A Distributed Converging Overland Flow Model
3. Application to Natural Watersheds

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The proposed distributed converging overland flow model is utilized to predict surface runoff from three natural agricultural watersheds. The Lax-Wendroff scheme is used to obtain numerical solutions. For determination of the kinematic wave friction relationship parameter a simple relation between the parameter and topographic slope is hypothesized. The simple relation contains two constants which are optimized for each watershed by the Rosenbrock-Palmer optimization algorithm. The model results are in good agreement with runoff observations from these watersheds. It is shown that if the model structure is sound, it will suffice to optimize model parameters on hydrograph peak only even for prediction of the entire hydrograph. The model results suggest that a distributed approach to kinematic wave modeling of watershed surface runoff is potentially promising and warrants further investigation.

INTRODUCTION

In part 1 of this series we developed mathematical solutions for nonlinear watershed runoff dynamics. In part 2 we described the effect of infiltration on the runoff process. In the present paper, the concluding part of the series, we apply the proposed model to natural agricultural watersheds and examine its predictive performance in relation to surface runoff observations from these watersheds.

In part 1 we noted that analytical solutions are not feasible for time-varying (complex) rainfall input. The most practical method is to utilize hybrid solutions. For a complete discussion on hybrid solutions, see the work by *Singh* [1974, 1975*a*]. We will only give numerical solutions here. The coupling of the continuity equation and the kinematic approximation to the momentum equation [*Singh*, 1974] yields

$$\frac{\partial h}{\partial t} + n\alpha(x)h^{n-1}\frac{\partial h}{\partial x} + h^n\frac{\partial \alpha(x)}{\partial x} = q(x, t) + \frac{\alpha(x)h^n}{(L - x)} \qquad (1)$$

where h is the depth of flow; L is the length of the converging section; $q(x, t)$ is the rate of effective lateral inflow per unit area, varying in time and space; x is a space coordinate; t is a time coordinate; n is an exponent fixed at 1.5 [*Singh*, 1975*b*]; and $\alpha(x)$ is the kinematic wave friction relationship parameter, varying in space. The Lax-Wendroff scheme [*Houghton and Kasahara*, 1968], which has been successfully used in many investigations on kinematic wave modeling of watershed runoff [*Kibler and Woolhiser*, 1970; *Singh*, 1974, 1975*a*], is formulated to solve (1).

We can write

$$\frac{\partial h}{\partial t} = -\alpha(x)nh^{n-1}\frac{\partial h}{\partial x} - \frac{\partial \alpha(x)}{\partial x} + q(x, t) + \frac{\alpha(x)h^n}{(L - x)} \qquad (2)$$

Expanding $h(x, t + \Delta t)$ by Taylor series, we get

$$h(x, t + \Delta t) = h(x, t) + \Delta t\frac{\partial h}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 h}{\partial t^2} + H_{OT} \qquad (3)$$

where H_{OT} denotes higher-order terms. Differentiating (2) with respect to t, we get

$$\frac{\partial^2 h}{\partial t^2} = -\alpha(x)\frac{\partial}{\partial x}\left(nh^{n-1}\frac{\partial h}{\partial t}\right) - \frac{\partial \alpha(x)}{\partial x}nh^{n-1}\frac{\partial h}{\partial t} + \frac{\partial q(x, t)}{\partial t} + \frac{\alpha(x)nh^{n-1}}{(L - x)}\frac{\partial h}{\partial t} \qquad (4)$$

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Inserting (2) and (4) into (3) and neglecting H_{OT}, we obtain

$$\begin{aligned} h(x, t + \Delta t) &= h(x, t) + \Delta t\left[-n\alpha(x)h^{n-1}\frac{\partial h}{\partial x}\right. \\ &\left. - h^n\frac{\partial \alpha(x)}{\partial x} + q(x, t) + \frac{\alpha(x)h^n}{(L - x)}\right] + \frac{(\Delta t)^2}{2}\left[-\alpha(x)\frac{\partial}{\partial x}\right. \\ &\cdot\left(nh^{n-1}\frac{\partial h}{\partial t}\right) - \frac{\partial \alpha(x)}{\partial x}nh^{n-1}\frac{\partial h}{\partial t} \\ &\left. + \frac{\partial q(x, t)}{\partial t} + \frac{\alpha(x)nh^{n-1}}{(L - x)}\frac{\partial h}{\partial t}\right] \end{aligned} \qquad (5)$$

Writing (5) in a compact form, we get

$$\begin{aligned} h(x, t + \Delta t) &= h(x, t) + \left[-\alpha(x)\frac{\partial h^n}{\partial x} - h^n\frac{\partial \alpha(x)}{\partial x}\right. \\ &\left. + q(x, t) + \frac{\alpha(x)h^n}{(L - x)}\right]\left\{\Delta t + (\Delta t)^2\frac{n}{2}h^{n-1}\right. \\ &\left.\cdot\left[-\frac{\partial \alpha(x)}{\partial x} + \frac{\alpha(x)}{(L - x)}\right]\right\} + \frac{(\Delta t)^2}{2} \\ &\cdot\left\{\frac{\partial q(x, t)}{\partial t} - \alpha(x)\frac{\partial}{\partial x}\left[nh^{n-1}\left(-\alpha(x)\frac{\partial h^n}{\partial x}\right.\right.\right. \\ &\left.\left.\left. - h^n\frac{\partial \alpha(x)}{\partial x} + q(x, t) + \frac{\alpha(x)h^n}{(L - x)}\right)\right]\right\} \end{aligned} \qquad (6)$$

