The Calculus of Finite Differences

by

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ABSTRACT

Mathematicians use divided difference equations to solve problems which have only a discrete set of possible values. This research is concerned with the application of difference equations to curve fitting data by means of splines. There is a paper written by four math professors at Texas A&M which concerns itself with this data fitting problem. In particular, a theorem in this paper states conditions for when a complex data function has a unique best fit from a spline space. Central to the proof of this theorem was the necessity of deciding when certain determinants were positive using divided difference equations. The purpose of my research was to broaden the classes of matrices for which the determinant is positive which, in turn, would broaden the classes of functions for which one could obtain a unique piecewise spline.

ACKNOWLEDGEMENT

I would like to sincerely thank Dr. Joseph D. Ward for helping me this year with my project. Without his wisdom to guide me, I could not have completed my research. I deeply appreciate him for making my first experience with mathematical research rewarding.

This paper is dedicated to

My parents

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Introduction

Some problems in calculus may conveniently be assumed to have only a discrete set of possible values. For example, in economics such a variable is time. Important ecnomic quantities, such as National Income, are ordinarily available only at certain time periods. In calculus, when problems like this arise, mathematicians use difference equations to derive mathematical models. A difference equation is simply an equation relating the values of a function y and one or more of its differences. The methods for solving difference equations are for the most part analogous to solving differential equations.

The purpose of this paper will be in the application of difference equations to data analysis. For instance, in a missile tracking operation, data naturally arises from radar readings of a missile flight. For various reasons, one would like to "fill in" this data to obtain a continuous flight pattern for the missile (in particular, this would help to predict where the missile would land). See Figure 1. One tries to curve fit this data by means of splines. Splines are piecewise polynomials with "knot" sequence, that is, between

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The citations on the following pages follow the style of the Siam Journal of Numerical Analysis.

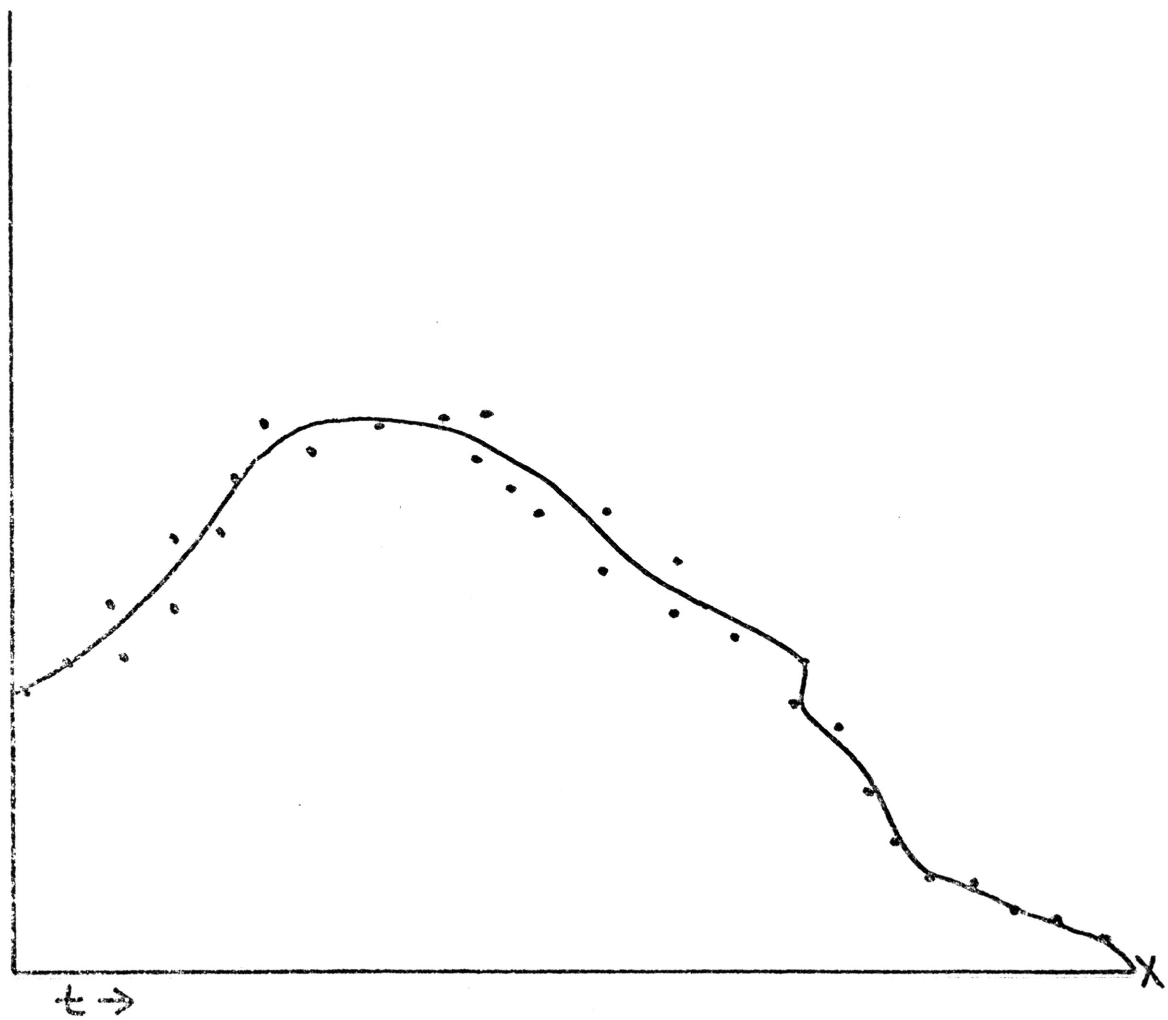

FIG. 1. Missile Flight Data

any two consecutive "knots" the spline is a polynomial, between another consecutive pair of "knots" the spline might be a different polynomial, but the different polynomials connect at the knots. For curve fitting, splines are favored over polynomials for computational purposes. One wishes to keep the number of knots small, so the location of knots is important.

There is a theorem, known as Theorem II, written in a paper by four math professors here at Texas A&M concerning this data fitting problem, which gives conditions for when a convex (or concave) data has a unique best fit from a spline space. (Consult bibliography). Central to the proof of this theorem was the necessity of deciding when certain determinants were positive using divided difference equations. The purpose of this research project is to broaden the classes of matrices for which the determinant is always positive.

This is important because by doing this, one can hope to broaden the classes of functions for which a unique spline may be obtained. One would like to know that there is a unique spline because uniqueness makes it easier to compute a spline by using a standard minimization method, such as the steepest descent method.

