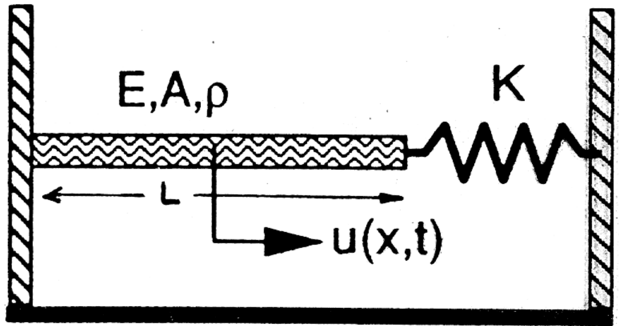
uis San Andres

‡5 (8 minutes)

ıdamental natural frequency
of the bar shown in the
as uniform properties ρ,A,E
is K.
modes method with an
function Ψ(x) for the

Find Natural frequency



SOLUTION: Find Approximate natural frequency
Use Assumed Modes method.

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 dx \; ; \; V = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{1}{2} K\, U^2_{(L,t)} \qquad (1)$$

Assume modes: let $u(x,t) = \psi(x)\, U(t)$ (2)

where $\psi(0) = 0$ by essential B.C. (3)

$\psi(x) \in C^1(0,L)$

Then

$$T = \frac{1}{2}\left[\int_0^L \rho A \psi^2 dx\right] \dot{U}^2 = \frac{1}{2} M_{eq}\, \dot{U}^2$$

$$V = \frac{1}{2}\left[\int_0^L AE(\psi')^2 dx + K\psi^2_{(L)}\right] U^2 = \frac{1}{2} K_{eq}\, U^2 \qquad (4)$$

Equivalent mass and stiffness are

$$M_{eq} = \int_0^L \rho A \psi^2 dx, \quad K_{eq} = \int_0^L EA\left(\frac{d\psi}{dx}\right)^2 dx + K\psi^2_{(L)}, \text{ and } \omega_n = \sqrt{\frac{K_{eq}}{M_{eq}}}$$

Select $\psi(x) = x/L$ satisfying (3)

$$M_{eq} = \rho A L/3 \; ; \; K_{eq} = \frac{EA}{L} + K$$

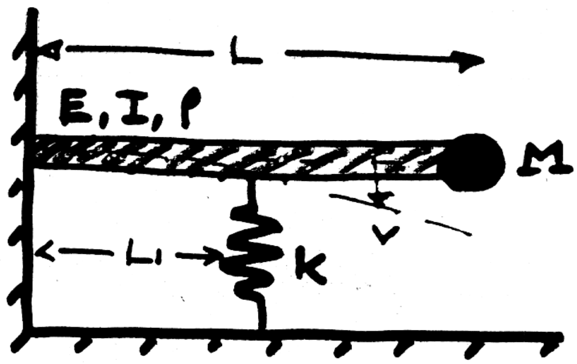
$$\omega_n = \left[\frac{EA/L + K}{\rho A L/3}\right]^{1/2}$$

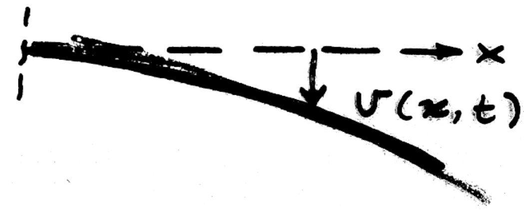
Rayleigh-Ritz Method.

Assumed Modes Method for CONTINUOUS SYSTEMS:

Multiple Degree of Freedom ANALYSIS

Consider the lateral vibrations of the following system:





The total kinetic energy and potential energy of the system are:

$$T = \frac{1}{2} M \dot{v}^2(L,t) + \frac{1}{2} \int_0^L \rho A \left(\frac{\partial v}{\partial t}\right)^2 dx \qquad (1)$$

$$V = \frac{1}{2} K v^2(L_1,t) + \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx \qquad (2)$$

Now, let's assume the displacement vector $v(x,t)$ is given by

$$v(x,t) = \sum_{i=1}^{n} \psi_i(x)\, V_i(t) \qquad (3)$$

where the set $\{\psi_i(x)\}_{i=1}^{n}$ is a set of

Assumed Modes functions

ADMISSIBLE to the problem, i.e.

$$\psi_i \in \mathbb{C}^2 (x \in [0,L]) \tag{4}$$

ψ_i satisfy <u>essential</u> boundary conditions

(for the example $\left.\begin{array}{l}\psi(0)=0 \\ d\psi/dx(0)=0\end{array}\right\}$ only!

