NOTES 2

DERIVATION OF THE CLASSICAL REYNOLDS EQUATION FOR THIN FILM FLOWS

The lecture presents the derivation of the Reynolds equation of classical lubrication theory. Consider a liquid flowing through a thin film region separated by two closely spaced moving surfaces. The fluid pressure does not vary across the film thickness and fluid inertia effects are ignored. From the momentum transport and continuity equations, the analysis leads to a single elliptic differential equation, namely Reynolds Eqn., for the generation of hydrodynamic pressure in the film flow region. Appropriate boundary conditions, either pressure or flow conditions known, are noted for solution of Reynolds Eqn. A brief description of lubricant cavitation follows. Formulas for fluid mean velocities and wall shear stress differences are also derived. Appendices detail the one-dimensional fluid flow analysis of pressure generation and load capacity in plane slider bearings, Rayleigh step bearings and simple squeeze film dampers.

Nomenclature

h	Film thickness
P	Hydrodynamic pressure
P_{sat}	Liquid saturation pressure or dissolved gases saturation pressure
M_x, M_z	$\int_0^h \rho V_x \, dy$, $\int_0^h \rho V_z \, dy$. Mass flow rates per unit length
V_x, V_y, V_z	Fluid velocities along x, y, z directions
$\bar{V}_x, \bar{V}_z$	$\frac{M_x}{\rho_A h}, \frac{M_z}{\rho_A h}$. Mean flow velocities
U_* , V_*	Characteristic fluid speeds – along & across film thickness $V_* = U_* C/L_*$
t	Time
x, y, z	Coordinate system on plane of bearing
ρ	Fluid density
ρ_A	Average fluid density across film thickness
μ	Fluid absolute viscosity
τ_{xy}, τ_{zy}	$\mu \frac{\partial V_z}{\partial y}, \mu \frac{\partial V_z}{\partial y}$. Fluid shear stresses across film.
$\Delta\tau_{xy}$, $\Delta\tau_{zy}$	$\left\langle \tau_{xy} \right\|_{y=0}^{y=h}$, $\left\langle \tau_{zy} \right\|_{y=0}^{y=h}$. Wall shear stress differences

Figure 2.1 depicts a typical thin film geometry with the *{x, y, z}* as a coordinate system in the plane of the thin film bearing and with the *y* axis directed across the film thickness *h(x, z, t)*. The flow of a Newtonian, inertialess, isoviscous fluid in the thin film region is governed by the following equations:

continuity (mass conservation): $$\frac{\partial(\rho V_x)}{\partial x}+\frac{\partial(\rho V_y)}{\partial y}+\frac{\partial(\rho V_z)}{\partial z}=0 \quad (2.1)$$

(*x*,*z*) –momentum transport:

$$0=-\frac{\partial P}{\partial x}+\mu\frac{\partial^2 V_x}{\partial y^2} \quad (2.2)$$

$$0=-\frac{\partial P}{\partial z}+\mu\frac{\partial^2 V_z}{\partial y^2} \quad (2.3)$$

with the pressure $P=f(x, z, t)$ not varying across the film thickness.

