THE VIBRATION ABSORBER

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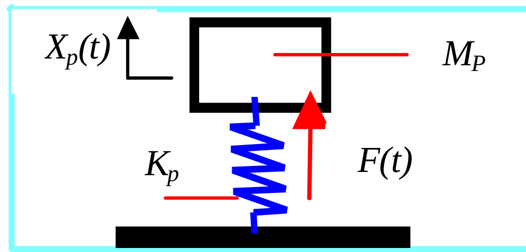
MEEN 363 - 617

Preamble - A NEED arises:

Consider the periodic forced response of a primary system (Kp-Mp) defined by

$$K_p := 1 \times 10^5 \cdot \frac{lbf}{in} \qquad M_p := 10^3 \cdot lb$$



Its natural frequency is

$$\omega_{np} := \left(\frac{K_p}{M_p}\right)^{.5}$$

$$\omega_{np} = 196.491 \frac{rad}{sec}$$

The EOM (from SEP) for periodic force excitation with magnitude Fo and frequency Ω is:

$$M_p \cdot \frac{d^2}{dt^2} X_p + K_p \cdot Y_p = F_o \cdot \cos(\Omega \cdot t) \quad \textbf{[1]}$$

Let:

$$F_o := 1000 lbf$$

The solution of [1] is of the form $X_p(t) = Z_{p.} \cdot \cos(\Omega \cdot t)$ **[2]**

Substitution of [2] into [1] gives:

$$\left(K_p - \Omega^2 \cdot M_p\right) \cdot Zp = F_o$$

$$Zp = \frac{F_o}{\left(K_p - \Omega^2 \cdot M_p\right)}$$

or

$$Z_p(r) := \frac{\frac{F_o}{K_p}}{\left(1 - r^2\right)} \quad \textbf{[3]}$$

with: $r = \dfrac{\Omega}{\omega_{np}}$ as the **frequency ratio**

Thus, the periodic force response of the system (Kp,Mp) is :

$$X_p(t) = \frac{\delta_s}{\left(1 - r^2\right)} \cdot \cos(\Omega \cdot t) = \frac{\delta_s}{\left|1 - r^2\right|} \cdot \cos(\Omega \cdot t + \phi) \quad \textbf{[4]}$$

with $\delta_s := \dfrac{F_o}{K_p}$

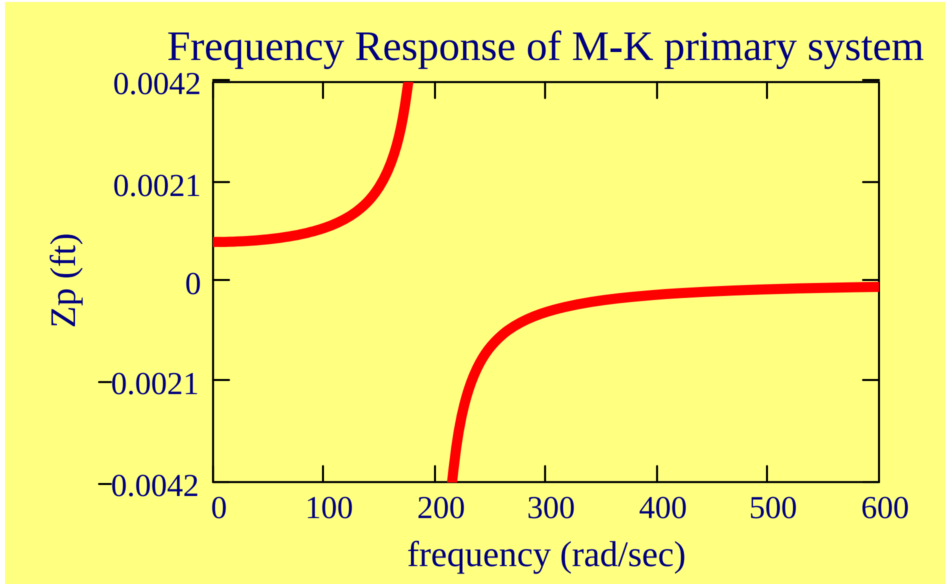
and the phase angle ϕ is zero degrees for excitation frequencies (Ω) < the natural frequency, ϕ = -180 deg for Ω > natural frequency, and ϕ=-90 degrees for Ω = natural frequency.

