Determination of Stress Concentration Factors in Flat Plates with Holes Using Finite-Element Analysis Methods

by

Roger Paul Young

Maritime Systems Engineering

Submitted in Partial Fulfillment of the Requirements of the University Undergraduate Program

1981-1982

Approved by:

Mr. Ralph Davis

April 1982

ABSTRACT

The importance of stress concentration factors in machines and stuctures led to the development of various experimental and analytical techniques for determining them. One of these methods is the application of finite element analysis to the various structures. A common structural element in a piece of machinery is a flat plate with one or more bolt holes, that are often off center.

Using certain assumptions from S. P. Timoshenko's Strength of Materials and the finite element method outlined in J. S. Przemienieck's, Theory of Matrix Stuctural Analysis, a computer program was written to help determine these stress concentration factors. The program consists of a main program that generates the nodal points, several subroutines that develop the stiffness matrix, a subroutine that transforms the element stiffness matrix into a system stiffness matrix, a subroutine that inverts a part of the system matrix, and a subroutine that multiplies the system deformations by another portion of the stiffness matrix.

The stresses computed with the system do not correlate with other theoretical values. This inconsistency is believed to be caused by an error in the transformation of the element stiffness matrices into the system stiffness matrix. When this problem is adjusted, the program should give valid stress concentrations.

ACKNOWLEDGEMENTS

I extend my thanks and gratitude to all the personnel in the computer center who had the time and patience to answer my many questions. I would also like to express my appreciation to all of the faculty and staff at Texas A&M at Galveston for the learning experience over the last four years. My special thanks go to my two advisors, Dr. James M. Nash and Mr. Ralph Davis for sharing their knowledge with me and for the encouragement to extend my studies in finite element analysis.

TABLE OF CONTENTS

TABLE OF ILLUSTRATIONS

INTRODUCTION

The theoretical calculation of the average stress in a plate of uniform cross-section is quite simple. However, if there is an abrupt change in the cross-sectional area of a plate, the stress concentration is far above the average stress. The stress concentration divided by the average stress is the stress concentration factor for that plate. The theoretical calculations of these stress concentration factors only exists in a few of the simplest cases. The majority of the information of the information has been obtained by experimentation. These stress concentration factors can also be determined by finite element methods with practically no limitation on the geometry of the plate, or discontinuity in the cross-section.

Finite element analysis separate a continuum into a number of finite element, in this case triangles. These elements are assumed to be connected only at the nodal points. A system is found to determine the stiffness matrix for each element, and then matrix structural analysis methods can be used to solve for the stress concentrations.

PROGRAMING PROCEDURES

MAIN PROGRAM

The first step in the programing procedure was to determine how to divide the plate into various triangular elements. If possible, it would be convenient to make this process variable to accommodate various sizes of plates and various hole diameters while minimizing the number of inputs and restrictions. It was decided to redefine the plate by drawing an imaginary "circle" around the hole with a relatively large enough diameter such that the stresses along the circumference of the "circle" would not be influenced by the hole (see Figure 1). It was hypothesized that if the area of stress could be broken up as in Figure 1, then the plate could be divided into two circles of different diameters as in Figure 2. The diameter of the outer circle, as determined by S. P. Timoshenko should be five times larger than the diameter of the hole. This is a definite limitation on the range of specimens that can be analyzed: however, by using this restriction the only inputs necessary to determine each nodal point are: 1) the distance from the center of the hole to the edge of the plate (variable name in program, SPR), 2) the distance from the center of the hole to the middle of the plate, or to the other edge of the plate if only one hole is being analyzed (SPL), 3) the diameter of the hole itself (D) and 4) the number of angular divisions of the mesh (NUMANG).

The cartesian coordinate system for determining the nodal points is located at the center of the hole with the x-axis directed perpendicular to the direction of stress, and y-axis directed tangential to the direction of stress. Each nodal point is defined by two numbers in the array POINT(I,J,K): POINT(I,J,1) being the abscissa and POINT(I,J,2)

