went back and compared them with what I had done for other teachers. This ongoing analysis was more likely to uncover some hidden patterns and themes among these teaching tapes. I also tried to connect these patterns, themes or events to those of other relevant studies. During the process of data collection and analysis, I relied on conceptual memos as an analytic technique. I recorded and discussed teachers' response differences, emergent themes, and everything that I thought critical. As Miles and Huberman (1984) suggested, "Memos are always conceptual in intent. They do not just report data, but they tie different pieces of data together in a cluster, or they show that a particular piece of data is an instance of a general concept" (p.69). This ongoing data analysis helped me modify the new focus and hypothesis in the follow-up observation. After the second wave of observation had been done, I had a larger sense of teachers' instructional differences and clear ideas of the themes. As a result, I narrowed my participants to eight teachers whose teaching had obvious patterns and differences. Four of these teachers were from Texas and four from Delaware. Meanwhile, I started contacting these teachers to interview them with the MSMP project leader's help, which resulted in six teachers who agreed to be interviewed. As a result, the participants were finally narrowed to six teachers.

During the third wave of observation, with the focus on the identified difficulties and errors and the critical emergent themes, I investigated these six teachers' videos with a process of natural inquiry. I used a simple coding scheme to code those sightings related to these foci for each teacher (see Appendix 2). The contextual information such as start and end time, teaching methods, teacher and students' main responses were recorded. After the location, classification, and cross-class and cross-teacher comparisons, I obtained vivid teacher portraits under different topics. For example, concerning the written representation "½ ≠ 2/4" that drew my attention from one teachers' board, I collected the relevant teacher responses throughout these video-tapes. As a result, both insights and issues concerning this theme were identified for teaching equivalent fractions. During this wave of observation, I also purposely selected one or two teaching sightings where students' learning difficulties or errors occurred for the upcoming teacher interviews. In addition, during the observation of teacher responses, I also had an eye on student learning which was mainly addressed in the fourth wave of observation.

The fourth wave of observation was to identify students' responses corresponding to teachers' instruction. Since the MSMP project video camera generally followed teachers rather than certain students during the video-taping process, it caused the difficulties with my justification of individuals' cognitive gains. During the observation, I captured individual students' cognitive change as complete as possible. Meanwhile, I employed a global assessment on the effect of teacher instruction through overtime comparison and cross-class comparison. In other words, I compared students' responses during launch, exploration, and summary periods within a class; I also compared students' reported strategies during the summary part across different classes. These global assessments along with individual student cognitive gains were direct evidences of teacher classroom instruction effects.

3.5.2 Qualitative interviewing

A qualitative interview method was used in this study. According to Rubin and Rubin (2005), qualitative interviewing allows researchers to explore more complex questions "in rich and realistic detail" (p.2) and to explore new areas and unravel intriguing puzzles. In-depth qualitative interviewing has three shared characteristics: (a) it is built on a naturalistic, interpretive philosophy; (b) it is an extension of ordinary conversation; and (c) there is a partner relationship between interviewer and interviewees (Rubin & Rubin, 2005).

Preparation for the interview. After teachers agreed to be interviewed, I emailed them and scheduled the interview day. I then started the preparation for the interview. First, I went through teachers' video transcriptions and data analysis several times. I organized these in an interview packet which included (a) TRED, (b) teachers' particular teaching strategies, (c) problematic verbal or written representations, and (d) unclear context information that I needed to know from teachers. In addition, I created the identified video clips by using AVS Video Converter 5.5 and AVS Video Remaker 2.2 software and I sent them to these teachers' email. The designed case and T or F questions were also sent to them before interviews. The careful and detailed preparation was for the purpose of achieving a successful interview. As Rubin and Rubin (2005) pointed out, successful interviews "are rooted in the interviewees' first-hand experiences and form the material that researchers gather and synthesize" (p. 13). With careful preparation, all six teachers were successfully interviewed - three Texas teachers were interviewed face to face while the other there teachers (one Texas and two Delaware) who were living at a distance were interviewed by phone. These interviews lasted about 40-90 minutes. The interviews in this study had three main characteristics: (a) teachers as conversional partners, (b) a thick description, and (c) responsive interviewing (Rubin & Rubin, 2005).

Teachers as conversational partners. During my interview, I tried to build a conversational relationship between my participants and me. In other words, the teachers in my study were "conversational partners" (Rubin & Rubin, 2005, p.14) rather than objects to be tested. I showed my respect and appreciation to my participants before, during, and after my interviews. I created a comfortable and natural atmosphere during our conversations by phone or face-to-face. The trustworthy relationships were also built through some seemingly tiny things. For example, teachers were very busy and it was hard to schedule an interview especially during high-stakes testing period. On the scheduled day that I visited teacher Kathleen, she unfortunately fell down and hurt her back 10 minutes before our interview. She suggested doing the interview before going to see the doctor because she would have no time right after this day – the state test was coming. I insisted on canceling this interview even though I really needed the interview results. These types of small details built good relationships between these teachers and me. As a result, teachers in this study were actually my *conversational partners.* According to Rubin and Rubin (2005), this type of relationship has three advantages: (a) it emphasizes the active role of the interviewee in shaping the discussion and in guiding the research path. During the interview, with the main questions in my mind, I adapted myself to teachers' conversation flow and allowed them to sufficiently express themselves; (b) it suggests a pleasant and cooperative experience, as both interviewer and interviewee work together to achieve a shared understanding. During the interview, teachers were patient with my rephrasing of their words to ensure my understanding and our interview was also full of laughter; and (c) it highlights the uniqueness of interviewees including his or her distinct knowledge and different responsive ways. During the interview, teachers used particular ways such as drawing pictures, or telling stories to show their understanding of my questions.

A thick description. The purpose of my interview was also for obtaining teachers' "thick descriptions" (Rubin & Rubin, 2005, p.13) concerning teachers' perspectives of teaching equivalent fractions. According to Rubin and Rubin, "thick description" meant depth, detail, and richness. During the interviews, even teachers were conversational partners with active roles, our conversation had obvious foci. As previously mentioned, the interview in this study included three main parts: (a) video clip, (b) a designed case, and (c) T or F questions. Therefore, the main procedures of our conversation followed the three steps.

First, we discussed teachers' video clips and then extended the discussion to the whole lessons. The main questions were from the interview packet that I prepared ahead of time for each teacher. For example, we talked about teacher perspectives about the learning difficulties, the common errors, teachers' interesting and even problematic representations. For example, I discussed with teachers who verbally repeated "doubling" and who wrote "$1/2 \neq 2/4$" on the board concerning their specific concerns of acting in those particular ways. Sometimes, I even brought certain events to other teachers and asked their comments. Based on teachers' responses, I asked teachers probing questions and follow-up questions. However, these questions were for the purpose of a deep understanding rather than a challenge or an argument with teachers. I tried to avoid making judgments but ensured correct information from teachers. As a result, I either rephrased teachers' words to check my understanding or asked questions such as "Do you mean …? " or "Could you explain a little bit more?" These questions functioned as member checking and improved the credibility of the interview results.

Concerning the designed case, I provided different detailed designs for each teacher. For those teachers whose classes included such events, I could have already discussed this issue with them in the first part. As a result, we might skip this part. With regard to those teachers who wanted to explain more, I also carefully listened to their words or observed their drawings. As to those teachers whose students did not make such errors, I told teachers these were students' invented strategies without saying they were correct or wrong. I then asked teachers' responses to this case. When teachers could not recognize the mistakes and agreed with these strategies, I brought out alternative (correct) representations and asked teachers' further opinions. In contrast, when teachers did recognize students' mistakes, I asked them probing and follow-up questions such as the underlying reasons for these mistakes, their possible responses, and their views of the influences of these mistakes. With these main questions, probes, and follow-ups (Rubin & Rubin, 2005), I gradually narrowed our conversation focus until reaching a thick description with the depth, detail, and richness.

Responsive interviewing. The interview in this study was also a dynamic and iterative process. According to Rubin and Rubin (2005), this approach was called responsive interviewing. "In this responsive interviewing model, *analysis is not a one-time task, but an ongoing process.* Interviews are systematically examined - analyzed - immediately after they are conducted, to suggest further questions and topics to pursue" (p.15).

In this study, a three-week period elapsed from the first teacher interview to the last one. All the interviews were recorded. After each interview, I immediately transcribed the conversation the same day or the next morning to ensure reliability. I then went through a similar process for data analysis. I analyzed the transcriptions and also combined it to match a teacher's own video tapes. Through the analysis, I found interview technique problems and I also modified interview focus for the follow-up ones. The following example of Mary's interview provides a sense of my interview dynamic.

Mary's interview was the first one among all the interviews with these teachers. This was a face to face interview and was conducted in the afternoon after Mary's whole work day. It lasted approximately one and a half hours. At first, she substantively discussed her opinion about student attention and motivation. She even sang the Rap song for me. However, when we finished the first two parts of the interview, more than one hour had already past. I saw Mary was yawning. I knew she was tired. As a result, the third part, T or F questions went very quickly. Mary read question by question. She sometimes doubted a certain one by saying "hum" and then passed it with very few explanations. As a result, she judged all these questions as "true". In fact, all of them were "false". After this interview, I transcribed the conversation and I also reflected upon the whole process. I found three problems: (a) the interview time lasted too long. It definitely should not exceed one hour, otherwise, teachers would tire and may not focus; (b) when a teacher was stuck on certain topics far from my research focus, I needed to draw the teacher back to our conversation track, otherwise I would rush through the T or F questions; (c) the quality of the voice recorder was poor even though the teacher's voice could be heard. Therefore, I needed to use a new one in the follow up interview. Most importantly, I doubted the interview results concerning the T or F questions because Mary was so tired at that time and she did not provide explanations. Even though I checked with her class videos and found much consistency between the interview and the observation, I still strongly felt that I needed a follow-up interview with this teacher for the credibility of my results. These problems from the first interview were attended to during the later interviews. This process – transcription, analysis, reflection, and modification – was applied for the whole interview process. In addition, with Mary's agreement, I specifically conducted a follow-up interview with only the T or F questions for about 25 minutes during the daytime. During this time, she provided detailed explanation. Even though the results were the same as the first one, the trustworthiness of this study was improved.

In addition, for improving the trustworthiness, I also conducted member checking. After all these interviews were transcribed, I sent them to each teacher while deleting my own thoughts, along with an appreciation letter. I highlighted several unclear places and asked teachers whether I misunderstood them. However, only one teacher responded and said it looked perfect.

3.5.3 Combine video and interview data

The video and interview data were not simply put together. As previously mentioned, I prepared for teacher interviews with video information. I listened to teachers' opinions about their video clips during interview. I also checked video tapes to ensure that my interviews were accurate. After all the interviews were complete, I also pulled out the video data and interview data and tried to find consistencies and gaps concerning each identified topic. To differentiate the interview from the video data, I highlighted the interview data in yellow. The findings from the comparisons and contrasts of video and interview data were also added to my memos. After examination of each teacher, I also searched for patterns across teachers and classes.

Another effort to combine both types of data was to repeatedly listen to the interview data until I was extremely familiar with them. In addition to the formal listening, I also took advantage of all possible opportunities such as driving, walking, eating, and waiting. As a result, when I observed teachers' videos, I could automatically recall what teachers said in their interview. In summary, I could naturally compare what teachers said and what they did because of the extent that I familiarized myself with these data. Through these efforts, the relationship between teacher knowledge, teacher instruction, and student learning were relatively clear and teacher instructional insights and issues also appeared obvious.

3.6 Data Report

This study used a case study report mode because case study provides thick description of context information which offers the transferability of the findings. Stake (2000) emphasized that case study should optimize the understanding of the case rather than generalization beyond. In this final report, I provided detailed context information for teacher instruction. I also connected teachers' responses to similar errors or difficulties across two lessons (see Appendix 3). Through the clusters of related sightings, I provided several portraits for each teacher under different themes. In addition, I also arranged different teachers' portraits under each theme. The purpose of arrangement was not to compare teachers but to clearly explicate the relationship between teaching and learning and to highlight the effective teaching strategies, which could provide insights for identifying a MKPT for teaching equivalent fractions.

4. RESULTS

In this section, I report *teacher responses to students' errors and difficulties* (TRED) and students' corresponding cognitive gains under six themes. The first four themes are the previously identified learning difficulties and common errors (see section 3.3.3). The last two themes, "Multiply by any number - 0/0?" and "Equivalent fractions and equivalence" emerged during data processing. Through these portraits of teaching examples with high or low quality, differences among teachers' instruction were observed that resulted in different learning outcomes for students. However, comparison is not the purpose of this study. The purpose is to reveal the relationship between teaching and learning through these lenses. Table 2 shows the teachers involved in each theme. Reading across themes provides the complete portrait for each teacher.

Table 2

Teachers involved in each theme

	Kathleen	Jennifer	Lisa	Mary	Barbara	Rose
What is your goal?	x			x	x	
Lines or pieces?		x		x		x
Are your really doubling?	x	x	x	x	x	x
What do you mean by adding ¾?	x	x	x	x	x	x
Multiply by any number - 0/0?	x	x	x	x	x	x
Equivalent fractions and equivalence	x	x	x			x

4.1 What Is Your Goal?

As discussed in Section 3, the fractional concept *whole* was part of the core knowledge for learning equivalent fractions. It was also a learning difficulty in Lesson 2.1 where the fund-raising goal $360 was the "whole" represented by the thermometer as the goal line. When using fraction strips to measure the fund-raising goal, students encountered a complex situation which included the thermometer, the fraction strips, and the "goal line". As a result, some students were confused by the real "goal". In the following section, I describe teachers' responses to this learning difficulty. Since Jennifer and Lisa's classes did not really work on the thermometer, students in these two classes did not encounter this type of difficulty. Jennifer did not require her students to use fraction strips because that was an advanced class according the later interview. Lisa mentioned that fraction strips could help students investigate but she forgot to ask students to take them out. The aim of the portraits for the other four classes is to highlight the relationship between teaching and learning. In addition, an emergent topic "1/2≠2/4" related to this theme is presented (see Table 3).

Table 3

Sub-theme and teachers in "What is your goal".

Section	Sub-theme	Teacher
4.1.1	Addressing the obstacle directly	Barbara /Kathleen
4.1.2	Confused by the obstacle	Mary
4.1.4	When "whole" is overemphasized: 1/2≠2/4	Kathleen/Lisa/Rose

4.1.1 Addressing the obstacle directly - Barbara and Kathleen

During the teacher interview, when I asked for teachers' opinions concerning learning difficulties in Lesson 2.1, both Kathleen and Barbara mentioned how to use fraction strips to measure the thermometer was a challenge for their students. In the video taped-lessons, these two teachers showed their emphasis on the "goal" or the "whole." They both addressed this issue before and during students' exploration. As a result, students in these two classes used different strategies and were able to share and prove their ideas.

Barbara's class. Barbara's lesson lasted approximately 35 minutes. She employed a cooperative learning method. As recalled by Barbara in the later interview, she knew using fraction strips to measure the thermometer was difficult because students' fraction strips did not directly work well. She said, for example, students could not simply use their fourths strip to get the correct answer 3/4 because the length of fourths strip was longer than the goal line. As a result, students had to figure out certain ways to make their fraction strips work. Barbara told me she had spent a lot of time with students on fraction strips before this lesson. Therefore, some students had already obtained the visual abilities to quickly identify which fraction strips might work better. In the following sections, I will describe how Barbara addressed this learning difficulty and provided students sufficient investigation opportunities in her video-taped lesson.

1) A critical review question. In the review, the teacher directly asked the whole class a question related to the length of "wholes." Three students answered this question with their own words.

T: Did the 1/4 on each of your fraction strips represent the same amount? Was it showing us exactly the same amount?

S1: No.

T: No? Why not? They were both 1/4.

S1: Because they were different lengths. So if you match them up together and you match the one fourth lines they would not be the same thing.

T: Excellent. Danny, would you want to add to that?

S2: Even though they both represent 1/4 of each strip, they don't equal the same length on the strips.

T: Ah, ok, they don't represent the same length of each strip.

S3: They both represent 1/4, but they are representing 1/4 like in different lengths.

T: Ok, good, 1/4 of different amounts, excellent, very nice.

The discussion of this review question - Did the 1/4 on each of your fraction strips represent the same amount - only took them 50 seconds. However, it helped students to retrieve their prior knowledge. That is, only a fraction of the same length represents the same amount. This prior knowledge contributed to students' new investigation because their fraction strips passed the goal line. As the teacher said in the interview, she purposely asked this question so that students might be aware of the situation such as "oh, what am I doing? Even these lines matched up, the wholes have different lengths!"

2) Addressing cognitive obstacles. During the exploration, this class employed group learning. Barbara walked around the classroom examining students' different strategies. Sometimes, she addressed students' cognitive obstacles and encouraged them to try different ways to work them out.

T: Hey, Which one are you trying first?

S: 2/3.

T: Ok.

S: That's the way.

T: Oh, good point, yeah, because the goal is where?

S: (point out)

T: So what's wrong with your fraction strip there?

S: Too long, it doesn't work.

T: So, ok, is there a way you can make it work?

S: Try to find the length that fit into it.

T: Ok, you can try that. You can certainly try that.

Barbara's questions "think, the goal is where" and "so what's wrong with your fraction strip" directly guided this student to see the relationship between the goal line and the length of their fraction strips, which also allowed him to see the problem. Barbara's guidance was not simply to reduce this student's difficulties but to point out a direction for his further exploration.

3) Encouraging different ways. During group exploration, some groups also tried numerical approaches. Barbara praised their solution but still encouraged them to see whether they could use fraction strips to work it out:

T: You all agree? So what teacher is right?

S: Ms. Mendoza. Because 90 goes into both these numbers. And 90 goes into the 270 three times and 90 goes in here four times, so it's 3/4.

T: OK, which fraction strip works? I want you to find a way to make that work too. I know that works, but I like you to show me a way that fraction strips work.… Figure it out. I'm going to come back, because I want to see how you are making it work, ok, because I know there is one way to do it, and you have a good strategy with that way, but I want to see if you find another way, ok, because sometimes, you might want to use the different model.

In the above sighting, even though students have solved this problem by using numbers, Barbara still challenged them to use fraction strips to work it out. This was different from some other classes where teachers were satisfied with students' one solution. Encouraging students to use fraction strips beyond numerical approaches would provide more learning opportunities especially in such a challenging situation.

4) Students' cognitive gains. As Barbara said in the interview, she had spent a lot of time with students on fraction strips before this lesson. As a result, some students had already obtained good visual abilities. During this lesson, some students came up with effective strategies by using fraction strips, which showed their clear understanding of the "goal":

T: Yes, How did you come up with that?

S: …Because I took one of the fraction strips and matched it up with this (Explaining with showing how to measure up with the thermometer on the book). But I knew that these weren't equal. So I fold this over, and so it breaks into 8 pieces instead of 9. So if you do that, then it's matched up with 6/9 which is actually 6/8. 6/8 is equal to 3/4, because 3/4 is just 6/8 in simplest form.

T: Excellent. That is a very, very nice answer. I love it Katy. I am going to have to have you share that because that was excellent.

In this example, the student used the ninths strip where the goal line matched up with 6/9. Since the fraction strip was longer than the goal, this student folded the last piece and the fraction strip turned out to be an eighths strip. As a result, the fraction 6/9 also turned out to be 6/8. Obviously, this student had a clear understanding of the concept fraction and was able to use this knowledge to construct her new understanding of equivalent fractions. In the summary part of the lesson, Barbara asked this student Katy to share her strategy with the whole class. It is reasonable to conclude that Katy's wise use of fraction strips together with other students' numerical ways provided the whole class sufficient opportunities to reach a deep level of understanding.

The effect of Barbara's addressing the learning obstacle in advance was also reflected by certain students' cognitive gains. In this class, when students had difficulties or wanted to explain their ideas, they tended to raise their hands and wait for the teacher. As a result, Barbara was busy with adapting her help to students' needs [1]. Among those students, there was an Asian boy Nelson who was not fluent in English. After the other students started the investigation for a while, Nelson raised his hand and told the teacher that he did not understand. The teacher then patiently restated the task for him including the "thermometer." Nelson then quickly provided his answer which was incorrect. Barbara made no judgment instead of saying: "How can they prove that? Like they had to go before the other teachers and convince them. How could they tell them?" Then

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[1] If this teacher had engaged more group discussion instead of providing immediate help, this class could have been more successful. For more information, see Ding, Li, Piccolo, & Kulm (2007).

Barbara left Nelson. About seven minutes later, Nelson again raised his hand and reported his new findings to the teacher. However, he agreed with both Christo's 2/3 and Mendoza's ¾. Barbara then asked him to explain his ideas. During the explanation, Nelson found the contradiction, and Barbara again suggested that he continue his work. In the summary part, Nelson volunteered to share his strategy which clearly showed his understanding. In fact, in the follow-up lesson (Lesson 2.2), Nelson showed increased interest in problem solving. When he explained the pattern for finding equivalent fractions, he even mentioned certain ideas in "generalized math." In a word, the change in Nelson's learning behaviors showed his cognitive gains under the Barbara's guidance.

Kathleen's class. Kathleen mainly used a directive teaching method. During her first class (Lesson 2.1), the partner work only lasted about 2 minutes of the total 31 minutes. In the first 8 minutes, they were actually reviewing homework. In the last 7 minutes, this class was discussing "1/2≠2/4" for the follow up lesson (elaborated upon later). As a result, this teacher actually spent 16 minutes on the new content. In general, this lesson went smoothly. Students were engaged, sometimes taking the lead in mathematical conversations. Kathleen's teaching language was extremely concise with very little repetition or rephrasing using clear logic. During the teacher interview, the teacher replayed her video clips and believed that class was a regular class. Even so, students in this regular class showed a good understanding of mathematics. The general impression of this teacher's classes was that she made mathematics easy to learn and the students were really doing math. During the interview, I mentioned my impression of her class. Surprisingly, this teacher said that she hated mathematics because her mother kept her practicing and practicing it when she was young. "That is why even though I do not like math, I want to teach math in a different way. I realize that if I myself do not understand math, I cannot teach it." Kathleen said her teaching style was to unpack the knowledge into small pieces (concepts) and she made sure her students really understand each piece. She said she was also aware that these pieces or concepts were all connected. As a result, it was reasonable to see that even though this teacher only spent about 16 minutes teaching the new content, her students demonstrated a good understanding of the lesson. In the following sections, I describe how this teacher unpacked this knowledge and how students showed their understanding.

1) Place your strips on top. In the introduction, Kathleen guided students to observe the thermometer and to place their fraction strips on top of the thermometer.

T: (11:15) Ok, listen to this question. If I was to use my fraction strips, can I get an equal measurement by placing my fraction strip on the goal of the thermometer that's given to you. I want you to take a few seconds and place your fraction strips on top.

S: …too long.

T: Ok, so even though they suggested that fraction strips may be handy or useful, if I place the smallest fraction strip which we created, does it fit exactly? No, so we are going to have to use our estimation tool which is part of your brain. So, looking at this picture, I want you and your partner to answer this question, letter A. You will write that on your spirals and we will discuss that and share your answers. So you won't necessarily use these (Showing the fraction strips). You have to estimate.

Kathleen's guidance - place your fraction strips on top of the thermometer – before students' investigation allowed students to see the difficulty hidden in this investigation. Her question "can I get an equal measurement by placing my fraction strip on the goal of the thermometer" helped students be aware of the relationship between the length of their fraction strips and the goal. The "equal measurement" also reminded students of the necessity of the "equal length" of the whole. As a result, this question led students to an action of examination by putting their fraction strips on the thermometer in their textbooks. This action assisted students in smoothly finding the problem with their fraction strips, that is, "too long."

Even though students in both Barbara and Kathleen's classes found the same problem – the fraction strip did not exactly fit the goal line - these two teachers used different teaching strategies. Barbara challenged her students "how can you make it work? That's your job." In contrast, Kathleen told her students that they did not necessarily have to use their fraction strips. They could use estimation tools which was part of their brain. In the later interview, Kathleen said "estimation" was part of mathematics. Visualization was one of estimation tool that she emphasized. She also said the thermometer showing the fund-raising process was a new type of content because traditionally, students were only required to solve problems such as "simplifying 2/4 to 1/2". When students were given a fraction strip longer than the goal line, especially a strip without labeling, student had no idea about how to use it as a measurement tool. She said many students could not figure out how to create a new fraction strip with the same length from the bottom to the goal line and then folded it into equal pieces. As a result, she did not require her students to stick with fraction strips. Instead, she pointed out another direction: estimation tools.

2) Students' cognitive gains. After the 2-minutes exploration, Kathleen asked the students to share their strategies. Her students came up with different ways to measure the fundraising process as shown by the thermometer. These strategies showed students' cognitive gains corresponding to this teacher's instruction.

Strategy 1:

T: (14:32) Ok, so let's share some different strategies that you used to answer this.

S: I say Ms. Mendoza.

T: Ms. Mendoze (wrote down "Mendoza" on the board).

S: Because her answer is right and it's simplified.

T: Her answer is right and it's simplified. Ok, as your teacher, I want you explain why you think it's right. Look at the picture shown and you have to explain to me because I don't understand.

S: One way is to add a fraction strip that was equal to it … (inaudiable).

T: So what you're saying is if I took the fraction bar and cut it up to this size, and broke that fraction bar evenly into four sections, you would have ¾ filled.

In this strategy, the student directly used the picture in the textbook. Since in the review part, Kathleen asked students to place their faction strips on the goal of the thermometer, these students noticed the extra part and were aware of the real "whole" they should use. As a result, this student said she would cut the size from the bottom to the goal line into four equal pieces and the shaded part turned out to be ¾.

As previously mentioned, Kathleen emphasized visual ability and estimation tools. When she tried to illustrate strategy 1, she drew another picture instead of using the thermometer directly (see Figure 12). She also suggested her students to draw at the same time.

T: Let me go ahead and copy this down and we'll have some more comments. Now go ahead and draw your thermometer. Here is your goal at 360. Because soon you guys will have problems where the thermometer won't even be there, you'll just kind of have to picture a thermometer. So be drawing your thermometer.

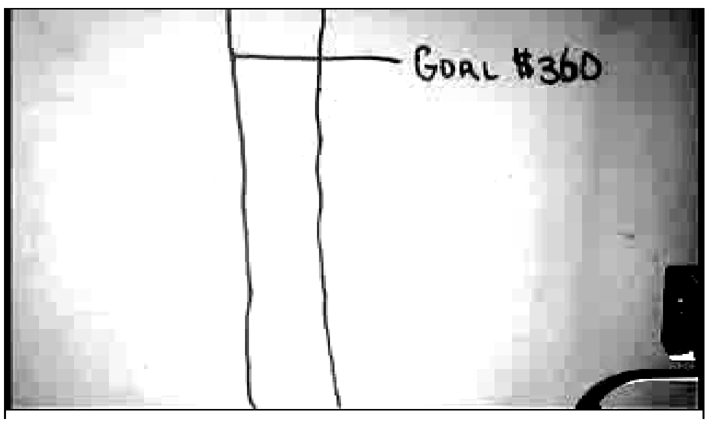

Figure 12. Kathleen drew a picture.

Requiring students to draw a fraction bar instead of directly using the thermometer provided students more opportunities to understand the importance of "goal." Kathleen again pointed out that the extra part above the goal line should be disregarded. Through the drawing process, students' understanding of the "goal" was possibly deepened. Based on this picture, Kathleen further guided her students: "But if you drew your thermometer and you would try to break it up, what's another strategy to figure out how much that base is, because he was saying if you took a fraction strip and fold it, that's good, that would work too. What do you think?" This question brought out two similar strategies - measuring the top part (1/4 part) first either with fingers or a piece of paper.

Strategy 2:

> S: Well, when I did it, I took from the goal to 4/4, that's about two fingers width. Then I took from ¾ to the top line of how much is reached and it's about the same.
>
> T: So you found that using the fraction strip was a little helpful if you just kind of estimate just a bit off on each section, it would work.

This student seemed to make a slip of the tongue. What he meant was that he first measured the unshaded part which was 2 fingers wide. He then measured the shaded part and he got three more 2-finger widths. As Kathleen pointed out, even though this student used finger strategy, it was the same idea as using fraction strips.

Strategy 3:

> S: I took my paper and just like marked through where the space of the white thing and just measured that …
>
> T: Right. So she took her paper and she marked. This part right there, this section, and she went on down and discovered that that was three more sections. So let's go ahead and draw this where we cut it in half and then cut each section in half again. And this part was shaded. Ok, if you said Mendoza was said that it was 3/4 of that goal, is that correct?
>
> S: Yes.

Students' invented strategies showed their clear understanding of the faction concept, which was related to Kathleen's prior instruction of addressing students' learning difficulties. As a result, students smoothly visualized the relationship between the numbers (270/360) and the pictorial representation (the shaded part out of the whole) which in turn helped them identify the equivalent fractions 3/4 and 6/8. Based on these, this class discussed the money corresponding to these fractions. They also got time to discuss why 3/4 and 6/8 were equivalent with both pictorial and numerical representations. To summarize, when the difficulty hidden in the thermometer was appropriately addressed by the teacher, students' understanding of the concept *fraction* - the core knowledge of equivalent fractions - was strengthened and their further learning was then made possible.

4.1.2 Confused by the obstacle – Mary

Compared with the above teachers, Mary is a teacher who emphasized students' motivation and attention. During the first interview with this teacher, she spent half of the time talking about how she thought students' attention was important and how she used various strategies such as singing rap songs to attract students' attention. Concerning the learning difficulties of Lesson 2.1, she went through the textbook that I brought to her. However, she quickly changed my topic. She told me that she did not like the CMP textbook at all:

> You know, you just got the bright color, bright paint, but if you look into the book, everything is just grey. It was just boring. People were not comfortable at all. You had to find a cartoon strip, or you had to find a song, you had to find a book, something to bring this whole lesson out.

As a result, Mary did not recall whether to identify the "goal" during the measurement was hard or not.

What did Mary do in the introduction/review part? The video-taped class was a regular one as indicated by Mary in the later interview. The lesson lasted for about 52 minutes. Mary spent about 8 minutes on the introduction part which included a cartoon and a poem. In the cartoon, two boys had a conversation: "What did you study in school today?" "Fractions." "What did you learn?" "1/10 of what I was supposed to." Both the teacher and students were laughing. The poem was saying that fractions were hard because of "that silly, little line". After the cartoon and the poem, the teacher directly brought students to problem 2.1 where she asked three students to go to the board and write up "three fourths", "six eighths", and "two thirds" with numbers, which took the class another 6 minutes. After that, they started the new investigation with the fraction strips.

Obviously, both the cartoon and the poem again reflected this teacher's emphasis of students' motivation and attention. They were also fun and related to fractions, that is, fractions were hard and students learn little in a class. However, to what extent did the cartoon and poem really contribute to the learning goal: compare fractions and then identify equivalent fractions? Writing "three fourths" and other words can help students see the three different answers in problem 2.1 clearly, however, does this really identify students' learning difficulties and warrant spending six minutes? In fact, the critical prior knowledge about the "goal" or "whole," which was also a learning difficulty, was not mentioned at all before students started their exploration. One of the interpretations was that the teacher was sure that her students mastered the concept very well and they were ready for the investigation. The other interpretation was that the teacher did not know students' learning difficulty at all. The following description for the exploration partly tells the answer.

Suggested three fraction strips and confused students. As previously mentioned, after the cartoon, poem and writing up the words, the class directly entered the exploration part. The teacher asked students to take out their fraction strips called by her as "number lines". She specifically guided students to use fourths, eighths, and thirds strips.

T: Let's take out our measuring tools here. Ok, take out your number lines and do some measuring and tell me, which one of these teachers you think is right based on the information you have received here. We got one number line that's looking at the what?

S: Fourths.

T: One number-line that's looking at the what?

S: Eighths.

T: And another number-line that's looking at the what?

S: Thirds.

T: So those are the three that you want to look at right now. Take those out, the ones that deal with your thirds, your fourths, and your eighths.

This teacher asked students only to use these three strips. To her, these fraction strips - fourths, eighths, and thirds - should work well and directly lead to an answer for these fractions ¾, 2/3, and 6/8. Put another way, this teacher assumed that if students simply put their fraction strips on the thermometer, they could tell their answers immediately.

However, the situation was not as simple as this teacher imagined. In fact, students' fraction strips were all longer than the thermometer. As a result, when the teacher checked students' process, they showed their confusion and uncertainty.

T: Who do you think is correct in making their announcement? … Who do you think is correct?

S1: Neither one.

T: Hum?

S1: None of them.

T: You don't think any of them is correct. What about you, David?

S2: Ms. Mendoza.

T: David thinks Ms. Mendoza who said we sold ¾ of the way. Ok, Joan, what about you?

S3: (No response)

T: Have you measured yet?

S3: No.

S4: I am measuring.

T: Ok, what are you measuring?

S5: I agree with Ms. Christos

T: Ok, so you have your thirds.

S5: I think it should be alright.

T: Mike, what do you think?

S6: Neither, none of them.

T: (walking to Mike) Ok, here is our thirds (put the thirds fraction strip on the student's textbook) and we have Ms. Christos who says that they have gotten 2/3 of their goal. And we have eighths, Mr. Park who says their what?

S6: 6/8

T: 6/8 of the way. (put eighths fraction strips on the student's textbook and then talk to another student) Ok, what about you? (Asking S7 and then turning to S6 again) And here is your thirds, your fourths.

S7: I think I'm confused because you can't really measure with them.

In the above conversation, teacher Mary asked seven students' responses. Only one of the students, David, got a correct answer. Since the teacher did not ask him to explain, no evidence could support how well this student understood. The rest of students said "neither one", or "Mr. Christos", or "not measuring yet".

Students' surprising comments. The last student in the above conversation raised his concern: "I think I am confused because you cannot measure with them." This comment seemed to surprise Mary because she left this student and walked toward the overhead while speaking to the whole class: "Can you measure them with those? Ok, once you chose, can you really measure with them? Hum?" She looked at the textbook for a while without saying anything. She then checked with the student who got the correct answer "Mendoza." She also checked with another student Michael who, however, raised an unexpected and more surprising comment:

T: Ok, Michael?

S7: It will be kind of hard because measuring things as long as this, so the goals is usually up here, but the goal is down here now.

Michael found the difference between today's investigation and the prior ones. That is, when they measured the thermometers in Investigation 1, their thermometers had the same length as that of their fund-raising goal. However, in this problem, the thermometer was longer than the goal line. In fact, Michael made a very critical comment which was also realized by many other teachers as reflected by other videos where students and teachers used various strategies for solving this problem.