The response Zp (amplitude and phase) as a function of the excitation frequency is:

$$Z_P(\Omega) := \frac{\delta_s}{\left(1 - \frac{\Omega^2}{\omega_{np}^2}\right)}$$

$$\delta_s = 8.333 \times 10^{-4}\,\text{ft}$$

$$\omega_{np} = 196.491\,\frac{\text{rad}}{\text{sec}}$$

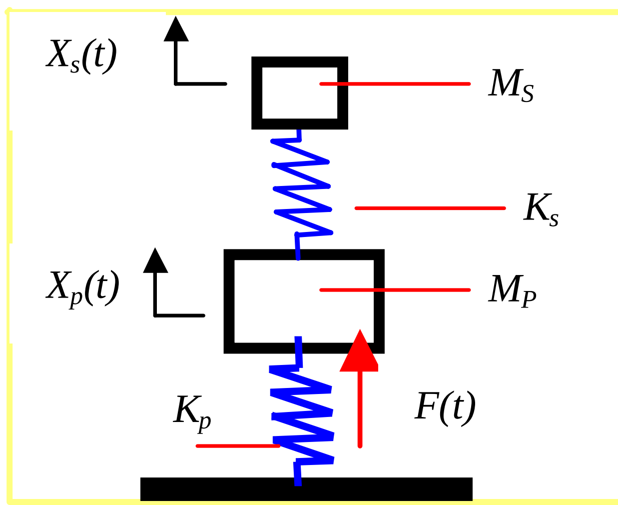


Note the very large amplitude of motions (unbounded) for excitation at the system natural frequency. In the graph above, Zp>0 means in phase with the external force, Zp<0 means 180 deg out of phase with external periodic force.

Clearly, the system cannot be operated at frequencies close (or at) the natural frequency. Since there is NO damping, the system will just fail b/c the amplitude of motion is just TOO LARGE!

Now, **consider a secondary (Ks-Ms) system** attached to the primary system (original).

In this case, the EOMs from the SEP are:



$$\begin{pmatrix} M_p & 0 \\ 0 & M_s \end{pmatrix} \cdot \frac{d^2}{dt^2}\begin{pmatrix} X_p \\ X_s \end{pmatrix} + \begin{pmatrix} K_p + K_s & -K_s \\ -K_s & K_s \end{pmatrix} \cdot \begin{pmatrix} X_p \\ X_s \end{pmatrix} = \begin{pmatrix} F_o \\ 0 \end{pmatrix} \cdot \cos(\Omega \cdot t) \qquad [5]$$

The system forced response is also periodic, i.e. with identical frequency as that of the periodic excitation force. Thus, let

$$\begin{pmatrix} X_p \\ X_s \end{pmatrix} = \begin{pmatrix} Z_p \\ Z_s \end{pmatrix} \cdot \cos(\Omega \cdot t) \qquad [6]$$

The combined system is 2-DOF. Thus, TWO natural frequencies (and natural modes) will appear.

Substitution of [6] into [5] leads to the algebraic set of equations:

$$\begin{pmatrix} K_p + K_s - \Omega^2 \cdot M_p & -K_s \\ -K_s & K_s - \Omega^2 \cdot M_s \end{pmatrix} \cdot \begin{pmatrix} Z_p \\ Z_s \end{pmatrix} = \begin{pmatrix} F_o \\ 0 \end{pmatrix} \qquad [7]$$

The **determinant** of the system of equations [7] is

$$\Delta(\Omega) = \left(K_p + K_s - \Omega^2 \cdot M_p\right) \cdot \left(K_s - \Omega^2 \cdot M_s\right) - K_s^{\,2} \qquad [8]$$

The solution to the algebraic system of equations [7] is simple, - use Cramer's rule, for example. The **response amplitudes for the primary and secondary masses** are:

$$Z_p(\Omega) = \frac{F_o \cdot \left(K_s - \Omega^2 \cdot M_s\right)}{\Delta(\Omega)} \qquad Z_s(\Omega) = \frac{F_o \cdot K_s}{\Delta(\Omega)} \qquad [9]$$

Note from Eq. [9] that **if**

$$\left(K_s - \Omega^2 \cdot M_s\right) = 0 \quad [10]$$

at a certain frequency $\Omega = \Omega_X$

then $Z_p(\Omega_X) = 0$ **That is, the motion of the primary system is zero (NULL)!**

A simple and practical SOLUTION:

A vibration absorber is a mechanical device which permits to reduce (even eliminate) amplitudes of vibration at certain excitation frequencies, in particular those at which the original system showed a highly undesirable response.