Following the notation in Figure 1, we can write (6) in finite difference form as

$$\begin{aligned} h_j^{i+1} &= h_j^i + \left[-\alpha_j^i\left(\frac{h_{j+1}^{in} - h_{j-1}^{in}}{2\Delta x}\right)\right. \\ &\left. - h_j^{in}\left(\frac{\alpha_{j+1}^i - \alpha_{j-1}^i}{2\Delta x}\right) + q_j^i + \frac{\alpha_j^i h_j^{in}}{(L - x_j^i)}\right] \\ &\cdot\left\{\Delta t + \frac{(\Delta t)^2}{2}nh_j^{i\,n-1}\left[-\frac{\alpha_{j+1}^i - \alpha_{j-1}^i}{2\Delta x}\right.\right. \\ &\left.\left. + \frac{\alpha_j^i}{(L - x_j^i)}\right]\right\} + \frac{(\Delta t)^2}{2}\left\{\frac{q_j^{i+1} - q_j^i}{\Delta t}\right. \\ &- \left[\alpha_j^i\frac{n}{\Delta x}\left(\frac{h_{j+1}^{i(n-1)} + h_j^{i(n-1)}}{2}\right)\left(-\left(\frac{\alpha_{j+1}^i + \alpha_j^i}{2}\right)\right.\right. \end{aligned}$$

$$
\cdot\left(\frac{h_{j+1}^{i\,n} - h_j^{i\,n}}{\Delta x}\right) - \left(\frac{h_{j+1}^{i\,n} + h_j^{i\,n}}{2}\right)\left(\frac{\alpha_{j+1}^{i} - \alpha_j^{i}}{\Delta x}\right)
+ \left(\frac{q_{j+1}^{i} + q_j^{i}}{2}\right) + \left(\frac{\alpha_{j+1}^{i} + \alpha_j^{i}}{2}\right)\left(\frac{h_{j+1}^{i\,n} + h_j^{i\,n}}{2}\right)
\cdot \frac{2}{[2L - (x_{j+1}^{i} + x_j^{i})]}\Bigg) - \alpha_j^{i}\frac{n}{\Delta x}
\cdot\left(\frac{h_j^{i\,(n-1)} + h_{j-1}^{i\,(n-1)}}{2}\right)\left(-\left(\frac{\alpha_j^{i} + \alpha_{j-1}^{i}}{2}\right)\right.
\cdot\left(\frac{h_j^{i\,n} - h_{j-1}^{i\,n}}{\Delta x}\right) - \left(\frac{h_j^{i\,n} + h_{j-1}^{i\,n}}{2}\right)\left(\frac{\alpha_j^{i} - \alpha_{j-1}^{i}}{\Delta x}\right)
+ \left(\frac{q_j^{i} + q_{j-1}^{i}}{2}\right) + \left(\frac{\alpha_j^{i} + \alpha_{j-1}^{i}}{2}\right)
\cdot\left(\frac{h_j^{i\,n} + h_{j-1}^{i\,n}}{2}\right)\left.\frac{2}{[2L - (x_j^{i} + x_{j-1}^{i})]}\right)\Bigg]\Bigg\} \quad (7)
$$

Assume that the depth of flow is to be determined at N nodal points. Then the depth of flow at nodal points $j = 1, 2, \cdots, (N - 1)$ will be computed by the scheme in (7) in conjunction with the following boundary conditions:

$$
h(0, t) = 0 \qquad h(x, t) = 0 \quad (8)
$$

Equation (8) represents an initially dry surface. The finite difference scheme of (7) is explicit, second order, and single step. The depth of flow at the downstream boundary ($j = N$) can be computed by the first-order scheme. That is,

$$
h(x, t + \Delta t) = h(x, t) + \Delta t\ \partial t/\partial t \quad (9)
$$

Substituting (2) into (9), we get

$$
h(x, t + \Delta t) = h(x, t) + \Delta t\left[-\alpha(x)\frac{\partial h^n}{\partial x} - h^n\frac{\partial \alpha(x)}{\partial x} + q(x, t) + \frac{\alpha(x)h^n}{(L - x)}\right] \quad (10)
$$

Writing the difference form of (10), we get

$$
h_N^{i+1} = h_N^{i} + \Delta t\left[-\alpha_N^{i}\left(\frac{h_N^{i\,n} - h_{N-1}^{i\,n}}{\Delta x}\right) - h_N^{i\,n}\left(\frac{\alpha_N^{i} - \alpha_{N-1}^{i}}{\Delta x}\right) + q_N^{i} + \frac{\alpha_N^{i} h_N^{i\,n}}{(L - x_N^{i})}\right] \quad (11)
$$

These numerical schemes can be combined with analytical solutions in an appropriate manner to yield hybrid solutions [*Singh*, 1974, 1975*a*]. It must be pointed out that the Lax-Wendroff scheme is one of the most popular numerical schemes for solving partial differential equations of hyperbolic type. Because of its explicit nature it is only conditionally stable. The criterion for its conditional stability is derived in Appendix A. The observance of this criterion automatically ensures its convergence. For an elaborate discussion on numerical stability and convergence, see *Mitchell* [1969] and *Smith* [1965].