Teacher was confused. The teacher heard this comment and immediately checked her fraction strips on the overhead: "Ok, if you would put your measuring tool on there, would it work out?" She looked at her fraction strips and seemed confused. She walked toward Michael to check his fraction strips: "Ok. Let me see. Where is your fourths? Let me look at your fourths." At this time, many students were off task. They did not talk but sat quietly with their backs against their chairs doing nothing

Mary was busy with counting Michael's fraction strip: "1/4, 2/4, half of the way…". She suddenly made an announcement to the class: "Michael made an excellent observation, Ok? Excellent observation. He was saying that we were used to our thermometers from the top to the bottom, our ruler is measuring from the top to the bottom, but on this particular one, our goal is where? Is it at the very tip? No, it's not. Ok? It's not at the tip." She then continued her work with Michael, "So, ok, let's look at it. If we were starting from here, about how much, let's see, do we have fourths? Ok, take out fourths. ok, and let's look at it. Let me look at it and see if this could come out to be close to the same."

The teacher persisted in checking the fourths strip. Obviously, she knew 3/4 was one of the correct answers and she assumed her fourths strip should work. However, Michael's fraction strip seemed not work at all. She then turned to another student and checked her book: "2/4, 3/4 …". She then left for the front without saying anything. She might find something wrong with the fourths strip. She might be curious about why the fourths strip did not fit the thermometer. She might even start doubting whether 3/4 was a correct answer. She checked with the student who got ¾ again in the front of the class:

T: David, what did you say that Ms. Mendoza was right? Why do you think Ms. Mendoza is right?

S: I measured it.

T: You measured it? Ok, alright.

Mary should have known that 3/4 was a correct answer. It is also reasonable to assume she could solve this problem in symbolic ways. Mary was able to manipulate these numbers as demonstrated by her later interview. For example, she quickly showed the process $\frac{270}{360} \div \frac{90}{90} = \frac{3}{4}$ to me. However, facing the fraction strips in that class, Mary seemed confused why the fourths strip could not demonstrate the correct answer 3/4 based her own "checking." She, therefore, turned to David, the student who provided the correct answer. However, this student did not give more hints except simply insisting that he measured with his fraction strip.

Teacher solved the problem. Mary spent her efforts on identifying what was really going on with the fraction strips. She at last found the problem but she did not know how to fix it or whether there was any way to fix it.

T: Ok, I know it's probably strange because, um, the point that Michael brought out which is that we're used to measuring from the top to the bottom, and here we have a goal that's in the different place. Hold on, Ok, yes sir.

S: (inaudible)

T: Ok, ok.

T: Alright. Ms. Mendoza is correct. Ok, Ms. Mendoza and, but not only Ms. Mendoza is correct, Mr Park is correct.

T: Ok, who else?

S: Mr. Park.

T: Why? If you take your measuring tools, what did Mr. Park say? It was what?

S: 6/8

T: If you take your thirds, I mean your fourths and eighths and put them together, ¾ of the way, and 6/8 of the way, what do you notice about the two?

S: They are equivalent…

T: They are equivalent. That word that we've learned is what?

S: Equivalent.

T: Equivalent.

As we see from the above conversation, Mary eventually solved the tough problem by simply confirming the correct answer "Alright, Ms. Mendoza is correct." Even though the fraction strip that they tried could not verify this conclusion, the teacher repeated this answer without any explanations. Moreover, the teacher offered this class another answer: "Ok, not only Ms. Mendoza is correct, Mr Park is also correct". Clearly, this statement was not examined yet. When the mathematical conversation about the relationship between the fraction strip, thermometer, and the goal could not continue any more, Mary also tried to find some hints from the student, David. However, students' understanding was usually incomplete. In fact, most of students appeared tired and had lost interest in this topic at that time. The only voice in that class was the teacher's voice. As a result, Mary quickly switched this topic "why 3/4 and 6/8 were correct" to another one "why 3/4 and 6/8 were equivalent."

Alternative solution: Misusing the goal. As previously mentioned, Mary was able to manipulate with numbers. After she talked "why 3/4 and 6/8 were equivalent," she came back to this original topic – "why 3/4 and 6/8 were both correct" by using money, during which, she made another mistake concerning the "goal". What follows was Mary's mistake causing the entire class to become completely lost.

1) Who misled whom? Mary suggested students think about money. At first, they discussed Ms. Mendoza's 3/4 and Mr. Park's 6/8. Even though the conversation appeared difficult – students had experienced confusion and most of them were already off task – the class got back on the right track. They use 360 as the "goal." However, when it came to Ms. Christos's 2/3, things changed.

T: Now, let's look at Ms. Cristos. We're looking at 360 as our total, Ok? And we want to divide 360 into what?

S: 3

T: 3 equal parts. Will it work?

S: (No response)

T: Hum?

S: (very low response)

(28:54-29:26. Mary was looking at her book without saying anything for about 32 seconds. At this time, some students were saying something inaudible.)

T: Ok, now. What about the 270 that we're earned? Will it work? Because Ms. Christos is saying what? Christors. The two thirds was really the simplest way to describe the progress already made (read textbook). 2/3.

(30:00-30:30. Mary was again looking at her textbook without saying anything for another 30 seconds).

T: What did you come up with for Ms. Christos?

S: 90

T: Ok, 90.

Interestingly, the mistake occurred naturally even though the teacher did not realize it. When Mary reminded students with both "What about the 270 that we learned" and Christos's argument of "2/3", one student provided "90" which was supposed to be the answer for $360 divided by 3. As a result, the students' wrong answer was immediately accepted by Mary, which in turn caused further confusion:

T: Now, break it into how many parts do we want [sic]?

S: 4

T: There are 4 parts, Ok, no, but not, not 4 parts, but there are not 4 parts. How, how many parts, Christos?

S: (no response)

T: Right there on the board. I have it. 2/3. Ok, so you've got 1/3 here, that will be what? 90. And 1/3 here, this would be what? 2/3, this would be what? 90. How much is 90 and 90?

Since the class just discussed Ms. Mendoza's 3/4 a few moments ago, some students might remember $90 represented ¼ of $360. As a result, when the teacher agreed with

"90" and then asked "how many parts do we want," a student provided the answer "4." As a result, Mary had to remind the class that they were talking about Christos whose statement was 2/3. Some students were still referring to the "goal" as "$360" while the teacher misused the goal as "$270" instead of "$360," the hidden conflict turned out to be a huge obstacle for their continued exploration concerning Christos's 2/3. Even on simple questions, some students seemed totally lost:

T: How much is 90 and 90?

S: 190

T: 90 plus 90 is what? One hundred and what?

S: 190.

T: No, no, no. 90 plus 90,

S: 199

T: No. one hundred and what?

S: 180.

T: 180, ok?

According to Mary, this was a regular six grade class. Students absolutely had the ability to calculate 90 plus 90. However, in the above conversation, this student could not think.

2) Students were struggling. After teacher Mary finally got the answer "180," she moved her discussion ahead with the assumption that students knew 180 represented 2/3 of 270:

T: Ok. 3/3 would be what?

S: 3/3 would be

T: What are you going to do here?

S: Put another 90.

T: Ok. Tell me how much? I'm writing upside down, so you have to work with me. How much would they be? (Sigh)

S: 270.

T: 270. Ok, have we made it to 270 a ways [sic] by going 2/3?

S: Yes.

T: No, no, no. When we got to 3/3, we made 270, so can we say that we can use 2/3 to describe how much we sold so far?

S: No.

T: No, we cannot. Ok, basically according to Ms. Christos's work here, we only sold how much?

S: 270.

T: No, how much? 2/3 would be what? 180. $180 which is not what we sold already. We sold how much? How much?

S: 270

T: So, instead of using 2/3, Ms. Christos should use what? What would it bring us, 270?

T: Hum? Hum?

S: ¾?

T: No. We are looking at one with thirds right now. ¾ is correct. But we are looking at the one with one with the thirds right now, Ok? I'm going to show you what I'm doing here.

Obviously, Mary expected students to see the following reasoning: Christo's statement 2/3 represented $180, which was not consistent with the textbook information "had raised $270." Therefore, Ms. Christo's 2/3 was wrong. However, students seemed to retain the information when they discussed Mendoza's statement 3/4 several minutes ago. That is, 3/4 of $360 was $270, exactly the raised money as the textbook said. As a result, when Mary discussed "$270" in Christos's case, students kept mentioning "3/4." At last, the teacher had to repeatedly remind these students that they were actually talking about Christos whose statement was 2/3.

3) Two missed hints. Teachers are human and sometimes will make errors in class. It should not be surprising to see that teachers make careless mistakes or slips of the tongue. However, if a teacher keeps making the same mistakes over time and has no ability to identify her errors even with obvious hints, it will be unnatural. In Mary's reasoning there was an obvious conflict, providing this teacher opportunities to realize her mistake. That is, the same "$270" sometimes represented the actual raised money and sometimes represented the fund raising goal while the textbook clearly stated that the goal was not reached yet. In other words, $270 represented both the part and the whole, which clearly should not be the same. However, even with such a clear inconsistency in the teachers' reasoning, she did not realize the mistake and correct herself. This mistake lasted more than 10 minutes until she went back to Mendoza's 3/4 and tried to ask students to calculate "270 ÷ 4":

> T: Alright, if we're looking at fourths, (drawing a number line), so this would be 1/4, this is what? 2/4, this is 3/4 and this is 4/4 which equals what?
>
> S: One whole.

T: Ok, how much, how much, how many equal parts will we have, to get $270?

S: (No response)

T: I mean $360, I am sorry, I'm saying $270.

Only now, this teacher seemed to realize her mistake. This is partially because "270÷ 4" was not as easy to be calculated as "270÷3." This actually provided the teacher another hint to recognize the mistake that she just made. However, she still did not recognize the mistake because she did not go back to point out that they used a wrong goal $270 for Ms. Christos's case. In addition, in the summary part, she still pointed out the number line with the whole clearly labeled as "$270" and asked students, "And we see why 2/3 is not a good what? One to use, was it?" Through the whole class instruction, Mary systematically used the goals "$270" for Ms. Christos and "$360" for Ms. Mendoza and Mr. Park. She also used "$270" as both the fund-raising goal and the actual raised money. As a result, a few students who tried to follow the teacher were struggling with the conflicting information provided by the teacher and the textbook while the rests lost interest and were then off task.

What did students learn in Mary's class? Clearly, students in Mary's class encountered difficulty when they used fraction strips to measure the thermometer because they found the length could not match up with the "goal." Mary tried to solve this problem but finally gave up. She then tried another way by using money where she misused the "goal" and, therefore, confused the entire class.

Using fraction strips to measure the thermometers was actually a common difficulty that occurred in other classes. This is because the fraction strips in this investigation were longer than the goal. As a result, students needed to figure out the "true" whole and further decide which fraction strips they could use. Even though there were no obvious fraction strips that matched perfectly, students in other classes figured out various strategies to make their fraction strips work. As previously mentioned, one student in Barbara's class used the ninths strip and covered the extra part and she finally came up with 6/8 as a correct answer. There were also some students who created new fraction strips with the same length as from the bottom to the goal line on the thermometer. Still, there were some students who used their fingers or pencils to measure the unshaded part first. In contrast, students in Mary's class learned nothing about using fraction strips to measure the fund-raising process of how $270 out of $360 was equal to ¾ and 6/8. After students pointed out the confusion about the "goal," Mary tried to work it out several times. She finally gave up after several tries because she herself was also lost. Mary then directly told the students that 3/4 and 6/8 both correct. As a result, fraction strips were only used to prove 6/8 and 3/4 were equivalent by lining them up.

Concerning symbolic approaches, since Mary misused the "goal" and confused the students, very few of them were engaged in this class. The teacher dominated the talk during most of the class time while students had no response at all. Mary tried many strategies including the cartoon, poem, and jokes to motivate students and she received poor results. The teacher even provided students some time to do exercises for renewing their energy:

> T: "Are you with me? I do not know if you're with me? Come on, here put up your back into it. Oh, did you all run for life today?
>
> S: I would like to play volleyball!
>
> T: Oh!

T: Let's give with it, Ok? Let's stand up and stretch for a second. Ok. We're going to do something now. You are falling down on me. Stand up and stretch silently. (Students stood up reluctantly) Katy, you need to stand up! Girl! Stand on up and stretch! Woo, woo! (Some students were turning around their body) Alright, let's sit on down.

The whole class was led by the teacher. Only a few students provided answers for the teacher's simple questions concerning "what," "how much" and ""how many". In the summary part, no student reported any strategy.

4.1.3 Summary of three teachers' instruction

In Problem 2.1, identifying the "goal" or "whole" in the complex situation where the thermometer, fraction strip, and the fundraising goal all had different lengths, was a difficult but critical concept for learning equivalent fractions. The three teachers, Barbara, Kathleen, and Mary, facing the same student difficulties responded differently. The biggest difference was that the first two teachers anticipated this error and addressed it from the very beginning while the third teacher did not anticipate it and, therefore, was confused during her teaching. As a result, students' experiences in these mathematics classes were also greatly different. Barbara and Kathleen's successful instruction provided students directions for exploration. Students in these two classes were engaged. They were busy with trying different ways, raising good questions, and sharing their ideas. In contrast, Mary did not know students' learning difficulty in lesson preparation even though she spent time finding cartoons and poems for attracting students' attention. As a result, when students' comments reminded her that the fraction strips did not work in the way that she imagined, she tried to fix the problem but she was ultimately lost. Her confusion made the mathematical conversation impossible and most of students were off task. In addition, when Mary later tried to use a "Money" strategy as an alternative solution, she made a big mistake by misusing the "goal" for about 10 minutes even though there were several obviously possible opportunities for her to correct the mistake, she still did not recognize them, which in turn made students struggle with balancing her misleading information and textbook information. Mathematics in this class appeared extremely difficult to the teacher, the student, and even the observer. Who matters in students' mathematical learning? The answer could be easily seen from the above portraits.

To compare fractions, the "whole" is a critical concept. During students' investigations with fraction strips and other concrete materials, it is appropriate and important to emphasize the size of the whole or the "goal". However, sometimes teachers overemphasize the "whole", resulting in obvious mathematical mistakes, which in turn may confuse or mislead students' understanding. The following section reports this emergent issue related to "whole" as it occurred in the video observation.

4.1.4 When "whole" is overemphasized: "1/2≠2/4"

Students' understanding of equivalent fraction was based on the concept *fraction*. The meaning of part-whole was viewed as a basis for learning equivalent fractions. As a result, when students tried to compare two fractions by using concrete representations such as fraction strips and chocolate bars, the "wholes" should be emphasized to be the same. For example, Barbara asked her students: "Did the 1/4 on each of your fraction strips represent the same amount? Was it showing us exactly the same amount?" It is true when the whole is not the same, 1/4 of this fraction strip will not equal to 1/4 of that fraction strip. However, when teachers overemphasize the "whole" in the symbolic representations, it might cause mistakes in mathematics notation.

Prove it to be true: 1/2≠2/4. As previously mentioned, Kathleen was highly aware of "whole." Her action of asking students to put fraction strips on top of the thermometer demonstrated her sensitivity to this issue. At the end of Kathleen's Lesson 2.1, she wrote 1/2≠2/4 on the board and asked her students to think about it and try to come up with a reason why this would be true. (see Figure 13)

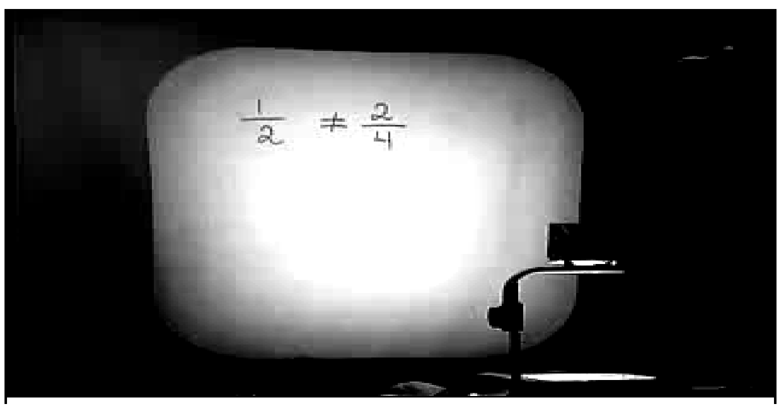

Figure 13. Prove it to be true.

Students were confused at first. They insisted that 1/2 should be 2/4.

S1: I think it is true because 2 divided by 4 is 2 [sic]. I think it is like some kind of rule, that is divide like 30 by 60, it's 2 [sic] and 50 by 25, it's 2.

T: Ok, what do you think?

S2: Maybe if you have like a pie or a something, and you divided it up and it's a half, and each of you get 1 piece. But then you divide it into 4, then each of you get 2.

S3: That's the same thing.

T: That's the same thing. What's your proof? Why it's not true? …

S4: Like simplify something, we need common factors, and "1" does not have any common factors.

T: That's not what I am looking for, but you got the right idea. What do you think?

S5: 2/4 is the 1/2 because if you divide 2/4 by 2, it will be 1/2.

T: Ok, I'm obviously getting you all to think a little bit more about the fractions when they are equivalent and when they are not. I'm just trying to start up an idea of what are going to do tomorrow.

In fact, what Kathleen wanted to emphasize was the "size" of the "whole". She wanted student to see when the wholes of two objects (A and B) were not the same size, 1/2 of object A would not equal to 1/2 of object B. She guided students to picture "half of an apple and half of a grape." Kathleen's attempt was correct. However, when she wrote it down as 1/2≠2/4, she made a mathematical mistake. This is because "1/2" is mathematically equal to "2/4" and "1/2=2/4" is mathematically correct.

Teacher and teaching resources. It is worthy of mention that Kathleen was not the only teacher who made such mistakes. I also found a similar question in Lisa's class. She wrote down "when 1/2 was not equal to 1/2" at the beginning of Lesson 2.2 (see Figure 14).

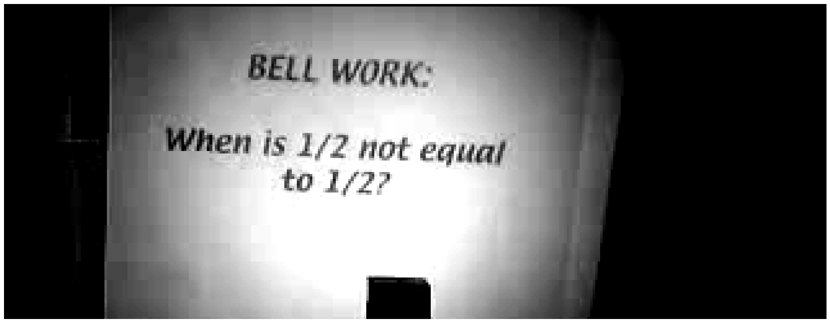

Figure 14. Lisa’s question.

The similar question was also asked by teacher Rose: “Is 1/3 always equal to 1/3? Is ¾ always ¾?” Students in Rose’s class clearly said “No, it depends on the size of whole.” Why did these teachers ask the similar questions? Did the CMP teacher guide mislead these teachers or did these teachers simply misinterpret teacher resources?

During Kathleen’s interview, I brought out the instructional part about “1/2≠2/4,” she was surprised and proud, “Oh, did I do this?” When I also mentioned the example of “half of grape and half of apple,” she laughed, “Oh! That is a good example.” I then asked her whether she designed this question by herself, she said, “No, I am not as smart as this. I definitely found it from some other place, maybe a teachers’ guide book or some internet resource, I do not remember now.” Interestingly, I examined the CMP teacher guide book (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998), and found a related suggestion right before Investigation 2. It says:

> Ask students to compare the fraction strips they folded to the preprinted strips.
>
> Is one half the same in both sets of strips?
>
> Why or why not? Students should understand that one half does not represent the same length on both strips because the strips are different lengths. However, in

> each case the symbol notation 1/2 represents one of two equal-size parts of a whole. (p.18j)

Obviously, the teacher guide book is correct. It suggested teachers ask questions about whether 1/2 of this strip represent the same amount as 1/2 of that strip. It did not say 1/2 may not equal 1/2. The CMP guide book also pointed out “in each case the symbol notation 1/2 represents one of two equal-size parts of a whole.” In other words, the mathematical value of “1/2” or the “ratio” in each case is the same. It is reasonable to assume teachers who used the CMP textbook were influenced by the CMP teacher guide book. However, the influences of this teacher resource on different teachers seemed greatly different. As previously mentioned, teacher Barbara’s question “Does 1/4 represent the same amount of each of your strips” is exactly the same idea as the teacher guide book. However, when Kathleen, Lisa, and Rose raised question such as “1/2≠2/4”, “when is 1/2 not equals to 1/2”, or “Is 1/3 always equals to 1/3”, they misinterpreted the teacher guide book. As a result, how should teachers use curriculum or teacher resources appropriately or how should curriculum influence teachers in a positive way is another question worthy of further exploration.

4.2 Lines or Pieces?

As analyzed in the Section 3, the learning difficulty in Lesson 2.2 was related to the transition from regional to number line representations. Since equivalent fractions in Lesson 2.2 were represented by a set of ruler-shape fraction strips which are number line-oriented, when students were asked to name these equivalent fractions, they were confused by counting lines or pieces. In this study, students in half of the video-taped classes did encounter this difficulty. Regarding those classes where this type of difficulty did not occur, I also interviewed the teachers concerning this issue. These teachers agreed that the "lines or pieces" issue was a learning difficulty but their class had spent time on it in their earlier learning.

The difficulty "lines or pieces" is not directly related to the learning goal of finding equivalent fractions. However, students need to name the fraction strips first. Only when they finished these labelings, could they move ahead to find patterns and to name more equivalent fractions. Therefore, how teachers were aware of and addressed this difficulty determined how far these classes reached. In the following sections, I provide three teachers' portraits. Jennifer addressed this difficulty and cleared the obstacle for students' follow-up investigation. Rose and Mary both noticed this difficulty but Rose's guidance was not effective while Mary's guidance was actually misleading.

4.2.1 Addressing obstacles in advance – Jennifer

One lesson or even one activity sometimes provides hints of why students learn a certain topic well. Jennifer's class is one of the examples. During the interview, I asked Jennifer to recall the biggest challenge for her students in understanding Lesson 2.2. Jennifer did not mention "lines or pieces." When I further asked her whether the "lines or pieces" difficulty occurred in her class, she told me her memory about this was blurry. This was reasonable. First, the video-clip that I sent to her was not related to this difficulty. Second, she is not teaching now and this class was about 4 years ago. Third, if this teacher did not particularly struggle with this type of student difficulty, she would not remember it. Jennifer's video-taped lessons confirmed the third interpretation.

Requiring a clear understanding from the very beginning. The textbook introduced the definition of "equivalent fractions" before Problem 2.2. To illustrate the definition, the textbook lined up three fraction strips with the same length – halves, fourths, and sixths - showing that ½, 2/4, 3/6 were equivalent fractions. (see Figure 15)

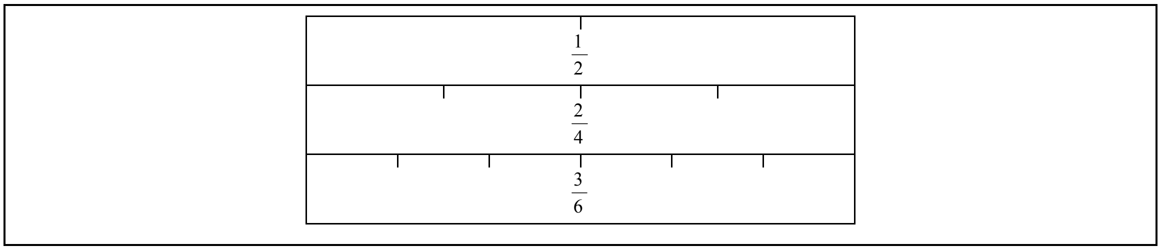

Figure 15. Fraction strips illustrate definition of equivalent fractions.

After a student read the definition, Jennifer guided the class to explain their understanding about how the fraction strips provided the equivalence among 1/2, 2/4 and 3/6:

> T: How are they showing that with those strips there? They have 3 fraction strips stocked on top of each other. And how are they saying that those strips are proving the equivalence of ½, 2/4 and 3/6. Daniel, can you explain that to us please?
>
> S: You have ½, this is right down in the middle and then you have 2/4 is right down in the middle, 3/6 is right down in the middle too. Besides, 3 is half of 6, and 2 is half of 4.

This student clearly described the location of 1/2, 2/4, and 3/6 – in the middle and lined up with each other. In addition, this student also provided a symbolic way to prove his conclusion. Jennifer confirmed this student's description. She, however, wanted to make sure her students understood this point appropriately:

T: Ok, if I sketch that real quick [sic], it looks like this (showing on overhead), right? And he's saying that they're all lined right above on the top of each other, and they are equivalent. When they say ½ or 2/4, what part are they talking about? What sections of these are they are looking at?

S: The middle.

It is true that this student might only look at the middle (the small dashes) and then got those equivalent fractions. This approach was not wrong. However, if students understood equivalent fractions only according to those middle lines, they would probably encounter difficulties when they needed to name fraction strips because they might be confused with counting "lines" or "pieces". Jennifer grasped the student's unclear verbal representation and asked further questions:

T: Are they just looking at this little, this like, hum, what part of this strip are they looking at? Just this middle?

S: Each half

T: Each half, what do you mean by that? Can you explain that a little bit more?

S: The one whole is divided into two equal…two parts… (Inaudible)

Only until the student showed his clear understanding, Jennifer stopped her probing. She then reminded the whole class to be aware of this issue during the follow-up investigation:

T: Ok, so if we look at those two halves, those two equal sections, they are all the same. You are right. And mainly as we go into the next few problems, we will be looking at the section. One thing that I want you to look at this kind of linear model, if this is 0, the very far left, and the very far right side is 1, then they're naming this "1/2" (She wrote fractions on the fraction strip on the board which looks like a number line). So from 0 to this point right here would be 1/2 of this strip. Ok, does that make sense? Ok, that's what I want you to think about today as we work through this problem.

Jennifer successfully grasped the important knowledge piece and ensured her students understood it appropriately and mathematically. She provided the students enough time to reflect on and elaborate their ideas rather than being rushed with introduction tasks and running into exploration.

Making connections between alternative representations. After the introduction, this class entered the exploration phase. Since this class employed cooperative learning, students at this period were supposed to do group learning. However, before students started group investigation, Jennifer used the 2/3 fraction strip (the first strip on the left side) (see section 3.3.2) as an example to check whether students were really ready for this task:

T: As we look at this fraction strip right here, how do they know that 2/3 was the fraction that they needed to use to label that first strip at the very top. What were they thinking when they decided that was 2/3? Think about it what we talked about numerator and denominator. Mike, can you help me out here? Why do you think they label that right here, 2/3?

S: Because the line for it is 1/3 and because … (inaudible).

T: So you're saying the line here is 1/3?

S: Yes. Because the beginning is 0 and the other side is 1 … And it's broken into three parts.

T: Very good.

This student clearly stated that he viewed this fraction strips as three parts rather than three lines. Put another way, he connected the denominator "3" with three equal parts of the thermometer. However, it was not clear whether all students had the same ability to visualize the ruler-like fractions strips divided into three equal parts. As a result, Jennifer, in the following sighting, wisely transferred these fraction strips (number-line oriented model) into real fractional bars (regional model) with only a little assistance – the extension of the lines to separate the pieces on the fraction strips completely:

T: We have this broken into 3 parts and if I extended this line down, it might even be easier for us to see those 3 parts. I kind of went around the fractions there, but here we are. We have 1, 2, 3 parts, that's where they came up with the thirds.

T: Now, did we count just the little dashes that had written on there? Do you see these little dashes? Did we say 1, 2 parts?

S: No.

With the extension of these lines, the ruler like or number line-oriented fraction strips turned out to be real fraction bars. (In fact, Kathleen also mentioned this method during her interview). Since students were more familiar with regional models, the fraction bars allowed students to see how these small dashes were actually related to those equal parts or equal pieces, which also allowed students to understand where 2/3 came from and what the mathematical meaning of 2/3 really was. As a result, Jennifer's extension of these dashes was likely to help students make sense of why they actually count pieces instead of lines.

T: I want you to be careful with that. As you answer these, we do not need to be just counting dashes. What are we actually looking for? Instead of dashes, what are we looking for?

S: the parts.

T: the actual pieces like this right here (show on overhead), ok?

T: You're going to be finding out what these question marks for both 2/3 and ¾. Are there any questions?

As we see, at the beginning of exploration, Jennifer spent almost three minutes to ensure students understood the task and she also equipped her students with the critical prior knowledge for the follow-up investigation. With a clear understanding of counting "lines" or "pieces", students in this class smoothly moved into new exploration of the learning goal. Jennifer's way provided other teachers a good sense of how to prepare students for exploration and how to improve the investigation quality in cooperative learning classes.

Students' cognitive gains. Problem 2.2 has three subquestions. Students were first asked to name three equivalent fractions for 2/3 or 3/4 by using fraction strips. These two subquestions requested students have a good understanding of "lines or pieces." Only after students finished these labelings, could they start the third question – to find patterns. Due to Jennifer's carefully designed instruction as described above, no students in the 10-minutes exploration phase appeared confused with counting lines or pieces. They spent most of the time looking for patterns. Jennifer walked around the classroom and several groups reported their findings to her. During the teacher- student communication, some other mathematical mistakes were identified and addressed by the teacher. All the students and the teacher were doing mathematics during this period.

Students' cognitive gains could be reported through two aspects in the summary phase of this class. One is the directive teaching effect on students' understanding of "lines or pieces" while the other is a global assessment demonstrating how far students reached the learning goal.

1) Students' clear understanding of "lines or pieces." Jennifer asked someone to come to the front to explain how he or she did the section for 2/3. Many hands were raised. One student named Josh came up.

S: I found how many sections there were.

T: Ok, and how did you do that?

S: I counted 1, 2, 3, 4, 5, 6 (counting sections on the overhead)

T: OK

S: Then I looked where the line was, where the question mark was. So I counted down again, 1,2,3,4 (counting sections on the overhead), 4.

T: Ok. So you counted all the sections that were to the left of that mark.

S: Yes.

T: Ok, great

The above conversation was student-centered. Josh clearly showed and explained his strategy – counting pieces rather than lines. The teacher said few words but simply listened and confirmed his ideas. However, when this student finished his report, the teacher's one-sentence restatement "so you counted all the sections that were to the left of that mark" was actually not a simple rephrase. It was a summary and refinement of this student's approach. It made this student's explanation more explicit for the rest of the class and more generalizable to other questions. The teacher's role in cooperative learning was, therefore, appropriately and sufficiently used (Ding, Li, Piccolo, & Kulm, 2007).

This student continued his work through the second equivalent fraction, during which his explanation was more impressive:

S: Ok. I counted down again. 1, 2, 3, 4, 5, 6, 7, 8, 9 (he suddenly turned to the teacher) and the reason that I counted it down was to find my denominator

T: Great. Thanks for explaining that.

S: And I looked where the mark was and I counted it down, 1, 2, 3, 4, 5, 6. Six. And that's my numerator

T: Ok

S: 6/8 (wrote it down)

When Josh tried to show how he got the second fraction, his explanation was in depth. He not only showed the procedures but also the reason of doing that. He connected the counting processes with fractional parts: the numerator and denominator. This type of explanation further demonstrated this student's clear understanding. Interestingly, since this student explained his counting process twice, he forgot the first counting result "9", and wrote the fraction as "6/8". This mistake confused Jennifer a bit. During the clarification of this mistake, the other students' responses showed their mental engagement:

T: So now we have 4/6 and 6/8. Ok, can you, when you counted the A2, I think I missed that. (For convenience, Jennifer labeled the three fraction strips on the left as A1, A2, and A3 and the other three on the right as B1, B2, and B3.)

S2: (to Josh)You said 9.

T: (to Josh)You said, (to the class) is that what he said? That's what I thought.

S3: Yes, he said 9 but he wrote 8.

S: Oh (changed to 6/9).

T: When you count A2, you counted 9 at the beginning, you might want to change it.

S: Sorry.

T: It's ok.

Clearly, when Josh showed his procedures, the other students were mentally involved. As a result, when Josh made a mistake and the teacher was not sure what this student previously said, the other students provided comments. Compare this sighting with Mary's class where students were entirely lost (see section 4.1.2), the above student comments demonstrated their understanding.

2) A global assessment. Students' clear understanding of "lines or pieces" allowed them to smoothly move from subquestions A and B to C. As a result, they were able to spend enough time on finding patterns and figuring out "why". In the summary part, three students came to the front and shared their patterns. Both additive and multiplicative approaches were offered and elaborated. The underlying reasons of these patterns were discussed. In addition, student's erroneous written explanation 2/3 x 2 = 4/6 received sufficient discussion where peers were greatly involved (elaborated upon later). Such a global view again provided a sense of students' cognitive gains in Jennifer's class.

4.2.2 Why so difficult – Rose

In contrast to Jennifer whose memory of student difficulty did not include "lines or pieces," Rose's response was exactly opposite. My first interview question with Rose was to ask her to recall the biggest challenge or difficulty concerning students' learning of equivalent fractions. What follows is her answer:

> T: What I see with that class is that they tend to count the number of lines instead of spaces. So they may mislabel the lines. Once they understand the spaces as the actual number of pieces, they do not have a problem with it. But initially students do have problems because they count lines.

In fact, my interview question about student difficulty was not limited to Lesson 2.2 but about equivalent fraction in a general sense. Even though, from Rose's perspective, the first challenging one was about "lines or pieces," Rose's answer might truly reflect the situation in the video-taped class. It could also be influenced by her accumulated teaching experiences over the years. Both interpretations reflected that the "line or pieces" issue did bother Rose's teaching. Why was Rose so concerned with this issue and why did her students always have this type of difficulty? Her video-taped lessons provided some answers. In the following parts, I present some related information.

Teacher's imprecise language in the prior lesson. As previously mentioned, the sets of fraction strips in Lesson 2.2 were the transition from region to number line representations of equivalence. As a result, students' difficulty about "lines or pieces" mainly occurred in this lesson rather than the prior one. However, when Rose reviewed fraction strips with students at the beginning of Lesson 2.1, her imprecise language also provides hints why counting "lines" or "pieces" appeared so hard for her students:

T: For this line right here, how did you label it, for the halves, how is it labeled?

S: 1/2.

T: Why did you choose 1/2?

S: (inaudible)

T: It's split in the middle, so it's divided into how many pieces?

S: 2

T: And this line represents how many of those two pieces?