For example, if zero amplitude vibration is desired for excitations at the natural frequency of the original system, the designer selects

$$\Omega_X = \omega_{np}$$

also known as a **TUNED ABSORBER**

Which then determines that the stiffness and mass of the secondary system should be such that:

$$\left(K_s - \Omega_X^{\,2} \cdot M_s\right) = 0 \qquad \frac{K_s}{M_s} = \omega_{np}^{\,2} \qquad [11]$$

i.e, the natural frequency of the seconday system MUST coincide with that of the original system

The components of the vibration absorver (secondary system) need NOT be the same size as the original system. In practice, **the magnitude of Ks and Ms are substantially smaller.**

Say for $a := 10$ $\quad K_s := \frac{K_p}{a} \quad M_s := \frac{M_p}{a}$

then $\left(\frac{K_s}{M_s}\right)^{.5} = 196.491\,\frac{rad}{sec}$ **equals** $\left(\frac{K_p}{M_p}\right)^{.5} = 196.491\,\frac{rad}{sec}$

and the system responses are

$$\Delta(\Omega) := \left(K_p + K_s - \Omega^2 \cdot M_p\right)\cdot\left(K_s - \Omega^2 \cdot M_s\right) - K_s^2$$

$$Z_p(\Omega) := \frac{F_o\cdot\left(K_s - \Omega^2\cdot M_s\right)}{\Delta(\Omega)} \qquad Z_s(\Omega) := \frac{F_o\cdot K_s}{\Delta(\Omega)}$$

thus, **for operation with excitation frequency** $\Omega := \omega_{np}$ $\quad \omega_{np} = 196.491\,\frac{rad}{sec}$

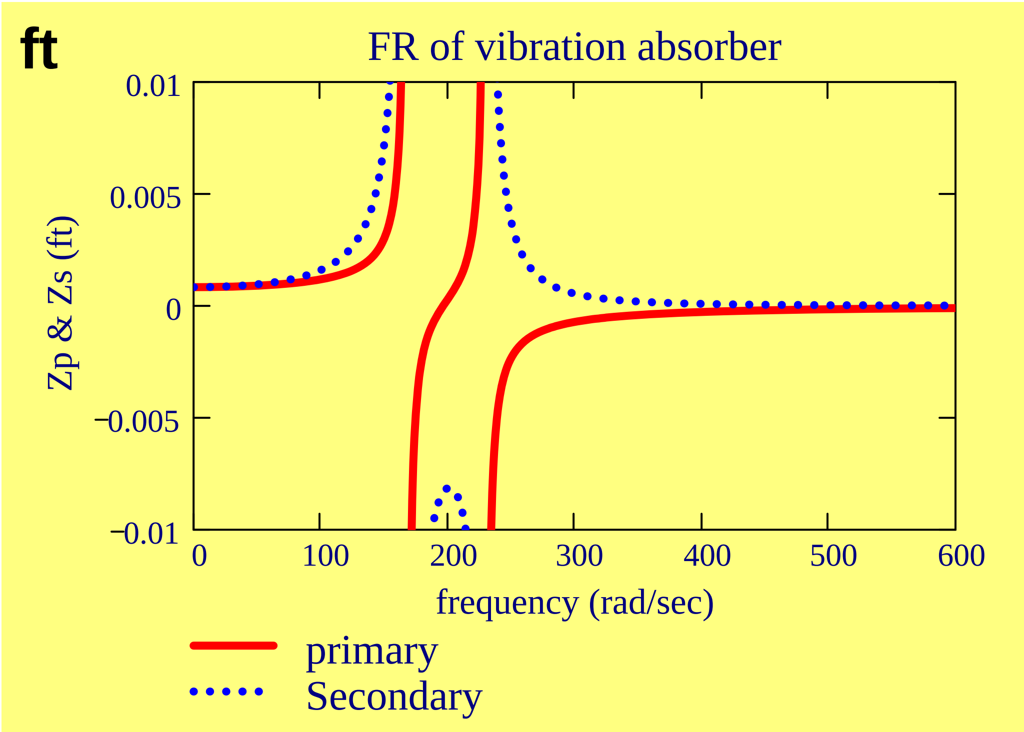
$$Z_p(\Omega) = 0\,ft \qquad Z_s(\omega_{np}) = -8.333\times10^{-3}\,ft = \frac{-F_o}{K_s}$$

Note

$a = 10$ $\quad \frac{Z_s(\omega_{np})}{\delta_s} = -10$

The SOFTER the secondary system is (Ks << Kp), the largest the motion of the secondary system at the desired frequency