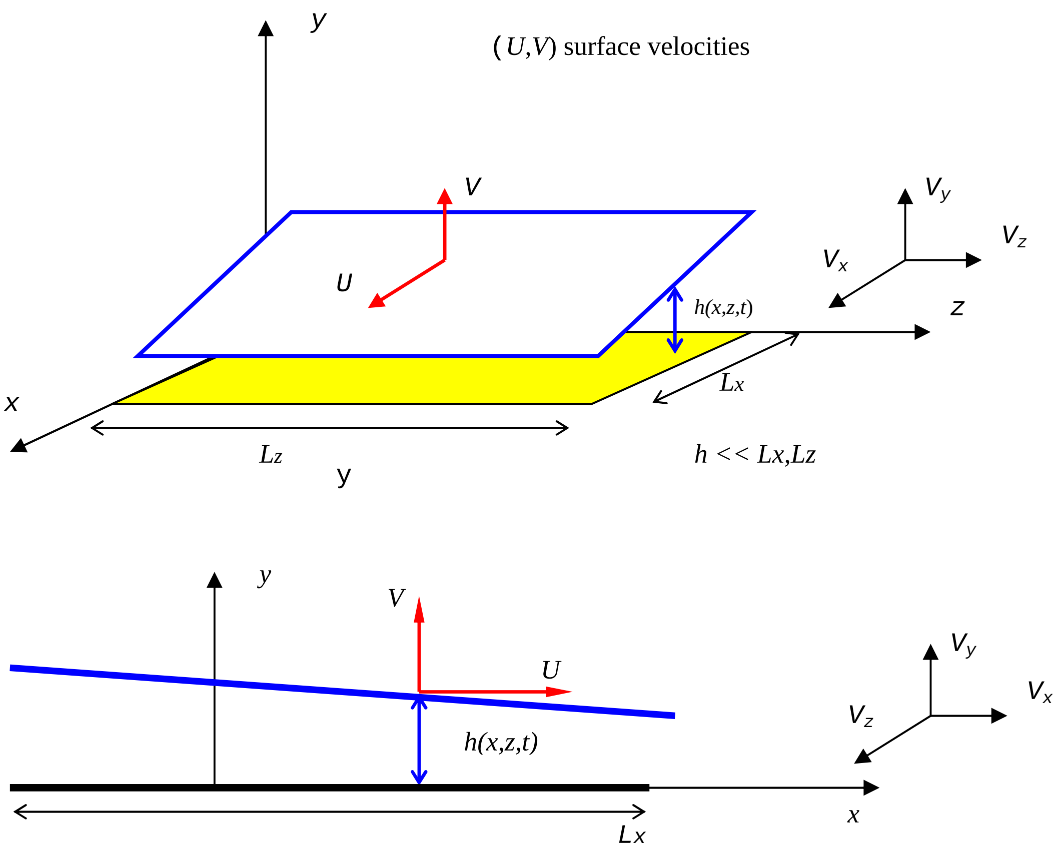


Figure 2.1 Geometry of fluid film bearing (*h* << L_x, L_z) and flow velocities

In the flow region of interest, the bottom surface *y=0* is stationary, while the top surface, *y=h(x,z,t)*, moves with velocity components *U* and *V* in the *x* and *y* directions, respectively. The lubricant adheres (**non-slips)** to the bounding surfaces. Thus, the boundary conditions for the fluid velocities are:

$$at\ \ y = 0,\ V_x = 0,\ V_z = 0,\ V_y = 0 \tag{2.4}$$

$$at\ \ y = h,\quad V_x = U,\quad V_z = 0,\quad V_y = V \tag{2.5}$$

From simple kinematics, the normal velocity *V* of the top surface equals to the temporal change in film thickness (*h*) plus the spatial change (advection) of the film thickness due to the lateral motion of the surface with velocity *U,* i.e.

$$V = \frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\frac{dx}{dt} = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} \tag{2.6}$$

Integration of the *x*- and *z*-momentum transport equations across the film (*y*-direction) is straightforward since the pressure (*P*) is constant across the film thickness. This procedure and application of the boundary conditions lead to:

$$V_x = \frac{1}{2\mu}\frac{\partial P}{\partial x}\left\{y^2 - y\,h\right\} + \frac{y}{h}U \tag{2.7}$$

$$V_z = \frac{1}{2\mu}\frac{\partial P}{\partial z}\left\{y^2 - y\,h\right\} \tag{2.8}$$

Note that the fluid velocities show the **superposition** of two distinct effects. The fluid moves due to an imposed pressure gradient **(*Poiseuille flow*)** and flows by a shear driven effect induced by the motion of the top surface **(*Couette flow*)**. The *Poiseuille flow* brings a parabolic velocity distribution across the film, while the *Couette flow* results in a linear velocity profile.

Figure 2.2 shows the (V_x/U) velocity profile for three pressure gradient conditions, $a=[(h^2/2\mu)(dP/dx)/U]$ = -5, 0 and 5, respectively. The first case corresponds to a pressure gradient decreasing in the direction of the shear induced flow, i.e. $dP/dx<0$, while the second case denotes pure shear flow, i.e. $dP/dx=0$. Note that a positive pressure gradient, $dP/dx > 0$, causes a region of back flow closest to the stationary surface $y=0$.