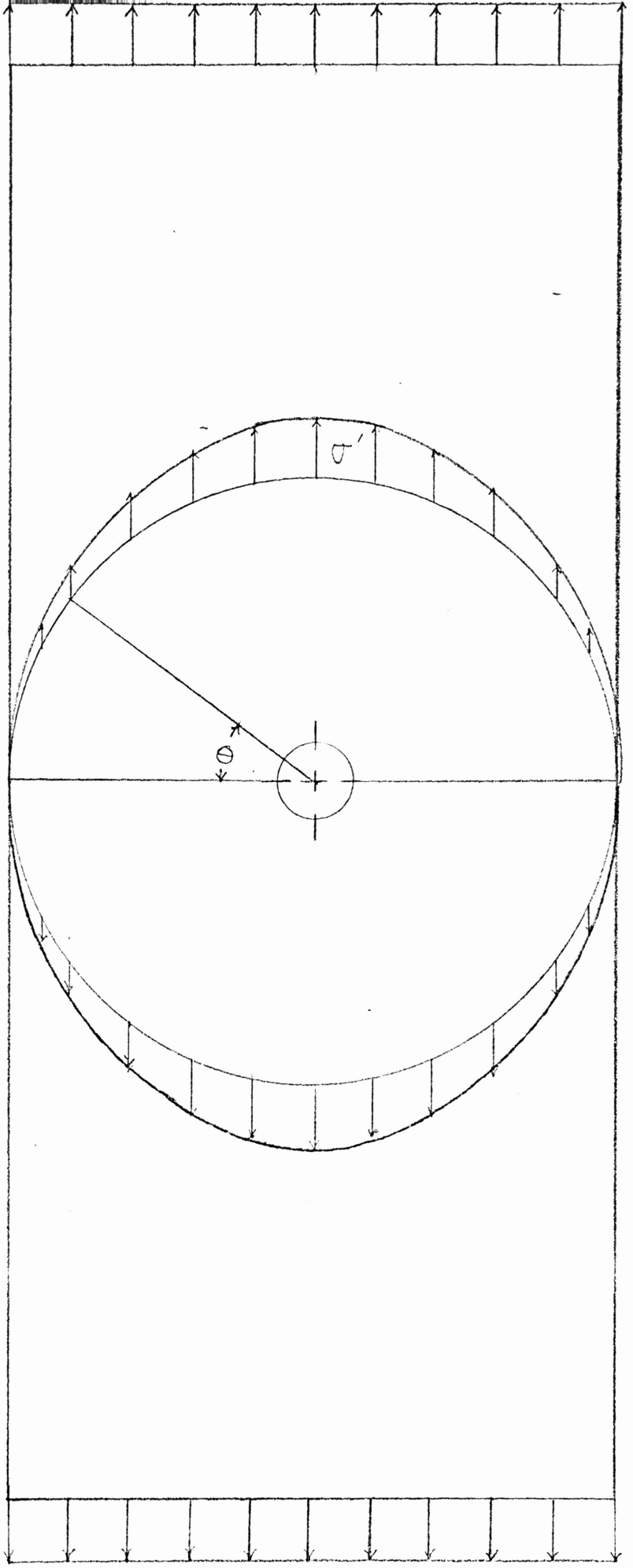


Figure 1
one Hole Specimen

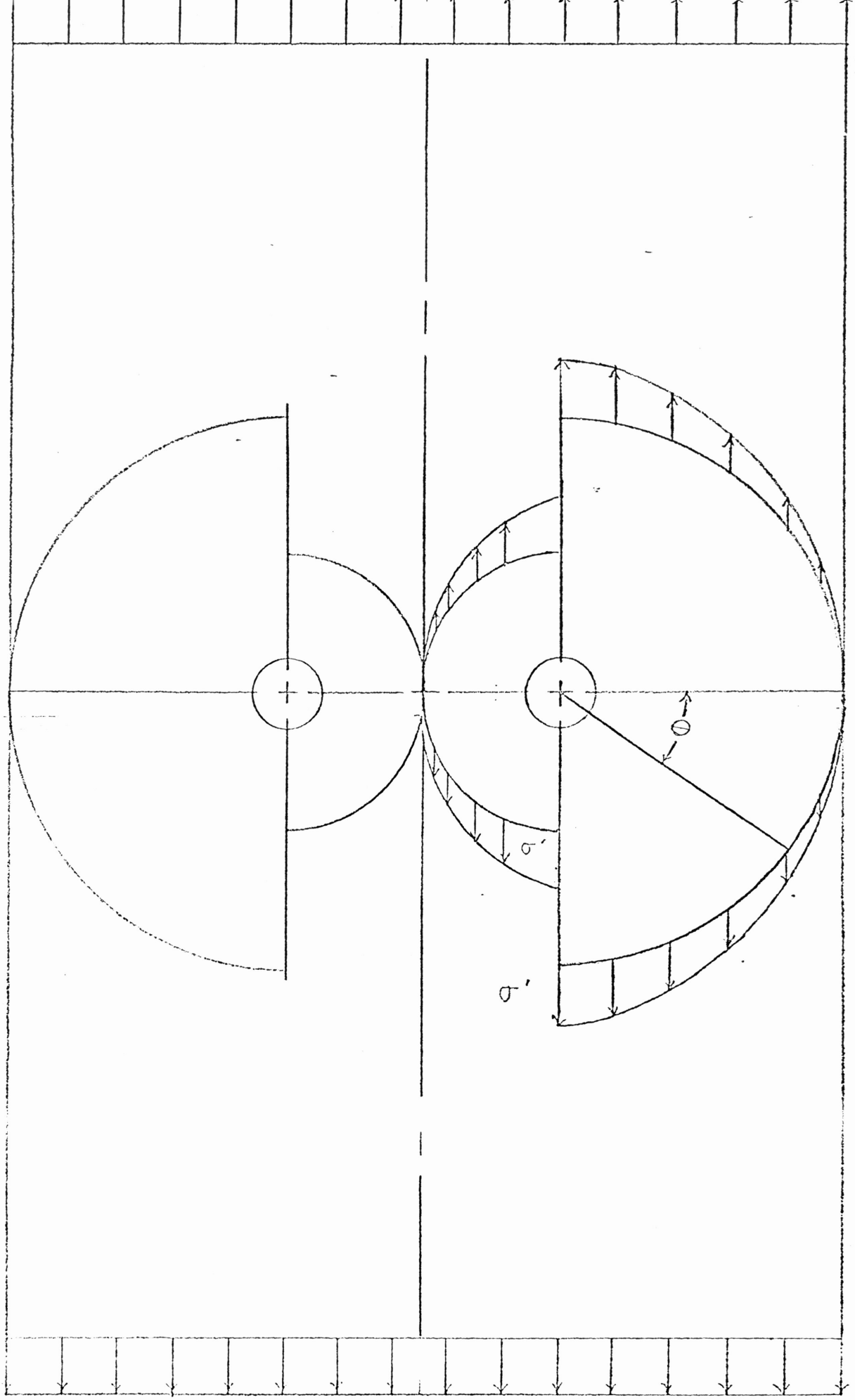


Figure 2
Two hole specimen

being the ordinate. POINT(I,J,1) and POINT(I,J,2) are determined by:

$$POINT(I,J,1) = R*COS(ANGN) \quad \text{(Eq. 1)}$$
$$POINT(I,J,2) = R*SIN(ANGN) \quad \text{(Eq. 2)}$$

where R is the distance between the nodal points and the origin, and ANGN is the angle between R and the positive x-axis. The initial value for R, where J=1, is D/2. The initial value for ANGN, when I=1, is zero. The subsequent values of R when J=J+1 are R+XSPACE, where:

$$XSPACE = 2*R*SIN(\pi/(2*NUMANG))$$

ANGN is held constant and each POINT(I,J,K) is calculated using Eq. 1 and Eq. 2 until R is greater than SPR. R is then set equal to SPR and POINT(I,J,K) is calculated using the same equations. ANGN is then changed, I=I+1, to ANGN + ANG where ANG = π/NUMANG. The same procedure for R is followed for the new ANGN until ANGN = $\pi/2$. The only difference for ANGN between $\pi/2$ and π is that SPR is replaced by SPL. When this process has been completed each value of POINT(I,J,K) is printed out for future use.

The main program is also used to call subroutines, TRIG, STFMX, INVERT and FORCE.

SUBROUTINE TRIG

The second step was to determine the relationship between a single element K, and the three nodal points. By using the system of obtaining the nodal points illustrated in the previous section, an algorithm could be made to determine the ordinate and the abscissa of each nodal point for each element. These values are then transfered down to the subroutine KAPA where the individual stiffness matrices are computed.

SUBROUTINE KAPA

The next step, and perhaps the most important, was to decide the method in developing the individual stiffness matrices of each element. The method outlined in J. S. Przemieniecki's book Theory of Matrix Structural Analysis was chosen. This method has no restrictions on the orientation of the local coordinate system.

