NOTES 5

DYNAMICS OF A RIGID ROTOR-FLUID FILM BEARING SYSTEM

Lecture 5 restates the analysis for static equilibrium in a journal bearing. Next, it considers the dynamics of the simplest rigid rotor bearing system supported on journal bearings. For small amplitude journal motions about an equilibrium position, the analysis proceeds to linearize the fluid film forces and introduces the concepts of bearing force coefficients, namely, 4 stiffnesses, 4 damping and 4 inertia coefficients. Formulas for the direct and cross-coupled stiffness and damping coefficients of a short length journal bearing are derived. The analysis focuses on the stability of the rigid rotor-bearing system to determine the threshold rotor speed at which the system loses its equivalent damping and develops ever growing motions at a whirl frequency that coincides with the rotor-bearing system natural frequency. The low and high magnitudes of the Sommerfeld number show whereas the system operates stably or not. The ½ whirl frequency ratio reveals a typical stability limit of lubricated journal bearings. The effect of rotor flexibility on further reducing the threshold speed of instability is noted since the rotor-bearing natural frequency is also lowered. An appendix provides a physical explanation of the follower force, induced by the cross-coupled stiffnesses, that drives the rotor bearing system into whirl. Remedies to avoid or delay the instability are given. Actual examples of instabilities and measurements in the author's laboratory make evident the harmful, potentially catastrophic, whirl instability. A list of industrial or commercial bearing configurations with noted advantages and disadvantages complements the lecture.

Nomenclature

C	Radial clearance [m]
C_{ij}	Bearing damping force coefficients, $i,j=X,Y$ [N.s/m]
$D=2R$	Bearing diameter
e	Journal eccentricity [m]
$\dot{e}, e\dot{\phi}$	V_r, V_t. Journal center radial and tangential velocities [m/s]
F	Fluid film force acting on journal surface [N]. $F=\sqrt{F_X{}^2+F_Y{}^2}=\sqrt{F_r{}^2+F_t{}^2}$
F_o	½ Static load [N]
h	Film thickness. $H=h/c$
K_{ij}	Bearing stiffness force coefficients, $i,j=X,Y$ [N/m]
K_e	Bearing equivalent stiffness [N/m]
K_{rot}	Elastic rotor stiffness (one side) [N/m]
L	Bearing axial length
M	½ Mass of rigid rotor [kg]
M_{ij}	Bearing added mass force coefficients, $i,j=X,Y$ [N.s^2/m]
$m_c=p_s{}^2$	Dimensionless critical mass
P	Hydrodynamic pressure [Pa]
R	Bearing radius [m]
r, t	Moving coordinate system
S	Sommerfeld number
t	Time (s)
u	Mass imbalance [kg]

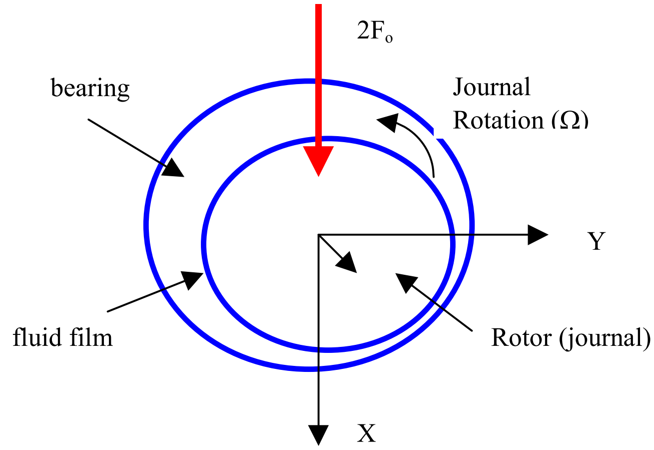
X, Y	Inertial coordinates system
$\Delta X, \Delta Y$	Small amplitude displacements in (X,Y) coordinate system
$\Delta e, e_o \Delta\varphi$	Small amplitude displacements in (r,t) coordinate system
ε	e/C. Journal eccentricity ratio
δ	u/C. imbalance parameter
ρ	Fluid density [kg/m^3]
σ	$\frac{\mu \Omega L R}{4 F_o}\left(\frac{L}{C}\right)^2$. Modified Sommerfeld number
μ	Absolute viscosity [N.s/m^2]
Γ	F_o/K_{rot}. Static (elastic) sag at rotor midspan [m]
ϕ	Journal attitude angle
Θ, θ	Circumferential coordinates
ω	Characteristic whirl frequency [rad/s]
ω_n	$(K_{eq}/M)^{1/2}$. Rotor-bearing system natural frequency [rad/s]
Ω	Journal rotational speed [rad/s]
Ω_s	Threshold speed of instability [rad/s]

Subscripts

a	Ambient value
o	Static or equilibrium condition
s, fs	Threshold of instability for rigid and flexible rotor
XX,XY,YX,YY	Indices of force coefficients in fixed (X,Y) coordinate system
rr,rt,tr,tt	Indices of force coefficients in moving (r,t) coordinate system

Equations of motion of a rigid rotor supported on plain journal bearings

Consider, as shown in Fig. 5.1, a symmetric rigid rotor of mass *2M* that carries a static load $(2F_o=W)$ along the *X* axis. Two identical plain journal bearings support the rotor. The equations of motion of the rotating system at constant rotational speed Ω are given by:



$$M\ddot{X} = F_X + M\,u\,\Omega^2 \sin(\Omega t) + F_o$$
$$M\ddot{Y} = F_Y + M\,u\,\Omega^2 \cos(\Omega t) \tag{5.1}$$

where u is the magnitude of the imbalance vector, $X(t)$ and $Y(t)$ are the coordinates of the rotor mass center, and (F_X, F_Y) are the fluid film bearing reaction forces.

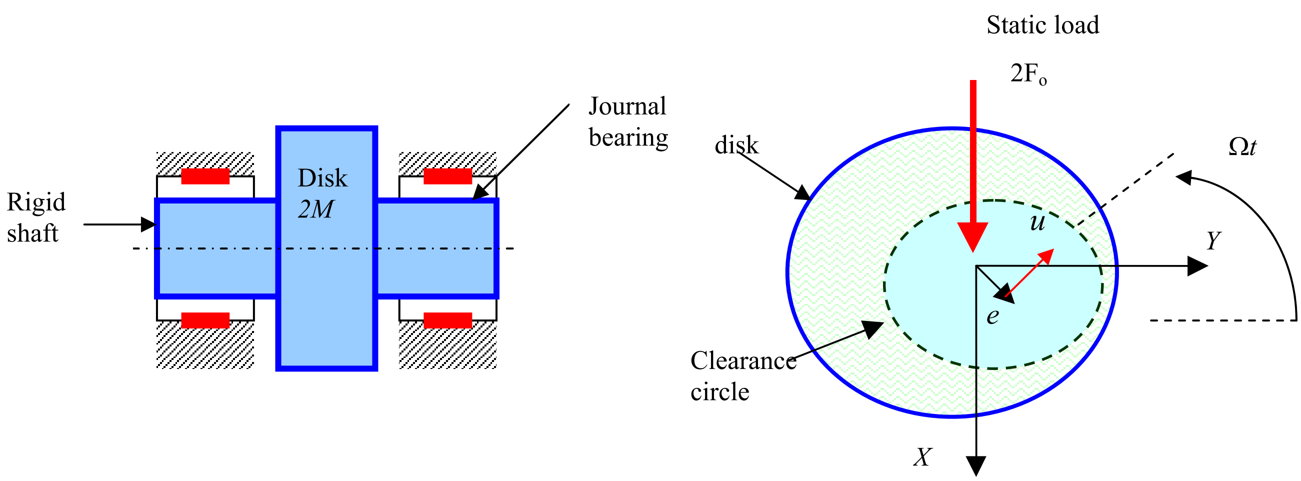


Figure 5.1. Rigid rotor supported on journal bearings.
(*u*) imbalance, (*e*) journal eccentricity

Since the rotor is rigid, the center of mass displacements are identical to those of the journal bearing centers, i.e.

$$X(t) = e_X(t), \;\; Y(t) = e_Y(t) \tag{5.2}$$

The rotor-bearing static equilibrium is defined by

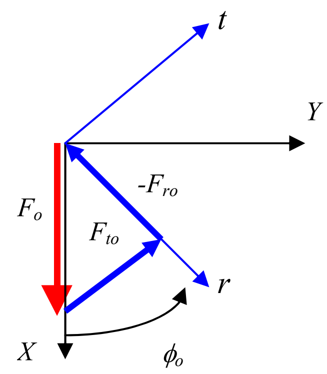
$$F_{X_O} = -F_o, \quad F_{Y_o} = 0, \quad \Rightarrow e_{X_O}, e_{Y_O} \text{ or } e_O, \phi_O \tag{5.3}$$

where (e_o, ϕ_o) are the static equilibrium journal eccentricity and attitude angle, respectively. The static fluid film reaction force components are such that:

$$F_o + F_{X_O} = 0 \quad \Rightarrow \quad F_o = F_{X_O} = F_{r_O} \cos\phi_O - F_{t_O} \sin\phi_O$$
$$F_{Y_O} = 0 \quad \Rightarrow \quad 0 = F_{Y_O} = F_{r_O} \sin\phi_O + F_{t_O} \cos\phi_O \tag{5.4}$$

Recall that $F = \sqrt{F_X{}^2 + F_Y{}^2} = \sqrt{F_r^2 + F_t^2}$

At equilibrium, the region of positive fluid film pressure extends from $\theta_1 = 0$ *to* $\theta_2 = \pi$. In a short length journal bearing, the radial and tangential components of the static fluid film force $\boldsymbol{F_o}$ are



$$F_{r_O} = -\frac{\mu R L^3 \Omega}{C^2} \frac{\varepsilon^2}{\left(1-\varepsilon^2\right)^2}; \quad F_{t_O} = +\frac{\mu R L^3 \Omega}{C^2} \frac{\pi \cdot \varepsilon}{4\left(1-\varepsilon^2\right)^{3/2}} \tag{5.5}$$

where $R = D/2$, L and C are the journal radius, axial length and radial clearance, respectively. $\varepsilon = e_o/C$ is the journal center eccentricity ratio, $\varepsilon < 1.0$; μ is the lubricant absolute viscosity, and $\Omega = (rpm\ \pi/30)$ is the rotor speed in rad/s. Figure 2 depicts the force components, radial and tangential, growing rapidly (nonlinearly) with the journal eccentricity e/C.

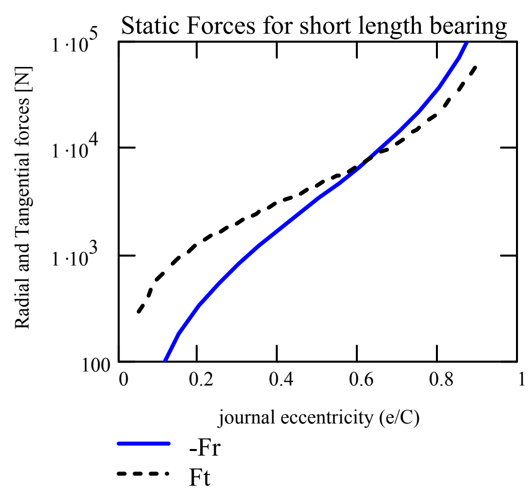


Figure 5.2. Radial and tangential forces in short length journal bearing

μ=0.019 Pa.s, L=0.05 m, C=0.1 mm,
Speed 3, 000 rpm
L/D=0.25

*

Note that the short length bearing forces are proportional to the lubricant viscosity and rotor surface speed (ΩR), the bearing length (L^3), and inversely proportional to the radial clearance (C^2). Most importantly, the bearing forces grow rapidly (non-linearly) with the journal eccentricity $(\varepsilon = e/C)$.

Each bearing reaction force balances a fraction of the applied static load $F_o = ½\, W$ for a symmetric rotor bearing system. Thus,

$$F_o = \left(F_{r_O}^2 + F_{t_O}^2\right)^{1/2} = \mu \Omega R L \left(\frac{L}{C}\right)^2 \frac{\varepsilon}{4} \frac{\sqrt{16\varepsilon^2 + \pi^2\left(1-\varepsilon^2\right)}}{\left(1-\varepsilon^2\right)^2} \tag{5.6}$$

The equilibrium attitude angle (ϕ_o) between the static load direction and the eccentricity vector is

$$\tan(\phi_o) = -\frac{F_{t_o}}{F_{r_o}} = \frac{\pi\sqrt{(1-\varepsilon^2)}}{4\cdot\varepsilon} \tag{5.7}$$

Note that as $\varepsilon \to 0$, $\phi_o \to \frac{1}{2}\pi$ (journal eccentricity is perpendicular to the static load direction), whereas $\varepsilon \to 1$, $\phi_o \to 0$ (journal eccentricity parallel or aligned to load direction).

The bearing design parameter is the modified Sommerfeld number (σ)

$$\frac{\mu \Omega L R}{4F_o}\left(\frac{L}{C}\right)^2 = \sigma = \frac{(1-\varepsilon^2)^2}{\varepsilon\sqrt{\{16\varepsilon^2 + \pi^2(1-\varepsilon^2)\}}} \tag{5.8}$$

For a rated operating condition, σ is known since the bearing geometry, speed, fluid type (viscosity) and load are known. Then Eqn. (5.8) gives a relationship to determine (iteratively) the equilibrium eccentricity ratio, $\varepsilon = e/c$, that generates the film reaction force balancing the applied static load F_o. Recall that,

Large Sommerfeld (σ) numbers (small load W, high speed Ω, large lubricant viscosity μ) determine small operating journal eccentricities or nearly centered operation, i.e. $\varepsilon \to 0.0$ and attitude angles approaching 90°; and

Small Sommerfeld (σ) numbers (large load W, low speed Ω, light lubricant viscosity μ) determine large operating eccentricities, i.e. $\varepsilon \to 1.0$ and attitude angle approaching 0°

Figures 4.6-8 in Lecture 4 depict the Sommerfeld number and attitude angle versus the journal eccentricity and the locus of the journal center within the clearance circle. The same figures are reproduced here in a smaller format.

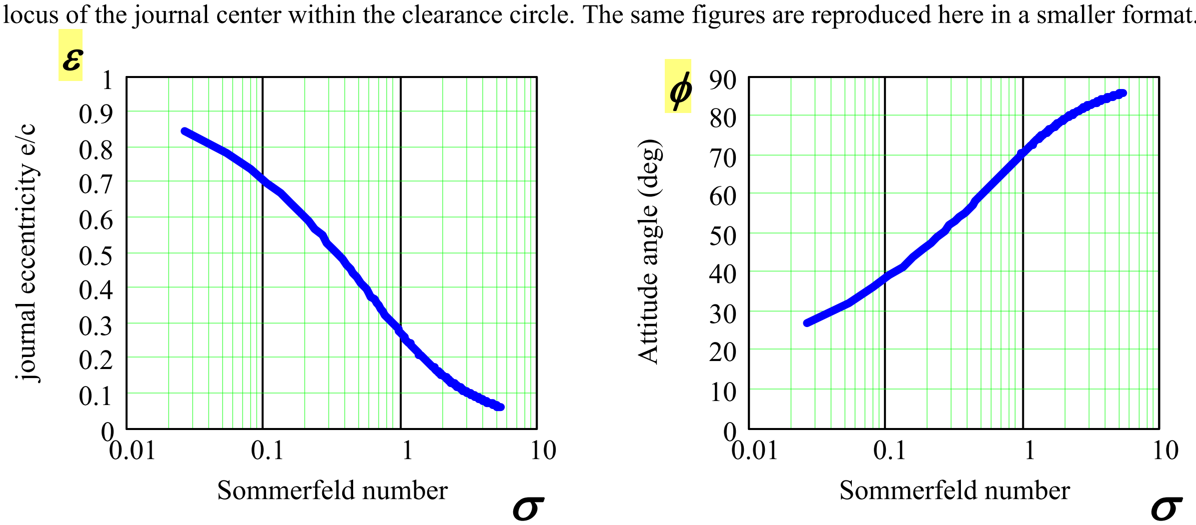


Fig. Eccentricity ratio (ε) and attitude angle (φ) versus Sommerfeld number (σ) in a short journal bearing

Consider, as represented in Figure 5.3, **small amplitude journal motions** about the equilibrium position. These motions are defined as

$$e_X = e_{X_O} + \Delta e_X(t),\;\; e_Y = e_{Y_O} + \Delta e_Y(t) \tag{5.9.a}$$

or

$$X = X_O + \Delta X(t),\;\; Y = Y_O + \Delta Y(t) \tag{5.9.b}$$

or conversely,

$$e(t) = e_O + \Delta e(t),\;\; \phi(t) = \phi_O + \Delta\phi(t) \tag{5.9.c}$$

with

$$\begin{aligned} \frac{dX}{dt} &= \dot{e}_X = \Delta\dot{e}_X.\quad \frac{dY}{dt} = \dot{e}_Y = \Delta\dot{e}_Y \\ \frac{d^2X}{dt^2} &= \ddot{e}_{\dot{X}} = \Delta\ddot{e}_{\dot{X}}.\quad \frac{d^2Y}{dt^2} = \ddot{e}_{\dot{Y}} = \Delta\ddot{e}_{\dot{Y}} \end{aligned} \tag{5.10}$$

The journal dynamic displacements in the *(r, t)* coordinate system are related to those in the *(X,Y)* fixed system by the linear transformation

$$\begin{bmatrix} \Delta e_X \\ \Delta e_Y \end{bmatrix} = \begin{bmatrix} \cos\phi_O & -\sin\phi_O \\ \sin\phi_O & \cos\phi_O \end{bmatrix} \begin{bmatrix} \Delta e(t) \\ e_O\,\Delta\phi(t) \end{bmatrix} \tag{5.11}$$

Similar relationships hold for the journal center velocities and accelerations.

