POROUS ELASTICITY

Lectures On The Elasticity Of Porous Materials As An Application Of The Theory Of Mixtures

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Preface

This work was originally planned as a textbook exploiting the structure of the Theory of Mixtures as the basis for the study of porous elasticity. The decision to write this book was made approximately thirty years ago! At that time, I was a faculty member in Mechanical Engineering at Rice University. It is an understatement to observe that it has taken awhile to complete, even partially, this project. I encountered a lot of diversions along the way. Not the least of which was an eight year period where I served as President of Texas A&M University. Prior to that time, I was a Dean of Engineering at Kentucky, an administrator at the National Science Foundation and a Provost and Interim President at Oklahoma State University. During my time as an administrator, I never lost my ambition to prepare this textbook. On occasion, during periods of relative calm or, at the other extreme, during periods of great stress, I would find comfort in returning to my manuscript. It would take someone not trained in Engineering to understand why I would find comfort thinking about this book when caught up in the tangles of university administration.

When I completed my term as President of Texas A&M University, I decided that I should spend the final years of my career back in the classroom teaching Mechanical Engineering and Mathematics. I have experienced the good fortune of two great and generous departments at Texas A&M willing to allow me to teach a variety of subjects in my areas of interests. It has been a joyous experience.

In addition to my teaching, I have revised my books on Vectors and Tensors, written with C.-C. Wang, and my Continuum Mechanics textbook. These books, of course, are out of print. This fact enabled me to arrange for their copyrights to be returned to the authors and to republish them informally by posting them on the web.[1] The reception to these free textbooks has encouraged me to complete, in some fashion, my textbook on porous elasticity. With this encouragement, I returned to a manuscript that has its roots in courses I taught at Rice University in the late 1970's.

One has to wonder if a book written over a thirty year period can make a contribution to knowledge in a subject studied by a large number of talented and knowledgeable investigators in the intervening years. I am not confident that I have made a contribution. The decision on that subject will need to be made by others.

Given the fact that it has not been used in the classroom and the fact that it has been typed by a less than competent proofreader (me!), it will come as a shock to no one that the manuscript is not free of a variety of errors. I can only hope they are minor. The readers of the posted versions of the textbooks mentioned above have assisted by emailing me about errors of all types in those works. This information has allowed me to make corrections and post updated versions several times over the last few years. For as long as body and mind will allow, I will follow the same practice with this work on porous elasticity.

[1] These books can be found at http://rbowen.tamu.edu/.

As a part of my teaching, I have had the opportunity to utilize computational tools that were not available when this book was conceived. In addition to the availability of a modern word processor with the capability to type mathematical equations that was unimaginable in 1980, I have had the pleasure of learning how to use MATLAB and, to a lesser extent, MAPLE. My ability with these packages is still evolving, but they were utilized during the preparation of this book. In that regard, I wish to express my appreciation to Dr. Waqar Malik for his patience and assistance during the years we have been colleagues in Mechanical Engineering at Texas A&M University. This book has benefitted in many ways from his service as my teacher as I tried to exploit these essential computer tools in my teaching and my writing.

This textbook was initially posted in late 2008. The update posted in February, 2014 mainly simply corrects typos. There is not new material added. However, in the first posting of this book, I did not include an index. This deficiency has been corrected with the February, 2014 posted. The quality of the index could be better, and I do remind the reader that the book is posted in .pdf format and that format allows for efficient electronic searches.

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Some Classical Porous Media Models

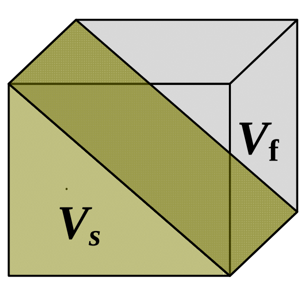
The objective of this introductory chapter is to set the stage for the porous media models to be developed and utilized in the later chapters. The basic approach in this chapter is to simply list the essential features of five rather well known porous media models. In this way, the reader can relate later developments to specific cases. In rough terms a *porous material* is a solid material containing one or more fluid constituents. The nature of the solid material defines various types of porous materials, as does the nature of the fluid constituents. For example, many classical porous media models assume that the solid is a rigid material. A rigid solid containing one incompressible fluid is one of the simplest types of porous material models one can formulate. In more interesting cases the solid is allowed to be an elastic material. If it is taken to be a linear elastic material, then one has one of the key assumptions built into *porous elasticity*. Within a given class of porous material models defined by an assumption about the solid behavior, one can construct other models by various assumptions about the fluids within the solid. The fluid can be incompressible or compressible. There can be multiple phases of incompressible or compressible fluids. Throughout this work we shall see various examples of all of these cases.

This Chapter contains a discussion of five porous media models. They are essentially classical in nature, each being well known and most having a large body of literature associated with them. These five models will be classified as follows:

1. A rigid solid containing one incompressible fluid.
2. A rigid solid containing one compressible fluid.
3. A rigid solid containing N incompressible fluids.
4. A linear elastic incompressible solid containing one incompressible fluid.
5. A linear elastic solid containing one compressible fluid.

1.1 Preliminary Definitions

First, we must introduce the concepts of *volume fraction*, *true density* and *bulk density*. These concepts arise for a special porous material consisting of *one* solid and *one* fluid. From the simple minded idea that within the solid material one can identify various volumes occupied by the solid and by the fluid, one can imagine the following geometric configuration within an elementary volume of the porous material.



In the figure, V_f denotes the volume occupied by the fluid constituent and V_s denotes the corresponding quantity for the solid. One might call these quantities the true volumes. The quantity V defined by

$$V = V_f + V_s \tag{1.1.1}$$

is the total or *bulk* volume. Given this simple geometric arrangement, the quantity φ_f defined by

$$\varphi_f = \frac{V_f}{V_f + V_s} = \frac{V_f}{V} \tag{1.1.2}$$

is the *volume fraction of the fluid*, and φ_s defined by

$$\varphi_s = \frac{V_s}{V_f + V_s} = \frac{V_s}{V} \tag{1.1.3}$$

is the *volume fraction of the solid*. It is trivially true from (1.1.2) and (1.1.3) that

$$\varphi_f + \varphi_s = 1 \tag{1.1.4}$$

The *porosity* of a porous material is the fraction of the elementary volume occupied by the fluid. It is given the symbol f and is defined by

$$f = \varphi_f = 1 - \varphi_s \tag{1.1.5}$$

If the solid is rigid, then the porosity f is a prescribed function of position. If m_f denotes the mass of the fluid constituent in the elementary volume, then γ_f denotes the *true density* of the fluid defined by

$$\gamma_f = \frac{m_f}{V_f} \tag{1.1.6}$$

Likewise, the *true density* of the solid is given by

$$\gamma_s = \frac{m_s}{V_s} \tag{1.1.7}$$

The *bulk density* of the fluid is defined by

$$\rho_f = \frac{m_f}{V} = \frac{m_f}{V_f}\frac{V_f}{V} = \gamma_f \varphi_f \tag{1.1.8}$$

where (1.1.2) and (1.1.6) have been used. Likewise, the bulk density of the solid is defined by

$$\rho_s = \frac{m_s}{V} = \frac{m_s}{V_s}\frac{V_s}{V} = \gamma_s \varphi_s \tag{1.1.9}$$

The velocity of the fluid is denoted by $\mathbf{v}_f$, and the velocity of the solid is denoted by $\mathbf{v}_s$. The product $\rho_f \mathbf{v}_f$ represents the mass flux passing a unit area. Its physical dimensions are mass per unit of time per unit of area. Therefore $\mathbf{v}_f = \rho_f \mathbf{v}_f / \rho_f$ represents the *bulk volume flux*, i.e. the bulk volume per unit of time per unit of area. Since, $\rho_f \mathbf{v}_f = \gamma_f \varphi_f \mathbf{v}_f$, the product $\varphi_f \mathbf{v}_f$ is the true volume per unit of time per unit of area. Likewise, the product $\varphi_f (\mathbf{v}_f - \mathbf{v}_s)$ is the true volume per unit of time passing an area moving with the solid. This product is called the *true volume flux* and is given the symbol $\mathbf{c}_f$, i.e.

$$\mathbf{c}_f = \varphi_f (\mathbf{v}_f - \mathbf{v}_s) \tag{1.1.10}$$

The quantity $\mathbf{c}_f$ has the physical dimensions of velocity and is called the *filtration velocity*. Also, sometimes it is called the *specific discharge*.

The simple ideas given above easily generalize to the case of a porous solid containing $N-1$ fluids. For each constituent, we introduce a volume fraction φ_a, where $a=1$ corresponds to the solid and $a=2,...,N$ corresponds to the fluids. As with equation (1.1.4),

$$\sum_{a=1}^{N} \varphi_a = 1 \tag{1.1.11}$$

As before, the *porosity* of a porous material is the fraction of the elementary volume occupied by all of the fluids. As a result,

$$f = \sum_{a=2}^{N} \varphi_a = 1 - \varphi_s \tag{1.1.12}$$

Also as before, if the solid is rigid the porosity *f* is a prescribed function of position. The bulk density ρ_a of the a^{th} constituent is related to its true density by the formula

$$\rho_a = \varphi_a \gamma_a \tag{1.1.13}$$

If $\mathbf{v}_a$ denotes the velocity of the a^{th} fluid, and $\mathbf{v}_s = \mathbf{v}_1$ the velocity of the solid, then

$$\mathbf{c}_a = \varphi_a(\mathbf{v}_a - \mathbf{v}_s) \tag{1.1.14}$$

is the *filtration velocity* for the a^{th} fluid.

1.2 Rigid Isotropic Solid Containing One Incompressible Fluid

As the first example of a classical porous media model, consider one consisting of a rigid porous solid containing one incompressible fluid. Since the fluid is incompressible, the true density γ_f can be taken to be a constant. The mathematical statement of the model requires that we give field equations and constitutive equations. Of course, a complete discussion of this model and the others in this chapter would also require a discussion of suitable boundary and, in some cases, initial data. Because of the special nature of this model, the only field equation is balance of mass and it takes the simple form

$$\operatorname{div}\mathbf{c}_f = 0 \tag{1.2.1}$$

The constitutive equation which completes the formulation of this model is known as *Darcy's law*. The formal statement is the following:

$$\mathbf{c}_f = -\frac{k}{\mu}\operatorname{grad}P_f \tag{1.2.2}$$

where $\mathbf{c}_f$ is the filtration velocity of fluid, P_f is the *pore pressure* of the fluid, k is a positive number known as the *permeability* of the solid and μ is a positive number known as the *viscosity* of the fluid. As will become clear later in this work, equation (1.2.2) is more than a pure constitutive equation. It involves constitutive equations which define the fluid and the solid, the equation of motion for the fluid and several other kinematic assumptions. In any case, equations (1.2.1) and (1.2.2) combine to yield a single partial differential equation for the pore pressure P_f. This equation is

$$\operatorname{div}\left(\frac{k}{\mu}\operatorname{grad}P_f\right) = 0 \tag{1.2.3}$$

If the ratio $\dfrac{k}{\mu}$ is a constant, the pore pressure is a solution of

$$\operatorname{div}(\operatorname{grad}P_f) = \Delta P_f = 0 \tag{1.2.4}$$

As the reader can observe, this model has the mathematical structure of the linear theory of heat conduction in a rigid solid. Darcy's Law plays the same role in the theory of porous materials as does Fourier's Law in the theory of rigid heat conduction.

1.3 Rigid Isotropic Solid Containing One Compressible Fluid

In this case, since the fluid is allowed to be compressible, the true density γ_f is not a constant. The field equation important to this model is balance of mass which takes the form

$$f\frac{\partial \gamma_f}{\partial t} + \operatorname{div}(\gamma_f \mathbf{c}_f) = 0 \tag{1.3.1}$$

The constitutive equations which complete this model are as follows:

$$\mathbf{c}_f = -\frac{k}{\mu}\operatorname{grad} P_f \tag{1.3.2}$$

$$P_f = P_f(\gamma_f) \tag{1.3.3}$$

and

$$f = f(\mathbf{x}) \tag{1.3.4}$$

where $\mathbf{x}$ is the position in the porous solid. Equation (1.3.2) is again Darcy's law with k representing the *permeability* of the solid and μ the *viscosity* of the fluid. Equation (1.3.3) is the pressure density relationship which comes from the thermodynamics of the compressible fluid in the solid pores. Finally, equation (1.3.4) represents the porosity distribution in the rigid solid. This distribution is a property of the solid. For simplicity, most models of rigid porous solids take the porosity to be a constant.

If (1.3.1), (1.3.2), (1.3.3) and (1.3.4) are combined, the result is

$$f(\mathbf{x})\frac{\partial \gamma_f}{\partial t} = \operatorname{div}\left\{\frac{k}{\mu}\gamma_f \operatorname{grad} P_f(\gamma_f)\right\} \tag{1.3.5}$$

Equation (1.3.5) is a nonlinear partial differential equation for the true density γ_f. Solutions of this partial differential equation have some features in common with solutions of the equations of nonsteady heat conduction.

1.4 Rigid Isotropic Solid Containing N - 1 Incompressible Fluids

If the model summarized in Section 1.2 is generalized to allow for N-1 incompressible fluids, we have a field equation representing balance of mass for each fluid of the form

$$\frac{\partial \phi_a}{\partial t} + \operatorname{div} \mathbf{c}_a = 0 \tag{1.4.1}$$

for a=2,3,...,N. Because the solid is taken to be rigid, equation (1.1.12) can be written

$$\sum_{a=2}^{N} \phi_a = f(\mathbf{x}) \tag{1.4.2}$$

Because $f(\mathbf{x})$ is a prescribed function for a rigid solid, equation (1.4.2) should be viewed as a constraint on the unknown volume fractions, $\varphi_2, \varphi_3, ..., \varphi_N$. The constitutive equations which complete this model are as follows:

$$\mathbf{c}_a = -\frac{k_a}{\mu_a} \operatorname{grad} P_a \tag{1.4.3}$$

for a = 2,3,..., N, and

$$P_a - P_2 = g_a(\varphi_3, \varphi_4, ..., \varphi_N) \tag{1.4.4}$$

for a = 3,...,N. In equations (1.4.3) and (1.4.4), P_a is the *pore pressure* for the a^{th} fluid, k_a is the *relative permeability* for the a^{th} fluid and μ_a is the *viscosity* for the a^{th} fluid. Equation (1.4.3) is an extension of Darcy's law which sometimes is given the name *Muskat's* law. Equation (1.4.4) is a *capillary pressure-volume fraction* which arises from a thermodynamic study of this model. The difference P_a-P_2 is the *capillary pressure* between the a^{th} fluid and the 2nd fluid. As will be illustrated later when this model is developed from the fundamentals of mixture theory, the choice of the 2nd fluid as a reference is arbitrary. Given the properties $f(\mathbf{x})$, k_a/μ_a, for a = 2, 3,..., N - 1, and g_a, for a = 3, 4,..., N, equations (1.4.1) through (1.4.4) represent 3N - 3 equations which, hopefully, contain solutions for the 3N - 3 unknowns $\varphi_2, \varphi_3, ..., \varphi_N, \mathbf{c}_2, \mathbf{c}_3, ..., \mathbf{c}_N,$ and $P_2, P_3, \ldots, P_N$.