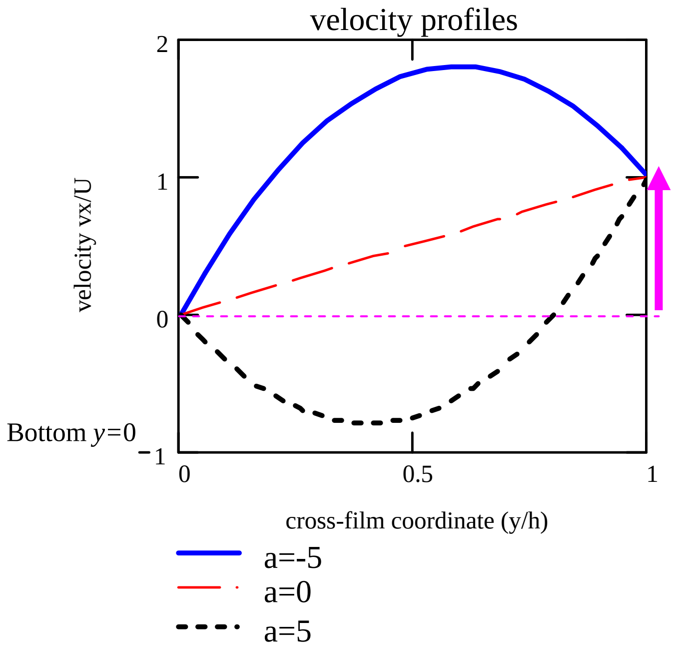


Figure 2.2. Dimensionless velocity profiles across film thickness. Pressure gradient varies, $a=[(h^2/2\mu)(dP/dx)/U]$

Mass flow rates across the film thickness and mean flow velocities in the *x- and z-* directions are defined as:

$$M_x = \int_0^h (\rho V_x)\,dy\ ,\ \ M_z = \int_0^h (\rho V_z)\,dy \qquad (2.9)$$

$$\overline{V}_x = \frac{M_x}{\rho_A\, h};\qquad \overline{V}_z = \frac{M_z}{\rho_A\, h} \qquad (2.10)$$

where $\rho_A = \frac{1}{h}\int_0^h \rho\, dy$ is an average fluid density across the film thickness. Note that if the fluid density is only a function of pressure, i.e. $\rho = \rho(P)$, and since the pressure does not vary across the film thickness, $\rho_A=\rho$. Barotropic[1] liquids and most gas fluid flows undergoing isentropic or adiabatic or isothermal processes show this type of relationship.

On the other hand, there are thin film flows where the fluid temperature changes dramatically across the film thickness. In this case not only viscosity variations but also density changes need be accounted for. The analysis of heavily loaded cylindrical and tilting pad bearings usually calls for the inclusion of thermal effects across the film: an energy transport equation must be used for adequate predictive analysis. Further details on the physical aspects and implementation of thermal effects are given later.

1 Barotropic fluid: one whose material properties depend on pressure only AND not temperature

Substitution of the velocity profiles, equations (2.7-8), into equations (2.9-10) gives, for an **isoviscous & barotropic** fluid, the following mass flow rates (per unit length) and average velocities as[2]:

$$M_x = \underbrace{-\frac{\rho h^3}{12\mu}\frac{\partial P}{\partial x}}_{Poiseuille\ flow} + \underbrace{\frac{\rho h U}{2}}_{Couette\ flow}\ ; \quad M_z = \underbrace{-\frac{\rho h^3}{12\mu}\frac{\partial P}{\partial z}}_{Poiseuille\ flow} \tag{2.11}$$

$$\bar{V}_x = -\frac{h^2}{12\mu}\frac{\partial P}{\partial x} + \frac{U}{2}; \qquad \bar{V}_z = -\frac{h^2}{12\mu}\frac{\partial P}{\partial z} \tag{2.12}$$

The mass flow rates (and mean velocities) are the superposition of the pressure flow (*Poiseuille flow*) and the shear flow (*Couette flow*) components. Note that the average shear driven fluid velocity ($\bar{V}_x$)equals 50% of the (top) surface speed *U*.