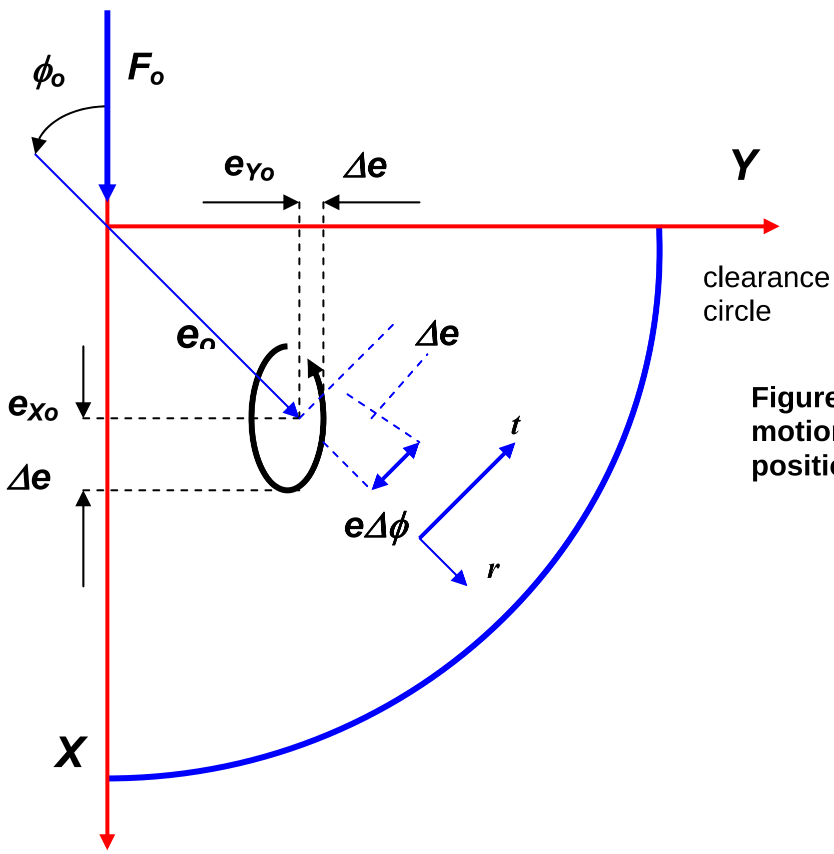


Figure 5.3. Small amplitude journal motions about a static equilibrium position

Note that the small amplitude motions assumption means $\Delta e_X, \Delta e_Y << C$, i.e., the journal dynamic displacements are much smaller than the bearing clearance.

The fluid film forces are general functions of the journal center displacements and velocities, i.e.

$$F_\alpha = F_\alpha \,[e_X(t), e_Y(t), \dot{e}_X(t), \dot{e}_Y(t)], \quad \alpha = X, Y \tag{5.12}$$

The assumption of small amplitude motions about an equilibrium position allows expressing the bearing reaction forces as a Taylor Series expansion around the static journal position *(e_{Xo}, e_{Yo})*, i.e.

$$F_X = F_{X_O} + \frac{\partial F_X}{\partial X}\Delta X + \frac{\partial F_X}{\partial Y}\Delta Y + \frac{\partial F_X}{\partial \dot{X}}\Delta \dot{X} + \frac{\partial F_X}{\partial \dot{Y}}\Delta \dot{Y} + \frac{\partial F_X}{\partial \ddot{X}}\Delta \ddot{X} + \frac{\partial F_X}{\partial \ddot{Y}}\Delta \ddot{Y}$$

$$F_Y = F_{Y_O} + \frac{\partial F_Y}{\partial X}\Delta X + \frac{\partial F_Y}{\partial Y}\Delta Y + \frac{\partial F_Y}{\partial \dot{X}}\Delta \dot{X} + \frac{\partial F_Y}{\partial \dot{Y}}\Delta \dot{Y} + \frac{\partial F_Y}{\partial \ddot{X}}\Delta \ddot{X} + \frac{\partial F_Y}{\partial \ddot{Y}}\Delta \ddot{Y} \tag{5.13}$$

Definition of dynamic force coefficients in fluid film bearings

Fluid film bearing stiffness *(K_{ij})* $_{ij=X,Y}$, damping *(C_{ij})* $_{ij=X,Y}$ and inertia force coefficients are defined as

$$K_{ij} = -\frac{\partial F_i}{\partial X_j} \quad ; C_{ij} = -\frac{\partial F_i}{\partial \dot{X}_j} \quad ; \; M_{ij} = -\frac{\partial F_i}{\partial \ddot{X}_j}; \; {}_{i,j=X,Y} \tag{5.14}$$

For example, $K_{XY} = -\partial F_X/\partial Y$ corresponds to a stiffness produced by a fluid force in the *X* direction due to a journal static displacement in the *Y* direction. By definition, this coefficient is evaluated at the equilibrium position with other journal center displacements and velocities equal to *zero*. The negative sign in the definition assures that a positive magnitude stiffness coefficient corresponds to a restorative force.

The coefficients *(K_{XX}, K_{YY})* are known as the **direct** stiffness terms, while the coefficients *(K_{XY}, K_{YX})* are referred as **cross-coupled**. Figure 5.4 provides an idealized representation of the bearing force coefficients as mechanical parameters.

Fluid inertia or added mass coefficients $M_{ij} = -\dfrac{\partial F_i}{\partial \ddot{X}_j}$; $_{i,j=X,Y}$ where $\{\ddot{X}, \ddot{Y}\}$ are journal center accelerations. Fluid inertia coefficients are of particular importance in superlaminar and turbulent flow bearings and seals handling liquids (large density). The inertia force coefficients or *apparent* masses have a sound physical interpretation and are always present in a fluid film bearing. Inertia coefficients are of large magnitude especially in dense liquids. However, the effect of inertia forces on the dynamic response of rotor-bearing systems is only of importance at large excitation frequencies (This fact also holds for most mechanical systems subjected to fast transient motions).

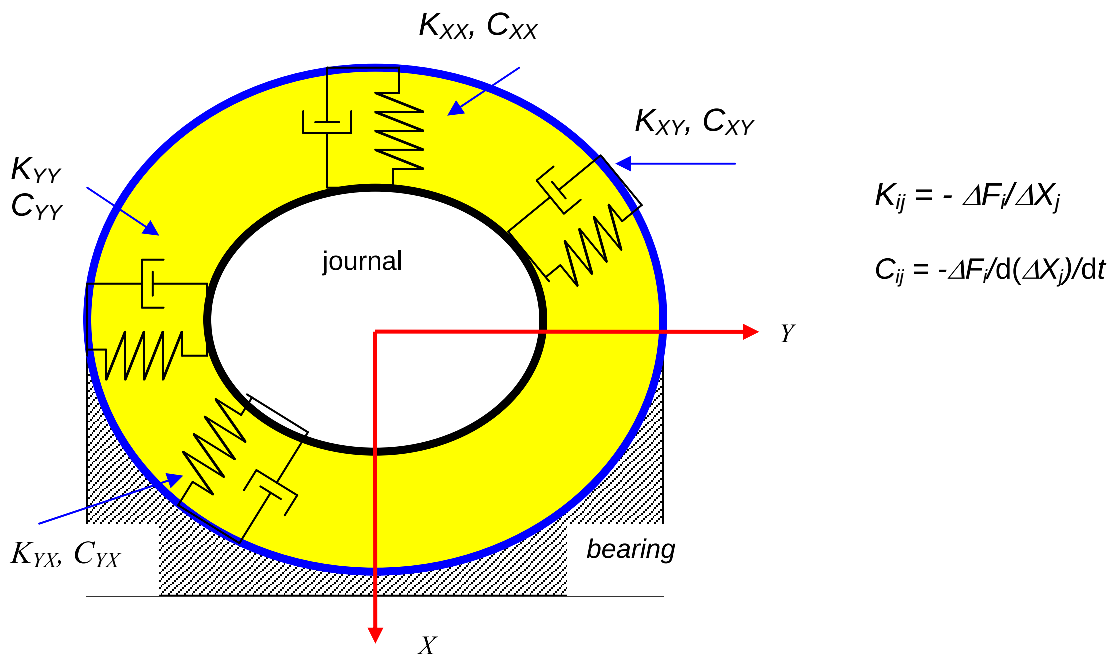


Figure 5.4. The "physical" representation of dynamic force coefficients in fluid film bearings

Note that the defined force coefficients allow the representation of the dynamic fluid film bearing (or seal) forces in terms of fundamental mechanical parameters *{K, C, M}*. However, this does not mean that these coefficients must be accordance with customary knowledge. For example, the "viscous" damping coefficients may be negative, i.e. non-dissipative, or the stiffness coefficients non restorative or non conservative.

Fluid film force coefficients in the radial and tangential directions *(r, t)* are also defined. Thus, the radial and tangential fluid film forces are expressed as (stiffness and damping for simplicity)

$$
\begin{aligned}
F_r &= F_{ro} + \frac{\partial F_r}{\partial e}\Delta e + \frac{\partial F_r}{e_o\,\partial \phi}\; e_o\,\Delta\phi \;+ \frac{\partial F_r}{\partial \dot{e}}\Delta\dot{e} + \frac{\partial F_r}{e_o \partial \dot{\phi}}\; e_o \Delta\dot{\phi} \\
&= F_{ro} - K_{rr}\;\Delta e - K_{rt}\; e_o\,\Delta\phi - C_{rr}\;\Delta\dot{e} - C_{rt}\; e_o\;\Delta\dot{\phi}
\end{aligned}
\tag{5.15a}
$$

$$
\begin{aligned}
F_t &= F_{to} + \frac{\partial F_t}{\partial e}\Delta e + \frac{\partial F_t}{e_o \partial \phi}\; e_o\;\;\Delta\phi + \frac{\partial F_t}{\partial \dot{e}}\Delta\dot{e} + \frac{\partial F_t}{e_o \partial \dot{\phi}}\; e_o\;\Delta\dot{\phi} \\
&= F_{to} - K_{tr}\;\Delta e - K_{tt}\; e_o\,\Delta\phi - C_{tr}\;\Delta\dot{e} - C_{tt}\; e_o \Delta\dot{\phi}
\end{aligned}
\tag{5.15b}
$$

Note that $\{\Delta\dot{e}, e_o\,\Delta\dot{\phi}\}$ are the journal center radial and tangential (small) velocities in the *(r, t)* coordinate system, respectively.

The relationship between the force coefficients in both coordinate systems is easily determined from equation (5.11) as:

$$\begin{bmatrix} K_{XX} & K_{XY} \\ K_{YX} & K_{YY} \end{bmatrix} = \begin{bmatrix} \cos\phi_o & -\sin\phi_o \\ \sin\phi_o & \cos\phi_o \end{bmatrix} \begin{bmatrix} K_{rr} & K_{rt} \\ K_{tr} & K_{tt} \end{bmatrix} \begin{bmatrix} \cos\phi_o & \sin\phi_o \\ -\sin\phi_o & \cos\phi_o \end{bmatrix}$$

$$\begin{bmatrix} C_{XX} & C_{XY} \\ C_{YX} & C_{YY} \end{bmatrix} = \begin{bmatrix} \cos\phi_o & -\sin\phi_o \\ \sin\phi_o & \cos\phi_o \end{bmatrix} \begin{bmatrix} C_{rr} & C_{rt} \\ C_{tr} & C_{tt} \end{bmatrix} \begin{bmatrix} \cos\phi_o & \sin\phi_o \\ -\sin\phi_o & \cos\phi_o \end{bmatrix} \tag{5.16}$$

Substitution of the force coefficient definitions (5.14) into equation (5.13) gives the following

$$\begin{pmatrix} F_X(t) \\ F_Y(t) \end{pmatrix} = \begin{bmatrix} F_{X_O} \\ F_{Y_O} \end{bmatrix} - \begin{bmatrix} K_{XX} & K_{XY} \\ K_{YX} & K_{YY} \end{bmatrix} \begin{pmatrix} \Delta X \\ \Delta Y \end{pmatrix} - \begin{bmatrix} C_{XX} & C_{XY} \\ C_{YX} & C_{YY} \end{bmatrix} \begin{pmatrix} \Delta \dot{X} \\ \Delta \dot{Y} \end{pmatrix} \tag{5.17}$$

And, the governing equations of motion for the rigid-rotor-bearing system, Eqn. (5.1) become

$$\begin{bmatrix} M & O \\ O & M \end{bmatrix} \begin{pmatrix} \Delta\ddot{X} \\ \Delta\ddot{Y} \end{pmatrix} + \begin{bmatrix} C_{XX} & C_{XY} \\ C_{YX} & C_{YY} \end{bmatrix} \begin{pmatrix} \Delta\dot{X} \\ \Delta\dot{Y} \end{pmatrix} + \begin{bmatrix} K_{XX} & K_{XY} \\ K_{YX} & K_{YY} \end{bmatrix} \begin{pmatrix} \Delta X \\ \Delta Y \end{pmatrix} = M\, u\, \Omega^2 \begin{pmatrix} \sin\Omega t \\ \cos\Omega t \end{pmatrix} \tag{5.18}$$

where $F_{Xo} = F_o = ½W$ and $F_{Yo} = 0$. These differential equations are linear and represent the rotor-bearing system dynamics for small amplitude motions about the equilibrium position.

Fluid inertia effects are altogether neglected in the traditional stability analysis of rotor-lubricated bearing systems.

Force coefficients for the short length journal bearing

The general definition of fluid film bearing dynamic force coefficients is above. The analytical derivation of these coefficients for the short length journal bearing follows.

The film thickness for an aligned cylindrical journal bearing is

$$h = C + e(t)\ \cos(\theta); \quad \theta = \Theta - \varphi \tag{5.19}$$

For small amplitude motions about the equilibrium position, $e(t) = e_o + \Delta e(t); \phi(t) = \phi_o + \Delta\phi(t)$, where Δe and $\Delta\phi$ are small radial and angular displacement quantities, respectively.

Eqn. (5.19) is rewritten with $\theta = \Theta - \phi_o$ as

$$h = C + (e_o + \Delta e)\{\cos\theta\ \cos\Delta\phi + \sin\theta\ \sin\Delta\phi\},$$

and, for small amplitude motions note that cos$(\Delta\phi)$~1, sin$(\Delta\phi)$ ~$\Delta\phi$. Then neglecting second order terms,

$$h = C + e_o \cos\theta + \Delta e \cos\theta + e_o \;\Delta\phi\; \sin\theta = h_o + h_1 \tag{5.20}$$

where,

$$h_0 = C + e_o \cos\theta;\quad h_1 = \Delta e \cos\theta + e_o \;\Delta\phi\; \sin\theta \tag{5.21}$$

are the ***zero$_{th}$-order*** and ***first-order or perturbed*** film thicknesses, respectively.

Recall that the Reynolds equation for the short length journal bearing model is[1]:

$$\frac{\partial}{\partial z}\left(\frac{h^3}{12\mu}\frac{\partial P}{\partial z}\right) = \frac{\partial h}{\partial t} + \frac{\Omega}{2}\frac{\partial h}{\partial \theta} \tag{5.22}$$

and,

$$\frac{\partial h}{\partial t} = \frac{\partial h_1}{\partial t} = \Delta\dot{e}\cos\theta + e_o\;\Delta\dot{\phi}\;\sin\theta$$

$$\frac{\partial h}{\partial \theta} = \frac{\partial h_o}{\partial \theta} - \Delta e \sin\theta + e_o\;\Delta\phi\;\cos\theta;\quad \frac{\partial h_o}{\partial \theta} = -e_o\;\sin\theta \tag{5.23}$$

Substitution of (5.23) into (5.22) gives:

$$\frac{\partial}{\partial z}\left(\frac{h^3}{12\mu}\frac{\partial P}{\partial z}\right) = \left\{\Delta\dot{e} + \frac{\Omega}{2}e_o\Delta\phi\right\}\cos\theta + \left\{e_o\left[\Delta\dot{\phi} - \frac{\Omega}{2}\right] - \Delta e\frac{\Omega}{2}\right\}\sin\theta \tag{5.24}$$

Integration of Eqn. (5.24) in the axial direction and applying the boundary conditions at the sides of the bearing, i.e. $P = P_a$ at $z = \pm \frac{1}{2} L$, gives:

$$P - P_a = \frac{6\mu}{C^3 H^3}\left[\left\{\Delta\dot{e} + \frac{\Omega}{2}e_o\Delta\phi\right\}\cos\theta + \left\{e_o\left[\Delta\dot{\phi} - \frac{\Omega}{2}\right] - \Delta e\frac{\Omega}{2}\right\}\sin\theta\right]\left(z^2 - \frac{L^2}{4}\right) \tag{5.25}$$

where $H = h/C$. Integration of the pressure field on the journal surface gives the *radial* and *tangential* components of the fluid film force, i.e.,

$$\begin{bmatrix} F_r \\ F_t \end{bmatrix} = 2\int_0^{L/2}\int_{\theta_1=0}^{\theta_2=\pi} P(\theta,z,t)\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} R\,d\theta\,dz \tag{5.26}$$

[1] This equation is valid for (L/D)<<0.50 and incompressible, isoviscous lubricants. No thermal effects are accounted for in this simple form of the classical Reynolds equation.

where the positive (uncavitated) pressure region lies between $\theta_1 = 0$ and $\theta_2 = \pi$ when P_a is set as zero (nil). Note that it is assumed the perturbed pressure field, due to small amplitude journal motions about the equilibrium position *(e_o, ϕ_o),* does not affect the extent of the steady state lubricant cavitation region, i.e. from 0 to π. This assumption is clearly void if the motions are large in character. By the way, the concept of linear force coefficients is also inadequate when motion amplitudes are large.