S: (inaudible)

T: It represents how many of those two pieces? This line right here represents how many of those two pieces? How many?

S: 1.

T: Thank you. So this becomes 1/2 because this strip has been split into 2 and this line represents 1 of the 2 pieces. Ok, remember last week when we worked on, what is the numerator, what is the denominator, what do they do, and what do they tell you, ok?

Consistent with interview, the teacher herself knew the importance of counting pieces and she also emphasized the meaning of denominator. However, in the above sighting, she repeatedly asked "this line represents how many of those two pieces". Is this language confusing and misleading? Line is line. How can lines represent pieces? Even though there was a relationship between lines and pieces, (from the first line and second line is the first piece), this type of language is imprecise. Obviously, when the teacher pointed out the middle line and asked students “this line represents how many of those two pieces”, what she actually meant was that there was only one piece out of two pieces from the beginning to this middle line. However, the same idea, when represented by the teacher Jennifer, was said in a different way: “So from 0 to this point here would be 1/2 of this strip.” Comparing two teachers’ verbal representations, it was predictable whose language might cause misconceptions and whose language might contribute more to students’ learning. As pointed out by Rose herself, students usually had difficulty with counting lines and pieces when they learn equivalent fractions. As a result, Rose’s question: “this line represents how many of those two pieces” blurred the difference between “lines” and “pieces”, which might mislead students to count “lines” when they name equivalent fractions in Lesson 2.2.

An obstacle for student exploration in Lesson 2.2. The difficulty about “lines or pieces” did occur in Rose’s Lesson 2.2. Since many students encountered this type of difficulty during the exploration part, Rose was busy with guiding students with “lines or pieces.” As a result, only a few students had time to look for patterns. Moreover, only a few students reported their finding related to the learning goal.

1) Did the teacher address the learning difficulty in the review? Even though Rose said in the interview that counting “lines” or “pieces” was the biggest challenge, she did not address this issue before students’ exploration. In the first 10 minutes, she guided students to (a) recall how they decide 3/4 and 6/8 were both correct in Lesson 2.1, (b) discuss “is 1/4 always equal to 1/4”, and (c) recall the prior identified rule “doubling” (elaborated upon later). In summary, Rose did not mention anything related to "lines or pieces."

2) What did this class do during the exploration? It is not unexpected that many students encountered the difficulty about "lines or pieces" in the exploration part which lasted about 18 minutes. During the first 15 minutes in this period, the teacher spent almost all her time checking students' counting and helping students with the "lines or pieces" issue. However, her guidance was simply telling without explanations.

T: You have 5 pieces, how come?

S: 1, 2, 3, 4, 5. Five.

T: Are you sure it's five pieces? What if you were to draw your line down, would you end up with 5 pieces?

S: No.

T: No, what would you end up with?

S: 6.

T: Be careful when you are doing the lines.

It was possible that Rose had discussed "lines or pieces" with students before, but some students in this class were still confused by this issue. Simply telling students not to count lines may not really help them connect the activity of "counting" with the meaning of fractions. It was also possible that in this exploration period, limited time did not allow this teacher to explain the whys to every student. However, this type of guidance could have been done earlier such as during review. In fact, Rose was knowledgeable of equivalent fractions and she was also aware of this type of students' difficulty. Her guidance, however, was not effective. During the exploration period, this class was mainly counting and labeling rather than looking for patterns. The teacher's most frequent language was: "How did you get 4/6?", "How did you pick up that fraction?", and "Counting it correctly. Check. Ok, Check." The following example was a typical sighting in this period:

T: How did you get the 4 out of 6?

S: 1, 2, 3, 4.

T: 4 pieces until you get to the question mark. And how did you get the 6?

S: (Not sure)

T: How did you get the 6? Daniel help.

S2: What did you count first, the spaces or the lines? …

T: So there's a total of 1, 2, 3, 4, 5, and 6… Right underneath, ok?

Only in the last three minutes of the exploration, three short sightings talking about "pattern" occurred: (a) One student reported the "doubling" strategy, an imprecise representation (discussed in a later section); (b) Rose helped a student with his mistake in the "addition" strategy; and (c) One student checked with Rose whether they really needed to find patterns. In summary no students really identified the patterns for obtaining equivalent fractions as observed from the video. Under this situation, this class entered the summary phase – reporting findings.

A global assessment of students' cognitive gains. The summary part reflects how far these students reached the learning goal - knowing the pattern of finding equivalent fractions and knowing why. This part lasted about 18 minutes. Rose first checked the equivalent fractions that student named on the strips. Then the teacher checked students' patterns. The assessment of student cognitive gains included two aspects:

1) Main patterns were about "addition". Students in this class mainly reported an addition strategy to find equivalent fractions: adding 2 to the numerator and adding 3 to the denominator. In fact, at the beginning of exploration period, there was one student who explained his finding by using the same pattern. The teacher then told him to work on fraction strips first. It might be ironic this class ended up with something that students had already known without much additional information. Even though it is not possible for teachers to help every student in class, the obvious contrast between the beginning and end with regard to that student, in addition to the actual learning goal - finding "multiplication" rather than "addition" pattern - still says something.

Why were so many students in this class stuck on additive thinking? The first possible reason was that the teacher overemphasized on "counting" because of her concern for the "lines or pieces" difficulty. During the exploration period, the teacher spent sufficient time checking students' fractions, asking them how they got their numerators and denominators. As a result, students' experiences of "counting and counting" were accentuated, which in turn reinforced their interpretations that fraction was a special format connecting to two separate counting experiences (D' Ambrosio & Mewborn, 1994; Moss & Case, 1999). Under this situation, students tended to view fractions as two separate parts: the numerator and denominator. This perspective had two possible consequences. On the one hand, students were more likely to pay attention to the ordered numerators such as 2, 4, 6, 8 and the ordered denominators, 3, 6, 9, 12, resulting in the pattern "adding 2" and "adding 3." One the other hand, students who viewed fractions as two separate parts could not easily see the "multiplication" pattern. This is because the "multiplication" pattern required students comparing the numerator and the denominator at the same time. For example, regarding 2/3 and 4/6, students need to compare, how 2 became 4 and meanwhile how 3 become 6. Otherwise, it was not possible for the students to find both the numerator and denominator times 2.

Why students were stuck on additive thinking might also have additional reasons. For example, students might be exhausted when they tried to find patterns because they struggled with labeling fractions, which may result in their superficial thinking. It was also possible that these students had no time to think of more patterns because the teacher asked for the summary at that time. Whatever the possible reasons were, the question is what made these students tend to think in such superficial ways? What made them run out of energy and time when they faced the most important learning content? Are there other possible ways to allow these students to learn more? What if the teacher addressed "lines or pieces" issue ahead of time and in depth like Jennifer?

2) *"Multiplication" pattern without justification.* In Rose's class, there were a few students who mentioned the "multiplication" pattern- multiplying the numerator and denominator by the same number – during the exploration phase. Therefore, at the end of this class, Rose led a discussion about this rule. Students even came out with many unusual fractions formed with huge numbers such as 9303/12404. However, the whole class did not mention why this rule worked at all. During the interview, the teacher said she did not discuss "why" because she would like the students to go home with the "why" question so that they could continue the discussion the next day. However, at the end of the video taped-lesson, the teacher said, "Unfortunately we've run out of time. I would explore more why multiplying the numerator and denominator by the same number and gives you equivalent fraction. Ok? They have to go." Obviously, the class ended up with

only a “rule” but no “why” because of the time issue. However, why could this teacher not touch the learning goal in a 50-mintue long class? In fact, this teacher herself had a good understanding of equivalent fractions as demonstrated in the interview. In addition, she emphasized the “multiplication” pattern (elaborated upon later). Why did Rose not use her knowledge in her teaching and why did her class not work in the way she expected?

4.2.3 Counting lines starting from “0” - Mary

Consistent with the portrait in the section “Where is your whole,” Mary emphasized students’ motivation and attention. Using her words, “you have to know students, otherwise you cannot teach.”

Learning difficulty from Mary’s point of view. Concerning the learning difficulty in Lesson 2.2, Mary’s answer was totally different from that of Jennifer and Rose. She first read through the introduction part: As you worked with your fractions strips, you found that some quantities can be described by several different fractions. In fact, any quantity can be described by an infinite number of different fractions! She then enthusiastically provided me a prolonged feedback:

> T: Ok, well, listen to that, ok, I am a sixth grader, am I familiar with quantity? Am I familiar with infinite numbers? Those words, my kids were really unfamiliar with. Ok? Was it my fault? No, it was not my fault. This was the vocabulary that they want to cover. And lots of vocabulary they used here was vocabulary that my students were not familiar with. So one of the things … before we started any unit, for CMP, my students were required to learn vocabulary. They will use that vocabulary and they were allowed to use those vocabulary

words for each unit … they probably took a vocabulary test to master some of those words. It was only in advance I could be sure that they have understood those words that they have been used to. Ok, equivalent fraction is one of the words that they have to learn. But "quantity" was the word that they were familiar with? "Infinite" was the word that they were familiar with? …

When I asked Mary how she made students understand the difficult vocabulary such as infinite number or quantity, she said she gave them a definition because she had a couple of dictionaries. She said she required students to write those definitions down and to discuss those definition terms and then gave them a test. Concerning the definition of equivalent fraction, Mary said she guided students to discuss it by using cookies. She said she remembered a lesson was making cookies and it was really fun.

Mary's opinion about "number line." Mary called the set of lined up fraction strips a "number line" in the video-taped lessons. During the interview, since Mary did not mention the "number line" as a learning difficulty, I brought it to her attention.

I: Making cookies is definitely more interesting than using number lines

T: Well, it could be fun when you fold fraction strips.

I: Yes

T: Making strips and folding it on lines. They got to be really fun with kids. They enjoy that.

I then described a sighting in her video - many students wrote the fractions in the center of their fraction strips instead of the folds of the lines; Mary had to spend a long time to correct them - Mary said she remembered that.

I: It is hard. There are so many lines. How can students know what fraction they should write?

T: Yes, it is not easy. I am trying to remember how I tried to guide. I remember I guided them how to count.

I: Yes, how did you teach them to count? What is an important thing? There are so many lines.

T: Yes, the difficult thing is to try to get them to remember "0" because they would start with the idea that this is 1, this is 2….Ok? No. this is 0. This is 0, your starting point.

Mary's way of counting lines starting from "0" was not wrong. It could give students a correct answer if students really followed this procedure and paid enough attention to it. However, as Mary mentioned in interview, "the difficult thing is try to get them to remember 0". Why it is hard for students to remember this seemly simple action "starting from 0"? Does the "starting from 0" strategy work well? To what extent were students struggling in these two lessons? The following video-taped portraits provided vivid answers.

"Lines are equal" – problematic language in Lesson 2.1. When teaching equivalent fractions, Mary believed that using fraction strips was sufficient. One of possible reasons is that, she thought fraction strips were lots of fun and students' learning "got to be really fun". In lesson 2.1, after Mary directly told students that both 3/4 and 6/8 were correct answers (see section 4.1.2), she guided students to see why these two fractions were equivalent by using fraction strips.

T: If I take my fourths and eighths, 3/4 of the way and 6/8 of the way, folding the same what? Have you notice that? If you look, they're lines are what?

S: equal

T: Equal. Ok? They are lines that are equivalent. They're equal. Does everyone see when I put them one under the other, or put one on the top of the other? Did you see that?

S: (No response)

T: I need to make sure that everyone notices that.

Mary attempted to point out that 3/4 and 6/8 were lined up on the fraction strips. Using her words in another context, 3/4 and 6/8 were folded in the same place. She guided students to compare the two lines, "Their lines are what?" Based on one student's answer "equal", Mary then repeated her statement "the lines are equivalent, that the lines are equal." This short sighting includes two problems with Mary's instruction. First, Mary misled student to pay attention to "lines" instead of "pieces" when they compare fractions. Focusing on lines was not wrong because these lines did line up with each other. However, it might cause student confusion in counting lines or pieces when they work on fraction strips. Moreover, it reflected the incomplete understanding of the definition of equivalent fractions, which, in sixth grade, based on "part-whole" relationship, meant "the same amount/quantity of a whole" (Lappan et al., 1998, p.20). As a result, which one was more accurate when representing the "amount", "lines", or "pieces?" The second problem with Mary's instruction in the above sighting was her imprecise language. What does it mean "the lines are equivalent?" Does it mean the two lines themselves have the same length? Since the language is confusing and misleading, when Mary asked student whether they saw that "equivalent lines," there was no response. Even Mary stated that she wanted to make sure every one understands, it was doubtful whether she was able to accomplish that with her imprecise mathematical language.

Misguidance in Lesson 2.2. Mary taught Lesson 2.2 right after Lesson 2.1 with a 5-minute break. The second Lesson only lasted 24 minutes. After she cleared the vocabulary such as "infinite", "Egyptian fractions", and "equivalent fractions" in the launch part of CMP textbook, she directly moved into Problem 2.2. What follows is her guidance before student exploration:

> T: First of all, you need to work and find out which strip is what? Ok, and how can you do that?
>
> S: I found out you can count the number of lines and that's …
>
> T: Jane said, what, say it loud Jane.
>
> S: Count the marks. That's the number you get, that's the numerator.
>
> T: The, the denominator.
>
> S: The denominator.
>
> T: The denominator, very good. If you count the lines on your strips that will tell you what your denominator is, ok?

Clearly, Mary asked students to count lines. She also emphasized that what they counted would be their denominator. Did this guidance work well? Could the students smoothly work with these fraction strips and then move to find patterns and figure out why? The following section shows students' corresponding responses.

Students' corresponding responses. After the teacher's guidance, students started their investigation. Mary also started helping those students who were confused. To show the commonality and frequency of students' confusion, I provide 4 sightings with the start time.

Sighting 1: (1:04:06-)

T: So you have that, this is two thirds (pointing at the first one on the textbook), what is this fraction strip going to be (pointing at the second one)?

S: um?

T: 1, 2, 3, 4, 5, 6 (counting for student) so? 6

S: 6.

T: Your next strip is what?

S: 6.

T: Ok, a sixths strip.

T: So what line is this?

S: Fourths.

T: You have to work with your partner and figure it out. Work together with your partner.

In this sighting, the student initially had no idea about counting. Teacher Mary counted for her, 1, 2, 3, 4, 5, 6 and told her that was 6. As a result, her simple question "your next strip is what" got a correct answer "sixths". Mary then suggested this student work with partner and figure out these labels.

Sighting 2 (1:05:18 -)

S: So I count from the first line?

T: No, the first one is 0

S: So 1, 2, 3, 4, 5, 6. So something is 6.

T: You need to be able to name each of these question marks, OK?

This student was not sure where he should count from. His conjecture “I count from the first line” was reasonable because if he counts, the first “object” would be naturally counted as “1”. However, when Mary told this student “this first one is 0”, will this statement contradict with a six grader’s common sense? If students were supposed to count and find the answer of “how many”, why is the first line counted as “0”? It was obviously there! Luckily, this student based on the teacher’s procedure: count all lines from “0” – find out the question mark – count again from “0”, he successfully got the correct answer.

Sighting 3 (1:05:37-)

S: I am missing.

T: Hum? You need to be able to name what each one of these fraction strips are, Ok? This one is 2/3, Ok? So this is how many? Let’s count. 1,

S: 2, 3, 4, 5, 6

T: Ok, so this is what kind of strip?

S: Sixths.

T: A sixths strip and what line is here? What would this fraction be?

S: 5

T: 5/6? Let’s count. 1, 2, 3… Are you sure 5?

S: 4

T: 4/6. It is equivalent to what?

S: 2/3

T: So you need to put down that this is sixths strip, and you need to find out what this strip is and what this strip is.

In this sighting, the student reported she was missing. Mary suggested counting. She counted “1” first – the hardest one - and then left the other “lines” for this student to count. As a six grader, students surly had the counting ability. She counted all lines correctly and answered the teacher’s obvious question “so this is what kind of strip” correctly. However, since Mary counted “1” from the second line for this student, did this student really understand that secret? Did this student really know where to count from and what to start with? Obviously, she did not know. When this student again counted for the numerator, she made a mistake because she counted the first line as “1” and got “5”. Mary’s again counted “1, 2, 3” and left the other lines for this student. Mary’s question “Are you sure 5?” was therefore easily answered.

Sighting 4 (1:07:40-)

T: Did you see what you’re doing?

S: Kind of.

T: Ok? This strip here is, a thirds strip. Ok? 1, 2, 3. So this is a thirds strip, you count the lines, ok? Now we want to find strips that are equal to 2/3, that’s why they have a question mark right under all of these because everything here will be equivalent, is equal to two thirds. Let’s find out what this strip is, count.

S: Count, from this (pointing at the first line)?

T: Yes.

S: (no response)

T: Count!

S: 1,

T: No, you always start here, this is 0.

T: So this is what? This is 1, 2, 3, 4, 5, 6 (counting with the student), ok? So this would be what?

S: 6

T: This is sixths strip and 2/3 is equal to what?

S: 4/6

T: Ok, 2/3 is equal to 4/6.

T: Ok, now you need to count the same way for these.

In the above sighting, the student had no idea about naming equivalent fractions on these strips. Again, Mary suggested counting. This student pointed at the first line and asked "Count, from this?" Mary agreed. However, which number should this student start with, 0 or 1? This student stopped. "Count!", Mary urged. The student then started with "1", a reasonable way as previously mentioned. "No, you always started here, this is 0." Mary's comments might confuse this student who possibly had these following thoughts: "The teacher just agreed with starting from here, but she says 'no' now. She said this first line was "0" but why not '1'?" At last, Mary counted all the six lines for the student who, therefore, got the correct answer "6" instead of "7".

The above sightings clearly showed the effects of "counting from 0." Students in this class just followed Mary's procedures without understanding. This was because the teacher Mary did not tell them why the first line should be counted as "0". As a result, students in Mary's class were seriously struggling with the "lines or pieces" issue. In addition, even if this strategy did work well, it would not allow these students to understand the meanings of fractions and equivalent fractions. Therefore, even though students with this strategy got correct answers, they might only reach a procedural understanding.

A Global assessment of students' cognitive gains. It could be easy to imagine what students really learned in this class. Since this class was full of "counting" and especially the teacher's counting, what students really did was to repeat the teacher's counting results and write the "given" fraction on their textbook. Concerning the learning goal of finding equivalent fractions, one student, at the last two minutes, correctly reported his "addition" pattern, which, however, was transferred into a mathematical mistake by the teacher (elaborated upon later). Another student told the teacher he multiplied numbers. Mary simply said, "You multiply? Oh! You write there!" and she then started announcing homework.

4.2.4 Summary

Regarding the same difficulty "line or pieces", the three teachers Jennifer, Rose, and Mary responded to it differently. Jennifer emphasized counting "pieces" and connected it to the concept fraction. Her extension of the "lines" changed the number line-oriented representation back into regional representation, a familiar model for most of those students. As a result, students in her class easily named the fractions and quickly moved into the exploration concerning the learning goal. This class spent sufficient time looking for patterns, explaining ideas, addressing mistakes, and discovering underlying mathematical reasons. Rose, a teacher with adequate mathematical knowledge and understanding of students, however, did not address this issue sufficiently. Even though she knew the student difficulty of “lines and pieces” and guided students to count spaces in her teaching, she did not think that “lines” and “pieces” made a big difference. If student could get correct answers, she thought both strategies would be fine. One evidence was her imprecise language of “this line represents of how many pieces”. The other evidence was her comments to Mary’s strategy of counting lines for “0” in the later interview. Rose said “counting lines” was fine because students could also get correct answers. Regarding Rose’s enacted teaching, since she did not address student difficulty ahead of time and in depth like Jennifer, she spent lots of time guiding students to label the fraction strips. She requested students show her the process of getting fractions. As a result, students in this class mainly experienced “counting” which was a low level of thinking, causing students’ superficial findings of “addition” pattern – and the lack of time to explore the underlying reasons for “multiplication” pattern. In contrast to both Jennifer and Rose, Mary emphasized the fun in mathematical learning. She noticed students’ difficulties in labeling fraction strips. As a result, she guided students to count lines but started from “0” rather than “1”. However, her strategy did not work well because some students were confused why the first line was not “1”. In addition, even though this strategy did work well in Mary’s other classes, “counting lines” was still misleading because it deemphasized the importance of the concept fraction in learning equivalent fractions, resulting in the disconnection between concrete representations and mathematics notations. In Mary’s video tapes, most students were confused by the issue of “lines or pieces.” With the teacher’s guidance, students were mainly requested to count, which was probably a first or second grade level activity.

4.3 Are You Really Doubling?

As previously analyzed (see section 3.3.3), one of the common errors in students learning was the "Doubling Error". This error occurred in both Lessons 2.1 and 2.2. When students tried to explain why 3/4 = 6/8, they made mistakes such as "$\frac{3}{4}\times 2=\frac{6}{8}$", "multiplying by 2", or "doubling 3/4". These types of errors also reflected students' learning difficulties. Additionally, it reflected students' superficial understanding of the "multiplication rule" and the underlying reasons in finding equivalent fractions. How teachers viewed and responded to this type of error and difficulty related to teachers' *Mathematical Knowledge for Teaching* (MKT). In this section, I first report teachers' knowledge related to this topic. I then provide teachers' effective strategies of addressing this type of error and difficulty. At last, I point out some issues concerning teachers' mathematical language. Since all teachers are involved in this section, for clarity, I provide Table 4 showing the specific teachers used in each sub-theme.

Table 4

Sub-theme and teachers in "Are you really doubling"

Section	Sub-theme	Teacher
4.3.1	Teacher knowledge	All
4.3.2	Addressing in depth	Jennifer/Barbara
4.3.3	Transiting between alternative representations	Kathleen/Lisa
4.3.4	Making connections with division	Kathleen
4.3.5	Issue of teacher Language	Rose/Mary

4.3.1 Teacher knowledge related to "Doubling Error"

Teacher responses to the designed case and T or F questions. Teachers' knowledge related to the "Doubling Error" was mainly reflected by their responses to the interview materials: the designed case and T or F questions (2) and (6). The related teaching tapes were also discussed and confirmed through conversations with these teachers. These evidences provide a general sense of teachers' knowledge in this study.

All teachers except Mary thought the student invented strategy $\frac{3}{4}\times 2=\frac{6}{8}$ in the designed case was definitely wrong because they believed it should be $\frac{3}{4}\times\frac{2}{2}=\frac{6}{8}$. These teachers pointed out 3/4×2 was equal to 6/4 rather than 6/8. They also pointed out that "2/2 =1" and "3/4 times one whole will not change the value". In fact, many teachers even mentioned the "identity property of multiplication". Similarly, they view the T or F (2) and (6) were wrong because of the same reason. In general, teachers showed good understanding of this issue.

Mary also knew $\frac{3}{4}\times\frac{2}{2}=\frac{6}{8}$. The only difference between her and the other teachers was that she thought $\frac{3}{4}\times 2=\frac{6}{8}$ was acceptable if students could explain what they meant. When I showed the designed case and asked her opinion, she said she would definitely praise students for the efforts they made on this method. "You could not tell them they were wrong because in many cases, they can do this," Mary argued. She also said if students could explain, she would allow them to share their strategies. What follows is part of our conversation during my second interview with Mary:

I: Some kids may say 3 times 2 is 6; and 4 times 2 is 8.

T: We always tell students we have to multiply by a whole number, the whole number has to equal “1”. So that’s why I would have 2/2. Remember, when the numerator and denominator are same, the fraction is equal to one whole. So what is your whole number here? The whole number would be 2 and when you break it down, it will be 2/2.

I: So students show you this strategy $\frac{3}{4}\times 2=\frac{6}{8}$, and then you ask them to explain. If they say this (pointing at “×2”) actually means 2/2, will you say this (pointing at “$\frac{3}{4}\times 2=\frac{6}{8}$”) is wrong?

T: No, this would be right. They know this has to be a whole number, the whole number has to be equal to 1, and then make fraction times fraction, so,

I: Same thing?

T: Same exact thing.

Mary would accept the invented strategy $\frac{3}{4}\times 2=\frac{6}{8}$ because of two reasons: (a) concern of student efforts or motivation, and (b) she was sort of confused with “whole number” and “one whole”. She viewed “×2” as the same thing as “×2/2”. This type of confusion was also found in her response to other interview questions such as T or F question (8) (elaborated in later sections). Similarly, Mary thought the T or F question (2) and (6) were both correct if a student could explain what they really meant.

Teacher analysis of student “Doubling Error”. Teachers’ understanding of the resources and the negative influence of this type of error partially demonstrated their knowledge of this topic. Through the interviews, teachers’ feedback showed their good understanding when the issue was purposely brought to their attention. In this section, I will only provide a general picture of teachers' analyses of "Doubling Error". Their detailed descriptions and analyses will be provided together with their teaching behaviors later. Since Mary did not view "Doubling Error" as a problem, she was not involved in the following part.

Concerning $\frac{3}{4}\times 2=\frac{6}{8}$, all the five teachers thought this error reflected that students did not truly understand $\frac{3}{4}\times\frac{2}{2}=\frac{6}{8}$. For example, Rose said students only got the rule - multiplying by the numerator and denominator by the same number but they did not know the whys behind the rule. They did not know why "× 2/2" would yield an equivalent fraction. Therefore, students tended to make errors like "× 2" because they did not see the difference. As a result, these teachers said they would guide their students to know that they were not multiplying by 2 but one whole. They would also point out the identity property of multiplication. Rose said even students knew 2/2 was one whole, it was still hard for them to understand 3/4 times one whole equal to 6/8 because the one whole was not really like "1" and 3/4 and 6/8 looked different. Therefore, Kathleen and Lisa said students also needed to see why $\frac{3}{4}\times\frac{2}{2}=\frac{6}{8}$ through pictures. These teachers believed that if they did not address this type of error, when students went to 7^{th} grade, they would be confused: "In sixth grade we did 'multiply by 2' and got equivalent fractions ($\frac{3}{4}\times 2=\frac{6}{8}$), but now we do this and get something different ($\frac{3}{4}\times 2=\frac{6}{4}$)".

Kathleen and Barbara pointed out the second explanation. That is, students might be influenced by whole number thinking. For example, students viewed "× 2/2" as the same thing as "× 2". This type of error might reinforce some misconceptions such as "6/8 was bigger than 3/4" because "multiplying by 2 provides a bigger number" did make sense to students. These two teachers mentioned that some of their students (not in the video-taped lessons in this study) thought in this way. In fact, even the teacher Rose made similar mistakes in both her video-taped lessons and the interview. During her teaching, she discussed student findings of equivalent fractions. She asked: "Did someone get something else? Something larger?" During the interview, she said when students continued "doubling", they would get bigger numbers. The "doubling" was Rose's imprecise language that actually meant "multiplying the numerator and denominator by 2" (discussed in section 4.3.5). Concerning the "bigger/smaller equivalent fraction" misconception (Hart, 1981; Post et al., 1984), I also found this phenomenon in another MSMP teacher's class whose video-tapes were not used in this study. In that class, since the teacher himself repeated "multiply to get a bigger fraction" and "simplify to get a smaller fraction", students also used the same terminology. Since "the bigger/smaller equivalent fractions" error did not occur frequently in the video-taped lessons in this study, I will not describe them in detail.

Regarding the source of $\frac{3}{4} \times 2 = \frac{6}{8}$, Rose provided the third explanation which was insightful. She said "I don't believe students truly understand the "=" sign either. At this age, it means 'the answer'. It is necessary to teach them that the values of the numbers on each side of the '=' sign are the same." When I asked Rose whether she had used the "="sign to address this type of error, she said, "No. With the fractions, I'm not sure that I have. Oh, that is a good idea though, I'll keep in my head." She further explained, since 3/4 and 6/8 were the same value, next time she would begin with the discussion of the "=" sign. She might ask her students how one side related to the other and whether they could show her the same value. She said when students found the left side was 6/4 rather than 6/8, they would go back to the identity property of 1. Rose's identification of the reason concerning the "equal sign" is meaningful because the "=" is a difficult concept (Ding, Li, Capraro, & Caprao, 2007) but a critical one that matters students' algebra learning (Knuth, Stephens, McNeil, & Alibali, 2006). Similarly, Lisa also mentioned that this type of error could be obstacles for students' later learning of algebra.

In general, when the "Doubling Error" was brought to teachers' attention in the interviews, they showed understanding of this issue. Except for teacher Mary who cared much more about student motivation and was somewhat unclear about the "whole number" and "one whole", all the other teachers stated that they would address this issue in various ways. As a result, how did teachers actually address this type of error and difficulty in the video-taped lessons? Did teachers' knowledge really contribute to their classroom instruction which in turn affected students' learning? What were effective strategies or the issues reflected in the video-taped lessons? These questions are answered in the following sections.

4.3.2 Addressing "Doubling Error" in depth

Consistent with their interviews, Jennifer and Barbara capitalized on student errors - both verbal and written formats – and inquired in a deep way.

Jennifer's class. As introduced in last section, Jennifer used cooperative learning employing her role appropriately. In both Lessons 2.1 and 2.2, she reviewed students' prior knowledge and addressed their learning difficulties: "what is your goal" and "lines or pieces." Therefore, students easily transitioned toward the learning goal. Jennifer walked around the classroom and checked students' thinking. She asked students to explain their ideas. In the summary part, she also asked students to present their strategies in the front of the class utilizing the overhead or chalk board. As a result, students had sufficient opportunities to express their strategies to the class. Since Jennifer had a good understanding of equivalent fractions, when students made the "Doubling Error," she was always able to quickly recognize it. Interestingly, she did not sufficiently discuss a student's error in Lesson 2.1. As a result, the same student made the same error in Lesson 2.2 where she successfully addressed this error in depth.

1) Mistake in Lesson 2.1. During the group exploration, Jennifer stopped by one group and checked students' work: One student Daniel told her both Mr. Park's 6/8 and Ms. Mendoza's ¾ were correct. He said, "Mr. Mendoza's 3/4 is right. Ms. Mendoza's is simplified form of 6/8." Jennifer grasped Daniel's word "simplified" and further asked:

T: Aha, what do you mean by simplified? Explain to me what you mean by that?

S: 6/8 divided by 2 is ¾.

T: Divided by 2?

S: Yes, like breaking it down.

T: Can you show me that on paper?

(This student was writing something on his paper. Jennifer was watching and waiting. The other two students in this group were playing with their pencils and they suddenly started discussing the "playing." Jennifer noticed that and stared at them for a while without their realization. At this time, Daniel finished his work on the paper.)

S: 6 divided by 2 is 3 and 8 divided by 2 is 4.

T: Cool, thank you for showing that to me.

In this sighting, Jennifer quickly grasped Daniels' verbal mistake "6/8 divided by 2 is ¾." She then asked him to show her what it meant "divided by 2" which included two possible interpretations: (a) Daniel meant the whole fraction 6/8 divided by 2; or (b) Daniel meant both the numerator and denominator divided by 2. If it turned out to be the first situation, Jennifer would definitely point it out because it would be incorrect. However, Daniel's explanation matched the second situation. Jennifer watched what he wrote on the paper – it should be correct– she praised him and moved ahead. Put another way, when Jennifer was sure this student's "dividing by 2" meant "both the numerator and denominator dividing by 2," she did not explicitly point out the verbal mistake - at least an imprecise verbal representation - for Daniel.

This video-clip was sent to Jennifer for interview. When we talked about this sighting, Jennifer laughed and told me, when she watched the video at this time, she was even more frustrated by the other two boys who were playing with their pencils in that group. She said in this sighting, she paid more attention to the other two boys instead of Daniel who made the mistake but corrected himself. Considering the particular context, it was understandable Jennifer did not address Daniel's mistake further because she was frustrated at that time and also because Daniel obviously explained it correctly.

However, did Daniel really "correct" his mistake? Could it be Daniel only tried another way to explain his idea? Since Jennifer did not explicitly point out "dividing by 2" was a verbal mistake, Daniel may never realize what was inappropriate in this representation. In addition, since the teacher also agreed with Daniel's explanation of the verbal mistake, Daniel might misinterpret that both representations were correct, or meant the same thing. As a result, could this insufficient discussion influence Daniel's later learning?

2) The same mistake. Jennifer's lesson moved into Lesson 2.2. Interestingly, during the summary part, the same student, Daniel, volunteered to report his strategy concerning how he found the equivalent fractions for 2/3. He went to the board and told Jennifer: "I use 2/3 and I multiply by 2." He then wrote down what he thought on the board: 2/3 ×2=4/6, a similar mistake as made in Lesson 2.1 as discussed above (see Figure 16).

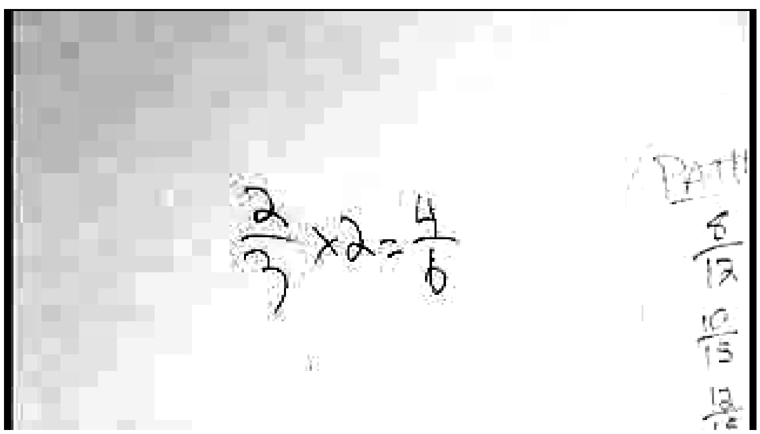

Figure 16. The "Doubling Error" in Jennifer's class.

Jennifer quickly recognized this error and said she had a question. Daniel immediately elaborated his idea: "2 times 2 to get 4" and "2 times 3 to get 6". It seemed to this boy, "$\frac{2}{3}\times 2$" and "$\frac{2}{3}\times\frac{2}{2}$" represented the same thing.

3) Dealing with the mistake –"10×2=20". Encountering the obvious mistake on the board, Jennifer then started her guidance:

T: Ok, now whenever we multiply like it, say 10 times 2, what does that equal?

S: 20

T: Now, what's, there is a word that says what I just did when we multiply 10 times 2, does anybody know the word that I am talking about that we use to describe when we multiply a number by 2? What are we doing to that number? 10 times 2 equal 20, what did I do to that number just now?

S2: Doubling

T: Doubling it, Ok?

T: My question is, when you multiply this by 2, did you just double that fraction?

S: Yes.

T: You did? Ok, what do we call these two? 2/3 and 4/6?

S: Equivalent fractions.