1.5 Linear Elastic Incompressible Isotropic Solid Containing An Incompressible Fluid

In this section, we see the first example where the solid is not rigid. It is allowed to be a deformable solid. In particular, it is allowed to undergo infinitesimal deformations as in the classical theory of linear elasticity. The assumption that the solid is incompressible means that the true density γ_s can be taken to be a constant in addition to our previous assumption that γ_f is constant. In this case one can show that the statement of balance of mass which replaces (1.2.1) is

$$\operatorname{div} \mathbf{c}_f = -\operatorname{div} \dot{\mathbf{w}}_s \tag{1.5.1}$$

where $\mathbf{w}_s$ denotes the *displacement* of the solid and $\dot{\mathbf{w}}_s$ denotes the time derivative of the displacement, i.e. the *velocity* of the solid. Because the solid is allowed to deform, one has an equation of motion of the form

$$\rho_s \ddot{\mathbf{w}}_s = -\operatorname{grad} P_f + (\lambda_s + \mu_s)\operatorname{grad}(\operatorname{div} \mathbf{w}_s) + \mu_s \Delta \mathbf{w}_s \tag{1.5.2}$$

where $\ddot{\mathbf{w}}_s$ is the *acceleration* of the solid, P_f is the *pore pressure* of the fluid, and λ_s and μ_s are *Lame' parameters* for the solid. As we shall see later, equation (1.5.2) involves constitutive assumptions which allow the equation of motion for the solid to be written in this special form. It turns out for this model that Darcy's law in the usual form

$$\mathbf{c}_f = -\frac{k}{\mu}\operatorname{grad} P_f \tag{1.5.3}$$

is valid.

1.6 Linear Elastic Isotropic Solid Containing A Compressible Fluid

In this section, we list the defining equations for a linear elastic isotropic solid which is compressible and also contains a single compressible fluid. As in the last section, the solid is allowed to undergo infinitesimal deformations as in the classical theory of linear elasticity. The equation which replaces (1.5.1) can be shown to be

$$\frac{\partial f \gamma_f}{\partial t} + \operatorname{div}(\gamma_f \mathbf{c}_f) = -\operatorname{div}(f \gamma_f \dot{\mathbf{w}}_s) \tag{1.6.1}$$

where $\mathbf{w}_s$ again denotes the *displacement of the solid* and $\dot{\mathbf{w}}_s$ denotes the time derivative of the displacement, i.e. the *velocity of the solid*. Because we are dealing with a two constituent mixture, the porosity f in (1.6.1) equals the volume fraction of the fluid ϕ_f. This result follows from equation (1.1.5). The equation of motion for the solid which replaces (1.5.2) turns out to be formally identical to (1.5.2), i.e.

$$\rho_s \ddot{\mathbf{w}}_s = -\operatorname{grad} P_f + (\lambda_s + \mu_s)\operatorname{grad}(\operatorname{div} \mathbf{w}_s) + \mu_s \Delta \mathbf{w}_s \tag{1.6.2}$$

As with equation (1.5.2), equation (1.6.2) involves constitutive assumptions which allow the equation of motion for the solid to be written in this special form. Again, it turns out for this model that Darcy's law in the usual form

$$\mathbf{c}_f = -\frac{k}{\mu}\operatorname{grad} P_f \tag{1.6.3}$$

remains valid. The additional constitutive equations which characterize this model are a *pressure density* relation of the form

$$P_f = P_f(\gamma_f) \tag{1.6.4}$$

and a *porosity-pore pressure-strain relation* of the form

$$f = f(P_f, \mathbf{E}_s) \tag{1.6.5}$$

where

$$\mathbf{E}_s = \tfrac{1}{2}(\text{grad}\,\mathbf{w}_s + \text{grad}\,\mathbf{w}_s{}^T) \tag{1.6.6}$$

is the *infinitesimal strain tensor* of the solid.

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2

Elements Of The Theory Of Mixtures

In order to fully understand the various assumptions build into the models of porous materials to be discussed in this work, it is helpful to understand some of the elements of the theory of mixtures. In this chapter, these elements will be summarized. The reader interested in an extensive discussion of this subject should consult references 1 through 4.

2.1. Kinematics

In this section we formulate the kinematics and the field equations for a mixture in a form which was popularized by Truesdell [Ref. 5].

Consider N continuous bodies $\mathscr{B}_1, \mathscr{B}_2, ..., \mathscr{B}_N$, each of which is visualized by the region it occupies in physical space. The deformation of each body is characterized by a deformation function $\boldsymbol{\chi}_a$ such that

$$\mathbf{x} = \boldsymbol{\chi}_a(\mathbf{X}_a, t) \tag{2.1.1}$$

where **x** is the *spatial position* at the time t of the particle of the a^{th} *constituent* which occupies the position $\mathbf{X}_a$ in a reference configuration. The *velocity* and *acceleration* of $\mathbf{X}_a$ at the time t are defined, respectively, by

$$\dot{\mathbf{x}}_a = \frac{\partial \boldsymbol{\chi}_a(\mathbf{X}_a, t)}{\partial t} \tag{2.1.2}$$

and

$$\ddot{\mathbf{x}}_a = \frac{\partial^2 \boldsymbol{\chi}_a(\mathbf{X}_a, t)}{\partial t^2} \tag{2.1.3}$$

The *gradient of the deformation* for $\mathbf{X}_a$ is a linear transformation defined by

$$\mathbf{F}_a = \mathrm{GRAD}\, \boldsymbol{\chi}_a(\mathbf{X}_a, t) \tag{2.1.4}$$

In (2.1.4), the notation GRAD denotes the gradient computed with respect to the position $\mathbf{X}_a$. The linear transformation inverse to $\mathbf{F}_a$ is

$$\mathbf{F}_a^{-1} = \mathrm{grad}\, \boldsymbol{\chi}_a^{-1}(\mathbf{x}, t) \tag{2.1.5}$$

where $\boldsymbol{\chi}_a^{-1}$ is the function inverse to $\boldsymbol{\chi}_a$. In (2.1.5) and later, the symbol grad denotes the gradient operation with respect to the spatial position **x**. The *velocity gradient* for $\mathscr{B}_a$ at (**x**,*t*) is defined by

$$\mathbf{L}_a = \operatorname{grad} \grave{\mathbf{x}}_a(\mathbf{x}, t) \tag{2.1.6}$$

An alternate expression for $\mathbf{L}_a$ is

$$\mathbf{L}_a = \grave{\mathbf{F}}_a \mathbf{F}_a^{-1} \tag{2.1.7}$$

where $\grave{\mathbf{F}}_a$ is the time derivative of $\mathbf{F}_a$.

Each constituent is assigned a density or, in the language of Chapter 1, a *bulk density*. For the a^{th} constituent the density is denoted by ρ_a. It is regarded as a function of (**x**,*t*). The *density of the mixture* at (**x**,*t*) is defined by

$$\rho(\mathbf{x}, t) = \sum_{a=1}^{N} \rho_a(\mathbf{x}, t) \tag{2.1.8}$$

The *mass concentration* of the a^{th} constituent at (**x**,*t*) is

$$c_a(\mathbf{x}, t) = \rho_a(\mathbf{x}, t) \big/ \rho(\mathbf{x}, t) \tag{2.1.9}$$

It follows from (2.1.8) and (2.1.9) that

$$\sum_{a=1}^{N} c_a(\mathbf{x}, t) = 1 \tag{2.1.10}$$

The *mean velocity*, or simply the *velocity of the mixture* at (**x**,*t*) is the mass-weighted average of the constituent velocities defined by

$$\dot{\mathbf{x}} = \frac{1}{\rho} \sum_{a=1}^{N} \rho_a \; \grave{\mathbf{x}}_a \tag{2.1.11}$$

If ψ is any function of (**x** ,*t*), the derivatives of ψ following the motions generated by $\grave{\mathbf{x}}_a$ and $\dot{\mathbf{x}}$ are, respectively,

$$\grave{\psi}_a = \frac{\partial \psi(\mathbf{x}, t)}{\partial t} + (\operatorname{grad} \psi(\mathbf{x}, t)) \; \grave{\mathbf{x}}_a \tag{2.1.12}$$

and

$$\dot{\psi} = \frac{\partial \psi(\mathbf{x},t)}{\partial t} + (\operatorname{grad} \psi(\mathbf{x},t))\dot{\mathbf{x}} \tag{2.1.13}$$

The *diffusion velocity* for the a^{th} constituent at ($\mathbf{x}$,t) is defined by

$$\mathbf{u}_a = \grave{\mathbf{x}}_a - \dot{\mathbf{x}} \tag{2.1.14}$$

It follows from (2.1.11), (2.1.14) and (2.1.8) that

$$\sum_{a=1}^{N} \rho_a \mathbf{u}_a = \mathbf{0} \tag{2.1.15}$$

The *true density* of the a^{th} constituent is denoted by γ_a and represents the mass of the a^{th} constituent per unit of volume of the a^{th} constituent. The quantity φ_a defined by

$$\varphi_a = \frac{\rho_a(\mathbf{x},t)}{\gamma_a(\mathbf{x},t)} \tag{2.1.16}$$

is the *volume fraction* of the a^{th} constituent. Physically φ_a represents the volume of the a^{th} constituent per unit volume of the mixture. Therefore,

$$\sum_{a=1}^{N} \varphi_a(\mathbf{x},t) = 1 \tag{2.1.17}$$

From its definition, it should be clear that φ_a, $a = 1,...,N$, measures a local structure of the mixture. For certain mixtures, it is reasonable to expect that this local structure would have no effect on the response of the mixture. Such mixtures are usually called *miscible* mixtures. An *immiscible* mixture is one where locally one can distinguish between mixture volumes and constituent volumes. A model of an immiscible mixture would necessarily allow the volume fractions to affect the mixture response. In cases where the first constituent is taken to be a solid and the remaining $N - 1$ constituents are fluids in the pores of the solid, the *porosity* of the solid is defined as in Chapter 1 by

$$f = \sum_{a=2}^{N} \varphi_a(\mathbf{x},t) = 1 - \varphi_1(\mathbf{x},t) \tag{2.1.18}$$

In Chapter 1, the *filtration velocities* were introduced. In the notation used in this Chapter, they are defined by the formulas

$$\mathbf{c}_a = \varphi_a(\grave{\mathbf{x}}_a - \grave{\mathbf{x}}_1) \tag{2.1.19}$$

Because the first constituent is normally the solid constituent, the filtration velocities measure the velocity of the a^{th} constituent relative to that of the solid.

In a theory of mixtures, the meaning of incompressibility can be slightly complicated. If the a^{th} constituent is *incompressible* then by definition, $\dot{\gamma}_a$ is zero. Physically, this means that γ_a can only depend on $\mathbf{X}_a$. It usually assumed that when the a^{th} constituent is incompressible that γ_a is a constant. In any case, the *mixture is incompressible* when every constituent in the mixture is incompressible. In this case, (2.1.16) and (2.1.17) imply a constraining relationship between the N bulk densities, ρ_a, $a=1, 2,..., N$.

2.2. Equations of Balance

In this section we shall briefly summarize the equations of balance and the resulting field equations which constitute the theory of mixtures [Ref. 1,2,3,4]. For simplicity, here the equations which will be given assume that chemical reactions are absent. If $\mathscr{V}$ is a fixed spatial region with surface $\partial\mathscr{V}$, then the equations of balance for the a^{th} constituent are as follows:

Balance Of Mass:

$$\frac{\partial}{\partial t}\int_{\mathscr{V}} \rho_a dv = -\oint_{\partial\mathscr{V}} \rho_a \dot{\mathbf{x}}_a \cdot d\mathbf{s} \tag{2.2.1}$$

Balance Of Linear Momentum**:**

$$\frac{\partial}{\partial t}\int_{\mathscr{V}} \rho_a \dot{\mathbf{x}}_a dv = -\oint_{\partial\mathscr{V}} \rho_a \dot{\mathbf{x}}_a (\dot{\mathbf{x}}_a \cdot d\mathbf{s}) + \oint_{\partial\mathscr{V}} \mathbf{T}_a d\mathbf{s} + \int_{\mathscr{V}} \rho_a \mathbf{b}_a dv + \int_{\mathscr{V}} \hat{\mathbf{p}}_a dv \tag{2.2.2}$$

where, for the a^{th} constituent, $\mathbf{T}_a$ is the *partial stress tensor* , $\mathbf{b}_a$ is the *external body force density* and $\hat{\mathbf{p}}_a$ is the *momentum supply*. It is the momentum supply which characterizes the mechanical interaction of the various constituents during diffusion.

Balance Of Moment Of Momentum:

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathscr{V}} \rho_a (\mathbf{x}-\mathbf{0})\times \dot{\mathbf{x}}_a dv &= -\oint_{\partial\mathscr{V}} \rho_a ((\mathbf{x}-\mathbf{0})\times \dot{\mathbf{x}}_a)(\dot{\mathbf{x}}_a \cdot d\mathbf{s}) \\ &+ \oint_{\partial\mathscr{V}} (\mathbf{x}-\mathbf{0})\times(\mathbf{T}_a d\mathbf{s}) + \int_{\mathscr{V}} \rho_a (\mathbf{x}-\mathbf{0})\times \mathbf{b}_a dv + \int_{\mathscr{V}} (\mathbf{x}-\mathbf{0})\times \hat{\mathbf{p}}_a dv + \int_{\mathscr{V}} \hat{\mathbf{m}}_a dv,\end{aligned} \tag{2.2.3}$$

where $\hat{\mathbf{m}}_a$ is the *momentum supply* for the a^{th} constituent.

