# Notes 10.

# Thermohydrodynamic Bulk-Flow Model in Thin Film Lubrication

## General flow characteristics in oil lubricated fluid film bearings

a) Incompressible liquids of large viscosity (mineral oils)
b) Dominance of shear driven (*Couette*) flow over pressure (*Poiseuille*) driven flow
c) Fluid inertia and flow turbulence are usually NOT important (low circumferential flow Reynolds numbers)
d) Heat transfer to bearing cartridge and to/from shaft are important along with mechanical deformations induced by temperature gradients
e) Fluid temperature gradient along axial plane is negligible
f) Thermal effects change the lubricant viscosity and operating clearance, thus affecting significantly to the bearing static load performance

Thermohydrodynamic analyses are important in heavily loaded hydrodynamic bearings such as pressure dam bearings, tilting pad bearings, etc. See Notes 7

## General flow characteristics in process fluid film bearings

applicable to damper annular seals and hybrid (hydrostatic + hydrodynamic) bearings

a) Process liquids have low viscosity (water, $R_{134}$, $LH_2$, LOx)
b) Material compressibility important, low bulk modulues ($LH_2$)
c) Large pressure drops along axial direction with significant mass flow rates (annular damper seals & hydrostatic bearings – up to 6,000 psig in cryogenic turbopumps)
d) Large heat capacity for transport of energy along axial direction
e) Large rotor speeds (up to 100 krpm) will induce large shear flow energy dissipation
f) Typically use macro-textured surfaces (roughened stator) to avoid generation of cross-coupled stiffness and to promote dynamic stability
g) Inlet fluid flow circumferential swirl is important (for rotordynamic stability)

These operation characteristics determine the need to account for

a) Flow turbulence (induced by shaft rotation and pressure driven flow conditions)
b) Fluid inertia effects – temporal and advective types.
c) Fluid properties depend on pressure and temperature (needs equation of state)
d) Adequate physics based modeling of machined surface texture (roughness)
e) Two-phase flow conditions under certain operating regimes

# Bulk-Flow Equations for Thin Fluid Films

The fluid flow within a thin film region, see Fig.1, is governed by the continuity (mass conservation), momentum and energy transport equations. In the flow region $\{x \in (0, \pi D), y \in (0, H(x,z,t)), z \in (0, L)\}$, the smallness of the film thickness allows a simplification of the general transport equations.

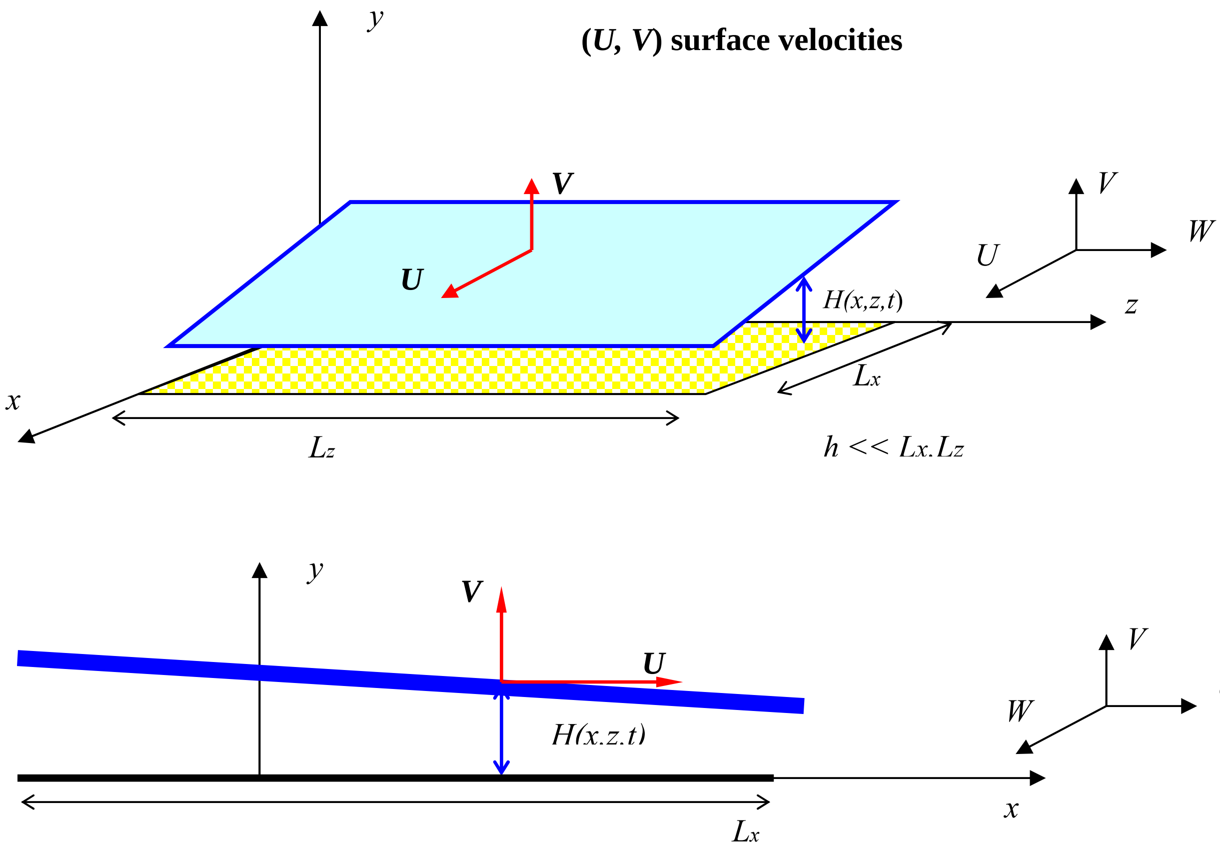



**Figure 1. Geometry of flow region in a fluid film bearing *($H$<<$L_x$,$L_z$)***

The coordinates in the plane of the bearing are circumferential ($x$=$R\theta$), axial ($z$), and across the film ($y$). Let $\{\tilde{U}, \tilde{V}, \tilde{W}\}, \tilde{P}, \tilde{T}$ be the fluid velocity field components along the *(x, y, z)* directions, the fluid pressure and its temperature, respectively.

The thin film fluid flow equations are (see Notes 8):

Mass conservation
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \tilde{U})}{\partial x} + \frac{\partial(\rho \tilde{V})}{\partial y} + \frac{\partial(\rho \tilde{W})}{\partial z} = 0 \qquad (1)$$

Circumferential-momentum transport $$\rho \frac{D\tilde{U}}{Dt} = -\frac{\partial \tilde{P}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \quad (2)$$

Axial-momentum transport $$\rho \frac{D\tilde{W}}{Dt} = -\frac{\partial \tilde{P}}{\partial z} + \frac{\partial \tau_{zy}}{\partial y} \quad (3)$$

Cross film momentum $$\frac{\partial \tilde{P}}{\partial y} = 0 \quad (4)$$

Energy-Transport *(*Bird et Al., 1960*)*

$$\rho C_p \frac{D\tilde{T}}{Dt} + \frac{\rho}{2}\frac{D}{Dt}\left(\tilde{U}^2 + \tilde{W}^2\right) = \frac{\partial}{\partial y}\left(K\frac{\partial \tilde{T}}{\partial y}\right) + \tilde{T}\beta_t \frac{D\tilde{P}}{Dt} - \left(\tilde{U}\frac{\partial \tilde{P}}{\partial x} + \tilde{W}\frac{\partial \tilde{P}}{\partial z}\right) + \frac{\partial}{\partial y}\left(\tilde{U}\tau_{xy} + \tilde{W}\tau_{zy}\right) \quad (5)$$

where $$\frac{D}{Dt} = \frac{\partial}{\partial t} + \tilde{U}\frac{\partial}{\partial x} + \tilde{V}\frac{\partial}{\partial y} + \tilde{W}\frac{\partial}{\partial z} \quad (6)$$

is the material derivative. $\{\rho, \mu, C_P, k, \beta_t\}$ represent the fluid properties of density, viscosity, specific heat, thermal conductivity, and volumetric expansion coefficient, respectively.

**In a turbulent flow, the effect of the turbulent mixing far outweighs the fluid molecular diffusivity. In consequence, the temperature raise by viscous dissipation tends to be distributed uniformly across the film thickness. Thus, temperature gradients across the film (*y*-dir) are confined to (very thin) boundary layers attached to the bearing and journal surfaces. The fluid velocity field presents the same characteristics in regions without reversed flow or recirculation.**

**Bulk-flow primitive variables** (velocities and temperature) represent average quantities across the film thickness, i.e.,

$$U = \frac{1}{H}\int_0^H \tilde{U}\,dy;\ W = \frac{1}{H}\int_0^H \tilde{W}\,dy;\ T = \frac{1}{H}\int_0^H \tilde{T}\,dy \quad \textbf{(7)}$$

Integration of Eqs. (1-5) across the film thickness renders the **bulk-flow equations (fully developed condition)**:

Continuity: $$\frac{\partial(\rho H)}{\partial t} + \frac{\partial(\rho H U)}{\partial x} + \frac{\partial(\rho H W)}{\partial z} = 0 \quad (8)$$

Circumferential momentum: $$\frac{\partial(\rho HU)}{\partial t}+\frac{\partial(\rho HU^2)}{\partial x}+\frac{\partial(\rho HUW)}{\partial z}=-H\frac{\partial P}{\partial x}+\tau_{xy}\Big|_0^H \tag{9}$$

Axial momentum: $$\frac{\partial(\rho HW)}{\partial t}+\frac{\partial(\rho HUW)}{\partial x}+\frac{\partial(\rho HW^2)}{\partial z}=-H\frac{\partial P}{\partial z}+\tau_{zy}\Big|_0^H \tag{10}$$

