MEEN 617 Handout #11

MODAL ANALYSIS OF MDOF Systems with VISCOUS DAMPING ^ Symmetric

Motion of a n-DOF linear system is described by the second order differential equations

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{C}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{F}_{(t)} \tag{1}$$

where $\mathbf{U}_{(t)}$ and $\mathbf{F}_{(t)}$ are n rows vectors of displacements and external forces, respectively. **M, K, C,** are the system (nxn) matrices of mass, stiffness, and viscous damping coefficients.

The solution to Eq. (1) is determined uniquely if vectors of initial displacements $\mathbf{U_o}$ and initial velocities $\mathbf{V_o} = \left({d\,\mathbf{U}}/{d\,t} \right)_{t=0}$ are specified.

For **free vibrations**, the external force vector $F_{(t)}=0$, and Eq. (1) reduces to

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{C}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{0} \tag{2}$$

A solution to Eq. (2) is of the form

$$\mathbf{U} = e^{\alpha t}\,\boldsymbol{\psi} \tag{3}$$

where in general α is a complex number. Substitution of Eq. (3) into Eq. (2) leads to the following characteristic equation:

$$\left(\alpha^2\,\mathbf{M} + \alpha\,\mathbf{C} + \mathbf{K}\right)\boldsymbol{\Psi} = \left[\mathbf{f}_{(\alpha)}\right]\boldsymbol{\Psi} = 0 \tag{4}$$

where $\left[\mathbf{f}_{(\alpha)}\right]$ is a nxn square matrix. The system of homogeneous equations (4) has a nontrivial solution if the determinant of the system of equation equals zero, i.e.

$$\Delta(\alpha)=\left|\mathbf{f}_{(\alpha)}\right|=0=c_0+c_1\,\alpha+c_2\,\alpha^2+c_3\,\alpha^3+\ldots c_{2n}\,\alpha^{2n} \tag{5}$$

The roots of the characteristic polynomial $\Delta(\alpha)$ given by Eq. (5) can be of three types:

a) **Real and negative**, $\alpha<0$, corresponding to over damped modes.

b) **Purely imaginary**, $\alpha=\pm i\,\omega$, for undamped modes.

c) **Complex conjugate pairs**[1] **of the form**, $\alpha=\zeta\omega\pm i\,\omega_d$, for under damped modes.

Clearly if the real part of any $\alpha>0$, it means the system is unstable.

The constituent solution, eq. (3), is then written as the **superposition of the roots** α_r and its associated vectors Ψ_r satisfying Eq. (4), i.e.

$$\mathbf{U}_{(t)}=\sum_{1}^{2n} C_r\,\mathbf{\Psi}_r\,e^{\alpha_r t} \tag{6}$$

or letting $\left[\mathbf{\Psi}\right]_{n\,x\,2n}=\left[\mathbf{\Psi}_1\,\mathbf{\Psi}_2\ldots\mathbf{\Psi}_{2n}\right]$ (7)

write Eq. (6) as

$$\mathbf{U}_{(t)}=\left[\mathbf{\Psi}\right]\left\{C_r\,e^{\alpha_r t}\right\} \tag{8}$$

[1] Only if system is defined by symmetric matrices. Otherwise, the complex roots may be not conjugate pairs.

However, a transformation of the form,

$$\mathbf{U}_{nx1} = [\boldsymbol{\Psi}]\ \mathbf{q}_{(t)_{2nx1}} \tag{9}$$

is not possible since this implies the existence of ***2n*- modal coordinates** which is not physically apparent when the **number of physical coordinates is only *n***.

To overcome this apparent difficulty, reformulate the problem in a slightly different form. Let **Y** be a *2n*- rows vector composed of the physical velocities and displacements, i.e.

$$\mathbf{Y} = \begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{bmatrix}, \text{ and } \mathbf{Q} = \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_{(t)} \end{bmatrix} \tag{10}$$

be a modified force vector. Then write $\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{C}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{F}_{(t)}$ as

$$\begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix} \begin{pmatrix} \ddot{\mathbf{U}} \\ \dot{\mathbf{U}} \end{pmatrix} + \begin{bmatrix} \mathbf{-M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \begin{pmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{F} \end{pmatrix} \tag{11.a}$$

or

$$\mathbf{A}\ \dot{\mathbf{Y}} + \mathbf{B}\ \mathbf{Y} = \mathbf{Q} \tag{11.b}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{-M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \tag{12}$$

A and **B** are 2nx2*n* matrices, in general symmetric if the **M, C, K** matrices are also symmetric.