Substitution of Eqn. (5.25) into Eqn. (5.26) and integration in the axial direction renders

$$-\begin{pmatrix} F_r \\ F_t \end{pmatrix} = \frac{\mu R L^3}{C^3} \int_0^{\pi} \left[\left\{ \Delta\dot{e} + \frac{\Omega}{2} e_0 \Delta\phi \right\} \frac{\cos\theta}{H^3} + \left\{ e_0 \left[\Delta\dot{\phi} - \frac{\Omega}{2} \right] - \Delta e \frac{\Omega}{2} \right\} \frac{\sin\theta}{H^3} \right] \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} d\theta \qquad (5.27)$$

However, the cubic term in the denominator *(H^3)* also depends on the perturbed journal center displacements. A first-order Taylor series expansion of this terms gives for $h/C=H$

$$h^{-3} = h_0^{-3} - 3 h_0^{-4} h_1 \qquad (5.28)$$

where $h_0 = C + e_o \cos\theta$; $h_1 = \Delta e \cos\theta + e_o\, \Delta\phi \sin\theta$. Substitution of Eqn. (5.28) into (5.27) and neglecting second-order terms, i.e. products of small quantities such as $\Delta e \cdot \Delta\phi$, etc., gives after some considerable algebraic manipulation

$$\begin{bmatrix} F_r \\ F_t \end{bmatrix} = \begin{bmatrix} F_{r0} \\ F_{t0} \end{bmatrix} - \frac{\mu R L^3}{C^3} \begin{bmatrix} J_3^{02} & J_3^{11} \\ J_3^{11} & J_3^{20} \end{bmatrix} \begin{pmatrix} \Delta\dot{e} \\ e\Delta\dot{\phi} \end{pmatrix} - \frac{\mu R L^3}{C^3} \frac{\Omega}{2} \begin{bmatrix} -J_3^{11} + 3\varepsilon J_4^{12} & J_3^{02} + 3\varepsilon J_4^{21} \\ -J_3^{20} + 3\varepsilon J_4^{21} & J_3^{11} + 3\varepsilon J_4^{30} \end{bmatrix} \begin{pmatrix} \Delta e \\ e\Delta\phi \end{pmatrix}$$

(5.29)

where $J_i^{kj} = \int_{\theta_1=0}^{\theta_2=\pi} \frac{(\sin\theta)^k (\cos\theta)^j}{H_0^i} d\theta$ are definite integrals and $H_o =(1 + \varepsilon \cos\, \theta)$.

The bearing **stiffness and damping force coefficients** are, from Eqn. (5.29), specified as

$$K_{rr}=\frac{\mu RL^3}{C^3}\frac{\Omega}{2}\left\{ -J_3^{11}+3\,\varepsilon\, J_4^{12}\right\}=\frac{\mu RL^3\,\Omega}{C^3}\frac{2\,\varepsilon\,(1+\varepsilon^2)}{(1-\varepsilon^2)^3}=-\frac{\partial F_r}{\partial e}$$

$$K_{tt}=\frac{\mu RL^3}{C^3}\frac{\Omega}{2}\left\{ J_3^{11}+3\,\varepsilon\, J_4^{30}\right\}=\frac{\mu RL^3\,\Omega}{C^3}\frac{\varepsilon}{(1-\varepsilon^2)^2}=-\frac{\partial F_t}{e_o\,\partial\varphi}$$

(5.30)

$$K_{rt}=\frac{\mu RL^3}{C^3}\frac{\Omega}{2}\left\{ J_3^{02}+3\,\varepsilon\, J_4^{21}\right\}=\frac{\mu RL^3\,\Omega}{C^3}\frac{\pi}{4\,(1-\varepsilon^2)^{3/2}}=-\frac{\partial F_r}{e_o\,\partial\varphi}$$

$$K_{tr}=\frac{\mu RL^3}{C^3}\frac{\Omega}{2}\left\{ -J_3^{20}+3\,\varepsilon\, J_4^{21}\right\}=-\frac{\mu RL^3\Omega}{C^3}\frac{\pi\,(1+2\,\varepsilon^2)}{4\,(1-\varepsilon^2)^{5/2}}=-\frac{\partial F_t}{\partial e}$$

$$C_{rr}=\frac{\mu RL^3}{C^3}J_3^{02}=\frac{\mu RL^3}{C^3}\frac{\pi}{2}\frac{(1+2\,\varepsilon^2)}{(1-\varepsilon^2)^{5/2}}=-\frac{\partial F_r}{\partial\dot{e}}$$

$$C_{tt}=\frac{\mu RL^3}{C^3}J_3^{20}=\frac{\mu RL^3}{C^3}\frac{\pi}{2\,(1-\varepsilon^2)^{3/2}}=-\frac{\partial F_t}{e_o\,\partial\dot{\phi}} \qquad (5.31)$$

$$C_{rt}=\frac{\mu RL^3}{C^3}J_3^{11}=\frac{\mu RL^3}{C^3}\frac{(-2\,\varepsilon)}{(1-\varepsilon^2)^2}=C_{tr}=-\frac{\partial F_r}{e_o\,\partial\dot{\phi}}=-\frac{\partial F_t}{\partial\dot{e}}$$

The $(\Delta\,\dot{e}\,,e_o\,\Delta\,\dot{\phi})$ correspond to the journal center radial and tangential velocities in the *(r, t)* coordinate system, respectively. Note that the stiffness coefficients $(K_{ij})_{ij=r,t}$ are proportional to the rotational speed (Ω) and fluid viscosity (μ). The damping coefficients $(C_{ij})_{ij=r,t}$ are not a a direct function of the angular speed but depend only on the fluid viscosity and the journal equilibrium position. Without journal rotation there cannot be a fluid film bearing stiffness.

Dimensionless Force Coefficients

The literature presents the force coefficients in dimensionless form according to

$$k_{ij}=K_{ij}\frac{C}{F_0};\;\; c_{ij}=C_{ij}\frac{C\,\Omega}{F_0} \qquad i,j=X,Y \qquad (5.32)$$

where F_o is the static load applied on each bearing (in the *X* direction). [Note that the total load $W=2F_o$ is shared by the two bearings in a symmetric rotor mount].

Recall that $F_o=\frac{\mu\,\Omega(L/C)^2\,LR}{4\sigma}$, where (σ) is the modified Sommerfeld Number defined as (See Notes 4)

$$\frac{\mu\Omega LR}{4F_o}\left(\frac{L}{C}\right)^2 = \sigma = \frac{\left(1-\varepsilon^2\right)^2}{\varepsilon\sqrt{\left\{16\varepsilon^2+\pi^2\left(1-\varepsilon^2\right)\right\}}} \tag{4.8}$$

Using the following definitions:

$$f_{ro} = -\frac{F_{ro}}{F_o} = \cos\phi_o = \frac{4\sigma\varepsilon^2}{\left(1-\varepsilon^2\right)^2}; \quad f_{to} = +\frac{F_{to}}{F_o} = \sin\phi_o = \frac{\pi\sigma\varepsilon}{\left(1-\varepsilon^2\right)^{3/2}} \tag{5.33}$$

the **dimensionless** force coefficients in the *(r, t)* coordinate system become,

$$\begin{gathered} k_{rr} = f_{ro}\frac{2\left(1+\varepsilon^2\right)}{\varepsilon\left(1-\varepsilon^2\right)}; \qquad c_{rr} = f_{to}\frac{2\left(1+2\varepsilon^2\right)}{\varepsilon\left(1-\varepsilon^2\right)} \\ k_{rt} = f_{to}\frac{1}{\varepsilon}; \qquad c_{tr} = c_{rt} = -f_{ro}\frac{2}{\varepsilon} \\ k_{tr} = -f_{to}\frac{\left(1+2\varepsilon^2\right)}{\varepsilon\left(1-\varepsilon^2\right)}; \\ k_{tt} = f_{ro}\frac{1}{\varepsilon}; \qquad c_{tt} = f_{to}\frac{2}{\varepsilon} \end{gathered} \tag{5.34}$$

Force coefficients in the *(X,Y)* coordinate system are easily obtained using the matrix transformation Eqn. (5.16). After a lengthy algebraic procedure,

$$k_{XX} = K_{XX}\frac{C}{F_o} = \frac{f_{ro}}{\varepsilon\left(1-\varepsilon^2\right)}\left\{f_{ro}^2+1+2\varepsilon^2\right\}$$

$$k_{YY} = K_{YY}\frac{C}{F_o} = \frac{f_{ro}}{\varepsilon\left(1-\varepsilon^2\right)}\left\{f_{to}^2+1-\varepsilon^2\right\}$$

$$k_{YX} = K_{YX}\frac{C}{F_o} = \frac{f_{to}}{\varepsilon\left(1-\varepsilon^2\right)}\left\{f_{ro}^2-1+\varepsilon^2\right\}$$

$$k_{XY} = K_{XY}\frac{C}{F_o} = \frac{f_{to}}{\varepsilon\left(1-\varepsilon^2\right)}\left\{f_{ro}^2+1+2\varepsilon^2\right\}$$

$$c_{XX} = C_{XX}\frac{C\Omega}{F_o} = \frac{2\cdot f_{t_o}}{\varepsilon\left(1-\varepsilon^2\right)}\left\{\left(2+\varepsilon^2\right)f_{ro}^2+1-\varepsilon^2\right\}$$

$$c_{YY} = C_{YY}\frac{C\Omega}{F_o} = \frac{2f_{to}}{\varepsilon\left(1-\varepsilon^2\right)}\left\{\left(2+\varepsilon^2\right)f_{t_o}^2-1+\varepsilon^2\right\}$$

$$c_{XY} = C_{XY}\frac{C\Omega}{F_o} = \frac{2\cdot f_{ro}}{\varepsilon\left(1-\varepsilon^2\right)}\left\{\left(2+\varepsilon^2\right)f_{t_o}^2-1+\varepsilon^2\right\} = c_{YX}$$

(5.35)

recall that the X-direction is along the static load F_o.

Figures 5.5 and 5.6 depict the dimensionless force coefficients, **stiffness and damping**, as functions of both the journal eccentricity and the modified Sommerfeld number (σ), respectively. Both representations are necessary since sometimes the journal eccentricity is known a priori while most often, the design parameter, i.e. the Sommerfeld number, is known in advance. In general, the physical magnitude of the stiffness and damping coefficients increases rapidly (nonlinearly) with the journal eccentricity (load too!).

Note that the dimensionless force coefficients do not represent the actual physical trends. For example, at e_o=0, $K_{XX}=K_{YY}=0$, but the dimensionless values $k_{XX}=k_{YY}$=0 in the figures show a definite value. This peculiar result follows from the definition of dimensionless force coefficients using the applied load (F_o). Thus, as $e_o \rightarrow 0$, the bearing load F_o is also nil.

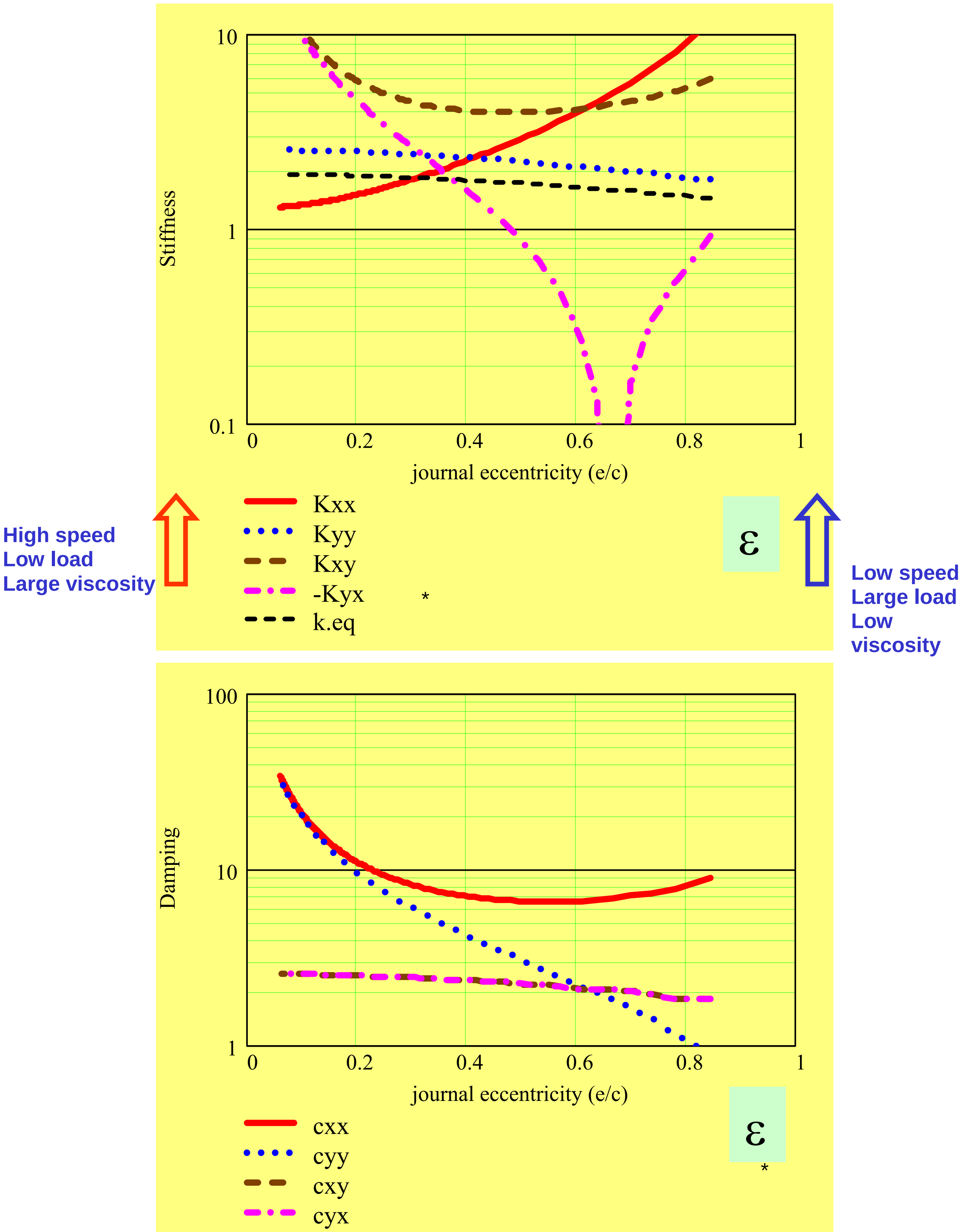
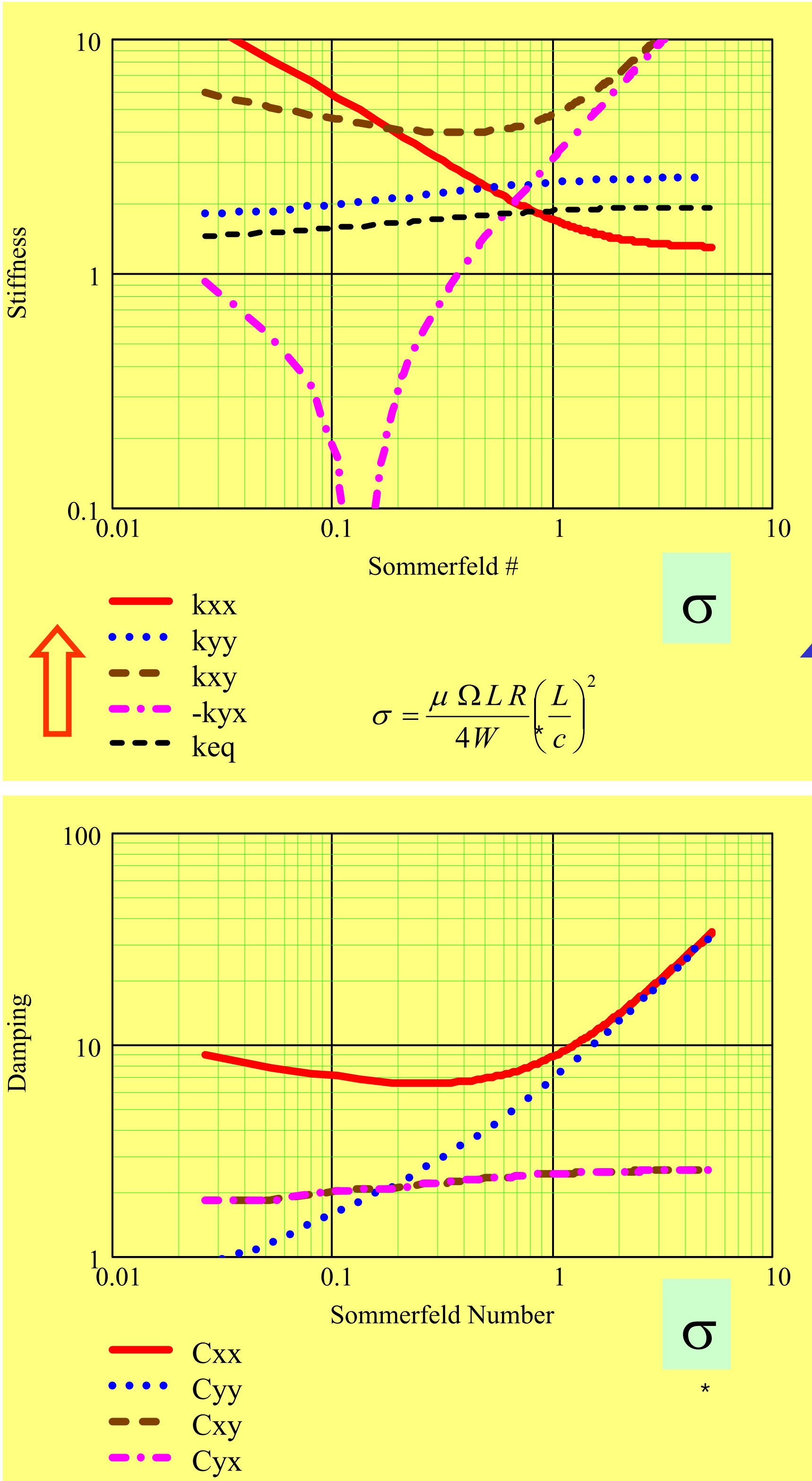