T: So I want you to think about that, he says we are doubling it, but he is calling them equivalent, are 10 and 20 equivalent?

S2: No.

T: 10 and 20, whole numbers. Not sure?

S: Not sure.

Jennifer brought out the whole number example "10×2=20" and tried to help her students make the connection between fraction and whole number operations. As mentioned in the later interview, Jennifer's purpose of using the "whole number" example was to connect this complex and new concept with what students were already familiar with:

T: Well, I think that makes it more concrete to them because they really understand the whole number operation. I think that is the best mathematical way to explain to them, you know. Because they have learned multiplying numbers in third grade, they have mastered that skill, so to use something that they have already mastered, to relate to the new concept. I think that is definitely important.

Jennifer wanted Daniel and other students to see the fact - when 10 was doubled, it would be 20, 10 and 20, however, were not equivalent. Jennifer grasped the basic mathematical concept "equivalence" and tried to guide students to see this conflict. Through relating to something that was concrete and mastered by students, Jennifer expected her students to make connections and realize why "$2/3 \times 2 = 4/6$" was not correct.

In fact, this type of strategy - connecting fraction to whole number - was also mentioned by another teacher Lisa who viewed the "whole number" as "foundation." Lisa described how she corrected students' mistakes such as $¾ \times 2=6/8$ in the interview (not observed from her video tapes). Lisa said she guided students to see it was not times "2 wholes" but times "2/2" which equals 1. What follows are her thoughts:

T: And we spend a lot of time about how 2/2 equals to 1, well, when I take 6 and multiply by 1, what do I get? I get 6 (wrote down: $6\times1=6$); when I take 4 and multiply by 1, what do I get? I get 4. (wrote down $4\times1=4$); when I take 5 and multiply by 1, what do I get? I get 5 (wrote down $5\times1=5$). Are they equivalent? Well, the answer is equivalent. What do I start with? I start with 6 and I end with 6. Ok, I take 3/4 and I multiply a number or a fraction that equals 1. What do I get? I tell you a number that is equivalent to ¾. …They can tell me, ok, if we multiply this by 2/2, that 2/2 equals to 1, just like I am doing 3/4 times 1, is just like doing 6 times 1, 4 times 1 and 5 times 1, and you get 6/8,

> so you know what you end up with here, or you are saying it is equivalent to what you started with.

To address the "Doubling Error" in equivalent fractions, both Jennifer and Lisa connected fraction with the whole number operation. The difference is, Jennifer tried to guide students to see why "double" was wrong through the example of "10×2=20" while Lisa tried to guide students to see why "×2/2 /× 1" was correct through the example of "6×1=6, 4×1=4, 5×1=5". Even though the focus looked different, Jennifer and Lisa both wanted their students to understand the same basic idea through the whole number operation, that is, every number times 1 (rather than "double") will not change the value, which is exactly the identity property of multiplication.

When Jennifer pointed out the whole number example "10×2=20" to illustrate the problem of the "Doubling Error," did her students really accept it? Daniel, still standing in front of the board, said he was not sure. At the same time, some other students who were sitting down raised their hands. Facing Daniel's "not sure", Jennifer did not stay with or repeat her "10×2=20" example. Instead, she told Daniel to continue showing his work. As a result, Daniel wrote down "2/3 × 3 = 6/9" and he said he would like to keep doing this by multiplying 2, 3, 4, 5, and so on.

4) Why 2/3 ×2 does not work - Using peer resources. Jennifer praised Daniel's efforts. She, however, did not let this error slide. She also did not draw this student back to the "whole number" idea or just depended on her own efforts. Instead, she switched the conversation flow by using peer resources to help Daniel:

T: Ok, what I want you to do is just turn around. And some of these guys are raising their hands to give you a comment or some advise if you want to call somebody and hear what they have to say.

Daniel, called upon someone and this student pointed out that it was not exactly 2/3 times 2 but times 2/2. He said he did not know what 2/3 ×2 would be, but it would not be 4/6. One interpretation here is that this student probably benefited from Jennifer's whole number example "10×2=20" during which Jennifer pointed out the idea "Doubling changes the value." As a result, even though this student did not know what 2/3 ×2 would be, he could still tell it would not be 4/6, which demonstrated his understanding of the "10×2=20" example. After several other students' comments, Daniel told Jennifer that he understood what the other students were talking about. He corrected his work on the board. Jennifer continued probing students' thinking asking why they should multiply by 2/2 instead of 2. Another student said if multiply by 2, it would be 4/3, an improper fraction, changing the value of 2/3.

5) Why 2/3 ×2/2 did work- Recapitalizing the role of the teacher. Based on students' discussion of why "2/3 ×2" did not work, Jennifer pointed students toward another direction: why "2/3 ×2/2" did work:

T: Ok, so here why is multiplying it by 2/2 ok in order to first get this equivalent fraction here?

S: 2/2 = 1

T: Ok, if we look at this 2/2, something that kind of like to do, just keep in my mind and as we did up here, we said, 2/3, oh 3/3, I am sorry, equals 1. What I kind of do sometimes is, for, when people are just learning this, say two halves, and then I put a little "1" around it to say "yes, it is twos, there are two twos there, but you are really multiplying this by "1" (Circled 2/2 with the shape of "1") Do you see how that 1, my beautiful drawing there? (see Figure17) (Daniel laughed).

T: Ok, so 2/3 is really multiplying by 1, so this value (circled 4/6) is still the same as which value, Daniel? The 4/6's value is still the same value as the?

S: the 2/3's value.

Based on Daniel's answer, Jennifer drew an arrow connecting 4/6 and 2/3. (see Figure17)

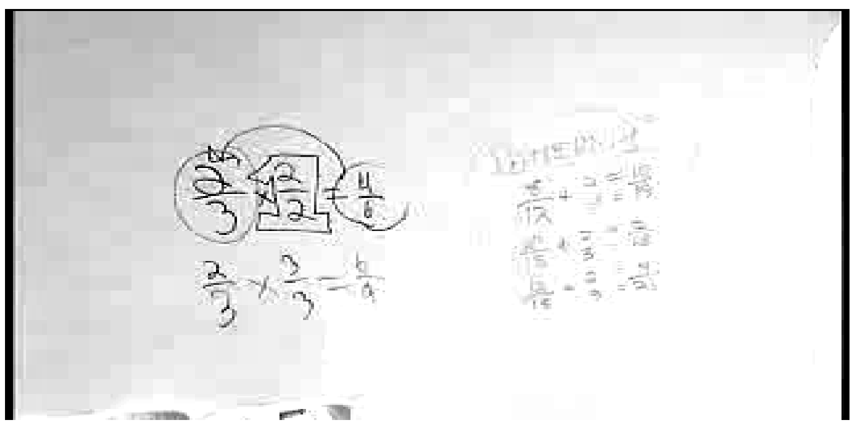

Figure 17. Jennifer's beautiful drawing.

5) Summary. Jennifer's class showed the effects of teacher instruction on student learning. It provided two insights concerning teacher's responses to student errors. First, when students made mathematical errors that were worthy of notice, teachers should not only check students' understanding but also make the errors explicit. For example, when Daniel in the first class said "dividing by 2", a verbal mistake similar to "÷2", Jennifer recognized it and checked his thinking. When the student correctly explained his idea on the paper, the teacher Jennifer just let it slide without pointing out "dividing by 2" was not accurate. As a result, when this student tried to report his idea in Lesson 2.2, he made a similar mistake. Therefore, when teachers try to address a student mistake, they should make the mistake overt. Otherwise, their instructional efforts might fail.

The second insight from Jennifer's class was the way to address student errors. Jennifer in Lesson 2.2 successfully capitalized on a student's error and addressed it in depth. She first connected the fractional error with whole numbers because she believed students had mastered whole number operations. However, Daniel was still not sure. One possible reason was that he was standing in the front and he might be somewhat uncomfortable. Another possible reason was that the fraction was abstract. It was not easy for this student to make connections between whole numbers and fractions, a point that was particularly argued by teacher Rose. As a result, Jennifer asked for peer contributions because there were some students who seemed to understand and raised their hands. It was possible that sometimes students might make ideas more understandable for their peers. Based on students' discussions about why it did not work – if you "double" a number, its value will change - Daniel experienced a cognitive change and he corrected his own mistake. At the end of the class, Jennifer recapitalized her role as a teacher by switching back the conversation flow. She humorously discussed why 3/4 ×2/2 did work by pointing out the basic idea "2/2=1" and the identity property of multiplication. As a result, the "Doubling Error", as a learning opportunity provided students various directions to make connections among mathematical ideas.

Barbara's class. Barbara's teaching style was similar to that of Jennifer. As previously discussed (see section 4.1.1), Barbara knew the learning difficulty and the critical prior knowledge piece for the exploration. In Lesson 2.1, her review question "does 1/4 represent the same amount of your fraction strip" directly addressed the students' learning obstacle. As a result, students in this class also had enough time to investigate the learning goal and explain their ideas to the teacher. Since Barbara had a firm understanding of equivalent fractions, she had clear expectations for student learning.

1) Are you really doubling? In Lesson 2.1, a group of students reported their findings to Barbara. One of students made a verbal mistake. Therefore, Barbara capitalized on this error insisting on asking them "Are your really doubling?" and "What are you really doing".

S: 6/8 equals 3/4.

T: How do you know that?

S1: To double it.

S2: 3 times 2 is 6.

T: You're not doubling, what are you doing?

S: 3 times 2 is 6, and 4 times 2 is 8.

T: But is that exactly multiplying a number by 2?

S: That's common denominator.

T: Wait, wait and think. Listen to my questions before you give me an answer. He says he's multiplying by 2. Are you really multiplying 3/4 by 2?

S: No. If you multiply 3/4 by 2, you get 1.5.

T: Right. So what are you really doing? Because you are right when you say you multiply by the three and multiply by the four. So what are you doing?

S: 3/4 is 6/8.

T: I know, but we are trying to figure out why.

S: You're just changing the numbers. It's the same fractions.

T: Yes, it's an equivalent fraction. You're right. But how are we changing the numbers? Think about it. I'll come back, ok? Talk to your table. Because you are not really doubling, I want you to think about what you are doing.

In this example, Barbara interacted with the whole group. Although she knew students' "Doubling Error" actually meant "multiplying both the numerator and denominator by 2," she did not let the error slide. She explained the "doubling" to these students as "multiplying by 2" and she asked them "Are you really multiplying 3/4 by 2?" Under her guidance, one student in the group discovered, if 3/4 was multiplied by 2, it would be 1.5, not the same value. Until this point in time - students knew why 3/4 ×2 was wrong - Barbara did not stop prompting students. She was persistent in engaging students in inquiry-based questioning and required the whole group to discuss and figure out what they were really doing. Apparently, Barbara had clear expectations for students. In this sighting, she expected them to find out that 3/4 was multiplying by 2/2 which was equal to 1 rather than 2. This type of expectation reflected this teacher's understanding and her attempt to address the "Doubling Error" in depth. Due to the limited class time, Barbara did not really return to this group as she said. As a result, it was necessary to see whether Barbara had the same expectation during her later teaching and how Barbara guided students to see what she expected in Lesson 2.2.

2) Interpret students' thinking based on expectation. Lesson 2.2 in Barbara's class went smoothly too. Students moved from labeling strips to looking for patterns very quickly. Barbara walked around the classroom and checked students' thinking during which she paid sufficient attention to students' verbal representations. There were several sightings where students explained their pattern as "times 2". Barbara then guided them "if times 2, that's 2/1". As a result, students were able to quickly realize that it should be 2/2. According to the interview, Barbara believed "times 2" demonstrated students superficial understanding of "why times 2/2". Additionally, "times 2" may reinforce students' misconceptions of "bigger equivalent fractions" as previously mentioned. Therefore, Barbara kept the "2/2" expectation in her mind and tried to ensure that students possess correct mathematical understanding.

However, to interpret students' thinking is not easy. Sometimes, students think in their own idiosyncratic ways which may greatly differ from what teachers expect. As a result, situations could arise where teachers misinterpret students' thinking and identify some mistakes that were actually correct. In the following example, one student got several equivalent fractions such as 8/12 and 16/24 for 2/3 in Lesson 2.2 follow-up. As a result, Barbara checked how this student got 16/24 and pointed out his "error":

T: How did you get that one? The 16/24.

S: I times 8 by 2 and I got 16; and then I times 8 by 3, I got 24. That will be 2/3.

If written down, this student's explanation would be $\frac{8\times 2}{8\times 3}=\frac{16}{24}$. This student multiplied the numerator and denominator of 2/3 both by "8". In another words, to get 16/24, he used 2/3 times 8/8. However, this regular order - 2/3 and 8/8 - was switched by this student. As a result, he used $\frac{8\times 2}{8\times 3}=\frac{16}{24}$, which might not be consistent with what the teacher expected: $\frac{2\times 8}{3\times 8}=\frac{16}{24}$.

When the student finished his explanation, he looked at the teacher and waited for her response. Barbara looked at the student's book for a while and said nothing. Then she asked: "Did you multiply them by the same thing?" Obviously, Barbara was expecting something like $\frac{2\times 8}{3\times 8}$, a format consistent with a classic description: Multiplying the numerator and the denominator by the same number. In this case, it was $\frac{\times 8}{\times 8}$. When the student said it backwards like $\frac{8\times 2}{8\times 3}$, and if the teacher only paid attention to the later part $\frac{\times 2}{\times 3}$, she might interpret that this student multiplied the numerator by 2 and the denominator by 3, an absolute mistake. As a result, Barbara raised the above question: "Did you multiply them by the same thing?"

This student was confused and looked at his book. He might keep thinking: "what did this teacher mean? I times 8/8 where 8 and 8 are obviously the same thing. But why did the teacher ask me this question? Where am I wrong?" At this time, the teacher provided the answer which was assumed to be the correct for the student. "You multiply by 2, then 8 times 2 is 16 and 12 times 2 is 24". If writing down this teacher's description, it would be $\frac{8\times 2}{12\times 2}=\frac{16}{24}$. Barbara might also experience a quick and hard process of constructing the understanding of this student's thinking. She finally figured out that this student used "8/12 times 2/2". Barbara was presented with three types of possible evidence: (a) This student said "times 2 … times 3"; (b) This students said "times 8"; and (c) This student wrote 8/12 before 16/24 on his textbook. Based on this evidence, Barbara conjectured that this student tried to say $\frac{8\times 2}{12\times 2}=\frac{16}{24}$ instead of $\frac{8\times 2}{12\times 3}=\frac{16}{24}$, a slip of the tongue.

T: You multiply them both by 2, 8 times 2 is 16, 12 times 2 is 24. You said 3, but is 12 times 3 is 24?

S: (Looked at the teacher and said nothing)

T: You wrote the right thing but you just said it differently. Ok?

S: Ok (Appeared confusing)

T: Ok, don't worry (Left).

This dramatic sighting shows that Barbara cared a lot about students' mathematical understanding including verbal representation. She always emphasized the point of "multiplying the numerator and denominator by the same number." As a result, when the student did not multiply by "the same number" – even in her point of view this was a slip of the tongue – she paid attention and pointed out this "error." In addition, this sighting shows that it is hard to interpret students' thinking. Teachers need to be extremely careful when they try to interpret what students think. Even a good teacher as Barbara, her own expectations caused the misinterpretation of student's thinking, which in turn caused the student's confusion. As a result, listening to students cautiously and asking follow-up questions such as "What do you mean…" or "Can you show me on your paper" could be helpful techniques for teachers to check students' mathematical thinking.

3) Using errors as springboards for inquiry. As described in the above sections, Barbara was a teacher who was sensitive to student mathematical representations and effective at using errors as learning opportunities for students' inquiry. The following sighting concerning the discussion of why 12/18 was equivalent to 2/3 sufficiently demonstrated how Barbara guided students to learning in depth.

To explain why 12/18 = 2/3, one student started from 6/9, and then she proved equivalent fractions by saying "6 times 2 and 9 times 3". Barbara immediately grasped this error and asked:

T: by 2/3? You think?

S1: Oh

T: You think that's going to give me what you said? (Write $\frac{6\times 2}{9\times 3}=\frac{12}{18}$ on the overhead and asked) You said 6 times 2 and 9 times 3?

S1: Oh, 9 times 2

T: (Still pointed at it and asked the class) Is that going to give me an equivalent fraction?

S: No.

T: Why won't it give me an equivalent fraction if I multiply by 2/3? Why is it not going to be an equivalent fraction?

Probably, this student really made a slip of the tongue here and she realized it quickly. She said "9 times 2". However, Barbara still grasped this "tiny" mistake as a counterexample and asked her class why times 2/3 was wrong.

S1: Because the numerator and denominator aren't the same.

T: Because the numerator and denominator aren't the same. Why do they need to be the same to give me an equivalent fraction? I know I heard a lot of people saying that was a pattern they saw.

S2: If you times 12 by 2, you get 24, and if you times 18 by 2, you get 36, and you get 24/36 which is equivalent.

T: Ok, he said if I did 12/18 times 2/2, he said it's 24/36, and you say that its equivalent? Yes, is that equivalent? Any people think it would be equivalent? Tommy, why do you think it's equivalent?

S3: I think it's equivalent because … any number times by 1, it will not change the number.

Three students answered Barbara's question. Why 2/3 was wrong. The first student simply said because the numerator and the denominator should be the same. The second student provided an example 12/18 × 2/2 = 24/36. And the third student, Tommy, provided the real reason: the identity property of multiplication – any number times by 1, the value will not change - which was the exact answer that Barbara wanted her students to discover. However, Barbara did not accept the third answer immediately. She asked scaffolding questions prompting more students to understand it:

T: But I did two haves. I did not see any "1" over there. Did I multiply by "1"?

S: No.

T: No?

T: Brown, what do you think?

S: No.

T: No. Tommy said that I multiplied 12/18 by "1". Is there anything these that looks like it could be "1"? Oh! I see lots of hands now. Katy, what do you think?

S: 2/2 equals to "1"

T: Ok, xxx, you agree with him?

S: Yes, I agree with this.

Barbara's purposeful confusing question "I did not see any '1' over there" showed that many students did not really understand Tommy's point because they agreed that there was no "1" there. It seemed students really needed a hint. Barbara therefore guided the class, "Is there anything now that looks like it could be 1?" This question provided students insights. Many students raised their hands and they finally saw the critical point, 2/2 = 1.

Looking back at the beginning of the sighting, what if teacher Barbara simply corrected the student's slip of tongue $\frac{6\times 2}{9\times 3}=\frac{12}{18}$ without discussing "why ×2/3 was wrong?" What if Barbara simply accepted Tommy's "any number times 1 will not change the number" without asking where the "1" was? If so, these thought-probing conversations would not exist. As a result, students might just simply memorize the rule – multiplying the numerator and denominator by the same number - without seeing the underlying reason concerning "1". In fact, during teacher interviews, many teachers pointed out that understanding equivalent fractions was extremely hard. Concerning finding equivalent fractions, many students could only remember the rule and simply use the rule to create equivalent fractions without knowing why. As previously mentioned, teacher Rose especially pointed out that even some students did know 2/2 equaled 1, and they did know every number times 1 equaled that number, when they facing ¾ ×2/2 = 6/8, they still could not understand why 3/4 and 6/8 were equivalent because in students' eyes, 2/2 did not look the like 1. In addition, 3/4 and 6/8 did not look like the same number.

Because of the difficulty and abstractness of equivalent fractions, Barbara grasped every opportunity and spent sufficient time on students' errors and difficulties. In the above sighting, even though the students said they saw "2/2 = 1", Barbara continued the discussion: Why do I have to multiply by numerator and denominator with the same number? Through the discussion, more students elaborated their thinking. As a result, three good explanations stood out:

S1: Well, I think that a number with the same numerator and denominator is "1". And if you multiply a number by 1, it will not change it.

S2: Also if you have 2/3, and say you times the numerator by 3 and the denominator by 2, before you had a fraction which wasn't a whole, and then you get 8/6, I mean you get 6/6, it will be a whole.

S3: Also, you have to have the same number because, then when you check it on the fraction strips, it wouldn't be lined up.

These students used basic mathematical principles (S1), provided counterexample as prove (S2), and connected mathematical ideas with concrete representations (S3). These explanations for the rule of finding equivalent fractions clearly demonstrated students' cognitive gains in Barbara's class.

Other teachers' classes. As previously mentioned, all teachers in this study knew the underlying reason behind the rule for finding equivalent fractions - multiplying the numerator and denominator by the same number. Put another way, they knew the basic ideas behind the rule, that is, $a/a=1$ and $b\times1=b$. (In this study, very few teachers realized $a/a=1$where $a\neq0$. This issue will be discussed later.) However, not all teachers in this study really addressed students' errors or difficulties in the same degree of depth as what Jennifer and Barbara reached during the enated classroom teaching.

Due to similar reasons as that of Jennifer and Barbara – addressing learning difficulties in advance –Kathleen's class also did lots of mathematics in two short classes (both about 30 minutes). However, Kathleen employed directive teaching, during which time she sometimes provided too many hints. The "Doubling Error" occurred once in her class when students proved 6/8 = 3/4:

S: You just divide 6, 8 by 2.

T: By 2?

S: (no response)

T: Ok, if, what (wrote down $\frac{6 \div 2}{8 \div 2} = \frac{3}{4}$ and circled $\frac{\div 2}{\div 2}$), you practiced before last year which was dividing by the wonderful one, so if you're dividing, what you're actually doing is dividing the top by 2 and the bottom by 2. Ok, because later on, when we work with fractions, if you divided by 2, it's a little different. But right now, we're dividing by one whole which is 2/2.

In this sighting, Kathleen immediately grasped the student's error and provided detailed explanation. Her mathematical language is quite concise and precise, which also demonstrated her in-depth knowledge of equivalent fractions. She knew the curriculum connection and the difference between "÷ 2" and "÷2/2". She also viewed division as one of the ways to find equivalent fractions. She is the only teacher who mentioned the connection between multiplication and division when finding equivalent fractions. (This theme will be discussed later). However, in this sighting, she directly told her students why it was wrong, what it should look like, and why it did matter, without really engaging them. The reason for providing many hints, according to her later interview, was that the class was her regular one. Even with these considerations, when Kathleen directly pointed out the issue, it is doubtful whether the majority of students really understood what she was saying. If Kathleen could unpack her language (Ball & Bass, 2000, 2003) and spend more time inquiring and engaging students in this error, students' understanding could reach a deeper level.

In the other three teachers' (Rose, Mary, and Lisa) classes, the "Doubling Error" did not occur. However, just because there are no errors, does not mean there are no difficulties. Since students in Mary's classes struggled with "counting", they were not really engaged in the investigation of the learning goals. Students in Rose's class also encountered the "counting" issue. However, Rose did discuss the patterns with her students. What they actually did was to repeatedly use the rule and produce lots of equivalent fractions without mentioning the basic ideas because this 50-minute long class ran out of time. Lisa discussed the "why" by using pictorial representations (discussed in section 4.3.3). However, she did not mention the basic ideas (e.g., identity property of multiplication) behind the rule. In other words, even though Lisa herself was clear about the basic principles and "foundations" (her words in interview), she did not employ that knowledge to address this learning difficulty in depth.

Summary. Teacher knowledge about the underlying reasons for the rule of finding equivalent fractions did not really make a significant difference in teacher interviews (Mary's knowledge seemed lacking). However, teachers' video tapes showed that not all teaching reached the same degree of depth. Classroom teaching is complex. Teachers should address students' prior knowledge difficulties ahead of time to ensure students have enough time to explore those tasks directly related to the learning goal. Barbara and Jennifer employed cooperative learning methods which provided students more opportunities to construct their understanding. During these processes, students actively explained their ideas which in turn provided the teachers opportunities to identify and address students' mistakes. In addition, classroom teaching is complex because of its contexts. Even as good teachers, their teaching may have problems. Jennifer in the first class was not explicit about students' errors; Barbara misinterpreted students' thinking; Kathleen sometimes provided direct explanations without any scaffolding questions. Even so, these teachers addressed student "Doubling Errors" in depth, allowing student to see the basic mathematical principles behind the rules.

4.3.3 Transiting between multiple representations

As Rose pointed out, students' abilities to abstract were different. Some students, even knowing "2/2=1" and "a×1=a", still had a hard time understanding equivalent fractions such as 3/4×2/2 = 6/8 because 2/2 did not look like 1 and 3/4 and 6/8 looked different. As a result, most of the teachers in this study believed that students need to "see" it. In the video tapes, concrete representations were also used to help students understand equivalent fractions. However, not all teachers could really help students transition from concrete to symbolic representations. For example, some teachers only asked students to compare their fraction strips. When students found out 3/4 and 6/8 lined up with each other and represented the same amount, some teachers believed that students really understood the reasons of "3/4=6/8" or "3/4 ×2/2 = 6/8". In fact, as pointed out by Leinhardt and Smith (1985), mapping between the numerical representation and regional representations of fractional equivalences was not at all straightforward for all students. Many of them could not see the connections between what they did with manipulatives and what they did with the symbols. In this study, both Kathleen and Lisa drew a semi-abstract representation in teaching equivalent fractions.

Their examples provided insights of using manipulative to improve students' mathematical understanding. However, the extent of these two teachers' transition from the concrete to the symbolic demonstrated differences.

Kathleen's class. Kathleen encouraged her students to use drawing in class. She said: "Visuals are very important especially when you get to upper math and calculus, you have to think abstractly, but to save you …, you have to create a picture in your mind or even on a sketched paper."

1) Both pictures and numbers. In Lesson 2.1, when she asked her students to prove 6/8=3/4, she drew her own picture instead of using the provided thermometer in the textbook. She guided students to prove it by using both pictures and numbers. When using the picture, a student said each piece needed to be cut into half. When using the numbers, a student said 6/8 should be divided by "2", a mistake immediately identified by Kathleen. Through the discussion with both ways, students were aware of the relationship between "cut into half" and the factor of "2" even in the first class.

2) *"×2/2 is actually splitting into 2".* In Lesson 2.2, when Kathleen guided students to see why the set of fractions 4/6, 6/9 and 8/12 were all equivalent to 2/3, she sufficiently used multiple representations. She again drew her own pictures. She wrote down 2/3 = 4/6, 2/3 = 6/9, and 2/3 = 8/12 in a vertical way. She then drew a rectangular picture to represent 2/3. After this, she asked her students to prove 2/3=4/6 (see Figure 18).

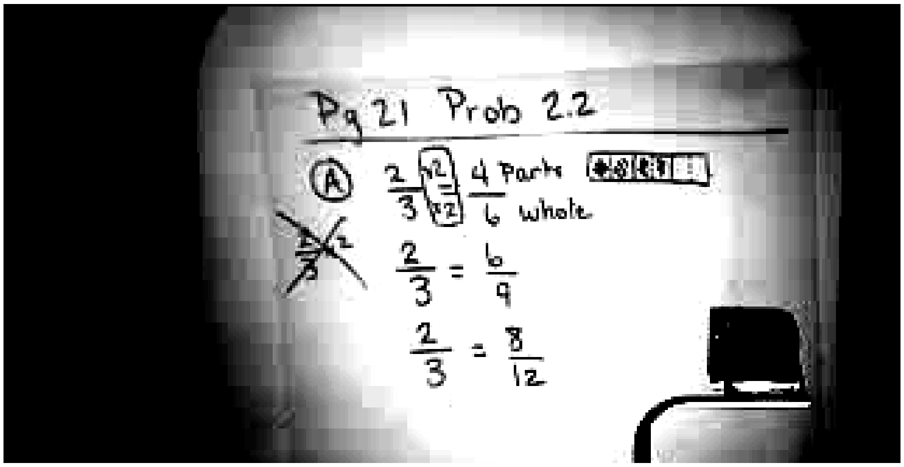

Figure 18. Prove 2/3 = 4/6.

T: Now you saw the picture that if I had 2/3, if I had 2/3 and I cut this up, how could I get 4/6? Ok, we're going to do it with a picture and we're going to work it with the numbers. What do you think?

S: You can multiply just like by the top and the bottom by 2.

T: Ok, (drew a rectangle after 2/3) so what you're actually doing is multiplying the top by 2 and the bottom by 2. Don't' get into the habit of just multiplying the top or just putting 2/3 × 2 (Marking $\frac{2}{3}\times 2$ with a big "×") because that's wrong. Ok? So we have 4 over 6.

As I previously mentioned, Kathleen sometimes provides too many hints. In the above part concerning the symbolic format, she warned students not to "× 2" directly. Even though this was a shortcoming, her action of drawing a rectangle and putting 2/2 inside of it was impressive. It emphasized that 2/2 was inseparable. It should not be viewed as two separate parts or two operations "× 2" and "× 2". It is a "whole", both visually and mathematically.

T: Now, show me with the picture

S: Draw the line to make each box into half

T: So if I multiplied the numerator by 2 and the denominator by 2 (Pointing at "× 2/2" inside the rectangle), what you're actually doing is you're splitting each one by 2, so now you have 4 parts, and 6 is the whole. And you agree that 2/3 is the same thing as 4/6.

Since the student suggested drawing lines to make each box of 2/3 into half, Kathleen then made her idea more explicit. She first drew students' attention to the symbolic representation - "× 2/2" inside a rectangle - and she verbally represented that symbolic format. She then clearly connected the symbolic representation to the action of drawing - to split each of boxes into 2. As a result, they got 4 shaded parts and 6 as a whole showing 4/6. As mentioned above, Kathleen's mathematical language was very concise, precise, and informative. In this sighting, her language was also very logical. With the seemingly simple guidance, the connection between symbolic and concrete: "× 2/2" is actually splitting into 2, was extremely clear. It was reasonable to assume most of the students in her class should be able to understand what "× 2/2"really meant and why "× 2/2" made equivalence in addition to those abstract explanations: "2/2=1" and "every number times 1 will not change the value".

3) Students' cognitive gains. Students' cognitive gains could be viewed in their proof of 2/3 = 6/9, during which students led the conversation flow. Concerning the numerical approach, students suggested multiplying each number by "3". The teacher then drew a rectangle after 2/3. Inside the box, she wrote "×3" and "×3" after the numerator and the denominator respectively. With the picture, students suggested drawing two lines to divide each square into three pieces. As a result, the symbol "×3/3" and the action of splitting each of the boxes into "3" was connected, the process of finding 6/9 from 2/3 were, therefore, meaningful.

4) Three lines and four dots: Sensitive to mathematics. With the same strategy, students proved why 2/3 = 8/12 by using "×4/4" and splitting each box into four pieces. It is worthy of mention that students in this class clearly knew that to get four pieces, they needed three lines. Kathleen herself also drew lines in a smart way. She said "middle, left, right" while drawing the three lines and produced four pieces. The "middle, left, and right" way seemed not a big deal. However, comparing with the "left, middle, right" way, the former one will produce a more accurate representation especially when a box needs to be divided into more pieces. Since the teacher is a model for students, when students observed their teacher's action, they would unconsciously learn the teacher's strategies. At last, this class came out with 8 pieces out of 12 pieces. How to shade these eight pieces additionally showed Kathleen's understanding of mathematics teaching. She did not shade all these eight pieces. For clarity, she used dots and lined up every four of them. As a result, eight dots stood in two lines (see Figure 19). This action seemed a trivial one; however, it showed Kathleen's clear understanding of the transition between symbolic and concrete representation- connecting "×4/4" with splitting each box into "4" pieces. Put another way, if Kathleen drew eight dots in one line, her students would not easily see each box was split into "4". This action demonstrated Kathleen's excellent mathematics sensitivity and her intention to teach students for understanding. Kathleen's sensitivity to mathematical representations partially answered the question: When "teaching for understanding" could really happen.

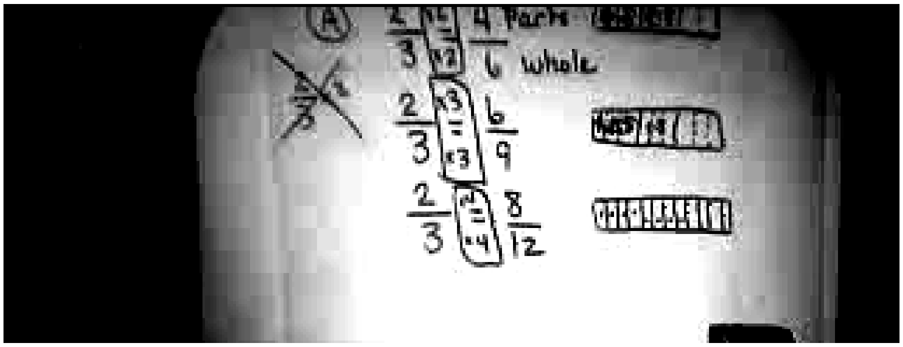

Figure 19. Transiting from between multiple representations –Kathleen.

Lisa's class. During Lisa's interview, she specifically emphasized understanding: "If I teach you something and you understand it, when you are thirty years old, you will still know it because it was not something you memorized it." In Lisa's point of view, the most important part of learning equivalent fractions was to understand the concept fraction because "students need prior knowledge in order for them to make sense." To understand fractions, Lisa believed students needed concrete representations. "They need to see it, they need to touch it, they need to manipulate it, they need to draw it. It has got to be very, very concrete."

In Lesson 2.2, when students in Lisa's class used the rule "multiplying the numerator and the denominator by the same number" and got numbers such as 300/400, 3000/4000, Lisa discussed the underlying reasons behind the rule with students. Similarly, Lisa also drew pictures and tried to transition between multiple representations. However, her techniques of using representations and the extent of "transition" that her students reached were different from that of Kathleen's class.

Drawing two rectangular bars. Different from Kathleen who used one fraction bar to represent two equivalent fractions, Lisa drew two rectangular bars on the board.

Regarding the “rectangle” shape, Lisa had a very interesting explanation during our conversation.

T: You know another thing, in the early grades, everything is Pizza and how to do fractions. Well if you get a Pizza and you need to divide it into, let’s say, 7. Well, it is not easy to divide a circle into 7 equal parts.

I: Definitely.

T: But if you draw a rectangle, they can easily, you know, even just use a ruler and draw 7 inch rectangle, and divided that into 7 sections, it is very easy for them to do that and to be accurate. So I tell them do not draw a circle, like forget the Pizza, forget the pie, we are doing a rectangular pizza. They do make rectangular pizzas.

Lisa’s point of view of rectangular pizza is highly consistent with that of Ball’s (1993) idea of “rectangular cookies” (p.182). It reflected her flexibility in using multiple representational models. As Lisa said, whatever the pizza’s shape was, the pizza was still a “whole”.

Reviewing fractional concept. Lisa drew two rectangular bars. She divided the first bar into 3 equal parts and shaded the first part. She then reviewed the concept fraction with students.