General Solution of the Homogeneous Equation

Before attempting to extend the classes of matrices for which the detrminant is positive, it is necessary to understand the general solution of the homogeneous equation for solving divided difference equations.

Suppose we are trying to find the general solution of the homogeneous equation

$$Y_{k+2} + a_1 Y_{k+1} + a_2 Y_k = 0. \qquad (2.1)$$

This reduces to the problem of finding two solutions.

It is easy to find solutions of the difference equation for which

$$y_k = m^k$$

where m is some suitably chosen constant different from zero. If this trial solution is substituted into (2.1), after division by the common factor m^k, one obtains,

$$m^2 + a_1 m + a_2 = 0.$$

This quadratic equation is called the auxiliary equation. Therefore, if m is a number satisfying the auxiliary equation, then $Y_k = m^k$ is the solution of the difference equation.

The auxiliary equation is a quadratic algebraic equation, and therefore has two nonzero roots, say m_1 and m_2.

There are three kinds of roots of a solution. They may be real and unequal, real and equal, or complex.

These cases will be considered separately.

Case 1. Suppose y and y' are two real and unequal solutions of the homogeneous difference equation

$$Y_{k+2} + a_1 Y_{k+1} + a_2 Y_k = 0 \qquad (2.2)$$

and let

$$Y = C_1 y + C_2 y'$$

where C_1 and C_2 are arbitrary constants. If

$$\begin{vmatrix} y_o & y_o' \\ y_1 & y_1' \end{vmatrix} = y_o y_1' - y_o' y_1 \neq 0 \qquad (2.3)$$

then Y is the general solution of (2.2).

Proof. It must be first shown that if y and y' are two solutions of linear homogeneous difference equations, then $C_1 y + C_2 y'$ is also a solution. This can easily be shown when n = 2. For if

$$(C_1 Y_{k+2} + C_2 Y'_{k+2}) + a(C_1 Y_{k+1} + C_2 Y'_{k+1}) + a_2(C_1 Y_k + C_2 Y'_k) = 0,$$

then this can be rewritten as

$$C_1(Y_{k+2} + a_1 Y_{k+1} + a_2 Y_k) + C_2(Y'_{k+2} + Y'_{k+1} + Y'_k).$$

Thus, $C_1 y$ and $C_2 y'$ are both solutions.

Now, it must be shown that if y is any solution of (2.2), then Y and y are identical. It suffices to show that Y and y are equal at k = 0 and k = 1. That is, the values of C_1 and C_2 must be determined so that $Y_o = y_o$ and $Y_1 = y_1$ for any choice of y_o and y_1. But,

$$Y_o = C_1 y_o + C_2 y'_o$$

$$Y_1 = C_1y_1 + C_2y'_1$$

so C_1 and C_2 must satisfy the equations

$$y_oC_1 + y'_oC_2 = y_o$$
$$y_1C_1 + y'_1C_2 = y_1$$

By hypothesis, the determinant formed by the coefficients of C_1 and C_2 is different from zero. Now a unique pair of values of C_1 and C_2 can be found for each choice of y_o and y_1 because a system of simultaneous equations has a unique solution if and only if (2.3) hold.

Since $m_1 \neq m_2$, the determinant of $\{\begin{smallmatrix}1 & 1\\ m_1 & m_2\end{smallmatrix}\}$ is not equal to zero. Thus the general solution of the homogeneous equation (2.1) is given by

$$Y_k = C_1m_1^{\,k} + C_2m_2^{\,k}. \tag{2.4}$$

End of proof.

<u>Case 2</u>. Now suppose the solutions $Y_k = m_1^{\,k}$ and $Y'_k = m_2k$ are real and equal. The solutions no longer form a fundamental set since the determinant is zero. Now it shall be shown that a solution is given by

$$Y'_k = km_1^{\,k} \tag{2.5}$$

To prove that if $m_1 = m_2$ is actually a solution, y' for y is substituted in the difference equation (2.1), and checked to see that the equation is satisfied:

$$Y'_{k+2}+a_1Y'_{k+1}+a_2Y'_k = (k+2)m_1^{\,k+2}+a_1(k+1)m_1^{\,k+1}+a_2km_1^{\,k}$$
$$= km_1^{\,k}(m_1^{\,2}+a_1m+a_2)+m_1^{\,k+1}(2m_1+a_1). \tag{2.6}$$

But the terms in the first parenthesis add to zero since m_1 is a root of the auxiliary equation. Furthermore, the sum of the roots of the auxiliary equation is $-a_1$. But, if $m_1 = m_2$, this sum is $2m_1$. Hence, $2m_1 + a_1 = 0$. This makes (2.6) identically zero, so (2.5) is a solution of the difference equation.

Case 3. Before considering the case of complex roots, some material needs to be reviewed. Recall that a+bi and a-bi are complex conjugates. The complex number a+bi can be represented graphically, using a rectangular coordinate system by the point P with coordinates (a,b). See Figure 2. Since $a = r\cos\theta$ and $b = r\sin\theta$, it follows that $a + bi = r(\cos\theta + i\sin\theta)$. Now the identity $\sin^2\theta + \cos^2\theta = 1$ shows that $a^2 + b^2 = r^2$ or

$$r = \sqrt{a^2 + b^2} \quad . \qquad (2.7)$$

Hence, θ may be taken as the unique angle, such that

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}} \qquad \sin\theta = \frac{b}{\sqrt{a^2+b^2}} \quad -\pi<\theta\leq \pi. \quad (2.8)$$

The de Moivre's theorem states that if n is any positive integer,

$$\{r(\cos\theta+i\sin\theta\}^n = r^n(\cos n\theta + i\sin n\theta).$$

Note that complex roots of a quadratic equation always occur in conjugate pairs. Therefore, if m_1 and m_2 are complex roots of the auxiliary equation, then $m_1 \neq m_2$. Thus, the only difficulty with the general solution (2.4)