For free vibrations, **Q=0**, and a solution to Eq. (11.b) is sought of the form:

$$\begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{bmatrix} = \mathbf{Y} = \mathbf{\Phi}\, e^{\alpha t} \tag{13}$$

Substitution of Eq. (13) into Eq. (11.b) gives:

$$\left[\alpha\,\mathbf{A} + \mathbf{B}\right]\mathbf{\Phi} = 0 \tag{14}$$

which can be written in the familiar form:

$$\mathbf{D}\,\mathbf{\Phi} = \frac{1}{\alpha}\mathbf{\Phi} \tag{15}$$

where

$$\mathbf{D} = \mathbf{-B}^{-1}\mathbf{A} = \begin{pmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-K}^{-1} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{C} \end{pmatrix}, \text{ or } \mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{-K}^{-1}\mathbf{M} & \mathbf{-K}^{-1}\mathbf{C} \end{pmatrix} \tag{16}$$

with **I** as the nxn identity matrix. From Eq. (15) write

$$\left[\mathbf{D} - \frac{1}{\alpha}\mathbf{I}\right]\mathbf{\Phi} = \left[\mathbf{f}_{(\alpha)}\right]\mathbf{\Phi} = 0 \tag{17}$$

The eigenvalue problem has a nontrivial solution if

$$\Delta(\alpha) = \left|\mathbf{f}_{(\alpha)}\right| = 0 \tag{18}$$

From Eq. (18) determine 2n eigenvalues $\{\alpha_r\}$, r=1, 2.., 2n and associated eigenvectors $\{\mathbf{\Phi}_r\}$. Each eigenvector must satisfy the relationship:

$$\mathbf{D}\,\mathbf{\Phi}_r = \frac{1}{\alpha_r}\mathbf{\Phi}_r \tag{19}$$

and can be written as $\mathbf{\Phi_r} = \begin{bmatrix} \mathbf{\Psi_r^1} \\ \mathbf{\Psi_r^2} \end{bmatrix}$ where $\mathbf{\Psi_r}$ is a nx1 vector satisfying:

$$\begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{-K^{-1}M} & \mathbf{-K^{-1}C} \end{pmatrix} \begin{bmatrix} \mathbf{\Psi_r^1} \\ \mathbf{\Psi_r^2} \end{bmatrix} = \frac{1}{\alpha_r} \begin{bmatrix} \mathbf{\Psi_r^1} \\ \mathbf{\Psi_r^2} \end{bmatrix} \qquad (20)$$

from the first row of Eq. (20) determine that:

$$\mathbf{I}\,\mathbf{\Psi_r^2} = \frac{1}{\alpha_r}\mathbf{\Psi_r^1} \quad \text{or} \quad \mathbf{\Psi_r^1} = \alpha_r\,\mathbf{\Psi_r^2} \qquad (21)$$

and from the second row of Eq. (20) with substitution of the relationship in Eq. (21) obtain

$$\left[\left(\mathbf{-K^{-1}M}\right)\alpha_r - \left(\mathbf{K^{-1}C}\right) - \mathbf{I}\frac{1}{\alpha_r} \right] \mathbf{\Psi_r^2} = \mathbf{0} \qquad (23)$$

for r=1, 2,….2n. Note that multiplying Eq. (23) by ($-\alpha_r\,\mathbf{K}$) gives

$$\left[\mathbf{M}\,\alpha_r^2 + \mathbf{C}\,\alpha_r + \mathbf{K} \right] \mathbf{\Psi_r^2} = \mathbf{0} \qquad (4)$$

i.e. the **original eigenvalue problem**. Solution of Eq. (23) delivers the ***2n-eigenpairs***

$$\left(\alpha_r\,;\, \mathbf{\Phi_r} = \begin{bmatrix} \alpha_r \mathbf{\Psi_r} \\ \mathbf{\Psi_r} \end{bmatrix} \right)_{r=1,2,..2n} \qquad (24)$$

In general, the j-components of the eigenvectors Ψ_r are complex numbers written as

$$\mathbf{\Psi_r}_j = a_{rj} + i\,b_{rj} = \delta_{rj}\,e^{i\phi_{rj}} \quad j=1,2\ldots n$$

where δ and φ are the magnitude and the phase angle.

Note: for viscous damped systems, not only the amplitudes but also the phase angles are arbitrary. However, the ratios of amplitudes and phase differences are constant for the elements of the eigenvectors $\mathbf{\Psi_r}$

i.e. $\left(\delta_j / \delta_k\right) = \text{const}_{jk}$ and $\left(\phi_j - \phi_k\right) = \text{const}_{jk}$ for $j, k = 1, 2, ..N$

A constituent solution of the homogeneous equation (**free vibration** problem) is then given as:

$$\mathbf{Y} = \begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{bmatrix} = \sum_{r=1}^{2n} C_r \, e^{\alpha_r t} \, \mathbf{\Phi_r} \tag{25}$$