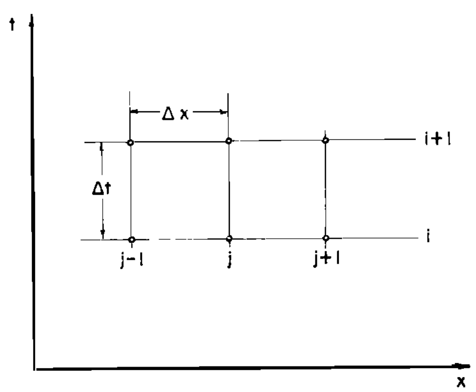

Fig. 1. Notation for finite difference scheme.

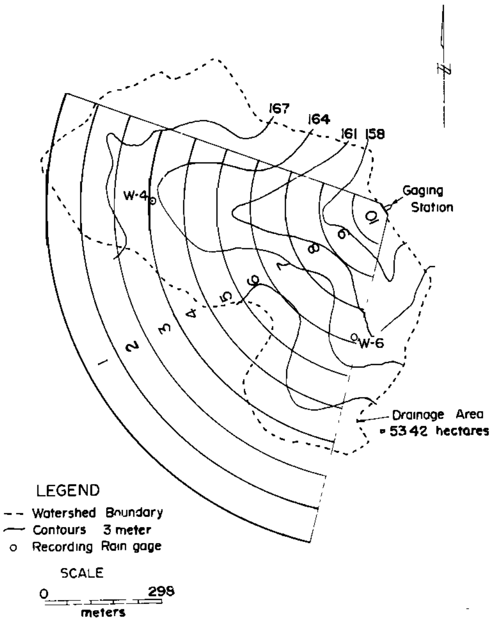

Fig. 2. Watershed W-2, Riesel (Waco), Texas.

APPLICATION TO NATURAL WATERSHEDS

The distributed converging overland flow model was applied to three natural agricultural watersheds near Riesel (Waco), Texas. They include watershed W-2, 53 ha in area, as is shown in Figure 2; watershed W-16, 17 ha in area, as is shown in Figure 3; and watershed G, 1772 ha in area, as is shown in Figure 4. Deep fine-textured granular slowly permeable alkaline throughout and slow internal drainage are typical charac-

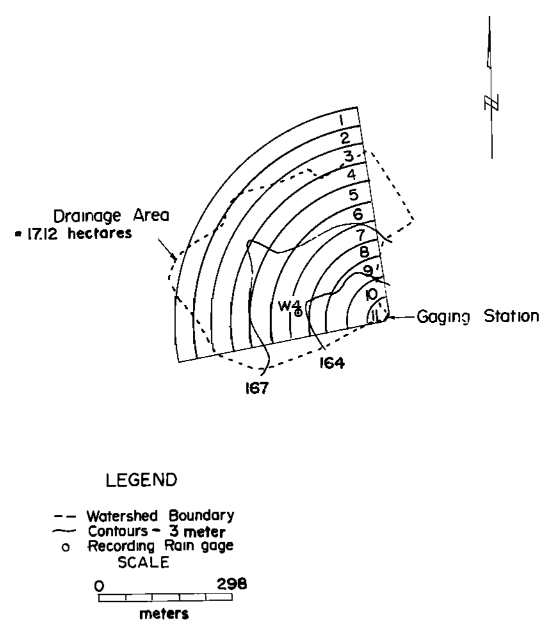

Fig. 3. Watershed W-6, Riesel (Waco), Texas.