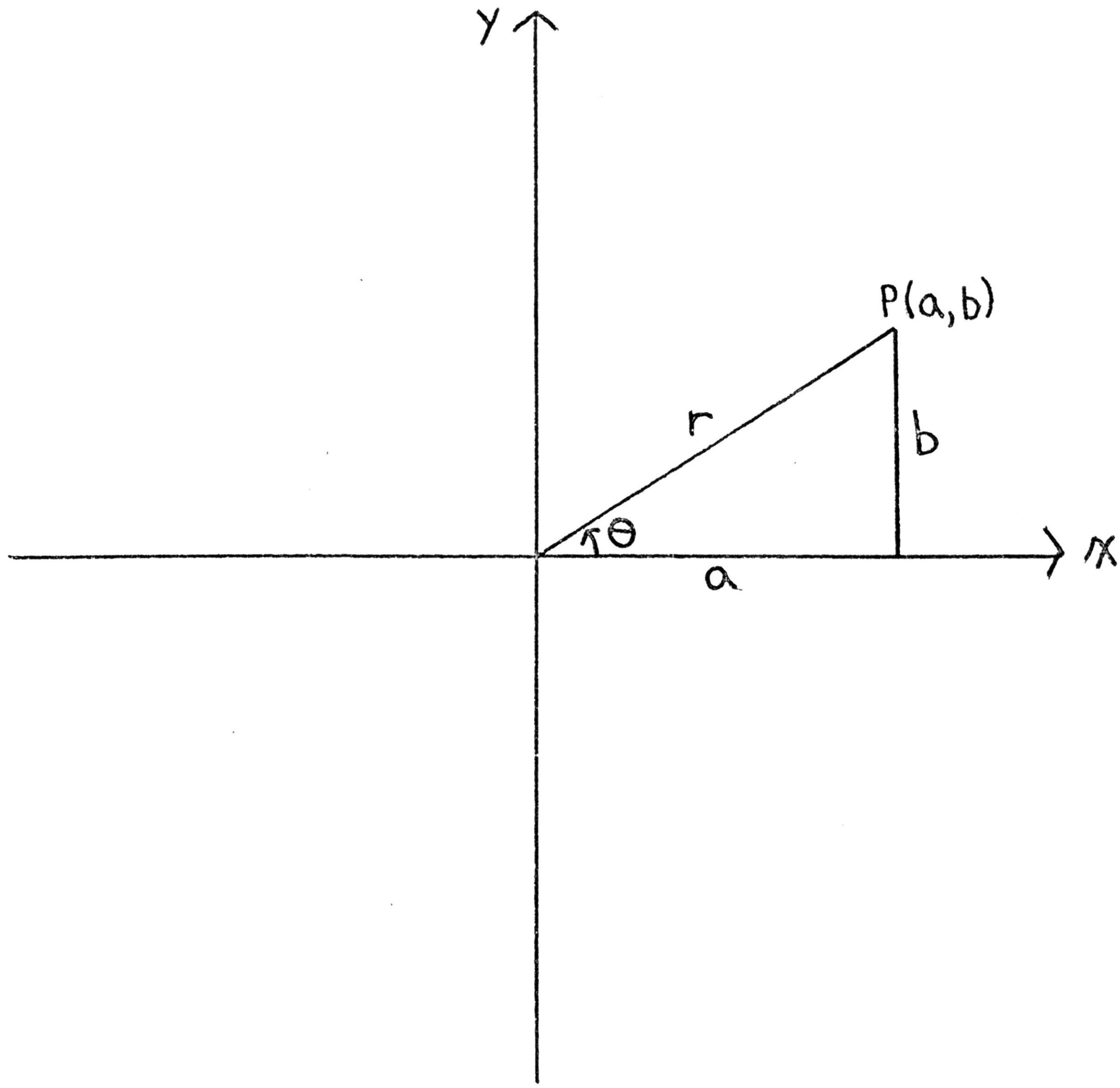

FIG. 2. Graphic Representation of a Complex Number

is that it may be a complex number if m_1 and m_2 are complex. Consequently, it must be shown that if C_1 and C_2 are complex conjugates, Y_k is always a real number. To prove this, write all conjugate numbers in polar form.

Proof. The roots of the auxiliary equation are complex conjugates and hence have polar form,

$$m_1 = r(\cos\theta + i\sin\theta) \quad m_2 = r(\cos\theta - i\sin\theta). \qquad (2.9)$$

Assuming C_1 and C_2 are complex conjugates,

$$C_1 = a(\cos B + i\sin B) \qquad C_2 = a(\cos B - i\sin B).$$

By the de Moivre's theorem

$$M^k{}_1 = r^k(\cos k\theta + i\sin k\theta) \quad m^k{}_2 = r^k(\cos k\theta - i\sin k\theta).$$

Therefore, using the rules for multiplying complex numbers, Y_k is simplified as follows:

$$\begin{aligned} Y_k = C_1 m_1{}^k + C_2 m_2{}^k &= ar^k\{\cos(k\theta+B) + i\sin(K\theta+B)\} \\ &= ar^k\{\cos(k\theta+B) - i\sin(k\theta+B)\} \\ &= 2ar^k\cos(k\theta+B). \qquad (2.10) \end{aligned}$$

This is a real number as claimed.

End of proof.

The numbers r and θ are determined from (2.9) by writing the roots of the auxiliary equation in polar form to find r and θ. The constants a and B in (2.10) take the place of C_1 and C_2. If 2a is denoted by A, the general solution of the homogeneous difference equation when the auxiliary equation has complex roots can be written in the form,

$$Y_k = Ar^k \cos (k\theta+B)_1 ,$$

where A and B are arbitrary constants. (Goldberg, 1958)

Fours on the Main Diagonal

As mentioned in the introduction, central to the proof of Theorem II was the necessity of proving the following proposition.

Proposition 1. Let $A = (a_{i,j})$ be a tridiagonal NxN real matrix, with positive diagonal entries. Then if

$$a_{n,n-1}a_{n-1,n} \leq a_{n,n}a_{n-1,n-1}(1+\pi^2/4N^2)/4 \qquad (3.1)$$

for n=2, . . ., N, it follows that determinant A> 0. This inequality was motivated by a tridiagonal matrix with 2's along the main diagonal and -1's along the remaining two diagonals.

This proposition was proven in the same paper as Theorem II, and it guarantees that Theorem II can be utilized for any tridiagonal matrix which satisfies conditon (3.1). Furthermore, the tridiagonal matrices are important because they correspond to curve fitting data by a linear spline.

The first step to broadening the classes of matrices for which the determinant >0, is to see if condition (3.1) holds true for other constants along the main diagonal, and for any nonconstant entries along the other two diagonals. For purposes of illustration, let us try the constant four along the main diagonal instead of two.

