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Fundamental natural frequency of the bar shown in the as uniform properties $\rho, A, E$ is $K$.

Solutions: Find approximate natural frequency using assumed modes method.

$$T = \frac{1}{2} \int_0^L \left( \frac{\partial^2 u}{\partial t^2} \right)^2 \, dx \quad \text{and} \quad V = \frac{1}{2} \int_0^L EA \left( \frac{\partial u}{\partial t} \right)^2 \, dx + \frac{1}{2} K \dot{U}(L,t) \tag{1}$$

Assume modes: Let \( u(x, t) = \psi(x) U(t) \) \tag{2}

where \( \psi(0) = 0 \) by essential B.C. \( \psi(x) \in C^1(0, L) \)

Then

$$T = \frac{1}{2} \left[ \int_0^L EA \psi^2 \, dx \right] \dot{U}^2 = \frac{1}{2} \text{Meq} \dot{U}^2 \tag{4}$$

$$V = \frac{1}{2} \left[ \int_0^L EA (\psi^2 \dot{x} + K \psi^2(x)) \, dx \right] \dot{U}^2 = \frac{1}{2} \text{Keq} \dot{U}^2$$

Equivalent mass and stiffness are

$$\text{Meq} = \int_0^L EA \psi^2 \, dx, \quad \text{Keq} = \int_0^L EA (\frac{\partial \psi}{\partial x})^2 \, dx + K \psi^2(L), \text{and} \ \omega_n = \sqrt{\frac{\text{Keq}}{\text{Meq}}}$$

Select \( \psi(x) = x/L \) satisfying \( \psi(0) = 0 \)

$$\text{Meq} = EAL/3 \quad \text{and} \quad \text{Keq} = \frac{EA}{L} + K$$

$$\omega_n = \left[ \frac{EA/L + K}{EAL/3} \right]^{1/2}$$
Rayleigh–Ritz Method.

Assumed Modes Method for Continuous Systems: Multiple Degree of Freedom Analysis

Consider the vibrations of the following system:

The total kinetic energy and potential energy of the system are:

\[ T = \frac{1}{2} M \dot{U}^2(L,t) + \frac{1}{2} \int_0^L PA \left( \frac{\partial U}{\partial t} \right)^2 dx \]  

\[ V = \frac{1}{2} K U^2(L,t) + \frac{1}{2} \int_0^L EI \left( \frac{\partial^2 U}{\partial x^2} \right)^2 dx \]  

Now, let's assume the displacement vector \( U(x,t) \) is given by

\[ U(x,t) = \sum_{i=1}^{\infty} \psi_i(x) \nu_i(t) \]
where the set \( \{ \psi_i(x) \}_{i=1}^n \) is a set of assumed modes functions ADMISSIBLE to the problem, i.e.

\[
\psi_i \in C^2( x \in [0, L])
\] (a)

\( \psi_i \) satisfy essential boundary conditions (for the example \( \psi(0) = 0 \) only!

\[
d\psi/dx(0) = 0
\]

AND: \( \psi_i(t) \), \( i = 1, 2, ..., n \) are discrete displacements associated with each mode shape.

Linear independence of the shape functions \( \psi \) is not required. (later explained)

Note that

\[
\ddot{\psi} \frac{d\psi}{dt} = \sum_{i=1}^{n} \dot{\psi}_i \psi_i(t)
\] (b)

\[
\ddot{\psi} \frac{d^2\psi}{dx^2} = \sum_{i=1}^{n} \frac{d^2\psi_i}{dx^2} \psi_i(t)
\]
Substitution of (3) into (1) & (2) gives the following approximate $T$ & $V$.

\[
T = \frac{1}{2} M \sum_{i}^{n} \psi_{i}^{2} V_{i}^{2} + \frac{1}{2} \int_{0}^{L} PA \left( \sum_{i}^{n} \psi_{i} V_{i} \right) \left( \sum_{j}^{n} \psi_{j} V_{j} \right) dx \tag{6}
\]

\[
V = \frac{i}{2} K \sum_{i}^{n} \psi_{i}^{2} V_{i}^{2} + \frac{i}{2} \int_{0}^{L} EI \left( \sum_{i}^{n} \psi_{i}'' V_{i} \right) \left( \sum_{j}^{n} \psi_{j}'' V_{j} \right) dx \tag{7}
\]

where $\psi_{i}'' = \frac{d^{2}\psi_{i}}{dx^{2}}$.

Now using: \[ \sum_{i}^{n} \int_{0}^{L} \psi_{i} = \sum_{i} C_{i} \int_{0}^{L} \]

we get from (6) & (7)
\[ t = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ M \Psi_i^2(x) \delta_{ij} + \int_{0}^{L} \rho A \Psi_i \Psi_j \, dx \right\} v_i^2 v_j \]  \\

where \[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

and

\[ V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{ k \Psi_i^2(x) \delta_{ij} + \int_{0}^{L} EI \Psi_i'' \Psi_j'' \, dx \right\} v_i v_j \]  \\