Energy transport:

$$C_p\left[\frac{\partial(\rho HT)}{\partial t}+\frac{\partial(\rho HUT)}{\partial x}+\frac{\partial(\rho HWT)}{\partial z}\right]+\frac{1}{2}\left[\frac{\partial(\rho HV_t^2)}{\partial t}+\frac{\partial(\rho HUV_t^2)}{\partial x}+\frac{\partial(\rho HWV_t^2)}{\partial z}\right]$$

$$=Q_S+T\beta_t H\frac{\partial P}{\partial t}+(T\beta_t-1)H\left(U\frac{\partial P}{\partial x}+W\frac{\partial P}{\partial z}\right)+R\Omega\,\tau_{xy}\Big|^H \tag{11}$$

Where $V_t=\sqrt{U^2+W^2}$ is the bulk-flow speed, and $Q_S=h_B(T-T_B)+h_J(T-T_J)$ (12)
is the heat flowing from the film to the bounding (bearing and journal) surfaces at temperatures $T_B$ and $T_J$. Above, $h_B$ and $h_J$ denote heat transfer convection coefficients to the bearing and journal surfaces. The fluid properties (density, viscosity and specific heat) depend on the fluid thermo physical state, i.e., functions of the fluid pressure and temperature.

$$\rho=\rho(P,T),\;\; \mu=\mu(P,T),\;\; C_p=C_p(P,T),...,etc. \tag{13}$$

From the bulk-flow theory for turbulence in thin film flows, the **wall shear stress differences** are (Hirs, 1973, Launder and Leschziner, 1978):

$$\tau_{xy}\Big|_0^H=-\frac{\mu}{H}\left(k_x U-k_J\frac{R\Omega}{2}\right);\quad \tau_{zy}\Big|_0^H=-\frac{\mu}{H}(k_z W);\quad \tau_{xy}\Big|^H=\frac{H}{2}\frac{\partial P}{\partial x}+\frac{\mu}{4H}\left[Uk_B-(U-R\Omega)k_J\right] \tag{14}$$

where the turbulent flow shear parameters $(k_x,k_z)$ and $(k_J,k_B)$ are local functions of the Reynolds numbers and friction factors based on the Moody friction factor. See Notes 8.

Note that for the volumetric expansion coefficient, $\beta_t=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_\rho$

$$\beta_t T=\begin{cases}0, & \text{for incompressible liquids}\\ 1, & \text{for ideal gases}\end{cases} \tag{15}$$

For example, $\beta_t T$ for $LH_2$ is not in the range of 0 to 1.

Substitution of the bulk-flow momentum Eqs. (9-10) into Eq. (11) and using the mass conservation principle, Eq. (8), renders a more suitable form of the energy transport equation :

$$C_p\left[\frac{\partial(\rho H T)}{\partial t}+\frac{\partial(\rho HU T)}{\partial x}+\frac{\partial(\rho HWT)}{\partial z}\right]+Q_S=$$

$$T\beta_t H\left(\frac{\partial P}{\partial t}+U\frac{\partial P}{\partial x}+W\frac{\partial P}{\partial z}\right)+R\Omega\,\tau_{xy}\Big|^{H}-U\,\tau_{xy}\Big|_0^H-W\,\tau_{zy}\Big|_0^H \qquad \textbf{(16a)}$$

And after substitution of the wall shear stress differences:

$$C_p\left[\frac{\partial(\rho H T)}{\partial t}+\frac{\partial(\rho HU T)}{\partial x}+\frac{\partial(\rho HWT)}{\partial z}\right]+Q_S=$$

$$T\beta_t H\left(\frac{\partial P}{\partial t}+U\frac{\partial P}{\partial x}+W\frac{\partial P}{\partial z}\right)+H\frac{\Omega R}{2}\frac{\partial P}{\partial x}+\frac{\mu}{H}\left[k_x\left(U^2+W^2+\frac{\Omega R}{2}U\right)+k_J\,\Omega R\left(\frac{\Omega R}{4}-U\right)\right] \qquad \textbf{(16b)}$$

This equation shows the energy transport and balance in the fluid film as:

**CONVECTION + DIFFUSION=COMPRESSION WORK + DISSIPATION**
**(Energy Disposed) = (Energy Generated)**

In annular seals and hydrostatic bearings, the variation of temperature along the axial direction and the energy needed for compression work are retained since the pressure drops across a seal or bearing can be quite large. These conditions differentiate this development from conventional THD analyses of incompressible fluid film journal bearings, for example (see Notes 7)

## Dimensionless Bulk-Flow Equations

Define dimensionless coordinates

$$\overline{x}=\frac{x}{R};\quad \overline{z}=\frac{z}{R};\quad h=\frac{H}{c_*};\quad \tau=\omega t;$$

and flow variables

$$u=\frac{U}{U_*};\quad w=\frac{W}{U_*};\quad p=\frac{P}{P_*};\quad \overline{T}=\frac{T}{T_*}; \qquad \textbf{(17)}$$

and properties

$$\overline{\rho}=\frac{\rho}{\rho_*};\quad \overline{\mu}=\frac{\mu}{\mu_*};\quad \overline{C_p}=\frac{C_p}{C_{p*}};\quad \overline{\beta}_t=\beta_t T_*$$

with

$$U_*=\frac{c_*^2 P_*}{\mu_* R} \qquad (18)$$

as a **characteristic flow speed due to pressure**. The subscript * denotes characteristic values. In dimensionless form, the flow equations in the film lands become:

Continuity

$$\frac{\partial}{\partial x}(\bar{\rho}hu)+\frac{\partial}{\partial z}(\bar{\rho}hw)+\sigma\frac{\partial\left(\bar{\rho}h\right)}{\partial\tau}=0; \qquad (19)$$

Circumferential Momentum

$$-h\frac{\partial p}{\partial x}=\frac{\bar{\mu}}{h}\left(k_x u-k_J\frac{\Lambda}{2}\right)+\mathrm{Re}_s\frac{\partial}{\partial\tau}\left(\bar{\rho}hu\right)+\mathrm{Re}_p^*\left[\frac{\partial}{\partial x}\left(\bar{\rho}hu^2\right)+\frac{\partial}{\partial z}\left(\bar{\rho}huw\right)\right] \qquad (20)$$

Axial-Momentum

$$-h\frac{\partial p}{\partial z}=\frac{\bar{\mu}}{h}\left(k_z w\right)+\mathrm{Re}_s\frac{\partial}{\partial\tau}\left(\bar{\rho}hw\right)+\mathrm{Re}_p^*\left[\frac{\partial}{\partial x}\left(\bar{\rho}huw\right)+\frac{\partial}{\partial z}\left(\bar{\rho}hw^2\right)\right] \qquad (21)$$

Energy Transport

$$\bar{C}_p\,\mathrm{Re}_s\frac{\partial}{\partial\tau}\left(\bar{\rho}h\bar{T}\right)+\bar{C}_p\,\mathrm{Re}_p^*\left[\frac{\partial}{\partial x}\left(\bar{\rho}hu\bar{T}\right)+\frac{\partial}{\partial z}\left(\bar{\rho}hw\bar{T}\right)\right]= \qquad (22)$$

$$\mathrm{Re}_p^*\,\bar{Q}_s+E_c\left\{\bar{\beta}_t h\left(\sigma\frac{\partial p}{\partial\tau}+u\frac{\partial p}{\partial x}+w\frac{\partial p}{\partial z}\right)\bar{T}+h\frac{\Lambda}{2}\frac{\partial p}{\partial x}+\frac{\bar{\mu}}{h}\left[k_x\left(v_t^2+\frac{1}{2}u\Lambda\right)+k_J\left(\frac{1}{4}\Lambda^2-u\Lambda\right)\right]\right\}$$

The **dimensionless flow parameters** are

$$\sigma=\frac{R\,\omega}{U_*};\quad \Lambda=\frac{R\,\Omega}{U_*};\quad E_c=\frac{U_*^2}{T_* C_{p*}};$$

$$\mathrm{Re}_S=\frac{\rho_*\,\omega c_*^2}{\mu_*}=\sigma\,\mathrm{Re}_p^*;\quad \mathrm{Re}_p=\frac{\rho_* U_* c_*}{\mu_*};\quad \mathrm{Re}_p^*=\mathrm{Re}_p\left(\frac{c_*}{R}\right) \qquad (23)$$

The frequency $(\sigma)$ and speed $(\Lambda)$ numbers denote the importance of squeeze film and shear flow effects relative to the pressure induced flow, respectively. The reference Reynold numbers $(\mathrm{Re}_p)$ denotes the ratio of fluid advection forces to viscous flow induced forces due to pressure. Recall that in hydrostatic bearings and annular seals, the large pressure differentials can generate flow turbulence even without journal rotation. The Eckert number $(E_c)$ denotes the ratio of kinetic energy to heat convection in the fluid film. The ratios $(\mathrm{Re}_p/E_c)$ or $(\mathrm{Re}_S/E_c)$ represent the effect of heat convection relative to shear dissipation.

## APPENDIX. Heat Transfer Convection Coefficients to Bearing and Journal

In the bulk-flow model, the heat transfer from the fluid film to the bounding surfaces is:

$$Q_S = h_B\left(T - T_B\right) + h_J\left(T - T_J\right) \tag{A.1}$$

where, $h_B$ and $h_J$ are the heat transfer convection coefficients to the bearing and journal surfaces, respectively. The Reynolds-Colburn analogy between fluid friction and heat transfer determines the heat convection coefficients (Holman, 1986).

The average heat transfer over the entire laminar/turbulent boundary is:

$$S_t\, \wp_r^{2/3} = \frac{f}{2} \tag{A.2}$$

where:

$$S_t = \frac{h_t}{\rho\, C_p\, V_t} \quad \text{(Stanton number)} \tag{A.3}$$

$$\wp_r = \frac{C_p\, \mu}{k} \quad \text{(Prandtl number)} \tag{A.4}$$

$$f = a_m\left[1 + \left(c_m \frac{r}{H} + \frac{b_m}{R_e}\right)^{e_m}\right] \tag{A.5}$$

is the Fanning friction factor based on Moody friction diagram. From the relationships above, the heat transfer convection coefficient is:

$$h_t = \frac{1}{2}\frac{\rho\, C_p\, V_t\, f}{\wp_r^{2/3}} \tag{A.6}$$

and by analogy,

$$h_B = \frac{1}{2}\frac{\rho\, C_p\, V_B f_B}{\wp_r^{2/3}}; \quad h_J = \frac{1}{2}\frac{\rho\, C_p\, V_J f_J}{\wp_r^{2/3}} \tag{A.7}$$

Where ($V_B$, $V_J$) and ($f_B$, $f_J$) are the fluid velocities and friction factors relative to the bearing and journal surfaces, respectively.