Let A be a tridiagonal matrix with fours along the main diagonal. See Figure 3. First we must discover a general equation for the detAn. This can be achieved by studying the first few determinants, and watching for a general pattern to occur.

$$\det A_1 = 4$$

$$\det A_2 = 16-a_{21}a_{12}$$

$$\det A_3 = -a_{23}a_{32}A_{n-2}+4A_{n-1}$$

$$\det A_4 = -a_{34}a_{43}A_{n-2}+4A_{n-1}$$

so it follows that

$$\det A_n = 4A_{n-1}-a_{n-1,n}a_{n,n-1}A_{n-2}.$$

Now we can define $A_o = 1$, because

$$A_2 = 16-a_{21}a_{12} = 4(4) - (1)a_{21}a_{12}.$$

Let $f = a_{n-1,n}a_{n,n-1}$ so that the equation becomes

$$A_n-4A_{n-1}+fA_{n-2} = 0 .$$

This can now be rewritten as

$$A_n-2A_{n-1}+A_{n-2}-2A_{n-1}+2A_{n-2} = gA_{n-2} = 0. \quad (3.2)$$

Using the definition of difference equations,

$$A_n-2A_{n-1}+A_{n-2} = \Delta^2 A_{n-2}$$

and

$$-2A_{n-1}+2A_{n-2} = -2\Delta A_{n-2}$$

so equation (3.2) is now equal to

$$\Delta^2 A_{n-2}-2\Delta A_{n-2} = C_n A_{n-2} = -w^2 A_{n-2}.$$

Before showing that each $A_n > 0$, $n = 1, \ldots, N$, we motivate a key equation (3.6) with the following

$$\begin{pmatrix} 4 & a_{12} & 0 & 0 & 0 & \cdots \\ a_{21} & 4 & a_{23} & 0 & 0 & \cdots \\ 0 & a_{32} & 4 & a_{34} & 0 & \cdots \\ 0 & 0 & a_{43} & 4 & a_{45} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

FIG. 3. Tridiagonal Matrix with 4's Along the Main Diagonal

observation.

Suppose that u and v solve the problem

$$u''(x) = f(x)\,u(x) + 2u'(x) \quad u(0) = 1 \qquad u'(0) = 1$$

$u'(o) = 3$ because $\det A_1 - \det A_o = 3$.

Now $u''(x)$ can be rewritten as

$$u''(x) = (f(x)-w^2+w^2)u(x)+2u'(x).$$

Let

$$v''(x) - = w^2 v(x)+2v'(x) \qquad v(o)=1 \quad v'(o)=3$$

Then, $W = u-v$ satisfies

$$w''(x)-2w'(x) = -w^2 w(x)+(f(x)+w^2)u(x) \qquad w(o)\ w'(o) = 0 \tag{3.3}$$

Solving the homogeneous equation

$$w''(x)-2w'(x)+w^2 w(x) = 0$$

the auxiliary equation becomes

$$m^2-2m+w^2 = 0.$$

Utilizing the quadratic equation, one finds that the roots m_1, and m_2 are $1 \pm \sqrt{1-w^2}$.

Therefore, using the form of a homogeneous differential equation,

$$W(x) = C_1 e^{(\sqrt{1-w^2}+1)x} + C_2 e^{(-\sqrt{1-w^2}+1)x} \tag{3.4}$$

Next, Green's function must be utilized to obtain a particular solution to (3.3).

Green's function is defined as

$$G(x,u) = \begin{cases} -y_1(u)y_2(x) \;/\; W(y_1,y_2)(X), & 0\leq u\leq X, \\ -y_1(x)y_2(u) \;\; W(y_1,y_2)(X) & X\leq u\leq 1, \end{cases}$$

where y_1 and y_2 are solutions of the homogeneous equation (3.4) which satisfies the boundary conditions at $X=0$, and $W(y_1,y_2)$ is the Wronskian of y_1 and y_2.

Thus a particular solution takes the form,

$$\phi\;(x) = \int_o^1 G(x,u)\; f(u)du.$$

Now, the Wronskian of y_1 and y_2 from (3.4) is found to be equal to $-2\sqrt{1-w^2}$. Plugging into Green's function our solution is

$$W(X) = \int_o^x \frac{\left[e^{(\sqrt{1-w^2}+1)t}e^{(-\sqrt{1-w^2}+1)x}\right] - \left[e^{(\sqrt{1-w^2}+1)x}e^{(-\sqrt{1-w^2}+1)t}\right]}{-2\sqrt{1-w^2}}$$

Now if we let

$$a(x-t) = \frac{e^{(-\sqrt{1-w^2}+1)(x-t)} - e^{(\sqrt{1-w^2}+1)(x-t)}}{-2\sqrt{1-w^2}}$$

then it can be rewritten as

$$a(x-t) = \frac{e^{\alpha(x-t)} - e^{\beta(x-t)}}{\alpha - \beta}$$

where $\alpha = -\sqrt{1-w^2}+1, \quad \beta = \sqrt{1-w^2}+1$

so $a(x) = \dfrac{e^{\alpha x}-e^{\beta x}}{\alpha - \beta}$ and $a'(x) = \dfrac{\alpha e^{\alpha x}-\beta e^{\beta x}}{\alpha - \beta}$.

Thus $a(o) = o$ and $a'(o) = 1$.

Now we can say that

$$\Delta^2 a_{n-2} - 2\Delta a_{n-2} = -w^2 a_{n-2}.$$

Now returning to our proof, let (V_n), (a_n), $n=0,\ldots N$ satisfy

$$\Delta^2 V_n - 2\Delta V_n = V_n - 4V_{n-1} + 4V_{n-2} = -w^2 V_{n-2}, \quad V_o = 1, V_1 = 4$$

and

$$\Delta^2 a_{n-2} \Delta a_{n-2} = -w^2 a_{n-2}, \quad a_o = 0,\ a_1 = 1$$

which we have just shown above.

Now let $W_n = U_n - V_n$, so that

$$\Delta^2 W_n - 2\Delta W_{n-2} = -w^2 W_{n-2} + (C_n + w^2) U_{n-2},\ W_o = W_1 = 0.$$

We claim that the sequence defined by

$$X_n = \sum_{k=0}^{n-1} a_{n-1-k} (C_{k+2} + W^2)\ U_k,\ n \geq 1, \quad X_o = 0. \tag{3.6}$$

equals W_n. Indeed, $X_1 = 0$ since $a_o = 0$, and we have only to prove (3.5) holds. A direct calculation for $n \geq 2$ gives that $W_n = X_o$. This calculation is analogous to the procedure used in the paper by the four math professors found in the introduction (consult bibliography), and because this calculation is complex, I leave it to the reader to accept this point.

Now solving for A_n. Suppose $A_n = R^n$, where R is a complex number. So,

$$R^n - 2R^{n-1} + R^{n-2} - 2(R^{n-1} - R^{n-2}) = -w^2 R^{n-2}$$

$$R^n - 4R^{n-1} + 4R^{n-2} = -w^2 R^{n-2}$$

$$R^2 - 4R + 3 = -w^2$$ is the auxiliary equation.