Later on, in the study of turbulence in fluid film bearings, the mean flow velocities will be renamed as **bulk-flow velocity components.**

Integration of the mass flow conservation equation (2.1) across the film thickness *(h)* gives,

$$\int_0^h \frac{\partial \rho}{\partial t} dy + \int_0^h \frac{\partial\left(\rho V_x\right)}{\partial x} dy + \int_0^h \frac{\partial\left(\rho V_y\right)}{\partial y} dy + \int_0^h \frac{\partial\left(\rho V_z\right)}{\partial z} dy = 0 \tag{2.13}$$

and using Leibniz's integration formulae[3]:

$$\int_0^{h(\zeta)} \frac{\partial g}{\partial \zeta}\, dy = \frac{\partial}{\partial \zeta}\left\{\int_0^{h(\zeta)} g\, dy\right\} - g(\zeta, h)\frac{\partial h}{\partial \zeta} \tag{2.14}$$

renders

$$\begin{aligned} &\frac{\partial}{\partial t}\int_0^h \rho\, dy - \rho_{(h)}\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h \rho V_x\, dy - \rho_{(h)} V_{x(h)}\frac{\partial h}{\partial x} + \\ &+ \rho_{y=h} V_{y=h} - \rho_{y=0} V_{y=0} + \frac{\partial}{\partial z}\int_0^h \rho\, V_z\, dy - \rho_{(h)} V_{z(h)}\frac{\partial h}{\partial z} = 0 \end{aligned} \tag{2.15}$$

---

2 The fluid viscosity is assumed uniform across the film thickness. This assumption is not valid in thin film flows with significant temperature gradients across the film.

3 Application of eqn. (2.14) requires of continuity of the function *h(ζ)*

The non-slip boundary conditions for the velocities at the bottom and top surfaces, eqns. (2.4,2.5), are

$$V_{x(0)} = 0,\ V_{x(h)} = U;\quad V_{z(0)} = V_{z(h)} = 0;\qquad V_{y=0} = 0,\ V_{y=h} = V = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x},$$

And applied into equation (2.15). With the definition of mass flow rates *(M_x, M_z)*, the conservation of mass equation becomes,

$$\frac{\partial(\rho h)}{\partial t} - \rho_{(h)}\frac{\partial h}{\partial t} + \frac{\partial M_x}{\partial x} - \rho_{(h)} U \frac{\partial h}{\partial x} + \rho_{(h)}\left\{\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x}\right\} + \frac{\partial M_z}{\partial z} = 0$$

Simplifying like-terms leads to the conservation of mass equation across the film thickness:

$$\frac{\partial(\rho h)}{\partial t} + \frac{\partial M_x}{\partial x} + \frac{\partial M_z}{\partial z} = 0 \qquad (2.16)$$

or in terms of the mean flow velocities:

$$\frac{\partial}{\partial t}\{\rho h\} + \frac{\partial}{\partial x}\{\rho h \bar{V}_x\} + \frac{\partial}{\partial z}\{\rho h \bar{V}_z\} = 0 \qquad (2.17)$$

Note that to arrive to the equations above, the common lubrication assumption $C/L << 1$ is not needed. Thus, equation (2.16) is valid for any type of fluid flow bounded between two surfaces.

Substitution of the mass flow rates across the film, $M_x = -\frac{\rho h^3}{12\mu}\frac{\partial P}{\partial x} + \frac{\rho h U}{2};\quad M_z = -\frac{\rho h^3}{12\mu}\frac{\partial P}{\partial z}$

Into the conservation of mass equation (2.16) renders the **Reynolds equation of classical lubrication theory, i.e.**

$$\frac{\partial}{\partial t}\{\rho h\} + \frac{1}{2}\frac{\partial}{\partial x}\{\rho h U\} = \frac{\partial}{\partial x}\left\{\frac{\rho h^3}{12\mu}\frac{\partial P}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{\frac{\rho h^3}{12\mu}\frac{\partial P}{\partial z}\right\} \qquad (2.18)$$

The terms on the right hand side of Reynolds equation represent the flow due to pressure gradients. The left hand side shows the flows induced by normal (squeeze) motions of the bounding surface and the shear induced flow by the (top) surface sliding with velocity *U*.