The archival literature presents many other formulas – empirically based - for turbulent flow heat transfer coefficients (Holman, 1986). These formulas depend on the heat transfer process, for example a constant wall temperature or a constant heat flux magnitude and for a fully developed condition or one of evolving thermal flow conditions. Eq. (A.6) is used because of its simplicity and ability to include surface texturing effects (through the friction factor).

# Numerical Analysis of Unsteady Turbulent Bulk-Flow in Fluid Film Bearings

Consider the unsteady flow fully developed turbulent bulk-flow in the thin film lands of a fluid film bearing or an annular seal. The governing equations are:

Continuity:

$$\frac{\partial}{\partial x_i}(\rho H U_i)+\frac{\partial}{\partial t}(\rho H)=0 \qquad i=1(x),2(z) \tag{1}$$

Momentum:

$$-H\frac{\partial P}{\partial x_i}=\frac{\mu\left(k_i U_i - k_J\frac{\bar{\Lambda}_i}{2}\right)}{H}+\frac{\partial}{\partial t}(\rho H U_i)+\frac{\partial}{\partial x_i}(\rho H U_i U_j) \tag{2}$$

*i,j=1(x),=2 (y)*

Energy:

$$C_p\left[\frac{\partial}{\partial t}(\rho H T)+\frac{\partial}{\partial x_i}(\rho H U_i T)\right]+Q_S=\beta_t H T\left(\frac{\partial P}{\partial t}+U_i\frac{\partial P}{\partial x_i}\right)+\Omega R\frac{H}{2}\frac{\partial P}{\partial x}$$
$$+\frac{\mu}{H}\left[k_x\left(U_x^2+U_y^2+\frac{\Omega R U_x}{2}\right)+k_J\left(\frac{\Omega^2 R^2}{4}-U_x\Omega R\right)\right] \tag{3}$$

where *i, j = x, y* are the circumferential and axial coordinates[1]; $\bar{\Lambda}_x=\Omega R$ and $\bar{\Lambda}_y=0$ denote journal surface speeds in the *x*- and *y*-directions. $k_x, k_y, k_J$ are the turbulent flow shear parameters. The fluid properties; namely density, viscosity, specific heat, and thermal expansion coefficient, are thermodynamic variables, i.e.

$$\rho=\rho(P,T);\mu=\mu(P,T) \qquad C_p=C_p(P,T);\beta_t=\beta_t(P,T)$$

In Eq. (3), $Q_S=h_B(T-T_B)+h_J(T-T_J)$ is the heat flow conducted into the bearing and journal surfaces. The film thickness *H* in an aligned journal is

$$H=c_{(x,y)}+e_X\cos\theta+e_Y\sin\theta \tag{4}$$

where *c(x,y)* is the bearing or seal radial clearance function, and $\left(e_{X(t)},e_{Y(t)}\right)$ are time dependent journal center displacements along the inertial coordinates *(X,Y)*. See Figure 2 for a schematic view of the coordinate systems, eccentric journal (rotor) and a bearing (pad).

[1] Note the change in notation with coordinate *y* replacing *z* for the axial direction. Velocity *V* is in the axial direction. The discrepancy in notation will be fixed in the near future.

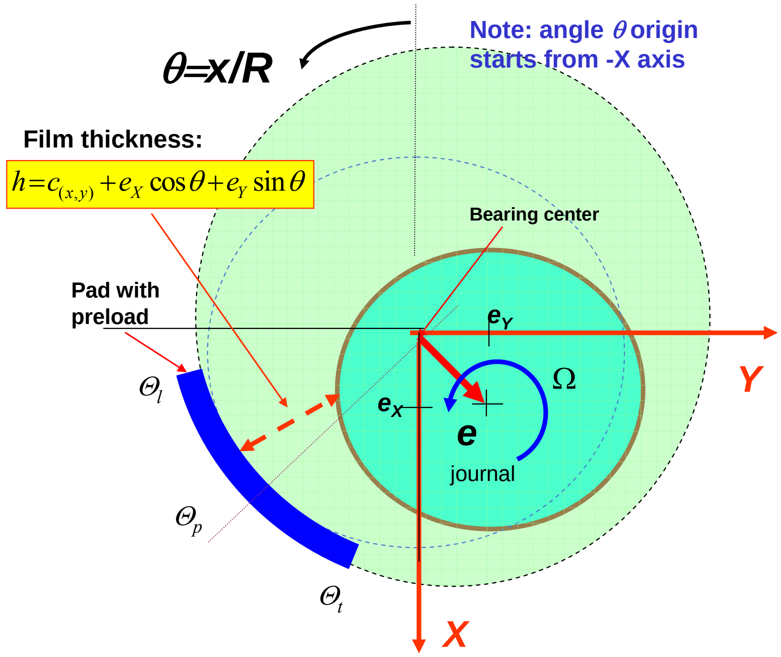



**Figure 2. Depiction of bearing pad, eccentric journal, and coordinate system**

## Dimensionless equations of motion

*Continuity:*

$$\frac{\partial}{\partial x}\left(\bar{\rho}\,h\,u_x\right)+\frac{\partial}{\partial y}\left(\bar{\rho}\,h\,u_y\right)+\sigma\frac{\partial}{\partial \tau}\left(\bar{\rho}\,h\right)=0 \qquad (5)$$

*Circumferential momentum:*

$$-h\frac{\partial p}{\partial x}=\frac{\left(k_x u_x-k_J\frac{\Lambda}{2}\right)\bar{\mu}}{h}+\mathrm{Re}_S\frac{\partial}{\partial \tau}\left(\bar{\rho}\,hu_x\right)+\mathrm{Re}_p^*\left[\frac{\partial}{\partial x}\left(\bar{\rho}\,hu_x^2\right)+\frac{\partial}{\partial y}\left(\bar{\rho}\,hu_xu_y\right)\right] \qquad (6.a)$$

*Axial momentum:*

$$-h\frac{\partial p}{\partial y}=\frac{\left(k_yU_y\right)\bar{\mu}}{h}+\mathrm{Re}_S\frac{\partial}{\partial \tau}\left(\bar{\rho}\,hu_y\right)+\mathrm{Re}_p^*\left[\frac{\partial}{\partial x}\left(\bar{\rho}\,hu_xu_y\right)+\frac{\partial}{\partial y}\left(\bar{\rho}\,hu_y^2\right)\right] \qquad (6.b)$$

where: $x=x/R$ ; $y=y/R$ ; $t=\tau/\omega$ ; $h=H/c_*$

$u_x=U_x/V_*$ $u_y=U_y/V_*$ $\Lambda=\Omega R/V_*$ ; $V_*=\frac{c^2P_{sa}}{\mu_*R}$ ; $\sigma=\omega R/V_*$

$p=(P-P_a)/P_{sa}$ ; $\bar{\rho}=\rho/\rho_*$ ; $\bar{\mu}=\mu/\mu_*$

$\mathrm{Re}_p^*=\mathrm{Re}_p\frac{c_*}{R}=\frac{\rho_*}{\mu_*}\frac{V_*}{R}c_*^{\,2}$ is a typical advection flow Reynolds number,

$\mathrm{Re}_S = \frac{\rho_*}{\mu_*}\omega c_*^2 = \mathrm{Re}_p^* \sigma$ is a typical squeeze film Reynolds number and $\omega$ is a characteristic whirl frequency (typically equal to the shaft rotational speed).

The flow domain is divided into a number of **staggered control volumes (*CVs*)**, as shown in Figure 3. Each control volume encloses a particular flow variable (circumferential and axial velocities, pressure and temperature) as a nodal quantity denoted by its **P** value. The boundaries of the *CV* are surfaces through which flow comes in or out. The control volumes are surrounded by nodal variables denoted as **E**ast, **W**est, **N**orth and **S**outh. The notation defines with lower-case the fluxes (mass, energy or momentum) through the surfaces of the *CVs,* i.e. **e**ast, **w**est, **n**orth and **s**outh.

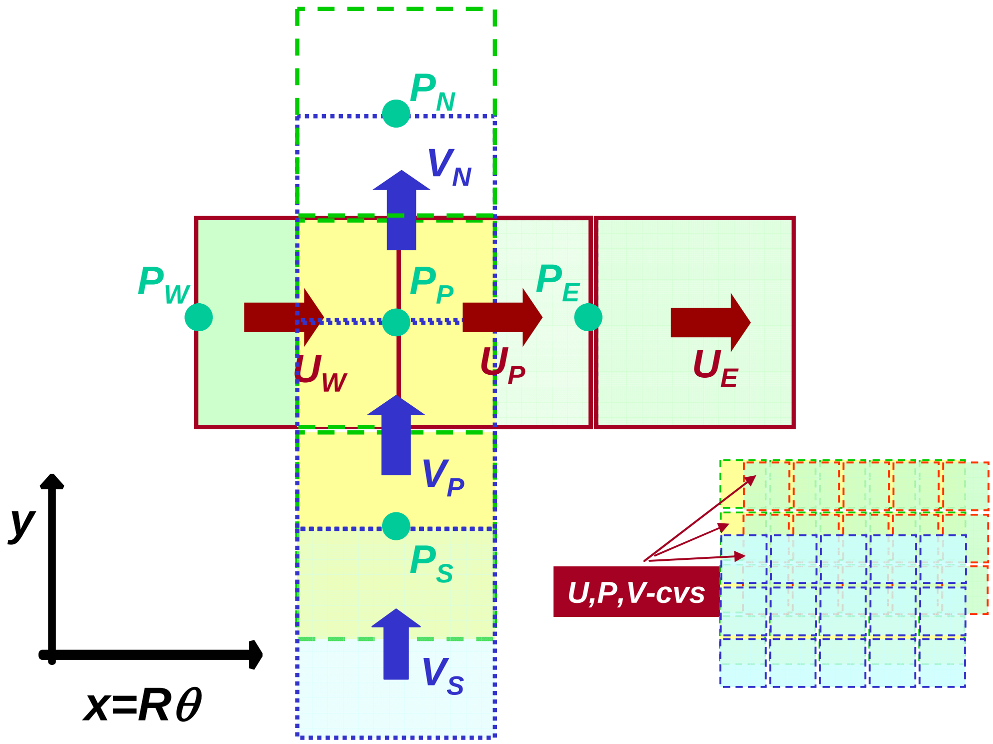
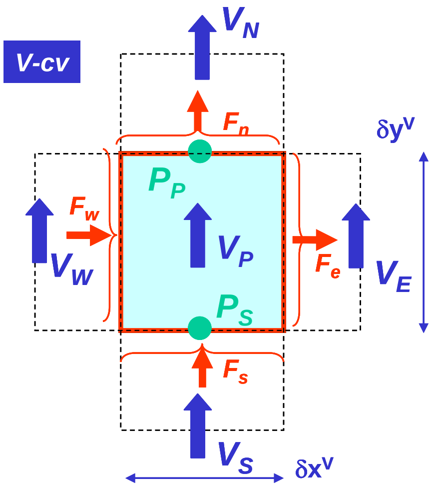