Then the roots are

$$R = 2 \pm \sqrt{1 - w^2}\ .$$

Solving for V_n.

$$V_n - 4V_{n-1} + (4+w^2) = 0$$

The auxiliary equation can be expressed as

$$r^2 - 4r + (4+w^2).$$

Using the quadratic equation, the roots become equal to $2 \pm iw$. Hence, our solution becomes equal to

$$C_1 \text{Real}(2+iw)^n + C_2 \text{Imaginary}(2+iw)^n .$$

Solving for C_1 and C_2 using the initial conditions that $V_o=1$, and $V_1=4$, we find,

$$1 = C_1(1)$$

$$4 = C_1(2) + C_2(w)$$

Therefore, $C_1 = 1$, and $C_2 = \frac{2}{w}$

Thus, our solution for V_n is equal to

$$\text{Re}(1+iw)^n + 2\text{Im}(2+iw)^n/w.$$

Suppose now we want to find a condition for w, so that the determinant will always be greater than zero. See Figure 4. We know that $2 + iw = \rho e^{i\theta}$ from complex variables, and hence $(\rho_e{}^{i\theta})^n = \rho^n e^{in\theta}$.

From the diagram

$$\tan\theta = \frac{w}{2} .$$

We want $2N\tan^{-1}\ \theta \leq \pi/2$

so

$$\tan^{-1}\ \theta \leq \pi/4N = W.$$

Therefore, since $NW \leq \pi/4$, we have $V_n > 0$ and $a_n > 0$ for $n = 1, ... N$. Finally, replacing w by $\pi/4N$, we can

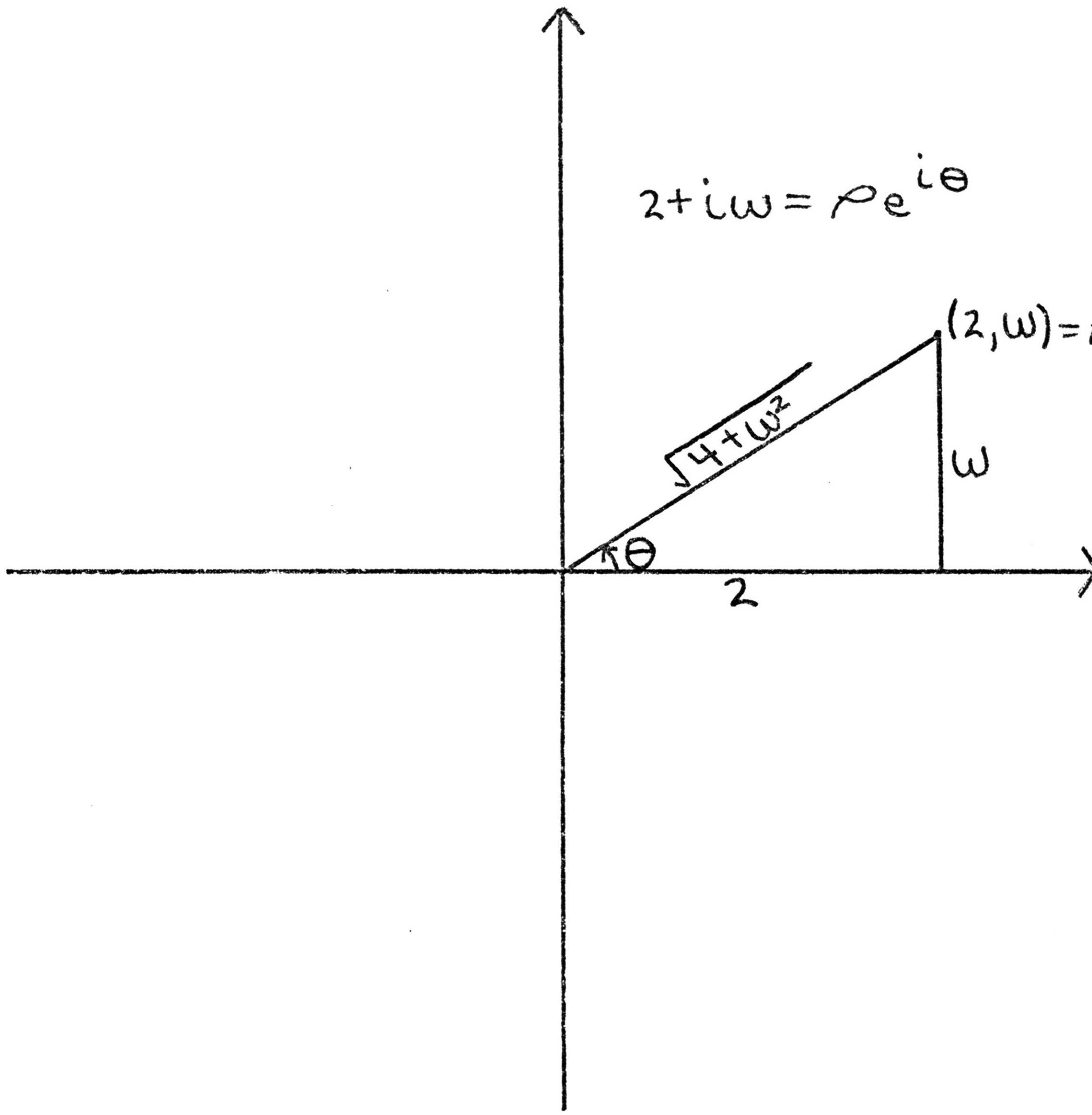

FIG. 4. Graphic Representation of

a Complex Number

conclude that $W_n<0$ is impossible and hence $U_n=W_n+V_n>0$, $n=1, \ldots . N$. Thus, we conclude that changing the two on the main diagonal did not change the validity of the original hypothesis.

This same analysis would probably hold true for any k along the main diagonal. If we find a general equation for the nth determinant it turns out to be

$$\det A_n = K_{\det} A_{n-1} - a_{n-1,n} a_{n,n-1} \det A_{n-2}.$$

Notice, however, that this equation does not correspond to any first or second difference. This would cause the computation to become very long and involved. Thus, the best we can do is to make a reasonable conjecture based on our previous study of the original hypothesis (3.1), and this conjecture would be that for any k along the main diagonal, hypothesis (3.1) would hold true.

Tridiagonal Case

The next phase of this project was to determine when a general tridiagonal matrix (i.e. when the diagonals are each a constant) has a positive determinant. The first step in determining when a constant tridiagonal matrix has a positive determinant is to derive a divided difference equation for the determinant of any NxN matrix. See Figure 5.