Balance Of Energy:

$$\frac{\partial}{\partial t}\int_{\mathscr{V}} \rho_a(\varepsilon_a + \tfrac{1}{2}\grave{\mathrm{x}}_a^2)dv = -\oint_{\partial\mathscr{V}} \rho_a(\varepsilon_a + \tfrac{1}{2}\grave{\mathrm{x}}_a^2)(\grave{\mathbf{x}}_a \cdot d\mathbf{s}) + \oint_{\partial\mathscr{V}} \grave{\mathbf{x}}_a \cdot (\mathbf{T}_a d\mathbf{s})$$
$$-\oint_{\partial\mathscr{V}} \mathbf{q}_a \cdot d\mathbf{s} + \int_{\mathscr{V}} \rho_a \grave{\mathbf{x}}_a \cdot \mathbf{b}_a dv + \int_{\mathscr{V}} \rho_a r_a dv + \int_{\mathscr{V}} (\hat{\varepsilon}_a + \grave{\mathbf{x}}_a \cdot \hat{\mathbf{p}}_a)dv, \qquad (2.2.4)$$

where, for the a^{th} constituent ε_a is the *partial internal energy density* , $\mathbf{q}_a$ is the *partial heat flux vector* , r_a is the *heat supply density* and $\hat{\varepsilon}_a$ is the *energy supply*. For the mixture, the equations of balance take the following forms:

Balance Of Linear Momentum:

$$\sum_{a=1}^{N} \hat{\mathbf{p}}_a = \mathbf{0} \qquad (2.2.5)$$

Balance Of Moment Of Momentum:

$$\sum_{a=1}^{N} \hat{\mathbf{m}}_a = \mathbf{0} \qquad (2.2.6)$$

and

Balance of Energy:

$$\sum_{a=1}^{N} (\hat{\varepsilon}_a + \grave{\mathbf{x}}_a \cdot \hat{\mathbf{p}}_a) = 0 \qquad (2.2.7)$$

2.3. Field Equations

Associated with the integral statements of balance given in the last section, one can derive field equations which reflect locally the statements of balance of mass, momentum, moment of momentum and energy. These equations take the following forms:

Balance Of Mass**:**

$$\frac{\partial \rho_a}{\partial t} + \operatorname{div}(\rho_a \grave{\mathbf{x}}_a) = 0 \qquad (2.3.1)$$

Balance Of Linear Momentum:

$$\rho_a \grave{\grave{\mathbf{x}}}_a = \operatorname{div}\mathbf{T}_a + \rho_a \mathbf{b}_a + \hat{\mathbf{p}}_a \qquad (2.3.2)$$

Balance Of Moment Of Momentum:

$$\mathbf{T}_a - \mathbf{T}_a^{\mathrm{T}} = \hat{\mathbf{M}}_a \tag{2.3.3}$$

where $\hat{\mathbf{M}}_a$ is the skew-symmetric linear transformation constructed from the vector $\hat{\mathbf{m}}_a$.

Balance Of Energy:

$$\rho_a \grave{\varepsilon}_a = \operatorname{tr}(\mathbf{T}_a^T \mathbf{L}_a) - \operatorname{div} \mathbf{q}_a + \rho_a r_a + \hat{\varepsilon}_a \tag{2.3.4}$$

Equations (2.2.5), (2.2.6) and (2.2.7) are the field equations which govern the mixture as a whole. Equivalent forms of these equations, respectively, can be shown to be

$$\rho \ddot{\mathbf{x}} = \operatorname{div} \mathbf{T} + \rho \mathbf{b} \tag{2.3.5}$$

$$\mathbf{T} = \mathbf{T}^T \tag{2.3.6}$$

and

$$\rho \dot{\varepsilon}_{\mathrm{I}} = \operatorname{tr} \sum_{a=1}^{N} (\mathbf{T}_a^T \mathbf{L}_a) - \operatorname{div} \mathbf{k} - \sum_{a=1}^{N} \mathbf{u}_a \cdot \hat{\mathbf{p}}_a + \rho r \tag{2.3.7}$$

where

$$\mathbf{T} = \sum_{a=1}^{N} (\mathbf{T}_a - \rho_a \mathbf{u}_a \otimes \mathbf{u}_a) \tag{2.3.8}$$

is the *stress tensor for the mixture*,

$$\mathbf{b} = \frac{1}{\rho} \sum_{a=1}^{N} \rho_a \mathbf{b}_a \tag{2.3.9}$$

is the *external body force density for the mixture*,

$$\varepsilon_{\mathrm{I}} = \frac{1}{\rho} \sum_{a=1}^{N} \rho_a \varepsilon_a \tag{2.3.10}$$

is the *inner part of the internal energy density for the mixture*,

$$\mathbf{k} = \sum_{a=1}^{N} (\mathbf{q}_a + \rho_a \varepsilon_a \mathbf{u}_a) \tag{2.3.11}$$

is one of the many possible *heat fluxes* one can define for the mixture and

$$r = \frac{1}{\rho}\sum_{a=1}^{N} \rho_a r_a \tag{2.3.12}$$

is the *heat supply density for the mixture*. For later use, it is useful to introduce the *inner part of the stress* $\mathbf{T}_{\mathrm{I}}$ defined by

$$\mathbf{T}_{\mathrm{I}} = \sum_{a=1}^{N} \mathbf{T}_a \tag{2.3.13}$$

Note, from (2.2.6) and (2.3.3), that $\mathbf{T}_{\mathrm{I}}$ is symmetric.

It is useful to note at this point that if the N equations (2.3.1) are summed the result takes the form

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \dot{\mathbf{x}}) = 0 \tag{2.3.14}$$

after (2.1.8) and (2.1.11) are used.

2.4. Balance of Mass-Special Forms

Given equation (2.2.1) and the corresponding field equation (2.3.1), it is useful to deduce special forms of the field equations governing balance of mass appropriate for some of the special cases which will be considered later. A simple rearrangement of (2.3.1) yields

$$\grave{\rho}_a + \rho_a \operatorname{div} \grave{\mathbf{x}}_a = 0 \tag{2.4.1}$$

If the a^{th} constituent is *incompressible*, then $\grave{\gamma}_a = 0$. In this special case, (2.4.1) can be reduced to

$$\grave{\varphi}_a + \varphi_a \operatorname{div} \grave{\mathbf{x}}_a = 0 \tag{2.4.2}$$

where (2.1.16) has been used. An alternate form of (2.4.2) is

$$\frac{\partial \varphi_a}{\partial t} + \operatorname{div}(\varphi_a \grave{\mathbf{x}}_a) = 0 \tag{2.4.3}$$

It is a simple exercise to show that (2.4.3) contains (1.2.1), (1.4.1) and (1.5.1) as special cases. For example, if $N = 2$ and both constituents are incompressible, then (2.4.3) is equivalent to the following two equations:

$$\frac{\partial \varphi_1}{\partial t} + \operatorname{div}(\varphi_1 \grave{\mathbf{x}}_1) = 0 \tag{2.4.4}$$

and

$$\frac{\partial \varphi_2}{\partial t} + \operatorname{div}(\varphi_2 \grave{\mathbf{x}}_2) = 0 \tag{2.4.5}$$

If (2.1.17) is utilized, the sum of (2.4.4) and (2.4.5) can be written

$$\operatorname{div}(\varphi_2(\grave{\mathbf{x}}_2 - \grave{\mathbf{x}}_1) = -\operatorname{div} \grave{\mathbf{x}}_1 \tag{2.4.6}$$

Given the definition of filtration velocity (2.1.19), it should be clear that (2.4.6) yields (1.5.1).

Next consider the case of a two constituent mixture where the first constituent is a rigid solid and the second constituent is a compressible fluid. Without loss of generality, the velocity of the rigid solid can be taken to be zero. It follows from (2.3.1) that for the rigid solid, the only implication of balance of mass is that φ_1 can only depend upon position $\mathbf{x}$. The exact form of this dependence is a property of the rigid body. For example, one might have a rigid body for which the porosity is a constant. It follows from (2.1.18) that the porosity f is given by a prescribed function of $\mathbf{x}$. For the compressible fluid, (2.3.1) can be written

$$f(\mathbf{x})\frac{\partial \gamma_2}{\partial t} + \operatorname{div}(\varphi_2 \gamma_2 \grave{\mathbf{x}}_2) = 0 \tag{2.4.7}$$

where (2.1.16) and (2.1.18) have been used. Because the velocity of the solid is zero, equation (2.1.19) allows (2.4.7) to be written

$$f(\mathbf{x})\frac{\partial \gamma_2}{\partial t} + \operatorname{div}(\gamma_2 \mathbf{c}_2) = 0 \tag{2.4.8}$$

Equation (2.4.8) is the result (1.3.1).

The reader can easily verify that in the case where the solid is allowed to deform along with the fluid, the field equation which governs balance of mass for the fluid which replaces (2.4.7) can be written in the form of (1.6.1).

As the examples suggest, frequently we shall select the first constituent to be a solid and allow the remaining *N - 1* constituents to be fluids. When one of the constituents is a solid, it is sometimes convenient to utilize its reference configuration to generate *material* forms of the field equations. The results of this argument for balance of mass will be briefly given here. In order to use a suggestive notation, the subscript *1* which designates the solid constituent will be replaced by the symbol *s*. Given, (2.4.1) for the solid, it is an elementary exercise to integrate the result to obtain

$$\rho_s \left|\det \mathbf{F}_s\right| = \rho_{s_R} \tag{2.4.9}$$

where ρ_{s_R} is the density of the solid in its reference configuration. For the fluid constituents, $a = 2,\ldots,N$, define *N - 1 contents* by

$$m_a = \rho_a \left|\det \mathbf{F}_s\right| \tag{2.4.10}$$

Physically, m_a represents the mass of the a^{th} fluid per unit of solid reference configuration volume. The notation $\overset{o}{m}_a$ will be used to denote material differentiation following the motion of the solid, i.e. from (2.1.13)

$$\overset{o}{m}_a = \frac{\partial m_a(\mathbf{x},t)}{\partial t} + \operatorname{grad} m_a(\mathbf{x},t)\cdot \dot{\mathbf{x}}_s \tag{2.4.11}$$

If $\mathbf{v}_a$ is the velocity of the a^{th} fluid relative to the solid defined by $\mathbf{v}_a = \dot{\mathbf{x}}_a - \dot{\mathbf{x}}_s$, then the quantities $\mathbf{j}_2, \mathbf{j}_3, \ldots, \mathbf{j}_N$ are defined by

$$\mathbf{j}_a = \mathbf{F}_a^{-1}\left(\dot{\mathbf{x}}_a - \dot{\mathbf{x}}_s\right) = \mathbf{F}_a^{-1}\mathbf{v}_a \tag{2.4.12}$$

Given the definitions (2.4.10), (2.4.11) and (2.4.12), it is now possible to start with equation (2.4.1) and derive

$$\overset{o}{m}_a + \operatorname{Div}(m_a \mathbf{j}_a) = 0 \tag{2.4.13}$$

for the material forms of balance of mass for the fluid constituents. In equation (2.4.13), Div denotes the divergence operator with respect to the material coordinates of the solid. It is possible to show that the product $m_a \mathbf{j}_a$ represents the mass flux of the a^{th} fluid into a material element for the solid constituent. This mass flux is measured per unit of area of the reference configuration for the solid. Balance of mass statements of the form (2.4.13) first appear in the work of Biot [Ref. 6,7,8,9].

2.5. Balance of Linear Momentum-Special Forms

It is useful at this point to illustrate how (2.3.2) can be specialized so as to recover some of the results listed in Chapter 1. Essentially, Darcy's law is a special case of the statement of balance of linear momentum. The various assumptions which must be made in order to go from (2.3.2) to Darcy's law are rather strong and need justification within the context of a acceptable constitutive theory for mixtures. Elements of this theory will be presented in Sections 2.6 and 2.7.

As the first example, consider a rigid solid containing an incompressible fluid. The equation of motion (2.3.2) for the fluid can be written

$$\rho_f \ddot{\mathbf{x}}_f = \operatorname{div} \mathbf{T}_f + \rho_f \mathbf{b}_f + \hat{\mathbf{p}}_f \tag{2.5.1}$$

We shall make the following purely formal assumptions:

i. Neglect the acceleration of the fluid.

ii. Replace the partial stress $\mathbf{T}_f$ by

$$\mathbf{T}_f = -\varphi_f P_f \mathbf{I} \tag{2.5.2}$$

where P_f is the fluid pore pressure and φ_f is the volume fraction of the fluid. Note that in this case φ_f is also the porosity of the solid.

iii. Replace the momentum supply $\hat{\mathbf{p}}_f$ by

$$\hat{\mathbf{p}}_f = -\xi \,\grave{\mathbf{x}}_f + P_f \operatorname{grad} \varphi_f \tag{2.5.3}$$

where ξ is a positive material constant. It is usually called the *drag coefficient*. Given these three assumptions, (2.5.1) can be written

$$\varphi_f \grave{\mathbf{x}}_f = -\frac{\varphi_f^2}{\xi}(\operatorname{grad} P_f - \rho_f \mathbf{b}_f) \tag{2.5.4}$$

If the external body force $\mathbf{b}_f$ is zero, then (2.5.4) is Darcy's law (1.2.2). In the case under discussion, the filtration velocity is $\varphi_f \grave{\mathbf{x}}_f$ (see (1.1.10)) and the ratio k/μ is identified as the ratio φ_f^2/ξ. Other assumptions will produce (2.5.4). Clearly, if any constant is added to the right side of (2.5.2), then (2.5.4) is again obtained. If the body force is not zero but is given by a gravitational potential ν by the relationship,

$$\mathbf{b}_f = -\operatorname{grad} \nu \tag{2.5.5}$$

then Darcy's law can be written

$$\varphi_f \grave{\mathbf{x}}_f = -g \frac{\varphi_f^2}{\xi} \operatorname{grad} \Phi \tag{2.5.6}$$

where g is the gravitational constant, and Φ is the *pressure head* defined by

$$g\Phi = \frac{P_f}{\gamma_f} + \nu \tag{2.5.7}$$

While it is not necessarily clear at this point, the simplifying assumptions which produced (2.5.4) are also consistent with balance of linear momentum for the mixture. Because the solid has been assumed to be rigid, it is a material with *internal constraints*. Such materials, as we shall see later, are characterized by certain indeterminacies in their constitutive equations.

Equation (2.5.3) deserves some additional discussion. The term $-\xi \grave{\mathbf{x}}_f$ represents a force on the fluid because of its relative motion between itself and the solid (which is at rest). The term $P_f \operatorname{grad}\varphi_f$ is slightly more difficult to motivate at this point. The volume fraction φ_f is the porosity in this case of a two constituent mixture. Therefore, $\operatorname{grad}\varphi_f$ results from a spatial distribution of porosity. The product $P_f \operatorname{grad}\varphi_f$ is therefore a force on the fluid caused by the changes in porosity as the fluid moves through the pores of the solid. The origin of such a force is somewhat easier to motivate after one has available a constitutive theory for mixtures. In Section 2.7 it is shown how this force arises. In particular, we shall show that (2.5.3) is very special. Some motivation for the product $P_f \operatorname{grad}\varphi_f$ can be found in reference 10.