**Figure 3. Depiction of staggered control volumes for integration of flow equations**



Axial velocity control volume

## Integration of Axial Momentum Transport Equation

Integrate the axial momentum transport equation (6.b) over the axial velocity $V$-control volume.

**7**

$$\int_w^e\int_s^n -h\frac{\partial p}{\partial y}\,dx\,dy = \int_w^e\int_s^n \frac{(k_y u_y)\bar{\mu}}{h}\,dx\,dy + \int_w^e\int_s^n \left[\mathrm{Re}_S\frac{\partial}{\partial \tau}\left(\bar{\rho}hu_y\right) + \mathrm{Re}_p^*\left[\frac{\partial}{\partial x}\left(\bar{\rho}hu_xu_y\right) + \frac{\partial}{\partial y}\left(\bar{\rho}hu_y^2\right)\right]\right]dxdy \quad (7)$$

Consider the following approximations for the various terms in Eq. (7),

$$\int_w^e\int_s^n -h\frac{\partial p}{\partial y}dxdy = \int_w^e -h_p^V\left[\delta p\right]_s^n dx \simeq h_p^V\left(p_S - p_P\right)\delta x^V \quad (8.a)$$

i.e., assume the pressure is uniform over the ***s**outh* and ***n**orth* faces of the control volume, with an average uniform film thickness evaluated at the center of the *V-control volume*.

$$\int_w^e\int_s^n \frac{k_y\bar{\mu}\,u_y}{h}\,dx\,dy \simeq \left(\frac{k_y\bar{\mu}}{h}\right)_P^V V_P\,\delta x^V\delta y^V \quad (8.b)$$

i.e., assume an average film thickness, viscosity and turbulent shear coefficient $k_y$ for the whole control volume.

For the momentum flux terms, assume uniform circumferential flows $\left(\bar{\rho}hu_x\,\delta y^V\right)$ across the ***e**ast* and ***w**est* faces,

$$\int_w^e\int_s^n \frac{\partial}{\partial x}\left(\bar{\rho}hu_xu_y\right)dxdy = \int_S^n\left(\bar{\rho}hu_xu_y\right)_w^e dy \simeq \left(\bar{\rho}hu_xu_y\right)_w^e\delta y^V \quad (8.c)$$

And, a uniform axial flow across the ***s**outh* and ***n**orth* faces of the $V$-control volume.

$$\int_w^e\int_s^n \frac{\partial}{\partial y}\left(\bar{\rho}hu_yu_y\right)dxdy = \int_w^e\left(\bar{\rho}hu_yu_y\right)_S^n dx \simeq \left(\bar{\rho}hu_yu_y\right)_S^n\delta x^V \quad (8.d)$$

For the temporal (unsteady) term,

$$\int_w^e\int_s^n \frac{\partial}{\partial \tau}\left(\bar{\rho}hu_y\right)dxdy = \frac{\partial}{\partial \tau}\int_w^e\int_s^n\left(\bar{\rho}hu_y\right)dxdy \quad (8.e)$$

Since the control volume size is fixed in space. Thus, Eq. (7) over the $V$-control volume becomes:

$$h_P^V\left(p_S-p_P\right)\delta x^V \simeq \left(\frac{k_y\bar{\mu}}{h}\right)_P^V V_P\,\delta x^V\delta y^V + \mathrm{Re}_S\frac{\partial}{\partial\tau}\iint\left(\bar{\rho}hu_y\right)dxdy + \mathrm{Re}_p^*\left[\left(\bar{\rho}\,hu_xu_y\right)_w^e\delta y^V + \left(\bar{\rho}\,hu_yu_y\right)_s^n\delta x^V\right] \tag{9}$$

Before proceeding further, integration of the continuity equation (1) over the *V*-control volume gives

$$\left(\bar{\rho}\,hu_x\right)_w^e\delta y^V + \left(\bar{\rho}\,hu_y\right)_s^n\delta x^V + \sigma\frac{\partial}{\partial\tau}\int_w^e\int_S^n\left(\bar{\rho}\,h\right)dxdy \simeq 0 \tag{10}$$

The flow rates across the faces (***e,n,s,w***) of the control volume are denoted by

$$F_e^V = \left(\bar{\rho}\,hu_x\right)^e\delta y^V\,;\quad F_w^V = \left(\bar{\rho}\,hu_x\right)^w\delta y^V \tag{11}$$

$$F_n^V = \left(\bar{\rho}\,hu_y\right)^n\delta x^V\,;\quad F_S^V = \left(\bar{\rho}\,hu_y\right)^s\delta x^V$$

With these definitions, Eq. (10) becomes

$$F_e^V - F_w^V + F_n^V - F_s^V + \sigma\frac{\partial}{\partial\tau}\int_w^e\int_s^n\left(\bar{\rho}\,h\right)dxdy = 0 \tag{12}$$

which establishes a balance of flows (in and out) through the *V-CV* faces and equaling the rate of fluid mass accumulation within the *CV*.

The momentum flux terms in Eq. (9) are treated using the **upwind scheme** of Launder and Leschziner (1978). This scheme establishes a selection of velocity based on whether the flow is into or out of the face of a control volume. For example:

$$\left(\bar{\rho}\,hu_xu_y\right)^e\delta y^V = F_e^V\,u_y^e = \begin{vmatrix} F_e^V\,V_P & \text{if} & F_e^V > 0 \\ F_e^V\,V_E & \text{if} & F_e^V < 0 \end{vmatrix} \qquad \text{where } F_e^V = \bar{\rho}\left(hu_x\right)^e\delta y^V \tag{a}$$

That is, if flow leaves the ***e***-face, $F_e>0$, it *carries* the upstream velocity, $V_P$. On the other hand, if flow comes into the *e*-face, it carries the downstream velocity, $V_E$. This procedure is known as **UPWINDING**.

Define the following operator,

$$[\![a,0]\!] = \max\left(a,0\right) \tag{b}$$

Then, statement (a) can be conveniently written as

$$F_e^V u_y^e = [\![F_e^V,0]\!]\,V_P - [\![-F_e^V,0]\!]\,V_E$$

Hence, the momentum fluxes are written as:

$$\left(\bar{\rho}\,hu_xu_y\right)^w\delta y^V = F_w^V\,u_y^w = [\![F_w^V,0]\!]V_W - [\![-F_w^V,0]\!]V_P$$

$$
\begin{aligned}
\left(\bar{\rho}\, h u_x u_y\right)^e \delta y^v &= F_e^V u_y^e = \left[\!\left[ F_e^V, 0 \right]\!\right] V_P - \left[\!\left[ -F_e^V, 0 \right]\!\right] V_E \\
\left(\bar{\rho}\, h u_y u_y\right)^s \delta x^V &= F_s^V {}_s u_y^s = \left[\!\left[ F_s^V, 0 \right]\!\right] V_S - \left[\!\left[ -F_s^V, 0 \right]\!\right] V_P \\
\left(\bar{\rho}\, h u_y u_y\right)^n \delta x^V &= F_n^V u_y^n = \left[\!\left[ F_n^V, 0 \right]\!\right] V_P - \left[\!\left[ -F_n^V, 0 \right]\!\right] V_N
\end{aligned}
\tag{13}
$$

The differences in momentum fluxes in Eq. (9) become:

$$
\begin{aligned}
\left(\bar{\rho} h u_x u_y\right)_w^e \delta y^V + \left(\bar{\rho} h u_y u_y\right)_s^n \delta x^V &= F_e^V u_y^e - F_w^V u_y^w + F_n^V u_y^n - F_s^V u_y^s \\
&= V_P \left( \left[\!\left[ F_e^V, 0 \right]\!\right] + \left[\!\left[ -F_w^V, 0 \right]\!\right] + \left[\!\left[ F_n^V, 0 \right]\!\right] + \left[\!\left[ -F_s^V, 0 \right]\!\right] \right) \\
&\quad - \left[\!\left[ -F_e^V, 0 \right]\!\right] V_E - \left[\!\left[ F_w^V, 0 \right]\!\right] V_W - \left[\!\left[ -F_n^V, 0 \right]\!\right] V_N - \left[\!\left[ F_s^V, 0 \right]\!\right] V_S
\end{aligned}
\tag{14}
$$

Let,

$$
a_E^V = \mathrm{Re}_p^* \left[\!\left[ -F_e^V, 0 \right]\!\right]; \quad a_W^V = \mathrm{Re}_p^* \left[\!\left[ F_w^V, 0 \right]\!\right] \qquad a_N^V = \mathrm{Re}_p^* \left[\!\left[ -F_n^V, 0 \right]\!\right]; \quad a_S^V = \mathrm{Re}_p^* \left[\!\left[ F_s^V, 0 \right]\!\right]
\tag{15}
$$