Let us compute the first few determinants to see if a general pattern occurs.

det $A_o = 1$ (an assumption to be checked later).

det $A_1 = a$

det $A_3 = -bcdetA_{n-2}+adetA_{n-1}$

det $A_4 = -bcdetA_{n-2}+adetA_{n-1}$.

Thus, it can be generalized that

$$detA_n = adetA_{n-1}-bcdetA_{n-2}. \qquad (4.1)$$

Now, $A_o = 1$ must be confirmed. Plugging into the formula for the determinant of A_2 using $a = 4$, $b = 1$, and $c = 2$, one finds

$$detA_2 = \begin{vmatrix} 4 & 1 \\ 2 & 4 \end{vmatrix} = 16-2 = 14$$

$$= -2 \times 1(1) + 4(4)$$

$$= -bcdetA_{n-2}+adetA_{n-1},$$

thus the det $A_o = 1$.

The auxiliary equation (2.1) from the equation (4.1)

$$\begin{pmatrix} a & b & 0 & 0 & 0 & \cdots \\ c & a & b & 0 & 0 & \cdots \\ 0 & c & a & b & 0 & \cdots \\ 0 & 0 & c & a & b & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

FIG. 5. Tridiagonal Matrix with a's Along the Main Diagonal

is $$m^2 - am + bc = 0.$$

To solve this quadratic equation, we must consider the three special cases.

Case 1. Suppose the roots are real and unequal. Utilizing the quadratic equation, the roots are,

$$\frac{a \pm \sqrt{a^2 - 4bc}}{2}$$

where $r_1 = \frac{a + \sqrt{a^2 - 4bc}}{2}$ and $r_2 = \frac{a - \sqrt{a^2 - 4bc}}{2}$.

These roots satisfy the equation

$$Y_k = C_1 r_1^k + C_2 r_2^k \quad . \qquad (4.2)$$

Solving for the initial conditions with $A_o = 1$, and $A_1 = a$,

$$1 = C_1 + C_2$$

$$a = C_1 r_1 + C_2 r_2.$$

Using Cramer's Rule to solve two simultaneous equations, we find that

$$C_1 = \frac{\begin{vmatrix} 1 & 1 \\ a & r_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix}} \quad \text{and } C_2 = \frac{\begin{vmatrix} 1 & 1 \\ r_1 & a \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix}}$$

so $C_1 = \frac{r_2 - a}{r_2 - r_1} \quad r_1^k$ and $C_2 = \frac{a - r_1}{r_2 - r_1}$

hence,

$$Y_k = \frac{r_2 - a}{r_2 - r_1} r_1^k + \frac{a - r_1}{r_2 - r_1} r_2^k.$$

Simplifying $r_2 - r_1$, we find

$$r_2 r_1 = \frac{a}{2} - \frac{\sqrt{a^2-4bc}}{2} - \frac{a}{2} - \frac{\sqrt{a^2-4bc}}{2}$$
$$= -\sqrt{a^2-4bc}$$

Thus,

$$Y_k = \frac{1}{-\sqrt{a^2-4bc}} \left[(r_2-a)r_1^k + (a-r_1)r_2^k \right] \quad (4.3)$$

Similarly, simplifying r_2-a, we find

$$r - a = \frac{a-\sqrt{a^2-4bc}}{2} - \frac{2a}{2}$$
$$= \frac{-a-\sqrt{a^2-4bc}}{2}$$
$$= r_1 .$$

Finally, simplifying $a-r_1$, we find

$$a - r_1 = \frac{2a}{2} - \frac{a+\sqrt{a^2-4bc}}{2}$$
$$= \frac{a-\sqrt{a^2-4bc}}{2}$$
$$= r_2 .$$

Substituting into (4.2), one finds

$$Y_k = \frac{1}{-\sqrt{a^2-4bc}} - (r_1^{k+1}) + (r_2^{k+1})$$

Looking at Y_k, it is obvious that if

$$\frac{a+\sqrt{a^2-4bc}}{2} > \frac{a-\sqrt{a^2-4bc}}{2}$$

then $\det A_n > 0$.

Consequently, if

$$a+\sqrt{a^2-4bc} > a - \sqrt{a^2-4bc}$$

then the following steps directly follow:

$$\sqrt{a^2-4bc} > -\sqrt{a^2-4bc}$$
$$a^2-4bc > -(a^2-4bc)$$
$$a^2-4bc > a^2+4bc$$
$$2a^2 > 8bc$$
$$a^2 > 4bc \quad .$$

Thus, if $a^2>4bc$, the $\det A_n > 0$.

Let us now consider one example. Suppose a=2, b=0, and c=0. Then,

$$Y_k = \frac{-1}{2}\left(\left(\frac{-2-\sqrt{4}}{2}\right)^{k+1} + \left(\frac{2-\sqrt{4}}{2}\right)^{k+1} \right)$$
$$= -\frac{1}{2}\left(-(2^{k+1}) \right)$$
$$= 2^k \quad .$$

Notice that $a^2>4bc$, because $2^2>0$, so the determinant should always be positive, and we found it is.

$2^k>0$ for all $k = 1, 2, 3, \ldots\ldots$

<u>Case 2</u>. Suppose the roots are real and equal. The auxiliary equation of (4.1) is still

$$m^2-am+bc = 0.$$

Using the quadratic equation, the roots equal

$$\frac{a\pm\sqrt{a^2-4bc}}{2}$$
$$= \frac{a}{2}$$

since $a^2-4bc=0$ by assumption. Recall from section two of this paper, that the general solution for this case is,

$$Y_k = (C_1+C_2k)\left(\frac{a}{2}\right)^k .$$

Using the initial conditions, $a_0=1$, and $a_1=a$, C_1 and C_2 are as follows,

$$1 = C_1$$

$$a = \frac{a}{2} C_1 + \frac{a}{2} C_2 \quad .$$

Plugging in 1 for C_1, C_2 becomes also equal to 1.

Thus,

$$Y_k = (1+k) \ (\frac{a}{2})^k \quad .$$

Therefore, when $a^2=4bc$ the determinant >0 if $a \geq 0$, and the determinant >0 when k is even if $a < 0$. Notice that in the case where the roots are real, the determinant is always positive when $a^2 > 4bc$. This is a slightly weaker assumption than the hypothesis (3.1) of proposition 1.

Next, let us work two examples.

Example 1. Suppose a = 2, b = 1, and c = 1. The roots are equal to $\frac{2 \pm \sqrt{4-4}}{2}$, so roots = 1.