2.6. Entropy Inequality

In order to complete the elements of the theory of mixtures, one must have a formal statement of the entropy inequality appropriate for such theories. In the notation of Section 2.2, the entropy inequality is the statement that

$$\sum_{a=1}^{N}\left\{\frac{\partial}{\partial t}\int_{\mathscr{V}}\rho_a\eta_a dv+\oint_{\partial\mathscr{V}}\rho_a\eta_a(\grave{\mathbf{x}}_a\cdot d\mathbf{s})+\oint_{\partial\mathscr{V}}\frac{1}{\theta_a}\mathbf{q}_a\cdot d\mathbf{s}-\int_{\mathscr{V}}\left(\rho_a r_a/\theta_a\right)dv\right\}\geq 0 \qquad (2.6.1)$$

where η_a is the *entropy density* for the a^{th} constituent and θ_a is its *temperature*. The local version of (2.6.1) is

$$\sum_{a=1}^{N}\left(\rho_a\grave{\eta}_a+\operatorname{div}\left(\mathbf{q}_a/\theta_a\right)-\rho_a r_a/\theta_a\right)\geq 0 \qquad (2.6.2)$$

If (2.3.4) is used to eliminate each r_a from (2.6.2), the result can be written

$$\sum_{a=1}^{N}\frac{1}{\theta_a}\left\{-\rho_a(\grave{\psi}_a+\eta_a\grave{\theta}_a)+\operatorname{tr}(\mathbf{T}_a^T\mathbf{L}_a)-\frac{\mathbf{q}_a}{\theta_a}\cdot\operatorname{grad}\theta_a+\hat{\varepsilon}_a\right\}\geq 0 \qquad (2.6.3)$$

where ψ_a is the *Helmholtz free energy density* for the a^{th} constituent defined by

$$\psi_a=\varepsilon_a-\theta_a\eta_a \qquad (2.6.4)$$

Most models of mixtures are constrained in some fashion. One of the most common constraints is for the constituent temperatures to be constrained to have the same value. If we adopt this constraint here and we denote the common value by θ, then (2.6.3) can be shown to reduce to

$$-\sum_{a=1}^{N} \grave{\Psi}_a - \rho\eta\dot{\theta} - \operatorname{tr}\sum_{a=1}^{N}(\rho_a \mathbf{K}_a \mathbf{L}_a) - \frac{\mathbf{h}}{\theta}\cdot \operatorname{grad}\theta - \sum_{a=1}^{N} \grave{\mathbf{x}}_a \cdot \hat{\mathbf{p}}_a \geq 0 \tag{2.6.5}$$

where θ is the *temperature of the mixture*,

$$\Psi_a = \rho_a \psi_a \tag{2.6.6}$$

is the *Helmholtz free energy* of the a^{th} constituent per unit of mixture volume,

$$\eta = \frac{1}{\rho}\sum_{a=1}^{N} \rho_a \eta_a \tag{2.6.7}$$

is the *entropy density for the mixture*,

$$\mathbf{K}_a = \frac{1}{\rho_a}(\Psi_a \mathbf{I} - \mathbf{T}_a^T) \tag{2.6.8}$$

is the *chemical potential* for the a^{th} constituent and

$$\mathbf{h} = \sum_{a=1}^{N}(\mathbf{q}_a + \rho_a \eta_a \theta \mathbf{u}_a) \tag{2.6.9}$$

is another possible heat flux vector for the mixture. It is a consequence of the single temperature constraint that the energy supplies $\hat{\varepsilon}_a$ do not appear in the entropy inequality (2.6.5). It turns out to be a consequence of this fact that these energy supplies are almost indeterminate in so far as the constitutive theory is concerned. The "almost" qualifier arises because the overall statement of balance of energy (2.2.7) must be satisfied. This requirement enters into the model formulation by the requirement that (2.2.7), or, equivalently, (2.3.7) control balance of energy for the mixture. In the additional special case where the mixture is *isothermal*, the entropy inequality (2.6.5) reduces to

$$-\sum_{a=1}^{N} \grave{\Psi}_a - \operatorname{tr}\sum_{a=1}^{N}(\rho_a \mathbf{K}_a \mathbf{L}_a) - \sum_{a=1}^{N} \grave{\mathbf{x}}_a \cdot \hat{\mathbf{p}}_a \geq 0 \tag{2.6.10}$$

When the mixture is constrained to be isothermal, the entropy inequality does not involve the heat flux vector **h**. This quantity is indeterminate in so far as the entropy inequality is concerned. It is allowed to assume any value consistent with the energy equation (2.3.7). In effect, what we have concluded is that for an isothermal model one simply does not utilize the energy equation.

Two additional constraints which will arise in this work are the case of a rigid solid and the case of an incompressible mixture. For the case of a rigid solid, the first constituent is only allowed to undergo rigid body motions. Without loss of generality, one can simply assume the rigid solid is at rest. It is possible to conclude that for a rigid solid, the momentum supply $\hat{\mathbf{p}}_s$ is unconstrained by the entropy inequality and can, thus, take on any value consistent with balance of linear momentum for the mixture in the form (2.2.5). A more interesting constraint arises when one assumes that all of the constituents in the mixture are incompressible [Ref. 11]. In this case, it turns out that the volume fractions, the constituent velocity gradients and the constituent velocities are no longer independent. This assertion follows from the following formulas:

$$\varphi_s \left|\det \mathbf{F}_s\right| = \varphi_{s_R} \tag{2.6.11}$$

and

$$\varphi_s + \sum_{a=2}^{N} \varphi_a = 1 \tag{2.6.12}$$

Equation (2.6.11) is simply (2.4.9) specialized for an incompressible solid. Of course, equation (2.6.12) is (2.1.17). Equations (2.6.11) and (2.6.12) establish a mathematical link between $\left|\det \mathbf{F}_s\right|$ and the volume fractions of the *N - 1* fluids. It is this relationship which causes the terms in (2.6.10) to be related. The exact form of this relationship follows by differentiation of (2.6.12) and use of (2.4.1). The result of this calculation is

$$\sum_{a=1}^{N} (\varphi_a \operatorname{tr} \mathbf{L}_a + \operatorname{grad} \varphi_a \cdot (\grave{\mathbf{x}}_a - \grave{\mathbf{x}}_s)) = 0 \tag{2.6.13}$$

Given (2.6.13), it easily follows that (2.6.10) can be replaced by

$$-\sum_{a=1}^{N} \grave{\Psi}_a - \operatorname{tr} \sum_{a=1}^{N} (\rho_a \mathbf{K}_a - \varphi_a \lambda \mathbf{I}) \mathbf{L}_a - \sum_{a=1}^{N} (\grave{\mathbf{x}}_a - \grave{\mathbf{x}}_s) \cdot (\hat{\mathbf{p}}_a - \lambda \operatorname{grad} \varphi_a) \geq 0, \tag{2.6.14}$$

for any scalar multiplier λ. It should be mentioned that equation (2.2.5) was also used in the derivation of (2.6.14).

As a final result in this section, there is an inequality implied by (2.6.1) which will be useful later. If the *N* equations (2.2.4) are summed over the constituents and (2.2.7) is used the result can be written

$$\sum_{a=1}^{N}\frac{\partial}{\partial t}\int_{\mathscr{V}}\rho_a(\varepsilon_a-\theta_o\eta_a+\tfrac{1}{2}\grave{x}_a^2)dv=-\sum_{a=1}^{N}\oint_{\partial\mathscr{V}}\rho_a(\varepsilon_a-\theta_o\eta_a+\tfrac{1}{2}\grave{x}_a^2)(\grave{\mathbf{x}}_a\cdot d\mathbf{s})$$
$$+\sum_{a=1}^{N}\oint_{\partial\mathscr{V}}\left\{\mathbf{T}_a^{\mathrm{T}}\grave{\mathbf{x}}_a-\mathbf{q}_a\cdot(1-\theta_o/\theta)\right\}\cdot d\mathbf{s}+\sum_{a=1}^{N}\int_{\mathscr{V}}\left\{\rho_a r_a(1-\theta_o/\theta)+\rho_a\grave{\mathbf{x}}_a\cdot\mathbf{b}_a\right\}dv \tag{2.6.15}$$
$$-\theta_o\left\{\sum_{a=1}^{N}\frac{\partial}{\partial t}\int_{\mathscr{V}}\rho_a\eta_a dv+\sum_{a=1}^{N}\oint_{\partial\mathscr{V}}(\rho_a\eta_a\grave{\mathbf{x}}_a+\frac{1}{\theta}\mathbf{q}_a)\cdot d\mathbf{s})-\sum_{a=1}^{N}\int_{\mathscr{V}}(\rho_a r_a/\theta)dv\right\}$$

where θ_o is any positive constant. If (2.6.1) is used, then (2.6.15) yields the following inequality:

$$\sum_{a=1}^{N}\frac{\partial}{\partial t}\int_{\mathscr{V}}\rho_a(\varepsilon_a-\theta_o\eta_a+\tfrac{1}{2}\grave{x}_a^2)dv\le-\sum_{a=1}^{N}\oint_{\partial\mathscr{V}}\rho_a(\varepsilon_a-\theta_o\eta_a+\tfrac{1}{2}\grave{x}_a^2)(\grave{\mathbf{x}}_a\cdot d\mathbf{s})$$
$$+\sum_{a=1}^{N}\oint_{\partial\mathscr{V}}\left\{\mathbf{T}_a^{T}\grave{\mathbf{x}}_a-\mathbf{q}_a\cdot(1-\theta_o/\theta)\right\}\cdot d\mathbf{s}+\sum_{a=1}^{N}\int_{\mathscr{V}}\left\{\rho_a r_a(1-\theta_o/\theta)+\rho_a\grave{\mathbf{x}}_a\cdot\mathbf{b}_a\right\}dv. \tag{2.6.16}$$

2.7 Rigid Isotropic Solid Containing Incompressible Fluids

It is not the purpose of this work to make the reader an expert on the theory of mixtures. The introduction of this subject in this chapter is intended to give the reader an indication of the formal basis for the models to be discussed later. Toward this goal it is instructive to illustrate how one of the simplest porous media models can be viewed as a consequence of the theory of mixtures. This simple model is the one summarized in Section 1.4. It consists of a rigid isotropic solid containing *N - 1* incompressible fluids. Of course, it is understood that N is an integer greater than or equal to two.

For simplicity, the porosity of the solid will be taken to be constant in this discussion. As a result, it follows from (2.1.18) that the *N - 1* volume fractions obey

$$f=\sum_{a=2}^{N}\varphi_a(\mathbf{x},t)=constant \tag{2.7.1}$$

Because of the relationship (2.7.1), it is often the case for this model that the volume fractions $\varphi_a, a=2,...,N$, are replaced as kinematic variables by the *saturations* defined by

$$s_a=\frac{\varphi_a}{f} \tag{2.7.2}$$

for $a = 2,...,N$. While some formal advantages arise by the introduction of these quantities, it is not necessary to complicate the notation by their introduction here. Because the solid is rigid, without loss of generality its velocity can be taken to be zero. This fact allows the entropy inequality (2.6.14) to be written

$$-\sum_{a=2}^{N} \grave{\Psi}_a - \operatorname{tr}\sum_{a=2}^{N}(\rho_a \mathbf{K}_a - \varphi_a \lambda \mathbf{I})\mathbf{L}_a - \sum_{a=2}^{N} \grave{\mathbf{x}}_a \cdot (\hat{\mathbf{p}}_a - \lambda \operatorname{grad}\varphi_a) \geq 0, \tag{2.7.3}$$

where, without loss of generality, the free energy $\Psi_1 = \Psi_s$ is taken to be a constant. The constitutive equations which we shall take as defining the mixture are as follows:

$$\Psi_a = \Psi_a(\varphi_c) \tag{2.7.4}$$

$$\hat{\mathbf{p}}_a - \lambda \operatorname{grad}\varphi_a = \mathbf{e}_a(\varphi_c, \operatorname{grad}\varphi_c, \grave{\mathbf{x}}_b) \tag{2.7.5}$$

and

$$\rho_a \mathbf{K}_a - \varphi_a \lambda \mathbf{I} = \mathbf{R}_a(\varphi_c) \tag{2.7.6}$$

where *a = 2,3,..., N*. Our notational convention is such that the subscripts *a* and *b* range from *2,..., N*. The subscripts *c*, *d* and *e* range from *3,..., N*. As an arbitrary choice, equation (2.7.1) has been used to eliminate φ_2 as a constitutive variable. Clearly, more general constitutive equations could have been written for this mixture. The reader interested in a somewhat more general discussion of this model should consult reference 11. The parameter λ appears in (2.7.5) and (2.7.6) because of the incompressible constraint. The special dependence reflects the arbitrariness which is intrinsic in the entropy inequality (2.7.3).

If (2.7.4) through (2.7.6) are substituted into the inequality (2.7.3), the result can be written

$$\begin{aligned}
&-\operatorname{tr}(\mathbf{R}_2(\varphi_c)\mathbf{L}_2) - \operatorname{tr}\sum_{d=3}^{N}((\mathbf{R}_d(\varphi_c) - \varphi_d \frac{\partial \Psi_{\mathrm{I}}(\varphi_c)}{\partial \varphi_d}\mathbf{I})\mathbf{L}_d) \\
&- \grave{\mathbf{x}}_2 \cdot (\mathbf{e}_2(\varphi_c, \operatorname{grad}\varphi_c, \grave{\mathbf{x}}_b) + \sum_{d=3}^{N} \frac{\partial \Psi_2(\varphi_c)}{\partial \varphi_d} \operatorname{grad}\varphi_d) \\
&-\sum_{d=3}^{N} \grave{\mathbf{x}}_d \cdot (\mathbf{e}_d(\varphi_c, \operatorname{grad}\varphi_c, \grave{\mathbf{x}}_b) - \frac{\partial \Psi_{\mathrm{I}}(\varphi_c)}{\partial \varphi_d} \operatorname{grad}\varphi_d + \sum_{e=3}^{N} \frac{\partial \Psi_{\mathrm{d}}(\varphi_c)}{\partial \varphi_e} \operatorname{grad}\phi_e) \geq 0,
\end{aligned} \tag{2.7.7}$$

where the quantity Ψ_{I} is the inner *part of the Helmholtz Free Energy* of the mixture defined by

$$\Psi_{\mathrm{I}} = \sum_{a=1}^{N} \Psi_a \tag{2.7.8}$$

As explained in reference 11, the inequality (2.7.7) implies the following restrictions:

$$\mathbf{R}_2(\varphi_c) = \mathbf{0} \tag{2.7.9}$$

$$\mathbf{R}_d(\varphi_c) = \varphi_d \frac{\partial \Psi_{\mathrm{I}}(\varphi_c)}{\partial \varphi_d}\mathbf{I} \tag{2.7.10}$$

for $d = 3,\ldots, N$, and

$$-\sum_{a=2}^{N} \grave{\mathbf{x}}_a \cdot \hat{\mathbf{f}}_a(\varphi_c, \operatorname{grad}\varphi_c, \grave{\mathbf{x}}_b) \geq 0 \tag{2.7.11}$$

where

$$\hat{\mathbf{f}}_2 = \mathbf{e}_2(\varphi_c, \operatorname{grad}\varphi_c, \grave{\mathbf{x}}_b) + \sum_{e=3}^{N} \frac{\partial \Psi_2(\varphi_c)}{\partial \varphi_e} \operatorname{grad}\varphi_e \tag{2.7.12}$$

and

$$\hat{\mathbf{f}}_d = \mathbf{e}_d(\varphi_c, \operatorname{grad}\varphi_c, \grave{\mathbf{x}}_b) - \frac{\partial \Psi_{\mathrm{I}}(\varphi_c)}{\partial \varphi_d} \operatorname{grad}\varphi_d + \sum_{e=3}^{N} \frac{\partial \Psi_d(\varphi_c)}{\partial \varphi_e} \operatorname{grad}\varphi_e \tag{2.7.13}$$

where $d = 3,\ldots,N$.