Figure 5.5. Short length journal bearing (dimensionless) stiffness and damping force coefficients vs. journal eccentricity (*ε*)



Low speed
Large load
Low viscosity

High speed
Low load
Large viscosity

Figure 5.6. Short length journal bearing (dimensionless) stiffness and damping force coefficients vs. modified Sommerfeld number (σ)

Dynamic force Coefficients for journal centered operation, i.e. static load=0

As the journal center approaches the bearing center, $e_o \rightarrow 0$, and from the formulas,

$$K_{rr} = K_{tt} = C_{rt} = C_{tr} = 0$$

$$\bar{k} = K_{rt} = -K_{tr} = \frac{\mu R\, L^3 \Omega}{C^3}\frac{\pi}{4} = \frac{\Omega}{2}\bar{c}; \quad \bar{c} = C_{tt} = C_{rr} = \frac{\mu\, R L^3}{C^3}\frac{\pi}{2} \tag{5.36}$$

At $e \rightarrow 0$, $\phi_o = 90^o$, so the force coefficients in the *(X,Y)* system are given as:

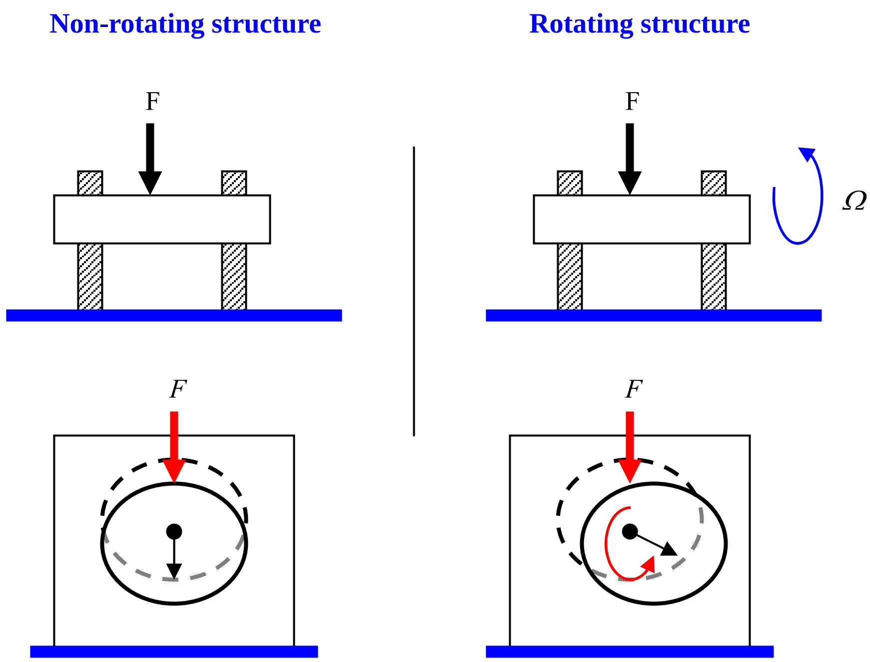
$$\begin{bmatrix} K_{XX} & K_{XY} \\ K_{YX} & K_{YY} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \bar{k} \\ -\bar{k} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & +\bar{k} \\ -\bar{k} & 0 \end{bmatrix}$$

$$\begin{bmatrix} C_{XX} & C_{XY} \\ C_{YX} & C_{YY} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{c} & 0 \\ 0 & \bar{c} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \bar{c} & 0 \\ 0 & \bar{c} \end{bmatrix}$$

Hence

$$K_{XY} = -K_{YX} = \bar{k} = \frac{\mu \Omega R L^3}{C^3}\frac{\pi}{4} = \frac{\Omega}{2}\bar{c}; \quad C_{XX} = C_{YY} = \bar{c} = \frac{\mu R L^3}{C^3}\frac{\pi}{2} \tag{5.37}$$

Thus, at the centered journal position the bearing offers no direct (support) stiffness but only cross-coupled support. A small static load applied on the bearing will cause a journal displacement in a direction orthogonal (perpendicular) to the load. This phenomenon is found in nearly all fluid film bearings of rigid geometry.



The significance of cross-coupling effect from fluid film journal bearings

Stability analysis of rigid rotor supported on plain journal bearings

For small amplitude journal motions about the equilibrium position *(e_o, ϕ_o)*, the equations of motion of a rigid rotor supported on (linear) fluid bearings are:

$$\begin{bmatrix} M & O \\ O & M \end{bmatrix}\begin{pmatrix} \Delta\ddot{X} \\ \Delta\ddot{Y} \end{pmatrix} + \begin{bmatrix} C_{XX} & C_{XY} \\ C_{YX} & C_{YY} \end{bmatrix}\begin{pmatrix} \Delta\dot{X} \\ \Delta\dot{Y} \end{pmatrix} + \begin{bmatrix} K_{XX} & K_{XY} \\ K_{YX} & K_{YY} \end{bmatrix}\begin{pmatrix} \Delta X \\ \Delta Y \end{pmatrix} = M\,u\,\Omega^2 \begin{pmatrix} \sin\Omega\, t \\ \cos\Omega\, t \end{pmatrix} \tag{5.38}$$

Introduce the dimensionless variables:

$$\Delta x = \frac{\Delta X}{C}, \quad \Delta y = \frac{\Delta Y}{C}, \quad \tau = \Omega t, \quad \delta = \frac{u}{C} \tag{5.39}$$

where C is the bearing radial clearance and Ω is the journal or rotor speed (regarded as invariant). Substitution of Eqn. (5.39) into (5.38) gives:

$$p^2 \begin{bmatrix} \Delta x'' \\ \Delta y'' \end{bmatrix} + \begin{bmatrix} c_{XX} & c_{XY} \\ c_{YX} & c_{YY} \end{bmatrix}\begin{bmatrix} \Delta x' \\ \Delta y' \end{bmatrix} + \begin{bmatrix} k_{XX} & k_{XY} \\ k_{YX} & k_{YY} \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = p^2\,\delta \begin{bmatrix} \sin(\tau) \\ \cos(\tau) \end{bmatrix} \tag{5.40}$$

where $(\,')=\dfrac{d}{d\tau}$; $p^2 = \dfrac{C\,M\,\Omega^2}{F_o}$ is a dimensionless mass, and $k_{ij} = K_{ij}\,(C/F_o)$, $c_{ij} = C_{ij}\,(C\Omega/F_o)$ are the dimensionless dynamic force coefficients.

It is of interest to study if the rotor-bearing system is **stable** for small amplitude journal center motions (perturbations) about the equilibrium position. To this end, set the imbalance parameter $\delta = 0$ in the equations above to obtain,

$$p^2\begin{bmatrix}\Delta x''\\ \Delta y''\end{bmatrix}+\begin{bmatrix}c_{XX} & c_{XY}\\ c_{YX} & c_{YY}\end{bmatrix}\begin{bmatrix}\Delta x'\\ \Delta y'\end{bmatrix}+\begin{bmatrix}k_{XX} & k_{XY}\\ k_{YX} & k_{YY}\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \tag{5.41}$$

If the rotor-bearing system is to become **unstable**, this will occur at a **threshold speed of rotation** (Ω_s) and the rotor will perform (undamped[2]) orbital motions at a whirl frequency (ω_s). These motions, satisfying equation (5.42), are of the form:

$$x = A\,e^{j\omega_s t} = A\,e^{j\bar{\omega}\tau}\ ;\ y = B\ e^{j\omega_s t} = B\,e^{j\bar{\omega}\tau};\quad j=\sqrt{-1} \tag{5.42}$$

where $\bar{\omega} = \omega_s / \Omega_s$ is known as the **whirl frequency ratio**, i.e. the ratio between the rotor whirl or precessional frequency and the rotor onset speed of instability.

Substitution of Eqn. (5.42) into (5.41) leads to:

$$\begin{bmatrix}-p_s^2\bar{\omega}_s^2+k_{XX}+j\bar{\omega}_s c_{XX} & k_{XY}+j\bar{\omega}_s c_{XY}\\ k_{YX}+j\bar{\omega}_s c_{YX} & -p_s^2\bar{\omega}_s^2+k_{YY}+j\bar{\omega}_s c_{YY}\end{bmatrix}\begin{bmatrix}A\\ B\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \tag{5.43}$$

The determinant of the system of equations must be zero for a non-trivial solution of the homogenous system of equations, i.e.

$$\begin{aligned}\Delta = \left(-p_s^2\bar{\omega}_s^2+k_{XX}+j\bar{\omega}_s c_{XX}\right)\cdot\left(-p_s^2\bar{\omega}_s^2+k_{YY}+j\bar{\omega}_s c_{YY}\right)\\ -\left(k_{YX}+j\bar{\omega}_s c_{YX}\right)\cdot\left(k_{XY}+j\bar{\omega}_s c_{XY}\right)=0\end{aligned} \tag{5.44}$$

After a rather lengthy algebraic manipulation, the real and imaginary parts of Δ above render,

$$p_s^2\bar{\omega}_s^2 = k_{eq} = \frac{k_{XX}c_{YY}+k_{YY}c_{XX}-c_{YX}k_{XY}-c_{XY}k_{YX}}{c_{XX}+c_{YY}} = \frac{C\,M\omega_S^2}{F_o} \tag{5.45}$$

and

$$\bar{\omega}_s^2 = \frac{\left(k_{eq}-k_{XX}\right)\left(k_{eq}-k_{YY}\right)-k_{XY}\cdot k_{YX}}{c_{XX}c_{YY}-c_{XY}c_{YX}} = \left(\frac{\omega_s}{\Omega_s}\right)^2 \tag{5.46}$$

For a given value of journal eccentricity (ε_o), i.e. a given Sommerfeld number (σ), one evaluates Eqn. (5.45) to obtain the dimensionless equivalent stiffness k_{eq}, and then (5.46) to obtain the whirl frequency ratio $\bar{\omega}_S$. This substitution then yields $p_s^2 = k_{eq}/\bar{\omega}_S^2$ (system critical mass) which in turn renders the onset speed of instability Ω_s.

[2] Recall that in a second order mechanical system an equivalent damping ratio>0 causes the damping or attenuation of motions induced by small perturbations. A damping equal to zero produces sustained periodic motions without decay or growth and indicates the threshold between stability and instability (amplitude growing motions).

Figures 5.7 and 5.8 depict the whirl frequency ratio $\bar{\varpi} = \omega_s / \Omega_s$ and the dimensionless threshold speed of instability *(p_s)* versus both the journal eccentricity and Sommerfeld number, respectively. Note that for near centered journal operation, i.e. large Sommerfeld numbers, the whirl frequency is 0.50, i.e. half-synchronous whirl.

Other important information is also obtained. If one assumes that the current (operating) rotational speed Ω is the onset speed of instability, then from the relations above, the magnitude of ½ system mass *(M)* is obtained, and which would make the rotor-bearing system become unstable. This mass is known as the **critical mass,** *M_c,* and corresponds to the limit mass which the system can carry dynamically. If the total mass is equal or larger than twice M_c, then the system will be unstable at the rated speed Ω ([3]).

The whirl frequency ratio, $\bar{\omega} = \omega_s / \Omega_s$, is the ratio between the rotor whirl frequency and the *onset speed of instability*. Note that this ratio, as given by Eqn. (5.46), depends only on the fluid film bearing characteristics and the equilibrium eccentricity, and it is independent of the rotor characteristics (rotor mass and flexibility).

The parameter k_{eq} is a journal bearing (dimensionless) **equivalent stiffness** and depicted in Figures 5.5 and 5.6. From the definitions of threshold speed and whirl ratio, $p_s^2 = M\,\Omega_s^2 \left(C/F_o\right)$ and $\bar{\varpi}_s = \omega_s / \Omega_s$, then

$$M\,\omega_s^2 = k_{eq}\left(\frac{F_o}{C}\right) = K_{eq}$$

Thus, the whirl or precessional frequency is given by

$$\omega_s = \sqrt{\frac{K_{eq}}{M}} = \omega_n \tag{5.47}$$

i.e., the whirl frequency equals the **natural frequency** of the rigid rotor supported on journal bearings.

For operation close to the concentric position, $\varepsilon_o \to 0$, i.e. large Sommerfeld numbers (no load condition), the force coefficients are, see Eqn. (5.37),

$$k_{XX} = k_{YY} = 0; \quad c_{XX} = c_{YY}; \quad k_{XY} = -k_{YX}; \quad c_{XY} = c_{YX} = 0 \tag{5.37}$$

$$k_{eq} = \left(k_{XX}c_{XX} + c_{XY}k_{XY}\right)/c_{XX} = 0$$

and

$$\frac{\omega_s}{\Omega_s} = \frac{k_{XY}}{c_{XX}} = 0.50 \quad \text{as } \varepsilon \to 0 \tag{5.48}$$

[3] Recall that each bearing carries half the static load, and also half the dynamic or inertia load *($2.M_c C\,\Omega^2$).*

The 0.5 magnitude for whirl frequency ratio *(WFR)* (or *50% whirl* as is called in industry) is a characteristic of hydrodynamic plain journal bearings. It shows us that at the onset of instability the rotor whirls at its natural frequency, which equals to 50% of the rotor speed. Furthermore, under no externally applied loads, $F_o=0$, as in vertically turbomachinery, the bearing possesses no support stiffness, i.e. $K_{eq}=0$ and the system natural frequency (ω_n) is zero, i.e. the rotor-bearing system whirls at all speeds.

Note that if $k_{XY} = 0$, i.e. the fluid film bearing does not show cross-coupled effects, then the *WFR* = 0, i.e. no whirl occurs and the system is ALWAYS stable. (Asymmetrical) cross-coupled stiffnesses are thus responsible for the instabilities so commonly observed in rotors mounted on journal bearings.

If the whirl frequency ratio is 0.50, then the maximum rotational speed that the rotor-bearing system can attain is just,

$$\Omega_{\max} = \frac{\omega_s}{0.50} = 2\omega_s = 2\omega_n \tag{5.49}$$

i.e., twice or two times the natural frequency (or observed rigid rotor critical speed).

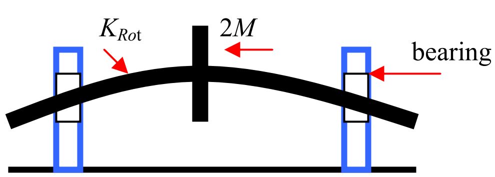
Figures 5.7, 5.8, and 5.9 show, respectively, the whirl frequency ratio, the dimensionless critical mass parameter (p_s), and the dimensionless critical mass $(p_s)^2$ versus the Sommerfeld number and operating journal eccentricity. The results show that a rigid-rotor supported on plain journal bearings is always **STABLE** for operation with journal eccentricity ratios ε > 0.75 (small *Sommerfeld numbers*) for all *L/D* ratios. Note that the critical mass and the whirl ratio are relatively insensitive for operation with eccentricities $\varepsilon_o < 0.50$.

Keep in mind that increasing the rotational speed of the rotor-bearing system determines larger Sommerfeld numbers, and consequently, operation at smaller journal eccentricities for the same applied static load. Thus, operation at ever increasing speeds will eventually lead to a rotor dynamically unstable system as the results show.

Effects of Rotor Flexibility

A similar analysis can be performed considering rotor flexibility. This analysis is more laborious though straightforward. The analysis shows that the whirl frequency ratio is not affected by the rotor flexibility. However, the onset speed of instability decreases dramatically!

The relationship for the **threshold speed of instability of a flexible rotor** is:



$$p_{sf}^2 = \frac{p_s^2}{1 + k_{eq}\left(\frac{\Gamma}{C}\right)} \tag{5.50}$$

where the sub index f denotes the flexible rotor, K_{rot} is the rotor stiffness on each side of the center disk, and $\Gamma = F_o / K_{rot}$ is the rotor **static sag** or **elastic deformation** at midspan.

The elastic shaft and bearing are mounted in series, i.e. the bearing and shaft flexibilities add (reciprocal of stiffnesses), and thus the equivalent system stiffness is lower than that of the bearings alone, and therefore the system natural frequency is lower.

Figure 5.10 shows the threshold speed of instability (p_{sf}) for a flexible rotor mounted on plain short length journal bearings. Note that the more flexible the rotor is, the lower the threshold speed of instability. If the fluid film bearings are designed too stiff (small Sommerfeld numbers), then the natural frequency of the rotor-bearing system is just $(K_{rot}/M)^{0.5}$, irrespective of the bearing configuration.

Postscript

See the Appendix to these notes for further understanding on the nature of the cross-coupled coefficients driving the whirl motion.

The MATHCAD programs attached include the algebraic formulas for evaluation of the bearing force coefficients in actual applications.