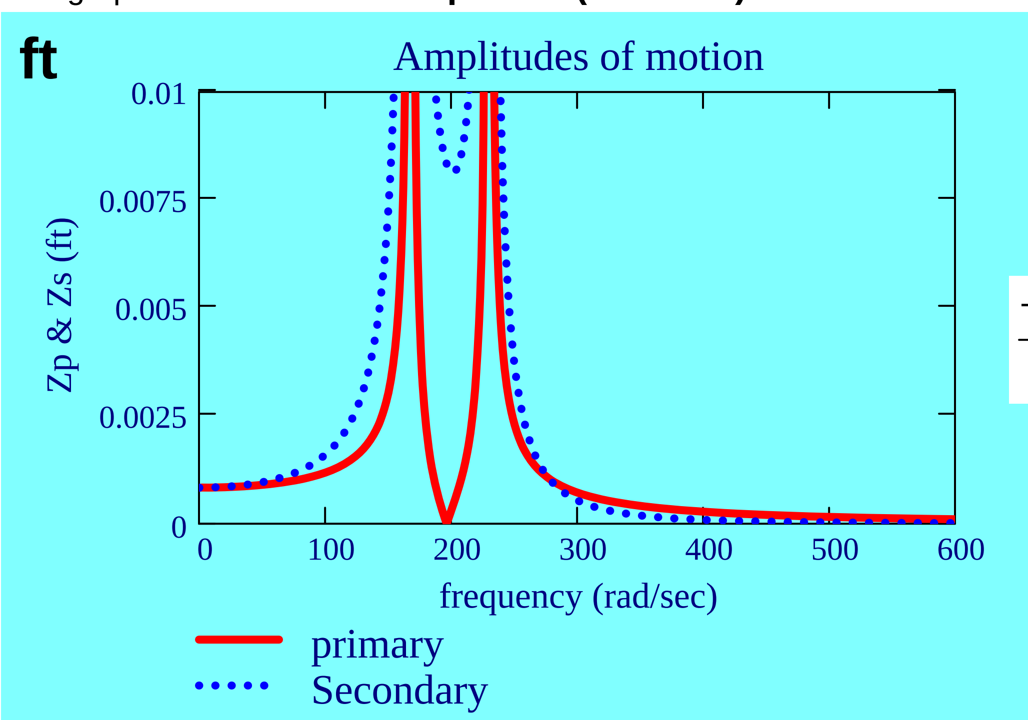
The graph below shows the **FRF (amplitude and phase)** of the vibration absorber:



$$\frac{K_p}{K_s} = a = 10$$

Note the null amplitude of motion for primary system excited at the ORIGINAL system natural frequency. In the graph, Zp,Zs >0 means in phase with the external force, Zp, Zs <0 means 180 deg out of phase with external periodic force.

The graph belos shows the **amplitude (absolute) of** the vibration absorber:



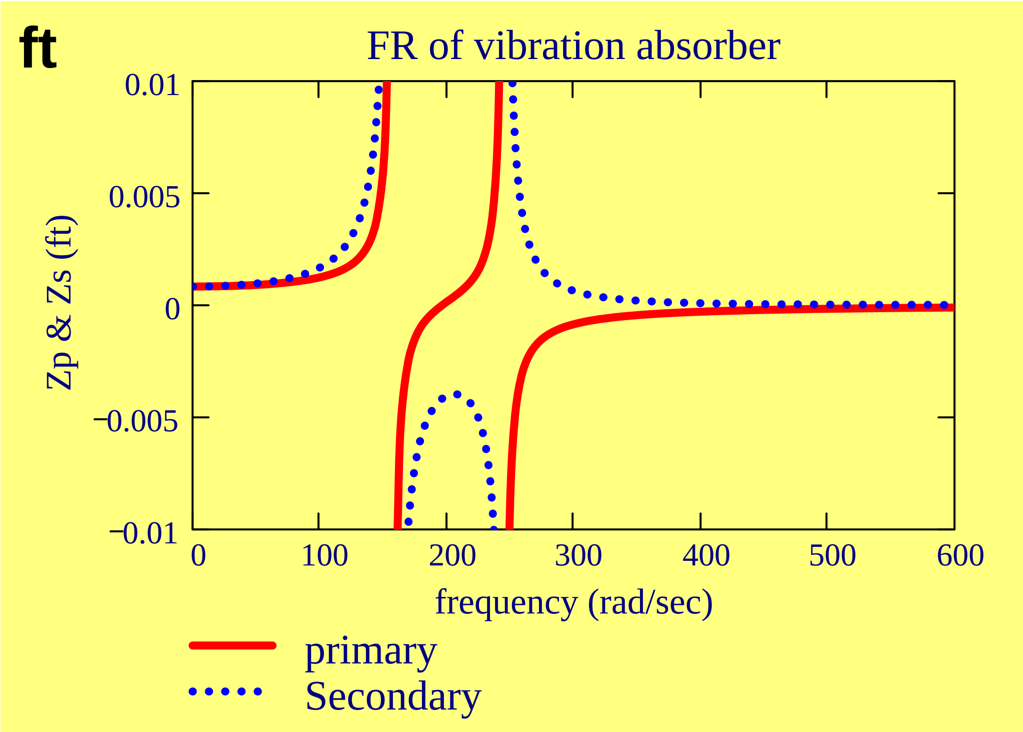
$$\frac{K_p}{K_s} = a = 10$$

$$\frac{-F_o}{K_s} = -8.333 \times 10^{-3}\,\mathrm{ft}$$

$$\omega_{np} = 196.491\,\frac{\mathrm{rad}}{\mathrm{sec}}$$

Note the amplitude of motion is zero for the primary system when excited at its original natural frequency. The secondary system does have a large amplitude of motion and is out of phase 180 degrees with the excitation force.
Note that the addition of the secondary (K-M) system renders a 2-DOF system with two natural frequencies, one above and one below the original natural frequency.
In general, the smaller the magnitude of the secondary stiffness and mass, the larger the amplitude c motion for the secondary system since it is extremely flexible. The system natural frequencies (1 and also tend to approach that of the original natural frequency