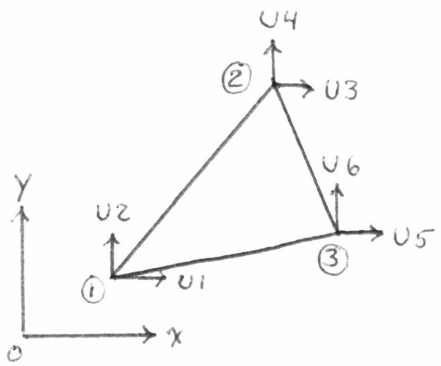


Figure (3)

Triangular Plate Element

Node (1) = (X1,Y1)
Node (2) = (X2,Y2)
Node (3) = (X3,Y3)

XIJ = XI-XJ

YIJ = YI-YJ

The triangular element is situated as shown in Figure (3). The element stiffness matrix is formulated on the basis of the difference between each nodal coordinate and not on the basis of Node (1) being below and to the left of Node (2). This indicates that the orientation of the element with respect to the local coordinate system is unimportant as long as the abscissa and ordinate of each nodal point of the triangle is known. In addition, the displacements U1, U3, and U5 will remain parallel with the x-axis, and the displacements U2, U4, and U6 will remain parallel to the y-axis.

The element stiffness matrix (ELSTF) was determined by:

$$K \simeq \int b^T \chi b \, dV \qquad \text{(Eq. 3)}$$

where b represents a matrix of the exact strains due to a unit diplacement,

b^T representing the transpose of b, and χ is a matrix of the form:

$$\chi = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2v)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2v)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2v)/2 \end{bmatrix}$$

where E is Young's modulus and v is Poisson's ratio. For convenience, the stiffness matrix was seperated into two parts:

$$K = K_n + K_s$$

where K_n represents the stiffness due to normal stresses and K_s represents stiffness due to shearing stresses. After the integration of Eq. 3, K_n and K_s are given by:

$$K_n = N \begin{bmatrix} Y32^2 & & & & & \\ -v*Y32*X32 & X32^2 & & \text{SYMMETRIC} & & \\ -Y32*Y31 & v*X32*Y31 & Y31^2 & & & \\ v*Y32*X31 & -X32*X31 & -v*Y31*X31 & X31^2 & & \\ Y32*Y21 & -v*X32*Y21 & -Y31*Y21 & v*X31*Y21 & Y21^2 & \\ -v*Y32*X21 & X32*X21 & v*Y31*X21 & -X31*X21 & -v*Y21*X21 & X21^2 \end{bmatrix}$$

$$K_s = N \begin{bmatrix} X32^2 & & & & & \\ -X32*Y32 & Y32^2 & & \text{SYMMETRIC} & & \\ -X32*X31 & Y32*X31 & X31^2 & & & \\ X32*Y31 & -Y32*Y31 & -X31*Y31 & Y31^2 & & \\ X32*X21 & -Y32*X21 & -X31*X21 & Y31*X21 & X21^2 & \\ -X32*Y21 & Y32*Y21 & X31*Y21 & -Y31*Y21 & -X21*Y21 & Y21^2 \end{bmatrix}$$

$$M = \frac{Et}{4(A123)(1-v^2)}$$

$$N = \frac{Et}{8(A123)(1+v)}$$

$$A123 = ((X32)(Y21)-(X21)(Y32))/2$$

where t is the thickness of the plate and all other variables are as defined earlier. As stated previously the orientation of the triangular element has no effect on the computation of the stiffness matrix: therefore, if the same local coordinate system is used for each element, the transformation matrix λ, is not necessary. This local coordinate system is the same system used in the main program to generate the nodal points.

SUBROUTINE STFMX

After each element stiffness matrix is computed, they must be correlated into the system stiffness matrix (STF). In M.M.467, Matrix Computer Analysis of Structures, Dr. James M. Nash illustrated a technique of systematically placing components of the element stiffness matrices into the system stiffness matrix. This technique requires the formation of a "Topology Matrix" (ITOPO), which must be read from a data file. The "Topology Matrix" will be a two dimensional matrix that has a number of columns equal to the number of element coordinates (NELC) and a number of rows equal to the number of elements in the system (NEL). Before the "Topology Matrix" can be formed, the system coordinates must be decided upon. If all of the system coordinates are situated in the positive x-direction and the positive y-direction, all of the elements of the "Topology Matrix" will be positive. It is necessary to number the vertical system coordinates along the x-axis last because the forces at these coordinates are the unknown forces the program will determine.

The deflection at these coordinates are assumed to be zero, along with the horizontal deflection on the y-axis if the hole is centered in the plate. The procedures for developing the "Topology Matrix" will be discussed later in the section titled SAMPLE PROBLEM.

SUBROUTINE INVERT

Once the system stiffness matrix has been correlated, a portion of the stiffness matrix needs to be inverted as shown below.

$$\begin{bmatrix} STF \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

P = Known Forces
F = Unknown Forces
u = Unknown Displacements
u_o = Displacements assumed to be zero

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{bmatrix} u \\ \hline u_o \end{bmatrix} = \begin{bmatrix} P \\ \hline F \end{bmatrix}$$

$$A_{11}\, u + A_{12}\, u_o = P \longrightarrow u = A_{11}^{-1} P$$

$$A_{21}\, u + A_{22}\, u_o = F$$

The horizontal partitioning of the stiffness matrix is located directly below the last row of the known forces (N). Since the matrix A_{11} needs to be a square matrix the vertical partition is located an equal number of columns over. First INVERT transfers the elements of the stiffness matrix to the A matrix. Second, the known forces are read in from a data file and placed next to the elements of the stiffness matrix in the A matrix. INVERT then inverts the matrix A, using Gauss-Jordan elimination while checking that the pivot element is not zero (EPSILN). In the process of inverting the matrix, INVERT multiplies the forces by the inverted matrix A, to obtain the unknown displacements.