Thus, the fluid flow in thin film geometries is reduced a single differential equation of elliptic type for the **hydrodynamic** pressure field *P(x, z, t)*. Appropriate boundary conditions for the pressure are required on the closure or boundaries of the flow domain as discussed later.

Once the pressure field is obtained, i.e., eqn. (2.18) solved, the fluid film velocity components are evaluated from equations (2.7, 2.8), i.e.

$$V_x = \frac{1}{2\mu}\frac{\partial P}{\partial x}\{y^2 - y\,h\} + \frac{y}{h}U \text{ , and } V_z = \frac{1}{2\mu}\frac{\partial P}{\partial z}\{y^2 - y\,h\}$$

It is of importance to evaluate the wall shear stresses *(τ_{xy}, τ_{zy})* at the bottom and top bearing surfaces. In the *x*- and *z*- directions,

$$\tau_{xy} = \mu\frac{\partial V_x}{\partial y} = \frac{1}{2}\frac{\partial P}{\partial x}\{2y - h\} + \frac{\mu}{h}U; \quad \tau_{zy} = \mu\frac{\partial V_z}{\partial y} = \frac{1}{2}\frac{\partial P}{\partial z}\{2y - h\} \tag{2.19}$$

and at the bottom y=0 surface and top y=h surface,

$$\langle\tau_{xy}\big|_{y=0} = \mu\frac{\partial V_x}{\partial y}\big|_{y=0} = -\frac{h}{2}\frac{\partial P}{\partial x} + \frac{\mu}{h}U; \quad \langle\tau_{zy}\big|_{y=0} = \mu\frac{\partial V_z}{\partial y}\big|_{y=0} = -\frac{h}{2}\frac{\partial P}{\partial z} \tag{2.20}$$

$$\langle\tau_{xy}\big|_{y=h} = \mu\frac{\partial V_x}{\partial y}\big|_{y=h} = +\frac{h}{2}\frac{\partial P}{\partial x} + \frac{\mu}{h}U; \quad \langle\tau_{zy}\big|_{h=0} = \mu\frac{\partial V_z}{\partial y}\big|_{y=h} = +\frac{h}{2}\frac{\partial P}{\partial z} \tag{2.21}$$

The wall shear stresses show clearly distinct functions for the pressure and shear driven flows. Figure 2.3 depicts in schematic form typical shear stress distributions due to *Poiseuille* and *Couette* type flows.

The **wall shear stress differences** are

$$\Delta\tau_{xy} = \langle\tau_{xy}\big|_{y=0}^{y=h} = h\frac{\partial P}{\partial x}; \quad \Delta\tau_{zy} = \langle\tau_{zy}\big|_{y=0}^{y=h} = h\frac{\partial P}{\partial z} \tag{2.22}$$

and since the pressure gradients are related to the mean flow velocity components, equations (2.12), then

$$\frac{\partial P}{\partial x} = -\frac{12\mu}{h^2}\left(\bar{V}_x - \frac{U}{2}\right); \quad \frac{\partial P}{\partial z} = -\frac{12\mu}{h^2}\left(\bar{V}_z\right); \tag{2.23}$$

The **wall shear stress differences** are, in terms of the mean flow velocity components,

$$\begin{aligned}\Delta\tau_{xy} &= \langle\tau_{xy}\big|_{y=0}^{y=h} = h\frac{\partial P}{\partial x} = -\frac{12\mu}{h}\left(\bar{V}_x - \frac{U}{2}\right); \\ \Delta\tau_{zy} &= \langle\tau_{zy}\big|_{y=0}^{y=h} = h\frac{\partial P}{\partial z} = -\frac{12\mu}{h}\left(\bar{V}_z\right)\end{aligned} \tag{2.24}$$

The wall shear stress differences appear as simple functions of the mean flow velocity components. Thus, there is no need to know with detail the velocity profiles across the thin film. Note that equation (2.24) shows a local (quasi-static) balance of forces, pressure gradient forces equal to the wall shear induced surface forces. Later, the formulas above will aid in the analysis of turbulent flows in thin film bearings.