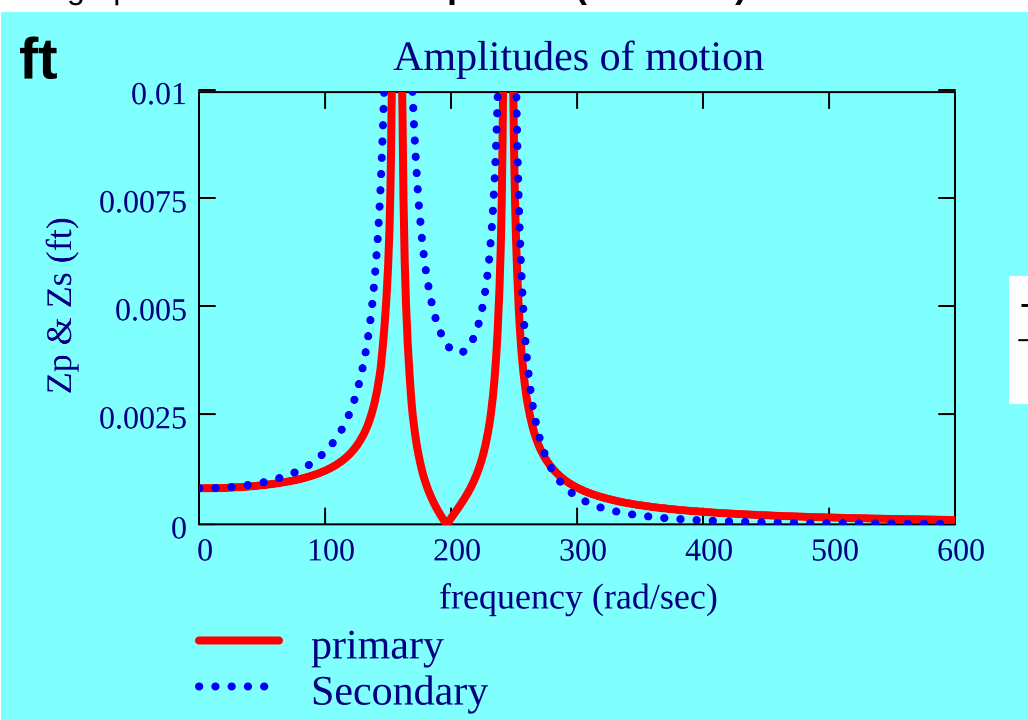
The graph below shows the **FRF (amplitude and phase)** of the vibration absorber:



$$\frac{K_p}{K_s} = a = 5$$

Note the null amplitude of motion for primary system excitated at the ORIGINAL system natural frequency. In the graph, Zp,Zs >0 means in phase with the external force, Zp, Zs <0 means 180 deg out of phase with external periodic force.

The graph belos shows the **amplitude (absolute) of** the vibration absorber:

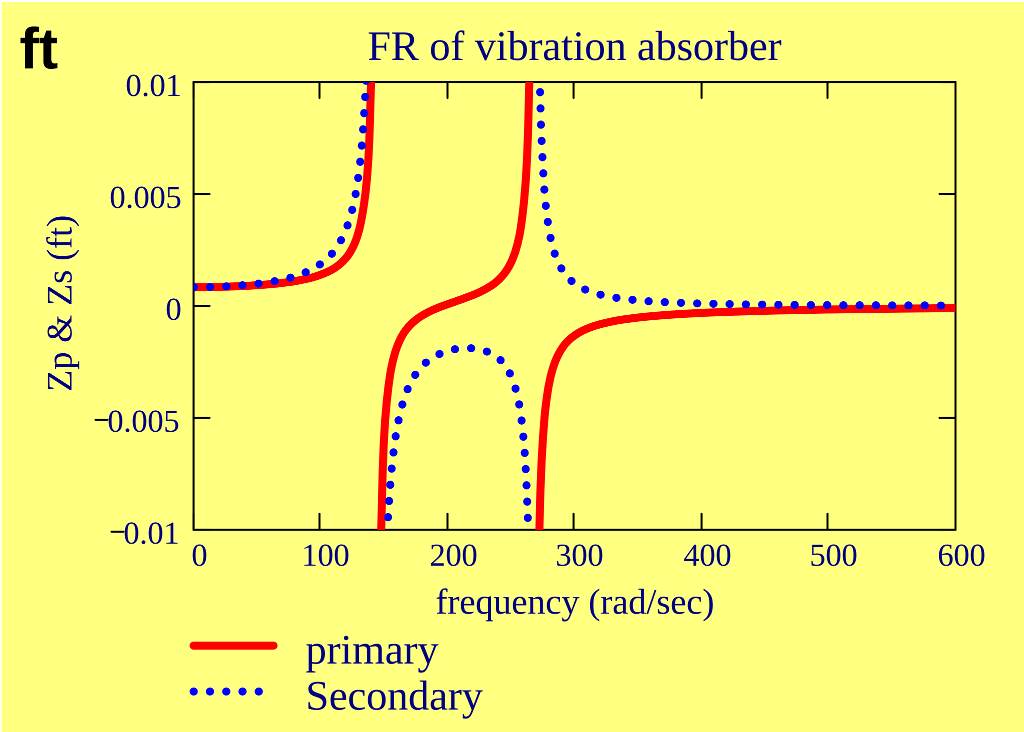


$$\frac{K_p}{K_s} = a = 5$$

$$\frac{-F_o}{K_s} = -4.167 \times 10^{-3}\,\text{ft}$$

$$\omega_{np} = 196.491\,\frac{\text{rad}}{\text{sec}}$$

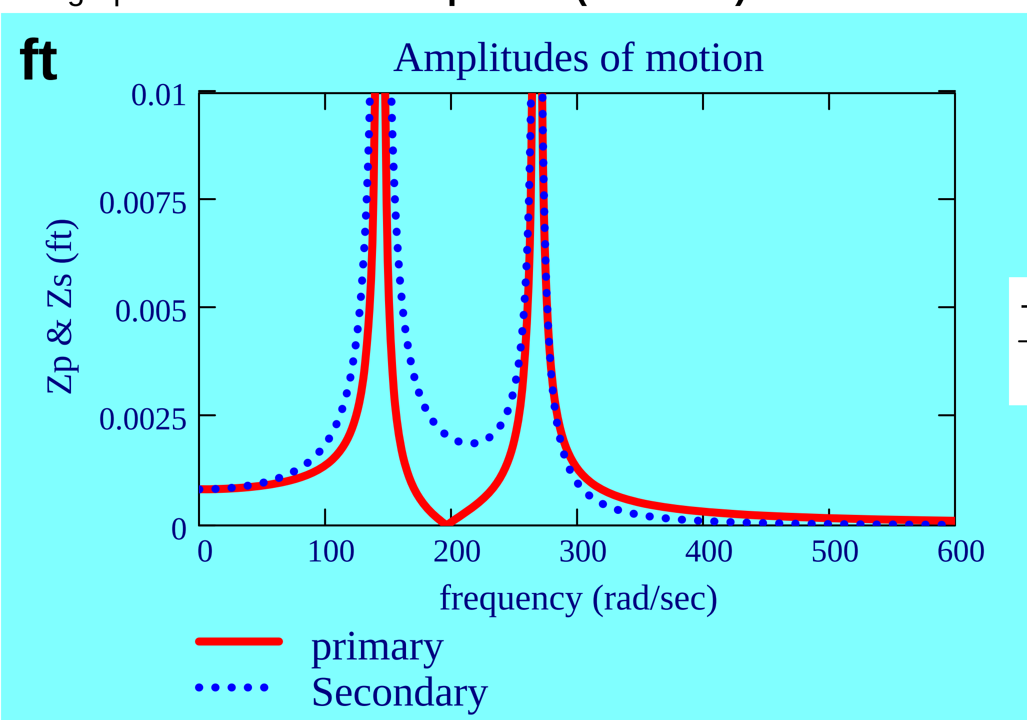
The graph below shows the **FRF (amplitude and phase)** of the vibration absorber:



$$\frac{K_p}{K_s} = a = 2.5$$

Note the null amplitude of motion for primary system excited at the ORIGINAL system natural frequency. In the graph, Zp,Zs >0 means in phase with the external force, Zp, Zs <0 means 180 deg out of phase with external periodic force.