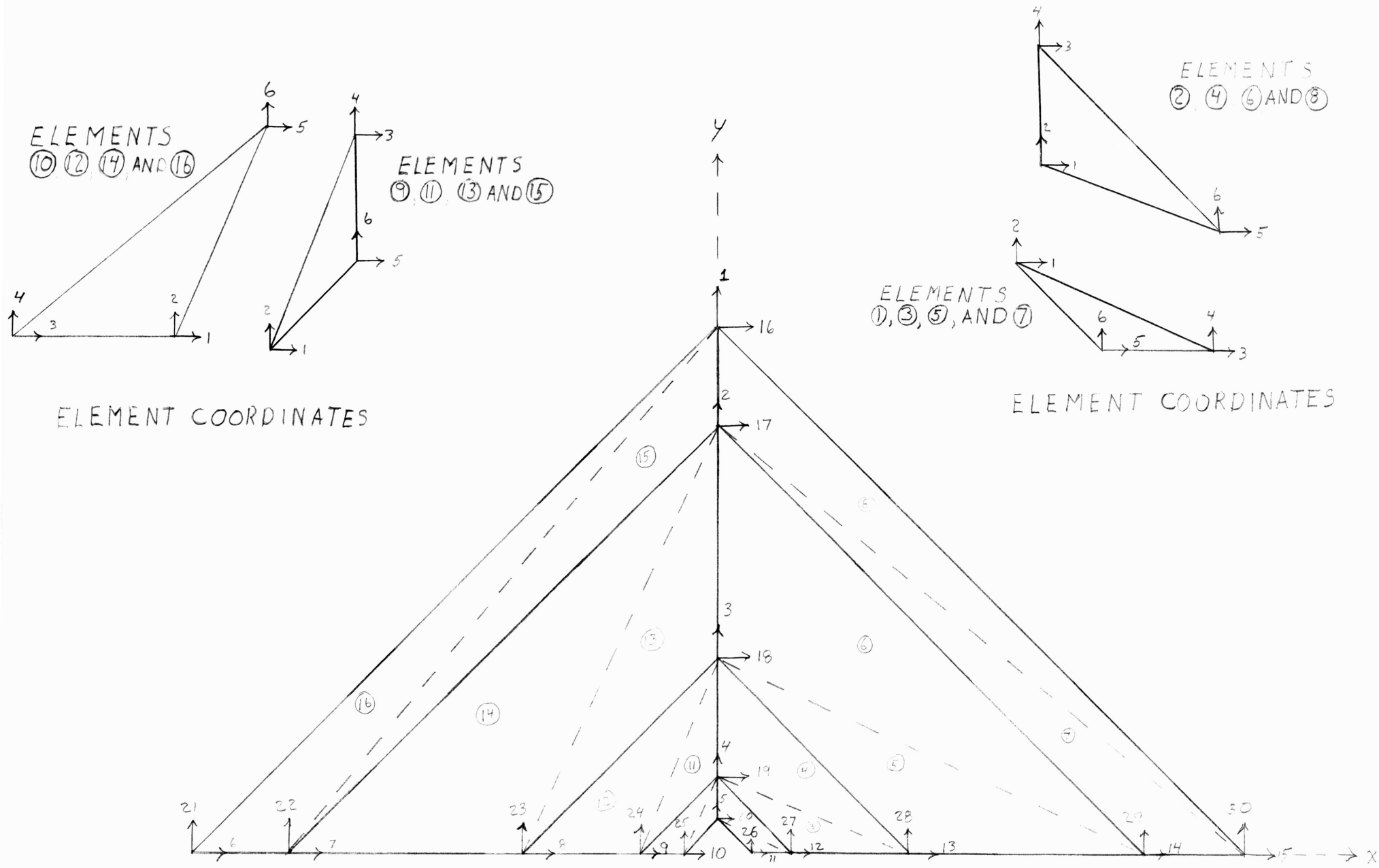
SUBROUTINE FORCE

Now that the unknown diplacements are known, they are multiplied by the portion of the stiffness matrix that was not inverted, and the unknown forces are printed out. From these forces and the original forces the stress concentration factor can be found.

SAMPLE PROBLEM

The first input into the system is the distance from the center of the hole to the edge of the plate, 4.0 inches. The next input is the distance from the center of the hole to the other edge of the plate (since this is a one hole example), 4.0 inches. The diameter of the hole is 0.5 inches and the number of angular divisions is 2. The program will then print out the points of each node. From these a composite of the system can be drawn, as in Figure (4). From Figure (4) the "Topology Matrix can be formed.

To produce the "Topology Matrix" each element is considered individually. If the element coordinates are superimposed on the system coordinates, the number of the system coordinate that corresponds to the element coordinate is placed in the "Topology Matrix" in the (Member Number, Element Coordinate) position. For example, if the element coordinates for element (1) where superimposed on the system coordinates, element coordinate 1 corresponds to system coordinate 20. Therefore 20 should be placed in the (1,1) position of the "Topology Matrix". This procedure is continued for each element coordinate and for each individual element, until the entire



Example Problem

Figure 4

ELEMENT COORDINATES

MEMBER NUMBER	1	2	3	4	5	6
1	20	05	12	27	11	26
2	20	05	19	04	12	27
3	19	04	13	28	12	27
4	19	04	18	03	13	28
5	18	03	14	29	13	28
6	18	03	17	02	14	29
7	17	02	15	30	14	29
8	17	02	16	01	15	30
9	10	25	19	04	20	05
10	10	25	09	24	19	04
11	09	24	18	03	19	04
12	09	24	00	23	18	03
13	08	23	17	02	18	03
14	08	23	07	22	17	02
15	07	22	16	01	17	02
16	07	22	06	21	16	01

"TOPOLOGY MATRIX"

"Topology Matrix" is completed. The "Topology Matrix" for this system is listed above. The "Topology Matrix" is then stored in a data file for the computer to read.

The next data file that needs to be computed is the known forces file. The only forces necessary to compute, are the vertical forces on the circumference of the "circle" and not on the x-axis. These are computed by integrating the function of stress over the surface area. From Figure (1), the function of stress is:

$$\sigma' = \sigma \sin\theta$$

$$F = \int_A t\sigma \sin\theta \, da = \int_{\theta_1}^{\theta_2} t(\sin)r\,d\theta = -tr\sigma\cos\Big|_{\theta_1}^{\theta_2}$$

where σ is the uniaxial stress applied to the plate, and θ_1 and θ_2 are the limits of integration. The θ_1 and θ_2 values are the half angle between angular divisions. For this problem, θ_1 and θ_2 would be $\pi/4$ and $3\pi/4$ respectively. The vertical forces that would normally be applied on the circumference of the "circle" and on the x-axis, are picked up at the nearest node along the circumference of the "circle".

The only force for this system is at coordinate 1, with a magnitude of $2tr\sigma$. The force matrix is set up to have six different loading conditions, for various values of σ. Therefore, the force matrix for this system will be:

		LOADING CONDITION					
		1	2	3	4	5	6
	1	2000.0	4000.0	8000.0	12000.0	16000.0	2000.0
	2	0.0	0.0	0.0	0.0	0.0	0.0
FORCE	3	0.0	0.0	0.0	0.0	0.0	0.0
	4	0.0	0.0	0.0	0.0	0.0	0.0
AT	5	0.0	0.0	0.0	0.0	0.0	0.0
	6	0.0	0.0	0.0	0.0	0.0	0.0
COORDINTE	7	0.0	0.0	0.0	0.0	0.0	0.0
	8	0.0	0.0	0.0	0.0	0.0	0.0
NUMBER	9	0.0	0.0	0.0	0.0	0.0	0.0
	10	0.0	0.0	0.0	0.0	0.0	0.0
	11	0.0	0.0	0.0	0.0	0.0	0.0
	12	0.0	0.0	0.0	0.0	0.0	0.0
	13	0.0	0.0	0.0	0.0	0.0	0.0
	14	0.0	0.0	0.0	0.0	0.0	0.0
	15	0.0	0.0	0.0	0.0	0.0	0.0