Let's define:

\[ M_{ij} = M \Psi_i^2(x) \delta_{ij} + \int_{0}^{L} \rho A \Psi_i \Psi_j \, dx \]  \\
i, j = 1, 2, \ldots, n

\[ K_{ij} = k \Psi_i^2(x) \delta_{ij} + \int_{0}^{L} EI \Psi_i'' \Psi_j'' \, dx \]  \\

so the system mass and stiffness elements

\[ \text{symmetric: } M_{ij} = M_{ji} \text{ i.e. symmetric, positive-definite} \]

\[ K_{ij} = K_{ji} \]
and write (8) & (9) as:

\[ T = \frac{1}{2} \sum_i \sum_j M_{ij} \dot{\mathbf{y}}_i \dot{\mathbf{y}}_j \]  
\[ V = \frac{1}{2} \sum_i \sum_j K_{ij} \mathbf{y}_i \mathbf{y}_j \]  

or letting

\[ \mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & \cdots \\ M_{21} & M_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \]

\[ \mathbf{M} = \mathbf{M}^T \]  

\[ \mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & \cdots \\ K_{21} & K_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \]

\[ \mathbf{K} = \mathbf{K}^T \]

be the system mass and stiffness matrix and

\[ \mathbf{y}^T = \{ y_1, y_2, \ldots, y_n \} \]

we write (12) & (13) as:

\[ T = \frac{1}{2} \mathbf{y}^T \mathbf{M} \dot{\mathbf{y}} \]  
\[ V = \frac{1}{2} \mathbf{y}^T \mathbf{K} \mathbf{y} \]
Now we have \( n \) DoF's: \( v_1, v_2, \ldots, v_n \)
so we can use Lagrange's equation of motion as:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_q} \right) + \frac{\partial V}{\partial q_q} = Q_q = 0
\]

\( q = 1, 2, \ldots, n \)  \hspace{1cm} (18)

to determine the eqn. of motion.

From (18) \( V = \frac{1}{2} \sum_i \sum_j K_{ij} \dot{v}_i \dot{v}_j \)

\[
\frac{\partial V}{\partial \ddot{q}_q} = \frac{1}{2} \sum_i \sum_j K_{ij} \left( \frac{\partial \ddot{v}_i}{\partial \ddot{q}_q} \dot{v}_j + \frac{\partial \ddot{v}_j}{\partial \ddot{q}_q} \dot{v}_i \right)
\]  \hspace{1cm} (a)

since \( K_{ij} \neq f_n(\ddot{q}_q) \)

and using: \( \frac{\partial \ddot{v}_i}{\partial \ddot{q}_q} = \delta_{iq} = \begin{cases} 1 & \text{if } i = q \\ 0 & \text{if } i \neq q \end{cases} \)  \hspace{1cm} (6)

we get:

\[
\frac{\partial V}{\partial \ddot{q}_q} = \frac{1}{2} \sum_i \sum_j K_{ij} \left( \delta_{iq} \dot{v}_j + \delta_{jq} \dot{v}_i \right)
\]

\[
= \frac{1}{2} \sum_j K_{qj} \dot{v}_j + \frac{1}{2} \sum_i K_{iq} \ddot{v}_i
\]

but since \( K_{qi} = K_{iq} \) (symmetric)
Then:
\[ \frac{\partial V}{\partial \dot{q}_i} = \frac{1}{2} \sum_j K_{qj} \ddot{y}_j + \frac{1}{2} \sum_i K_{qi} \ddot{y}_i \]  

\( i \) and \( j \) are "dummys" here

so we express (c) on:
\[ \frac{\partial V}{\partial \dot{q}_i} = \frac{1}{2} \sum_j K_{qj} \ddot{y}_j + \frac{1}{2} \sum_j K_{jq} \ddot{y}_j \]  

\[ \frac{\partial V}{\partial \dot{q}_i} = \sum_j K_{qj} \ddot{y}_j = 1K \ddot{y} \]  

Similarly we can show that:
\[ \frac{\partial T}{\partial \dot{q}_j} = \sum_i M_{qj} \dot{y}_i = M \ddot{y} \]  

and
\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) = \sum_i M_{qj} \dddot{y}_i = M \dddot{y} \]  

Here we have used \( M^T = M \)
Thus, the equations of motion are equal to:

\[ IM \dddot{\gamma} + IK \ddot{\gamma} = 0 \quad (22) \]

or

\[ \sum \dot{M}_{ij} \dddot{\gamma}_j + \sum \dot{K}_{ij} \ddot{\gamma}_j = 0; \quad (23) \]

Now, for the problem of interest, the elements of the mass and stiffness matrices are:

\[ M_{ij} = \int_{0}^{L} cA \psi_i \psi_j \, dx + M \psi_i \psi_j (L) \delta_{ij} = M_{ji} \]

\[ K_{ij} = \int_{0}^{L} EI \dddot{\psi}_i \dddot{\psi}_j \, dx + K \psi_i (L) \psi_j (L) \delta_{ij} = K_{ji} \]

\[ i, j = 1, 2, \ldots, n \]

**Exercise:** What are the equations of motion if \( IM \) and \( IK \) are not symmetric?
**Summary:**

Using the assumed mode method, a continuous system can be modeled as an n-DOF discrete system with equations $M\ddot{\mathbf{y}} + K\mathbf{y} = \mathbf{0}$.

This was the way done in the past to handle continuous systems of complex shapes.

**Disadvantages:**

1) Difficult to find more than one shape function (assumed mode).

2) $\{\psi_i\}_{i=1}^n$ need to be defined over entire domain, and

3) $M$ and $K$ matrices are usually full of elements.

If $\{\psi_i\}_{i=1}^n$ are linearly independent, the problem is very much reduced since in this case the mass and stiffness matrices are diagonal!
Here, we mean linear independence of the shape functions in the following way:

\[
\int_0^L \mathbf{p}_i \mathbf{u}_j \, dx = 0 \quad \text{if} \quad i \neq j
\]

\[
\int_0^L \varepsilon \mathbf{r}_i \mathbf{r}_j \, dx = 0 \quad \text{if} \quad i \neq j
\]

i.e. the *inner products* are zero!

This is very difficult to achieve! for most cases.

\[
\left[ \begin{array}{c}
\mathbf{M} \\
\mathbf{K}
\end{array} \right] \quad : \quad \left[ \begin{array}{c}
\mathbf{M} \\
\mathbf{K}
\end{array} \right]
\]