The graph belos shows the **amplitude (absolute) of** the vibration absorber:

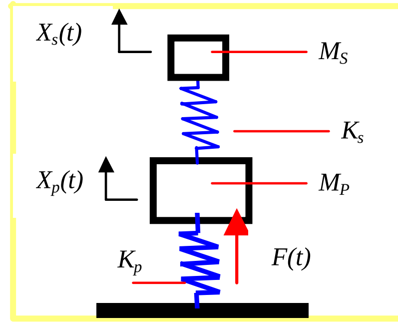


$$\frac{K_p}{K_s} = a = 2.5$$

$$\frac{-F_o}{K_s} = -2.083 \times 10^{-3}\,\mathrm{ft}$$

$$\omega_{np} = 196.491\,\frac{\mathrm{rad}}{\mathrm{sec}}$$

Design of vibration absorber



The equations of motion for the 2-DOF system are:

$$\begin{pmatrix} M_P & 0 \\ 0 & M_S \end{pmatrix} \cdot \frac{d^2}{dt^2} \begin{pmatrix} X_P \\ X_S \end{pmatrix} + \begin{pmatrix} K_P + K_S & -K_S \\ -K_S & K_S \end{pmatrix} \cdot \begin{pmatrix} X_P \\ X_S \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \cdot \sin(\Omega \cdot t) \qquad [1]$$

where primary system has::

$M_P := 50 \cdot kg$ $\quad K_P := 20000 \cdot \frac{N}{m}$ $\quad F := 1000 \cdot N$

$\omega_n = 20 \frac{rad}{s}$ $\quad \omega_n := \left(\frac{K_P}{M_P}\right)^{.5}$

Ranges of excitation frequency:

$\omega_{min} := 16 \cdot \frac{rad}{s}$ $\quad \omega_{max} := 24 \cdot \frac{rad}{s}$

The system response is of the form:

$$\begin{pmatrix} X_P \\ X_S \end{pmatrix} = \begin{pmatrix} Z_P \\ Z_S \end{pmatrix} \cdot \sin(\Omega \cdot t) \qquad [2]$$

Substitution of Eq. [2] into [1] leads to:

$$\begin{pmatrix} K_P + K_S - \Omega^2 \cdot M_P & -K_S \\ -K_S & K_S - \Omega^2 \cdot M_S \end{pmatrix} \cdot \begin{pmatrix} Z_P \\ Z_S \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \qquad [3]$$

1) A **tuned absorber** is designed so that $Z_P = 0$ i.e. no motion of the primary mass.

for operation at **Ω=ωn** $\quad \Omega = \left(\frac{K_P}{M_P}\right)^{.5} = \left(\frac{K_S}{M_S}\right)^{.5}$

Thus, from the first of eqns (3)

$$Z_S = \frac{-F}{K_S} \qquad [4]$$

Design absorber (select Ks & Ms) that satisfy operation within range of excitation frequencies:

$\omega_p := \left(\frac{K_P}{M_P}\right)^{.5}$

The determinat of the system of equations [3] is

$$\Delta(\Omega) = \left(K_P + K_S - \Omega^2 \cdot M_P\right) \cdot \left(K_S - \Omega^2 \cdot M_S\right) - K_S^{\,2} \qquad [5]$$

$\omega_p = 20 \frac{rad}{s}$

Let: $\lambda = \Omega^2 \cdot \frac{M_P}{K_P}$ $a = \frac{K_s}{K_P} = \frac{M_s}{M_P}$ stiffness ratio = mass ratio for tuned absorber

Expand Eq. (5), i.e. the characteristic equation:

$$\Delta(\lambda) = \left(1 + a - \lambda^1\right)\cdot(1-\lambda)\cdot a - a^2 = a\cdot\left[\left(1 + a - \lambda^1\right)\cdot(1-\lambda) - a\right]$$

$$\Delta(\lambda) = a\cdot\left(1 + a - \lambda - \lambda - \lambda\cdot a + \lambda^2 - a\right) = a\cdot\left(1 - 2\cdot\lambda - \lambda\cdot a + \lambda^2\right)$$

$$\Delta(\lambda) = \lambda^2 - \lambda\cdot(2+a) + 1 = 0 \quad [6]$$

$$(2+a)^2 - 4 = 4\cdot a + a^2$$

The roots of the characteristic equations are:

lowest: $\lambda_1(a) := \frac{(2+a) - \left(4\cdot a + a^2\right)^{.5}}{2}$ **highest** $\lambda_2(a) := \frac{(2+a) + \left(4\cdot a + a^2\right)^{.5}}{2}$

Let

$$\lambda_{min} := \left(\frac{\omega_{min}}{\omega_n}\right)^2 \qquad \lambda_{max} := \left(\frac{\omega_{max}}{\omega_n}\right)^2$$

Given the max and min values then, from eqn. (6)

$$a\_(\lambda) := \frac{\left(1 + \lambda^2\right)}{\lambda} - 2$$

$a_{max} := a\_\left(\lambda_{min}\right)$ $a_{max} = 0.202$

$a_{min} := a\_\left(\lambda_{max}\right)$ $a_{min} = 0.134$

for $a := a_{min}$

$$\lambda_1(a) := \frac{(2+a) - \left(4\cdot a + a^2\right)^{.5}}{2} \qquad \lambda_2(a) := \frac{(2+a) + \left(4\cdot a + a^2\right)^{.5}}{2}$$