When the "Topology Matrix" and the force matrix have been completed the program is ready to run. All manual inputs will be prompted by the computer. The phrases the computer uses to prompt the user corresponding to the inputs are as follows:

INPUT THE DISTANCE FROM THE CENTER OF THE HOLE TO THE EDGE OF THE PLATE.
4.0 (CR)

INPUT THE DISTANCE FROM THE CENTER OF THE HOLE TO THE MIDDLE OF THE PLATE.
4.0 (CR)

INPUT THE DIAMETER OF THE HOLE.
0.5 (CR)

INPUT THE NUMBER OF ANGULAR DIVISIONS OF THE MESH, UP TO 8. THE LARGER THE NUMBER OF DIVISIONS, THE MORE ACCURATE THE ANSWER WILL BE.
2 (CR)

Each point is now being printed out.

```
INPUT THE VALUES FOR POISSON RATIO, YOUNG'S MODULS AND THE THICKNESS
OF THE PLATE.
    0.3            (CR)
29000000.0         (CR)
   0.125           (CR)
```

The stiffness matrix for each element is now being printed out. The "THE STIFFNESS MATRIX FOR ELEMENT K IS", is printed above each element stiffness matrix for the convenience of the user.

```
INPUT THE NUMBER OF KNOWN FORCES (THE NUMBER OF SYSTEM COORDINATES
NECESSARY TO INVERT).
     15          (CR)

INPUT AN EPSILON VALUE APROXIMATELY SIX ORDERS OF MAGNITUDE LESS THAN
THE STIFFNESS MATRIX.
  .000001        (CR)
```

The unknown forces for each system coordinate is then listed. The approximate stress adjacent to the hole σ_m, can be computed by dividing the force at the adjacent nodes (in this case 25 or 26) by the thickness of the plate t, and half the distance to the nearest node along the x-axis. The stress concentration factor is equal to

$$\text{STRESS CONCENTRATION FACTOR} = \frac{\sigma_m}{\sigma}$$

where σ is the original stress used to compute the original forces.

INTERPRETATION OF DATA

The forces along the x-axis for the example mentioned above do not agree with estimated theoretical values. The reason for this variation is not yet apparent. The problem appears to be in the formation of the "Topology Matrix", or an undetected error in the STFMX has not yet been determined. The main program and the rest of the subroutines appear to work. The output for each nodal point and each stiffness matrix has been checked by hand calculations. Subroutines INVERT and FORCE have been unloaded into another file, the input formats

changed and a small workable matrix was assessed to check the output. The results demonstrate that the remaining subroutines work. With these checks and the use of computer traces, the only problem is the formation of the system stiffness matrix. When the error in STFMX is found and corrected, the program should give reasonable output.

BIBLIOGRAPHY

Baumeister, Theodore, Eugene A. Avallone, and Theodore Baumeister III, Eds. Mark's Standard Handbook for Mechanical Engineers. New York: McGraw-Hill Book Company, 1978.

Bray, Elliott, Computer Center Director at Texas A&M University at Galveston. Interview. Galveston, Texas.

Davis, Ralph, Department Head of Marine Engineering at Texas A&M University at Galveston. Interview. Galveston, Texas.

Nash, James M., Professional Engineer. Interview. Houston, Texas.

Przemieniecki, J. S. Theory of Matrix Stuctured Anlaysis. New York: McGraw-Hill Book Company, 1968.

Singer, Ferdinand L. Strength of Materials. New York: Harper & Row, Publishers, 1962.

Timoshenko, S. P. Strength of Materials - Part II: Advanced Theory and Problems. New York: D. Van Norstrand Company,Inc., 1957.