T: Ok, we’re going to divide this into thirds. Right there, look ok? Ok, tell me what this is.

S: 1/3

…

T: We talked about this at the very beginning. You remember we talked about what the numerator is and what the denominator is. What did this denominator tell me?

S: Number of equal parts in the fraction.

T: Very good. Number of equal parts in the whole. What does this "1" tell me?

S: How many we're talking about.

T: Very good. We were just dealing with one of them. We were only looking at one of them. So this tells me that I have one out of the three equal in the whole.

As previously mentioned, Lisa believed the concept fraction was the most important prior knowledge for students to learn equivalent fractions. What she did in the class was also consistent with what she said in the interview: "They have to know a fraction is part of a whole. That is very basic. If they cannot understand a fraction is a part of a whole, then they kind of messed up both." Lisa specifically emphasized "part" and "whole". Therefore, she reviewed the meaning of the numerator and denominator before the discussion of equivalent fractions.

Why "×5/5"? After reviewing the concept fraction, the basis of equivalent fractions, Lisa moved to her original question - why multiply the numerator and denominator by the same number? She explored this question through the example 1/3 ×5/5 = 5/15. To understand Lisa's teaching effects, it is necessary to examine how she tried to reach her goal step by step.

1) Starting from the denominator. Lisa pointed at the fraction 1/3 and she began from the denominator "3".

T: Ok, let's say we multiply this 3 by 5, what does that give us?

S: 15

T: So what does that 15 tell me now? What does that 15 mean?

S: It means it takes 15 equal parts to equal that whole.

T: Ok, here we go. So that means I have to take each one of these and turn it into what?

S: 5

T: Not 5. Oh, yeah, I am sorry, you're right. 5. Because we want how many are in the whole?

S: 15

T: (Drawing lines) … We pretend, we are pretending I can draw…So we are going to say 15 equal parts, right? So we multiply the bottom by 5, and we got 15.

Obviously, concerning the two equivalent fractions 1/3 and 5/15, Lisa tried to guide students to see how the denominator 3 changed into 15, the result of multiplying the bottom by 5. However, Lisa focused on the "result" (15) rather than the "process" (multiplying the bottom by 5). What follows is the evidence. After the class figured out they should turn the whole into 15 equal parts, Lisa asked: "So that means I have to take each one of these and turn it into what?" When students answered with "5", the teacher immediately denied it because she was anticipating the answer "15", the number of the total equal parts. She then refined her question as "how many are in a whole", which led the anticipated answer "15". Based on the discussion, Lisa also divided each of the three sections into 5. However, she did not point out she was "splitting each section into 5".

Instead, she said she turned the whole into 15 pieces. In a word, she connected the operation "multiplying the bottom by 5" with the result "15" rather than the action of "splitting each section into 5". Lisa's emphasis on the result "15" or the fact "15 equal pieces" might draw students' attention away from the process - splitting each section into 5 pieces or multiplying the bottom by 5 - resulting in the disconnection between the concrete (splitting each into 5) and the symbolic (×5/5) representations.

2) Continuing with the numerator. Lisa's focus while working with the numerator was similar to that of denominator. She was guiding students to see the "results" of multiplying 1 by 5 rather than the meaning of the "process".

T: So what should we do here?

S: Multiply 1 by 5

T: So what does this 5 tell us now?

S: shading 5

T: This is the total number we're going to look at, so 1, 2, 3, 4, 5 (shaded 5 pieces).

The process of "multiplying 1 by 5" was actually corresponding to the action of "turning 1 piece into 5 pieces", which had already been done during the discussion with the denominator. However, when students mentioned "shading 5", they focused more on the results "5 pieces" and skipped the process of "splitting". In other words, the symbol "×5" in the numerator was not explicitly connected to the action of drawing.

3) Comparing the shaded amount. So far on the board, the first rectangular bar was shaded with 1 piece while the second bar was shaded with 5 pieces. These two shaded parts had the same amount (see Figure 20):

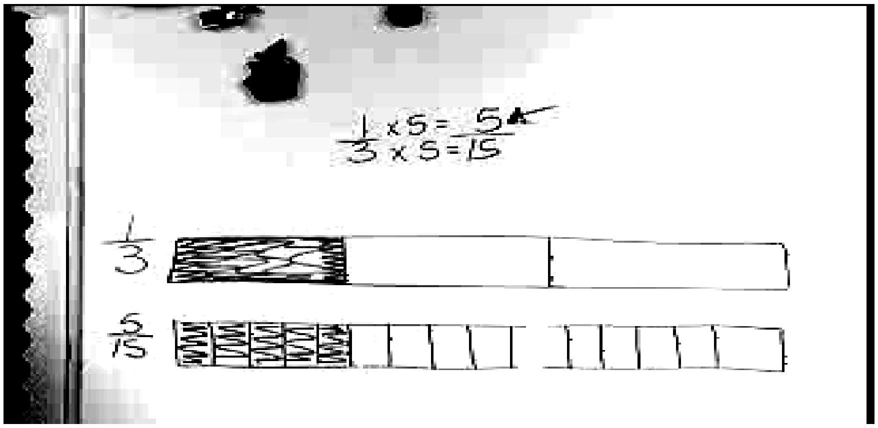

Figure 20. Transiting between multiple representations – Lisa.

Lisa pointed to these two bars and further guided her students:

T: What do you think?

S: equal

T: Equal? Do you see why it works? What does this fraction represent right here? Name this fraction for me in numerical form.

S: 5/15

T: The 15 tells us we have how many equal parts in the whole?

S: 15.

T: 15 equal parts in the whole? And we're going to deal with 5 of them. 1, 2, 3, 4, 5.

T: This one tells me I have 3 equal parts in the whole and I'm going to deal with one of them. So tell me about 1/3 and 5/15.

S: They are equivalent.

T: Very good, they are equivalent.

In this conversation, Lisa wanted her students to reason in the following ways: (a) the two shaded parts represented 1/3 and 5/15 respectively; (b) the two shaded parts had the same amount; and (c) the two fractions were equivalent. This reasoning process matched well with Post et al.'s (1985) findings of students' thinking characteristics in learning equivalent fractions. According to Post et al., Lisa's classroom teaching could result in students' understanding of equivalent fractions. This is because, Lisa reviewed fractional concepts with students, therefore, students could transit (a) from the embodiments to the symbol of 1/3, and (b) from the embodiments to the symbol of 5/15. In addition, Lisa guided students to compare the amount of shaded parts, which might ensure students to see (c) the embodiments of 1/3 and 5/15 had the same amount (Post et al, 1985). As a result, students were able to understand "1/3 = 5/15". However, according to Leinhart & Smith (1985), learning equivalent fractions was not as simple as what Post et al. (1985) concluded. In Lisa's example, even though students recognized 1/3=5/15, when they encounter the symbolic format $\frac{1\times 5}{3\times 5}=\frac{5}{15}$, students may not be able to figure out what $\frac{\times 5}{\times 5}$ meant. Put another way, even though student saw the concrete materials, they might not understand the connection between the pictorial and the symbolic representations of equivalence. In contrast, Kathleen in her class purposely used a rectangle to encircle $\frac{\times 2}{\times 2}$. She then clearly connected this operation $\frac{\times 2}{\times 2}$ with the action of "splitting each box into 2", resulting in a new part and a new whole, which exactly corresponded to the operation results: a new numerator and a new denominator. As a result, even though both Kathleen and Lisa used drawing (concrete representations) to help students understand the underlying reasons of the rule for finding equivalent fractions, students in Kathleen's class probably have more possibilities to reach a higher level of understanding.

Summary. All teachers in this study used concrete representations such as fraction strips or number lines in their textbooks. However, when teachers tried to guide students to see the underlying reasons behind the rule or pattern for finding equivalent fractions, teachers' guidance methods were different. Barbara and Jennifer placed much more emphasis on the basic mathematical ideas. Their expectations of students intensely reflected on their responses to student "Doubling Errors" (see section 4.3.2). In contrast, Kathleen and Lisa guided students to understand the "rule" by drawing pictures. These two teachers spent enough time transitioning between multiple representations. Obviously, Kathleen's class tended to reach a higher level of understanding. Rose and Mary did not have time to sufficiently discuss the "rule" even thought these teachers themselves knew how and why this "rule" works.

4.3.4 Making connections to division

The concept of "equivalent fractions" is abstract. It is hard for many students to really understand the underlying reasons behind the rules or patterns. As a result, students tend to make mistakes such as "Doubling Error". To address this type of error or difficulty, teachers in this study made various connections between the multiplication pattern and other concepts. For example, some teachers connected fractions to whole number ideas (e.g., Jennifer's 10×2=20, Lisa's 4×1=4, 5×1=5, and 6×1=6) while other teachers compared equivalent fractions with fractional multiplication that students would learn in 7th graders (e.g., Lisa and Rose pointed out $\frac{3}{4}\times 2=\frac{3}{4}\times\frac{2}{1}=\frac{6}{4}$). Even though teachers did make efforts to locate equivalent fractions within a concept map, all these connections were still limited within the operation of “multiplication”. As a result, another important link was missing, that is, the link between multiplication and division in obtaining equivalent fractions.

As pointed out by Leinhardt and Smith (1985), some textbooks and teachers mainly viewed the particular way to obtain equivalent fractions as multiplication - raising a fraction in higher terms - rather than division. In general, division was used to simplify a fraction into its lower terms. As a result, the symmetry of multiplication and division in finding equivalent fractions was overlooked. Some teachers even misunderstood that finding equivalent fractions and reducing fractions were opposite (Leinhart & Smith, 1985).

Overemphasis on the “multiplication” approach to obtain equivalence also occurred in this study. First, the CMP teacher guide book only suggested teachers anticipate the “multiplication” pattern (Lappan et al., 1998). Second, among all the six teachers video-taped lessons, only one teacher, Kathleen, explicitly pointed out that both multiplication and division could be used to find equivalent fractions. With regard to the other teachers, even though they demonstrated their knowledge of the “division” way, no one clearly pointed out it as an alternative approach to obtain equivalent fractions for their students. In the following section, I will describe how the teacher, Kathleen, introduced this idea to her class. Before the description of Kathleen, I will briefly mention how such an opportunity concerning “division” was missed in other teachers’ classes, which might provide a general sense of the significance for this section.

A missed opportunity in other teachers' classes. During the learning of equivalent fractions, many students knew about "simplifying". If students raised this type of method during their explanation, this actually provided teachers good opportunities to help students make connection between multiplication and division ways in finding equivalent fractions. However, some teachers did not grasp this type of opportunity.

1) Barbara's class. From the above descriptions of Barbara's class, one can easily find that Barbara was an effective teacher who had the ability to address student learning errors and difficulties in depth. However, Barbara did not show an awareness of the relationship between multiplication and division in finding equivalent fractions.

In a previous sighting (see section 4.3.2), Barbara asked her students to explain why 12/18 and 2/3 were equivalent. One student figured out that she was dividing 12 and 18 both by 6. Written down this student words were $\frac{12 \div 6}{18 \div 6} = \frac{2}{3}$. Facing this unexpected students' strategy, Barbara immediately provided the following response to her class:

T: How did you think, the person who named 12/18 to begin? How did you get it? You said something about doing multiplication table. Did anybody else multiply by something to get them?

Obviously, this student's strategy – from 12/18 to 2/3 by using division – was not what Barbara anticipated. She reminded this student "you said you are doing multiplication tables" and then changed the direction of discussion back to multiplication: "Does anybody else multiply by something to get them". Even though it eventually resulted in a very nice discussion where Barbara used student errors as springboards for inquiry, the opportunity of using "division" to obtain equivalent fractions was obviously missed in this class.

2) Mary's class. Mary's class did not really have time to talk about the strategies for finding equivalent fractions. The following sighting was from Lesson 2.1 where one student tried to explain why Mr. Park's 6/8 was correct.

S: Ok, if you know that ¾ is already, if you know it's right, you can just simplify the 6/8.

T: Simplify the 6/8, and if you simplify 6/8, what would it be equal to?

S: 3/4.

T: 3/4? Very good. What did you use to simplify with?

S: Divide the numerator by 2 and the denominator by 2

T: Yes, you are correct. You are correct. You are correct.

This student clearly saw the relationship between 3/4 and 6/8. He also pointed out the way of "dividing" to get equivalent fractions. However, Mary only raised one question "what did you simplify with" without any real probing questions. In fact, when explaining why 3/4 and 6/8 were equivalent, Mary herself mainly depended on lining up the fraction strips and comparing the "folds". In this sighting, when the student posed this mathematical strategy, she did not make any connection between multiplication and division. Probably, "division" in this student's eyes, was still only connected to "simplifying".

Using "division" as a way to find equivalent fractions – Kathleen's class. In Lesson 2.1, Kathleen discussed equivalent fractions with division. In that sighting, Kathleen asked students to prove 6/8 and 3/4 were equivalent by using both pictures and numbers (see section 4.3.2). She wrote down $\frac{6}{8}=\frac{3}{4}$ and one student told her to divide 6/8 by 2. As a result, Kathleen grasped this mistake and told this student that she was actually dividing the top by 2 and the bottom by 2. She then wrote down $\frac{6 \div 2}{8 \div 2} = \frac{3}{4}$ and circled $\frac{\div 2}{\div 2}$.

She reminded her students that they were actually dividing by one whole which is 2/2. As I discussed before, Kathleen in this sighting directly provided a lot of information to students. If she could unpack these knowledge pieces and prompt her students to figure out these answers, the teaching effect would have been better. However, if we pay more attention to the content rather than the method in this sighting, there is no doubt Kathleen noticed "division" as an alternative way to find equivalent fractions and she also exposed this strategy to her students.

A clear effort on "division". In Lesson 2.2, Kathleen again demonstrated an awareness of using "division" to find equivalent fractions. In the following sighting, she spent enough time with students on this issue**:**

1) Pointing out a common mistake. After Kathleen discussed the "multiplication" pattern – multiplying the numerator and the denominator by the same number, she raised a difficult point, called by Kathleen as a "common mistake", to her students:

T: Now, one of the common mistakes that I see I would like you to write down… If I was to ask you if 1/2 equals 2/4, that's usually not a problem. Ok? But if I asked you if 2/4 equals (wrote down 2/4), um, here, let's do this one, what comes after 2/4.

S: 3/6.

T: You can say 3/6, ok? So we know these three were equal to each other. But for some reason, when students see 2/4 = 3/6, if they don't see a pattern, they think it's wrong. Ok?

The common mistake, according to Kathleen was that, when students first glance at the problem 2/4 = 3/6, they may not be able to find the pattern – the numerator and the denominator both by 1.5. This relationship was not straightforward. There were two possible reasons that student could not find the relationship. First, some students may not be familiar with decimals, they could not visualize 2×□=3 and 4×□=6. Second, students may be influenced by "whole number thinking" (Cramer & Lesh, 1988; Post et al., 1988) and may think only a "whole" number could go into the box. This second conjecture can be substantiated by two pieces of evidence: (a) The CMP teacher guide book specifically emphasized "any whole number" in the following suggestion: "multiply the numerator and the denominator of the original representation of the fraction by the same number – any whole number – to obtain an equivalent fraction" (Lappan et al., 1998, p.30b); (b) Another teacher, Jennifer, was the only teacher who correctly answered the first T or F question (1) – to find equivalent fractions, we can multiply the numerator and denominator by the same number – as False. However, Jennifer's explanation was somewhat problematic. She said, this question was wrong because we only can multiply the numerator and denominator by a nonzero "whole" number (About "0", I will discuss later). As a result, the teacher's and the guide book's emphasis of "whole number" provided the clues for why some students thought 2/4 ×□/□=3/6 could not be solved, and why students might think 2/4 and 3/6 were not equivalent.

2) A try: 2/4 = 5/10. To illustrate this difficulty or mistake, Kathleen provided two fractions and asked students whether they were equal at first sight.

T: Let me show you another one. That one's (pointing to 2/4=3/6) not too horrible, but, if I had 2/4 and I had 5/10, (write 2/4 = 5/10), first sight, it's kind of hard to tell, but after you look at it for a while, are they equal?

S: yes

T: How come?

S: They're both halves.

T: They're halves. Can you all see that is halves?

S: Yes.

Fortunately, Kathleen's fractions 2/4 and 5/10 were easy to be dealt with because they were both equal to half. Students recognized this point quickly. When Kathleen asked them "how come", these students said "they are halves". As a result, student in Kathleen's class did not really use the "multiplication" way to think about 2/4 × □/□ =5/10. Instead, they used an opposite way "division" through simplifying the higher-termed fractions (2/4 and 5/10) into a lower-termed one (1/2), the exact way as Kathleen anticipated.

3) Make "division" approach clear to students. Kathleen then used another example, a harder one for students to try. Based on this example, she made the connection between "multiplication" and "division" in finding equivalent fractions explicit.

T: Let's see if there's that's a little harder (Tried to think of an example).

S: If you …(inaudible suggestion).

T: Ok, so let's do, um, ok, let's do 2/6, and we want to know if it is equal to 5/15.

S1: No.

S2: Yes.

S3: No.

T: See, at first sight, your brain wants to say no because you do not see the pattern yet. Now, give yourself a few more seconds, what is the pattern?

S: They are both 1/3.

T: So if I cannot make an equivalent fraction by multiplying. I can also make an equivalent fraction by?

S: Dividing

T: Dividing or simplifying. So 2/6 can be divided by what? (wrote "÷" after 2 and 6 respectively) (see Figure 21)

S: "2"

T: 2. (wrote "÷ 2" and "÷2") You see 1/3 (surround 1/3 with a rectangle)? And this can be divided by what (wrote "÷" after 5 and 15 respectively)

S: 5

T: 5 (wrote "÷ 5", "÷5" and "1/3"). So you see 1/3 (surround 1/3 with a rectangle). So watch out for that one (see Figure 21), that's the most common mistake when you're looking for equivalent fractions.

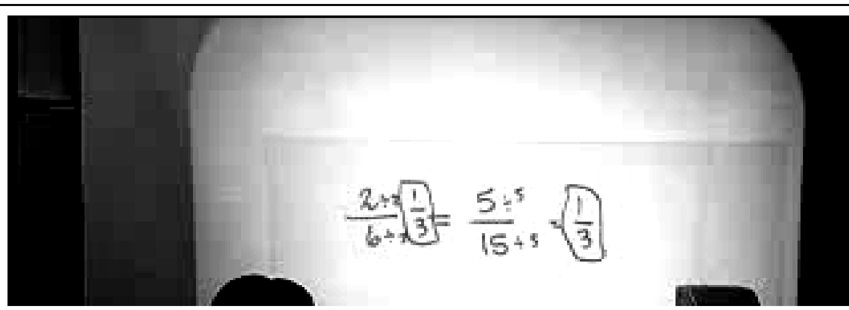

Figure 21. Using division to find equivalent fractions

Clearly, this example 2/6 = 5/15 was an emergent one that Kathleen figured out during the teaching context. Her awareness of the symmetry between multiplication and division in finding equivalent fractions was not limited to this class. From Lesson 2.1 where Kathleen asked students to prove 6/8 = 3/4 to Lesson 2.2 Kathleen purposely pointed out “division” as an alternative way to find equivalent fractions, Kathleen showed her comprehensive understanding of this topic. In fact, during the interview, when I asked why she mentioned division, Kathleen said all these concepts or operations were related. She even provided me other examples such as how she helped her students to make connections among fractions, decimals and percents.

Whose students understand better? Kathleen was the only teacher in this study who fully realized the connection between multiplication and division approaches to finding equivalent fractions. All the other teachers only focused on “multiplication” when addressing students’ errors and difficulties. For example, in Barbara’s class, when a student provided the “division” way as an explanation for finding equivalent fractions, she stopped him and changed the class conversation flow into “multiplication” as she planned to do. As a result, when teachers overemphasized the use of multiplication to obtain equivalent fractions, they only told half the story and provided incomplete information to students (Leinhardt & Smith, 1985), which might cause consequences for students’ incorrect inferences and reinforcement of misconceptions (Resnick, 1980). Obviously, students in Kathleen’s class gradually constructed a complete concept net where multiplication and division were both ways to obtain equivalent fractions and division was not only related to “reducing or simplifying”. Kathleen’s method not only provided an example for those teachers who wanted to teach for understanding but also raised questions for curriculum designers, teacher educators, and researchers concerning how to provide teachers correct directions and more opportunities to teach for understanding.

4.3.5 Issues of teacher's mathematical language

In this study, all teachers' classroom teaching touched the "pattern" or "rule" of finding equivalent fractions. Some of them had time to explore the whys behind the rule while others only had time to discover the "rule" itself. During the process of working on the rule – multiplying the numerator and denominator by the same number - some teachers paid detailed attention to students' verbal and symbolic representations and tried hard to ensure them to be accurate while others' own verbal representations were probably misleading or confusing. In this section, I provide Rose's and Mary's examples with the purpose of alerting teachers to be aware of their mathematical language.

Dublin (Doubling) is the capital of Ireland. Rose demonstrated her good understanding of equivalent fractions in the interview. She was clear about the difference between "×2/2" and "×2". She knew the underlying reasons for the rule - the identity of the multiplication property. She also emphasized the understanding of the concept fraction in order to comprehend equivalent fractions. Concerning the designed case: "Doubling Error" (3/4×2), Rose was the only teacher who pointed out that student misconception of the equal sign was one of the reasons for this mistake. She said she would like to address this type of error starting from a discussion of the equal sign. In general, Rose is a knowledgeable teacher and eager to learn. However, there was a gap between her teacher knowledge as demonstrated by the interview and her enacted teaching. Specially, Rose was not cognizant of her possible misleading verbal representations. In this section, I provide her “Dublin” metaphor.

1) Metaphor of “Dublin”. In Lesson 2.1, Rose and her students discussed why 3/4 and 6/8 were equivalent. They came out 3 times 2 equals 6 and 4 times 2 equals 8. As a result, Rose concluded,

> T: So you’re going to do these by multiplying everything by 2, which is same as what? … When you multiply by 2, it’s the same thing as doubling. I tell people you go to Ireland, want to see the capital of Ireland is Dublin. So when you play in that way, you are busy with doing doubling (Dublin).

According to the teacher, when students were multiplying the numerator and denominator by 2, they were doubling the numerator and the denominator. Therefore, Rose said students were busy with doubling. She specifically provided students the “Dublin” metaphor to help students memorize the pattern of “doubling”. However, the “doubling” was actually a common mistake made by many students. The erroneous written format “$3/4 \times 2$” was exactly consistent with “doubling 3/4”. As a result, when Rose kept emphasizing “doubling” to students, her metaphor and the repeating of “doubling” might have negatively influenced students and caused students’ “Doubling Errors” such as $3/4 \times 2$ and other misconceptions such as 6/8 was lager than ¾ because “doubling” usually made a larger number.

2) Teacher’s continuing emphasis on “doubling”. One of the learning goals in Lesson 2.2 was to find the pattern or rule for obtaining equivalent fractions. In this class, Rose asked her students to recall the related information before the exploration:

T: We said to make equivalent fractions yesterday, we were supposed to do what? You remember?

S: Dividing by 2.

T: Dividing by 2. You left us with something else yesterday.

S: Doubling.

T: And the question I left with you is when it comes to equivalent fractions, do we always just multiply the numerator and denominator by 2? Today's investigation has some rulers for you.

This short conversation was to help students recall the rule of "doubling". Rose asked student how they could obtain equivalent fractions, one student answered with "dividing by 2", a good opportunity for Rose to help students make connections between "division" and "multiplication" strategies as previously mentioned. However, the teacher ignored it because she was anticipating the answer of "doubling". When the student figured out the answer "doubling", Rose provided further elaboration: "multiply the numerator and denominator by 2". Considering Rose's efforts on emphasizing "doubling", it will be interesting to see how students respond to the teacher's instruction and whether the teacher's imprecise representation had negative influences on students' representations.

3) Students were saying "doubling". During exploration, Rose walked around the classroom and one student reported his pattern: "I am doubling it". Rose then asked this student to provide further explanation: "How did you double it? How did you double your fractions?" It seemed students in this class had already memorized the word "doubling". Now the teacher and the students could have conversations by using "doubling".

During the summary part, after students' mainly provided the pattern of "addition" Rose tried to guide them to see the pattern of "multiplication". As a result, students found the rule of "doubling" and the teacher then restated the metaphor of "Dublin".

> T: Bryan, when I walked around, you had some very, very large equivalent fractions. Do you still have them? You got rid of them all? … Bryan had 8/12. and before he erased, the next one he had was 16/24. And the next one he had was 32/48. His were getting big. Do you know what Bryan was doing? Can you tell me what Bryan was doing before he erased to go to the smaller numbers? Can anybody look at Bryan's pattern up at the top? What was he doing?
>
> S: He was doubling.
>
> T: He was doubling them. Michel said yesterday, he went to Ireland, right? He's in Dublin. The capital of Ireland is Dublin. That's what Bran was doing, he was doubling. He was busy with doubling.
>
> T: Are they equivalent fractions?
>
> S: yes.

In the above sighting, students obviously mastered the "doubling" pattern. They summarized Rose's description of Bryan's strategy as "doubling", which showed the influence of the teacher's language. However, as previously mentioned, this type of imprecise verbal representation is likely to cause mistakes such as "3/4 ×2=6/8", which in turn reinforces the misconception of "a bigger equivalent fraction". In fact, Rose did make such errors in the above sighting -"He was getting big" and "go to the smaller numbers" – which corresponded to the context of "doubling". As a result, it was doubtful that whether the students in Rose's class would also produce and entrench such misconceptions over time.

4) Why "Dublin". During the teacher interview, Rose provided the reason why she used the metaphor of "Dublin" which again brought out the above misconception.

> T: I tried to relate things to children. I let them know that I brought up equivalent fractions in order, but they just continued multiplying by 2, which is double, so it is playing on words with Dublin of Capital, and then I let them, then you have to ask them, can I multiply by 3, can I multiply by 4, because they get their bigger numbers and they continue double, double, and double.
>
> I: Otherwise, they think they only can double, right?
>
> T: That's, many of them think that, can you only use 2? So you listen to what they have to say and then I do not tell them they are wrong. But I may look at the fact that they are doubling things. So, eventually, I will get them to the point that they do not have to double, double, and double. Again, you take away what you gave and you remove from what you have.

In this conversation, Rose explained her purpose of the metaphor "Dublin", that is, she wanted to "relate things to children". Put another way, she was trying to help students remember the rule. However, in the above explanation, when she mentioned if student continued doubling they would get bigger numbers, she again exposed her own misconception showing the negative influence of "Doubling". In addition, when Rose mentioned that they were not just doubling, but also multiplying by 3, 4, and 5, she was systematically creating further confusion. As the teacher Barbara commented, "doubling" definitely confused students.

The Greatest Common Factor. Another example concerning the issue of teachers' mathematical language was Mary's *Greatest Common Factor (GCF),* a terminology showing her blurred understanding.

1) Summarize student's pattern with GCF. During Lesson 2.1, Mary guided her students to find another equivalent fraction for ¾ and 6/8 through using fraction strips.

T: Ok. Now if I were to ask you to take out your strips could you find another strip for me that would be equivalent to a ¾ strip and 6/8 strip? …Ok? We have ¾, we have 6/8, is there other one we can use? Lee?

S: 12/16

T: 12/16? So you have a ruler that goes up to sixteenths already?

S: I just…

T: Why do you know that? Tell me why you know that?

S: You can multiply the numerator and the denominator by 2.

T: Ok, so you're using your *GCF* right now to help you? Very good.

In the above sighting, the student stated that he used the "multiplication" rule to get the equivalent fractions. However, Mary shortened it to "GCF" (*Greatest Common Factor) which* was related to "multiply the numerator and the denominator both by 2" in this context. Put another way, the part "$\frac{\times 2}{\times 2}$" in $\frac{6 \times 2}{8 \times 2} = \frac{12}{16}$ inspired Mary to think of the terminology of "GCF". According to Mary's logic, if multiply the numerator and denominator both by 3, or 4, she will get the new GCF 3 or 4 for the fraction 6/8. As a result, GCF includes any number used to find equivalent fractions.

2) A comparison with Lisa's GCF. Interestingly, the GCF also occurred in Lisa's class where a student made a similar mistake but the teacher Lisa showed her clear understanding about this terminology.

T: What patterns do you see? Kevin?

S1: I multiply the numerator times 2 and the denominator times 2.

T: Ok, he says to find an equivalent fraction, you should multiply the numerator by 2 and the denominators by 2. More ideas? Jack?

S2: Find the great common factor

T: Find the greatest common factor? And?

S2: And multiply the number you got times the numerator and the denominators.

T: Ok, So far, I have, you can multiply the numerator and denominator by 2; multiply by the greatest factor. Susan?

S3: Multiplying the numerator and denominator by any given number. They need to be the same.

T: Multiplying the numerator and denominator by any given number

S3: Or any number.

T: Any number, any given number. Sounds good.

T: What do you think about those? What do you think about Susan's? Does hers cover what Jake said and what Kevin said?

S: Yes.

When the student in Lisa's class mentioned GCF, he probably made the same mistake as Mary, that is, he called any number multiplied to get equivalent fractions as "GCF". He also might really mean to use GCF. Whatever this students tried to mean, teacher Lisa clearly understand what GCF meant. She also made it understandable that GCF was not equal to any number.

3) Unsure of what GCF meant. During the interview, Mary mentioned GCF. When I tried to ask further questions to claify what GCF meant, she was lost.

T: I tried to remember everything we did. I tried to remember what patterns you saw, what've gotten funny there. By then, we will look at what we've done. We are looking at greatest common factor.

I: Yes, someone mentioned that.

T: Yes, I remember somebody told me about greatest common factor. It helps you figure out the equivalence thing, hum, the numerator and denominator equal 1, ok, yeah.

I: What do you mean numerator and denominator equal 1?

T: Hum (stop about 10 seconds), say, for instance, no, I'm thinking of something else right now. That's wrong, that's wrong. Finding a common (10 second), let me think for while…

In this interview, Mary pointed out "the numerator and denominator equal 1" was *Greatest Common Factor* (GCF). According to Mary, all $\frac{\times 2}{\times 2}, \frac{\times 3}{\times 3}, \frac{\times 4}{\times 4}$ were called GCF. The question is, what is greatest common factor? What does "greatest" mean? What does "common" mean? What does "factor" mean? In equivalent fractions, Greatest Common Factor (GCF) is most specifically related to "simplifying" or "dividing". To simplify 40/80, we can divide both the numerator and denominator by 40 to get ½. As Leinhart and Smith (1985), using "division" to reduce (simplify) fractions was restricted because there would be several common factors and only one GCF. During the interview, when I asked Mary what she meant “the numerator and denominator equal 1”, she realized something was wrong with her language. She said she was thinking of something else but she was not sure about that. In fact, Mary was talking about the pattern of “multiplication” but she misused the terminology in simplifying fractions, an alternative way to obtain equivalent fractions through “division”. Mary’s inappropriate use of the terminology GCF reflected her lack of clarity about the relationship between the multiplication and division strategies for finding equivalent fractions. Her summary for students’ patterns with this GCF was misleading. Therefore, teachers should be aware of their mathematical language and pay attention to their verbal representations.

4.3.6 Summary

To find equivalent fractions, students not only need to know the rule but also the underlying reasons behind the rule. Due to the abstractness of fraction and the influence of whole number thinking, students tend to make “Doubling Errors” such as “$\frac{3}{4}\times 2=\frac{6}{8}$” or saying “double 3/4”. To address this type of error and difficulty, teachers need to allow students to understand why $\frac{3}{4}\times\frac{2}{2}=\frac{6}{8}$ works. In other words, students need to understand basic mathematical ideas such as “2/2 = 1” and “Every number multiplied by 1 will not change the value”. However, not every student can reach the same cognitive level. Teachers also need to use multiple representations to help student “see” why “×2/2” yields a correct answer and to transition students’ understanding from the concrete to the symbolic. In addition, finding equivalent fractions involves not only multiplication but also division strategies. For example, “×2/2” and “÷2/2” are the same process but reciprocal operations. As a result, teachers should be aware of such symmetry and make connections between multiplication (to obtain higher-term equivalent fractions) and division (to obtain lower-term equivalent fractions). Only through these efforts on "Doubling error and difficulty" – addressing in depth, transitioning between representations, and making connection to division - can students be able to construct deep understanding of equivalent fractions. Finally, teachers need to pay attention to their verbal representations and to use accurate mathematical language in order to avoid misleading students.

4.4 What Do You Mean by Adding ¾?

Another common error occurred when students tried to find equivalent fractions by using the "addition" pattern. When students were asked to explain their patterns, some of them wrote down $\frac{3}{4}+\frac{3}{4}=\frac{6}{8}$ instead of $\frac{3+3}{4+4}=\frac{6}{8}$. Similarly, some students said they were "adding 3/4" instead of "adding 3 to the numerator and 4 to the denominator". Teachers in this study responded to this type of error and difficulty in different ways, which demonstrated their different understanding of equivalent fractions. Both the video-tapes and teacher interviews provided insights for addressing this error and also reflected some issues. In this section, I first describe the evidence of teacher knowledge. I then provide how teachers respond to this type of error and difficulty differently. Since some teachers concerned the efficiency of the "addition" pattern, I arranged this theme at the end of this section (see Table 5).

Table 5

Sub-theme and teachers in "What do you mean by adding 3/4".

Section	Sub-theme	Teacher
4.4.1	Teacher knowledge	All
4.4.2	Teacher responses in classrooms	Jennifer/Barbara/Mary
4.4.3	Do you have a quicker way	Rose/Kathleen/Barbara

4.4.1 Teacher knowledge

Teacher knowledge about the erroneous formats such as ¾ + ¾ = 6/8 was examined through the teacher interview which included a designed case and two T or F questions. Teachers' responses to these questions not only showed their understanding of equivalent fractions but also of the related concepts of addition of fraction, equivalence, and ratios.

Designed case. The designed case was to ask teachers' opinions about student invented strategy ¾ + ¾ = 6/8 in finding equivalent fractions. This question relates to teacher knowledge of the concept of fraction and the addition of fractions. All the teachers except Mary thought this as an error. These teachers said that students were confused by "equivalent fractions" and "addition of fraction". They pointed out 3/4 + 3/4 was equal to 6/4 rather than 6/8. Mary did not recognize this mistake and she said she would like to praise her students' efforts. During the teacher interview, Rose and Lisa especially emphasized the importance of the concept fraction in the learning of equivalent fractions.