Thus $Y_k = (1+k)(1)^k$, which is always > 0.

Example 2. Suppose a = -2, b= -1, and c = -1. The roots are equal to $\frac{-2 \pm \sqrt{4-4}}{2}$, so roots = -1.

Thus, $Y_k = (1+k)(-1)^k$, which is positive when k is even.

Case 3. Suppose the roots are complex. Again, the auxiliary equation of (4.1) is given by

$$m^2 - am + bc = 0.$$

Using the quadratic equation, the roots equal

$$\frac{a \pm \sqrt{a^2-4bc}}{2} \quad .$$

We are now assuming that $4bc > a^2$ so that conjugate roots are obtained.

From section two,

$$r = \sqrt{(\frac{a}{2})^2 + (\frac{(4bc-a^2)^{\frac{1}{2}}}{2})^2}$$

$$= \sqrt{\frac{a^2}{4} + \frac{4bc-a^2}{4}}$$

$$= \sqrt{\frac{4bc}{4}}$$

$$= \sqrt{bc} \quad .$$

Also from section two,

$$\cos\theta = \frac{a}{2}\left(\frac{1}{\sqrt{bc}}\right) = \frac{a}{2\sqrt{bc}}$$

$$\sin\theta = \frac{\sqrt{4bc-a^2}}{2}\left(\frac{1}{\sqrt{bc}}\right) = \frac{\sqrt{4bc-a^2}}{2bc} .$$

Recall the solution for complex roots is given by

$$Y_k = A_r{}^k \cos(k\theta+B)$$

where

$$2a = A \quad \text{and } C_1 = a,\ C_2 = B.$$

Thus,

$$Y_k = 2C_1 r^k \cos(k\theta+C_2) . \qquad (4.4)$$

Solving for C_1 and C_2 with initial conditions $a_o = 1$, $a_1 = a$,

$$1 = 2C_1 \cos C_2$$

which simplifies to

$$\tfrac{1}{2} = C_1 \cos C_2 .$$

We also know by our initial conditions that

$$a = 2C_1\sqrt{bc}\ \cos(\theta+C_2)$$

which simplifies as follows:

$$\frac{a}{2\sqrt{bc}} = C_1\cos(\theta+C_2)$$

$$= C_1(\cos\theta\cos C_2-\sin\theta\sin C_2) \text{ from trigonometry}$$

$$= C_1\left[\frac{a}{2\sqrt{bc}}\cos C_2-\frac{\sqrt{4bc-a^2}}{2\sqrt{bc}}\sin C_2\right]$$

which finally simplifies to

$$a = C_1\left[a\cos C_2-\sqrt{4bc-a^2}\ \sin C_2\right]. \qquad (4.5)$$

See Figure 6.

Using the fact that $\cos\theta = \frac{\text{side adjacent}}{\text{side opposite}}$, we find

$$\cos C_2 = \frac{1}{2C_1}. \qquad (4.6)$$

Using the Pythagorean Theorem,

$$y = \sqrt{4c_1^2-1}$$

Using the fact that $\sin\theta = \frac{\text{side opposite}}{\text{hypotenuse}}$,

$$\sin C_2 = \frac{\sqrt{4C_1^2-1}}{2C_1}.$$

Substituting into (4.4), the equation becomes

$$a = C_1\left[a\frac{1}{2C_1} - (4bc-a^2)^{\frac{1}{2}}\ \frac{(4C_1^2-1)^{\frac{1}{2}}}{2C_1}\right]$$

$\cos C_2 = 1/2C_1$

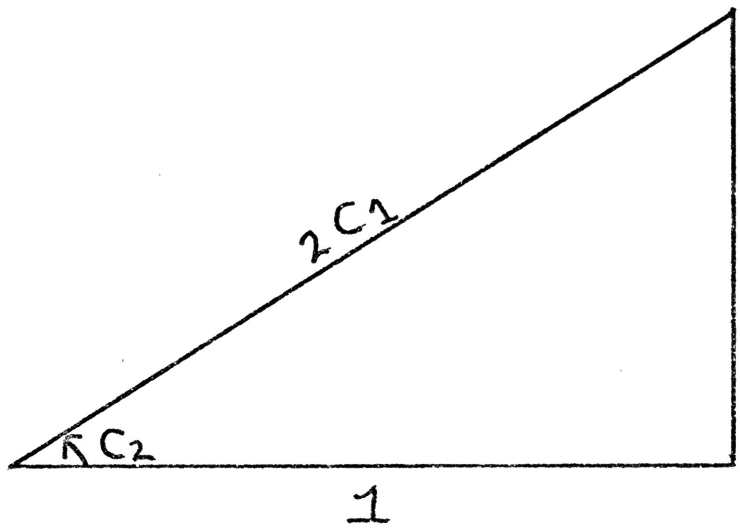

FIG. 6. Finding the Cosine of C_2

or

$$a = (4bc-a^2)^{\frac{1}{2}} (4C_1{}^2-1)^{\frac{1}{2}}.$$

Upon squaring both sides

$$a^2 = (4bc-a^2)(4C_1{}^2-1).$$

Solving for C_1,

$$\frac{a^2}{4bc-a^2} = 4C_1{}^2-1 .$$

Multiplying by $\frac{1}{4bc-a^2}$ we find

$$\frac{4bc-a^2}{4bc-a^2} + \frac{a^2}{4bc-a^2} = 4C_1{}^2.$$

Simplifying,

$$\frac{4bc}{4bc-a^2} = 4C_1{}^2$$

and $$C_1{}^2 = \frac{bc}{4bc-a^2}$$

so $C_1 = \pm\sqrt{\frac{bc}{4bc-a^2}}$.

Substituting into (4.6), one can solve for C_2.

$$\cos C_2 = \frac{1}{2\left(\sqrt{\frac{bc}{4bc-a^2}}\right)} .$$

Rearranging,

$$1 = \frac{2\sqrt{bc}}{\sqrt{4bc-a^2}} \cos C_2$$

so
$$\frac{\sqrt{4bc-a^2}}{2\sqrt{bc}} = \cos C_2$$

hence,
$$C_2 = \text{arc cos}\left(\frac{\sqrt{4bc-a^2}}{2\sqrt{bc}}\right).$$

Returning to (4.4)

$$Y_k = 2C_1\sqrt{bc}^k \cos(k\theta+C_2) \qquad k = 0, 1, 2, \ldots$$

Now we must consider two cases, when $C_1>0$, and when $C_1<0$.