Let the (roots) α_r be written in the form

$$\alpha_r = \zeta_r \, \omega_r + i \, \omega_{d\,r} \tag{26}$$

and write Eq. (25) as

$$\mathbf{Y} = \begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{bmatrix} = \sum_{r=1}^{2n} C_r \, \mathbf{\Phi_r} \; e^{\left(-\zeta_r \, \omega_r + i \, \omega_{d r}\right) t} \tag{27}$$

and since $\mathbf{\Phi_r} = \begin{bmatrix} \alpha_r \mathbf{\Psi_r} \\ \mathbf{\Psi_r} \end{bmatrix}$, the vector of displacements is just

$$\mathbf{U} = \sum_{r=1}^{2n} C_r \, \mathbf{\Psi_r} \; e^{\left(-\zeta_r \, \omega_r + i \, \omega_{d r}\right) t} \tag{27}$$

ORTHOGONALITY OF DAMPED MODES

The eigenvalues α_r and corresponding eigenvectors $\mathbf{\Phi_r}$ satisfy the equation:

$$\alpha_r \; \mathbf{A}\,\mathbf{\Phi_r} + \mathbf{B}\,\mathbf{\Phi_r} = 0 \qquad (28)$$

Consider two different eigenvalues (not complex conjugates):

$\{\alpha_s ; \mathbf{\Phi_s}\}$ and $\{\alpha_q ; \mathbf{\Phi_q}\}$, then **if** $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$ and $\mathbf{B} = \mathbf{B}^T$ (a symmetric system), it is easy to demonstrate that:

$$\left(\alpha_s - \alpha_q\right)\, \mathbf{\Phi_s^T}\; \mathbf{A}\,\mathbf{\Phi_q} = \mathbf{0}$$

and infer $\mathbf{\Phi_s^T}\,\mathbf{A}\,\mathbf{\Phi_q} = \mathbf{0}$; $\mathbf{\Phi_s^T}\,\mathbf{B}\,\mathbf{\Phi_q} = \mathbf{0}$ for $\alpha_s \neq \alpha_q$

(29)

At this time, construct a modal damped matrix $\mathbf{\Phi}_{(2n\mathrm{x}2n)}$ formed by the columns of the modal vectors $\mathbf{\Phi_r}$, i.e.

$$\mathbf{\Phi} = \left[\mathbf{\Phi}_1 \quad \mathbf{\Phi}_2 \quad \ldots \quad \mathbf{\Phi}_n \quad .. \quad \mathbf{\Phi}_{2n-1} \quad \mathbf{\Phi}_{2n}\right] \qquad (30)$$

And write the **orthogonality property** as:

$$\mathbf{\Phi}^{\mathbf{T}}\mathbf{A}\,\mathbf{\Phi} = \boldsymbol{\sigma} \qquad \mathbf{\Phi}^{\mathbf{T}}\mathbf{B}\,\mathbf{\Phi} = \boldsymbol{\beta} \qquad (31)$$

Where $\boldsymbol{\sigma}$ and $\boldsymbol{\beta}$ are $(2n\mathrm{x}2n)$ diagonal matrices.

Now, in the general case, the equations of motion on the physical coordinates are of second order and given by:

$$\mathbf{M}\,\ddot{\mathbf{U}} + \mathbf{C}\,\dot{\mathbf{U}} + \mathbf{K}\,\mathbf{U} = \mathbf{F}_{(t)} \qquad (1)$$

With the definition $\mathbf{Y}=\begin{bmatrix}\dot{\mathbf{U}}\\ \mathbf{U}\end{bmatrix}$, Eqs. (1) are converted into $2n$ first order differential equations:

$$\mathbf{A}\,\dot{\mathbf{Y}}+\mathbf{B}\,\mathbf{Y}=\mathbf{Q}=\begin{bmatrix}\mathbf{0}\\ \mathbf{F}_{(t)}\end{bmatrix} \tag{32}$$

Where $\mathbf{A}=\begin{bmatrix}\mathbf{0} & \mathbf{M}\\ \mathbf{M} & \mathbf{C}\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}\mathbf{-M} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{bmatrix}$ (12)

To uncouple the set of $2n$ first-order equations (32), a solution of the following form is assumed:

$$\mathbf{Y}_{(t)}=\sum_{r=1}^{2n}\mathbf{\Phi}_{\mathbf{r}}\,z_{r(t)}=\mathbf{\Phi}\,\mathbf{Z}_{(t)} \tag{33}$$

Substitution of Eq. (33) into Eq. (32) gives:

$$\mathbf{A}\ \mathbf{\Phi}\ \dot{\mathbf{Z}}+\mathbf{B}\ \mathbf{\Phi}\ \mathbf{Z}=\mathbf{Q} \tag{34}$$