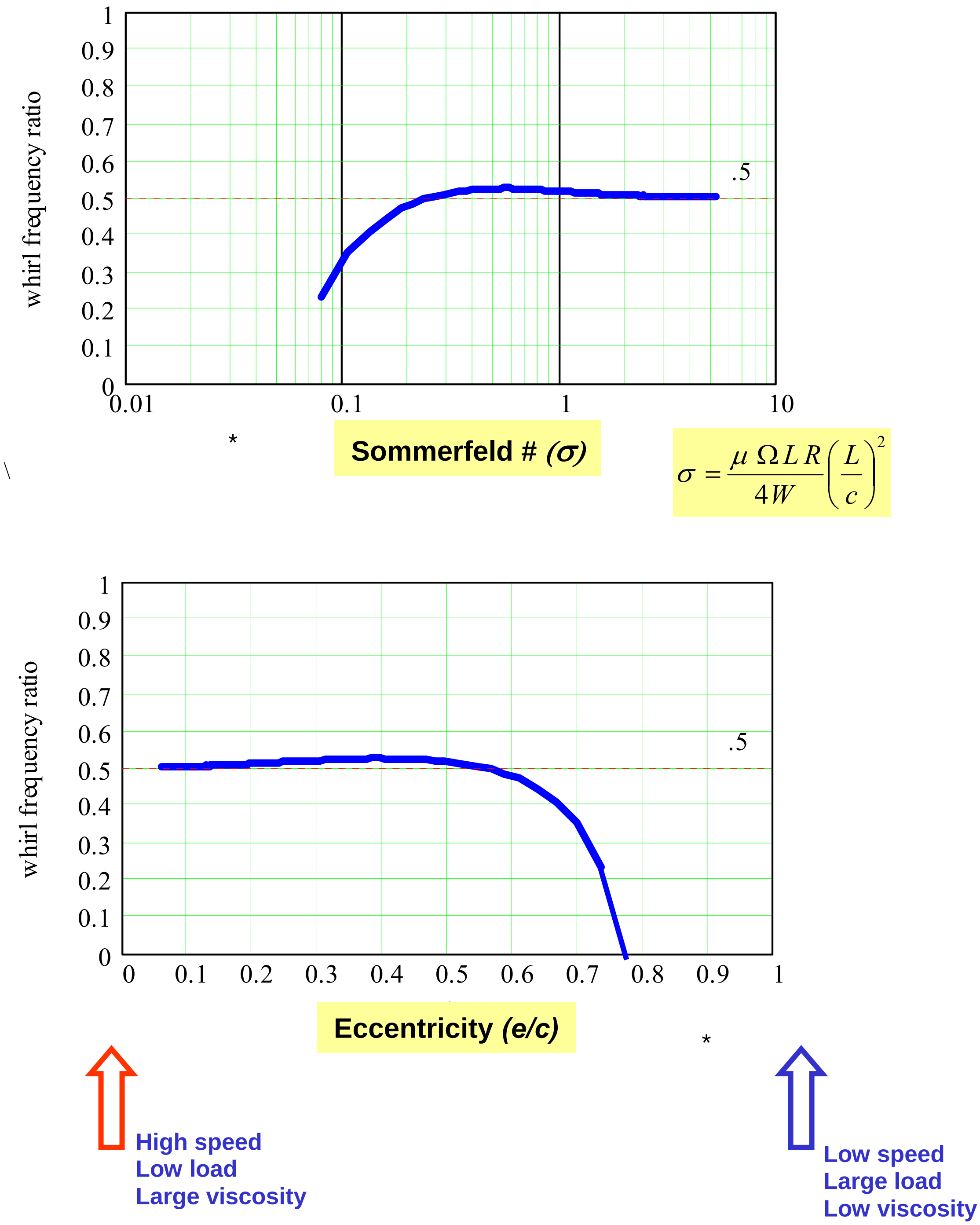
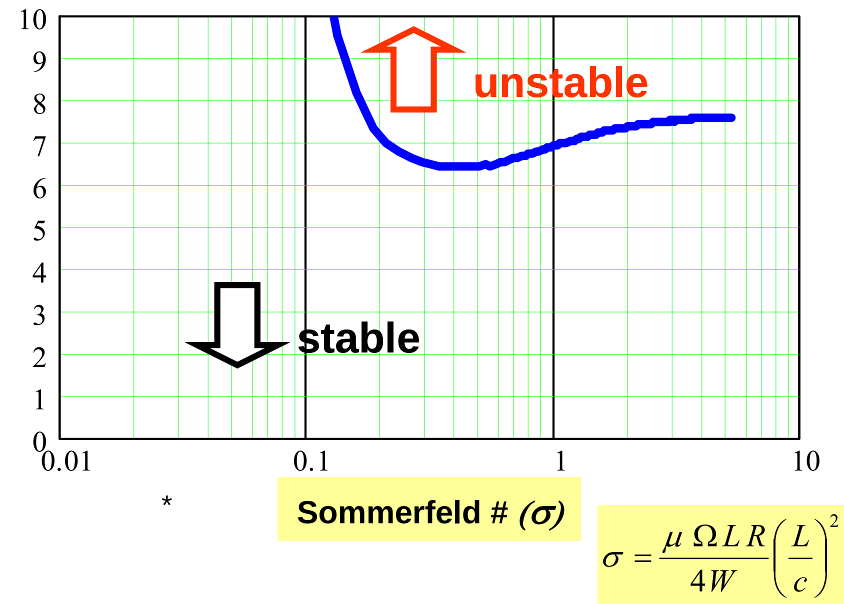


Figure 5.7. Whirl frequency ratio vs. modified Sommerfeld number (σ) and journal eccentricity (ε)



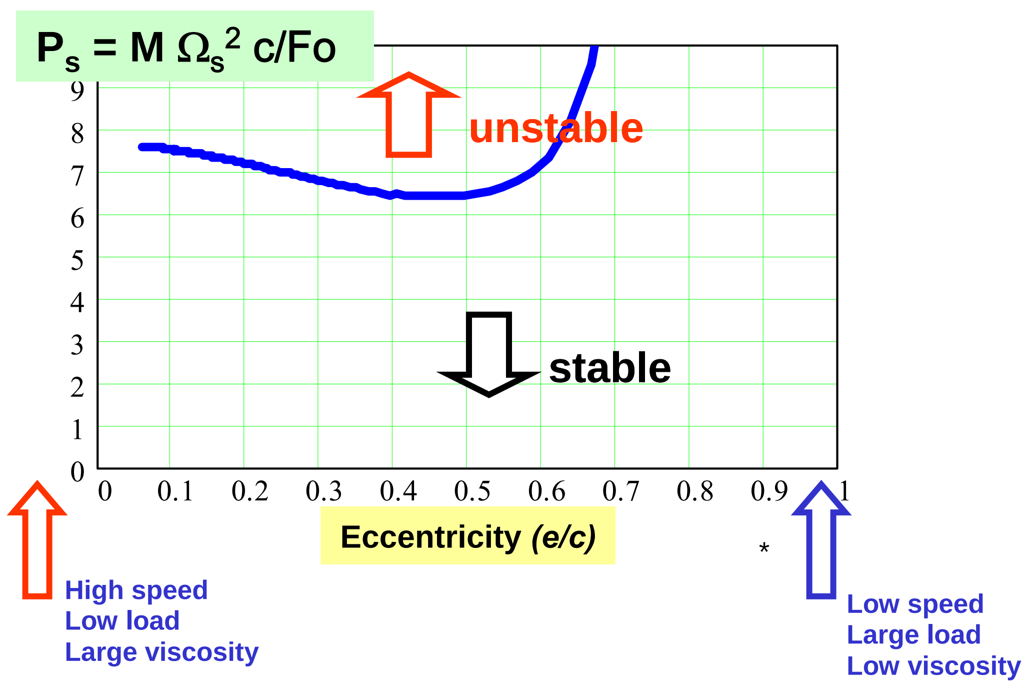


Figure 5.8. Dimensionless threshold speed of instability (p_s) vs. modified Sommerfeld number (σ) and journal eccentricity (ε)

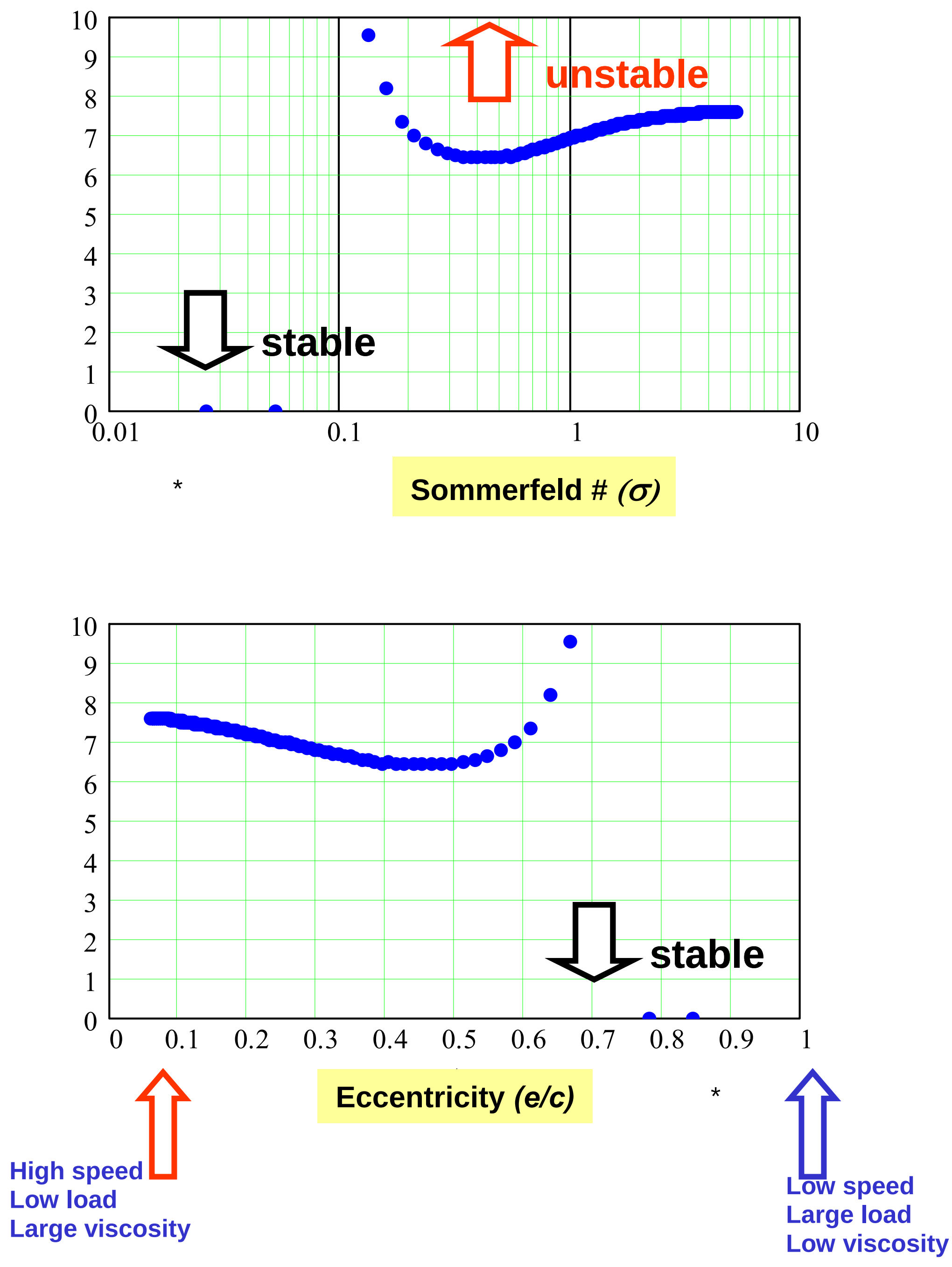
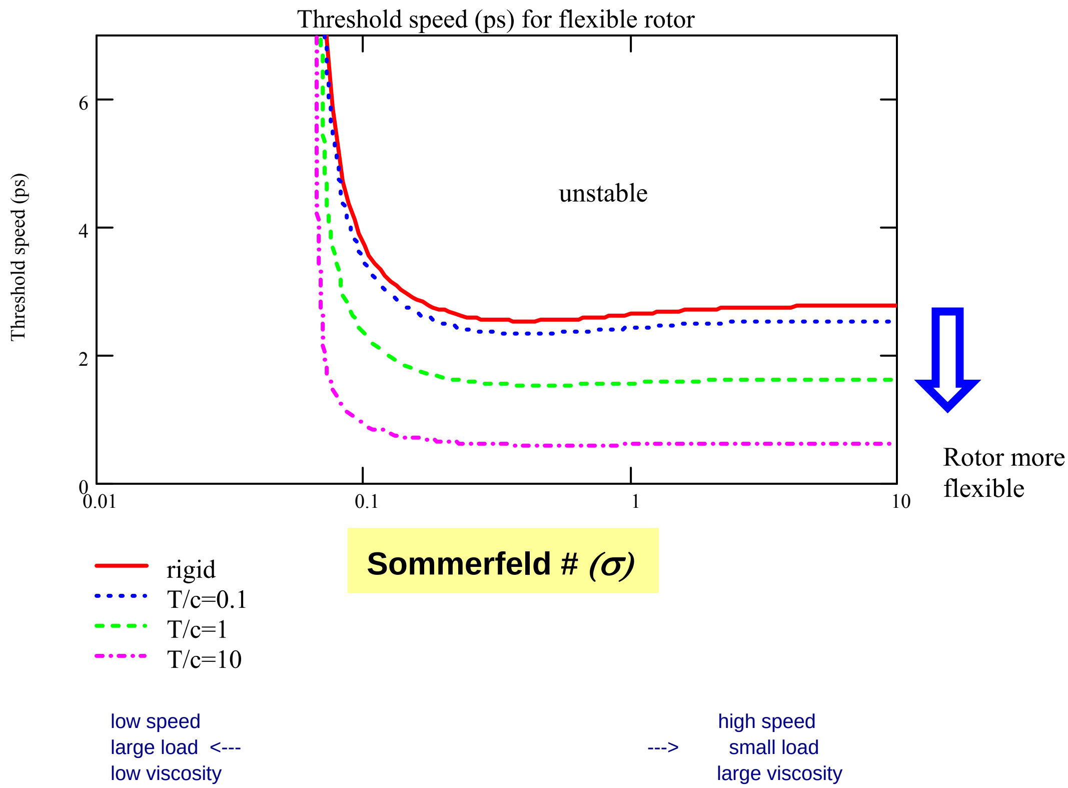


Figure 5.9. Dimensionless critical mass (m_c=p_s^2) vs. modified Sommerfeld number (σ) and journal eccentricity (ε).



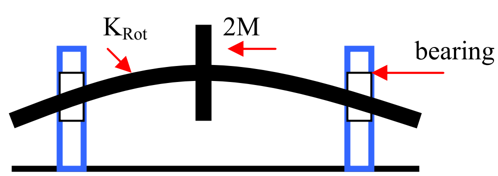


Figure 5.10. Dimensionless threshold speed of instability (p_s) for flexible rotor vs. modified Sommerfeld number (σ). Static sag (Γ/c) varies

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[1] Turbomachinery Rotordynamics, (chapter 3), D. Childs, John Wiley & Sons, Inc., 1993.
[2] Rotordynamics of Turbomachinery, J.M. Vance, J., Wiley Inter-Science Pubs., 1988.
[3] Self-Excited, Stationary Whirl Orbits of a Journal in a Sleeve Bearing, J. Lund, Ph.D. Thesis, Rensselaer Polytechnic Institute, Troy, N.Y., 1966.
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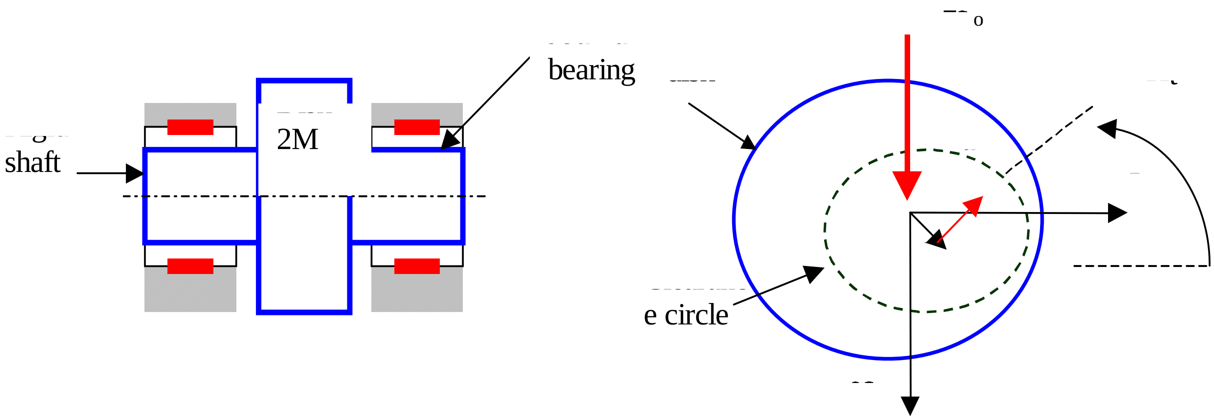
[5] A Table of the Journal Bearing Integrals, J.F. Booker, ASME Journal of Basic Engineering, pp. 533-535, 1965.
[6] Tribology – Friction, Lubrication & Wear, A. Szeri, Hemisphere Pubs, 1980.