Using the **identities**:

$$
\left[\!\left[ a, 0 \right]\!\right] = \frac{1}{2}\left[ a + |a| \right]; \quad \left[\!\left[ -a, 0 \right]\!\right] = \frac{1}{2}\left[ -a + |a| \right]
$$

The following relationship for the RHS of Eq. (14) is obtained:

$$
\begin{aligned}
F_e^V + \left[\!\left[ -F_e^V, 0 \right]\!\right] - F_w^V + \left[\!\left[ F_w^V, 0 \right]\!\right] + F_n^V + \left[\!\left[ -F_n^V, 0 \right]\!\right] + F_s^V + \left[\!\left[ -F_s^V, 0 \right]\!\right] &= \left( F_e - F_w + F_n - F_s \right)^V \\
&\quad + \frac{1}{\mathrm{Re}_p^*} \left[ a_E^V + a_W^V + a_S^V + a_N^V \right]
\end{aligned}
\tag{16.a}
$$

And using the discrete form of the continuity equation, Eq. (12),

$$
\begin{aligned}
&F_e^V + \left[\!\left[ -F_e^V, 0 \right]\!\right] - F_w^V + \left[\!\left[ F_w^V, 0 \right]\!\right] + F_n^V + \left[\!\left[ -F_n^V, 0 \right]\!\right] + F_s^V + \left[\!\left[ -F_s^V, 0 \right]\!\right] = \\
&-\sigma \frac{\partial}{\partial \tau} \int_w^e \int_s^n \left( \bar{\rho} h \right) dx dy \quad + \quad \frac{1}{\mathrm{Re}_p^*} \sum_{nb} a_{nb}^V
\end{aligned}
\tag{16.b}
$$

where ***nb*** refers to the neighbor nodes ***(e, w, n, s)*** on each of the surfaces bounding the control volume. Substitution of Eq. (16.b) into the axial momentum equation (14) gives:

$$
\begin{aligned}
h_P^V \left( p_S - p_P \right) \delta x^V &= \left( \frac{k_y \bar{\mu}}{h} \right)_P^V \delta x^V \delta y^V + \mathrm{Re}_s \frac{\partial}{\partial \tau} \int \int \left( \bar{\rho} h u_y \right) dx dy \\
&\quad + \mathrm{Re}_p^* \left[ V_P \left( \left[\!\left[ F_e^V, 0 \right]\!\right] + \left[\!\left[ -F_w^V, 0 \right]\!\right] + \left[\!\left[ F_n^V, 0 \right]\!\right] + \left[\!\left[ -F_s^V, 0 \right]\!\right] \right) \right] \\
&\quad - \mathrm{Re}_p^* \left( \left[\!\left[ -F_e^V, 0 \right]\!\right] V_E - \left[\!\left[ F_w^V, 0 \right]\!\right] V_W - \left[\!\left[ -F_n^V, 0 \right]\!\right] V_N - \left[\!\left[ F_s^V, 0 \right]\!\right] V_S \right)
\end{aligned}
\tag{17}
$$

And substituting Eq. (16.b),

$$h_P^V\left(p_S-p_P\right)\delta x^V+\sum_{nb}a_{nb}^V V_{nb}=\left[\left(\frac{k_y\bar{\mu}}{h}\right)_P^V+\sum_{nb}a_{nb}^V\right]V_P+\mathrm{Re}_s\frac{\partial}{\partial\tau}\int\int\left(\bar{\rho}hu_y\right)dxdy$$
$$-V_P\,\mathrm{Re}_s\frac{\partial}{\partial\tau}\int\int\left(\bar{\rho}h\right)dxdy \qquad (18)$$

Hence, the **difference form** of the **axial momentum transport equation** is:

$$h_P^V\left(p_S-p_P\right)\delta x^V+\sum_{nb}a_{nb}^V V_{nb}=\left[\left(\frac{k_y\bar{\mu}}{h}\right)_P^V+\sum_{nb}a_{nb}^V\right]V_P+\mathrm{Re}_s\left[\left(\bar{\rho}h\right)_P^V\delta x^V\delta y^V\right]\frac{\partial}{\partial\tau}V_P \qquad (19)$$

A suitable approximation for the unsteady term (time derivative) is needed. An **implicit first-order scheme** is used, i.e.,

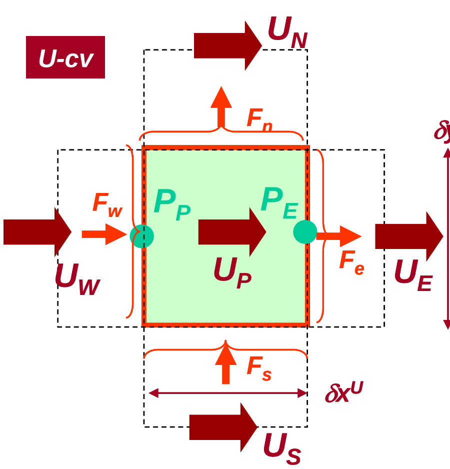
$$\frac{\partial V_P}{\partial\tau}\approx\frac{V_P-V_P^*}{\Delta\tau} \qquad (*)$$

where $V_P$ is the axial velocity at time $t+\Delta t$, and $V_P^*$ is the axial velocity at time *t*, respectively. For the scheme to be implicit, all field variables (velocity and pressure) in Eq. (19) must be evaluated also at time $t+\Delta t$.

Finally, the **discrete form of the axial momentum transport equation** is

$$h_P^V\left(p_S-p_P\right)\delta x^V+\sum_{nb}a_{nb}^V V_{nb}+\frac{\mathrm{Re}_s\left(\bar{\rho}h\right)_P^V\delta x^V\delta y^V}{\Delta\tau}V_P^*=a_P^V V_P \qquad (20)$$

where $$a_P^V=\left(\frac{k_y\bar{\mu}}{h}\right)_P^V\delta x^V\delta y^V+\sum_{nb}a_{nb}^V+\mathrm{Re}_s\left(\bar{\rho}h\right)_P^V\frac{\delta x^V\delta y^V}{\Delta\tau}$$




Circumferential velocity control volume

## Integration of Circumferential Momentum Transport Equation

Integration of the circumferential momentum transport Eq. (6.a) over the *U-velocity control volume*, and using the continuity equation to simplify some terms (same as for the *V* transport equation) leads to the following algebraic equation:

$$h_P^U\left(p_P-p_E\right)\delta y^U+\sum_{nb}a_{nb}^U U_{nb}+\left[\frac{k_J\overline{\mu}}{h}\frac{\Lambda}{2}\right]_P^U\delta x^U\delta y^U+\frac{\mathrm{Re}_s}{\Delta\tau}\left(\overline{\rho}h\right)_p^u U_P^*\,\delta x^U\delta y^U=a_P^U\,U_P \quad (21)$$

where $U_p^*=U$ at time $t$, and

$$a_P^U=\sum_{nb}a_{nb}^U+\left(\frac{k_x\overline{\mu}}{h}\right)_P^U\delta x^U\delta y^U+\mathrm{Re}_s\left(\overline{\rho}h\right)_P^U\frac{\delta x^U\delta y^U}{\Delta\tau}$$

$$a_E^U=\mathrm{Re}_p^*\left[\!\left[-F_e^U,0\right]\!\right];\quad a_W^U=\mathrm{Re}_p^*\left[\!\left[F_W^U,0\right]\!\right];\ a_N^U=\mathrm{Re}_p^*\left[\!\left[-F_n^U,0\right]\!\right];\quad a_S^U=\mathrm{Re}_p^*\left[\!\left[F_s^U,0\right]\!\right]$$

$$F_e^U=\left(\overline{\rho}hu_x\right)^e\delta y^U\ ;\quad F_w^u=\left(\overline{\rho}hu_x\right)^w\delta y^U\ ;\ F_n^u=\left(\overline{\rho}hu_y\right)^n\delta x^U\ ;\quad F_s^u=\left(\overline{\rho}hu_y\right)^s\delta x^U$$

Hence, the **difference equations** for fluid momentum transport are

**Circumferential momentum in *U-CV***

$$h_P^U\left(p_P-p_E\right)\delta y^U+\sum_{nb}a_{nb}^U U_{nb}+S_P^U+S_\tau^U U_P^*=a_P^U\,U_P \quad (22)$$

**Axial-momentum in *V-CV***:

$$h_P^V\left(p_S-p_P\right)\delta x^V+\sum_{nb}a_{nb}^V V_{nb}+S_P^V+S_\tau^V\,V_P^*=a_P^V\,V_P \quad (23)$$

where:

$$a_P^V=\left[\frac{k_y\overline{\mu}}{h}\right]_P^V\delta x^V\delta y^V+\sum_{nb}a_{nb}^V+S_\tau^V\ ;\quad S_\tau^V=\mathrm{Re}_s\left(\overline{\rho}h\right)_P^V\frac{\delta x^V\delta y^V}{\Delta\tau}$$

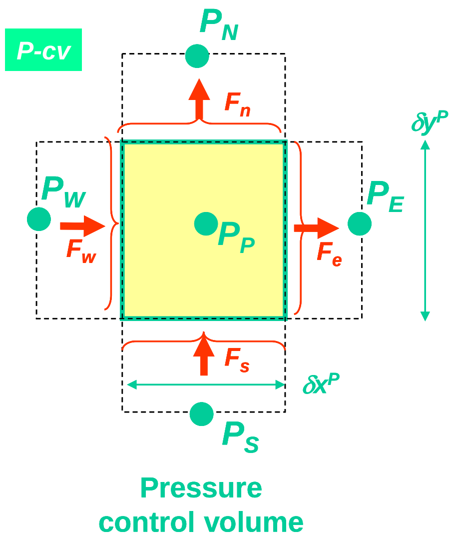
$$a_P^U=\left[\frac{k_x\overline{\mu}}{h}\right]_P^U\delta x^U\delta y^U+\sum_{nb}a_{nb}^U+S_\tau^U\ ;\quad S_\tau^U=\mathrm{Re}_s\left(\overline{\rho}h\right)_P^U\frac{\delta x^U\delta y^U}{\Delta\tau}$$

$$S_P^U=\left[\frac{k_J\overline{\mu}}{h}\frac{\Lambda}{2}\right]_P^U\delta x^U\delta y^U\ ;\quad S_P^V=0$$