Rose believed that this error showed students' weak understanding of denominator. Every year students in her class made this type of mistake and she had to spend lots of time telling them what the denominator meant. According to Rose, the emphasis on denominator could contribute to students' understanding of equivalent fractions and addressing this type of error as well.

> T: When you were dealing with equivalent fractions, you were doing 3/4 = 6/8; both the numerator and the denominator change because if done, you changed the number of pieces of the whole as divided by 2. When you add it, you do not change the piece that has been divided, they are still fourths. So if you add ¾ and another ¾, you get 6 of the things we call fourths. The children will call them eighths.

To help student understand equivalent fractions, she also grasped "denominator" and analyzed the meaning of addition of fractions:

> T: We talk at length about how is it possible to take two items that are cut into fourths", like ¾ and ¾, and all of a sudden they become eighths??? … Sometimes I said that you cannot add apple and oranges. If you have 3 oranges, then you have 2 apples, then 3 oranges. You cannot add them until they become fruit.

Lisa said she would like to start from the real life example like eating sandwich and guide students to see why 3/4 + 3/4 = 6/8 was wrong.

> T: I always start with something like, ok, if I have half pieces of a sandwich, and I eat that half, but the other half I have left over. And I go back and I eat the other half, what have I eaten? And they will tell you that you have eaten the

whole sandwich. Ok, that's correct. Now, let's write it mathematically and you will see how I eat the whole sandwich. So if I do 1 plus 1 is 2, and 2 plus 2 is 4, 2/4 (wrote on the paper: 1/2 + ½ = 2/4). Is that a whole sandwich? And they will say no… And they figure it out, that is "1". My denominator is still 2 because I did not change the sandwich from halves to fourths. So I started with half and I ended with half. So I talked them about the sandwich and they would say, "Ok, I know, 2/2 is 1, and you eat the whole sandwich."

Lisa said after students understood the sandwich example (1/2 + 1/2 ≠ 2/4), she would like to relate it to this error: 3/4 + 3/4 = 6/8. She would also draw pictures to show why 3/4 and 6/8 were equivalent:

T: I will use something they understand and they can relate. Ok, because 3/4 and 3/4, you are still dealing with fourths. Not eighths, but fourths. So then we draw 3/4 and 6/8 to see why they are equal to one another.

T or F questions. Two questions (4) and (7) were related to this theme. Question (4) says, 3/4, 6/8, 9/12, and 12/16 are equivalent fractions because $\frac{3}{4}+\frac{3}{4}=\frac{6}{8}+\frac{3}{4}=\frac{9}{12}+\frac{3}{4}=\frac{12}{16}$. This question was similar to the above designed case but was more complex. It was inspired by a student's mistake from a MSMP video that was not included in this study. When asked to find equivalent fractions, this student's verbal representation was correct but he made errors in her written format: he kept adding 3 to the top and 4 to the bottom of these fractions. In fact, this student combined 3/4 + 3/4 = 6/8 and 6/8 + 3/4 = 9/12 into one form: 3/4 + 3/4 = 6/8 + 3/4 = 9/12. Thus, he made two mistakes here. First, he viewed $\frac{3}{4}+\frac{3}{4}=\frac{6}{8}$ as the same representation as $\frac{3+3}{4+4}=\frac{6}{8}$, a similar mistake as the above one in the designed case. Second, when the student combined the forms into one, he made the similar mistake as “15 + 20 = 35 + 5 = 40 + 1 = 41”, a mistake related to equivalence.

Again, all teachers except Mary in this study correctly answered this question. They answered it as “False”. However, they only identified the first mistake. No teacher recognized there was something wrong with equivalence (elaborated upon later).

Question (7) was purposely used to examine whether teachers really had a complete understanding of “fraction addition”, a related concept to “equivalent fractions”. This question was modified from Borasi’s (1987) misleading suggestion, that is, the real life situations such as games could prove the erroneous format $\frac{2}{3}+\frac{5}{7}=\frac{7}{10}$ could be actually true. The six teachers’ understanding fell into three levels: (a) deep understanding. Jennifer and Kathleen insisted this statement to be wrong. They analyzed that students were confused by fraction addition and ratio. They said, as to fraction addition, students could not solve it in this way; (b) partially understanding. Barbara and Rose doubted this statement without insisting that it was wrong. Rose also mentioned ratio. She said she would like students to solve the ratio problem with words. However, she finally agreed with this written format. She said: “Particularly with the kids I teach, I will not do that 2/3 + 5/7 = 7/10. I will tell them it works in this case. In any other problems they have to do, it will be wrong.” Barbara did not mention ratio but emphasized that students needed to be able to explain to her; otherwise, it would be wrong. Clearly, these two teachers’ understanding of fraction addition was not complete;

(c) misunderstanding. The other two teachers Mary and Lisa totally agreed with this. Mary was not sure about it at first. After thinking it over, she said, “Ok, I am convinced. I will tell my kids: ‘you are right, you are right’ ”. Lisa also truly believed this statement was correct. Moreover, she showed me an example, that is, “11 + 2 = 1” was true. She said “When 11 o’clock and you add 2 hours, and you end up with 1 o’clock”. I asked Lisa, if first graders wrote the format like “11+2=1”, whether she would agree. She said, if student could explain to her, she would say it was correct.

In general, the designed case and two T or F questions show teachers’ understanding of the concept fraction and equivalent fractions. Most of the teachers immediately recognized that 3/4 + 3/4 = 6/8 was incorrect. However, when they encountered the related concepts such as equivalence, addition of fractions, and ratio problems, some teachers showed weakness in their knowledge.

4.4.2 Teacher responses in classrooms

In the enacted teaching context, teachers’ awareness of this type of error and difficulty was displayed differently.

Being sensitive to adding 2/3. Both Barbara and Jennifer paid attention to the error of “adding a fraction” in finding equivalent fractions. In Barbara’s class, even though such errors were not observed, Barbara reminded her class “Ok, remember, you are not actually adding 2/3, you are adding 2 to the numerator and 3 to the denominator.” In Jennifer’s class, some students showed their difficulty in verbal representations of this adding pattern. For example, one student said, “I did 2/3 plus 2/3, but it’s not really plus 2/3. But it’s numerator plus numerator and denominator plus the denominator”. This student recognized that he was not really adding a fraction. However, his words "numerator plus numerator and denominator plus the denominator" was still inappropriate. Since Jennifer was always aware of this type of error and difficulty, students in her class showed clear understanding. What follows is an example where a student reported his pattern of "addition":

S: 8/12 plus and then I added the numerator, not really the fractions 2/3 but 2 and 3 (wrote down + 2 and +3 respectively without fraction line) (see Figure 22)

T: Why do you say not really the fraction 2/3?

S: Because 8/12 plus 2/3 it isn't the same.

T: Ok, great, thank you very much

S: And then that equal 10/15

T: So you adding the numerator, or the original numerator 2 to the new numerator 8, and the original 3 to the new one 12, ok.

S: And did that again, (write down 10/15 + 2 3 = 12/18)

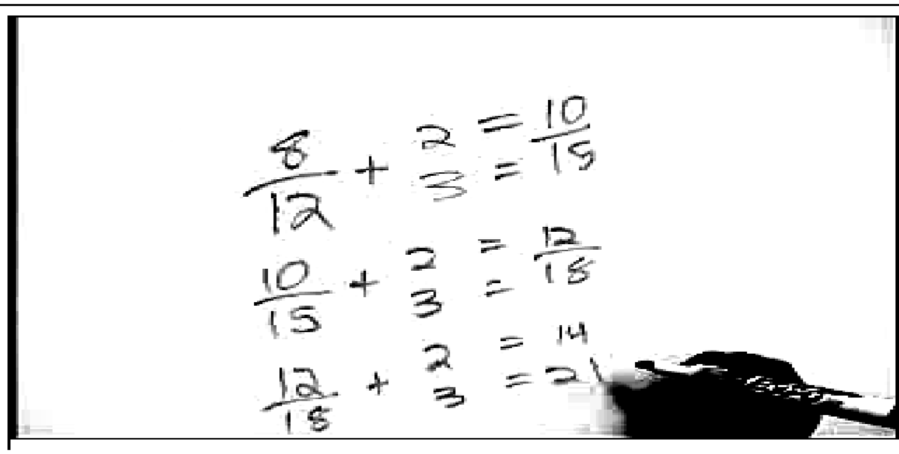

Figure 22. A clear understanding of addition pattern in Jennifer's class.

Teacher making mistakes – Mary. Consistent with the interview, Mary, the teacher who agreed with student invented strategy ¾ + ¾ = 6/8 in the designed case, made similar mistakes in her teaching practices. As previously introduced, Mary's class was busy with counting and labeling fraction strips rather than looking for patterns, therefore, her class did not really discuss either addition or multiplication patterns. As a result, there were not many related mistakes shown on the video tapes. The following sighting was the only one where the class discussed the addition pattern.

T: Ok, so it's like 2/3 …If I multiply 2/3 to everyone, um, to everyone of these, or add 2/3 to everyone of these, 4/6 plus 2/3, I came out with what? 8/9. 8/9 plus 2/3, I come out with what? 10 what? 10/12. No, 6/9, I'm sorry. 8/12. 8/12 plus 2/3 will give me? What's the next? 8 plus 2 is what? 10. 12 plus 2 is what? 15.

T: You notice I can keep going? I keep going on what?

S: Add 2/3

T: There you go! There you go!

In the above short conversation, Mary's language was confusing and misleading. First, she was trying to guide students with the pattern "adding 2/3", that is, 4/6 + 2/3 = 6/9, 6/9 + 2/3 = 8/12, 8/12+ 2/3 =10/15. This was incorrect. Second, she made a computational error. According to her pattern, 4/6 + 2/3 should be 6/9. However, she said 8/9. Even though she corrected herself, her mathematical language was confusing. Mary's classroom teaching showed her knowledge weakness. When a teacher is lacking deep understanding of certain concepts, he or she may have limited ability to anticipate students' possible mistakes or difficulties and address them appropriately.

No errors occurring. For the other three teachers – Rose, Lisa, and Kathleen – the error of adding a fraction did not appear in their enacted lessons. One of the interpretations was these teachers had complete understanding of fraction concepts and equivalent fractions as demonstrated in the teacher interview. They might have already addressed this type of error before. Another interpretation was related to the complexity of the teaching context. For example, Rose actually viewed the "repeating adding" pattern was a problem because she emphasized multiplication patterns (discussed this in section 4.4.3). She, therefore, might not want to spend too much time on the addition pattern. Concerning Lisa and Kathleen's enacted lessons, students mainly reported multiplication patterns. As a result, these teachers may not have had a chance to address the errors such as "adding 2/3" in their classes. However, if teachers are really sensitive to the "adding 2/3" error in finding equivalent fractions, will they at least remind their students to be aware of that or will they even provide some designed cases for students' discussion and exploration?

4.4.3 Do you have a quicker way?

As mentioned above, teacher Rose even viewed the correct "addition" pattern (e.g., $\frac{3+3}{4+4}=\frac{6}{8}$) as a big concern because she thought $\frac{3\times 2}{4\times 2}=\frac{6}{8}$ should work in a more efficient way. This actually brought up another topic: student's multiplicative thinking. Since this topic is also related to "addition pattern", I describe it in this section even though it is somewhat unrelated to the "adding 3/4 error". Another reason to place this topic here is because the "multiplicative thinking" topic is based on the comparison of the two patterns - multiplication and addition - in finding equivalent fractions. As a result, the discussion under this topic also acts as a summary of the above two sections: “Are you really multiplying by 2” and “what do you mean by adding 3/4”.

Multiplicative thinking was an important thinking skill. Therefore, even though students could obtain equivalent fractions by using both patterns, teachers should point out the multiplicative way works quicker. In fact, these two patterns $\frac{3+3}{4+4}=\frac{6}{8}$ and $\frac{3\times 2}{4\times 2}=\frac{6}{8}$ are related. This is because “3+3” is the same thing as “3×2” and “4+4” is the same thing as “4×2”. No one will deny that multiplication is a convenient way for repeated addition. Teachers should grasp various opportunities to cultivate students’ multiplicative thinking. What follows are clear examples.

Rose’s class. After students provided both multiplication and addition patterns, Rose raised questions to the class: “Which pattern do they show better? Do they show the add “2” add “3” better or do they look at the other pattern that Martin looked about and Nichole is talking about. Which is what? The multiplication pattern.”

Why did Rose guide students to compare these two patterns? The interview provided the hints. During the interview, when I asked Rose’s opinion about students’ common errors, she raised the “addition” pattern – repeated addition.

I: So what kind of common errors are there in students’ learning of equivalent fractions?

T: One of them is repeated addition.

I: Repeated addition?

T: Right. They once started with, they saw the pattern 2/3, 4/6, and 6/9, they added 2 to the top and added 3 to the bottom. In another one, they added 3 to the top and 4 to the bottom, and they continued with addition.

I: But it is not wrong, right?

T: No, it will work.

I: Yes, it works.

T: Yes, it works because multiplication is repeated addition.

In Rose's mind, the first impression about the common mistake was the "addition" pattern. In fact, Rose also agreed that "repeated adding" as a pattern to find equivalent fractions was not wrong. It would actually work. However, Rose still thought it was not an effective strategy. She suggested teachers moving students from addition to multiplication.

I: So you think this is not a good strategy?

T: The good strategy that works is only when you have the first fraction and you can continue it. The strategy that you do not want to use is to pick fractions and then go in order. But you have to work them from repeated addition because the majority of my class will do repeated addition initially. You have to move them toward multiplying.

I: So like you move them from additional thinking to multiplicative thinking?

T: Right. They will not realize that. The repeated addition will work. It is just not, if you want them to know ¾ is equal to what over 100, I do not want them to add ¾ until they get something over 100 because it will take them too much time.

I: They kept adding many times and then got the big number.

T: Right. You have to show them that it will work but it is not efficient.

Kathleen's class. Kathleen also emphasized students' multiplicative thinking. She employed a directive teaching method. During the summary part, when one student explained the strategy of "wonderful 1", Kathleen then pointed out that multiplying by "wonderful 1" was more convenient than the pattern of addition. After that, another student provided her pattern of "addition". Kathleen, therefore, compared these two patterns again and underscored the importance of multiplication:

S: For 2/3, you just add 3 to the denominator and for ¾, I add 4 to the denominator.

T: Ok, there is another pattern. Here, if I had 6, then I had 9, then I had 12, what pattern was there? Or what pattern did we find there?

S: Plus 3.

T: Ok, so there were other patterns. Which one's going to help you all the time? Because what if you don't have more than one number? You just have two numbers. How could you compare them to make sure they're equivalent?

S: The wonderful 1.

T: The wonderful 1, but that was a pattern to?

S: Divide fractions.

T: Yes, so you can simplify the fractions.

In this sighting, Kathleen pointed out that multiplication was more convenient than addition. Noticeably, she specifically mentioned "division" when they discussed the "wonderful 1", which again proved that this teacher tended to make connections between multiplication and division as previously discussed (see section 4.3.4). As a result, it is reasonable to conjecture that students in her class were more likely to obtain multiplicative thinking with deep understanding.

Barbara's class. Barbara employed cooperative learning. During group exploration, she was constantly guiding students not only to use additive but also multiplicative way to find equivalent fractions. The following intervention sighting is a typical one:

T: How's it going?

S: Add 3.

T: Good for you… Excellent. What do you think would be another one?

S: 10/15.

T: Ok, how did you get that one?

S: Because there is a pattern that going up by 2 on these and by 3 on these.

T: Ok, good.

T: Do you think could it be any other operation besides adding?

S: You can times it by 2, you can multiply it.

T: (Left) Ok, good. (Stopped) What would you multiply by to get that 10/15 if you started from 2/3?

S: 5.

T: Yes, good. (Left but is stopped again) 5 or 5/5?

S: 5/5

T: Ok, cool, good job.

In this sighting, the student first reported the addition pattern “adding 2 and adding 3”. Barbara then guided her whether there could be another operation besides adding. This student immediately thought of “multiplication”. She, however, made a computational error. When she corrected the mistake “2” as “5”, Barbara caught another mistake, the same mistake as “Doubling Error”. Barbara’s question “5 or 5/5” accurately moved this student from additive to multiplicative thinking while reinforcing the basic idea of identity property of multiplication.

Interestingly, during the interview, I asked Barbara why she emphasized the multiplication pattern, she thought for a while and told me that teachers should always encourage students to solve problems from different ways. It seemed Barbara did not explicitly notice the importance of multiplicative thinking. However, since she was actually guiding students to try another operation besides adding, students themselves probably realized the importance of multiplication. As a result, there were no students reporting the pattern of addition. This class was, therefore, able to sufficiently discuss the errors and difficulties related to multiplicative thinking.

All these three teachers Rose, Barbara, and Kathleen moved students’ thinking from addition to multiplication even though Barbara’s purpose was for students to develop multiple solutions. Students who received this type of instruction over time would be more likely to develop their mathematical thinking.

4.4.4 Summary

In this section, I discussed the common error “3/4 + ¾ = 6/8” in the addition pattern for finding equivalent fractions. Teachers in general had an understanding of this type of error and difficulty. However, knowledge difference still exists which causes their instructional differences. Students' cognitive gains were found in some teachers' classes. However, multiplication pattern, compared with the addition pattern, was more efficient. As a result, it was necessary to move students from additive thinking to multiplicative thinking.

4.5 Multiplying by the Same Number 0/0?

4.5.1 Why this theme?

This is an emergent theme that occurred through the data processing. The learning goal of Lesson 2.2 is to find equivalent fractions. That is, students are expected to find patterns and then to know why. Students in all the six video-taped classes found the rule "multiplying the numerator and denominator by the same number" which was exactly consistent with the CMP teacher guide book: "You want students to propose the idea that you can multiply the numerator and the denominator of the original representation of the fraction by the same number – any whole number – to obtain an equivalent fraction" (Lappan et al., 1998, p.30b). This rule works "effectively" because it can help students find equivalent fractions. This rule also works "safely" because it will not result in mistakes during students' work.

However, when I observed these videos and heard teachers repeating this rule without something that I expected, I was amazed. "Something" that had been engrained in my mind since I was an elementary student is "except 0". According to my personal experience, whether as a Chinese student or as a Chinese elementary mathematics teacher, we called this rule "the basic property of fraction" (Fen Shu de Ji Ben Xing Zhi). That is, to multiply or divide the numerator and denominator by the same number (except 0), the value of the fraction will not change.

Compared with the rule that I learned, the suggestion from the CMP teacher guide book provides incomplete and misleading information. It has three shortcomings: (a) it only pointed out multiplication strategies without mentioning “division”; (b) it explicitly explained the “same number” as “any whole number”. However, should it necessarily be “whole” number? (c) it emphasized “any” whole number. However, does “0” work? The first two shortcomings were discussed in earlier sections and thus will not be discussed here. Regarding the third one, I clearly remembered that “except 0” or “nonzero number” was emphasized by Chinese textbooks and Chinese teachers because of the basic mathematics principle “0 cannot be a divisor”. In fact, “0 cannot be a divisor”, “0 as a divisor was undefined”, or “0 as a divisor has no meaning” was believed as a dogma in Chinese school mathematics and every student even in early grades is expected to understand this. As a result, “0 cannot be a divisor” would be automatically considered for the mathematical precision reasons. Obviously, my study was not a cross-country study, therefore, I will not involve any Chinese teachers in my study. However, my Chinese background provided me with sensitivity to “0” and I was curious because no teachers in my study emphasized it. I was even more surprising after I carefully examined the CMP teacher guide book, where the rule was clearly stated without any mention of “0”. Under this situation, I modified my interview questions by adding the ones concerning “0/0” and brought them to the teachers’ attention during the interviews.

4.5.2 Interview materials concerning 0/0

To examine teacher knowledge, I designed three T or F questions (see Figure 23)

(1) To find equivalent fractions we can multiply the numerator and denominator by the same number. ……………………………………………………………..(   )

(5) $\frac{3\times 0}{4\times 0}=\frac{3\times 1}{4\times 1}=\frac{3\times 2}{4\times 2}=\frac{3\times 3}{4\times 3}$ …………………………………………………… (.)

(8) $\frac{0}{0}=\frac{1}{1}=\frac{2}{2}=\frac{3}{3}=\frac{4}{4}$, because all of these fractions equal to one whole…...(   )

*Figure 23.*True or False questions concerning 0/0.

The T or F question (1) was the same rule that was provided by the CMP teacher guide book and also repeated by teachers in their enacted lessons. The purpose of this question was to examine whether teachers were able to recognize "any number" should exclude "0". To decrease the possibility that teachers could not identify the "0" issue because their attention was drawn away by the "rule" for finding equivalent fractions, I provided question (5) and (8) where "0/0" was explicitly presented. Question (5) was an elaboration of the rule in question (1), providing the hints for teachers to go back to identify the problem in question (1). Question (8) again contained 0/0 in a much more obvious way reminding teachers that both question (1) and (5) were wrong. As a result, the three questions were triangulated for the examination of teachers' knowledge about "0/0" and for the purpose of finding out why no teacher mentioned "except 0" in the teaching context.

4.5.3 Teacher responses to "0/0"

All six teachers provided their answers and explanations for these three T or F questions through the interviews. In general, teachers in this study did not demonstrate their good understanding of "0" (see Table 6).

Table 6

Teacher responses to T or F questions

Name	(1)	(5)	(8)
Kathleen	T	F 0/0 is not possible	F 0/0 is not possible
Jennifer	F nonzero whole number	F 0/0 is not possible	F 0/0 is not possible
Lisa	T	F…………….0/0=0	F…………….0/0=0
Mary	T	T…………….0/0=0	T…………….0/0=0
Barbara	T	F……………..0/0=0	F……………..0/0=0
Rose	T	F……………..0/0=0	F……………...0/0=0

With regard to question (1), very few teachers (except Jennifer) recognized that "0" could be a hidden issue in the rule of finding equivalent fractions. Concerning question (5) and (8), when the "0/0" was brought to these teachers' attention, all of them except Mary correctly answered these two questions as "False". However, their explanations were not satisfying. No teacher recognized why "0/0" was a problem. There were two levels of understanding indicated by teachers' explanations of why 0/0 was wrong: (a) a correct explanation without certain depth – 0 as a divisor is undefined. Two teachers, Kathleen and Jennifer, reached this level; (b) a wrong explanation - 0/0=0. The other four teachers including Mary believed that 0/0=0. However, Mary's incorrect understanding for question (5) and (8) was caused by her confusion with other concepts rather than "0/0". In the following sections, I provide teachers' responses to "0/0" based on two levels of understanding.

A correct explanation without certain depth - "0/0" is undefined. Both Kathleen and Jennifer recognized that "0" cannot be a divisor. They said "it is just impossible to divide 0" and "it is just mathematically inaccurate to divide by 0". However, they could not remember why. What follows are the interviews concerning 0/0.

Jennifer's responses. Regarding the first question - finding equivalent fractions, we can multiply the numerator and denominator by the same number - Jennifer pointed out the problem with "any number":

I: About the first question, why do you think it is false?

T: Because it cannot just be any number. It has to be a whole non-zero number. I probably said that aloud in some point. But I think you can figure out saying multiplying by the same number. But you also need to know it cannot be any number, it cannot be zero, it has to be a whole number. And one of the things, I will prefer to say, like, one whole, like 2/2, 3/3, 4/4, I like to say it in that way. You know, people will probably say that is true, but when you really look at it deeper it definitely looks different.

I: So you think the "same number" is a problem. You cannot just simply say "same number", right?

T: Yes, that is correct. I also say that, I think probably, I might say this to my class because it is easy to say that, I think that is a common error that teachers would like to say.

Jennifer is the only teacher that identified the problem of “any number”. She said it should not be “any number” but “any nonzero whole number”. Even though her idea about “whole number” was incomplete as previously discussed (see section 4.3.4, e.g., 2/4 ×□/□=3/6), her answer of “nonzero” is correct. Interestingly, Jennifer also admitted that she probably said this to her class because it was easy to say. She pointed out that many people would also claim this statement was true if they did not look at it in a deeper way. She mentioned that it might be a common error that teachers made during their lessons.

Regarding question (5) and (8), Jennifer also judged them as False without any hesitation. She explained “0” could not be a divisor because it was not possible. However, when I asked her why it was not possible, Jennifer said she did not know, it was just mathematically not accurate. What follows is our conversation:

T: Number 5 is false because the 0, that’s not even possible.

I: 0 not even possible?

T: Right, you cannot divide by 0, so, you will get an error.

I: So, why 0 cannot be a divisor?

T: That’s just mathematically not correct. You cannot do that. I do not know, you cannot divide something by 0.

I: Yes, we learned that in elementary school.

T: Yes, you know, you just cannot do that.

T: The last one (question 8) is false because 0 cannot be a divisor. It is just mathematically not accurate.

Kathleen's responses. Kathleen was the other teacher who correctly understood the "0/0". She, however, could not explain why "0" cannot be a divisor either. On question 8, she first drew several rectangles with the same length to represent 1/1, 2/2, 3/3, 4/4. She said it was true because all of these were one whole. However, she skipped 0/0. Therefore, I did some further investigating:

I: (pointing at 0/0) How will you draw this picture?

T: Nothing. I will leave it blank. Oh, that's a good question. 0/0? Is that one whole?

I: I do not know.

T: I do not think so. That will be good to show that it's not possible. This is not 1.

When I asked Kathleen how to draw 0/0, at first she said she would leave it bank, but then immediately recognized that she could not draw a picture to show 0/0. She said it was not possible. To make certain of her understanding, I posed her a complex question:

I: (Pointing at the blank box she drew before) Students say, see there is nothing, nothing out of nothing, that is one whole. Is that one whole?

T: Because this is also 0 divided by 0. And that's not right.

I: 0/0, what's the value?

T: In higher math, there is no value. You cannot divide by 0.

Kathleen insisted on her answer: 0/0 was not possible. She also pointed out that in higher math, there is no value. When she mentioned "higher math", I interpreted that she wanted to use the term "undefined", but she did not say that. I kept asking questions to see whether she knew why 0 could not be a divisor.

I: Ok, we cannot divide by 0. Why?

T: I do not know. I cannot remember (laughing). Let me think. Yes, because there is nothing to be divided by. There is no standard, you know, there is no possibility.

I: So you think 0/0, that's not possible

T: That's not possible. Well, they get a, what did they call?

I: Undefined, you mean?

T: Right! Thank you! Thank you! (Laughing) What is that word? I haven't used it for a long time!

Kathleen was searching hard for the word "undefined", a terminology used in advanced math. However, she still did not explain why "0 as a divisor" was undefined.

A wrong explanation - 0/0=0. The other four teachers were sure that 0/0 had a value, that is, 0. Therefore, they answered question (5) and (8) as "False" except Mary. In fact, Mary thought about it in this way at first but she changed her mind after she rethought about the problem. According to these teachers, 0/0=0, therefore, $\frac{3\times 0}{4\times 0} = 0$.

Since $\frac{3\times 0}{4\times 0}$ =0 while $\frac{3\times 1}{4\times 1}=\frac{3\times 2}{4\times 2}=\frac{3\times 3}{4\times 3}=\frac{3}{4}$, and $0\neq\frac{3}{4}$, therefore, question (5) was False.

Regarding question (8), they thought $\frac{0}{0} = 0$ while $\frac{1}{1}=\frac{2}{2}=\frac{3}{3}=\frac{4}{4}$ =1, obviously, $0\neq 1$, therefore, they also answered it as "False". In contrast, Mary answered these two questions as "True" mainly due to her confusion with "one whole" and "whole number" instead of knowledge of "0/0". For example, when she answered question (8), she wrote down 0/0=0, 1/1=1, 2/2=2, 3/3=3, and 4/4=4, she said 0, 1, 2, 3, and 4 were all whole numbers. Therefore, she changed her line of thinking and agreed these questions were both correct. In the following sections, I provide teachers' typical responses at this level of understanding.

Lisa's responses.

T: #5 is false because 0. From there on, it will be true (covering $\frac{3\times 0}{4\times 0}$)

I: Could you please explain why "0" is wrong here?

T: Yes, because we talked about how "0" is nothing. If my whole is nothing, if my whole is "0", is nothing. So it cannot be equal to 3/3, which is 1.

I: Did you tell your students nothing is equal to 0?

T: We talked about how denominator is "0" and if something cost 0, cost nothing, it is free. They have already known 0/0, which equals to 0. We talked about how you have to multiply the numerator and denominator by a fraction equal to 1, and we talked about how the fraction can be made up by any number greater than 0 because 0/0 is 0.

T: The last one, I put false because here, 0/0, that's equal to 0. (write down 0/0=0)

To Lisa, the problem in 0/0 was not about the divisor but because of 0/0 itself. She said if the denominator was "0", it meant nothing. As a result, she told her students not to multiply the numerator and denominator by "0" because 0/0 was 0. Lisa even summarized a rule for her students "to multiply by any number greater than 0". Obviously, "greater than" was another problem because "less than 0" would yield a correct answer. For example, when multiplying the numerator and denominator both by "-2", it still works.

Rose's responses. Rose also argued that 0/0 = 0 because she thought 0 was nothing. Nothing out of nothing equals to nothing. Regarding T or F question (5), she directly pointed out the problem of "0":

T: The only problem is to times 0.

I: Times 0?

T: That would have messed them up because 3 times 0 is 0 and 4 times 0 is 0 and how could that equal 3/4?

I: Why does it not equal ¾?

T: Well, 3 times nothing is nothing. 4 times nothing is nothing.

I: so it will be 0/0

T: Right, so how could it equal ¾?

I: What will it be?

T: It will be 0!

To Rose, 0/0 was absolutely equal to 0. Since $\frac{3\times 0}{4\times 0}=\frac{0}{0}=0$ while $\frac{3\times 1}{4\times 1}=\frac{3}{4}$, question (5) was "False". Moreover, Rose pointed out that students had not formally learned this "Property of 0" yet. "The property of 0", according to Rose, was to multiply the numerator and denominator both by 0, and it should be "0", similar to the whole number operation:

T: And I think you have to deal with the property of 0. So that's a different math concept that they have not been formally introduced to.

I: Introduced to what?

T: The concept multiplying by 0. You know they know the whole number and they know 0 times any number is 0, but when you write it in equivalent fractions, I cannot imagine why they will even think of using 0.

I: Right, they may not use.

T: I cannot even imagine why because I never have had a child to do that. They started from 1. 0? No.

Interestingly, in the above conversation, Rose realized some problems with multiplying "0/0" because according to her experience, students would not use "0". She said she could not imagine why some student would think of using 0.

Regarding 0/0 is a whole, Rose rejected this idea because of the same concern, 0/0 was not one whole but equal to 0.

T: Oh, 0/0? I never consider 0 out 0 as one whole.

I: Why?

T: It got nothing.

I: So nothing out of nothing is that one whole?

T: It is nothing. That is 0. I do not having anything. I got nothing.

Summary of teachers' responses. In general, teachers' responses to 0/0 or "0 as a divisor" in this study were not unexpected because prior studies had already pointed out this phenomenon (e.g., Ball, 1999). In Ball's (1999) study, for "7/0", some teachers said it was zero; whereas some teachers said "0" could not be a divisor because it was undefined but without knowing why. The other teachers could not recall a rule at all and left this problem unsolved. Clearly, teachers' understanding about "0/0" in this study was basically consistent with that of Ball's study. In fact, the questions (5) and (8) provided the hints for teachers to find the inaccuracy in question (1). However, no teachers went back to point out the problem even after they had done the other two questions, which showed that most teachers in this study were lacking mathematical sensitivity and reflective thinking.

4.5.4 Does 0/0 matter in the understanding of equivalent fractions?

Since my study was about equivalent fractions, it seemed unnecessary to expand upon and discuss teacher responses to 0/0. However, "0 cannot be a divisor" is one of the mathematical principles and fundamental ideas that teachers and students should know. Lacking this type of knowledge will inhibit teachers' sensitivity to school mathematics. For example, they may not be able to identify incomplete information provided by curricula (Leinhardt & Smith, 1985). As previously mentioned, the suggestion by the CMP teacher guide book "multiply the numerator and the denominator of the original representation of the fraction by the same number – any whole number – to obtain an equivalent fraction" (Lappan et al., 1998, p.30b) was actually incomplete and misleading information. Facing the language "any whole number", teachers with complete understanding of fundamental mathematical ideas such as 0/0 may raise two arguments: (a) Can a fraction multiply by 1.5/1.5? Mathematically, it can. However, 1.5 is not a "whole" number. (b)Can a fraction multiply by 0/0? Mathematically it cannot. However, "0" is a "whole" number. However, when a teacher lacks this type of knowledge, it is hard for them to be sensitive to this type of imprecise mathematical language concerning the rule for finding equivalent fractions. Thus, teachers' weak understanding of basic ideas may hinder their deep understanding of equivalent fractions, which in turn causes procedurally-based classroom instruction.

4.6 Equivalent Fractions and Equivalence

4.6.1 Origination of this theme

This second emergent theme in this study was about the concept of "equivalence", a fundamental mathematical idea reflected in equivalent fractions. Since U.S. students had a weak understanding of this concept (e.g., Ding et al., 2007; Knuth et al., 2006), it is meaningful to examine teachers' understanding of equivalent fractions with an eye on the concept of "equivalence".

The direct inspiration of this theme was two MSMP videos where a teacher and a student made mistakes about equivalence. The teacher and the student were not from the same class and these two videos were not part of this study. However, these errors were transformed into two interview questions.

The student mistake occurred during his explanation about how he found five equivalent fractions for 3/4: 6/8, 9/12, 12/16, 15/20, and 18/24. This student wrote down $\frac{3}{4}+\frac{3}{4}=\frac{6}{8}+\frac{3}{4}=\frac{9}{12}+\frac{3}{4}=\frac{12}{16}+\frac{3}{4}=\frac{15}{20}+\frac{3}{4}=\frac{18}{24}$. As previously mentioned, this format contains two mistakes: (a) repeated adding ¾ and (b) equivalence. Concerning (b) equivalence, the student combined two wrong number sentences, that is, he combined $\frac{3}{4}+\frac{3}{4}=\frac{6}{8}$ and $\frac{6}{8}+\frac{3}{4}=\frac{9}{12}$ into one number sentence: $\frac{3}{4}+\frac{3}{4}=\frac{6}{8}+\frac{3}{4}=\frac{9}{12}$. However, this teacher was not aware of his mistake.