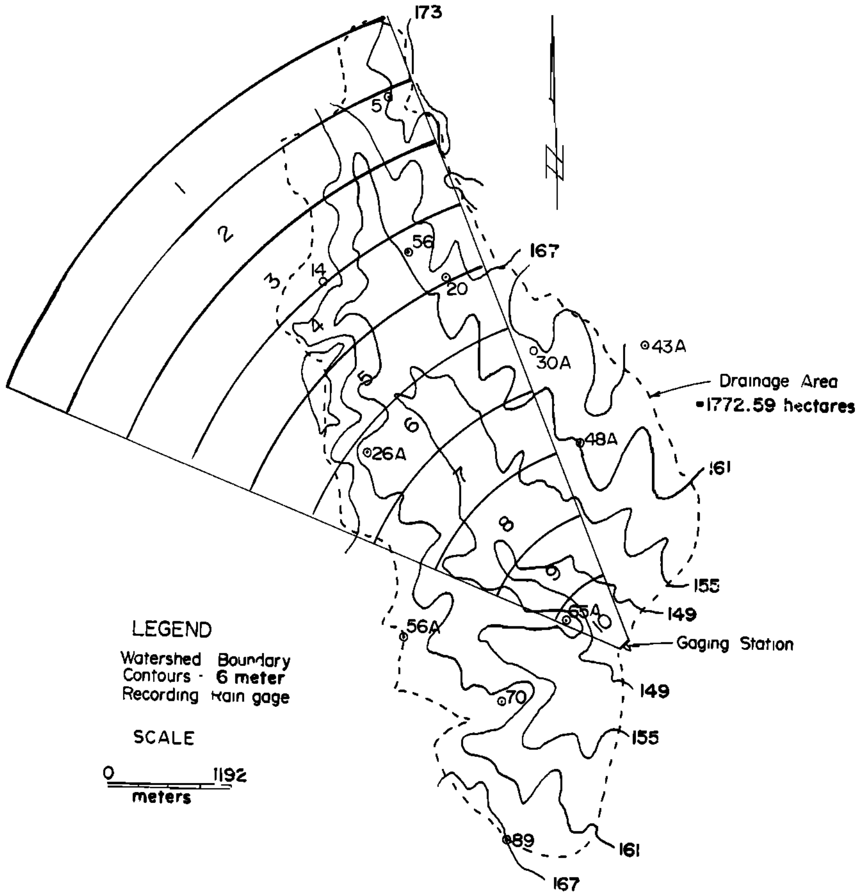

Fig. 4. Watershed G, Riesel (Waco), Texas.

teristics of the soils in these watersheds. The dominance of Houston black clay is notable. These soils are also noted for the formation of large extensive cracks upon drying. Surface drainage is usually good, but no well-defined drainageways exist on these watersheds. Normally, water is drained by rills and poorly defined field gullies.

Most of the time these watersheds are covered with agricultural crops. Because of the low permeability of the soils, these watersheds respond rapidly to rainfall and produce quickly rising hydrographs. For the rainfall events that were considered in this study most of the rainfall was observed as surface runoff, and infiltration was only minor. For a more complete discussion of these watersheds and rainfall-runoff data thereon, see publications of the U.S. Department of Agriculture on hydrologic data of experimental agricultural watersheds in the United States. These publications appear almost every year and contain, on the average, one event per watershed.

Determination of rainfall excess. Rainfall excess forms input to the model. We do recognize that the concept of rainfall excess is more an artifice than a reality. The processes of rainfall, infiltration, and runoff occur concurrently in nature. Simultaneous consideration of these distinct processes injects intractable complexity in runoff modeling. It is therefore not surprising that despite this recognition a great many investigators have utilized this artificial notion of rainfall excess in their investigations on rainfall-runoff modeling and that a great many continue to do so even today; in addition, very little attention has been paid to this fundamental problem.

For simplicity we ourselves adhered to the traditional practice. Infiltration was determined by Philip's equation [*Philip,* 1957], which can be written as

$$f = a + bt^{-0.5} \tag{12}$$

where f is the infiltration loss rate, t is time, and a and b are parameters dependent on soil characteristics and initial soil moisture conditions. These parameters vary from one storm to another for the same watershed and from one watershed to another for the same storm. However, parameter a was considered roughly equivalent to steady infiltration and was thus rendered amenable to determination from physical characteristics of the soil. Parameter b was allowed to vary with each rainfall episode, antecedent soil moisture conditions thus being accounted for. Parameter b was estimated for each storm by Newton's algorithm [*Conte,* 1965], subject to the preservation of the volume continuity of flow. In a recent study [*Singh,* 1974] these parameters were specified for all available events on these watersheds. We utilized these results in the present study.

Geometric representation. The objective is to transform the geometry of a natural watershed into a simpler geometry having a similar hydrologic response. The only perfect representation of a watershed is, of course, the watershed itself. In studies of the response characteristics of the linearly converging section [*Woolhiser,* 1969] it was suggested that such a geometry might be a useful abstraction of a watershed regardless of its complexity. This hypothesis was later incorporated in a study by *Singh* [1974, 1975*a*] and was found promising. Therefore the linearly converging section of a cone, as is shown in Figure 5, was utilized to represent the geometry of a natural watershed. From this figure it is apparent that the converging section geometry has four geometric parameters including $L(1 -$

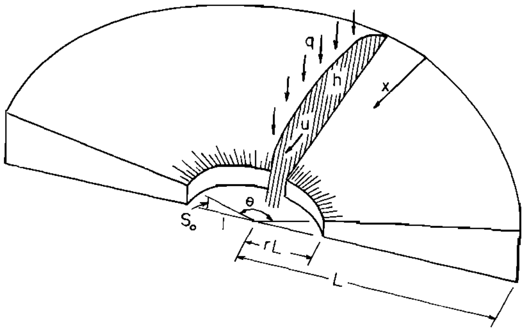

Fig. 5. Geometry of converging overland flow model.

r), r, θ, and S_0, where $L(1 - r)$ is the length of flow, S_0 is the slope, r is the parameter related to the degree of convergence, and θ is the interior angle. Because of the radial symmetry, θ does not affect the relative response characteristics. It is necessary only to preserve the watershed area and is therefore dependent on L and r. The converging section geometry possesses some interesting properties.