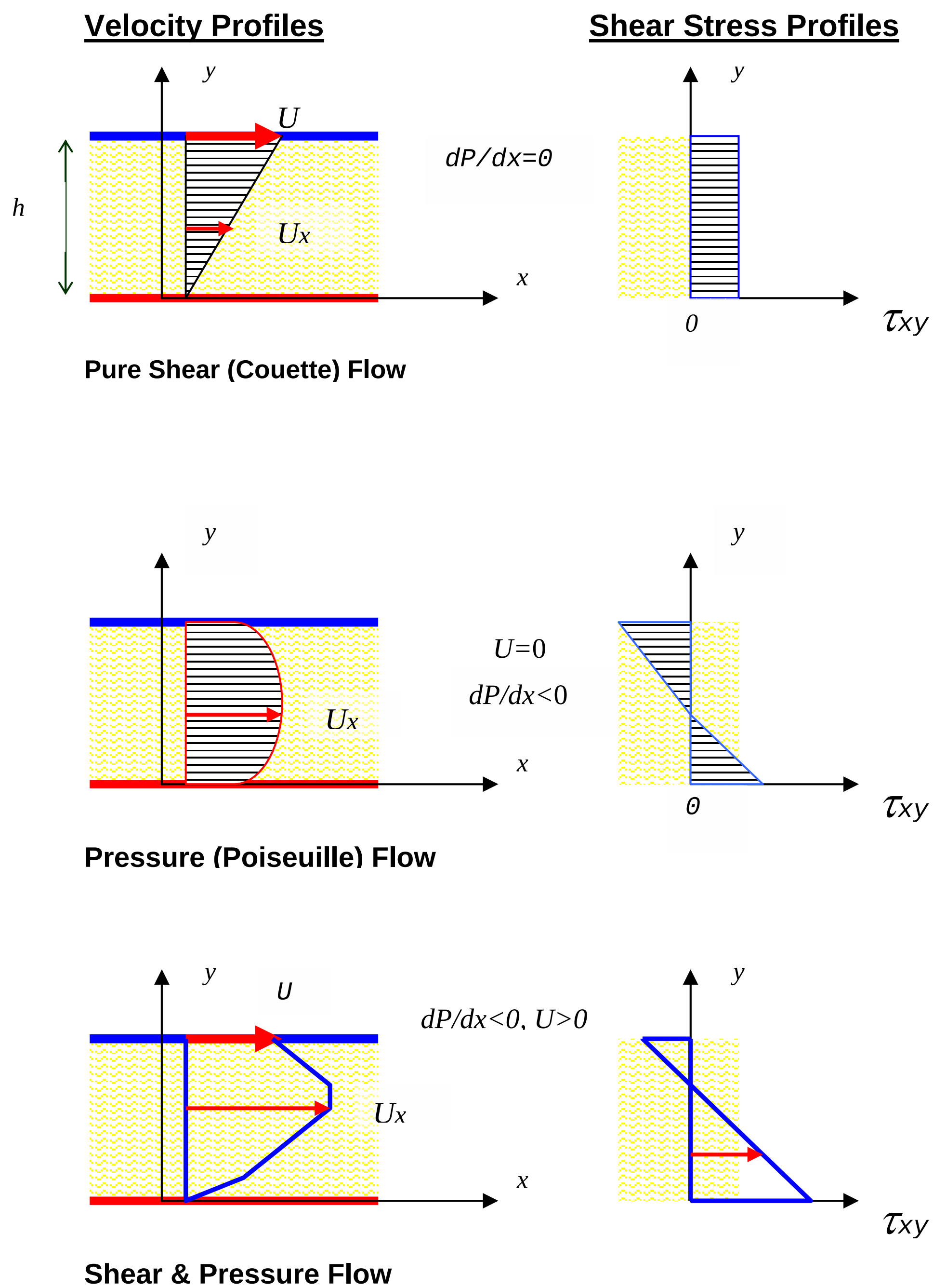


Figure 2.3 Velocity and shear stress profiles in a fluid film bearing

Boundary Conditions for the Reynolds equation

The Reynolds equation governing the generation of hydrodynamic pressure in the fluid film bearing is of elliptic type. Consequently, appropriate boundary conditions are needed over the entire closure or boundary $\Gamma(x, z)$ of the flow domain $\Omega(x, z)$.