Given (2.7.9) and (2.7.10), it follows from (2.7.6) that

$$\rho_2 \mathbf{K}_2 = \varphi_2 \lambda \mathbf{I} \tag{2.7.14}$$

and

$$\rho_d \mathbf{K}_d = \varphi_d \left(\lambda + \frac{\partial \Psi_{\mathrm{I}}(\varphi_c)}{\partial \varphi_d}\right)\mathbf{I} \tag{2.7.15}$$

for $d = 3,\ldots, N$. Given (2.7.12) and (2.7.13), it follows from (2.7.5) that

$$\hat{\mathbf{p}}_2 = \lambda \operatorname{grad}\varphi_2 - \sum_{e=3}^{N} \frac{\partial \Psi_2(\varphi_c)}{\partial \phi_e} \operatorname{grad}\phi_e + \hat{\mathbf{f}}_2(\varphi_c, \operatorname{grad}\varphi_c, \grave{\mathbf{x}}_b) \tag{2.7.16}$$

and

$$\hat{\mathbf{p}}_d = \left(\lambda + \frac{\partial \Psi_{\mathrm{I}}(\varphi_c)}{\partial \varphi_d}\right) \operatorname{grad}\varphi_d - \sum_{e=3}^{N} \frac{\partial \Psi_d(\varphi_c)}{\partial \varphi_e} \operatorname{grad}\varphi_e + \hat{\mathbf{f}}_d(\varphi_c, \operatorname{grad}\varphi_c, \grave{\mathbf{x}}_b) \tag{2.7.17}$$

for $d = 3,\ldots, N$. It is convenient to define fluid *pore pressures* by

$$P_2 = \lambda \tag{2.7.18}$$

and

$$P_d = \lambda + \frac{\partial \Psi_{\mathrm{I}}(\varphi_c)}{\partial \varphi_d} \tag{2.7.19}$$

for $d = 3,..., N$. With these definitions, the *capillary pressures* , defined by $P_d - P_2$, are given by

$$P_d - P_2 = \frac{\partial \Psi_{\mathrm{I}}(\varphi_c)}{\partial \varphi_{\mathrm{d}}} \tag{2.7.20}$$

for $d = 3,..., N$. Equation (2.7.20) establishes within the thermodynamics of the theory of mixtures the special constitutive relation (1.4.4). Next we shall illustrate how one can find the result (1.4.3) within the results of this section.

First, it is helpful to extract a physical meaning for the $N-1$ quantities, $\hat{\mathbf{f}}_a$. These quantities are restricted by the inequality (2.7.11). When viewed as a function of the $N-1$ velocities, $\grave{\mathbf{x}}_b$, the quantity Φ defined by

$$\Phi(\grave{\mathbf{x}}_b) = -\sum_{a=2}^{N} \grave{\mathbf{x}}_a \cdot \hat{\mathbf{f}}_a(\varphi_c, \operatorname{grad}\varphi_c, \grave{\mathbf{x}}_b) \tag{2.7.21}$$

is a minimum at $\grave{\mathbf{x}}_b = \mathbf{0}$. It is elementary to use this observation to prove that

$$\hat{\mathbf{f}}_a(\phi_c, \operatorname{grad}\phi_c, \mathbf{0}) = \mathbf{0} \tag{2.7.22}$$

and

$$\sum_{a=2}^{N}\sum_{b=2}^{N} \mathbf{a}_a \cdot \left(\frac{\partial \mathbf{f}_a(\varphi_c, \operatorname{grad}\varphi_c, \mathbf{0})}{\partial \grave{\mathbf{x}}_b} \mathbf{b}_b \right) \le 0 \tag{2.7.23}$$

for all vectors $\mathbf{a}_a, a = 2,3,..., N$ and $\mathbf{b}_b, b = 2,3,..., N$. Equation (2.7.22) shows that the quantities $\hat{\mathbf{f}}_a$, for $a = 2, 3,..., N$, vanish when the fluid constituents have zero velocity. It is, therefore, natural to call these quantities, the *diffusion forces*. The definitions (2.7.18) and (2.7.19) allow (2.7.16) and (2.7.17) to be written

$$\hat{\mathbf{p}}_b = P_b \operatorname{grad}\varphi_b - \sum_{e=3}^{N} \frac{\partial \Psi_b(\varphi_c)}{\partial \varphi_e} \operatorname{grad}\varphi_e + \hat{\mathbf{f}}_b(\varphi_c, \operatorname{grad}\varphi_c, \grave{\mathbf{x}}_b) \tag{2.7.24}$$

In addition, equations (2.7.14), (2.7.15), (2.7.18) and (2.7.19) combine with (2.6.8) to yield

$$\mathbf{T}_b = -\varphi_b P_b \mathbf{I} + \Psi_b \mathbf{I} \tag{2.7.25}$$

If (2.7.24) and (2.7.25) are substituted into (2.3.2), balance of linear momentum for the b^{th} fluid takes the following form:

$$\rho_b \grave{\grave{\mathbf{x}}}_b = -\varphi_b \operatorname{grad} P_b + \rho_b \mathbf{b}_b + \hat{\mathbf{f}}_b \tag{2.7.26}$$

It is instructive at this point to attempt to find the results of Section 2.5 in the results obtained in this section. In Section 2.5, we assumed that the mixture consisted of a rigid solid and a single incompressible fluid. Therefore, $N = 2$. In this case, the constitutive equation (2.7.4) reduces to

$$\Psi_2 = constant \tag{2.7.27}$$

Without loss of generality, this constant can be taken to be zero. Therefore, (2.7.24) and (2.7.25) reduce to

$$\hat{\mathbf{p}}_2 = P_2 \operatorname{grad} \varphi_2 + \hat{\mathbf{f}}_2(\grave{\mathbf{x}}_2) \tag{2.7.28}$$

and

$$\mathbf{T}_2 = -\varphi_2 P_2 \mathbf{I} \tag{2.7.29}$$

Equation (2.7.29) is the result (2.5.2). Equation (2.7.28) reduces to (2.5.3) if the diffusion force $\hat{\mathbf{f}}_2(\grave{\mathbf{x}}_2)$ takes the special form

$$\hat{\mathbf{f}}_2(\grave{\mathbf{x}}_2) = -\xi\, \grave{\mathbf{x}}_2 \tag{2.7.30}$$

Clearly, (2.7.30) obeys (2.7.22). Given (2.7.30), it easily follows from (2.7.23) that the material coefficient ξ, the *drag coefficient*, cannot be negative. In fact, this result is sufficient to satisfy the residual inequality (2.7.11). The reader will recognize (2.7.30) as the linearized constitutive equation for the drag force which arises when one assumes small departures from the state $\grave{\mathbf{x}}_2 = \mathbf{0}$ and that the rigid solid is isotropic.

Finally, it is instructive return to the case of an arbitrary number of incompressible fluids and to illustrate how the result (1.4.3) arises as a special case of the results given here. The reader should note in passing that (1.4.1) is a simple rearrangement of (2.4.3), and (1.4.4) has been established with the thermodynamic result (2.7.20). Equation (1.4.3) follows from (2.7.26) if one neglects the constituent accelerations, the external body forces and adopts the following constitutive equations for the diffusion forces $\hat{\mathbf{f}}_b(\grave{\mathbf{x}}_c)$:

$$\hat{\mathbf{f}}_b(\grave{\mathbf{x}}_c) = -\xi_b\, \grave{\mathbf{x}}_b \tag{2.7.31}$$

With these specializations, (2.7.26) can be written

$$\varphi_b \grave{\mathbf{x}}_b = -\frac{\varphi_b^2}{\xi_b} \operatorname{grad} P_b \tag{2.7.32}$$

where the coefficient ξ_b has been assumed to be nonzero. Note that (2.7.23) shows that each ξ_b must obey the inequality

$$\xi_b \geq 0 \tag{2.7.33}$$

Because the filtration velocity equals $\varphi_b \grave{\mathbf{x}}_b$ in this case, it is easily seen that (2.7.32) is equivalent to the result (1.4.3). One simply has to identify the coefficient $\frac{\varphi_b^2}{\xi_b}$ as the ratio $\frac{k_b}{\mu_b}$.

As indicated above, the reader interested in additional information about how one models incompressible porous materials by use of the theory of mixtures should consult reference 11.

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3

Porous Elasticity Models

In this chapter the mathematical models which form the basis of the theory of porous elasticity will be given. For the most part, the basis of the models, as implied by the theory of mixtures, will not be given. References will be given where the reader can find this fundamental material.

3.1. Immiscible Mixtures

As indicated in Section 2.1, a mixture is immiscible if volume fractions affect the response. In this section, an example of an immiscible mixture is briefly considered. The model is that of an immiscible compressible mixture whose constituents are a single solid and $N-1$ fluids. For the second, the immiscible mixture is taken to be incompressible. This example is discussed in detail in references 1.

An immiscible compressible mixture having a single temperature θ is defined by the following constitutive equations:

$$\Psi_a = \Psi_a(\theta, \mathbf{F}_s, \rho_b, \varphi_b) \tag{3.1.1}$$

and

$$\begin{aligned}(\eta_a, \hat{\mathbf{p}}_a, \mathbf{K}_a, \hat{\mathbf{M}}_a, \mathbf{h}, \grave{\varphi}_a) = \\ f_a(\theta, \operatorname{grad}\theta_a, \mathbf{F}_s, \operatorname{GRAD}\mathbf{F}_s, \rho_b, \operatorname{grad}\rho_b, \varphi_b, \operatorname{grad}\varphi_b, \grave{\mathbf{x}}_b - \grave{\mathbf{x}}_s).\end{aligned} \tag{3.1.2}$$

In (3.1.1) and (3.1.2), the subscript *s* denotes the first constituent, the solid. In addition, it is understood in writing these equations that the index *a* ranges from 1 to N while the indices *b* and *c* range from 2 to *N*. Note that (3.1.2) shows that the volume fractions have been treated as internal state variables. This assumption is sufficiently general to allow one to treat, for example, the volume fractions as determined by the state of the mixture or as given constants. Both special cases occur in the classical literature on porous elasticity. If (3.1.1) and (3.1.2) are required to be consistent with the entropy inequality (2.6.5), the following results are obtained [Ref. 1]:

$$\rho\eta = -\frac{\partial \Psi_{\mathrm{I}}}{\partial \theta} \tag{3.1.3}$$

$$\rho_s \mathbf{K}_s = -\mathbf{F}_s \frac{\partial \Psi_{\mathrm{I}}}{\partial \mathbf{F}_s}^T \tag{3.1.4}$$

$$\mathbf{K}_c = \mu_c \mathbf{I} \tag{3.1.5}$$

and

$$-\mathbf{m}\cdot \operatorname{grad}\theta/\theta - \sum_{c=2}^{N}(\grave{\mathbf{x}}_c - \grave{\mathbf{x}}_s)\cdot \hat{\mathbf{f}}_c - \sum_{c=2}^{N}\sigma_c\omega_c \geq 0 \tag{3.1.6}$$

where

$$\mu_c = \frac{\partial \Psi_{\mathrm{I}}}{\partial \rho_c} \tag{3.1.7}$$

$$\sigma_c = -\frac{\partial \Psi_{\mathrm{I}}}{\partial \varphi_c} \tag{3.1.8}$$

$$\begin{aligned}\hat{\mathbf{f}}_c = \hat{\mathbf{p}}_c - \mu_c \operatorname{grad}\rho_c + \sigma_c \operatorname{grad}\varphi_c + (\mathbf{F}_s^{-1})^{\mathrm{T}}\left\{\frac{\partial \Psi_c}{\partial \mathbf{F}_s}\left[\operatorname{GRAD}\mathbf{F}_s\right]\right\} \\ + \sum_{b=2}^{N}\frac{\partial \Psi_c}{\partial \rho_b}\operatorname{grad}\rho_b + \sum_{b=2}^{N}\frac{\partial \Psi_c}{\partial \varphi_b}\operatorname{grad}\varphi_b\end{aligned} \tag{3.1.9}$$

and

$$\mathbf{m} = \mathbf{h} + \theta\sum_{a=1}^{N}\frac{\partial \Psi_a}{\partial \theta}\mathbf{u}_a \tag{3.1.10}$$

Of course, the heat flux vector **h** is defined by (2.6.9). In (3.1.9) a special notation has been used in the term involving $\operatorname{GRAD}\mathbf{F}_s$. In components, this notation means the following:

$$\frac{\partial \Psi_c}{\partial \mathbf{F}_s}\left[\operatorname{GRAD}\mathbf{F}_s\right] = \frac{\partial \Psi_c}{\partial \mathrm{F}_{s_{jk}}}\frac{\partial \mathrm{F}_{s_{jk}}}{\partial X_L}\mathbf{i}_L \tag{3.1.11}$$

where $\mathbf{i}_L$ denotes the L^{th} basis vector in a rectangular Cartesian coordinate system. Finally, as follows from (3.1.2), the quantity ω_c in (3.1.6) is the function whose value is $\grave{\varphi}_c$, *i.e.*,

$$\grave{\varphi}_c = \omega_c(\theta, \operatorname{grad}\theta, \mathbf{F}_s, \operatorname{GRAD}\mathbf{F}_s, \rho_b, \operatorname{grad}\rho_b, \varphi_b, \operatorname{grad}\varphi_b, \grave{\mathbf{x}}_b - \grave{\mathbf{x}}_s) \tag{3.1.12}$$

Next, certain thermodynamic equilibrium results, which follow from (3.1.6), will be recorded. For a model employing internal state variables, there are several natural definitions of equilibrium. The reader interested in a discussion of these possibilities can consult Bowen [Ref. 1, 2, 3]. Here, equilibrium is a state in which the temperature gradient is zero, the constituent velocities are all equal and the N - 1 quantities σ_c , c = 2,...,N, defined by (3.1.8) all vanish. It shall be assumed that the N - 1 equations (3.1.8) can be inverted to obtain

$$\varphi_b = \varphi_b(\theta, \mathbf{F}_s, \rho_c, \sigma_c) \tag{3.1.13}$$

The assumed inversion implies that the $(N-1)\times(N-1)$ symmetric matrix

$$\frac{\partial \sigma_c}{\partial \varphi_b} = -\frac{\partial^2 \Psi_{\mathrm{I}}}{\partial \varphi_b \partial \varphi_c} \tag{3.1.14}$$

is nonsingular for all values of its argument. Given (3.1.13), it follows that in the equilibrium state the volume fractions are determined by the temperature and the deformation of the solid. By the kind of argument mentioned in Section 2.7, it is possible to show from (3.1.6) that, in the equilibrium state defined here, $\mathbf{m}$, $\hat{\mathbf{f}}_c$ and ω_c each vanish [Ref. 1].