$Z_s := \frac{-F}{a\cdot K_P}$

$Z_p := 0\cdot m$

$$\omega_1 := \left(\lambda_1(a)\cdot\frac{K_P}{M_P}\right)^{.5} \qquad \omega_2 := \left(\lambda_2(a)\cdot\frac{K_P}{M_P}\right)^{.5}$$

$\omega_{min} = 16\frac{rad}{s}$

$Z_s = -0.372\,m$ $\omega_1 = 16.667\frac{rad}{s}$ $\omega_2 = 24\frac{rad}{s}$

$\omega_{max} = 24\frac{rad}{s}$

$K_{min} := a \cdot K_P$ $\quad M_{min} := a \cdot M_P$

$K_{min} = 2.689 \times 10^3 \frac{N}{m}$ $\quad M_{min} = 6.722\,kg$

for

$a := a_{max}$

$Z_s := \frac{-F}{a \cdot K_P}$

$$\lambda_1(a) := \frac{(2+a) - \left(4 \cdot a + a^2\right)^{.5}}{2} \qquad \lambda_2(a) := \frac{(2+a) + \left(4 \cdot a + a^2\right)^{.5}}{2}$$

$$\omega_1 := \left(\lambda_1(a) \cdot \frac{K_P}{M_P}\right)^{.5} \qquad \omega_2 := \left(\lambda_2(a) \cdot \frac{K_P}{M_P}\right)^{.5}$$

$Z_p := 0 \cdot m$

$Z_s = -0.247\,m$

$\omega_1 = 16 \frac{rad}{s}$ $\quad \omega_2 = 25 \frac{rad}{s}$

$\omega_{min} = 16 \frac{rad}{s}$

$\omega_{max} = 24 \frac{rad}{s}$

$K_{max} := a \cdot K_P$ $\quad M_{max} := a \cdot M_P$

DESIGNER selects secondary system with since natural frequencies are outside operating range

$a_{max} = 0.202$

$K_{max} = 4.05 \times 10^3 \frac{N}{m}$

$M_{max} = 10.125\,kg$

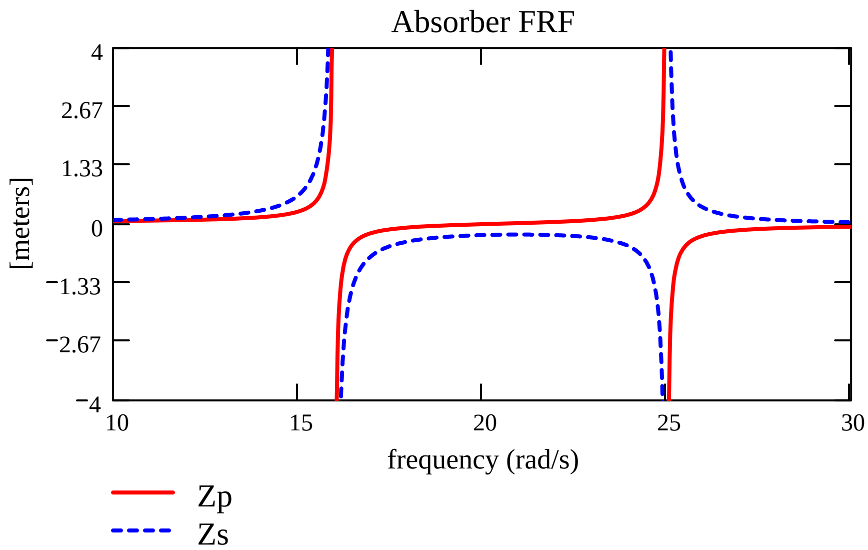
Build Absorber FRF

Now let's graph the amplitude and phase lag of frequency response function for both absorbers (primary and secondary mass motions):
Values >0 mean phase lag of 0 degrees, values <0 mean phase lag of -180 degrees with respect to forcing function.
Passing through the natural frequencies gives a phase lag of -90 degrees.
Amplitudes become unbounded while crossing the system natural frequencies.

$$\omega_{nP} = 20\,\frac{rad}{s}$$

$$a := a_{max}$$

$$a = 0.202$$



$$Z_{s\omega p} := \frac{-F}{a \cdot K}$$

$$Z_{s\omega p} = -0.247\,m$$

$$\omega_1(a) = 16\,\frac{rad}{s}$$

$$\omega_2(a) = 25\,\frac{rad}{s}$$

==================

$$\omega_{min} = 16\,\frac{rad}{s}$$

$$\omega_{max} = 24\,\frac{rad}{s}$$