First, note that a Newtonian fluid is a material not able to sustain tension. A liquid will cavitate (evaporate) before it reaches its absolute zero pressure. This occurs at its saturation pressure (P_{sat}). Also, if there are dissolved gases (most likely air) in the liquid, these will be released at their saturation pressure ($\approx P_{ambient}$) and the fluid could not undergo a further reduction in pressure. Hence, the liquid pressure needs to be greater than its saturation pressure (> zero absolute) everywhere in the flow domain.

$$P(x, z, t) > 0 \text{ in } \Omega(x, z) \tag{2.25}$$

Most fluids under normal working conditions can sustain small levels of tension. Sometimes if the actions imposed on the fluid are very fast, then the liquid is able to support large tensile stresses over short periods of time. The likelihood of fluid tensile stresses is (usually) not accounted for in classical hydrodynamic lubrication theory.

The brief discussion above points out to a more complicated process yet to be well understood (and modeled) in fluid film bearing performance. The distinctions made call for two **different** types of **cavitation regimes**:

a) **Gaseous cavitation** when dissolved gases in the lubricant come out of solution. Thus,

$$P \geq P_{saturation\ gases} \approx P_{ambient} \tag{2.26a}$$

b) **Vapor cavitation** when a pure liquid reaches its saturation pressure and evaporates (a phase change)

$$P \geq P_{saturation\ liquid} \tag{2.26b}$$

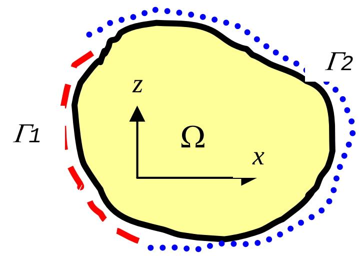
The saturation pressure of most liquids usually amounts to a minute fraction of one atmosphere, i.e. it is very close to zero absolute.

Finally, note that Reynolds equation is not valid within the fluid cavitation region since, for its derivation, the fluid is regarded as a single-phase component. Furthermore, no fluid phase-changes are accounted for within the flow region when deriving Reynolds equation.

A later section (See Notes 6-a) describes in detail the phenomenon of lubricant cavitation, including a (well accepted) sound physical model and a discussion on whether the cavitation zone actually includes lubricant vapor or released gases. The phenomenon of air entrainment in fluid film bearings, and in particular squeeze film dampers, is of utmost importance under dynamic force operating conditions. This aspect of modern lubrication will also be considered in detail later (See Notes 13).

Other type of fluid "cavitation" arises due to thermal heating as the fluid flows through thin film thickness, i.e., the fluid **flashes** (boils) when reaching its critical temperature. This condition is prevalent in many face seal applications handling volatile fluids, water included.

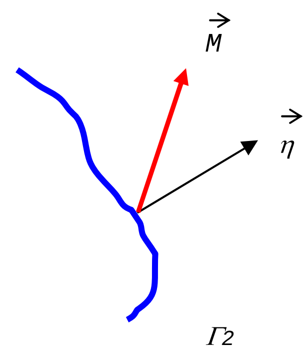
Consider the **boundary** Γ of the flow domain Ω to be composed of two separate regions ($\Gamma = \Gamma_1 \cup \Gamma_2$).



Along Γ_1 the pressure is known (essential or Dirichlett type boundary condition), i.e.

$$P = P_*(\Gamma_1) \quad on \quad \Gamma_1 \tag{2.27}$$

And, along Γ_2 the mass flow rate leaving through the boundary is known, i.e.,



$$M_\eta = \boldsymbol{M}_{\eta *} \tag{2.28}$$

where $M_\eta = M_x \eta_x + M_z \eta_z$

$$M_\eta = -\frac{\rho h^3}{12\mu}\frac{\partial P}{\partial \eta} + \frac{\rho U h}{2}\eta_x \tag{2.29}$$

with (η_x , η_z) as the components of the outward normal vector to the boundary Γ_2.

A typical boundary condition of this type occurs when the fluid film bearing has symmetry in the *z*-plane (axial). Since $M_z = 0$, then,

$$\frac{\partial P}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{2.30}$$