1. Assume $C_1>0$ so $-\pi/2 < k\theta + C_2 < \pi/2$.

Substituting into (4.4) we find Y_k will be greater than zero whenever

$$\pi\left(\frac{4n-1}{n}\right) < k \text{ arc cos } \frac{a}{2\sqrt{bc}} + \text{arc cos } \frac{\sqrt{4bc-a^2}}{2\sqrt{bc}} < \pi\frac{4n+1}{2} \quad .(4.7)$$

2. Assume $C_1<0$ and use the same procedure as part 1.

Now we find that Y_k is greater than zero whenever

$$\pi\frac{2n-1}{2} < k \text{ arc cos } \frac{a}{2\sqrt{bc}} + \text{arc cos } \frac{\sqrt{4bc-a^2}}{2\sqrt{bc}} < \Pi\frac{2n+1}{2} \quad (4.8)$$

Both of these cases are true for $n = 0, 1, 2, \ldots$.

Thus we have found that the determinant of (4.1) is greater than zero when:

1. $a^2>4bc$, for real and unequal roots,
2. $a^2 = 4bc$, for real and equal roots provided k is even, if $a<0$,
3. (4.7) and (4.8) hold for the complex roots.

$$\begin{pmatrix} 4 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 4 & 2 & 1 & 0 & \cdots \\ 0 & 1 & 4 & 2 & 1 & \cdots \\ 0 & 0 & 1 & 4 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

FIG. 7. Five Diagonal Matrix with

0,1,4,2,and 1 along the diagonals

Five Diagonal Case

The last phase of this project was to determine when a general five diagonal (i.e. when the diagonals are each a constant) matrix has a positive determinant. The five diagonal matrices correspond to forming quadratic polynomials to estimate a data fit. This is helpful because using quadratic functions lessens the number of knots needed in a spline.

Before attacking the general five diagonal case, let us consider the following specific case, where the five constants are 0,1,4,2, and 1. See Figure 7. By using the same method as in the tridiagonal case, we find that the equation for the n^{th} determinant is given by the following,

$$U_n = 4U_{n-1}-2U_{n-2}+U_{n-3}.$$

In the case where n=1,2,3,4, the determinants are respectively 4, 13, 49, and 174. We wish to make the claim that $U_n \geq 2U_{n-1}$ and show this true by induction.

Proof. Suppose $U_n \geq 2U_{n-1}$ for n-3, n-2, and n-1

$$\begin{aligned} U_n &= 4U_{n-1}-2U_{n-2}+U_{n-3} \\ &\geq 4U_{n-1}-2U_{n-2} \\ &= 2U_{n-1}+2(U_{n-1}-U_{n-2}) \\ &\geq 2U_{n-1} \end{aligned}$$

End of proof.

Next, we replaced the five diagonals by the constants o,c,a,b,c. See Figure 8. The general equation for the determinant was found to be

$$U_n = c^3U_{n-3} - bc\ U_{n-2} + a\ U_{n-1}.$$

The auxiliary equation is

$$m^3 - am^2 + bcm - c^3 = 0.$$

This auxiliary equation can be transformed into $x^3 + sx + t = 0$, by substituting for y the value x+a/3. This method for solving cubic equations was found in the <u>Standard Mathematical Tables, 21st Edition</u>. Now $s = 1/3\ (3bc-a^2)$ and $t = 1/27\ (-2a^3+9abc-27c^3)$.

For a solution, let,

$$A = \sqrt[3]{\frac{-s}{2} + \sqrt{\frac{s^2}{4} + \frac{t^3}{27}}} \qquad B = \sqrt{\frac{-s}{2} - \sqrt{\frac{s^2}{4} + \frac{a^3}{27}}}$$

then the values of X will be given by

$$X = A + B$$

$$X = \frac{-A+B}{2} + \frac{\pm A-B}{2} \quad \sqrt{-3}$$

$$X = \frac{-A+B}{2} \quad \frac{-A-B}{2} \quad \sqrt{-3}$$

(5.1)

If -a, bc, and -c are real, then:

If $s^2/4 + t^3/27 > 0$ there will be one real root and two conjugate imaginary roots,

If $s^2/4 + t^3/27 = 0$ there will be three real roots of which at least two are equal.

If $s^2/4 + t^3/27 < 0$ there will be three real and unequal roots.

$$\begin{pmatrix} a & b & c & 0 & 0 & \cdots \\ c & a & b & c & 0 & \cdots \\ 0 & c & a & b & c & \cdots \\ 0 & 0 & c & a & b & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

FIG. 8. Five Diagonal Matrix with

o, c, a, b, and c along the diagonals

We know that the general solution of a difference equation is of the form (2.4), so we find that,

$$y_o = C_1+C_2+C_3 = 1$$
$$y_1 = C_1r_1 +C_2r_2+C_3r_3 = a$$
$$y_2 = c_1r_1{}^2+C_2r_2{}^2+C_3r_3{}^2 = a^2-bc$$

where r is given by (5.1).

When we attempted to solve for C_1, C_2, and C_3, to obtain conditions on a, b, and c for when the determinant is positive, the calculations became "messy," and it was impossible to get any satisfactory conditions in the time we had to work on this problem. Thus, this part of our research was disappointing. In fact, it becomes impossible for a polynomial of degree ≥ 5 to have a solution which follows a general formula by a well known theorem in algebra known as the insolvability of the quintic. Therefore, if we had tried to solve a matrix with nine constants, the effort would have proved worthless in the amount of time we had to spend on this project.

CONCLUSION

First of all, this research has shown that we extended the classes of matrices for which the determinant is always positive to all constant tridiagonal matrices. Actually, Proposition 1 was sufficient to conclude when the determinants of constant tridiagonal matrices are positive, but we were able to weaken the condition in Proposition 1. Furthermore, we also showed that when there are fours along the main diagonal of any tridiagonal matrix, the determinant will be positive whenever Proposition 1 holds true. In fact, we feel safe in saying that for any constant k on the main diagonal of a tridiagonal matrix, the determinant will be greater than zero, under the assumptions of Proposition 1. However, we were not able to specifically prove this, so it will hopefully be attacked by future research.

Secondly, we were able to give conditions for when the determinant of a five diagonal matrix is positive only for specific matrices. When we attempted to generalize the five diagonal case, the computation became too complex for us to solve in the allotted time period. Thus, this is also a problem I leave to future research.

Thirdly, because we did extend the classes of matrices for which the determinant is always positive, we can speculate that Theorem II can now be generalized to include piecewise convex or concave functions. This is important for mathematicians who are trying to curve fit data.

Finally, although our research did not result in startling conclusions, I would like to say that it was a very enlightening and immeasurable experience.

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