Stability and Imbalance Response of a Jeffcott-Rotor Supported on Short Length Journal Bearings

(c) Dr. Luis San Andres UT/2000, TAMU/2006 Extended with eigen analysis: 10/00 TAMU



(u) imbalance, (e) journal eccentricity

DATA for rotor

$W_T := 600 \cdot 4.448 \cdot N$ Rotor weight

Gravitational acceleration $g = 9.807\, m\, s^{-2}$

$W := \frac{W_T}{2}$ $W = 1.334 \times 10^3 N$ Static load per bearing

$M := \frac{W}{g}$ $M = 136.071\, kg$ 1/2 rotor mass

BEARING GEOMETRY, OIL viscosity and Operating conditions:

Top shaft speed for analysis

$RPM\_max := 12000$

$D := 0.15 \cdot m$ journal diameter

$L := 2 \cdot 0.025 \cdot m$ bearing length

$c := 0.060 \cdot mm$ radial clearance

$\mu_R := 2.5 \cdot 10^{-6} \cdot 6894.757 \cdot \frac{N \cdot s}{m^2}$ fluid viscosity at reference etmperature: $T_R := 50$ $\alpha := 0.030$

Oil viscosity-temperature coefficient (1/deg C)

$T := 60$ degC operating (feed) temperature

$k_{shaft} := 10^8 \cdot \frac{N}{m}$ 1/2 shaft stiffness $a := 0.2 \cdot c$ Amplitude of imbalance on rotor disk

Static shaft deflection due to rotor weight = % of clearance

$\delta := \frac{W}{k_{shaft}}$ $\Gamma := \frac{\delta}{c}$ $\Gamma = 0.222$ $1N = 1\, kg\, m\, s^{-2}$

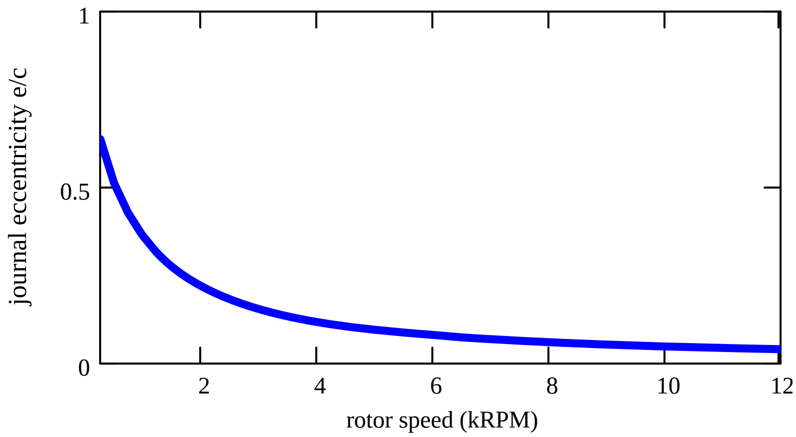
$\delta = 1.334 \times 10^{-5} m$ if Γ > 1 then rotor is quite flexible

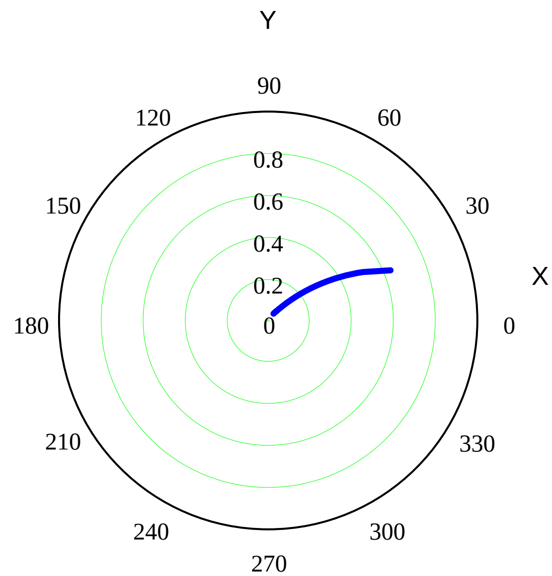
$\mu := \mu_R \cdot e^{-\alpha \cdot (T - T_R)}$ $\mu = 0.013 \frac{N \cdot s}{m^2}$ Operating viscosity

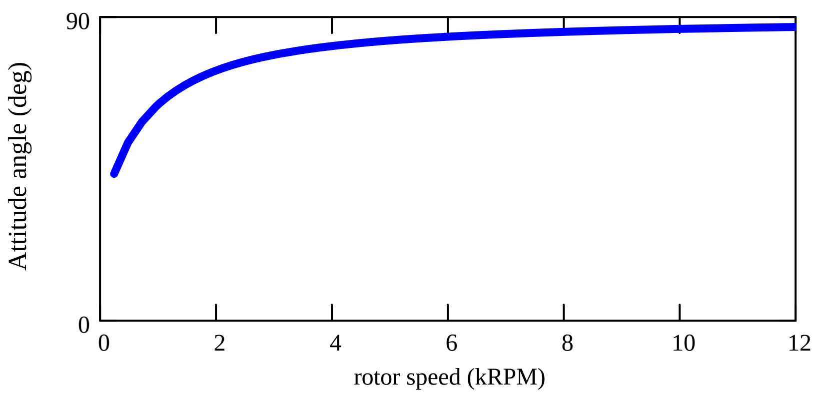
▸ Static performance

Journal eccentricity ratio and attitude angle for STATIC equilibrium

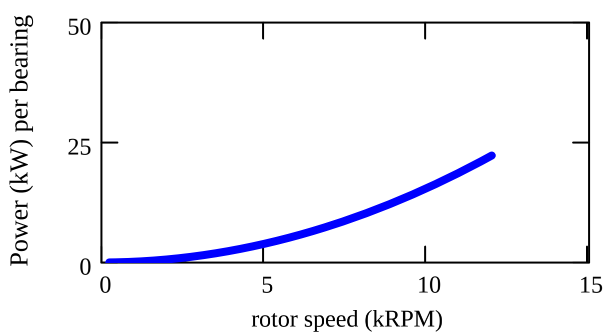
Short bearing solution for an isoviscous-incompressible lubricant



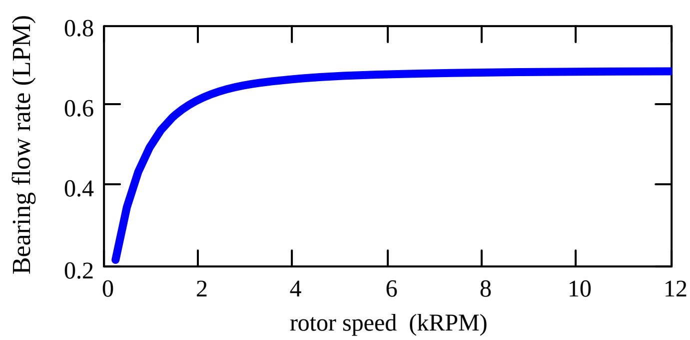




Power loss (kW) / bearing

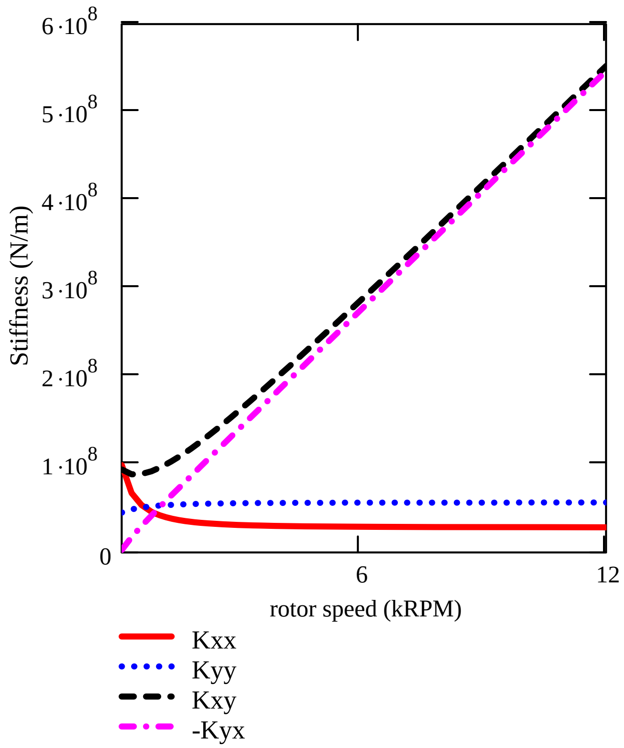


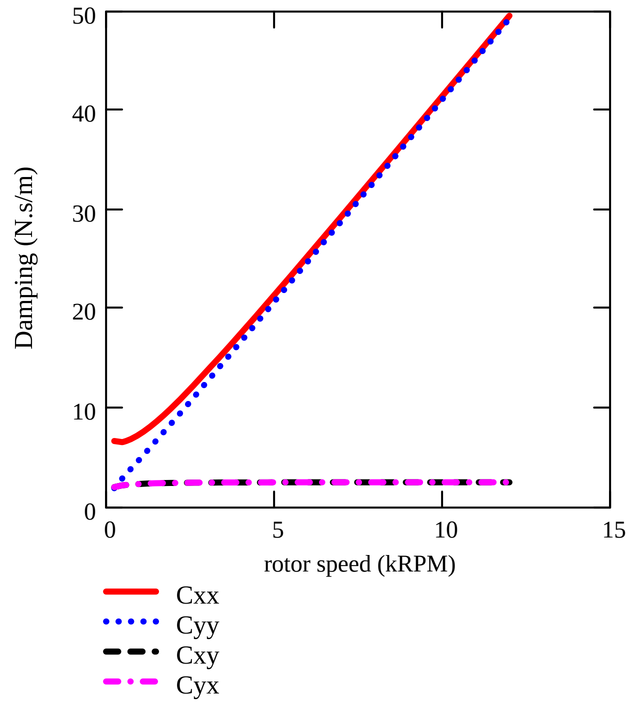
Side flow rate / bearing (LPM)



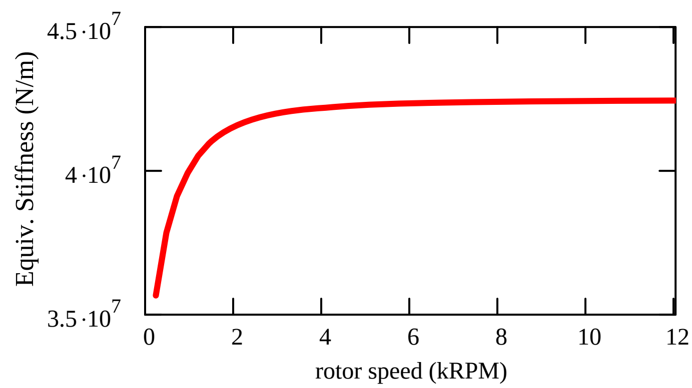
JB force coeffs & stability

Stiffness and damping force coefficients vs shaft speed

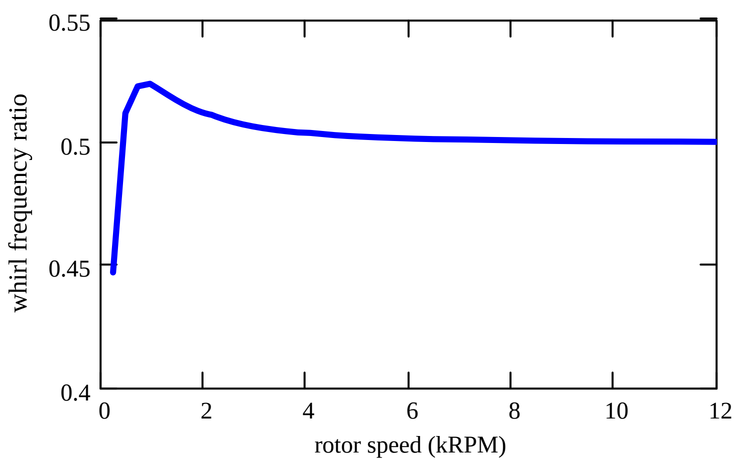




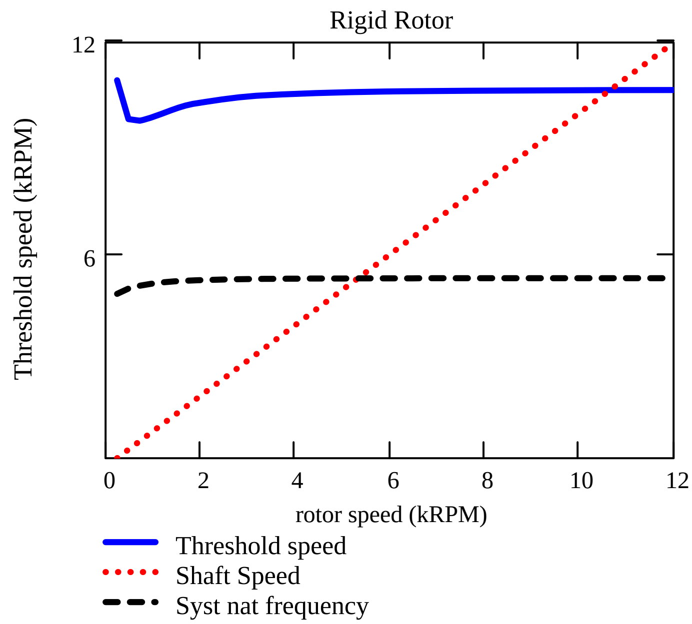
Equivalent bearing stiffness for rigid rotor



Whirl frequency ratio

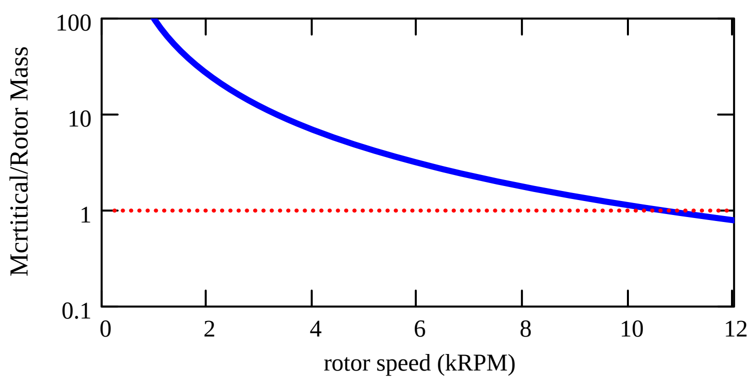


Threshold speed of instability(rpm)



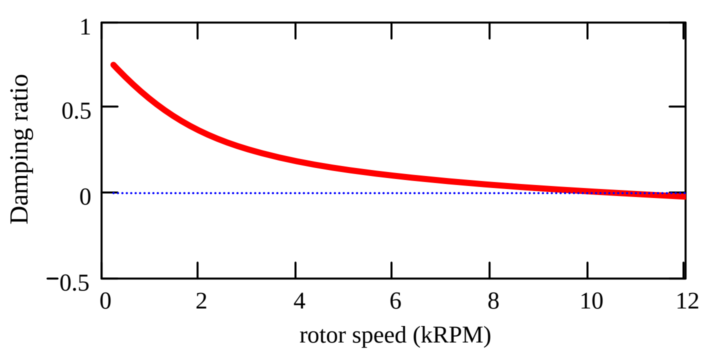
Critical Rotor Mass for rigid rotor

Recall rotor mass: $2 \cdot M = 272.142\ kg$



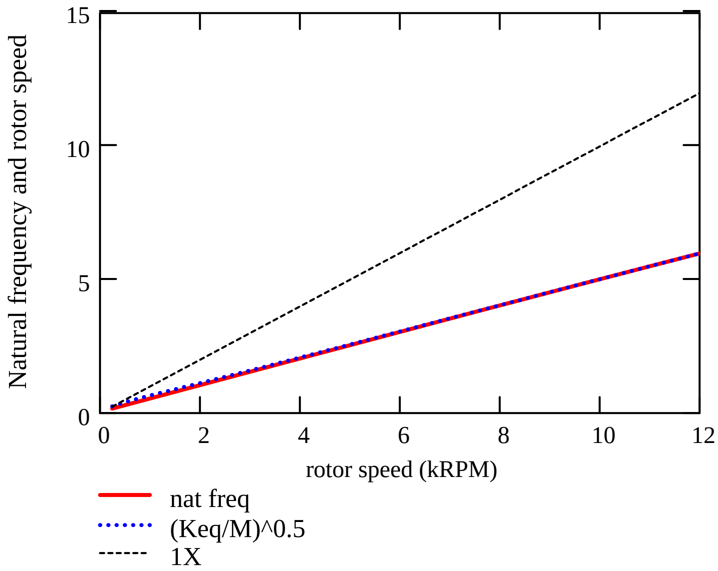
System goes unstable when ratio < 1

Eigenvalues rig rotor



damping ratio for system,

Search for intersection of real part with Y=0 axis to determine thrshold speed of instability.



natural frequency - from eigenvalue analysis and from (Keq/M) formula

Effect of shaft flexibility on threshold speed of instability:

$k_{shaft} = 1 \times 10^{8}\,\frac{N}{m}$

Static Deflection (Γ=δ/c) and Stiffness of ***Flexible*** shaft $\Gamma = 0.222$

$\delta = 1.334 \times 10^{-5}\,m$

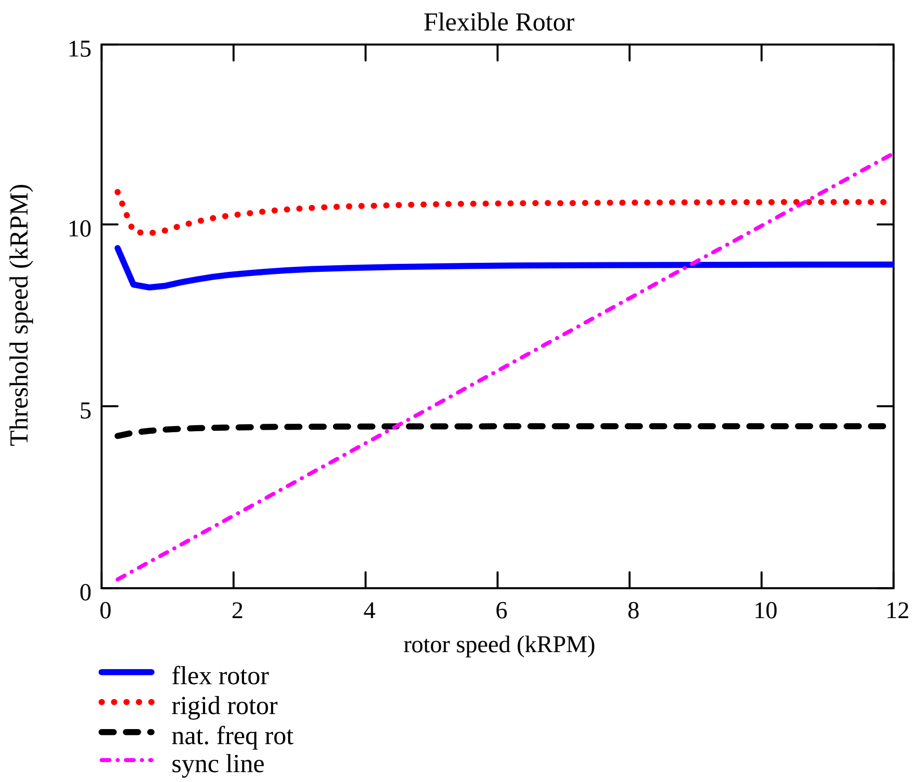
threshold speed of instability

$$\Omega_{tf_n} := \frac{\Omega_{t_n}}{\left(1 + k_{eq_n} \cdot \Gamma\right)^{\frac{1}{2}}}$$