Premultiply this equation by $\mathbf{\Phi}^{\mathbf{T}}$ and use the orthogonality property[2] of the damped modes to get:

$$\left(\mathbf{\Phi}^{\mathbf{T}}\mathbf{A}\ \mathbf{\Phi}\right)\dot{\mathbf{Z}}+\left(\mathbf{\Phi}^{\mathbf{T}}\mathbf{B}\ \mathbf{\Phi}\right)\mathbf{Z}=\mathbf{\Phi}^{\mathbf{T}}\mathbf{Q} \tag{35}$$

or

$$\boldsymbol{\sigma}\ \dot{\mathbf{Z}}+\boldsymbol{\beta}\ \mathbf{Z}=\mathbf{G}=\mathbf{\Phi}^{\mathbf{T}}\ \mathbf{Q} \tag{36}$$

Eq. (36) represents a set of $2n$ uncoupled first order equations:

$$\sigma_1\,\dot{z}_1+\beta_1\,z_1=g_1\left(t\right)$$

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[2] The result below is only valid for symmetric systems, i.e. with **M,K** and **C** as symmetric matrices. For the more general case, see the textbook of **Meirovitch** to find a discussion on LEFT and RIGHT eigenvectors.

$$\sigma_2\,\dot{z}_2+\beta_2\,z_2=g_2\left(t\right)$$

……

$$\sigma_{2N}\,\dot{z}_{2N}+\beta_{2N}\,z_{2N}=g_{2N(t)} \qquad (37)$$

where $\sigma_r=\mathbf{\Phi}_\mathbf{r}^\mathbf{T}\,\mathbf{A}\,\mathbf{\Phi}_\mathbf{r}\,;\;\beta_r=\mathbf{\Phi}_\mathbf{r}^\mathbf{T}\,\mathbf{B}\,\mathbf{\Phi}_\mathbf{r}=-\alpha_r\,\sigma_r$, r=1, 2..2N

$$\alpha_r=-\beta_r/\sigma_r \qquad (38)$$

since $\alpha_r\,\mathbf{A}\,\mathbf{\Phi}_\mathbf{r}+\mathbf{B}\;\mathbf{\Phi}_\mathbf{r}=\mathbf{0}$. In addition,

$$g_r=\mathbf{\Phi}_\mathbf{r}^\mathbf{T}\,\mathbf{Q}(t) \qquad (39)$$

Initial conditions are also determined from $\mathbf{Y}_o=\begin{bmatrix}\dot{\mathbf{U}}_o\\ \mathbf{U}_o\end{bmatrix}$ with the transformation

$$\boldsymbol{\sigma}\,\mathbf{Z_0}=\mathbf{\Phi}^\mathbf{T}\,\mathbf{A}\,\mathbf{Y_0} \qquad (40.a)$$

$$Z_{o_r}=\frac{1}{\sigma_r}\mathbf{\Phi}_\mathbf{r}^\mathbf{T}\,\mathbf{A}\,\mathbf{Y_o} \qquad r=1,2,\ldots 2n \qquad (40.b)$$

The general solution of the first order equation $\sigma_r\,\dot{z}_r+\beta_r\,z_r=g_r\left(t\right)$, with initial condition $z_{r(t=0)}=z_{o_r}$, is derived from the Convolution integral

$$z_r=z_{o_r}\;e^{\alpha_r t}\,\frac{1}{\sigma_r}\int_0^t g_{r(\tau)}\,e^{\alpha_r(t-\tau)}d\tau \qquad (41)$$

with $\alpha_r=-\beta_r/\sigma_r$

Once each of the $z(t)$ solutions are obtained, then return to physical coordinates to get:

$$\mathbf{Y}_{(t)} = \begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{U} \end{bmatrix} = \sum_{r=1}^{2n} \mathbf{\Phi}_{\mathbf{r}} \, z_{r(t)} = \mathbf{\Phi} \, \mathbf{Z}_{(t)} \qquad (33=43)$$

and since $\mathbf{\Phi}_{\mathbf{r}} = \begin{bmatrix} \alpha_r \mathbf{\Psi}_{\mathbf{r}} \\ \mathbf{\Psi}_{\mathbf{r}} \end{bmatrix}$, the physical displacement dynamic response is given by:

$$\mathbf{U}_{(t)} = \sum_{r=1}^{2n} \mathbf{\Psi}_{\mathbf{r}} \, z_{r(t)} \qquad (44)$$

and the velocity vector is correspondingly equal to:

$$\dot{\mathbf{U}}_{(t)} = \sum_{r=1}^{2n} \alpha_r \, \mathbf{\Psi}_{\mathbf{r}} \, z_{r(t)} \qquad (45)$$

See the accompanying MATHCAD® worksheet with a detailed example for discussion in class.