AND : $\nu_i(t)$, $i=1,2,\dots n$ are discrete displacements associated with each mode shape.

Linear independence of the <u>shape</u> functions ψ is not required. (later explained)

Note that
$$\dot{v} = \frac{\partial v}{\partial t} = \sum_i^n \psi_i \, \dot{\nu}_i(t) \tag{5}$$

$$\frac{\partial^2 v}{\partial x^2} = \sum_i^n \frac{d^2\psi_i}{dx^2} \cdot \nu_i(t)$$

Substitution of (3) into (1) & (2) gives the following approximate T & V.

$$T = \frac{1}{2} M \sum_{i}^{n} \psi_i^2(L)\, \dot{\nu}_i^2 + \frac{1}{2} \int_0^L \rho A \underline{\left(\sum_i^n \psi_i \dot{\nu}_i\right)\left(\sum_j \psi_j \dot{\nu}_j\right)}\, dx \tag{6}$$

$$V = \frac{1}{2} K \sum_{i}^{n} \psi_i^2(L_1)\, \nu_i^2 + \frac{1}{2} \int_0^L EI \left(\sum_i^n \psi_i'' \nu_i\right)\left(\sum_j \psi_j'' \nu_j\right) dx \tag{7}$$

where $\psi_i'' = \dfrac{d^2\psi_i}{dx^2}$.

now using: $\displaystyle\int_0^L \sum_i = \sum_i \int_0^L$

we get from (6) & (7)

$$T \cong \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ M \underbrace{\psi_i^2(L)\,\delta_{ij}} + \int_0^L \rho A\, \psi_i\, \psi_j\, dx \right\} \dot{\nu}_i\, \dot{\nu}_j \tag{8}$$

where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

and

$$V \cong \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ k\, \psi_i^2(L)\, \delta_{ij} + \int_0^L EI\, \psi_i''\, \psi_j''\, dx \right\} \nu_i\, \nu_j \tag{9}$$

Let's define:

$$M_{ij} = M\, \psi_i^2(L)\, \delta_{ij} + \int_0^L \rho A\, \psi_i\, \psi_j\, dx \tag{10}$$

$i, j = 1, 2, \ldots n$

$$K_{ij} = k\, \psi_i^2(L_1)\, \delta_{ij} + \int_0^L EI\, \psi_i''\, \psi_j''\, dx \tag{11}$$

as the system mass and stiffness elements

Note: $M_{ij} = M_{ji}$

$K_{ij} = K_{ji}$

i.e $\Rightarrow$ symmetric positive-definite r d write (8) & (9) as:

$$T \simeq \frac{1}{2} \sum_i^n \sum_j^n M_{ij} \dot{v}_i \dot{v}_j \qquad (12)$$

$$V \simeq \frac{1}{2} \sum_i \sum_j K_{ij} v_i v_j \qquad (13)$$

or letting $\mathbb{M} = \begin{bmatrix} M_{11} & M_{12} & \cdots & \\ M_{21} & & & \\ & & & M_{nn} \end{bmatrix}$ $\qquad \mathbb{M} = \mathbb{M}^T$

(14)

$$\mathbb{K} = \begin{bmatrix} K_{11} & K_{12} & & \\ K_{21} & & \ddots & \\ & & & K_{nn} \end{bmatrix} \qquad \mathbb{K} = \mathbb{K}^T$$

be the system mass and stiffness matrices and

$$\mathbb{v}^T = \{ v_1 \ v_2 \ \ldots \ v_n \} \qquad (15)$$

we write (12) & (13) as:

$$T \simeq \frac{1}{2} \dot{\mathbb{v}}^T \mathbb{M} \dot{\mathbb{v}} \qquad (16)$$

$$V \simeq \frac{1}{2} \mathbb{v}^T \mathbb{K} \mathbb{v} \qquad (17)$$

Now we have n DOF's: $\nu_1, \nu_2, \dots \nu_n$ so we can use Lagrange's equation of motion as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\nu}_q}\right) + \frac{\partial V}{\partial \nu_q} = Q_q = 0 \qquad (18)$$

$q = 1, 2, \dots n$

to determine the eqn. of motion.