The mistake made by the teacher occurred during a student's verbal explanation of why ¾ was correct and why 2/3 was wrong in Lesson 2.1. This student correctly represented his idea while the teacher made a mistake. This student said, "360 divided by 4 is 90, 90 times 3 is 270; 360 divided by 3 is 120, 120 times 2 is 240." Symbolically, this student was saying: (a) $360 \div 4 = 90$; $90 \times 3 = 270$; and (b) $360 \div 2 = 120$; $120 \times 3 = 240$. However, the teacher wrote down the student's words on the blackboard as: (a) $360 \div 4 = 90 \times 3 = 270$; and (b) $360 \div 2 = 120 \times 3 = 240$. This teacher combined the student's two number sentences into one. As a result, she made a similar error as the student discussed above.

Both the student and the teacher showed a weak understanding of equivalence. Their mistakes might be influenced by their linearity of thinking and speaking (Sáenz-Ludlow & Walgamuth, 1998) which corresponded with the linear way of writing. Based on the video evidence in other teachers' classes, I designed two T or F questions to examine teacher knowledge of equivalence in this study (see Figure 24).

(3) 270/360 = ¾ because $270 \div 3 = 90 \times 4 = 360$ or $360 \div 4 = 90 \times 3 = 270$ ………(   )

(4) ¾, 6/8, 9/12, and 12/16 are equivalent fractions because

$$\frac{3}{4}+\frac{3}{4}=\frac{6}{8}+\frac{3}{4}=\frac{9}{12}+\frac{3}{4}=\frac{12}{16}$$ ……………………………………………………(   )

Figure 24. True or False questions concerning equivalence.

4.6.2 Teacher knowledge about equivalence

Teachers' responses to the above T or F questions reflected their knowledge of and sensitivity to "equivalence". In general, teachers in this study had a better understanding of equivalence than "0/0". However, the situation was still not promising. At most, half of these teachers had a clear understanding. In question (4), no teachers pointed out the "equivalence" mistake. In question (3), teachers' responses demonstrated two levels of understanding.

Correct understanding. Rose and Jennifer showed better understanding than Barbara who provided the correct answer in a follow-up call.

Rose's responses. Rose knew the equal sign well. As I mentioned before, Rose was the only teacher who pointed out that students' weak understanding of the equal sign could be one of the reasons for the "Doubling Error". What follows is her elaboration about the problems that her students encountered with the "=".

I: Could you please explain more about the equal sign?

T: Ok, the children say 2 + 3 = 5, 5 + 3 = 8, the equal sign to them means the answer. They do not understand 5 + 3 is the same value as 8.

I: So they see the equal sign as an operator, to do something.

T: Right. 5 + 3 give them an answer 8. It does not mean the value on the right is the same value as the left. They truly do not understand that. The equal sign means the answer.

I: Means the answer. Yes. Not the relationship.

T: Yes, it is not the relationship between the left site and the right site.

Clearly, Rose had a very clear understanding of the equal sign. She knew students viewed the equal sign as an "answer" rather than relationship between the left side and the right side. Rose told me she brought up the "equal sign" issue to students every year as they discussed many different mathematical topics not just algebraic expressions. She said she tried to get them to understand what's on the right was the same as what's on the left. When I asked her whether she had used the equal sign to address student errors in the teaching of equivalent fractions, she said she did not do that but she would like to try in the future. She said, "But I can tell you for sure that I will try that when I teach fractions."

Since Rose did not have time to pre-read the T or F questions that I sent her through the email before our interview, I read these questions to her. When I read the question (3), Rose immediately identified the problem about equivalence:

T: Ok, so they did not stop and put a period?

I: No period.

T: Ok, you go back and call it very sloppy math. It is the equal sign again.

I: Equal sign again. Ok.

T: You need to stop the child and make them two different equations. 270÷3=90, 90×4=360.

Jennifer's responses. Jennifer also had a good understanding of equivalence. She recognized the student's error and pointed out where the mistake was: "270 divided by 3 does not equal 90 multiplied by 4, or 360. So that sign of equivalent is not accurate." Jennifer also attributed student errors to the teacher model effect:

I: Ok, what do you think the reason here, why students make this kind of error?

T: Hum, I am not sure. Probably their teacher wrote that? I do not know. I just think that when you write it, I do not know. I would think they probably have a model for them. I really would ever… I cannot think of a time that I saw my students write something like that.

I: Because you never do this.

T: Yes, that's true. I do not think that I ever do that because that is confusing. But I think that it has to be a model for them or maybe they came up with something.

Jennifer thought about teachers as students' models whose representations would affect students' learning. She said she never wrote in this way, therefore, she could not remember if her students made such errors. Jennifer also provided another interpretation why she did not see sixth graders make this type of linear mistake.

T: But most sixth graders do not write like that. You usually do not write like that until you get algebra or line type of things.

I: So do you think this will influence students' later algebra learning?

T: Yes. My students do not learn algebra. They are in sixth grade. But I think once you look at algebra, it is always about this type of things like equations. They learn more linear math. Well, I guess what I am saying is, in sixth grade they were doing things more vertically, not as much as linearly, like writing the equations problems, so I just did not see this type of error in sixth grade classrooms.

Jennifer agreed that this type of error would negatively influence student algebraic learning. She mentioned even though students in sixth grade did not make many mistakes like this, when they learn algebra, they would do more things like equation. As a result, students' written format would have to change from vertically to linearly, which might cause more mistakes about "equivalence".

Barbara's response. Barbara's response to the T or F question (3) was interesting. She provided me the answer via a follow-up call, 5 minutes after our interview. Since both Barbara and Rose did not have time to pre-read the questions, I read these questions slowly and clearly over the phone to them. Rose immediately identified the error while Barbara did not. She spent a long time considering why students explained 270/360=3/4 in this way rather than dividing both the numerator and denominator by 90. She said that however, the student's way was also correct. I reminded her that the student explained in one number sentence, she ended up the conversation saying this statement could be true. After the interview was completed, she called me back and said after she thought it over and wrote it down, she realized it was incorrect

There are two interpretations of Barbara's interview of T or F question (3). First, it might reflect the limitation of my interview methods. Since I employed a phone interview with teachers who lived at a distance and some teachers did not have time to pre-read these questions, there could have been some difficulties in our conversation. Barbara might have misunderstood my question at first. In retrospect, she recognized what the question was about and then changed her answer. The second interpretation was that, compared with Rose's immediate response, Barbara's knowledge of and sensitivity to equivalence did not reach the same level as that of Rose.

Lacking understanding. The other three teachers, Kathleen, Lisa, and Mary, did not find the mistake in T or F (3). Even though Kathleen and Lisa answered these two questions as False, their concerns were not about "equivalence". Mary did not find this mistake either. Instead, she accepted it as True without any hesitation.

Mary's responses. Mary read this question and connected it to the fundraising story on the textbook: 360 ÷4= 90, that's 1/4 of the fraction strip which represents $90; 90 ×3=270, that's 3/4 of the fraction strip which represents $270. $270 was exactly the fund-raising goal. This is why Mary said it was correct.

Lisa's responses. Lisa thought this student cross-multiplied or used the butterfly method backwards. She, therefore, provided a new "cross-multiply":

T: (Read question) Ok, because it is backwards, so I put false because 270÷3 = 90×4 = 360, that's wrong. Because 270×4 = 1080÷360 =3, so, it's basically (wrote down $\frac{270}{360} = \frac{3}{4}$ and crossed two lines) should cross multiply.

I: Go this way.

T: They are actually sixth graders and seventh graders.

I: Is there a name say butterfly?

T: Yes, butterfly is cross multiply.

Kathleen's responses. Kathleen also interpreted that this student was cross-multiplying, a strategy that was strongly against. She said if students showed a short-cut such as butterfly to her, she would like to say it was wrong. She said students needed to explain to her and demonstrate their understanding.

Summary. Teacher responses in the interview provided a general sense of their understanding of equivalence. Even though this type of issue was brought to teachers' attention, most of them did not demonstrate a complete understanding. Concerning T or F (4), teachers focused all their attention on the "repeated adding of 3/4" rather than the linear mistake, as a result, no teacher identified the equivalence issue. Finally in T or F (3), less than half of the teachers demonstrated a sensitivity toward “equivalence”.

4.6.3 Teacher responses in classrooms

Teachers who had this type of knowledge may not necessarily always use it effectively in their classroom teaching. In the following section, I provide teacher Jennifer’s example to show the complexity of teaching context and the existing issues in classroom practices.

Being aware of the “=”. Jennifer’s understanding of the equal sign can be seen in this classroom sighting. When the students found three equivalent fractions: 2/3, 4/6, and 6/9 and wrote them separately on the board, Jennifer asked the class whether there were additional ways to express these fractions as equivalent fractions. As a result, some students thought of using the “=” and connected them as: 6/9 = 4/6 = 2/3. Jennifer observed this fraction notation and asked further: “Does it matter which order I write them in?” Students answered: “Not really”. In this short sighting, Jennifer’s two questions reminded her students that the “=” represented equivalence and it could be used to connect fractions with the “same” values (equivalent fractions). Her question about the order also reminded students that what was on the left and right sides of the equal sign were the same.

Ignoring equivalence mistake. Even though Jennifer had a firm understanding of “equivalence”, she did not always address students’ obvious errors in her class. In the following sighting, a student went to the board and shared her explanation about why 270/360=3/4.

S: 270 and 360, (writing down) 270 + 360

T: Can you explain why? Where did you get these numbers?

S: This is the fraction like 270/360, and I just put them in that one, and I divided it by 10, (writing down) 270 + 360 ÷ 10 = 27 + 36 (see Figure 25)

…

S: 27/9 = 3, 36/9 = 4. And I put that in a fraction and I got 3/4.

T: You used the money to help you figure it out. Thanks for showing all of your work. Good job.

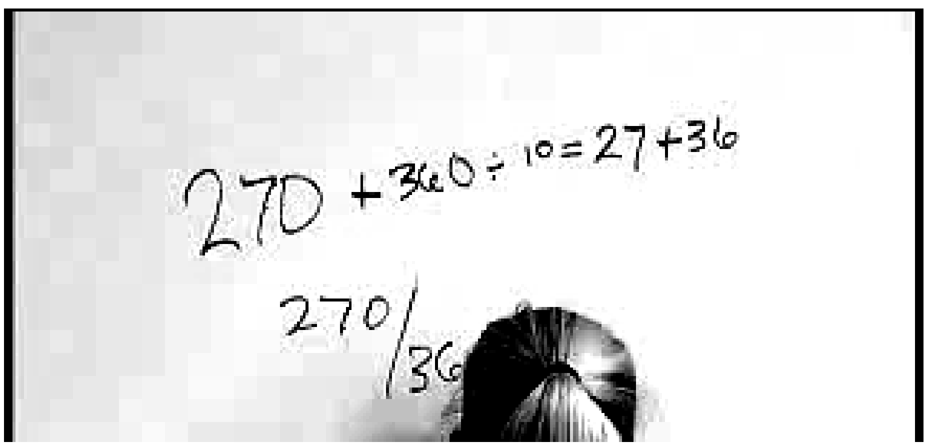

Figure 25. Is it equivalent?

Apparently, this student understood why 270/360 = 3/4, and also explained correctly in oral language such as "270 and 360", "I divided by 10"; however, her natural language influenced her symbolic representation. She put "+" while saying "and". What she did in written symbols was consistent with her oral language order. As a result, she used 270 + 360 to represent 270/360. She, therefore, wrote erroneously: 270 + 360 ÷ 10 = 27 + 36. However, is this true? Is the value on the left and right sides really equal?

Obviously, this student tried to express (270 + 360) ÷ 10 =27 + 36 or $\frac{270 \div 10}{360 \div 10} = \frac{27}{36}$. If Jennifer was really sensitive to this type of error and used the equal sign to prompt this student to self correct his mistake, this student might have a better understanding of equivalence. However, in the above sighting, Jennifer said nothing and just praised this student's work.

4.7 Summary

In this part, I reported teachers' responses to the identified student learning difficulties and common errors in learning equivalent fractions. The first difficulty was the fractional concept *whole* (fundraising "goal" in this study), a base of the core knowledge for learning equivalent fractions. Since Lesson 2.1 provided a complex context, it is hard for many students to correctly identify the "whole" for their further investigation. The second difficulty occurred in the transition from regional to number-line models. The set of ruler-like fraction strips in Lesson 2.2 had the property of both types of representations. As a result, when students tried to name equivalent fractions on their fraction strips, some students had no idea about counting lines or pieces. Teachers who were cognizant of and addressed these types of difficulties in advance, cleared the learning obstacles and provided students' more time to work on investigations connected to the learning goal. In contrast, teachers who did not realize these difficulties early in the lesson, spent much more time on these prior knowledge issues during the lesson, resulting in students' low cognitive gains on the specific learning goals.

There were two common errors, the "Doubling Error" and "repeatedly adding a fraction", that students used in finding equivalent fractions. Both errors reflected students' weak understanding of why the "pattern" or "rule" works. Teachers in this study employed different strategies to address these errors. With the "Doubling Error", some teachers addressed them in depth allowing students to see the basic mathematical principles behind the rule (e.g., 2/2=1, every number multiplied by 1 will not change the value); some teachers helped students transition from concrete to abstract representations, allowing students to see what the rule (e.g., $\frac{\times 2}{\times 2}$) really meant (e.g., "splitting each section into 2 pieces"); other teachers realized both multiplication (raising to higher-terms) and division (simplifying to lower-terms) were both strategies for finding equivalent fractions and they helped students make connection between them. Regarding the "repeated adding a fraction" mistake, some teachers paid attention to it in their classrooms. Since most of teachers purposely cultivated students' multiplicative thinking, they did not really spend enough time to prevent this type of error. Teachers' knowledge concerning this error as reflected in teacher interviews was much deeper than that in their enacted lessons.

During the examination of teachers' TRED, I combined resources such as curriculum, prior studies, other MSMP videos that were not used in this study, and teacher perspectives in the interviews. As a result, two critical mathematical issues emerged. One was concerning the basic idea that "0 cannot be a divisor" while the other, the fundamental concept of "equivalence". Even though these two themes were not directly associated with the topic "student errors and difficulties in equivalent fractions", I reported them in the last sections due to their mathematical significance.

5. DISCUSSION AND CONCLUSION

In this section, I first provide a *Mathematical Knowledge Package for Teaching Equivalent Fractions* (MKPT) as identified from this study, combined with the previous research. Beyond this topic - equivalent fractions - I discuss my findings about how teachers' MKT contributes to their TRED, resulting in different cognitive gains. Looking closer at the MKT components, I argue that SCK – using knowledge in the context- is the most critical one. However, even though some teachers in this study had knowledge of CCK and PCK, they did not demonstrate SCK in their classroom instruction. As a result, I discuss the issue of what enables SCK with an emphasis on strong connections in teachers' knowledge bases. Finally, I discuss the issue of mathematical sensitivity mainly based on the emergent data during the research.

5.1 A Mathematical Knowledge Package for Teaching Equivalent Fractions

In this study, I qualitatively researched six teachers' classroom teaching practices concerning equivalent fractions. Through the process of naturalistic inquiry of the videos and through negotiation of the meaning with teachers, I developed a deep understanding of how to effectively teach equivalent fractions. Based on these practice-based data, a MKPT for equivalent fractions was constructed. The knowledge package combined both characteristics of Ball and others' MKT (e.g., Ball, 2006; Hill et al., 2005) and Ma's (1999) knowledge package. This is because (a) the MKPT was developed from the real teaching contexts as Ball et al. intended and (b) the MKPT provides a knowledge package for teaching equivalent fractions with depth, breadth, and thoroughness. This MKPT transferred the practitioner knowledge (the insights and issues from six teachers' classroom teaching) into professional knowledge, which is storable and sharable (Hiebert et al, 2002). As a result, it serves the global aims of mathematics educational reform –to improve the teaching profession (Hiebert et al., 2002). Figure 26 provides the MKPT for teaching equivalent fractions.

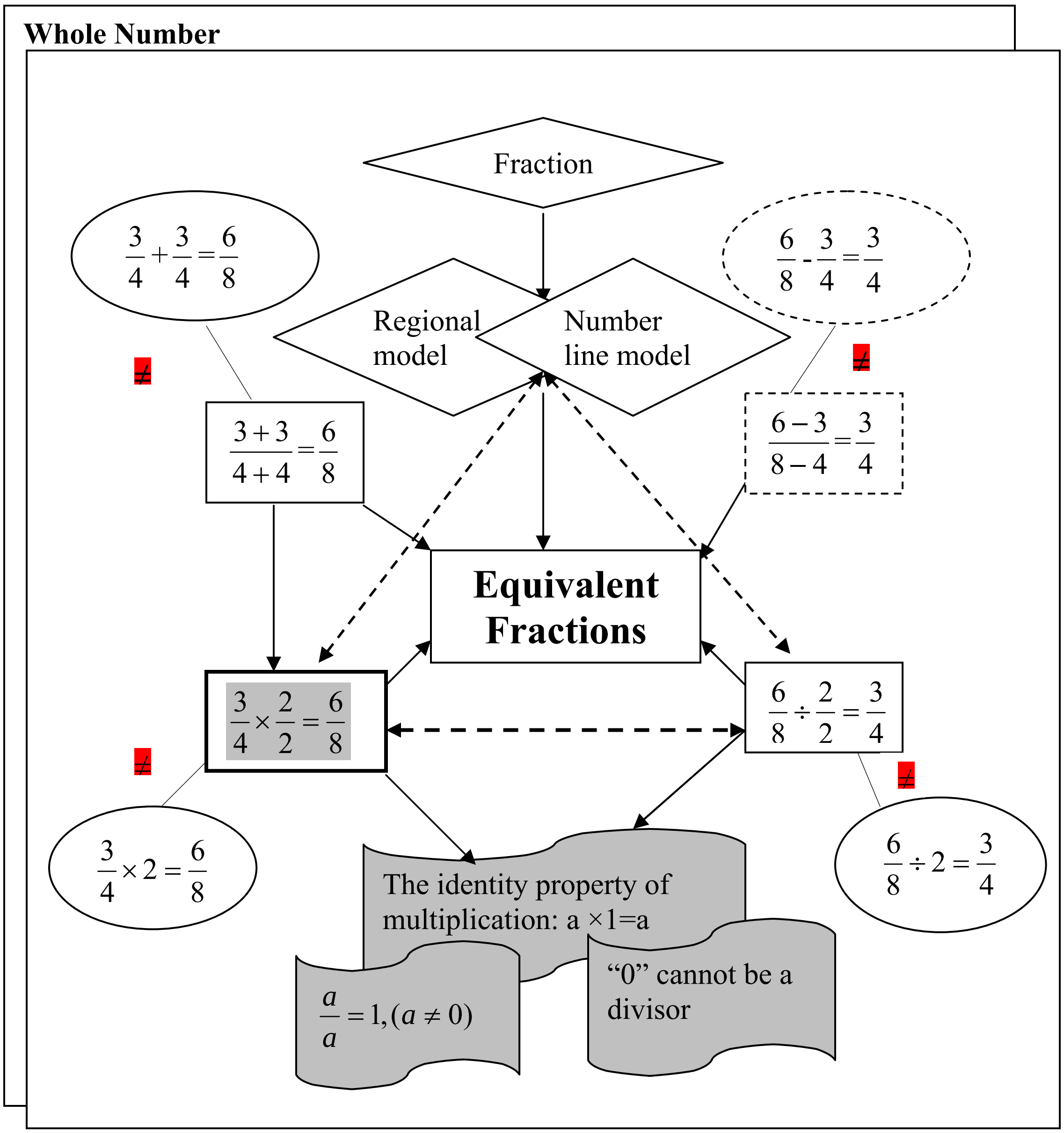

Figure 26. A MKPT for teaching equivalent fractions.

Note. All the pieces in this MKPT are connected. For clarity, I only show the more obvious or important connections for teaching equivalent fractions. Dashes represent the connections that are usually missing.

5.1.1 The explanation of MKPT for teaching equivalent fractions

Students learn fractions or equivalent fractions based on their prior knowledge of whole numbers. Students' whole number thinking has both positive and negative influences on their learning of equivalent fractions. Teachers should be aware of such influences during the teaching of equivalent fractions.

With regard to equivalent fractions, teachers who successfully taught Lessons 2.1 and 2.2 in this study, addressed the concept fraction and used non-numerical representations – regional model and number line model (see Figure 26, ◊-shape pieces). The concept fraction is the core knowledge for equivalent fraction (Leinhardt & Smith, 1985). In Lesson 2.1, some teachers grasped questions such as "what is your goal" (whole) during students' investigations by using fraction strips. With a better understanding of this prior knowledge, students were able to move smoothly into the exploration of equivalent fractions. The two non-numerical representations should be integrated. Teachers should help students transition between these two models. However, mapping students thinking from a regional model to a number-line model is not easy for students (Leinhardt & Smith, 1985). In this study, when labeling the number line–oriented fraction strips, students were not sure whether to count "lines or pieces", which was a typical error during students' transition from regional to number line representations (Chazan & Ball, 1999). Teachers should allow students to see "why to count pieces". Strategies such as the extension of the small dashes can easily change the number-line mode back into the regional one and make connections between both representations. In general, these two pieces of prior knowledge - the concept fraction and non-numerical representations - need to be addressed in advance.

To find equivalent fractions, students can use different strategies. For example, to find an equivalent fraction for ¾ or 6/8, students can start from either one to obtain the other. During this process, all four operations may be involved (see Figure 26, □-shape pieces). In this study, among these strategies, the "multiplication" rule – multiply the numerator and denominator both by the same number - was used most often ($\frac{3}{4}\times\frac{2}{2}=\frac{6}{8}$), which reflects both effective and less appropriate aspects. The effective one is when teachers emphasized students' multiplicative thinking (rather than $\frac{3+3}{4+4}=\frac{6}{8}$). The less appropriate strategy is when teachers disconnected an important connection between multiplication and division (with $\frac{6}{8}\div\frac{2}{2}=\frac{3}{4}$). In this study, the successful teachers moved students from "addition" pattern to the "multiplication" rule. However, most of the classes only discussed the "multiplication" approach. In fact, both multiplication and division were alternative ways to obtain equivalent fractions. Teachers should make connections between both approaches. If the "division" approach is taught to simplify fractions, students' understanding will be incomplete (Leinhardt & Smith, 1985). Concerning the "subtraction" strategy ($\frac{6-3}{8-4}=\frac{3}{4}$), aside from one teacher mentioning it in one class, it did not occur in this study. All these four strategies are connected.

Concerning the strategies for finding equivalent fractions, students have common errors in their written formats. For example, students may have been confused $\frac{3}{4}\times\frac{2}{2}=\frac{6}{8}$ with $\frac{3}{4}\times 2=\frac{6}{8}$ or $\frac{3+3}{4+4}=\frac{6}{8}$ with $\frac{3}{4}+\frac{3}{4}=\frac{6}{8}$ (see Figure 26, ○-shape pieces). Teachers should be highly aware of these types of errors and address them whenever they occur. Otherwise, students will be confused by fractional operations and some misconceptions such as “a bigger equivalent fraction” will be reinforced.

Student errors concerning the “rules” or “patterns” reflect both their weak understanding of equivalent fractions and learning difficulties. For example, concerning the multiplication pattern (e.g., $\frac{3}{4}\times\frac{2}{2}=\frac{6}{8}$), students may not understand the mathematical principles behind the rule. Therefore, teachers need to address this type of error in depth. Thus, teachers need to help students understand 2/2 = 1, and any number times 1 will not change the value. Beyond this example, students need to see that they could multiply both the numerator and denominator by the same number “except 0”. As a result, three basic principles should be brought to students’ attention: (a) a ×1=a, (b) a/a=1 (a≠0), and (c) 0 cannot be a divisor (see Figure 26, the highlighted part).

Since fractions are abstract, some students may not really understand the basic ideas behind the rules. As a result, teachers also need to use concrete representations (e.g., regional and number-line modes) to help students understand equivalent fractions. However, using non-numerical representations does not simply mean to compare fraction strips or draw pictures. Teachers need to transition students from concrete representations to symbolic ones. In this study, one teacher used “drawings” explicitly connecting “$\times\frac{2}{2}$” with “splitting each section into 2 pieces”. This type of connection that can effectively transition students from concrete to symbolic representations (Leinhardt &Smith, 1985), is generally missing in current classroom teaching (see Figure 26).

In summary, teachers who understand such a MKPT in teaching equivalent fractions will be aware of the critical cognitive pieces and address student errors and difficulties in depth and breadth with connectedness and thoroughness.

5.1.2 A starting point for developing professional knowledge

The components of the above MKPT for teaching equivalent fractions were identified from the enacted teaching context, transferring teachers' practitioner knowledge into professional knowledge (Hiebert et al., 2002). According to Hiebert et al., (2002), practitioner knowledge means the knowledge obtained from teaching contexts. It is very useful because it is detailed, concrete, and specific, and aligned and integrated with practice. However, this type of knowledge is not shareable because it is built in concrete contexts, this is why "good teaching" is not teachable. In other words, even though most teachers can readily provide examples, it is not obvious that they can transform their classroom-based knowledge (practitioner knowledge) into theories of teaching (professional knowledge). As a result, when the researchers and teachers negotiate those meanings and interpretations of real classroom examples, the practitioner knowledge will be developed into professional knowledge which is storable, sharable and has a broad application. In this sense, my case study of teaching equivalent fractions serves as a window on how teachers and researchers can collaborate to examine classroom teaching examples as a central way to develop professional knowledge.

The MKPT in this study was only about equivalent fractions, one of the critical focal points in school mathematics (NCTM, 2006). This package, offering pedagogically useful information for teaching practice serves as a window for this field to collaboratively develop more MKPT for teaching the other focal points. Even though "the search to delineate what it is that teachers do need to know in order to teach has been a challenging one" (Ball et al., 2001, p.440), continuing efforts on those critical concepts beginning with equivalent fractions will gradually established a sound knowledge base for teachers' professional development.

5.1.3 The need for improving MKPT

Even though the above MKPT, grounded from real classrooms, has transferability among similar teaching contexts (Lincoln & Guba, 1985), for the purpose of professional knowledge, it requires a mechanism for verification and improvement (Hiebert et al, 2002). According to Hiebert et al (2002), professional knowledge needs to be accurate, verifiable, and continually improving. Since there is no guarantee that the knowledge generated at local sites is absolutely correct or useful, the knowledge drawn from particular cases needs verification in order to become a base of professional knowledge. As a result, the above MKPT needs a continual evaluation in different local contexts. For example, researchers and teacher educators could observe those teachers who are interested in using this type of MKPT in order to modify this knowledge package in the future. Collaboration is as a central way for developing professional knowledge (Hiebert et al, 2002). With repeated observation over multiple trials, this MKPT can yield trustworthy knowledge.

5.2 MKT, TRED, and Student Cognitive Gains

The MKPT provided in the above section cannot ensure good teaching and learning because how teachers use such a knowledge package in the enacted teaching contexts makes the difference.

From this case study, the relationship between teachers' MKT, TRED, and students' cognitive gains appeared complex. Other than one teacher who obviously lacked knowledge, all the other teachers seemed to understand equivalent fractions well. However, their responses to student errors and difficulties were greatly different, which resulted in different levels of student cognitive gains. Why is it that teachers who know mathematics cannot teach mathematics effectively? What kind of mathematical knowledge can really contribute to teachers' classroom instruction? Before the discussion about how teachers' MKT influences the teaching and learning, it is necessary to look closely at the components of MKT and to locate these components in this study.

5.2.1 The components of MKT- CCK, PCK, and SCK

According to Ball (2006) (see section 2.2.1), teachers' MKT includes CCK, PCK, and SCK. CCK was related to the subject matter; PCK is the type of knowledge that teachers used to prepare for their classroom instruction; SCK is the type of knowledge that used in the teaching context or the use of the knowledge during the teaching process. In this study, teachers' CCK was mainly demonstrated by the T or F questions (see Appendix 4). PCK was reflected by teachers' discussions in the interviews about student learning difficulties and common errors. Finally, teachers' SCK could be directly observed from their classroom instruction (TRED) from the video tapes (see Appendix 5).

CCK and PCK in this study. As previously mentioned, the eight T or F questions included two parts: (a) the questions associated with equivalent fractions; and (b) the other emergent questions beyond equivalent fractions such as “0/0”, equivalence, and mathematics notation, which related to basic mathematical ideas and mathematics sensitivity. Results ranged from one teacher correctly answering all the eight questions to one teacher answering none correctly. However, looking more closely, five of the teachers’ understanding of equivalent fractions did not show a big difference. All of these five teachers recognized the common errors 3/4 ×2=6/8 and 3/4 + 3/4 = 6/8 immediately. In addition, they provided excellent strategies about how to address these common errors if they occurred in their class. As a result, these teachers’ CCK and PCK concerning equivalent fractions reached a similar level. However, teachers’ knowledge on topics beyond equivalent fractions did demonstrate a great difference. Concerning 0/0, even though the five teachers answered it as False, their understanding were different: (a) a correct explanation even though it lacked of depth - 0 as a divisor was undefined; and (b) 0/0=0 – a wrong explanation. Interestingly, even though there were obvious hints for teachers to recognize the hidden “0/0” in the multiplication rule, only one teacher recognized it, which showed teachers’ superficial understanding and low mathematics sensitivity in a general sense. Similarly, teachers’ understanding about the “=” and “2/3 + 5/7 = 7/10” in the question about math notation and “ratio” was also different. In summary, the five teachers’ CCK and PCK concerning the topic of equivalent fractions itself had no obvious difference. However, their CCK about these basic ideas, related concepts, and mathematical notations reached different levels.

SCK- Using knowledge in the context. SCK is directly associated with teacher' classroom instruction (Ball, 2006). In this study, teachers' SCK is reflected by TRED. Since teaching practice is embedded with both regularities and uncertainties (Ball & Bass, 2000), teachers' SCK is especially useful to equip teachers with the ability to respond to those uncertainties - to size up students' thinking and address students' alternative representations mathematically and pedagogically during the teaching context (Ball, 2006). In this study, teachers' TRED showed differences which also demonstrated their different SCK in the classrooms. In other words, teachers in this study had different abilities to use their mathematical knowledge in different ways. Three of the teachers' classroom instruction showed high quality, which also demonstrated their high level of SCK in the teaching context. In contrast, the teachers whose SCK was lower exhibited lower quality of classroom instruction.

5.2.2 What enables SCK in the teaching context?

Even though Ball and her colleagues (e.g., Ball, 2006; Hill et al.; 2005) provided the theory of MKT, how these components – CCK, PCK, and SCK - really contribute to classroom teaching is not clear. Specifically, there is no disagreement that teachers' SCK - how to use mathematical knowledge in the context – is the most important. However, where does teachers' SCK come from? What enables teachers SCK to occur in the classroom context? Why do CCK and PCK transfer to SCK, or why do they not? Through the portraits provided in the Section 4, the relationships between teachers' MTK, TRED, and student cognitive gains revealed certain patterns. In this study, teachers' MKT (mainly CCK and PCK) was mirrored by teachers' responses to the T or F questions combined with their discussions in the interview. TRED in the classrooms reflected their teaching quality (also SCK). The comparison between these aspects provide some insights concerning how teachers MKT contributes to their classroom teaching, specifically, what enables teachers' SCK in the teaching context.

A precondition for SCK – when teacher lack CCK and PCK. Teachers' CCK and PCK concerning certain topics are the preconditions for possible SCK to arise in the classroom context. Teachers who lack common content knowledge cannot identify students' errors that occur in the classroom, or may even cause teachers' own flawed SCK, resulting in problematic teaching. For example, one teacher did not recognize the student error of $3/4 \times 2 = 6/8$ and $3/4 + 3/4 = 6/8$ during the interview. She thought they were correct not only because of her concern about motivation but also her confusion with "whole number" and "one whole". She said that multiplying by "2/2" or "2" were the same because "2" was a whole number. In the teaching context, this teacher summarized students' addition patterns by saying they were repeatedly "adding 3/4". Therefore, teachers who lack CCK and PCK concerning certain topics cannot obtain SCK in the context. Their teaching may mislead and confuse students.

Why do CCK and PCK not transfer to SCK? Teachers who possess CCK and PCK do not necessarily teach well. Put another way, CCK and PCK may not smoothly transfer to SCK in the classroom context. In this study, five teachers had similar levels of knowledge about equivalent fractions as demonstrated by the interview. For example, one teacher knew the difficulty of "lines or pieces" and complained that it was the biggest challenge. Moreover, she understood the underlying reasons for the multiplication rule. However, during the teaching context, she did not address the "line or pieces" difficulty ahead of time. She spent lots of time guiding students with counting and labeling in the exploration part. As a result, this class ran out of time and did not discuss the underlying reasons for the multiplication rule. This example showed that teachers' CCK and PCK may not necessarily transfer to teachers' SCK. One of possible reasons is due to the complexity of teaching context.

1) Complex teaching context. Knowing well does not equal teaching well. Since classroom teaching is complex, various context factors may inhibit teachers' use of knowledge. The above example reflected the importance of addressing critical prior knowledge ahead of time. In this study, the three successful teachers addressed this type of difficulty in advance. As a result, students in these classes smoothly moved into the new investigation of the learning goal. These teachers, therefore, had time to discuss the mathematical principles behind the multiplication rule. Therefore, when to address the difficulties related to prior knowledge may be one of the factors influencing teachers' SCK in the investigation on learning goal. Concerning addressing of student errors, some researchers have also raised the concern of "context". For example, O'Connor (2001) suggested teachers not address errors during the exploration part but in the summary part (This issue will be discussed later).

Another possible factor that seems to influence teachers' SCK is related to students' thinking characteristics. When teachers encounter more than 20 students in the classroom; they need to adapt their instruction to students' thinking which, however, is almost impossible to be accurately monitored (Chi, Siler, & Jeong, 2004). Teachers tend to interpret students' thinking according to their own expectations (O'Connor, 2001). In this study, even effective teachers misinterpreted a student's explanation and made that student somewhat confused. At that moment, the teacher's CCK and PCK did not transfer to an effective use of her SCK.

2) Behind the context factors. It is true that classroom context is complex which might inhibit teachers' transfer of CCK and PCK to SCK. The example of running out of time and thus being unable to explain the underlying reasons for the multiplication rule showed the importance of time allotment. However, why did this teacher not address student difficulty in advance? Why did she spend most of the class time helping students with counting? What was the learning goal for that class? What was the critical prior knowledge in this teacher's eyes? What was the connection between the critical knowledge and the new investigation in terms of this teacher's point of view? In considering these questions, the context factor becomes less important and the answers return to teacher knowledge. It is reasonable to assume that this teacher, even though she knew about the student difficulty based on her experiences, she was not clear about the connections between the knowledge pieces and the ones that weigh the most (Ma, 1999). As a result, when a teacher cannot grasp the core knowledge and address students' learning obstacles for the follow-up exploration, the teacher's knowledge base demonstrated a lack of strong connections.