NEARLY RIGID ROTOR

Natural Frequency (rpm) of ***Flexible*** Shaft

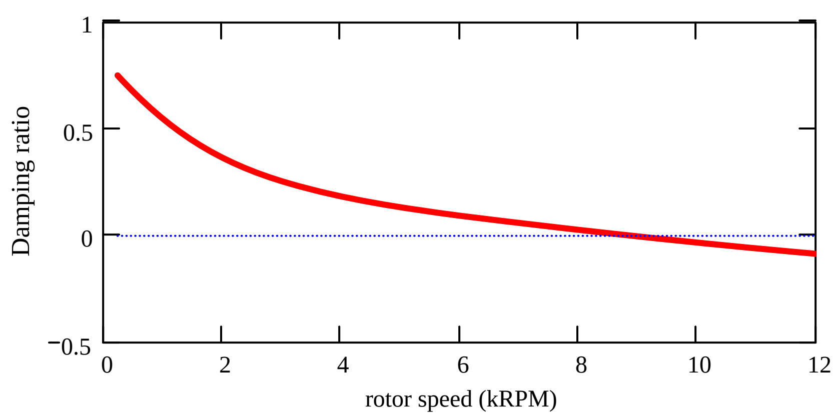
$$\omega_{n_n} := WFR_n \cdot \Omega_{tf_n}$$



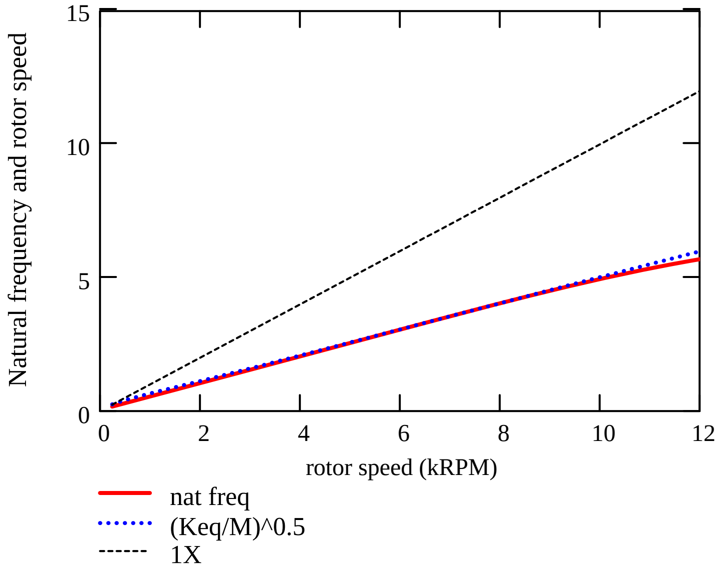
$\Gamma = 0.222$

The threshold speed of instability is lower for the flexible rotor than for the rigid rotor model.

Eigenvalues: flexible rotor



Damping ratio of flex rotor system



Natural frequency - flex rotor and simple formula (rigid rotor)

$\omega_{elastic} = 8.186 \times 10^{3}$ RPM

Synchronous imbalance response of flexible rotor

The equations of motion for both rotor and journal bearings are given below. The coordinates of rotor and disk motion have their origin at the static equilibrium position. No damping at rotor midspan, no mass lumped at the bearings.

$$m \cdot \left(\frac{d^2 \cdot X}{dt^2}\right) + k_{shaft} \cdot (X - x) = m \cdot a \cdot \omega^2 \cdot \cos(\omega \cdot t)$$

$$m \cdot \left(\frac{d^2 \cdot Y}{dt^2}\right) + k_{shaft} \cdot (Y - y) = m \cdot a \cdot \omega^2 \cdot \sin(\omega \cdot t)$$

ROTOR

$$\left(C_{I,J}\right) \cdot \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} + \left(K_{I,J}\right) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = -k \cdot \begin{pmatrix} x - X \\ y - Y \end{pmatrix}$$

MASSLESS BEARINGS

$a = 1.2 \times 10^{-5}$ m

imbalance displacement

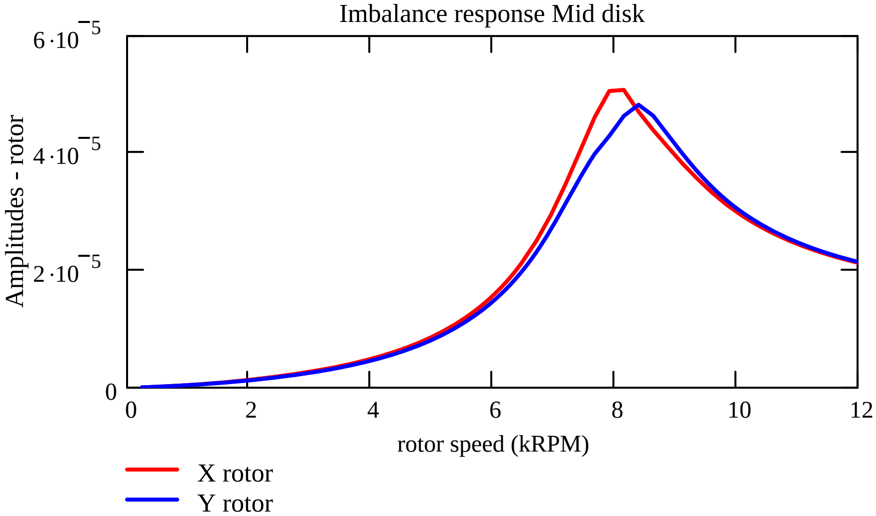
$\frac{a}{c} = 0.2$

The rotor disk (X,Y) and journal center displacements (x,y) are synchronous with the imbalance excitation, i.e.

$$X = X_c \cdot \cos(\omega \cdot t) + X_s \cdot \sin(\omega \cdot t) \qquad Y = Y_c \cdot \cos(\omega \cdot t) + Y_s \cdot \sin(\omega \cdot t)$$

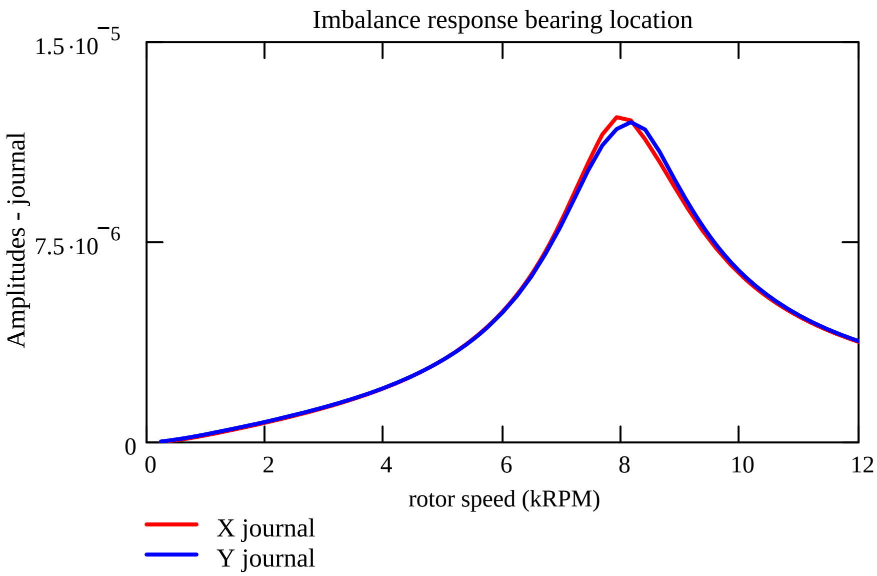
$$x = x_c \cdot \cos(\omega \cdot t) + x_s \cdot \sin(\omega \cdot t) \qquad y = y_c \cdot \cos(\omega \cdot t) + y_s \cdot \sin(\omega \cdot t)$$

Synchronous response



amplitudes in [m]

Amplitudes of motion at rotor midspan



$c = 6 \times 10^{-5}\,m$

$a = 1.2 \times 10^{-5}\,m$

Amplitudes of motion at bearings

Exercise: Calculate the major and minor axes of the ellipses describing the (X,Y) motions. See Appendix A of Childs' Rotordynamics Book:.

Notes: You could update this program to account for
a) bearing mass MB, a fraction of total rotor mass,
b) introduce damping at the rotor midspan, Cs.

Effect of shaft flexibility on threshold speed of instability:

$k_{shaft} = 1 \times 10^{7} \frac{N}{m}$

Static Deflection (Γ=δ/c) and Stiffness of ***Flexible*** shaft $\Gamma = 2.224$

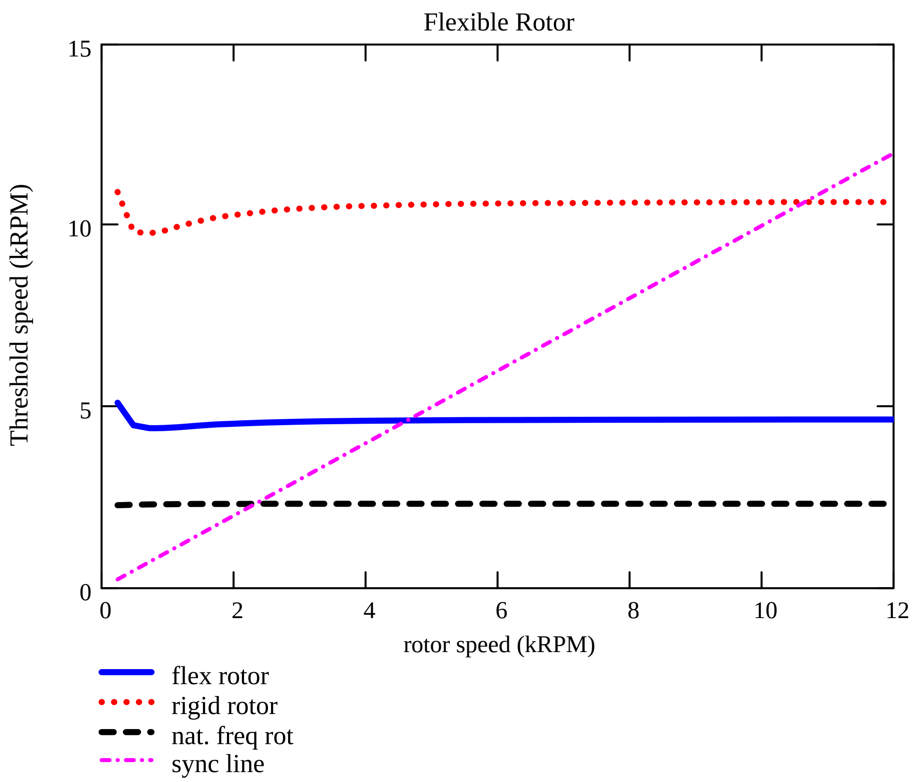
$\delta = 1.334 \times 10^{-4}\, m$

threshold speed of instability $\Omega tf_n := \dfrac{\Omega t_n}{\left(1 + keq_n \cdot \Gamma\right)^{\frac{1}{2}}}$

FLEXIBLE ROTOR

Natural Frequency (rpm) of ***Flexible*** Shaft

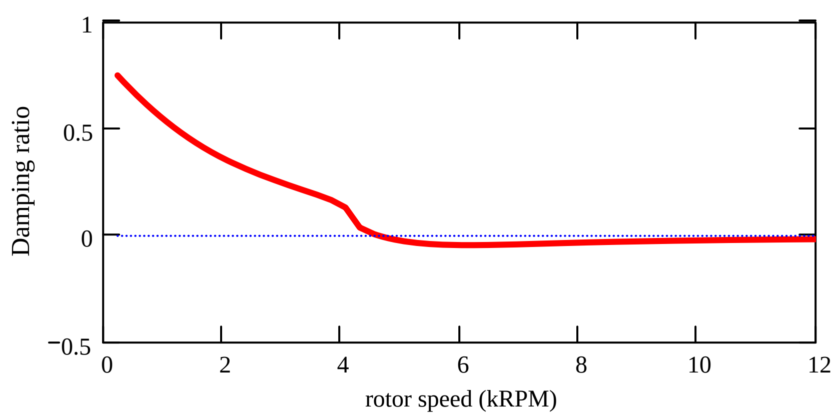
$\omega n_n := WFR_n \cdot \Omega tf_n$



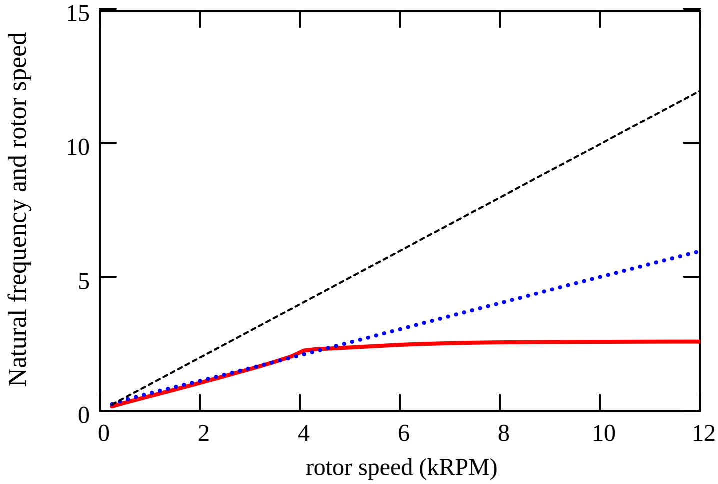
$\Gamma = 2.224$

The threshold speed of instability is lower for the flexible rotor than for the rigid rotor model.

Eigenvalues: flexible rotor



Damping ratio of flex rotor system



Natural frequency - flex rotor and simple formula (rigid rotor)

$\omega_{elastic} = 2.589 \times 10^{3}$ RPM

Synchronous imbalance response of flexible rotor

The equations of motion for both rotor and journal bearings are given below. The coordinates of rotor and disk motion have their origin at the static equilibrium position. No damping at rotor midspan, no mass lumped at the bearings.

$$m \cdot \left(\frac{d^2 \cdot X}{dt^2}\right) + k_{shaft} \cdot (X - x) = m \cdot a \cdot \omega^2 \cdot \cos(\omega \cdot t)$$

$$m \cdot \left(\frac{d^2 \cdot Y}{dt^2}\right) + k_{shaft} \cdot (Y - y) = m \cdot a \cdot \omega^2 \cdot \sin(\omega \cdot t)$$

ROTOR

$$\left(C_{I,J}\right) \cdot \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} + \left(K_{I,J}\right) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = -k \cdot \begin{pmatrix} x - X \\ y - Y \end{pmatrix}$$

MASSLESS BEARINGS

$a = 1.2 \times 10^{-5}$ m

imbalance displacement

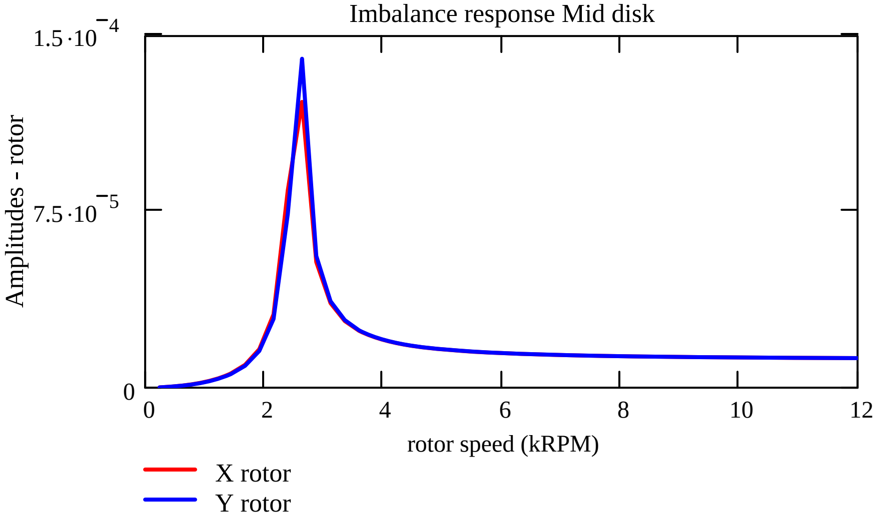
$\frac{a}{c} = 0.2$

The rotor disk (X,Y) and journal center displacements (x,y) are synchronous with the imbalance excitation, i.e.

$$X = X_c \cdot \cos(\omega \cdot t) + X_s \cdot \sin(\omega \cdot t) \qquad Y = Y_c \cdot \cos(\omega \cdot t) + Y_s \cdot \sin(\omega \cdot t)$$

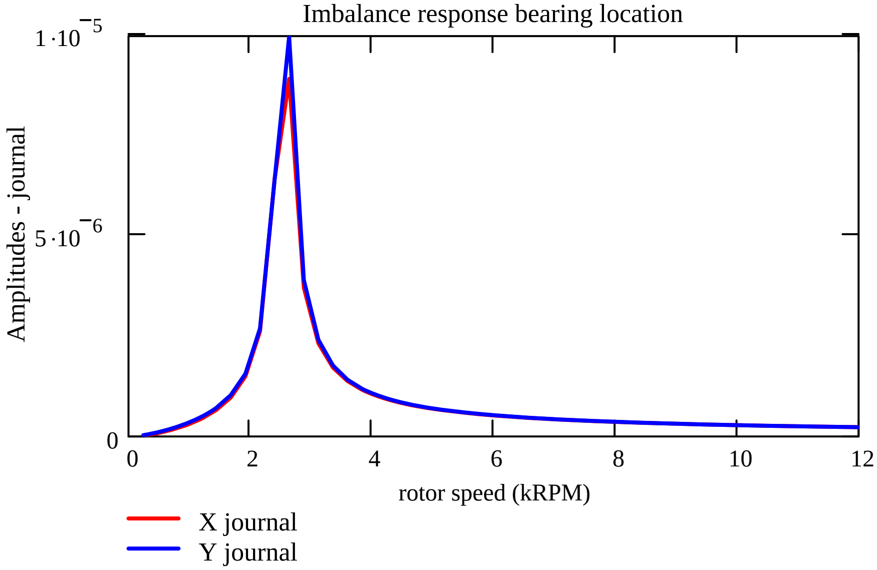
$$x = x_c \cdot \cos(\omega \cdot t) + x_s \cdot \sin(\omega \cdot t) \qquad y = y_c \cdot \cos(\omega \cdot t) + y_s \cdot \sin(\omega \cdot t)$$

Synchronous response



amplitudes in [m]

Amplitudes of motion at rotor midspan



$c = 6 \times 10^{-5}$ m

$a = 1.2 \times 10^{-5}$ m

Amplitudes of motion at bearings

Exercise: Calculate the major and minor axes of the ellipses describing the (X,Y) motions. See Appendix A of Childs' Rotordynamics Book:.