In general:

$$F_e^r=\left(\overline{\rho}hu_x\right)_e^r\delta y^r\ ;\quad F_w^r=\left(\overline{\rho}hu_x\right)_w^r\delta y^r\ ;$$
$$F_n^r=\left(\overline{\rho}hu_x\right)_n^r\delta x^r\ ;\quad F_s^r=\left(\overline{\rho}hu_x\right)_s^r\delta x^r$$

where $r=U$ or $V$




## Derivation of the pressure correction equation

Integration of mass conservation Eq. (5) over the pressure control volume (*P-CV)* leads to:

$$F_e^P - F_w^P + F_n^P - F_s^P + \sigma \frac{\partial}{\partial \tau} \int_w^e \int_s^n \left(\overline{\rho} h\right) dx\, dy = 0 \quad (24)$$

where the flow rates across the faces of the *CV* are:

$$\begin{aligned} F_e^P &= \left(\overline{\rho} h u_x\right)_e^P \delta y^P = \left(\overline{\rho} h\right)_e^P U_P\, \delta y^P \ ; \\ F_w^P &= \left(\overline{\rho} h u_x\right)_w^P \delta y^P = \left(\overline{\rho} h\right)_w^P U_W\, \delta y^P \ ; \\ F_n^P &= \left(\overline{\rho} h u_x\right)_n^P \delta x^P = \left(\overline{\rho} h\right)_n^P U_N\, \delta x^P \ ; \\ F_s^P &= \left(\overline{\rho} h u_x\right)_s^P \delta y^P = \left(\overline{\rho} h\right)_s^P U_S\, \delta x^P \end{aligned} \quad (25)$$

The term containing the unsteady variation of fluid density and film thickness is approximated as:

$$\frac{\partial}{\partial \tau} \int_w^e \int_s^n \left(\overline{\rho} h\right) dxdy = \left[\frac{\partial}{\partial \tau}\left(\overline{\rho} h\right)_p\right] \delta x^P \delta y^P = \left[\overline{\rho}_P^P \frac{\partial h_P^P}{\partial \tau} + h_P^P \frac{\partial \overline{\rho}_P^P}{\partial \tau}\right] \delta x^P \delta y^P$$

which implies that the *P-CV* is fixed in space, with film thickness *h* and density $\overline{\rho}$ taken as uniform within the control volume.

The algebraic form of the continuity equation establishes the flow balance on a finite size control volume as:

$$F_e^P - F_w^P + F_n^P - F_s^P + \sigma\, \delta x^P \delta y^P \left[\overline{\rho}_P^P \frac{\partial h_P^P}{\partial \tau} + h_P^P \frac{\partial \overline{\rho}_P^P}{\partial \tau}\right] = 0 \quad (26)$$

Since $h = \overline{c}(x,y) + \varepsilon_{X(t)} \cos\theta + \varepsilon_{Y(t)} \sin\theta$, then:

$$\frac{\partial h_P^P}{\partial \tau} = \frac{\partial \varepsilon_X}{\partial \tau} \cos\left(\theta_P\right) + \frac{\partial \varepsilon_Y}{\partial \tau} \sin\left(\theta_P\right) = \dot{\varepsilon}_X \cos\theta_P + \dot{\varepsilon}_Y \sin\theta_P$$

Note that simultaneous solution of the rotor-bearing equations of motion determines the journal (shaft) center coordinates $\left(\varepsilon_X, \varepsilon_Y\right)_{(\tau)}$ and its time derivatives $\left(\dot{\varepsilon}_X, \dot{\varepsilon}_Y\right)$.

Incidentally, $$\frac{\partial \overline{\rho}_P^P}{\partial \tau} = \frac{\overline{\rho}_P^P - \overline{\rho}_P^{P*}}{\Delta \tau} \text{ where } \overline{\rho}_P^{P*} = \overline{\rho}(t) \text{ and } \overline{\rho}_P^P = \overline{\rho}\left(t + \Delta\tau\right)$$

The algebraic form of the continuity equation in the *P-CV* is thus

$$F_e^P - F_w^P + F_n^P - F_s^P + \sigma\,\delta x^p \delta y^p \left[ \overline{\rho}_P^P \dot{h}_P + h_P \frac{\overline{\rho}_P^P - \overline{\rho}_P^{P*}}{\Delta\tau} \right] = 0 \qquad (27)$$

**The pressure correction method** (Launder and Leschziner, 1978]

Let the flow variables be expressed as:

$$U = \overline{U} + u' \,;\; V = \overline{V} + v' \,;\; p = \overline{p} + p' \qquad (28)$$

where the current velocities $(U, V)$ satisfy the momentum equations but not the mass continuity equation. Above $u'$, $v'$, and $p'$ are correction fields. Substituting Eq. (28) into the momentum equations (22) and (23) leads to:

$$\begin{aligned} & h_P^U \left( \overline{p}_P - \overline{p}_E \right) \delta y^U + \sum_{nb} a_{nb}^U \overline{U}_{nb} + S_P^U + S_\tau^U U_P^* \\ & + h_P^U \left( p'_P - p'_E \right) \delta y^U + \sum_{nb} a_{nb}^U \overline{u}'_{nb} = a_P^U \overline{U}_P + a_P^u u'_P \end{aligned} \qquad (29)$$

Then

$$h_P^U \left( p'_P - p'_E \right) \delta y^U + \sum_{nb} a_{nb}^u u'_{nb} = a_P^U u'_P \qquad (30.a)$$

and identically, from the axial transport equation:

$$h_P^V \left( p'_S - p'_P \right) \delta x^V + \sum_{nb} a_{nb}^v v'_{nb} = a_P^V v'_P \qquad (30.b)$$

Introducing the **SIMPLEC procedure** (Van Doormal & Raithby, 1984]

$$\sum a_{nb}^U u'_{nb} \approx \sum a_{nb}^U u'_P ; \;\; \sum a_{nb}^V v'_{nb} \approx \sum a_{nb}^V v'_P \qquad (31)$$

Equations (30) become:

$$\begin{aligned} u'_P &= \frac{d_P^U \left( p'_P - p'_E \right)}{a_P^U - \sum a_{nb}^U} \,; \quad d_P^U = h_P^U \delta y^U \\ v'_P &= \frac{d_P^V \left( p'_P - p'_E \right)}{a_P^V - \sum a_{nb}^V} \,; \quad d_P^V = h_P^V \delta x^V \end{aligned} \qquad (32)$$

where

$$a_P^U = \left( \frac{k_x \overline{\mu}}{h} \right)_P^U \delta x^U \delta y^U + \sum a_{nb}^U + S_\tau^U \,; \quad S_\tau^U = \mathrm{Re}_s \left( \overline{\rho} h \right)_P^U \frac{\delta x^U \delta y^U}{\Delta\tau}$$

$$a_P^V = \left( \frac{k_y \overline{\mu}}{h} \right)_P^V \delta x^V \delta y^V + \sum a_{nb}^V + S_\tau^V \,; \quad S_\tau^V = \mathrm{Re}_s \left( \overline{\rho} h \right)_P^V \frac{\delta x^V \delta y^V}{\Delta\tau}$$

Let

$$D_P^U = \frac{d_P^U}{a_P^U - \sum a_{nb}^U} \,; \quad D_P^V = \frac{d_P^V}{a_P^V - \sum a_{nb}^V}$$

Substitution of the correction fields, Eq. **(28),** into the continuity Eq. (27) gives:

$$\left(\overline{F}_e+F'_e\right)^P-\left(\overline{F}_w+F'_w\right)^P+\left(\overline{F}_n+F'_n\right)^P-\left(\overline{F}_s+F'_s\right)^P+\sigma\,\delta x^P\delta y^P\left[\overline{\rho}_P h_P+h_P\frac{\overline{\rho}_P^{\,P}-\overline{\rho}_P^{\,P*}}{\Delta\tau}\right]\approx 0 \qquad (33)$$

where
$$\overline{F}_e=\left(\overline{\rho}\,h\,\overline{u}_x\,\delta y\right)_e^P=\left(\overline{\rho}\,h\right)_e^P\overline{U}_P\,\delta y^P$$
$$\overline{F}_s=\left(\overline{\rho}\,h\,\overline{u}_y\,\delta x\right)_s^P=\left(\overline{\rho}\,h\right)_s^P\overline{V}_P\,\delta x^P,\ \text{etc.} \qquad (34)$$

and
$$F'_e=\left(\overline{\rho}h\right)_e^P u'_P\,\delta y^P=\left(\overline{\rho}h\right)_e^P D_P^U\left(p'_P-p'_E\right)\delta y^P=a_e^P\left(p'_P-p'_E\right);\quad a_e^P=\left(\overline{\rho}h\right)_e^P\delta y^P D_P^U$$
$$F'_s=\left(\overline{\rho}h\right)_s^P v'_P\,\delta x^P=\left(\overline{\rho}h\right)_s^P D_P^V\left(p'_S-p'_P\right)\delta x^P=a_S^P\left(p'_S-p'_P\right);\quad a_S^P=\left(\overline{\rho}h\right)_s^P\delta x^P D_P^V \qquad (35)$$
$$F'_w=a_W^P\left(p'_W-p'_P\right);\quad F'_n=a_N^P\left(p'_P-p'_N\right)$$

Let
$$S_P^*=\overline{F}_e^P-\overline{F}_w^P+\overline{F}_n^P-\overline{F}_s^P+\sigma\,\delta x^P\delta y^P\left[\overline{\rho}_P\dot{h}_P+h_P\frac{\overline{\rho}_P^{\,P}-\overline{\rho}_P^{\,*}}{\Delta\tau}\right] \qquad (36)$$

Then, Eq. (33) becomes
$$F'_e-F'_w+F'_n-F'_s=-\overline{S}_P^* \qquad (37.a)$$

or
$$a_E^P\left(p'_P-p'_E\right)-a_N^P\left(p'_W-p'_P\right)+a_N^P\left(p'_P-p'_N\right)-a_S^P\left(p'_S-p'_P\right)\equiv-\overline{S}_P^* \qquad (37.b)$$

$$a_P^P\,p'_P=\sum_{nb}a_{nb}^P\,p'_{nb}-\overline{S}_P^* \qquad (37.c)$$

where
$$a_P^P=\left(a_E+a_W+a_N+a_S\right)^P=\sum_{nb}a_{nb}^P \qquad (38)$$

Note that if $p'_{nb}=p'_P=0$ then $S_P^*=0$ and mass continuity is satisfied. Thus, the momentum equations are also satisfied and the (current) flow field is considered as the solution to the fluid transport equations.