3) A weak foundation in CCK. Why teachers' CCK and PCK cannot flexibly transit to SCK was also possibly due to teachers' weak knowledge foundation. According to Ball, Hill, et al., (2005), to address student errors, teachers need to explain the basis and principles of algorithms related to a certain topic. As a result, to address student errors and difficulties in equivalent fractions, teachers need to explain the basic ideas such as the identity property of multiplication, a basic principle underlying the rule for finding equivalent fractions. However, do teachers who know these basic principles of equivalent fractions necessarily use them in the teaching context? In this study, all the teachers knew the basic idea about the "identity property of multiplication". However, only three classes discussed it. If the basic principle existed in the teachers' mind, why did they not use it? In fact, teachers' knowledge is stored in their long-term memory. During the teaching context, without clear clues to retrieve certain knowledge, only teachers whose knowledge base has strong connections can timely activate the information they need. As a result, some teachers in this study, even though they knew the basic principles behind the rule, were unable to activate that knowledge to expand students' understanding. These teachers allowed students to obtain many equivalent fractions by repeatedly using the rule. What the students did were actually simple computations – doing whole number multiplication by using the numerator and denominator respectively without learning the basic principle.

What does the "strong connections in teachers' knowledge base" mean? What is the evidence in this study? As previously mentioned, teachers' CCK in this study was mainly reflected by teacher responses to T or F questions concerning (a) equivalent fractions and (b) basic mathematics ideas which are not directly associated with equivalent fractions but are able to reflect one's mathematical sensitivity. As a result, even though teachers had similar understanding of equivalent fractions, their responses to those basic principles demonstrated a big difference. Clearly, the quality of classroom instruction was associated with teachers' knowledge of these basic ideas. Put another way, teachers' understanding of those seemingly unrelated basic mathematical ideas and their mathematical sensitivity made differences in their classroom instruction. Those teachers whose knowledge was limited to certain topics could not necessarily teach well.

From the perspective of constructivism, this phenomenon is reasonable. When teachers' knowledge is limited in scope, they cannot flexibly make connections among various concepts. Mathematics learning is a process of constructing knowledge over time. Students need to connect their knowledge with prior ones, eventually going back to the basic mathematical principles which are usually the simple and powerful ones. For example, the knowledge of 0, 1, and equivalence serve as the foundations of school mathematics (Ball, 1990; Knuth, Stephens, McNeil, & Alibali, 2006; Ma, 1999). As a result, during the teaching process, teachers need to help students make such connections with the basic ideas, using such ideas to illustrate the more complex ones. Before teachers provide such help, they have to activate their own knowledge about some basic ideas, which is also a process of unpacking teachers' knowledge (Ball & Bass, 2000). As a result, only those teachers whose knowledge bases have strong connections could easily activate the basic ideas even in the situation without obvious clues. Put another way, only teachers who have Profound Understanding of Fundamental Mathematics (PUFM) (Ma, 1999) could teach mathematics in a flexible and deeper way. Based on these conjectures, I argue that when a teacher's CCK is limited to certain topics but lacks of a strong mathematical foundation, they cannot flexibly make connections during their teaching. Teacher's narrow CCK inhibits teachers' abilities to address student errors and difficulties in depth.

When teachers have SCK - Teaching for understanding? When teachers' CCK and PCK transfer to their SCK in the context, they have the ability to teach for understanding. In this study, teachers' SCK contributes to TRED in three ways: (a) being sensitive to mathematical mistakes; (b) making connections with basic ideas; and (c) making connections with various related concepts. As a result, these three ways were integrated and demonstrated teachers' knowledge packages that had depth and breadth (Ma, 1999).

(1) Being sensitive to mathematical mistakes – mathematics precision. Teachers' SCK will equip teachers with the ability to identify students' mathematic mistakes quickly. In this study, the three successful teachers addressed students' mathematical mistakes immediately. One of the teachers was especially sensitive to students' oral language. Even in the first class, she grasped a student oral mistake "doubling" and persisted in questioning "Are you really doubling". In the second lesson, during the exploration part, she also quickly capitalized on each oral mistake such as "times 5" and immediately reminded the students with short questions such as "5 or 5/5". This teacher's sensitivity to mathematical language precision positively affected her students who in turn paid attention to the accuracy of representations too. This finding is not consistent with O'Connor (2001)'s suggestion: to address mistakes in the summary part rather than exploration period. In this study, all teachers agreed that mathematical errors needed to be addressed immediately. Otherwise, it might cause or reinforce student misconceptions (Resnick, 1980). As a result, I argue that there is no absolute rule about when to address mistakes. Whenever student errors occur, the most critical element is to make a judgment. Concerning those mathematical mistakes or the mistakes that may influence the learning goal, teachers should address them immediately and deeply. Teachers who have SCK understand that all mathematical knowledge pieces are connected. Learning is a process

of making connections between these knowledge pieces. If students' mathematical mistakes are left unaddressed, their knowledge structure will be flawed or fragmented. As a result, teachers should turn students' mistakes into learning opportunities for the reason of mathematical precision.

(2) Making connections with the basic ideas naturally – teaching in depth. Teachers' SCK also equip them with the ability to make connections among basic mathematical ideas. In this study, only one teacher correctly answered all the questions concerning the basic ideas. These questions demonstrated her deeper understanding and strong mathematical sensitivity. As a result, her profound understanding of this type of CCK transferred to her SCK, and equipped her with the ability to address students' errors and difficulties during the teaching context as demonstrated through the videos. For example, when students made the error $3/4 \times 2 = 6/8$ in the second lesson, she quickly grasped this error and provided the example "$10 \times 2 = 20$". She correctly used the analogy strategy by connecting fractions with whole numbers and wanted her students to understand when "doubling" a number, the value will change. In addition, she asked her class, "He says we are doubling it, but you are calling them equivalent, are 10 and 20 equivalent?" This type of question vividly reflects how the strong understanding of the concept of "equivalence" contributed to her teaching of equivalent fractions. Put another way, this teachers' CCK about these basic concepts of mathematics naturally transferred to SCK during that context, resulting in effective teaching.

(3) Making connections among various concepts- teach in breadth. Teachers' SCK also equips them with the ability to make various connections among related topics. Another teacher who demonstrated her strong knowledge base mentioned during the

interview that all concepts were connected. She connected fractions with “decimal”, “percent” and even “integers”; she insisted that “2/3 + 5/7 = 7/10” was wrong because she said it was about “ratio”, a related concept that students would learn in later grades. Concerning the multiplication rule, she also made various connections beyond connecting to the basic idea. As some teachers claimed in this study, even though students knew the basic idea - every number times 1 will not change the value – they still did not really understand why ¾ ×2/2=6/8.This is because students did not see the “1” in the format and they saw that 3/4 and 6/8 were different. Concerning a learning difficulty like this, only this teacher transferred students’ understanding from the concrete (drawing a line) to the symbolic (×2/2) (Leindhardt & Smith, 1985). She also emphasized both multiplication and division as alternative ways to find equivalent fractions. The breadth of her knowledge base helped her relate equivalent fractions to the other concepts easily and efficiently. In other words, her SCK provided her useful tools to help students construct various connections in their knowledge bases.

As Stigler and Hiebert (1999) defined, “understanding” is “making connections”. Teachers who have mathematical sensitivity and knowledge could automatically make connections among concepts and also help their students to construct deep understanding. As a result, teachers’ CCK and PCK, especially teachers’ deep understanding of basic mathematical ideas and relevant concepts, contribute to SCK, enabling them to flexibly use their mathematical knowledge to make differences in their classroom instruction.

5.2.3 What matters in students' mathematics learning?

Teaching and learning cannot be separated. In this study, all students used the same CMP textbook and they all studied the same two lessons about equivalent fractions. However, students' learning experiences and cognitive gains were greatly different. During the process of video observation, with an eye on students' learning, I was engaged in the same processes of either struggling or enjoying the lessons. As a result, two questions were echoing in my mind: Who made mathematics easy or difficult to learn? What was the intrinsic motivation?

Who made mathematics easy or difficult to learn? Many students do not like mathematics because they think it is too difficult. However, comparing students' responses and cognitive gains in this study, I found that mathematics was very easy to learn in some classes while extremely hard in the other classes.

Students in three teachers' classes were very engaged. All these classes moved smoothly to the exploration of the learning goal because the teachers addressed learning difficulties about prior critical knowledge. Students in these classes shared their strategies and explanations with the teachers. Some students raised good questions which showed their mental engagement. Mathematics appeared easy to be dealt with. In the summary part, students in all the three classes presented various concrete and symbolic strategies while knowing the basic mathematical ideas behind these rules.

In contrast, students in two other classes were mainly measuring or counting. Students in one class experienced a hard time counting "lines or pieces," but the teacher mainly examined students' labeling and asked them to show her the counting process. Students in this class mainly found the "addition" pattern. Even though this teacher discussed the multiplication rule, she did not mention the underlying reason. Students in the other class struggled while the teacher did not realize their learning difficulty and was confused by students' comments. In addition, she herself made mistakes. As a result, both the teacher and the students were lost in this class. In the second lesson, the teacher guided students to count "lines" instead of "pieces" without explanation. Therefore, many students were lost, forcing the teacher to count for the students during which time students only needed to repeat the counting results. In the end, this class mainly finished labeling the fractions strips without any real discussion about the patterns for finding equivalent fractions, no mention the underlying reasons.

Considering the effects of the teaching on the learning in this study, teachers' classroom instruction obviously made a difference in student cognitive gains. Some teachers made mathematics easy to learn without decreasing the level of challenge. Students in these classes were engaged in real cognitive activities and these mathematics classes appeared to be enjoyable experiences. In contrast, other teachers made mathematics hard even though they tried to decrease the level of mathematics complexity such as "counting for students". Students in these classes still struggled because of teachers' misguidance, mistakes, or confusion. As a result, these teachers made mathematics hard and students lost interest because these classes appeared as painful experiences.

What can really motivate students? Research has shown that making mathematics fun was central to U.S. beginning and experienced teachers' pedagogical reasoning (e.g., Ball, 1999). That is, mathematics is set of procedures which are not interesting. Students' interests need to be motivated and fun ways need to be used for attracting their attention to those procedures (Stigler & Hiebert, 1999) Assuming that mathematics is inherently boring and hard to learn, these teachers thought their role was to find ways to motivate or engage students or to search games to lighten load for students. One of the teachers in the present study reflected such a cultural belief. From using a cartoon or poem, to making jokes, to finally asking students to stand up and stretch their arms, all these activities were for the same purpose – to motivate students and draw their attention. Because of the concern for students' motivation, this teacher viewed students' errors as alternative strategies and suggested students' efforts should be praised. As previously mentioned, the concern of diminishing students' motivation reflects the U.S. cultural view of teaching and learning mathematics. However, since this teacher herself had a flawed knowledge base and she also made similar errors as her students, it is not clear whether her concern for motivation was related to her weak understanding which caused her to underestimate the importance of mathematical precision and incapability to recognize the negative influence of these errors on student future learning. Other teachers in this study also reflected such beliefs. During the interview, one teacher mentioned "You cannot teach students without fun." In her class, non-mathematics language seemed also to serve for student motivation.

However, can these fun activities or language really motivate students or engage students in the learning of mathematics? The results of this study told the answer. For example, when the students were confused by one of the teachers, they lost their interest in the class. Even though the teacher made jokes with the students, they never laughed. They sat and looked tired of either of the mathematics class or mathematics. What is real motivation? How can students be motivated and engaged in mathematical learning? As argued by von Glasersfeld (1996), the only source of real motivation is students' experiences of intellectual success and pleasure. In this study, even though successful teachers did not make any jokes or do non-mathematical activities, students were engaged. When the students really understood mathematics, they experienced the "intellectual success and pleasure". As a result, they were intrinsically motivated.

Summary. As Ball (1993) pointed out, teachers' own understanding matters in their ability to teach and their sensitivity to school mathematics, which determines students' learning opportunities and motivation and makes a difference in their learning outcomes (Ball & Bass, 2003a).

5.3 Cultivating Mathematical Sensitivity in School Mathematics

In this study, an emergent issue reflected by teachers' responses to the T or F questions and their class videos, is about mathematical sensitivity.

5.3.1 What is mathematical sensitivity?

In this study, "mathematical sensitivity" is different from but related to "mathematical knowledge". Teachers who have no mathematical knowledge will definitely have no mathematical sensitivity. However, teachers who have mathematical knowledge do not necessarily have the sensitivity. For example, in this study, all of the teachers recognized 0/0 was wrong, even though some of them thought that 0/0=0 when this "format" was clearly brought to their attention. However, when there was no clear clue about considering "0" only one teacher recognized the problem with "0/0". When later questions about the idea of 0/0 were raised, it was reasonable to expect teachers to find the error. However, no teacher went back to change their answer. This example showed differences in mathematical sensitivity among teachers. Mathematical sensitivity can equip teachers with the ability to quickly identify the hidden mathematical mistakes and imprecision. In addition, it helps them automatically grasp critical knowledge pieces to make connections.

5.3.2 Teachers' weak mathematical sensitivity

In general, teachers in this study demonstrated a weak mathematical sensitivity. There were three sources of evidence: (a) sensitivity to mathematical notation; (b) sensitivity to basic mathematical ideas; and (c) sensitivity to mathematical thinking skills.

A weak sensitivity to mathematical notation. The T or F questions, 2/3 + 5/7 = 7/10, entails the knowledge about "ratio" and the understanding of fractions. However, it also requires teachers' sensitivity to mathematical notation. Among the six teachers, only two of them identified it as false. The rest of the teachers thought it was appropriate under real life situations such as the game problem. During the interview, one teacher also provided me another example as an illustration that non-standard written representations could be correct.

Another example was provided by an effective teacher who emphasized that when comparing fractions, students should realize the "whole" needed be the same. This teacher said it was possible that "1/2 ≠ 2/4". Whatever the other alternative explanations, this representation was mathematically incorrect. There are four arguments. First, Number is an abstraction from objects. When we use notation such as 1/2=2/4, we automatically assume the "whole" is the same size. Put another way, when we compare the value of two numbers in the notation of “½ ○2/4”, we do not consider the concrete “whole” or “unit” any more. The value of “½” is always mathematically equal to “2/4”. A simple example is about whole number. The notation “1” is abstracted from various objects such as 1 apple, 1 grape, 1 pencil, and 1 car. After these process of abstraction, we reach mathematics agreements such as “1 = 1” or “1<2”.If a teacher raises a question such as “when is 1 equal to 1 but when not”, we will say he is misleading students. It is also strange for someone to argue that he cannot fill the circle “1 ○ 2” with either sign of “>, =. Or <” because he has a concern about “unit” and he cannot compare 1 apple and 2 grapes. Similarly, with regard to the notations in fractions, if teachers overemphasize the “whole” and argue that “½=2/4” is sometimes correct but sometimes not, it would possibly cause students confusion and trouble when they perform fraction operations or manipulate these fractional symbols. For example, students may argue that fractions such as 1/2, 1/3, and 1/4 are not comparable because they do not know the size of the “wholes”.

Fraction has multiple meanings (Behr, Lesh, Post, & Silver, 1983; Kieren,1980) and “Part-whole” is only one of them. When we consider the other meaning such as operator, ratio, or measure, we come out with other arguments. With regard to the example, “1/2 grape and 2/4 apple”, it is true that the two wholes (X and Y) are not the same size ($X \neq Y$). When we compare 1/2 grape and 2/4 apple, we actually compare 1/2X and 2/4 Y. Under this situation, the meanings of 1/2 and 2/4 are all “operators” rather than “part-whole relationship”. Since $X \neq Y$, we get $1/2X \neq 2/4Y$ rather than $1/2 \neq 2/4$, we can only say 1/2 grape is not the same amount as 2/4, but we can say 1/2 is not equal to 2/4. The third argument is to view 1/2 and 2/4 as ratios. With this meaning, 1/2 is always equal to 2/4 whatever the size of the "whole”. For example, if Mike ate1 out of 2 apples, and Jack ate 2 out of 4 grapes, we can say they both ate half of the fruits. The ratio is the same and we symbolically represent the relationship between these two ratios as 1/2 = 2/4. The fourth argument is to view 1/2 and 2/4 as measure. Since there is only one point on the number line corresponding to 1/2 and 2/4, these two numbers are mathematically equivalent.

It is true students learn equivalent fractions based on “part-whole” relationship and when teachers use manipulatives and multiple representations to compare fractions, they allow students to see the size of the “whole.” However, when teachers overemphasize the “whole” and ask them to prove expressions such as “1/2≠2/4” could be true or ask students questions such as “when is 1/2 was not equal to 1/2”, they may confuse students or cause trouble for students later learning. To use a real-life situation such as the “grape and apple” example to prove “1/2≠2/4” to be true may also decrease students’ mathematical sensitivity.

These mistakes or impreciseness with mathematical notations were negatively influenced by the real life situations. Mathematics notation has a fixed written format and rule of operation by definitions. Even though these mathematical mistakes probably reflected teachers’ flawed understanding, it also reflected their weak mathematical sensitivity to mathematical notation. Teachers who have high sensitivity to mathematical notation precision would identify such errors without providing such examples. In a word, they would not draw mathematics back into real life situations.

Teachers’ weak sensitivity to mathematical notation also could be seen from their responses to students’ written mistakes. As previously mentioned, mathematical knowledge does not mean mathematical sensitivity. Teachers who have certain knowledge do not necessarily point out the errors in students' symbolic representations. The teacher who had the highest mathematical knowledge also knew the "equivalence" concept as reflected by the interview. However, when a student wrote down 270 +360 ÷10 = 27 + 36 on the board, she did not point out the error. In fact, this student made another error, showing that the operational order sign such as "(   )" had no meaning to her. Surely, this mistake will not influence the student's understanding of equivalent fractions in this class. However, this type of mistake might be entrenched over time and the student's sensitivity to mathematics would also be damaged. Briefly, mathematical sensitivity relates to teacher knowledge. However, teacher knowledge cannot ensure good mathematical sensitivity. As a result, teachers should be careful when using mathematical notation. Meanwhile, they should also pay attention to students' verbal and written representations. Over time, both teachers and students' mathematical sensitivity could be cultivated.

A weak sensitivity to basic mathematical ideas. In this study, teachers in general had weak knowledge about some basic ideas such as "0" and the "=", which negatively inhibited their mathematical sensitivity. Since this study did not examine teachers other basic concepts or mathematical ideas, I cannot conclude that these teachers had a lack of knowledge and sensitivity to all basic mathematical principles. However, teachers' understanding of "0" was a well-known issue (Ball, 1990). My finding was consistent with that of Ball. The understanding or "=" issue was also another popular research topic. Prior studies mainly focused on students' learning throughout elementary to university level (e.g., Collis, 1974; Seo & Ginsburg, 2003; Wolters, 1991). When questions were raised for textbook effects (e.g, Ding, Li, Capraro & Capraro, 2007; McNeil et al. 2006;

Seo & Ginsburg, 2003) or teacher instructional effects (e.g., Denmark, Barco, & Voran, 1976; Sáenz-Ludlow & Walgamuth, 1998), very few researchers thought of the issue in terms of teacher's own knowledge about the "=". In this study, when these equivalence questions were brought to the teachers, their responses were striking. Half of these teachers did not recognize the problem related to this basic idea while another teacher provided the correct answer after the interview. Even though these teachers could solve problems such as 8 + 4 = □ + 5 (Falkner, Levi, & Carpenter, 1999), their responses in this study at least showed their low sensitivity to the concept of equivalence, which also demonstrated their weak understanding of the equal sign. The teachers who have low sensitivity to this type of basic ideas cannot automatically make connections between new contents and the mathematical foundation without clear stimulations or clues. As a result, even though they knew these knowledge pieces, they could not activate such information during their actual teaching. When teachers cannot make connections during their teaching, what they taught was procedural based. Meanwhile, when teachers cannot help students construct these types of connections during their learning, what students learned was also fragmented knowledge pieces without real understanding. As the other researchers (Ball, 2000; Ma, 1999) pointed out, teachers who lack mathematical knowledge and sensitivity, even though they claimed to teach for understanding, their classroom instruction would still be procedure focused.

When teachers' lack of mathematical sensitivity is due to their own flawed knowledge, the consequences might be much more striking. As previously mentioned, teachers in this study were lacking deep understanding of "0". The highest level of understanding is "0 cannot be a divisor because it was undefined" without knowing why it was undefined. In addition, all the four teachers who thought 0/0=0 confidently viewed it as "truth". They not only provided the explanation but also claimed that all their students knew this fact. As a result, it is hard to imagine how many students in the future would believe that 0/0=0. Prior studies mainly focused either on students' learning of the basic ideas (e.g., the research on the "=") or teachers' knowledge of the basic principles (e.g., the research related to "0"). Future research needs to make connections between both – to see how teachers' own knowledge or sensitivity of basic mathematical ideas affects students' corresponding understanding.

A weak sensitivity of multiplicative thinking. In this study, another issue related to mathematical sensitivity was concerning both teachers' and students' multiplicative thinking. Since multiplication is repeated addition, a convenient operation embodying the historical mathematical development and reflecting mathematical thinking, teachers should be sensitive to it and cultivate students' thinking skills. In this study, some teachers emphasized the multiplication pattern rather than addition pattern. One teacher even viewed students "repeated addition" as an error. She said that kind of operation was too slow and inconvenient. However, not all teachers fully realized the importance of "multiplicative thinking". In this study, even though most of teachers discussed the multiplication rule, they did not necessarily emphasize students' multiplicative thinking skills. One of the clues was from the teacher interview in which one teacher told me students should solve problems from multiple ways. She did not mention that multiplication was a more efficient way. As a result, it seemed this teacher was not sensitive to multiplicative thinking. Further evidence was observed from the videos. One teacher divided the fundraising goal into four pieces and guided students in a computational process involving additive thinking. Some teachers guided students to label equivalent fractions on the strips, then counted all the pieces in the whole, "1, 2, 3, 4, 5, …". No teacher asked questions such as "How many sections do we have? How many pieces in each section now? In total, how many pieces do we have?" In other lessons not used for this study, teachers missed opportunities to work with fraction strips to develop multiplicative thinking. For example, when a fraction strip was folded into six and then folded in half again, teachers could ask questions such as "Tell me how many pieces without counting", allowing students to think about "6×2". However, some teachers just asked students to display their fraction strips to count all the pieces. It seemed for these teachers, only the answers obtained by touching each of the pieces were the true ones. In fact, if we think about a real life situation, it is very common to see that both teachers and students tend to depend on their fingers to figure out problems such as "7+7" and got 14. In their mind, the fact of "7×2" was missing.

Summary. Ball et al (2001) reviewed prior studies and pointed out that U.S. teachers, elementary and secondary, preservice and experienced, had a weak mathematical sensitivity and understanding of fundamental mathematical ideas and relationships. In this study, I found both teachers and students encountered the same problem. The question is, why the teachers and students have not developed sensitivity toward mathematical notations, basic mathematical ideas, and mathematical thinking? Why is U.S. school mathematics lacking in mathematical sensitivity?

5.3.3 How to cultivate mathematical sensitivity in school mathematics

Why do teachers lack this type of knowledge and lose this type of mathematical sensitivity? What kind of new generation will be cultivated by teachers who lack this type of knowledge? Thinking about these questions, this phenomenon of the weak mathematical sensitivity in school mathematics is then more striking. Teachers' flawed knowledge or low mathematical sensitivity might be a learning outcome of the long accumulation process starting as elementary students. In fact, if even these teachers did not have a good knowledge base before they became teachers, they should still have opportunities to develop their knowledge because there were so many resources such as textbooks and teacher guide books. However, in this study, when I examined the curriculum materials, these teaching resources provided incomplete or imprecise information (Smith & Leinhardt, 1985). For example, concerning the hidden "0/0" in the multiplication rule, the CMP teacher guidebook did not provide any hints. In addition, as I analyzed before, the multiplication rule as provided by the CMP teacher guide book had other problems: it denied that the numerator and denominator could be multiplied by a rational number; and it only emphasized multiplication without mentioning division. As Ball et al (2000) pointed out, a precise definition should consider both students' understanding and mathematical precision. However, it seemed the CMP teacher guide book only considered students' learning but not mathematical precision. This is because this definition can help students effectively find equivalent fractions. Using this rule, students could find millions of correct answers. As one teacher said during her interview, "I cannot imagine why these student use "0" to find equivalent fractions! I never have students to do that!" It was true there were probably very few students who would really use “0/0” when they tried to find equivalent fractions. Therefore, teachers did not encounter such mistakes and thus had no sensitivity toward this issue. However, if we view mathematics learning as a process of constructing understanding and making connections, should we not provide any opportunity (even very tiny ones) to develop students’ sensitivity to the precision of mathematical language, basic mathematical ideas, and mathematical thinking? Concerning “0 cannot be a divisor” idea, even if it was not directly associated with equivalent fractions, if the textbook added two more words - “except 0” - or emphasized a “nonzero number” - it could lead a good discussion - without confusing students about the basic mathematical ideas, resulting in a deeper understanding of the multiplication rule and a growth of both teachers and students’ mathematical sensitivity.

What follows is an example to illustrate this suggestion. In the comparison study of students’ understanding of the “=”, Chinese students demonstrated statistically significant differences concerning the understanding of the equivalence concept (Ding, Li, Capraro, & Capraro, 2006). The researchers, therefore, examined three sets of dominant Chinese textbooks and teacher guide books from grade 1 to grade 6 with the purpose of finding the reasons for the difference. As a result, a clear answer occurred. That is, these curriculum materials not only provided the basic mathematical idea in a correct context but also provided teachers with detailed guidance as well as developed the concept of equivalence throughout the entire elementary years by using various opportunities. Concerning the weak mathematical sensitivity in this study, who should be responsible for it? Since the U.S. curriculum is not a centralized one, various textbooks and related resources exist. The textbook is not the only resource that teachers depend on. Some teachers tend to use self-created textbooks or online resources. As a result, how to improve the quality of curricula with the balance between mathematical precision and students' cognitive levels is an important but unsolved issue. In addition, should mathematics educators make more efforts to cultivate teachers and students' mathematical sensitivity to the basic mathematical ideas, notations, and thinking skills? Should students have to develop this type of sensitivity only when they encounter advanced mathematics? Curriculum designers, teacher educators, researchers, and teachers themselves should all cooperate to equip teachers and students with deep mathematical sensitivity and understanding.

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APPENDIX 1

TEACHER INTERVIEW MATERIALS

Video Clips

A Designed Case

Imagine that one of your students comes to you before the new class very excited. She tells you she has already figured out two ways of finding equivalent fractions. Using the example ¾ and 6/8, she explains her way and writes down her idea: the first way is to multiply the fraction by a whole number, for example, $\frac{3}{4} \times 2 = \frac{6}{8}$; and the second way is to add the numerator and the denominator by itself respectively, for example, $\frac{3}{4} + \frac{3}{4} = \frac{6}{8}$. How would you respond to this student?

True or False

(1) To find equivalent fractions we can multiply the numerator and denominator by the same number. ……………………………………………………………………………..( )

(2) To find equivalent fractions we can multiply the fraction by the same number………..( )

(3) 270/360 = ¾ because 270 ÷3 = 90 × 4 = 360……………………………………..…..( )

(4) ¾, 6/8, 9/12, and 12/16 are equivalent fractions because $\frac{3}{4}+\frac{3}{4}=\frac{6}{8}+\frac{3}{4}=\frac{9}{12}+\frac{3}{4}=\frac{12}{16}$ ...( )

(5) $\frac{3\times 0}{4\times 0}=\frac{3\times 1}{4\times 1}=\frac{3\times 2}{4\times 2}=\frac{3\times 3}{4\times 3}$ ………………………………………………………............ ( )

(6) $\frac{3}{4}\times 3=\frac{9}{12}$ …………………………………………………………………..………....... ( )

(7) In some real life situations, students' erroneous operations might be actually correct. Fore example, "$\frac{2}{3}+\frac{5}{7}=\frac{7}{10}$" represents: "if you won 2 out of 3 games yesterday, and 5 out of 7 games today, altogether you have won 7 out of 10 games, and not 29/21"……………...( )

(8) $\frac{0}{0}=\frac{1}{1}=\frac{2}{2}=\frac{3}{3}=\frac{4}{4}$, because all of these fractions equal to one whole…..................( )

APPENDIX 2

LOCATING STUDENTS' ERRORS AND DIFFICULTIES AS REFLECTED IN VIDEO TAPES

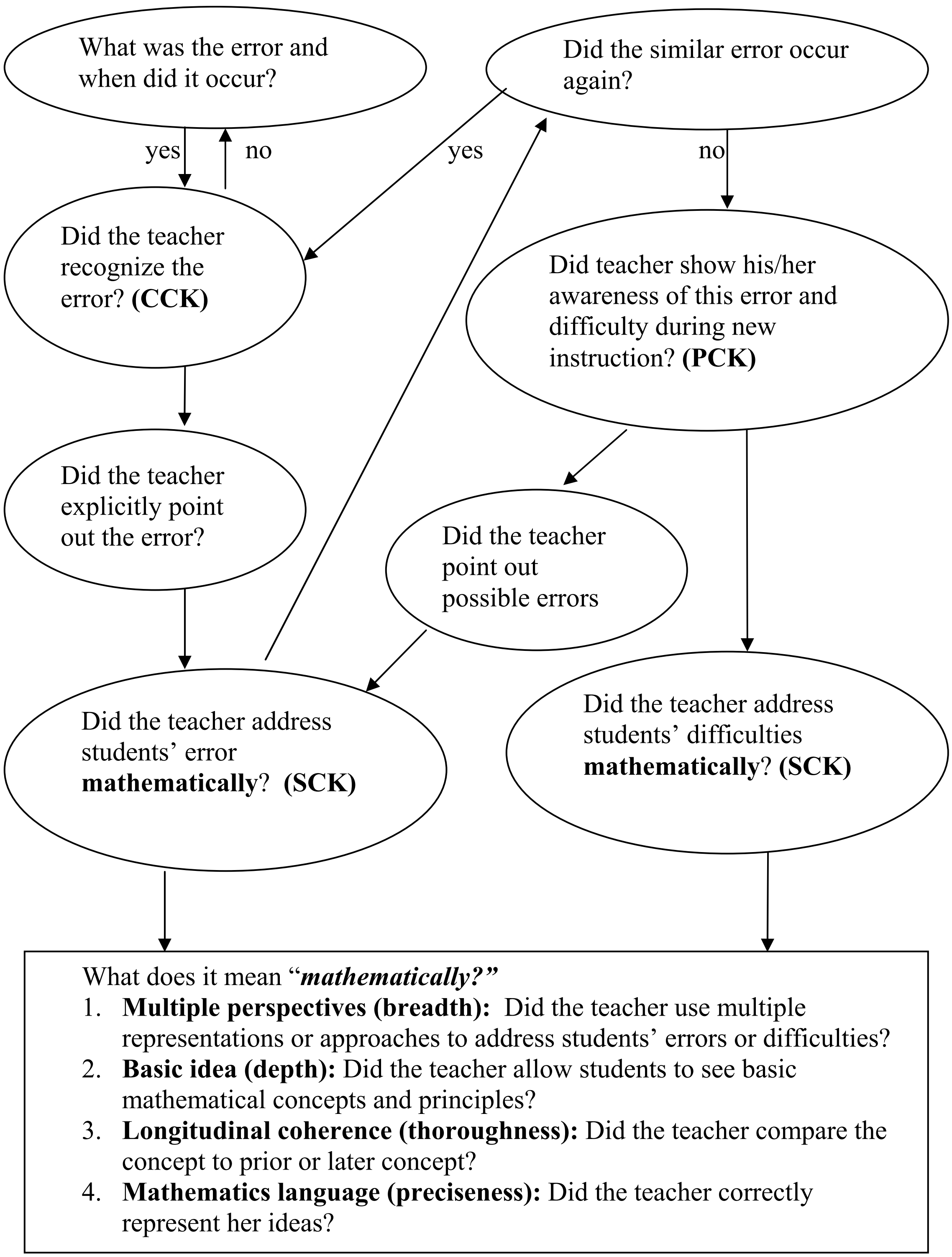
Teacher ___, Lesson ___, Total class time____

Sighting	Time		Type		Context							Description of Responses	
					Period				Format				
	Start	End	Error	Difficulty	a. Review	b. Intro.	c. Explore	d. Summary	a. Individual	b. Small group	c. whole class	T	S
1													
2													
3													
4													
5													
6													
7													
8													

APPENDIX 3

TEACHER RESPONSES TO STUDENTS' ERRORS AND DIFFICULTIES IN TWO LESSONS

Lesson 1 **Lesson 2**



APPENDIX 4

TEACHER RESPONSES TO T OR F QUESTIONS

Teacher	Basic ideas and related concepts					Equivalent fraction		
	(1) “Except 0	(5) 0/0	(8) 0/0	(3) “=”	(7) 2/3+5/7 (notation)	(2) Multiply by a “fraction”	(4) 3/4×3	(6) ¾ + 3/4
Jennifer	F	F	F	F	F	F	F	F
Kathleen	T	F	F	T	F	F	F	F
Barbara	T	F (0)	F (0)	F	F/T	F	F	F
Rose	T	F (0)	F (0)	F	F/T	F	F	F
Lisa	T	F (0)	F (0)	T	T	F	F	F
Mary	T	T (0)	T (0)	T	T	T	T	T

Note. All these questions were false. In the above table “F” represents teachers’ correct response. Concerning question (5) and (8), The F(0) means the teachers judged those statements as “False” but their explain was 0/0=0 which was wrong.

APPENDIX 5

TEACHERS INVOLVED AS HIGH/LOW LEVEL TEACHING EXAMPLE IN EACH SECTION

	What is your goal	Line or pieces	Are you really doubling	What do you mean by 3/4
Jennifer		high	high	high
Kathleen	high		high	
Barbara	high		high	high
Rose		low	low	
Lisa			low	
Mary	low	low	low	low

VITA

Name: Meixia Ding

Address: 4232 Harrington Tower, Teaching Learning and Culture, College Station, TX, 77843-4232

Email Address: amble007@tamu.edu

Education: Ph.D. (Curriculum & Instruction) Texas A&M University, U.S., 2007

M.S. (Ancient Chinese Bibliography) Nanjing University, China, 2003

Experience: 2005-2007 Graduate Assistant, Middle School Mathematics Project, Texas A&M University, U.S.

1995-2000 Mathematics Teacher, The Third Affiliated Elementary School with Nantong Normal School, China.