Zienhiewicz, O. C. The Finite Element Method in Stuctural and Continuum Mechanics. London: McGraw-Hill Publishing Company Limited, 1967.

```
C .....................STRESS COMMON BLOCK.............................C
C                                                                       C
      DIMENSION  POINT(16,20,2),ELSTF(200,6,6),STF(200,210),A(200,210),
     * P(30,6),RX(30)
      COMMON //  POINT,ELSTF,SPL,SPR,D,ANGN,
     * R,ANG,HANG,XSPACE,AR,STF,A,P,RX
      DOUBLE PRECISION POINT,ELSTF,SPL,SPR,D,ANGN,RX,SIGNI,SIGNJ,
     * R,ANG,HANG,XSPACE,AR,STF,A,EPSILN,SIMEQN,AKL,AIL,HOLD,PX,P
C                                                                       C
C .........................END COMMON.................................C
      WRITE(1,10)
10    FORMAT(//' THIS PROGRAM IS DESIGNED TO COMPUTE THE STRESS'/
     * ' CONCENTRATION FACTORS IN FLAT PLATES WITH 1 OR 2 HOLES USING'/
     * ' FINITE ELEMENT ANALYSIS BY J. S. PRZEMIENIECKI '//)
      WRITE(1,20)
20    FORMAT(//' INPUT THE DISTANCE FROM THE CENTER OF THE HOLE TO THE'/
     *' EDGE OF THE PLATE.  '//)
      READ (1,*) SPR
      WRITE(1,30)
30    FORMAT(//' INPUT THE DISTANCE FROM THE CENTER OF THE HOLE TO THE'/
     * ' MIDDLE OF THE PLATE. '//)
      READ(1,*) SPL
      WRITE(1,40)
40    FORMAT(//' INPUT THE DIAMETER OF THE HOLE. '//)
      READ(1,*) D
      WRITE(1,50)
50    FORMAT(//' INPUT THE NUMBER OF ANGULAR DIVISIONS OF THE MESH'/
     * ' UP TO 8.  THE LARGER THE NUMBER OF DIVISIONS THE'/
     * ' MORE ACCURATE THE ANSWER WILL BE.'//)
      READ(1,*) NUMANG
      ANGN = 0.0
      R = D/2.
      ANG = 3.141592653589793/NUMANG
      HANG = ANG/2
      INMANG = (NUMANG/2) + 1
      DO 80 I=1,INMANG
      J = 1
      POINT(I,J,1) = R*(DCOS (ANGN))
      POINT(I,J,2) = R*(DSIN (ANGN))
60    CONTINUE
      XSPACE = 2.*R*(DSIN(HANG))
      R = R+XSPACE
      J = J+1
      POINT(I,J,1) = R*(DCOS (ANGN))
      POINT(I,J,2) = R*(DSIN (ANGN))
      IF(R.GT.SPR)GO TO 70
      GO TO 60
70    POINT(I,J,1) = SPR*(DCOS (ANGN))
      POINT(I,J,2) = SPR*(DSIN (ANGN))
      R = D/2
      JMAX1 = J
      ANGN = ANGN+ANG
80    CONTINUE
      INANGP = INMANG+1
      NMANGP = NUMANG+1
```

```
      ANGN = 1.570796326794896+ANG
      DO 110 I=INANGP,NMANGP
      R = D/2
      J=1
      POINT(I,J,1) = R*(DCOS (ANGN))
      POINT(I,J,2) = R*(DSIN (ANGN))
90    CONTINUE
      XSPACE = 2*R*(DSIN (HANG))
      R = R+XSPACE
      J = J+1
      POINT(I,J,1) = R*(DCOS (ANGN))
      POINT(I,J,2) = R*(DSIN (ANGN))
      AR = DABS(R)
      IF(AR.GT.SPL)GO TO 100
      GO TO 90
100   POINT(I,J,1) = SPL*(DCOS (ANGN))
      POINT(I,J,2) = SPL*(DSIN (ANGN))
      ANGN = ANGN+ANG
      JMAX2 = J
110   CONTINUE
      WRITE(1,120)JMAX
120   FORMAT(//'JMAX IS = ',I4)
      DO 140 I=1,INMANG
      DO 140 J=1,JMAX1
      DO 140 K=1,2
      WRITE(1,130)I,J,K,POINT(I,J,K)
130   FORMAT(//' POINT (',I2,',',I3,',',I3,' ) IS ',D14.7)
140   CONTINUE
      DO 160 I=INANGP,NMANGP
      DO 160 J=1,JMAX2
      DO 160 K=1,2
      WRITE(1,150)I,J,K,POINT(I,J,K)
150   FORMAT(//' POINT (',I2,',',I3,',',I3,') IS ', D14.7)
160   CONTINUE
      CALL TRIG(JMAX1,JMAX2,INMANG,INANGP,NMANGP)
      NEL = (JMAX1-1)*NUMANG + (JMAX2-1)*NUMANG
      DO 200 L = 1,NEL
      WRITE(1,170)L
170   FORMAT(//' THE STIFFNESS MATRIX FOR ELEMENT ',I4,'   IS,'////)
      DO 190 IJ=1,6
      WRITE(1,180)( ELSTF(L,IL,IJ),IL = 1,6)
180   FORMAT(3X,D14.7,3X,D14.7,3X,D14.7,3X,D14.7,3X,D14.7,3X,D14.7,//)
190   CONTINUE
200   CONTINUE
      NSC = (2*JMAX2*(INMANG-1))+(2*JMAX1*INMANG)
      CALL STFMX(NSC,NEL)
      DO 220 N=1,NSC,6
      NP = N + 6
      WRITE(1,210)((STF(I,J),J=N,NP),I=1,NSC)
210   FORMAT(D14.7,3X,D14.7,3X,D14.7,3X,D14.7,3X,D14.7,3X,D14.7,//)
220   CONTINUE
      CALL INVERT(N,NPNV,NVECS,NPLUS1,SIMEQN)
      CALL FORCE(N,NPLUS1,NPNV,NSC)
      WRITE(1,230)
230   FORMAT(//////' THE UNKNOWN FORCES ARE AS FOLLOWS   '///)
      DO 250 IROW=NPLUS1,NSC
      WRITE(1,240)IROW,(P(IROW,ICOL),ICOL= NPLUS1,NPNV)
240   FORMAT('THE FORCE AT ',I4,'  IS  ',6(D11.4,2X)//)
```

```
250   CONTINUE
      CALL EXIT
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                     C
C               THIS SUBROUTINE CORRELATES THE INDIVIDUAL             C
C               POINTS INTO EACH ELEMENTAL TRIANGLE.                  C
C                                                                     C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                     C
      SUBROUTINE TRIG (JMAX1,JMAX2,INMANG,INANGP,NMANGP)
C                                                                     C
C .......................STRESS COMMON BLOCK.........................C
C                                                                     C
      DIMENSION  POINT(16,20,2),ELSTF(200,6,6),STF(200,210),A(200,210),
     * P(30,6),RX(30)
      COMMON //  POINT,ELSTF,SPL,SPR,D,ANGN,
     * R,ANG,HANG,XSPACE,AR,STF,A,P,RX
      DOUBLE PRECISION POINT,ELSTF,SPL,SPR,D,ANGN,RX,SIGNI,SIGNJ,
     * R,ANG,HANG,XSPACE,AR,STF,A,EPSILN,SIMEQN,AKL,AIL,HOLD,PX,P
C                                                                     C
C ...........................END COMMON.................................C
      DOUBLE PRECISION X1,X2,X3,Y1,Y2,Y3,
     * POIS,E,T
      WRITE(1,260)
260   FORMAT('INPUT THE VALUES FOR POISONS, ELASTICITY, AND THE '
     *'THICKNESS OF THE PLATE.')
      READ(1,*)POIS,E,T
      INANGM = INMANG-1
      JM1 = JMAX1-1