From (13) $V \simeq \frac{1}{2} \sum_i \sum_j K_{ij} \nu_i \nu_j$

$$\frac{\partial V}{\partial \nu_q} = \frac{1}{2} \sum_i \sum_j K_{ij}\left(\frac{\partial \nu_i}{\partial \nu_q} \nu_j + \frac{\partial \nu_j}{\partial \nu_q} \nu_i\right) \qquad (a)$$

since $K_{ij} \neq fn(\nu_q)$

and using: $\frac{\partial \nu_i}{\partial \nu_q} = \delta_{iq} = \begin{cases} 1 & \text{if } i = q \\ 0 & \text{if } i \neq q \end{cases} \qquad (b)$

we get:

$$\frac{\partial V}{\partial \nu_q} = \frac{1}{2} \sum_i \sum_j K_{ij}\left(\delta_{iq} \nu_j + \delta_{jq} \nu_i\right)$$

$$= \frac{1}{2} \sum_j K_{qj} \nu_j + \frac{1}{2} \sum_q K_{iq} \nu_i$$

but since $K_{qi} = K_{iq}$ (symmetric)

Then:

$$\frac{\partial V}{\partial \nu_q} = \frac{1}{2} \sum_j K_{qj}\, \nu_j + \frac{1}{2} \sum_i K_{qi}\, \nu_i \qquad (c)$$

i & j are "dumys" here

so we express (c) as:

$$\frac{\partial V}{\partial \nu_q} = \frac{1}{2} \sum_j K_{qj}\, \nu_j + \frac{1}{2} \sum_j K_{qj}\, \nu_j \qquad (d)$$

or $$\frac{\partial V}{\partial \nu_q} = \sum_j K_{qj}\, \nu_j = \mathbb{K}\, \boldsymbol{\nu} \qquad (19)$$

Similarly we can show that:

$$\frac{\partial T}{\partial \dot{\nu}_q} = \sum_j M_{qj}\, \dot{\nu}_j = \mathbb{M}\, \dot{\boldsymbol{\nu}} \qquad (20)$$

and $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\nu}_q}\right) = \sum_j M_{qj}\, \ddot{\nu}_j = \mathbb{M}\, \ddot{\boldsymbol{\nu}} \qquad (21)$$

here we have used $\mathbb{M}^T = \mathbb{M}$

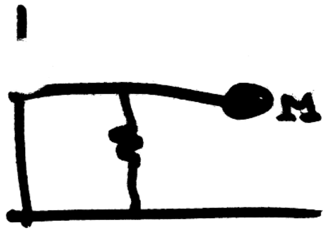
<u>Thus</u>, the equation of motion are equal to:

$$\mathbb{M}\ddot{\nu} + \mathbb{K}\nu = \mathbb{0} \qquad (22)$$

or

$$\sum_{i}^{n} M_{ij}\ddot{\nu}_j + \sum_{i}^{n} K_{ij}\nu_j = 0_i \qquad (23)$$

<u>Now</u>, for the problem of interest, the elements of the mass & stiffness matrices are:



$$M_{ij} = \int_0^L \rho A \phi_i \phi_j \, dx + M \phi_{i(L)} \phi_{j(L)} \delta_{ij} = M_{ji}$$

$$K_{ij} = \int_0^L EI \phi_i'' \phi_j'' \, dx + K \phi_{i(L)} \phi_{j(L)} \delta_{ij} = K_{ji}$$

$$i, j = 1, 2, \ldots n$$

<u>EXERCISE</u>: what are the eqns. of motion if $\mathbb{M}$ and $\mathbb{K}$ are not <u>symmetric</u>?

Summary:

Using the assumed modes method, a continuous system can be modeled as n-DOF discrete system with equations $\mathbb{M}\ddot{v} + \mathbb{K}v = \mathbb{Q}$

This was the way done in the past to handle continuous systems of complex shapes.

Disadvantages:

1) difficult to find more than one shape function (assumed mode)

2) $\{\psi_i\}_{i=1}^{n}$ need to be defined over entire domain, and

3) $\mathbb{M}$ and $\mathbb{K}$ matrices are usually full of elements.

If $\{\psi_i\}_{i=1}^{n}$ are Linearly independent the problem is very much reduced since in this case the mass and stiffness matrices are diagonal!

Here, we mean linear independence of the shape functions in the following way:

$$\int_0^L \rho A \psi_i \psi_j \, dx = 0 \quad \text{if} \quad i \neq j$$

$$\int_0^L EI \, \psi_i'' \psi_j'' \, dx = 0 \quad \text{if} \quad i \neq j \qquad (24)$$

i.e. the functional inner products are zero!

This is very difficult to achieve! for most cases.

$$\left[\ \diagdown\, M \,\diagdown\ \right] \;,\; \left[\ \diagdown\, K \,\diagdown\ \right]$$