Once the correction pressure field $p'_P$ is obtained, correction to the circumferential *u* and axial *v* fields are performed using Eqs. (27). In the numerical procedure, the pressure is typically **under-relaxed** as

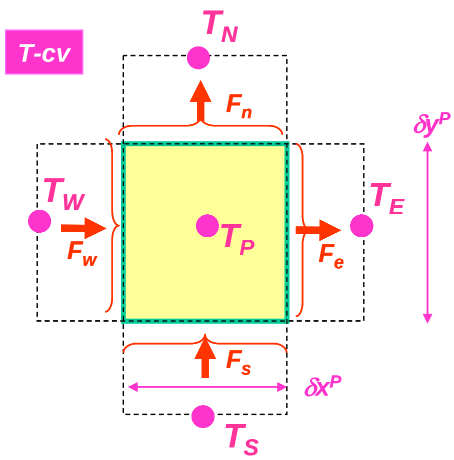
$$p^{new}=p^{old}+\alpha\,p'$$
$$U^{new}=U^{old}+u' \qquad (39)$$
$$V^{new}=V^{old}+v'$$

with $\alpha$ as a relaxation parameter, whose value is typically less than one.

## Discretization of Energy Transport Equation

The dimensionless equation for turbulent bulk-flow energy transport is (Yang, 1992):

$$\frac{\overline{C}_p \mathrm{Re}_s}{E_c}\frac{\partial}{\partial \tau}\left(\overline{\rho} h \overline{T}\right)+\frac{\overline{C}_p \mathrm{Re}_p^*}{E_c}\left[\frac{\partial}{\partial x}\left(\overline{\rho} h u_x \overline{T}\right)+\frac{\partial}{\partial y}\left(\overline{\rho} h u_y \overline{T}\right)\right]+\frac{\mathrm{Re}_p^*}{E_c}\overline{Q}_s$$
$$=\left(\sigma\frac{\partial p}{\partial \tau}+u_x\frac{\partial p}{\partial x}+u_y\frac{\partial p}{\partial y}\right)\overline{\beta}_t h\overline{T}+h\frac{\Lambda}{2}\frac{\partial p}{\partial x}+\frac{\overline{\mu}}{h}\left(k_x\left(u_x^2+u_y^2+\frac{1}{2}u_x\Lambda\right)+k_J\left(\frac{\Lambda^2}{4}-u_x\Lambda\right)\right) \quad (40)$$



Temperature control volume

where

$$x=x/R; \quad y=y/R; \quad t=\tau/\omega; \quad h=H/c;$$
$$u_x=U_x/U_*; \quad u_y=U_y/U_*; \quad \Lambda=\Omega R/U_*; \quad \sigma=\omega R/U_*;$$
$$T=T/T_*; \;\; p=\left(P-P_a\right)/P_{sa}; \quad \beta_t=\beta_t T_*$$

and

$E_c=U_*^2/\left(T_* C_p^*\right)$ is the Eckerd number

$\mathrm{Re}_p^*=\mathrm{Re}_p\; c/R=\left(\rho_*/\mu_*\right)\left(U_*/\mathrm{R}\right)c^2$ is a (modified) Reynolds number

$\mathrm{Re}_s=\left(\rho_*/\mu_*\right)\omega c^2$ is the squeeze film Reynolds number

$\overline{Q}_s=\overline{h}_B\left(\overline{T}-\overline{T}_B\right)+\overline{h}_J\left(\overline{T}-\overline{T}_J\right)$ is the heat flow to bearing and journal surfaces

Define the following source terms:

$$S_1=\left[\left(\sigma\frac{\partial p}{\partial \tau}+u_x\frac{\partial p}{\partial x}+u_y\frac{\partial p}{\partial y}\right)\overline{\beta}_t h; \quad S_2=h\frac{\Lambda}{2}\frac{\partial p}{\partial x}+\frac{\overline{\mu}}{h}\left(k_x\left(u_x^2+u_y^2+\frac{1}{2}u_x\Lambda\right)+k_J\left(\frac{\Lambda^2}{4}-u_x\Lambda\right)\right) \quad (41)$$

Integration of Eq. (40) over the temperature control volume (*T-CV*) leads to

$$\frac{\overline{C}_p \mathrm{Re}_s}{E_c}\int\int_{T-cv}\frac{\partial}{\partial \tau}\left(\overline{\rho} h \overline{T}\right)dxdy+\frac{\overline{C}_p \mathrm{Re}_p^*}{E_c}\left[\int_s^n\left(\overline{\rho} h u_x \overline{T}\right)_w^e dy+\int_w^e\left(\overline{\rho} h u_y \overline{T}\right)_s^n dx\right]+\frac{\mathrm{Re}_p^*}{E_c}\overline{Q}_s\,\Delta x\,\Delta y=\left[S_1\overline{T}_P+S_2\right]\left(\Delta x\,\Delta y\right)$$
(42)

Implementation of the upwind scheme for the thermal flux transport terms gives

$$\left(\overline{\rho} h u_x \overline{T}\right)^e\Delta y=F_e\overline{T}_e=\left[\!\left[F_e,0\right]\!\right]\overline{T}_P-\left[\!\left[-F_e,0\right]\!\right]\overline{T}_E$$
$$\left(\overline{\rho} h u_x \overline{T}\right)^w\Delta y=F_w\overline{T}_w=\left[\!\left[F_w,0\right]\!\right]\overline{T}_W-\left[\!\left[-F_w,0\right]\!\right]\overline{T}_P \quad (43)$$
$$\left(\overline{\rho} h u_y \overline{T}\right)^n\Delta x=\left[\!\left[F_n,0\right]\!\right]\overline{T}_P-\left[\!\left[-F_n,0\right]\!\right]\overline{T}_N$$

$$\left(\overline{\rho}\, h u_y \overline{T}\right)^s \Delta x = [\![ F_s, 0 ]\!] \overline{T}_S - [\![ -F_s, 0 ]\!] \overline{T}_P$$

where $F_e = \left(\overline{\rho}\, h u_x \Delta y\right)_e$; $F_w = \left(\overline{\rho}\, h u_x \Delta y\right)_w$; $F_n = \left(\overline{\rho}\, h u_y \Delta x\right)_n$; $F_s = \left(\overline{\rho}\, h u_y \Delta x\right)_s$ are the momentum fluxes through the control-volume faces. Using Eq. (43), the LHS of Eq. (42) is rewritten as

$$LHS_{Eq.42} = \frac{\overline{C}_P \operatorname{Re}_s}{E_c} \int\int_{T-cv} \frac{\partial}{\partial \tau}\left(\overline{\rho}\, h \overline{T}\right) dxdy + \left(\sum a_{nb}\right) \overline{T}_P - \sum a_{nb} \overline{T}_{nb} + \frac{\overline{C}_P \operatorname{Re}_p^*}{E_c}\left[F_e - F_w + F_n - F_s\right] \overline{T}_P + \frac{\operatorname{Re}_p^*}{E_c} Q_s \Delta x \Delta y \tag{44}$$

where $a_e = \frac{\overline{C}_P \operatorname{Re}_p^*}{E_c} [\![ -F_e, 0 ]\!]$; $a_w = \frac{\overline{C}_P \operatorname{Re}_p^*}{E_c} [\![ F_w, 0 ]\!]$; $a_n = \frac{\overline{C}_P \operatorname{Re}_p^*}{E_c} [\![ -F_n, 0 ]\!]$; $a_s = \frac{\overline{C}_P \operatorname{Re}_p^*}{E_c} [\![ F_s, 0 ]\!]$ (45)

The discrete form of the continuity equation in the *T-CV* gives

$$\left[F_e - F_w + F_n - F_s\right] = -\sigma \int\int_{T-cv} \frac{\partial}{\partial \tau}\left(\overline{\rho}\, h\right) dxdy \tag{46}$$

Substitution of Eq. **(46)** into **(44)** gives:

$$LHS_{Eq.42} = \frac{\overline{C}_P \operatorname{Re}_s}{E_c} \int\int_{T-cv} \frac{\partial}{\partial \tau}\left(\overline{\rho}\, h \overline{T}\right) dxdy + \left(\sum a_{nb}\right) \overline{T}_P - \sum a_{nb} \overline{T}_{nb} + \frac{\operatorname{Re}_p^*}{E_c} Q_s \Delta x \Delta y + \frac{\overline{C}_P}{E_c} \overline{T}_P \left[\operatorname{Re}_s \int\int_{T-cv} \frac{\partial\left(\overline{\rho}\, h\right)}{\partial \tau} dxdy - \operatorname{Re}_p^* \sigma \int\int_{T-cv} \frac{\partial}{\partial \tau}\left(\overline{\rho}\, h\right) dxdy\right] \tag{47}$$

Since $\operatorname{Re}_s = \operatorname{Re}_p^* \sigma$, the last two terms on the RHS of the previous expression add to zero; i.e., they satisfy the continuity equation. Then:

$$LHS_{Eq.42} = \frac{\overline{C}_P \operatorname{Re}_s}{E_c} \int\int_{T-cv} \overline{\rho}\, h \frac{\partial \overline{T}}{\partial \tau} dxdy + \left(\sum a_{nb}\right) \overline{T}_P - \sum a_{nb} \overline{T_{nb}} + \frac{\operatorname{Re}_p^*}{E_c} Q_s \Delta x \Delta y \tag{48}$$