1. Its discrete analog is, from a systems viewpoint, a system composed of a cascade of unequal nonlinear reservoirs.

2. Its response is similar to that of a cascade of planes of decreasing size.

3. The convergence may account for the concentration of runoff at the mouth of a natural watershed.

The converging section geometry has three geometric parameters, $L(1 - r)$, r, and θ, that need to be specified. Since the area of a watershed is usually known, only two parameters need to be estimated. The study [*Singh*, 1974] showed that for a watershed under consideration, parameter $L(1 - r)$ could be taken to be equal to the longest horizontal projection from the most remote portion of the watershed to the outlet and that parameter r could be taken to be equal to 0.01. Thus the topographic map of a watershed suffices to transform the natural geometry into a simpler geometry.

Choice of objective function. The following objective function, based on hydrograph peak, was used in the present study:

$$F = \min \sum_{j=1}^{M} [Q_{p_o}(j) - Q_{p_e}(j)]^2 \qquad (13)$$

where F is the objective function or error criterion, $Q_{p_o}(j)$ is the observed hydrograph peak for the jth event, $Q_{p_e}w)j)$ is the estimated hydrograph peak for the jth event, and M is the number of events in the optimization set. The choice of this objective function is based on the findings of *Kibler and Woolhiser* [1970] and *Singh* [1974, 1975*a*, *b*]. Besides its usefulness in flood studies and some statistical properties that it possesses, it has the advantage that it does not suffer from the timing errors that result from improper synchronization of rainfall and runoff observations.

Parameter optimization. A simple relation between parameter α and topographic slope was considered:

$$\alpha(x) = c_1 + c_2[S(x)]^{1/2} \qquad (14)$$

where $S(x)$ is the topographic slope, varying in space and c_1 and c_2 are parameters. These parameters will supposedly vary from one watershed to another. At present we can only hope to obtain them by the technique of optimization. The topograph slope varies in space, and so does parameter α correspondingly. The choice of this relation is based on recent studies conducted by *Singh* [1974, 1975*c*]. From a physical standpoint this relation will be valid if the Darcy-Weisback friction equation is used and the flow Reynolds number is high.

For computational purposes the converging section geometry was decomposed into several segments, for example, 10 segments for watershed W-2, 11 segments for watershed W-6, and 10 segments for watershed G, as is shown in Figures 2–4. For each segment the weighted slope is known from the topographic map. Two sets of rainfall-runoff events were selected for each of the three watersheds; one set was called the optimization set, implying that the events in that set were used for optimization only, and the other set was named the prediction set, implying that those events were used for hydrograph prediction only. These two sets were mutually exclusive; that is, they did not have any events in common. The optimization sets consisted of five events each on watersheds G, W-2, and W-6. The prediction sets consisted of three events each on watersheds G, W-2, and W-6. The constants were obtained by optimization over the optimization set for each watershed. The optimization was performed by the Rosenbrock-Palmer algorithm [*Rosenbrock*, 1960; *Himmelblau*, 1972], utilizing the objective function of (13). The optimized values of constants c_1 and c_2 were 3.3 and 4.95 for watershed G, 3.6 and 5.0 for watershed W-2, and 1.5 and 2.94 for watershed W-6.

Hydrograph prediction. Through the utilization of optimized values of constants c_1 and c_2, hydrograph predictions were made for the events in the prediction set of each watershed. Sample predicted hydrographs are shown in Figures 6–8. On comparing predicted runoff peaks with observed runoff peaks, we found that they were in reasonable agreement. Hydrograph time and shape characteristics were predicted quite well by the model, especially when its simplicity is considered. However, a few points prompt discussion.

1. In some cases the error in the prediction of the hydrograph peak was as high as about 50%, although in most cases it remained well below 20%. There might be several reasons for high prediction error. The following two reasons appear to be most prominent. First, the size of the optimization set is very small, and therefore we cannot hope to obtain representative values of constants c_1 and c_2, especially since the rainfall-runoff events for each watershed under consideration represent a long stretch of time, often 15 years or more. During this period of time several changes in land management and cropping pattern must have taken place on these watersheds. These changes can in no way be represented by such small samples as we have considered. Second, there is difficulty in determining rainfall excess, which in fact generated observed runoff. The determination of true rainfall excess seems to be the major problem in most rainfall-runoff models, and our model is no exception. Philip's equation, utilized in this study, is too simple to predict the time distribution of infiltration accurately, and then there is the difficulty of estimating its parameters. Our model was used primarily for its simplicity.

2. Figures 6–8 illustrate that our model predicts the time distribution of runoff quite well. We must note that the optimization of parameters c_1 and c_2 employed an objective function that was based on hydrograph peak only. Runoff timing was not considered explicitly, yet the hydrograph shape and time characteristics are well predicted. It seems to us that if the model structure is sound, it might suffice to perform optimization of parameters on some prominent characteristics of the runoff hydrograph even for the prediction of the entire hydrograph; therefore there is no need to consider the entire hydrograph explicitly in the optimization.