3.2 Immiscible Mixtures-Linearized Isotropic Models

In order to further simplify the formalities, it will be assumed that the solid is *isotropic.* With this assumption, the constitutive equations will be linearized about a thermodynamic equilibrium state of uniform temperature θ^+, uniform deformation $\mathbf{F}_s = \mathbf{I}$ for the solid, uniform bulk densities $\rho_c = \rho_c^+$ and uniform volume fractions calculated from (3.1.13) by the formulas

$$\varphi_b^+ = \varphi_b(\theta^+, \mathbf{I}, \rho_c^+, 0) \tag{3.2.1}$$

It is readily established that the state defined in this fashion is a solution of the field equations which govern the immiscible mixture if $\mathbf{b}_a = \mathbf{0}$, for a = *1,...,N,* and r = 0. It is convenient to refer to this state as the *static solution* of the field equations.

The equations which govern the immiscible mixture are linearized by requiring the temperature changes, the displacement gradient of the solid, the density changes for the fluids, and the departure from thermodynamic equilibrium to be small. As is customary, the *displacement* of the a^{th} constituent is defined by

$$\mathbf{w}_a(\mathbf{X}_a, t) = \boldsymbol{\chi}_a(\mathbf{X}_a, t) - \mathbf{X}_a \tag{3.2.2}$$

Given (3.2.2), the displacement gradient of the a^{th} constituent is defined by

$$\mathbf{H}_a = \mathrm{GRAD}\,\mathbf{w}_a(\mathbf{X}_a, t) \tag{3.2.3}$$

and, by (3.2.2) and (2.1.4),

$$\mathbf{H}_a = \mathbf{F}_a - \mathbf{I} \tag{3.2.4}$$

A formal linearization of the constitutive equations and the field equations involves expansions about the static solution

$$\begin{aligned}&\left(\theta, \operatorname{grad}\theta, \mathbf{F}_s, \operatorname{GRAD}\mathbf{F}_s, \rho_b, \operatorname{grad}\rho_b, \varphi_b, \operatorname{grad}\varphi_b, \ \grave{\mathbf{x}}_b - \grave{\mathbf{x}}_s\right)\\ &\quad = \left(\theta^+, \mathbf{0}, \mathbf{I}, \mathbf{0}, \rho_b^+, \mathbf{0}, \phi_b^+, \mathbf{0}, \mathbf{0}\right).\end{aligned} \tag{3.2.5}$$

Departures from this state are measured by a positive number ε defined by

$$\begin{aligned}\varepsilon^2 &= (\theta-\theta^+)^2 + \operatorname{grad}\theta\cdot\operatorname{grad}\theta + \operatorname{tr}(\mathbf{H}_s\mathbf{H}_s^{\mathrm{T}}) + \operatorname{GRAD}\mathbf{F}_s\cdot\operatorname{GRAD}\mathbf{F}_s + \sum_{b=2}^{N}(\rho_b-\rho_b^+)^2/\rho_b^{+2}\\ &+\sum_{b=2}^{N}\operatorname{grad}\rho_b\cdot\operatorname{grad}\rho_b + \sum_{b=2}^{N}(\varphi_b-\varphi_b^+)^2 + \sum_{b=2}^{N}\operatorname{grad}\varphi_b\cdot\operatorname{grad}\varphi_b + \sum_{b=2}^{N}(\ \grave{\mathbf{x}}_b - \grave{\mathbf{x}}_s)\cdot(\ \grave{\mathbf{x}}_b - \grave{\mathbf{x}}_s).\end{aligned} \tag{3.2.6}$$

Given the parameter ε, the formal expansion of Ψ_a which follows from (3.1.1) can be written

$$\begin{aligned}\Psi_a &= \Psi_a^+ + \alpha_a(\theta-\theta^+) + \iota_{as}(\operatorname{tr}\mathbf{E}_s) - \sum_{b=2}^{N}\iota_{ab}(\rho_b-\rho_b^+)/\rho_b^+ + \sum_{b=2}^{N}\left(\frac{\partial\Psi_a}{\partial\varphi_b}\right)^+(\varphi_b-\varphi_b^+)\\ &-\frac{1}{2}\nu_a(\theta-\theta^+)^2 - \tau_{as}(\theta-\theta^+)(\operatorname{tr}\mathbf{E}_s) + \sum_{b=2}^{N}\tau_{ab}(\theta-\theta^+)(\rho_b-\rho_b^+)/\rho_b^+\\ &+\sum_{b=2}^{N}\Theta_{ab}(\theta-\theta^+)(\varphi_b-\varphi_b^+) + \frac{1}{2}\lambda_{ass}(\operatorname{tr}\mathbf{E}_s)^2 - \sum_{b=2}^{N}\lambda_{asb}(\operatorname{tr}\mathbf{E}_s)(\rho_b-\rho_b^+)/\rho_b^+\\ &+\frac{1}{2}\sum_{b=2}^{N}\sum_{c=2}^{N}\lambda_{abc}(\rho_b-\rho_b^+)(\rho_c-\rho_c^+)/\rho_b^+\rho_c^+ + \mu_{ass}\operatorname{tr}(\mathbf{E}_s\mathbf{E}_s) + \sum_{b=2}^{N}\Gamma_{asb}(\operatorname{tr}\mathbf{E}_s)(\varphi_b-\varphi_b^+)\\ &-\sum_{b=2}^{N}\sum_{c=2}^{N}\Gamma_{abc}((\rho_b-\rho_b^+)/\rho_b^+)(\varphi_c-\varphi_c^+) + \frac{1}{2}\sum_{b=2}^{N}\sum_{c=2}^{N}\Phi_{abc}(\varphi_b-\varphi_b^+)(\varphi_c-\varphi_c^+) + O(\varepsilon^3),\end{aligned} \tag{3.2.7}$$

where the coefficients in (3.2.7) are material constants. These constants can be identified as partial derivatives of Ψ_a evaluated at the static solution. The complicated result (3.2.7) reflects our assumption that the solid is isotropic. It also reflects the various restrictions one would have developed from the concept of material frame indifference. This concept is explained in textbooks on Continuum Mechanics [Ref. 4]. In (3.2.7), the quantity $\mathbf{E}_s$ is the *infinitesimal strain tensor* for the solid defined by

$$2\mathbf{E}_s = \mathbf{H}_s + \mathbf{H}_s^T \tag{3.2.8}$$

By use of (2.7.8), it follows from (3.2.7) that Ψ_{I} is given by

$$\begin{aligned}\Psi_{\mathrm{I}} = {} & \Psi_{\mathrm{I}}^{+} + \alpha(\theta-\theta^{+}) + \iota_s(\operatorname{tr}\mathbf{E}_s) - \sum_{b=2}^{N}\iota_b(\rho_b-\rho_b^{+})/\rho_b^{+} - \frac{1}{2}\nu(\theta-\theta^{+})^2 - \tau_s(\theta-\theta^{+})(\operatorname{tr}\mathbf{E}_s) \\ & + \sum_{b=2}^{N}\tau_b(\theta-\theta^{+})(\rho_b-\rho_b^{+})/\rho_b^{+} + \sum_{b=2}^{N}\Theta_b(\theta-\theta^{+})(\varphi_b-\varphi_b^{+}) + \frac{1}{2}\lambda_{ss}(\operatorname{tr}\mathbf{E}_s)^2 \\ & - \sum_{b=2}^{N}\lambda_{sb}(\operatorname{tr}\mathbf{E}_s)(\rho_b-\rho_b^{+})/\rho_b^{+} + \frac{1}{2}\sum_{b=2}^{N}\sum_{c=2}^{N}\lambda_{bc}(\rho_b-\rho_b^{+})(\rho_c-\rho_c^{+})/\rho_b^{+}\rho_c^{+} \\ & + \mu_{ss}\operatorname{tr}(\mathbf{E}_s\mathbf{E}_s) + \sum_{b=2}^{N}\Gamma_{sb}(\operatorname{tr}\mathbf{E}_s)(\varphi_b-\varphi_b^{+}) - \sum_{b=2}^{N}\sum_{c=2}^{N}\Gamma_{bc}((\rho_b-\rho_b^{+})/\rho_b^{+})(\varphi_c-\varphi_c^{+}) \\ & + \frac{1}{2}\sum_{b=2}^{N}\sum_{c=2}^{N}\Phi_{bc}(\varphi_b-\varphi_b^{+})(\varphi_c-\varphi_c^{+}) + O(\varepsilon^3).\end{aligned} \tag{3.2.9}$$

In writing (3.2.9), the result (3.1.8) and the fact that the σ_a vanish at the static solution has been used. Also, the various material coefficients in (3.2.9) are defined in terms of those in (3.2.7) by a set of formulas like

$$\Phi_{bc} = \sum_{a=2}^{N}\Phi_{abc} \tag{3.2.10}$$

It follows from (3.2.9) and the thermodynamic results (3.1.3), (3.1.4), (3.1.7) and (3.1.8) that

$$\rho\eta = -\alpha + \nu(\theta-\theta^{+}) + \tau_s(\operatorname{tr}\mathbf{E}_s) - \sum_{b=2}^{N}\tau_b(\rho_b-\rho_b^{+})/\rho_b^{+} - \sum_{b=2}^{N}\Theta_b(\varphi_b-\varphi_b^{+}) + O(\varepsilon^2), \tag{3.2.11}$$

$$\begin{aligned}\rho_{s_R}\mathbf{K}_s = {} & -\iota_s\mathbf{I} - (\lambda_{ss}+\iota_s)(\operatorname{tr}\mathbf{E}_s)\mathbf{I} - 2(\mu_{ss}+\iota_s)\mathbf{E}_s + \sum_{b=2}^{N}\lambda_{sb}((\rho_b-\rho_b^{+})/\rho_b^{+})\mathbf{I} \\ & + \tau_s(\theta-\theta^{+})\mathbf{I} - \sum_{b=2}^{N}\Gamma_{sb}(\varphi_b-\varphi_b^{+})\mathbf{I} + O(\varepsilon^2),\end{aligned} \tag{3.2.12}$$

$$\rho_c^{+}\mu_c = -\iota_c - \lambda_{sc}(\operatorname{tr}\mathbf{E}_s) + \sum_{b=2}^{N}\lambda_{cb}(\rho_b-\rho_b^{+})/\rho_b^{+} + \tau_c(\theta-\theta^{+}) - \sum_{b=2}^{N}\Gamma_{cb}(\varphi_b-\varphi_b^{+}) + O(\varepsilon^2) \tag{3.2.13}$$

and

$$\sigma_c = -\Theta_c(\theta-\theta^{+}) - \Gamma_{sc}(\operatorname{tr}\mathbf{E}_s) + \sum_{b=2}^{N}\Gamma_{bc}(\rho_b-\rho_b^{+})/\rho_b^{+} - \sum_{b=2}^{N}\Phi_{cb}(\phi_b-\phi_b^{+}) + O(\varepsilon^2). \tag{3.2.14}$$

In writing (3.2.12), equation (2.4.9) has been used to introduce the reference density for the solid ρ_{s_R}. In addition, the quantity $\det\mathbf{F}_s$ has been replaced by the approximate expression

$$\det \mathbf{F}_s = 1 + \operatorname{tr} \mathbf{E}_s + O(\varepsilon^2) \tag{3.2.15}$$

It follows from (2.3.13), (2.6.8), (2.7.8) that

$$\mathbf{T}_{\mathrm{I}} = -\sum_{a=1}^{N} \rho_a \mathbf{K}_a + \Psi_{\mathrm{I}} \mathbf{I} \tag{3.2.16}$$

Given (3.1.5), this result becomes

$$\mathbf{T}_{\mathrm{I}} = -\rho_s \mathbf{K}_s - \sum_{b=2}^{N} \rho_b \mu_b \mathbf{I} + \Psi_{\mathrm{I}} \mathbf{I} \tag{3.2.17}$$

By use of the same approximations that yielded (3.2.9), (3.2.11), (3.2.12) and (3.2.13), it follows from (3.2.17) that

$$\begin{aligned}\mathbf{T}_{\mathrm{I}} = {} & (\Psi_{\mathrm{I}}^{+} + \iota_s + \sum_{c=2}^{N} \iota_c)\mathbf{I} + (\lambda_{ss} + \sum_{c=2}^{N} \lambda_{sc} + \iota_s)(\operatorname{tr} \mathbf{E}_s)\mathbf{I} + 2(\mu_{ss} + \iota_s)\mathbf{E}_s \\ & - \sum_{b=2}^{N} (\lambda_{sb} + \sum_{c=2}^{N} \lambda_{cb})((\rho_b - \rho_b^{+})/\rho_b^{+})\mathbf{I} - (\tau_s + \sum_{c=2}^{N} \tau_c - \alpha)(\theta - \theta^{+})\mathbf{I} \\ & + \sum_{b=2}^{N} (\Gamma_{sb} + \sum_{c=2}^{N} \Gamma_{cb})(\varphi_b - \varphi_b^{+}))\mathbf{I} + O(\varepsilon^2)\end{aligned} \tag{3.2.18}$$