      JM2 = JMAX2 - 1
      DO 270 J=1,INANGM
      DO 270 I=1,JM1
      JP = J+1
      IP = I+1
      X1 = POINT(JP,I,1)
      Y1 = POINT(JP,I,2)
      X2 = POINT(J,IP,1)
      Y2 = POINT(J,IP,2)
      X3 = POINT(J,I,1)
      Y3 = POINT(J,I,2)
      K = JM1*((J-1)*2)+((2*I)-1)
      CALL KAPA(X1,X2,X3,Y1,Y2,Y3,E,T,POIS,K)
270   CONTINUE
      DO 280 J=1,INANGM
      DO 280 I=1,JM1
      IP = I+1
      JP = J+1
      X1 = POINT(JP,I,1)
      Y1 = POINT(JP,I,2)
      X2 = POINT(JP,IP,1)
      Y2 = POINT(JP,IP,2)
      X3 = POINT(J,IP,1)
      Y3 = POINT(J,IP,2)
      K = (JM1*((J-1)*2))+(2*I)
      CALL KAPA(X1,X2,X3,Y1,Y2,Y3,E,T,POIS,K)
280   CONTINUE
      DO 290 J=1,INANGM
```

```
      DO 290 I=1,JM2
      IP = I+1
      JA = J+INANGM
      JP = J+INMANG
      X1 = POINT(JP,I,1)
      Y1 = POINT(JP,I,2)
      X2 = POINT(JA,IP,1)
      Y2 = POINT(JA,IP,2)
      X3 = POINT(JA,I,1)
      Y3 = POINT(JA,I,2)
      K = (INANGM*JM1*2)+(JM2*((J-1)*2))+((2*I)-1)
      CALL KAPA(X1,X2,X3,Y1,Y2,Y3,E,T,POIS,K)
290   CONTINUE
      DO 300 J=1,INANGM
      DO 300 I=1,JM2
      IP = I+1
      JA = J+INANGM
      JP = J+INMANG
      X1 = POINT(JP,I,1)
      Y1 = POINT(JP,I,2)
      X2 = POINT(JP,IP,1)
      Y2 = POINT(JP,IP,2)
      X3 = POINT(JA,IP,1)
      Y3 = POINT(JA,IP,2)
      K = (INANGM*JM1*2) + (JM2*((J-1)*2))+(2*I)
      CALL KAPA(X1,X2,X3,Y1,Y2,Y3,E,T,POIS,K)
300   CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                           C
C               THIS SUBROUTINE CALCULATES THE STIFFNESS                    C
C               MATRIX FOR EACH INDIVIDUAL ELEMENT.                         C
C                                                                           C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                           C
      SUBROUTINE KAPA(X1,X2,X3,Y1,Y2,Y3,E,T,POIS,K)
C                                                                           C
C .....................STRESS COMMON BLOCK.....................C
C                                                                           C
      DIMENSION  POINT(16,20,2),ELSTF(200,6,6),STF(200,210),A(200,210),
     * P(30,6),RX(30)
      COMMON //  POINT,ELSTF,SPL,SPR,D,ANGN,
     * R,ANG,HANG,XSPACE,AR,STF,A,P,RX
      DOUBLE PRECISION POINT,ELSTF,SPL,SPR,D,ANGN,RX,SIGNI,SIGNJ,
     * R,ANG,HANG,XSPACE,AR,STF,A,EPSILN,SIMEQN,AKL,AIL,HOLD,PX,P

C .........................END COMMON.............................C
      DOUBLE PRECISION X21,X31,X32,Y21,Y31,Y32,
     * A123,XM1,XM2,POIS,X1,X2,X3,Y1,Y2,Y3,E,T
      X21 = (X2-X1)
      X32 = (X3-X2)
      X31 = (X3-X1)
      Y21 = (Y2-Y1)
      Y32 = (Y3-Y2)
      Y31 = (Y3-Y1)
      A123 = ((X32*Y21)-(X21*Y32))/2
      XM1 = (E*T)/((4.*A123)*(1-(POIS**2)))
```

```
      XM2 = (E*T)/((8.*A123)*(1+POIS))
      ELSTF(K,1,1) = (XM1*(Y32**2))+(XM2*(X32**2))
      ELSTF(K,2,1) = (XM1*(-POIS*Y32*X32))+(XM2*(-X32*Y32))
      ELSTF(K,1,2) = ELSTF(K,2,1)
      ELSTF(K,2,2) = (XM1*(X32**2))+(XM2*(Y32**2))
      ELSTF(K,3,1) = (XM1*(-Y32*Y31))+(XM2*(-X32*X31))
      ELSTF(K,1,3) = ELSTF(K,3,1)
      ELSTF(K,3,2) = (XM1*(POIS*X32*Y31))+(XM2*(Y32*X31))
      ELSTF(K,2,3) = ELSTF(K,3,2)
      ELSTF(K,3,3) = (XM1*(Y31**2))+(XM2*(X31**2))
      ELSTF(K,4,1) = (XM1*(POIS*Y32*X31))+(XM2*(X32*Y31))
      ELSTF(K,1,4) = ELSTF(K,4,1)
      ELSTF(K,4,2) = (XM1*(-X32*X31))+(XM2*(-Y32*Y31))
      ELSTF(K,2,4) = ELSTF(K,4,2)
      ELSTF(K,4,3) = (XM1*(-POIS*Y31*X31))+(XM2*(-X31*Y31))
      ELSTF(K,3,4) = ELSTF(K,4,3)
      ELSTF(K,4,4) = (XM1*(X31**2))+(XM2*(Y31**2))
      ELSTF(K,5,1) = (XM1*(Y32*Y21))+(XM2*(X32*X21))
      ELSTF(K,1,5) = ELSTF(K,5,1)
      ELSTF(K,5,2) = (XM1*(-POIS*X32*Y21))+(XM2*(-Y32*X21))
      ELSTF(K,2,5) = ELSTF(K,5,2)
      ELSTF(K,5,3) = (XM1*(-Y31*Y21))+(XM2*(-X31*X21))
      ELSTF(K,3,5) = ELSTF(K,5,3)
      ELSTF(K,5,4) = (XM1*(POIS*X31*Y21))+(XM2*(Y31*X21))
      ELSTF(K,4,5) = ELSTF(K,5,4)
      ELSTF(K,5,5) = (XM1*(Y21**2))+(XM2*(X21**2))
      ELSTF(K,6,1) = (XM1*(-POIS*Y32*X21))+(XM2*(-X32*Y21))
      ELSTF(K,1,6) = ELSTF(K,6,1)
      ELSTF(K,6,2) = (XM1*(X32*X21))+(XM2*(Y32*Y21))
      ELSTF(K,2,6) = ELSTF(K,6,2)
      ELSTF(K,6,3) = (XM1*(POIS*Y31*X21))+(XM2*(X31*Y21))
      ELSTF(K,3,6) = ELSTF(K,6,3)
      ELSTF(K,6,4) = (XM1*(-X31*X21))+(XM2*(-Y31*Y21))
      ELSTF(K,4,6) = ELSTF(K,6,4)
      ELSTF(K,6,5) = (XM1*(-POIS*Y21*X21))+(XM2*(-X21*Y21))
      ELSTF(K,5,6) = ELSTF(K,6,5)
      ELSTF(K,6,6) = (XM1*(X21**2))+(XM2*(Y21**2))
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                      C
C               THIS SUBROUTINE COMBINES EACH ELEMENT STIFFNESS        C
C               MATRIX INTO THE SYSTEM STIFFNESS MATRIX.               C
C                                                                      C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                      C
      SUBROUTINE STFMX(NSC,NMBR)
C                                                                      C
C ......................STRESS COMMON BLOCK.........................C
C                                                                      C
      DIMENSION  POINT(16,20,2),ELSTF(200,6,6),STF(200,210),A(200,210),
     * P(30,6),RX(30)
      COMMON //  POINT,ELSTF,SPL,SPR,D,ANGN,
     * R,ANG,HANG,XSPACE,AR,STF,A,P,RX
      DOUBLE PRECISION POINT,ELSTF,SPL,SPR,D,ANGN,RX,SIGNI,SIGNJ,
     * R,ANG,HANG,XSPACE,AR,STF,A,EPSILN,SIMEQN,AKL,AIL,HOLD,PX,P