Considering its simplicity the distributed converging over-

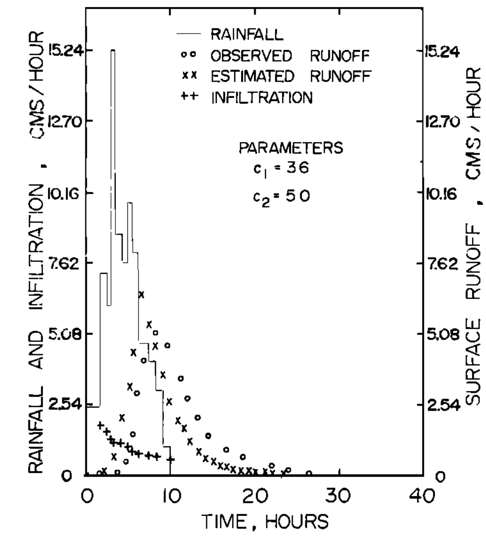

Fig. 6. Prediction of surface runoff hydrograph for rainfall event of April 24, 1957, on watershed W-2, Riesel (Waco), Texas.

land flow model appears to be a promising tool. There is, however, a need for exhaustive testing of the proposed model on a number of natural watersheds in a variety of physiographic and climatic settings. Another aspect would be to investigate the problem of determining constants c_1 and c_2 from physically measurable watershed characteristics. If the problem of a priori estimation of parameters c_1 and c_2 can be tackled, the utility of the proposed model will be greatly enhanced.

CONCLUSION

On the basis of the limited testing it can be safely argued that the proposed model is potentially promising and deserves further investigation. Its simplicity is sufficient to justify the above argument. The model is capable of predicting hydrograph peak, time, and shape characteristics. The model requires specification of parameters c_1 and c_2, which need further investigation.

APPENDIX A: STABILITY ANALYSIS

Stability is essential for the convergence of a difference scheme. In an unstable scheme, small numerical errors introduced in the computational method are amplified and dominate the solution. A linear stability analysis for the Lax-Wendroff scheme is given. Although the method is not rigorous for nonlinear equations, it does identify the unsuitability of the difference scheme and determine the appropriate step for conditional stability.

In a linear stability analysis it is assumed that instabilities first appear in a small region of space, so that if the coefficients of the derivative are smooth functions, they can be approximated as constants in this region. We write (1) in a linearized form as

$$\frac{\partial h}{\partial t} + n\alpha(x)\bar{h}^{n-1}\frac{\partial h}{\partial x} + h\bar{h}^{n-1}\frac{\partial \alpha(x)}{\partial x} = q(x, t) + \frac{\alpha(x)h\bar{h}^{n-1}}{(L-x)} \quad (A1)$$

where $\bar{h}$ is a constant. Now at any point (j, k) the numerical solution h_k^j equals the true solution $h(j\Delta x, k\Delta t)$ plus an error term ϵ_j^k. Thus we can write

$$h_j^k = h(j\Delta x, k\Delta t) + \epsilon_j^k \quad (A2)$$

Since the system of (A1) and (A2) is linear, it may suffice to consider only one term of the Fourier series expression for the error term. That is,

$$\epsilon_M^N = \epsilon_0 \exp\left[i(M\sigma\Delta x + N\gamma\Delta t)\right] \quad (A3)$$

where ϵ_0 is a constant, σ and γ are wave numbers in space and time, and $i = (-1)^{1/2}$. It is assumed that the errors are perturbations added to the solution of the linear system. If the linearized finite difference equation is written in terms of the correct solution plus the error term and then the exact solution is subtracted, a differential equation in the error terms can be obtained. That is,

$$\frac{\partial \epsilon}{\partial t} + n\alpha(x)\bar{h}^{n-1}\frac{\partial \epsilon}{\partial x} + \epsilon\bar{h}^{n-1}\frac{\partial \alpha(x)}{\partial x} = \frac{\alpha(x)\epsilon\bar{h}^{n-1}}{(L-x)}$$
$$g(x) = n\alpha(x)\bar{h}^{n-1} \qquad f(x) = \bar{h}^{n-1}\left[\frac{\partial \alpha(x)}{\partial x} - \frac{\alpha(x)}{(L-x)}\right] \quad (A4)$$

Equation (A4) can be written in a simplified form as

$$\frac{\partial \epsilon}{\partial t} + g(x)\frac{\partial \epsilon}{\partial x} + \epsilon f(x) = 0 \quad (A5)$$

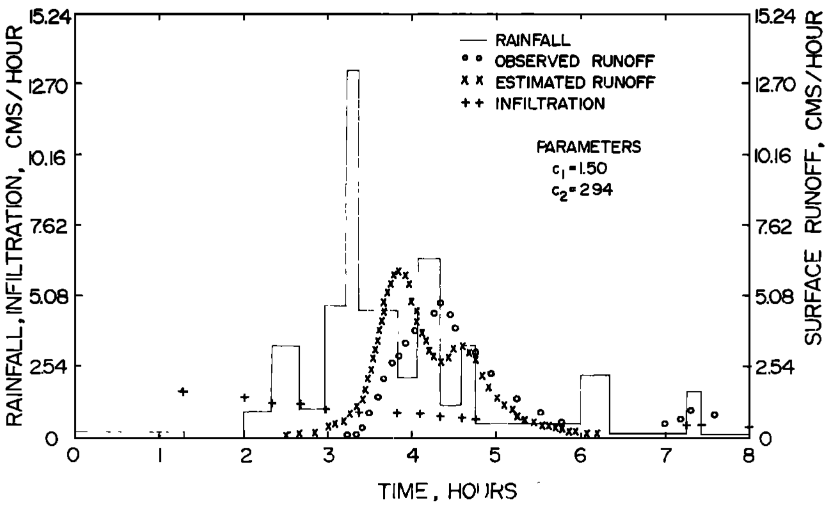

Fig. 7. Prediction of surface runoff hydrograph for rainfall event of March 29, 1965, on watershed W-6, Riesel (Waco), Texas.