Notes: You could update this program to account for
a) bearing mass MB, a fraction of total rotor mass,
b) introduce damping at the rotor midspan, Cs.

NOTES 5. APPENDIX B

OTHER TYPES OF LUBRICATED JOURNAL BEARINGS

Compressors, turbines, pumps, electric motors, electric generators and other rotating machines are commonly supported on fluid film bearings. In the past, the vast majority of these bearings were plain journal bearings. As machines have achieved higher speeds, rotor dynamic instability problems such as oil whirl have brought the need for other types of bearing configurations. Cutting axial grooves in the bearing to provide a different oil flow pattern across the lubricated surface generates some of these geometries. Other bearing types have various patterns of variable clearance (preload and offset) to create a pad film thickness that has strongly converging and diverging regions, thus generating a direct stiffness for operation even at the journal centered position. Various other geometries have evolved as well, such as the tilting pad bearings which allow each pad to pivot, and thus to take its own equilibrium position. This usually results in a strongly converging film region for each loaded pad and the near absence of cross-coupled stiffness coefficients.

TYPES OF HYDRODYNAMIC BEARINGS:

The Tables below list in a condensed form some of the advantages and disadvantages of various practical bearing configurations.

FIXED PAD NON-PRE LOADED JOURNAL BEARINGS

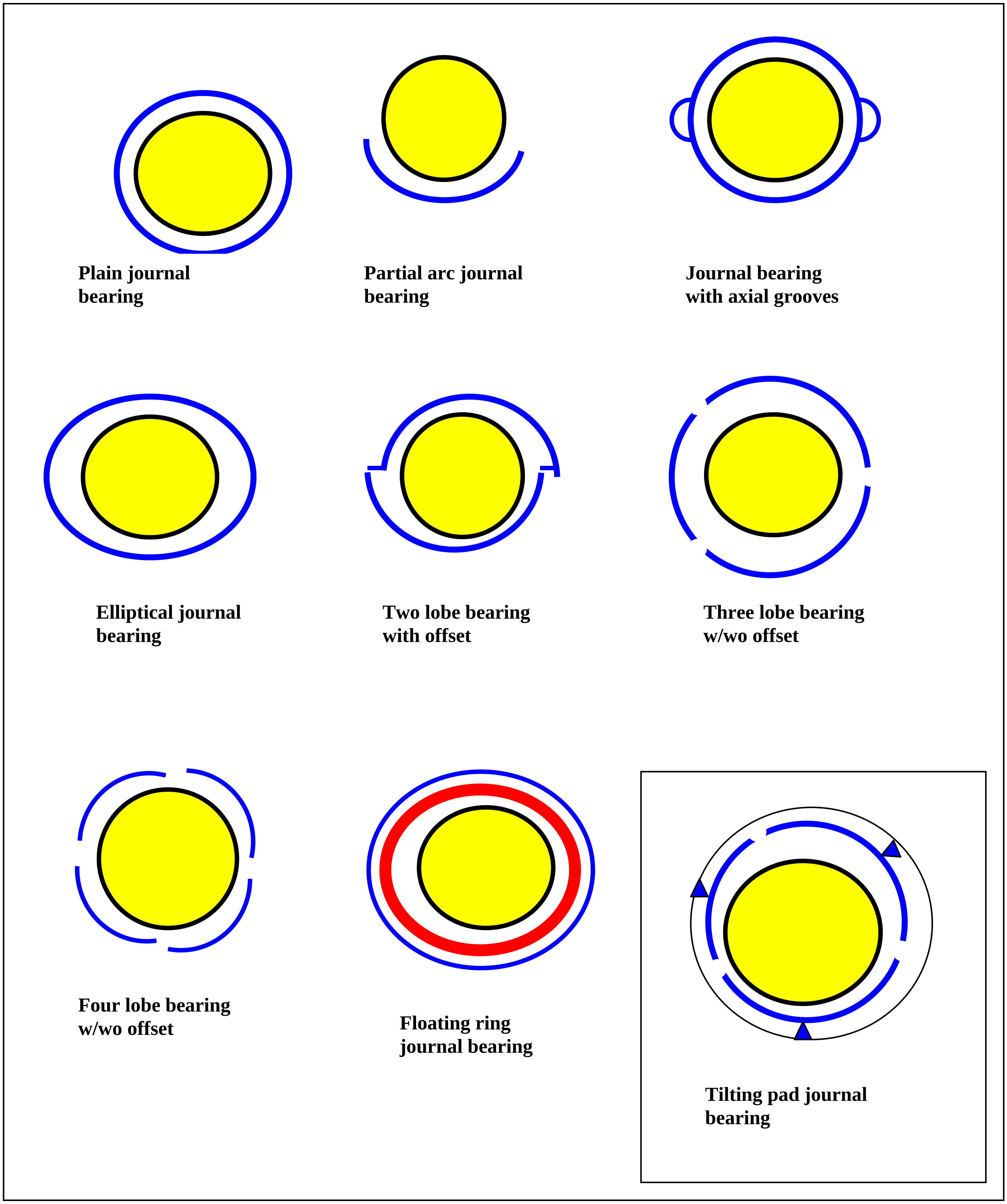
Bearing Type	Advantages	Disadvantages	Comments
Plain Journal	1. Easy to make 2. Low Cost	1. Most prone to subsynchronous whirl	Round bearings are nearly always “crushed” to make elliptical bearings
Partial Arc	1. Easy to make 2. Low Cost 3. Low horsepower loss	1. Poor vibration resistance 2. Oil supply not easily contained	Bearing used only in rather old machines
Axial Groove	1. Easy to make 2. Low Cost	1. Subject to oil whirl	Round bearings are nearly always “crushed” to make elliptical or multi-lobe
Floating Ring	1. Relatively easy to make 2. Low Cost	2. Subject to oil (two whirl frequencies from inner and outer films (50% shaft speed, 50% [shaft + ring] speeds)	Used primarily on high speed turbochargers for diesel engines and P/C vehicles
Elliptical	1. Easy to make 2. Low Cost 3. Good damping at critical speeds	1. Subject to oil whirl at high speeds 2. Load direction must be known	Probably most widely used bearing at low or moderate rotor speeds
Offset Half (With Horizontal Split)	1. Excellent suppression of whirl at high speeds 2. Low Cost 3. Easy to make	1. Fair suppression of whirl at moderate speeds 2. Load direction must be known	High horizontal stiffness and low vertical stiffness - may become popular - used outside U.S.
Three and Four Lobe	1. Good suppression of whirl 2. Overall good performance 3. Moderate cost	1. Some types can be expensive to make properly 2. Subject to whirl at high speeds	Currently used by some manufacturers as a standard bearing design

FIXED PAD JOURNAL BEARINGS WITH STEPS, DAMS OR POCKETS

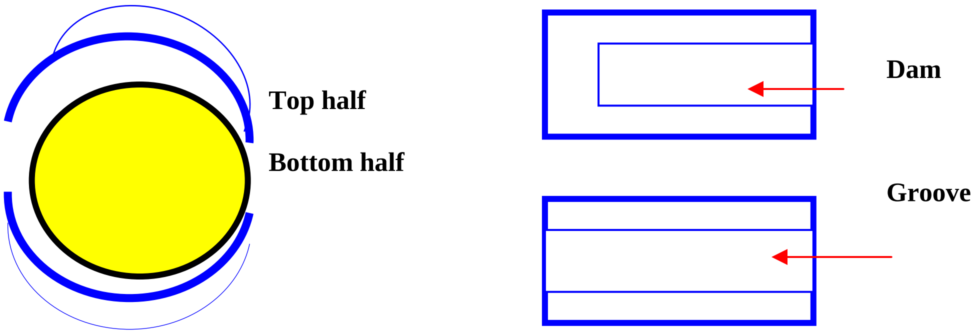
Bearing Type	Advantages	Disadvantages	Comments
Pressure Dam (Single Dam)	1. Good suppression of whirl 2. Low cost 3. Good damping at critical speeds 4. Easy to make	1. Goes unstable with little warning 2. Dam may be subject to wear or build up over time 3. Load direction must be known	Very popular in the petrochemical industry. Easy to convert elliptical over to pressure dam
Multi-Dam Axial Groove or Multiple-Lobe	1. Dams are relatively easy to place in existing bearings 2. Good suppression of whirl 3. Relatively low cost 4. Good overall performance	1. Complex bearing requiring detailed analysis 2. May not suppress whirl due to nonbearing causes	Used as standard design by some manufacturers
Hydrostatic	1. Good suppression of oil whirl 2. Wide range of design parameters 3. Moderate cost	1. Poor damping at critical speeds 2. Requires careful design 3. Requires high pressure lubricant supply	Generally high stiffness properties used for high precision rotors

NON-FIXED PAD JOURNAL BEARINGS

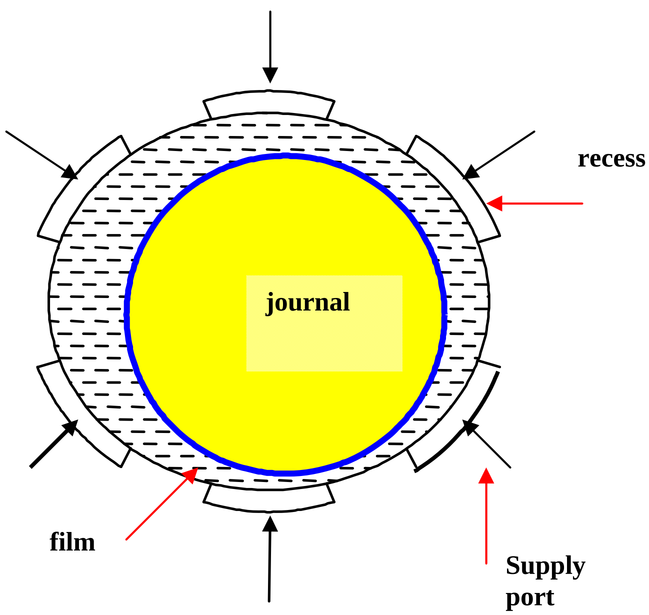
Bearing Type	Advantages	Disadvantages	Comments
Tilting Pad Journal bearing **Flexure pivot, tilting pad bearing**	1. Will not cause subsynchronous whirl (no cross coupling)	1. High Cost 2. Requires careful design 3. Poor damping at critical speeds 4. Hard to determine actual clearances 5. High horsepower loss 6. Load direction must be known	Widely used bearing to stabilize machines with subsynchronous non-bearing related excitations
Foil bearing	1. Tolerance to misalignment. 2. Oil-free	1.High cost. 2.Dynamic performance not well known for heavily loaded machinery. 3.Prone to subsynchronous whirl	Used mainly for low load support on high speed machinery (APU units).



Typical configurations of cylindrical journal bearings (1)

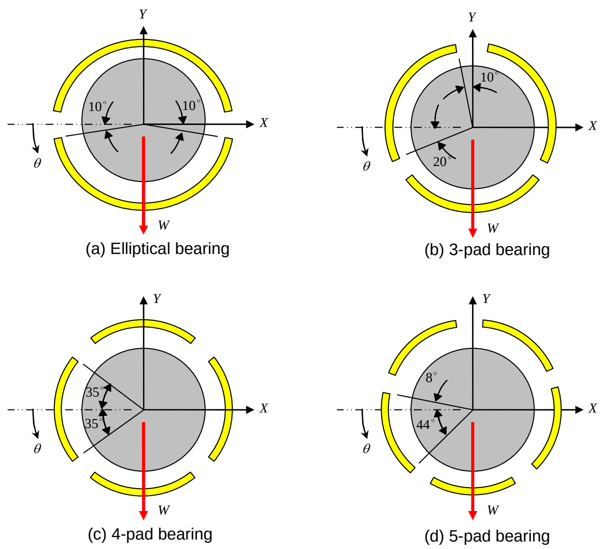


Pressure dam journal bearing



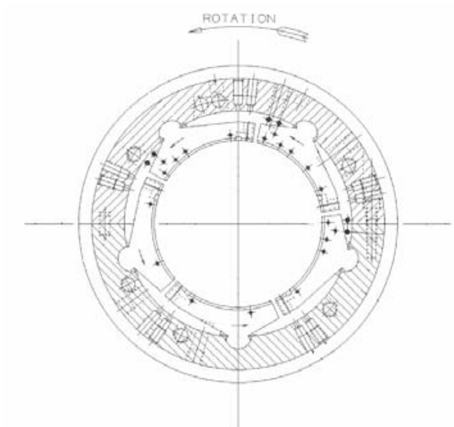
hydrostatic journal bearing

Typical configurations of cylindrical journal bearings (2)

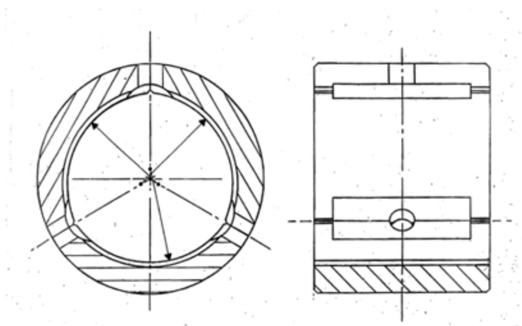


(a) Elliptical bearing (b) 3-pad bearing

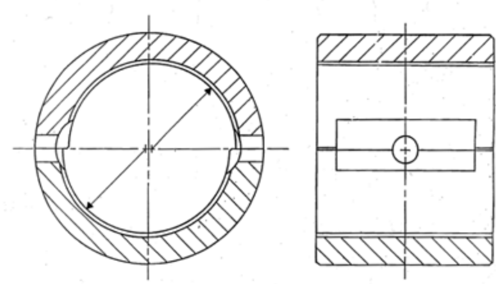
(c) 4-pad bearing (d) 5-pad bearing



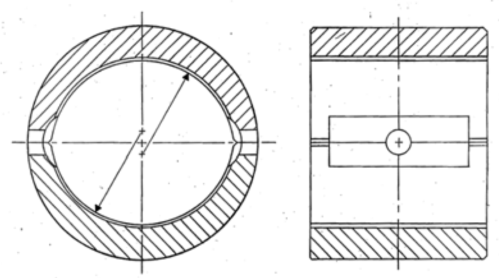
Tilting pad bearing



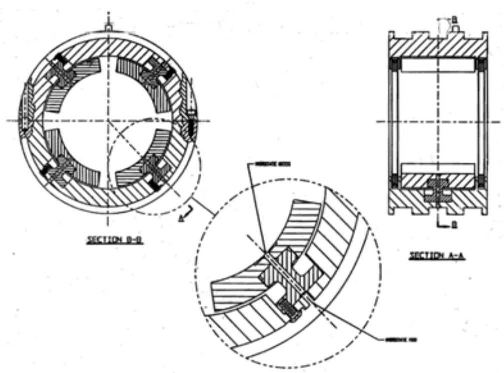
3-LOBE JOURNAL BEARING



OFFSET SPLIT BEARING



LEMON BORE BEARING



flexure pivot hydrostatic bearing

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[1] Design of Journal Bearings for Rotating Machinery, P. Allaire & R.D. Flack, Proc. of the 10th Turbomachinery Symposium, TAMU, pp. 25-45, 1981

[2] Fluid Film Bearing Fundamentals and Failure, F. Zeidan & B. Herbage, Proc. of the 20th Turbomachinery Symposium, TAMU, pp. 161-186. 1991.

[3] Fundamentals of Fluid Film Journal Bearing Operation and Modeling, M. He & J. Byrne, Proc. of the 34th Turbomachinery Symposium, TAMU, pp. 155-176, 2005