C                                                                      C
C ......................END COMMON..................................C
```

```
      DIMENSION ITOPO(200,6)
      NELC = 6
      READ(5,*)((ITOPO(I,J),J=1,6),I=1,NMBR)
      DO 310 ISC = 1,NSC
      DO 310 JSC = 1,NSC
      STF(ISC,JSC) = 0.0
310   CONTINUE
C     CONSTRUCT STF
320   MBR = 0
330   MBR = MBR+1
      IF(MBR-NMBR)340,340,410
340   L = 0
350   L = L+1
      IF(L-NELC)360,360,330
360   INDEX = ITOPO(MBR,L)
      IF (INDEX)370,350,370
370   ISC = IABS(INDEX)
      SIGNI = INDEX/ISC
      STF(ISC,ISC) = STF(ISC,ISC)+ELSTF(MBR,L,L)
      M = L
380   M = M+1
      IF(M-NELC) 390,390,350
390   JNDEX = ITOPO(MBR,M)
      IF(JNDEX) 400,380,400
400   JSC = IABS(JNDEX)
      SIGNJ = JNDEX/JSC
      STF(ISC,JSC) = STF(ISC,JSC)+SIGNI*SIGNJ*ELSTF(MBR,L,M)
      STF(JSC,ISC) = STF(ISC,JSC)
      GO TO 380
410   CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                           C
C          THIS SUBROUTINE INVERTS THE PORTION OF THE SYSTEM                C
C          STIFFNESS MATRIX NECESSARY TO FIND THE DEFORMATIONS.             C
C                                                                           C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                           C
      SUBROUTINE INVERT(N,NPNV,NVECS,NPLUS1,SIMEQN)
C                                                                           C
C .....................STRESS COMMON BLOCK.....................C
C                                                                           C
      DIMENSION  POINT(16,20,2),ELSTF(200,6,6),STF(200,210),A(200,210),
     * P(30,6),RX(30)
      COMMON //  POINT,ELSTF,SPL,SPR,D,ANGN,
     * R,ANG,HANG,XSPACE,AR,STF,A,P,RX
      DOUBLE PRECISION POINT,ELSTF,SPL,SPR,D,ANGN,RX,SIGNI,SIGNJ,
     * R,ANG,HANG,XSPACE,AR,STF,A,EPSILN,SIMEQN,AKL,AIL,HOLD,PX,P
C                                                                           C
C .........................END COMMON.............................C
      DIMENSION JC(200),IR(200)
      WRITE(1,420)
420   FORMAT(' INPUT THE NUMBER OF KNOWN FORCES (THE NUMBER '/
     *' OF SYSTEM COORDINATES NECESSARY TO INVERT)')
      READ(1,*)N
      DO 430 I=1,N
      DO 430 J=1,N
```

```
      A(I,J) = STF(I,J)
430   CONTINUE
      NVECS = 6
      NPNV = N+NVECS
      NPLUS1 = N+1
      READ(6,*)((A(I,J),J=NPLUS1,NPNV),I=1,N)
      WRITE(1,440)
440   FORMAT(/' INPUT AN EPSILON VALUE APPROXIMATELY SIX ORDERS '/
     * ' OF MAGNITUDE LESS THAN THE STIFFNESS MATRIX '//)
      READ(1,*)EPSILN
      NMINS1 = N-1
      DO 450 L=1,N
      JC(L)=0
450   CONTINUE
      DO 550 L=1,N
      DO 480 K=1,N
      IF (ABS(A(K,L))-EPSILN) 480,460,460
460   DO 470 KROW=1,L
      IF(JC(KROW)-K) 470,480,470
470   CONTINUE
      GO TO 500
480   CONTINUE
      WRITE(1,490)L,EPSILN
490   FORMAT(' MX SINGULAR AT PASS ',I3,'W.R.T. ',E12.5)
      SIMEQN = 0
      RETURN
500   JC(L) = K
      AKL = 1.0/A(K,L)
      DO 510 J = 1,NPNV
510   A(K,J) = A(K,J)*AKL
      A(K,L) = AKL
      DO 540 I=1,N
      IF (I-K) 520,540,520
520   AIL = A(I,L)
      DO 530 J=1,NPNV
      A(I,J) = A(I,J)-AIL*A(K,J)
530   CONTINUE
      A(I,L) = -AIL*AKL
540   CONTINUE
550   CONTINUE
      DO 560 I=1,N
      IRC = JC(I)
      IR(IRC) = I
560   CONTINUE
      DO 610 I=1,NMINS1
      IF (IR(I)-I) 570,610,570
570   IP1 = I+1
      DO 580 IRS=IP1,N
      IF (IR(IRS)-I) 580,590,580
580   CONTINUE
590   DO 600 J=1,NPNV
      HOLD = A(IRS,J)
      A(IRS,J) = A(I,J)
      A(I,J) = HOLD
600   CONTINUE
      IR(IRS) = IR(I)
      IR(I) = I
610   CONTINUE
```

```
      DO 660 J=1, NMINS1
      IF (JC(J)-J) 620,660,620
620   JP1 = J+1
      DO 630 JCS=JP1,N
      IF(JC(JCS)-J) 630,640,630
630   CONTINUE
640   DO 650 I=1,N
      HOLD = A(I,JCS)
      A(I,JCS) = A(I,J)
650   A(I,J) = HOLD
      JC(JCS) = JC(J)
660   CONTINUE
      RETURN
      END
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                           C
C              THIS SUBROUTINE MULTIPLIES THE PORTION OF THE                C
C              SYSTEM STIFFNESS MATRIX THAT WAS NOT INVERTED                C
C              BY THE SYSTEM DEFORMATINS TO OBTAIN THE UN-                  C
C              KNOWN SYSTEM FORCES.                                         C
C                                                                           C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C                                                                           C
      SUBROUTINE FORCE(N,NPLUS1,NPNV,NSC)
C                                                                           C
C .......................STRESS COMMON BLOCK.............................C
C                                                                           C

      DIMENSION  POINT(16,20,2),ELSTF(200,6,6),STF(200,210),A(200,210),
     * P(30,6),RX(30)
      COMMON //  POINT,ELSTF,SPL,SPR,D,ANGN,
     * R,ANG,HANG,XSPACE,AR,STF,A,P,RX
      DOUBLE PRECISION POINT,ELSTF,SPL,SPR,D,ANGN,RX,SIGNI,SIGNJ,
     * R,ANG,HANG,XSPACE,AR,STF,A,EPSILN,SIMEON,AKL,AIL,HOLD,PX,P
C                                                                           C
C .........................END COMMON.................................C
      DO 680 ICOL=NPLUS1,NPNV
      DO 680 IROW=NPLUS1,NSC
      PX = 0.0D0
      DO 670 MULT=1,N
      PX = PX + STF(IROW,MULT)*A(MULT,ICOL)
670   CONTINUE
      P(IROW,ICOL) = PX
680   CONTINUE
      RETURN
      END
```