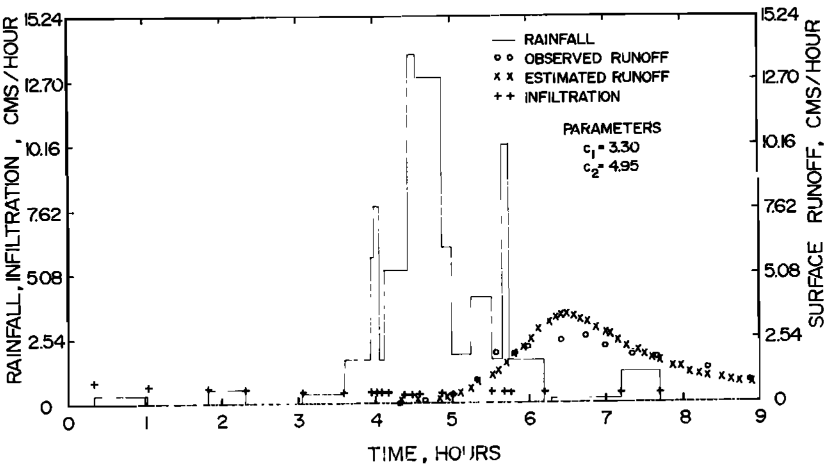

Fig. 8. Prediction of surface runoff hydrograph for rainfall event of March 29, 1965, on watershed G, Riesel (Waco) Texas.

Then we can write

$$\frac{\partial \epsilon}{\partial t} = -g(x)\frac{\partial \epsilon}{\partial x} - \epsilon f(x) \tag{A6}$$

$$\begin{aligned}\frac{\partial^2 \epsilon}{\partial t} &= -g(x)\frac{\partial}{\partial x}\left[-g(x)\frac{\partial \epsilon}{\partial x} - \epsilon f(x)\right] \\ &\quad - f(x)\left[-g(x)\frac{\partial \epsilon}{\partial x} - \epsilon f(x)\right] \\ &= g(x)\left[g(x)\frac{\partial^2 \epsilon}{\partial x^2} + \frac{\partial \epsilon}{\partial x} g'(x) + \frac{\partial \epsilon}{\partial x} f(x) \right. \\ &\quad \left. + \epsilon f'(x)\right] + f(x)\left[g(x)\frac{\partial \epsilon}{\partial x} + \epsilon f(x)\right]\end{aligned} \tag{A7}$$

From the Taylor series expansion of $\epsilon(x, t + \Delta t)$,

$$\epsilon(x, t + \Delta t) = \epsilon(x, t) + \Delta t \frac{\partial \epsilon}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 \epsilon}{\partial x^2} + O(\Delta t^3) \tag{A8}$$

Substituting (A6) and (A7) in (A8) and neglecting higher-order terms, we get

$$\begin{aligned}\epsilon(x, t + \Delta t) &= \epsilon(x, t) - \Delta t\left[g(x)\frac{\partial \epsilon}{\partial x} + \epsilon f(x)\right] \\ &\quad + \frac{(\Delta t)^2}{2}\left\{g(x)\left[g(x)\frac{\partial^2 \epsilon}{\partial x^2} + \frac{\partial \epsilon}{\partial x} g'(x)\right.\right. \\ &\quad \left.\left. + \frac{\partial \epsilon}{\partial x} f(x) + \epsilon f'(x)\right] + f(x)\left[g(x)\frac{\partial \epsilon}{\partial x} + \epsilon f(x)\right]\right\}\end{aligned} \tag{A9}$$

Writing (A9) in a simplified form, we get

$$\begin{aligned}\epsilon(x, t + \Delta t) &= \epsilon(x, t) + \left[-\Delta t + \frac{(\Delta t)^2}{2} f(x)\right] \\ &\quad \cdot \left[g(x)\frac{\partial \epsilon}{\partial x} + \epsilon f(x)\right] + \frac{(\Delta t)^2}{2} g(x) \\ &\quad \cdot \left\{g(x)\frac{\partial^2 \epsilon}{\partial x^2} + \frac{\partial \epsilon}{\partial x}[g'(x) + f(x)] + \epsilon f'(x)\right\}\end{aligned} \tag{A10}$$

Expressing (A10) in Lax-Wendroff finite difference form, we get

$$\begin{aligned}\epsilon_j^{k+1} &= \epsilon_j^k + \left[-\Delta t + \frac{(\Delta t)^2}{2} f(x)\right] \\ &\quad \cdot \left[g(x)\frac{\epsilon_{j+1}^k - \epsilon_{j-1}^k}{2\Delta x} + \epsilon_j^k f(x)\right] + \frac{(\Delta t)^2}{2} g(x) \\ &\quad \cdot \left\{g(x)\frac{\epsilon_{j+1}^k - 2\epsilon_j^k + \epsilon_{j-1}^k}{(\Delta x)^2} + \frac{\epsilon_{j+1}^k - \epsilon_{j-1}^k}{2\Delta x} \right. \\ &\quad \left. \cdot [g'(x) + f(x)] + \epsilon_j^k f'(x)\right\}\end{aligned} \tag{A11}$$