Because we are pursuing a linearization about the static solution described at the beginning of this section, in this uniform state everything is a constant. It is convenient, at this point, to simplify matters by making the formal assumption that the constant coefficients Ψ_a^{+}, ι_s and ι_b, for $b = 2,...,N$, are zero. The result is that the static solution has zero partial free energy Ψ_a, zero inner part of the free energy for the mixture Ψ_{I}, zero chemical potentials $\mathbf{K}_s$ and μ_c. It follow from (3.2.18) that $\mathbf{T}_{\mathrm{I}}$ vanishes at the static solution. In other words, there is no residual stress for the static solution. As a result of this simplification, equations (3.2.9), (3.2.12), (3.2.13) and (3.2.18) reduce to

$$
\begin{aligned}
\Psi_{\mathrm{I}} &= \alpha(\theta-\theta^{+}) - \frac{1}{2}\nu(\theta-\theta^{+})^{2} - \tau_{s}(\theta-\theta^{+})(\operatorname{tr}\mathbf{E}_{s}) \\
&\quad + \sum_{b=2}^{N}\tau_{b}(\theta-\theta^{+})(\rho_{b}-\rho_{b}^{+})\big/\rho_{b}^{+} \quad + \sum_{b=2}^{N}\Theta_{b}(\theta-\theta^{+})(\varphi_{b}-\varphi_{b}^{+}) \\
&\quad + \frac{1}{2}\lambda_{ss}(\operatorname{tr}\mathbf{E}_{s})^{2} - \sum_{b=2}^{N}\lambda_{sb}(\operatorname{tr}\mathbf{E}_{s})(\rho_{b}-\rho_{b}^{+})\big/\rho_{b}^{+} \\
&\quad + \frac{1}{2}\sum_{b=2}^{N}\sum_{c=2}^{N}\lambda_{bc}(\rho_{b}-\rho_{b}^{+})(\rho_{c}-\rho_{c}^{+})\big/\rho_{b}^{+}\rho_{c}^{+} + \mu_{ss}\operatorname{tr}(\mathbf{E}_{s}\mathbf{E}_{s}) \\
&\quad + \sum_{b=2}^{N}\Gamma_{sb}(\operatorname{tr}\mathbf{E}_{s})(\varphi_{b}-\varphi_{b}^{+}) - \sum_{b=2}^{N}\sum_{c=2}^{N}\Gamma_{bc}((\rho_{b}-\rho_{b}^{+})\big/\rho_{b}^{+})(\varphi_{c}-\varphi_{c}^{+}) \\
&\quad + \frac{1}{2}\sum_{b=2}^{N}\sum_{c=2}^{N}\Phi_{bc}(\varphi_{b}-\varphi_{b}^{+})(\varphi_{c}-\varphi_{c}^{+}) + O(\varepsilon^{3})
\end{aligned}
\tag{3.2.19}
$$

$$
\begin{aligned}
\rho_{s_R}\mathbf{K}_{s} = -\frac{\partial\Psi_{\mathrm{I}}}{\partial\mathbf{E}_{s}} &= -\lambda_{ss}(\operatorname{tr}\mathbf{E}_{s})\mathbf{I} - 2\mu_{ss}\mathbf{E}_{s} + \sum_{b=2}^{N}\lambda_{sb}((\rho_{b}-\rho_{b}^{+})\big/\rho_{b}^{+})\mathbf{I} \\
&\quad + \tau_{s}(\theta-\theta^{+})\mathbf{I} - \sum_{b=2}^{N}\Gamma_{sb}(\varphi_{b}-\varphi_{b}^{+})\mathbf{I} + O(\varepsilon^{2}),
\end{aligned}
\tag{3.2.20}
$$

$$
\begin{aligned}
\rho_{c}^{+}\mu_{c} = \rho_{c}^{+}\frac{\partial\Psi_{\mathrm{I}}}{\partial\rho_{c}} &= -\lambda_{sc}(\operatorname{tr}\mathbf{E}_{s}) + \sum_{b=2}^{N}\lambda_{cb}(\rho_{b}-\rho_{b}^{+})\big/\rho_{b}^{+} \\
&\quad + \tau_{c}(\theta-\theta^{+}) - \sum_{b=2}^{N}\Gamma_{cb}(\varphi_{b}-\varphi_{b}^{+}) + O(\varepsilon^{2}).
\end{aligned}
\tag{3.2.21}
$$

$$
\begin{aligned}
\mathbf{T}_{\mathrm{I}} &= (\lambda_{ss} + \sum_{c=2}^{N}\lambda_{sc})(\operatorname{tr}\mathbf{E}_{s})\mathbf{I} + 2\mu_{ss}\mathbf{E}_{s} - \sum_{b=2}^{N}(\lambda_{sb} + \sum_{c=2}^{N}\lambda_{cb})((\rho_{b}-\rho_{b}^{+})\big/\rho_{b}^{+})\mathbf{I} \\
&\quad -(\tau_{s} + \sum_{c=2}^{N}\tau_{c} - \alpha)(\theta-\theta^{+})\mathbf{I} + \sum_{b=2}^{N}(\Gamma_{sb} + \sum_{c=2}^{N}\Gamma_{cb})(\varphi_{b}-\varphi_{b}^{+}))\mathbf{I} + O(\varepsilon^{2})
\end{aligned}
\tag{3.2.22}
$$

The corresponding linearized expressions for $\mathbf{m}$, $\hat{\mathbf{f}}_c$ and ω_c turn out to be

$$
\mathbf{m} = -\kappa\,\mathrm{GRAD}\,\theta - \sum_{b=2}^{N}\varsigma_{b}\left(\frac{\partial\mathbf{w}_{b}}{\partial t} - \frac{\partial\mathbf{w}_{s}}{\partial t}\right) + O(\varepsilon^{2}), \tag{3.2.23}
$$

$$
\hat{\mathbf{f}}_{c} = -\gamma_{c}\,\mathrm{GRAD}\,\theta - \sum_{b=2}^{N}\xi_{cb}\left(\frac{\partial\mathbf{w}_{b}}{\partial t} - \frac{\partial\mathbf{w}_{s}}{\partial t}\right) + O(\varepsilon^{2}) \tag{3.2.24}
$$

and

$$\omega_c = \sum_{b=2}^{N}\sum_{d=2}^{N}\Lambda_{cb}\Phi_{bd}^{-1}\sigma_d + O(\varepsilon^2) \tag{3.2.25}$$

where the Φ_{bd}^{-1} are the inverse elements of the coefficients Φ_{bd} introduced in (3.2.10). It follows from (3.2.9) that the Φ_{bd} are given by

$$\Phi_{bd} = -\frac{\partial \sigma_b}{\partial \varphi_d} = \frac{\partial^2 \Psi_{\mathrm{I}}}{\partial \varphi_b \partial \varphi_d}. \tag{3.2.26}$$

The material constant κ in (3.2.23) is the *thermal conductivity* of the mixture, and the coefficients ξ_{cb} in (3.2.24) are the *drag coefficients*. As indicated in Section 3.1, the assumed inversion of (3.1.8) insures that the inverse elements Φ_{bd}^{-1} exist. Given (3.2.23), (3.2.24) and (3.2.25), the residual entropy inequality (3.1.6) yields

$$\frac{1}{\theta^+}\kappa\mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\sum_{b=2}^{N}(\varsigma_b/\theta^+ + \gamma_b)\mathbf{d}_b + \sum_{b=2}^{N}\sum_{c=2}^{N}\xi_{bc}\mathbf{d}_b\cdot\mathbf{d}_c \geq 0 \tag{3.2.27}$$

and

$$\sum_{c=2}^{N}\sum_{b=2}^{N}\sum_{d=2}^{N}\sigma_c\Lambda_{cb}\Phi_{bd}^{-1}\sigma_d \geq 0 \tag{3.2.28}$$

Equation (3.2.27) must hold for all vectors $\mathbf{a}$ and $\mathbf{d}_2,\ldots,\mathbf{d}_N$. Equation (3.2.28) must hold for all scalars $\sigma_2,\ldots,\sigma_N$.

3.3 Immiscible Mixtures-Field Equations for the Linearized Model

When one simply drops the terms $O(\varepsilon^3)$ in (3.2.19) and the terms $O(\varepsilon^2)$ in (3.2.11), (3.2.14), (3.2.23), (3.2.24), (3.2.25), (3.2.20) and (3.2.21), one has the constitutive equations appropriate for the linearized model. The field equations which govern motions near the static solution turn out to be

$$\begin{aligned}\rho_{s_R}\frac{\partial^2\mathbf{w}_s}{\partial t^2} &= (\lambda_{ss}+\mu_{ss})\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_s) + \mu_{ss}\,\mathrm{Div}(\mathrm{GRAD}\,\mathbf{w}_s)\\ &+\sum_{b=2}^{N}\lambda_{sb}\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_b) + (\alpha_s-\tau_s-\gamma_s)\,\mathrm{GRAD}\,\theta\\ &+\sum_{b=2}^{N}\Gamma_{sb}\,\mathrm{GRAD}\,\varphi_b - \sum_{b=2}^{N}\xi_{sb}\left(\frac{\partial\mathbf{w}_b}{\partial t}-\frac{\partial\mathbf{w}_s}{\partial t}\right)\end{aligned} \tag{3.3.1}$$

$$\rho_c^+ \frac{\partial^2 \mathbf{w}_c}{\partial t^2} = \sum_{b=2}^{N} \lambda_{cb} \operatorname{GRAD}(\operatorname{Div} \mathbf{w}_b) + \lambda_{sc} \operatorname{GRAD}(\operatorname{Div} \mathbf{w}_s) + (\alpha_c - \tau_c - \gamma_c) \operatorname{GRAD} \theta$$
$$+ \sum_{b=2}^{N} \Gamma_{cb} \operatorname{GRAD} \varphi_b - \sum_{b=2}^{N} \xi_{cb} \left(\frac{\partial \mathbf{w}_b}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t} \right) \tag{3.3.2}$$

$$c_{\mathrm{v}} \frac{\partial \theta}{\partial t} + \theta^+ \left((\tau_s - \alpha_s - \varsigma_s / \theta^+) \operatorname{Div} \frac{\partial \mathbf{w}_s}{\partial t} + \sum_{b=2}^{N} (\tau_b - \alpha_b - \varsigma_b / \theta^+) \operatorname{Div} \frac{\partial \mathbf{w}_b}{\partial t} - \sum_{b=2}^{N} \Theta_b \frac{\partial \varphi_b}{\partial t} \right)$$
$$= \kappa \operatorname{Div}(\operatorname{GRAD} \theta) \tag{3.3.3}$$

and

$$\frac{\partial \phi_c}{\partial t} = -\sum_{b=2}^{N} \Lambda_{cb} \left(\phi_b - \phi_b^+ + \sum_{d=2}^{N} \Phi_{bd}^{-1} \Gamma_{sd} \operatorname{Div} \mathbf{w}_s + \sum_{d=2}^{N} \sum_{e=2}^{N} \Phi_{bd}^{-1} \Gamma_{ed} \operatorname{Div} \mathbf{w}_e + \sum_{d=2}^{N} \Phi_{bd}^{-1} \Theta_d (\theta - \theta^+) \right) \tag{3.3.4}$$

where

$$\xi_{sb} = -\sum_{c=2}^{N} \xi_{cb} \tag{3.3.5}$$

$$\gamma_s = -\sum_{c=2}^{N} \gamma_c \tag{3.3.6}$$

and

$$\varsigma_s = -\sum_{c=2}^{N} \varsigma_c . \tag{3.3.7}$$

In addition, the coefficient c_{v} in (3.3.3) is the *specific heat at constant volume* for the mixture. This quantity is related to the coefficient ν in (3.2.19) by the formula

$$c_{\mathrm{v}} = \theta^+ \nu \tag{3.3.8}$$

Among the many formulas which were used to deduce (3.3.1) through (3.3.4) are the expressions

$$\left(\rho_b - \rho_b^+ \right) \big/ \rho_b^+ = -\operatorname{Div} \mathbf{w}_b + O(\varepsilon^2), \tag{3.3.9}$$

for $b = 2, \ldots, N$. These results follow by integration of (2.4.1) to obtain

$$\rho_b \left| \det \mathbf{F}_b \right| = \rho_b^+ \tag{3.3.10}$$

and the utilization of the following approximation for $\left|\det \mathbf{F}_b\right|$

$$\left|\det \mathbf{F}_b\right| = 1 + \operatorname{Div} \mathbf{w}_b + O(\varepsilon^2) \tag{3.3.11}$$

Equations (3.3.1), (3.3.2), (3.3.3) and (3.3.4) can be shown to have unique solutions for various initial-boundary value problems if the quadratic form Σ, defined by

$$\begin{aligned}\Sigma = \Psi_{\mathrm{I}} + \rho\eta(\theta-\theta^+) &= \frac{1}{2}\frac{c_{\mathrm{v}}}{\theta^+}(\theta-\theta^+)^2 + \frac{1}{2}\lambda_{ss}(\operatorname{tr}\mathbf{E}_s)^2 - \sum_{b=2}^{N}\lambda_{sb}(\operatorname{tr}\mathbf{E}_s)(\rho_b-\rho_b^+)/\rho_b^+ \\ &+ \frac{1}{2}\sum_{b=2}^{N}\sum_{c=2}^{N}\lambda_{bc}(\rho_b-\rho_b^+)(\rho_c-\rho_c^+)/\rho_b^+\rho_c^+ + \mu_{ss}\operatorname{tr}(\mathbf{E}_s\mathbf{E}_s) \\ &+ \sum_{b=2}^{N}\Gamma_{sb}(\operatorname{tr}\mathbf{E}_s)(\varphi_b-\varphi_b^+) - \sum_{b=2}^{N}\sum_{c=2}^{N}\Gamma_{bc}((\rho_b-\rho_b^+)/\rho_b^+)(\varphi_c-\varphi_c^+) \\ &+ \frac{1}{2}\sum_{b=2}^{N}\sum_{c=2}^{N}\Phi_{bc}(\varphi_b-\varphi_b^+)(\varphi_c-\varphi_c^+)\end{aligned} \tag{3.3.12}$$

is positive definite. Given (3.3.12), one can establish that

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{\mathscr{V}}\left(\Sigma + \frac{1}{2}\rho_{s_R}\frac{\partial \mathbf{w}_s}{\partial t}\cdot\frac{\partial \mathbf{w}_s}{\partial t} + \frac{1}{2}\sum_{b=2}^{N}\rho_b^+\frac{\partial \mathbf{w}_b}{\partial t}\cdot\frac{\partial \mathbf{w}_b}{\partial t}\right)dv = \oint_{\partial\mathscr{V}}(\mathbf{t} - \frac{\theta-\theta^+}{\theta^+}\mathbf{m})\cdot d\mathbf{s} \\ &-\int_{\mathscr{V}}\left(\sum_{c=2}^{N}\sum_{b=2}^{N}\sum_{d=2}^{N}\sigma_c\Lambda_{cb}\Phi_{bd}^{-1}\sigma_c\right)dv \\ &-\int_{\mathscr{V}}\left(\frac{1}{\theta^+}\kappa\,\mathrm{GRAD}\,\theta\cdot\mathrm{GRAD}\,\theta + \mathrm{GRAD}\,\theta\cdot\sum_{b=2}^{N}(\varsigma_b/\theta^+ + \gamma_b)\frac{\partial \mathbf{w}_b}{\partial t} + \sum_{b=2}^{N}\sum_{c=2}^{N}\xi_{cb}\frac{\partial \mathbf{w}_c}{\partial t}\cdot\frac{\partial \mathbf{w}_b}{\partial t}\right)dv\end{aligned} \tag{3.3.13}$$

where $\mathbf{t}$ is a vector defined by

$$\mathbf{t} = -(\rho_{s_R}\mathbf{K}_s - \alpha_s(\theta-\theta^+)\mathbf{I})\frac{\partial \mathbf{w}_s}{\partial t} - \sum_{b=2}^{N}(\rho_b^+\mu_b - \alpha_b(\theta-\theta^+))\frac{\partial \mathbf{w}_b}{\partial t} \tag{3.3.14}$$