The integral form of the energy transport Eq. (42) becomes

$$\frac{\overline{C}_P \operatorname{Re}_s}{E_c} \int\int_{T-c\forall} \overline{\rho}\, h \frac{\partial \overline{T}}{\partial \tau} dxdy + \left(\sum a_{nb}\right) \overline{T}_P - \sum a_{nb} \overline{T_{nb}} + \frac{\operatorname{Re}_p^*}{E_c} Q_s \Delta x \Delta y = \left(S_1 \Delta x \Delta y\right) \overline{T}_P + S_2 \Delta x \Delta y \tag{49}$$

Let $\frac{\partial \overline{T}}{\partial \tau} = \frac{\overline{T}_P - \overline{T}_P^*}{\Delta \tau}$ and $\overline{Q}_s = \overline{T}_P\left(\overline{h}_B + \overline{h}_J\right) - \left(\overline{h}_B \overline{T}_B + \overline{h}_J \overline{T}_J\right)$; with $\overline{T}_P^*$ as the film temperature in the previous time step. Then, the discrete form of Eq. (49) becomes:

$$\left[\sum a_{nb} + \frac{\overline{C}_P \operatorname{Re}_s}{E_c}\frac{\overline{\rho}_P h_P}{\Delta \tau}\Delta x\,\Delta y + \frac{\operatorname{Re}_P^*}{E_c}\left(\overline{h}_B + \overline{h}_J\right)\Delta x\,\Delta y - \left(S_1\,\Delta x\,\Delta y\right)\right]\overline{T}_P$$
$$= \sum a_{nb}\overline{T}_{nb} + S_2 \Delta x\,\Delta y + \frac{\overline{C}_P \operatorname{Re}_s}{E_c}\frac{\overline{\rho}_P h_P}{\Delta \tau}\Delta x\,\Delta y\,\overline{T}_P^* + \frac{\operatorname{Re}_P^*}{E_c}\left(\overline{h}_B \overline{T}_B + \overline{h}_J \overline{T}_J\right)\Delta x \Delta y \tag{50}$$

The algebraic form of the energy transport equation is finally written as:

$$a_p^T \overline{T}_P = a_E^T \overline{T}_E + a_W^T \overline{T}_W + a_N^T \overline{T}_N + a_S^T \overline{T}_S + S_c^T \tag{51}$$

where:

$$a_E^T = \frac{\overline{C}_P \operatorname{Re}_P^*}{E_c}\left[\!\left[-F_e^T, 0\right]\!\right];\quad a_W^T = \frac{\overline{C}_P \operatorname{Re}_P^*}{E_c}\left[\!\left[-F_w^T, 0\right]\!\right]$$
$$a_N^T = \frac{\overline{C}_P \operatorname{Re}_P^*}{E_c}\left[\!\left[-F_n^T, 0\right]\!\right];\quad a_S^T = \frac{\overline{C}_P \operatorname{Re}_P^*}{E_c}\left[\!\left[-F_s^T, 0\right]\!\right] \tag{52.a}$$

$$a_P^T = a_E^T + a_W^T + a_N^T + a_S^T + S_{P1}^T + \left[\!\left[S_{P2}^T, 0\right]\!\right] + S_{P3}^T \tag{52.b}$$

$$S_c^T = S_{C1}^T + S_{C2}^T + S_{C3}^T + S_{C4}^T + \left[\!\left[-S_{P2}^T, 0\right]\!\right]\overline{T}_P^* \tag{52.c}$$

$$S_{P1}^T = \frac{\operatorname{Re}_P^*}{E_c}\left(\overline{h}_B + \overline{h}_J\right)\Delta x \Delta y$$
$$S_{P2}^T = \overline{\beta}_{t_P} h_P \left[\frac{\sigma\left(p_P - p_P^*\right)}{\Delta \tau}\Delta x \Delta y + U_P\left(p_e - p_w\right)\Delta y + V_P\left(p_n - p_s\right)\Delta x\right] \tag{52.d}$$
$$S_{P3}^T = \frac{\overline{C}_P \operatorname{Re}_s}{E_c}\frac{\overline{\rho}_P h_P}{\Delta \tau}\Delta x \Delta y$$

$$S_{C1}^T = \left[\frac{\overline{\mu}}{h}\left(k_x\left(u_x^2 + u_y^2 + \frac{1}{2}u_x \Lambda\right) + k_J\left(\frac{\Lambda^2}{4} - u_x \Lambda\right)\right)\right]_P \Delta x\ \Delta y$$
$$S_{C2}^T = \frac{\Lambda}{2} h_P \left(p_e - p_w\right)\Delta y$$
$$S_{C3}^T = \frac{\operatorname{Re}_P^*}{E_c}\left(\overline{h}_B \overline{T}_B + \overline{h}_J \overline{T}_J\right)\Delta x\ \Delta y \tag{52.e}$$
$$S_{C4}^T = \frac{\overline{C}_P \operatorname{Re}_s}{E_c}\frac{\overline{\rho}_P h_P}{\Delta \tau}\Delta x \Delta y\,\overline{T}_P^*$$

Note the terms involving $\Delta\tau$ correspond to unsteady flow conditions. As for the source term -$(S_1\Delta x\Delta y)\overline{T}_P$:

1) $(-S_1\Delta x\Delta y)$ goes into $a_P$ if $(-S_1\Delta x\Delta y)>0$; or

2) $(S_1\Delta x\Delta y)T_P$ into the source term of the RHS if $(-S_1\Delta x\Delta y)<0$.

## Assembling and solving the flow equations

The generic algebraic form of the flow equations is

$$a_P^{\Phi}\Phi_P - a_E^{\Phi}\Phi_E - a_W^{\Phi}\Phi_W - a_S^{\Phi}\Phi_S - a_N^{\Phi}\Phi_N = S_P^{\Phi} \qquad (53)$$

where $\Phi$ is the flow variable and $S_P^{\Phi}$ is a source-like term. In Eq. (53) the nodal value (P) of the flow variable in the control volume is a function of its **four** neighbors (***E, W, N, S***) and the source term. Eq. (53) applied to all the *CV*s in the grid leads to a system of (penta) algebraic equations easily solved using efficient schemes for banded linear equations.

In particular, when the flow direction is well known as in annular seals (axial flow dominance), $a_N^{\Phi}=0$ and the flow equations reduce to the tri-diagonal form

$$a_P^{\Phi}\Phi_P - a_E^{\Phi}\Phi_E - a_W^{\Phi}\Phi_W = S_P^{\Phi} + a_S^{\Phi}\Phi_S \qquad (54)$$

where $\Phi_S$, the upstream value, is known from solution of the prior equation. Eq. (54) is readily solved using very-fast schemes such as the TDMA solver.

For unsteady journal (shaft) motions leading to unsteady flow conditions, at the current time (*t*), the algebraic flow equations are

$$a_P^{\Phi}\Phi_P - a_E^{\Phi}\Phi_E - a_W^{\Phi}\Phi_W - a_S^{\Phi}\Phi_S - a_N^{\Phi}\Phi_N = S_P^{\Phi} + B_p^{\Phi}\,\Phi_p^{(t-\Delta t)}\delta t \qquad (55)$$

where $\Phi = \{U, W, P, T\}$,and $\Phi_p^{(t-\Delta t)}$ is the value of the variable one time step before. That is, at each time step, the previous flow field must be known fully; in particular the one at the initial time when the solution procedure starts.

Once the solution of the set flow equations is obtained at time *t,* integration of the pressure field *P* over the journal (rotor) surface gives the components of the bearing or seal reaction force $(F_X, F_Y)$

$$\begin{Bmatrix} F_X \\ F_Y \end{Bmatrix}_{(t)} = \int_0^L \int_0^{2\pi R} P \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} R\, d\theta\, dz \tag{56}$$

These reaction forces are also known as (nonlinear) impedances since they depend on the journal center position and its velocity components, i.e.

$$F_X = f_X\left(X, Y, \dot{X}, \dot{Y}\right);\ F_Y = f_Y\left(X, Y, \dot{X}, \dot{Y}\right) \tag{57}$$

A rotordynamics model predicting the transient response of a rotor-bearing system needs to integrate the (nonlinear) bearing reaction forces at each time step. The typical equations of motion are of the form

$$\mathbf{M}\,\ddot{\mathbf{u}} + \left(\mathbf{C} + \Omega\,\mathbf{G_R}\right)\dot{\mathbf{u}} + \mathbf{K}\,\mathbf{u} = \mathbf{F_{ext}}\left(\mathbf{u}, \dot{\mathbf{u}}, t\right) + \mathbf{F_B}\left(\mathbf{u}, \dot{\mathbf{u}}, \Omega\right) \tag{58}$$

where $\Omega$ is the rotor speed, and **M, K, C** and $\mathbf{G_R}$ denote the system mass, stiffness, damping and gyroscopic matrices. The vector **u** represents rotor displacements (translations and rotations), $\mathbf{F_{ext}}\left(\mathbf{u}, \dot{\mathbf{u}}, t\right)$ contains the external forces such as weight and those due to mass imbalance, and $\mathbf{F_B}\left(\mathbf{u}, \dot{\mathbf{u}}, \Omega\right)$ corresponds to the bearing forces, for example those from Eqn. (57).

Note that the solution of the rotor-bearing system equations of motion, Eqn. (58), is linked to the solution of the bulk-flow equations for each bearing (or seal), Eqs. (55).

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The CFD method detailed above was quite popular in the 1980s and throughout the mid 1990s. The author published many papers related to the numerical solution of the flow field in bearings and seals dominated by flow turbulence and with fluid inertia effects.

Nowadays, however, CFD methods use efficiently non-staggered and non-structured grids and implement very fine meshes (large number of nodal points) without incurring into excessive computational costs. The CFD field has rapidly evolved thanks to the ever increasing speeds (and low cost) of personal computers. The governing equations remain unchanged, however.

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