Let $M = N = 0$ for the point (j, k) (which we can do with no loss in generality); then

$$\epsilon_j^{k+1} = \epsilon_0 \exp(i\gamma\Delta t) \qquad \epsilon_j^k = \epsilon_0$$

$$\epsilon_{j-1}^k = \epsilon_0 \exp(-i\sigma\Delta x) \qquad \epsilon_{j+1}^k = \epsilon_0 \exp(i\sigma\Delta x)$$

Substituting these expressions in (A11) and dividing by ϵ_0, we get

$$\begin{aligned}e^{i\gamma\Delta t} &= 1 + \left[-\Delta t + \frac{(\Delta t)^2}{2} f(x)\right] \\ &\quad \cdot \left[g(x)\frac{e^{i\sigma\Delta x} - e^{-i\sigma\Delta x}}{2\Delta x} + f(x)\right] + \frac{(\Delta t)^2}{2} g(x) \\ &\quad \cdot \left\{g(x)\frac{e^{i\sigma\Delta x} - 2 + e^{-i\sigma\Delta x}}{(\Delta x)^2} \right. \\ &\quad \left. + \frac{e^{i\sigma\Delta x} - e^{-i\sigma\Delta x}}{2\Delta x}[g'(x) + f(x)] + f'(x)\right\}\end{aligned} \tag{A12}$$

With appropriate trigonometric substitutions,

TABLE 1. Stability Criteria

$\sigma\Delta x$	sin $\sigma\Delta x$	cos $\sigma\Delta x$	Criterion
0	0	1	$\lvert 1 \rvert \leq 1$
$\frac{\pi}{2}$	1	0	$\left\lvert 1 - [g(x)]^2\left(\frac{\Delta t}{\Delta x}\right)^2 + [g(x)]^4\left(\frac{\Delta t}{\Delta x}\right)^4\right\rvert \leq 1$
π	0	−1	$\left\lvert 1 - 4[g(x)]^2\left(\frac{\Delta t}{\Delta x}\right)^2 + 4[g(x)]^2\left(\frac{\Delta t}{\Delta x}\right)^4\right\rvert \leq 1$
$\frac{3\pi}{2}$	−1	0	$\left\lvert -[g(x)]^2\left(\frac{\Delta t}{\Delta x}\right)^2 + [g(x)]^4\left(\frac{\Delta t}{\Delta x}\right)^4\right\rvert \leq 1$

$$e^{i\sigma\Delta t} = 1 + \left[-\Delta t + \frac{(\Delta t)^2}{2} f(x)\right] \cdot \left[ig(x)\frac{\sin\sigma\Delta x}{\Delta x} + f(x)\right] + \frac{(\Delta t)^2}{2} g(x) \cdot \{[g(x)/(\Delta x)^2](2\cos\sigma\Delta x - 2) + (i/\Delta x) \cdot \sin\sigma\Delta x[g'(x) + f(x)] + f'(x)\} \quad \text{(A13)}$$

For stability the quantity $e^{i\gamma\Delta t}$ must lie within the unit circle on the complex plane. The real part of (A13) is

$$1 + \left[-\Delta t + \frac{(\Delta t)^2}{2} f(x)\right] f(x) + \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{g(x)}{2} \cdot [g(x)(2\cos\sigma\Delta x - 2) + f'(x)(\Delta x)^2]$$

and the imaginary part is

$$\left[-\frac{\Delta t}{\Delta x} + \frac{(\Delta t)^2}{2\Delta x} f(x)\right][g(x)\sin\sigma\Delta x] + \frac{(\Delta t)^2}{\Delta x} g(x) \cdot \sin\sigma\Delta x[g'(x) + f(x)]$$

Squaring the real and imaginary parts and dropping the terms of smaller magnitudes and especially those involving $O(\Delta t)$, we get

$$\left|\left[1 + \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{[g(x)]^2}{2}(2\cos\theta - 2)\right]^2 + \left[\frac{\Delta t}{\Delta x} g(x)\sin\theta\right]^2\right| \leqq 1 \quad \text{(A14)}$$

where $\theta = \sigma\Delta x$. Equation (A14) gives the stability criterion. Let us consider the most critical condition, when the left-hand side of (A14) is evaluated at the values of $\sigma\Delta x$ shown in Table 1. From this analysis it is apparent that the criterion stated in Table 1 is satisfied when

$$\left(g(x)\frac{\Delta t}{\Delta x}\right)^2 \leqq 1 \qquad g(x)\frac{\Delta t}{\Delta x} \leqq 1$$

or

$$\Delta t/\Delta x \leqq [n\alpha(x)\bar{h}^{n-1}]^{-1} \quad \text{(A15)}$$

Equation (A16) shows that the point $(j, k+1)$ must lie within the zone of determinacy of the line from $(j-1, k)$ to $(j+1, K)$. The Lax-Wendroff scheme is linearly stable subject to condition (A15).

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