It is convenient for use later on to rearrange (3.3.14) and write the result as

$$\begin{aligned}\mathbf{t} = &-\left(\rho_{s_R}\mathbf{K}_s - \alpha_s(\theta-\theta^+)\mathbf{I} + \sum_{b=2}^{N}(\rho_b^+\mu_b - \alpha_b(\theta-\theta^+))\mathbf{I}\right)\frac{\partial \mathbf{w}_s}{\partial t} \\ &-\sum_{b=2}^{N}(\rho_b^+\mu_b - \alpha_b(\theta-\theta^+))\left(\frac{\partial \mathbf{w}_b}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t}\right)\end{aligned} \tag{3.3.15}$$

where the symmetry of $\mathbf{T}_{\mathrm{I}}$ has been used. It is easily seen from (3.2.19), (3.2.20) and (3.2.21) that (3.2.17) can be written

$$\mathbf{T}_{\mathrm{I}} = -\left(\rho_{s_R} \mathbf{K}_s - \alpha_s (\theta - \theta^+) \mathbf{I} + \sum_{b=2}^{N} (\rho_b^+ \mu_b - \alpha_b (\theta - \theta^+)) \mathbf{I} \right) + O(\varepsilon^2) \tag{3.3.16}$$

Therefore, consistent with the linearization adopted in this section, (3.3.15) can be replaced by

$$\mathbf{t} = \mathbf{T}_{\mathrm{I}} \frac{\partial \mathbf{w}_s}{\partial t} - \sum_{b=2}^{N} (\rho_b^+ \mu_b - \alpha_b (\theta - \theta^+)) \left(\frac{\partial \mathbf{w}_b}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t} \right) \tag{3.3.17}$$

Given (3.2.27) and (3.2.28), it follows from (3.3.13) that the following inequality must hold

$$\frac{\partial}{\partial t} \int_{\mathscr{V}} \left(\Sigma + \frac{1}{2} \rho_{s_R} \frac{\partial \mathbf{w}_s}{\partial t} \cdot \frac{\partial \mathbf{w}_s}{\partial t} + \frac{1}{2} \sum_{b=2}^{N} \rho_b^+ \frac{\partial \mathbf{w}_b}{\partial t} \cdot \frac{\partial \mathbf{w}_b}{\partial t} \right) dv \leq \oint_{\partial \mathscr{V}} (\mathbf{t} - \frac{\theta - \theta^+}{\theta^+} \mathbf{m}) \cdot d\mathbf{s} \tag{3.3.18}$$

The reader should note that (3.3.18) could have been derived by specializing (2.6.16) to the linearized case under discussion here.

The reader with experience in the classical theory of elasticity will recognize how (3.3.18) can be used to construct a uniqueness theorem if Σ is positive definite [Ref. 5,6]. If Σ is assumed to be positive definite, it is necessary and sufficient that the following restrictions hold for the material constants:

$$c_{\mathrm{v}} > 0, \tag{3.3.19}$$

$$\mu_s > 0 \tag{3.3.20}$$

and the $(2N-1) \times (2N-1)$ symmetric matrix

$$\begin{bmatrix}
\lambda_{ss} + \frac{2}{3}\mu_{ss} & \lambda_{s2} & \cdot & \cdot & \cdot & \cdot & \lambda_{sN} & \Gamma_{s2} & \cdot & \cdot & \cdot & \Gamma_{sN} \\
\lambda_{s2} & \lambda_{22} & \cdot & \cdot & \cdot & \cdot & \lambda_{2N} & \Gamma_{22} & \cdot & \cdot & \cdot & \Gamma_{2N} \\
\cdot & \cdot & \cdot & & & & & \cdot & \cdot & & & \cdot \\
\cdot & \cdot & & & \cdot & & & \cdot & & \cdot & & \cdot \\
\cdot & \cdot & & & & \cdot & & \cdot & & & \cdot & \cdot \\
\lambda_{sN} & \lambda_{N2} & \cdot & \cdot & \cdot & \cdot & \lambda_{NN} & \Gamma_{N2} & \cdot & \cdot & \cdot & \Gamma_{NN} \\
\Gamma_{s2} & \Gamma_{22} & \cdot & \cdot & \cdot & \cdot & \Gamma_{N2} & \Phi_{22} & & & & \Phi_{2N} \\
\cdot & \cdot & \cdot & & & & \cdot & \cdot & & & & \cdot \\
\cdot & \cdot & & \cdot & & & \cdot & \cdot & \cdot & & & \cdot \\
\cdot & \cdot & & & \cdot & & \cdot & \cdot & & \cdot & & \cdot \\
\cdot & \cdot & & & & \cdot & \cdot & \cdot & & & \cdot & \cdot \\
\Gamma_{sN} & \Gamma_{2N} & \cdot & \cdot & \cdot & \cdot & \Gamma_{sN} & \Phi_{N2} & \cdot & \cdot & \cdot & \Phi_{NN}
\end{bmatrix}$$

is positive definite. Among the implications of the positive definite nature of this matrix is that the $(N-1)\times(N-1)$ matrix $\left[\Phi_{bc}\right]$ is positive definite. In the following, we shall assume that Σ is positive definite and utilize the necessary restrictions on the properties of the mixture which appear in (3.3.12)

Equation (3.3.4) contains two special cases which need to be mentioned here. If the volume fractions $\varphi_2,\varphi_3,...,\varphi_N$ are taken to be constants, then (3.3.4) yields this special case by taking Λ_{cb} to be zero for $c,b=2,...,N$. It is convenient to refer to this case as the *frozen* one. The most interesting case arises when the volume fractions adjust instantly to their equilibrium values defined by the vanishing of the affinities $\sigma_2,\sigma_3,...,\sigma_N$. It is useful to view this case as one which also follows from (3.3.4). The coefficients Λ_{cb} have the physical dimension of inverse time. As a result, they can be thought of as defining characteristic times of relaxation for the volume fractions to their equilibrium values. If we formally allow these characteristic times to approach zero as a limit, (3.3.4) yields

$$\begin{aligned}\varphi_b-\varphi_b^+ &= -\sum_{d=2}^{N}\Phi_{bd}^{-1}\Gamma_{sd}\,\operatorname{tr}\mathbf{E}_s+\sum_{d=2}^{N}\sum_{e=2}^{N}\Phi_{bd}^{-1}\Gamma_{ed}\left((\rho_e-\rho_e^+)\big/\rho_e^+\right)\\ &\quad -\sum_{d=2}^{N}\Phi_{bd}^{-1}\Theta_d(\theta-\theta^+)\end{aligned} \tag{3.3.21}$$

where (3.3.9) and (3.2.8) have been used. Equation (3.3.21) gives the volume fractions of the fluid constituents in terms of the stress on the solid, the densities of the fluids and the temperature. When one assumes the validity of (3.3.21), the mixture is said to be in *shifting equilibrium.* Equation (3.3.21) can be used to eliminate the volume fractions from the other constitutive equations for this model. The results of this elimination are as follows:

$$\begin{aligned}\Psi_{\mathrm{I}} &= \alpha(\theta-\theta^+)-\frac{1}{2}\frac{\overline{c}_{\mathrm{v}}}{\theta^+}(\theta-\theta^+)^2-\overline{\tau}_s(\theta-\theta^+)(\operatorname{tr}\mathbf{E}_s)+\sum_{b=2}^{N}\overline{\tau}_b(\theta-\theta^+)(\rho_b-\rho_b^+)\big/\rho_b^+\\ &\quad+\frac{1}{2}\overline{\lambda}_{ss}(\operatorname{tr}\mathbf{E}_s)^2-\sum_{b=2}^{N}\overline{\lambda}_{sb}(\operatorname{tr}\mathbf{E}_s)(\rho_b-\rho_b^+)\big/\rho_b^+\\ &\quad+\frac{1}{2}\sum_{b=2}^{N}\sum_{c=2}^{N}\overline{\lambda}_{bc}(\rho_b-\rho_b^+)(\rho_c-\rho_c^+)\big/\rho_b^+\rho_c^+ +\mu_{ss}\operatorname{tr}(\mathbf{E}_s\mathbf{E}_s)\end{aligned} \tag{3.3.22}$$

$$\rho\eta=-\frac{\partial\Psi_{\mathrm{I}}(\theta,\mathbf{E}_s,\rho_b)}{\partial\theta}=-\alpha+\frac{\overline{c}_{\mathrm{v}}}{\theta^+}(\theta-\theta^+)+\overline{\tau}_s(\operatorname{tr}\mathbf{E}_s)-\sum_{b=2}^{N}\overline{\tau}_b(\rho_b-\rho_b^+)\big/\rho_b^+ \tag{3.3.23}$$

$$\rho_{s_R}\mathbf{K}_s=-\frac{\partial\Psi_{\mathrm{I}}(\theta,\mathbf{E}_s,\rho_b)}{\partial\mathbf{E}_s}=-\overline{\lambda}_{ss}(\operatorname{tr}\mathbf{E}_s)\mathbf{I}-2\mu_{ss}\mathbf{E}_s+\sum_{b=2}^{N}\overline{\lambda}_{sb}\left((\rho_b-\rho_b^+)\big/\rho_b^+\right)\mathbf{I}+\overline{\tau}_s(\theta-\theta^+)\mathbf{I} \tag{3.3.24}$$

$$\rho_c^+ \mu_c = \rho_c^+ \frac{\partial \Psi_{\mathrm{I}}(\theta, \mathbf{E}_s, \rho_b)}{\partial \rho_c} = \overline{\tau}_c(\theta - \theta^+) - \overline{\lambda}_{sc}(\operatorname{tr}\mathbf{E}_s) + \sum_{b=2}^{N} \overline{\lambda}_{cb}(\rho_b - \rho_b^+)\big/\rho_b^+ \tag{3.3.25}$$

and

$$\begin{aligned}\mathbf{T}_{\mathrm{I}} = {} & (\overline{\lambda}_{ss} + \sum_{c=2}^{N} \overline{\lambda}_{sc})(\operatorname{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s - \sum_{b=2}^{N}(\overline{\lambda}_{sb} + \sum_{c=2}^{N}\overline{\lambda}_{cb})((\rho_b - \rho_b^+)\big/\rho_b^+)\mathbf{I} \\ & -(\overline{\tau}_s + \sum_{c=2}^{N}\overline{\tau}_c - \alpha)(\theta - \theta^+)\mathbf{I}\end{aligned} \tag{3.3.26}$$

where

$$\overline{c}_{\mathrm{v}} = c_{\mathrm{v}} + \theta^+ \sum_{b=2}^{N}\sum_{d=2}^{N} \Theta_b \Phi_{bd}^{-1} \Theta_d \tag{3.3.27}$$

$$\overline{\tau}_s = \tau_s + \sum_{b=2}^{N}\sum_{d=2}^{N} \Theta_b \Phi_{bd}^{-1} \Gamma_{sd} \tag{3.3.28}$$

$$\overline{\tau}_c = \tau_c + \sum_{b=2}^{N}\sum_{d=2}^{N} \Theta_b \Phi_{bd}^{-1} \Gamma_{cd} \tag{3.3.29}$$

$$\overline{\lambda}_{ss} = \lambda_{ss} - \sum_{b=2}^{N}\sum_{d=2}^{N} \Gamma_{sb} \Phi_{bd}^{-1} \Gamma_{sd} \tag{3.3.30}$$

$$\overline{\lambda}_{sc} = \lambda_{sc} - \sum_{b=2}^{N}\sum_{d=2}^{N} \Gamma_{sb} \Phi_{bd}^{-1} \Gamma_{cd}, \tag{3.3.31}$$

and

$$\overline{\lambda}_{cb} = \lambda_{cb} - \sum_{e=2}^{N}\sum_{d=2}^{N} \Gamma_{ce} \Phi_{ed}^{-1} \Gamma_{bd}. \tag{3.3.32}$$

The coefficients (3.3.27) through (3.3.32) are the *shifting equilibrium* properties of the mixture. It is sometimes convenient to refer to the material coefficients introduced in (3.2.9) as the *frozen equilibrium* properties of the mixture.

The field equations in the shifting equilibrium case which replace (3.3.1) through (3.3.3) are easily shown to be

$$\begin{aligned}\rho_{s_R} \frac{\partial^2 \mathbf{w}_s}{\partial t^2} = {} & (\overline{\lambda}_{ss} + \mu_{ss})\operatorname{GRAD}(\operatorname{Div}\mathbf{w}_s) + \mu_{ss}\operatorname{Div}(\operatorname{GRAD}\mathbf{w}_s) + \sum_{b=2}^{N}\overline{\lambda}_{sb}\operatorname{GRAD}(\operatorname{Div}\mathbf{w}_b) \\ & +(\alpha_s - \overline{\tau}_s - \gamma_s)\operatorname{GRAD}\theta - \sum_{b=2}^{N}\xi_{sb}\left(\frac{\partial \mathbf{w}_b}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t}\right)\end{aligned} \tag{3.3.33}$$

$$\rho_c^+ \frac{\partial^2 \mathbf{w}_c}{\partial t^2} = \sum_{b=2}^{N} \bar{\lambda}_{cb} \,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_b) + \bar{\lambda}_{sc} \,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_s) + (\alpha_c - \bar{\tau}_c - \gamma_c)\,\mathrm{GRAD}\,\theta \\ -\sum_{b=2}^{N} \xi_{cb} \left(\frac{\partial \mathbf{w}_b}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t} \right) \tag{3.3.34}$$

and

$$\bar{c}_{\mathrm{v}} \frac{\partial \theta}{\partial t} + \theta^+ \left((\bar{\tau}_s - \alpha_s - \varsigma_s / \theta^+)\,\mathrm{Div} \frac{\partial \mathbf{w}_s}{\partial t} + \sum_{b=2}^{N} (\bar{\tau}_b - \alpha_b - \varsigma_b / \theta^+)\,\mathrm{Div} \frac{\partial \mathbf{w}_b}{\partial t} \right) \\ = \kappa\,\mathrm{Div}(\mathrm{GRAD}\,\theta) \tag{3.3.35}$$

Solutions of (3.3.33) through (3.3.35) must also obey the inequality (3.3.18), where Σ is again defined by $(3.3.12)_1$. Given (3.3.22) and (3.3.23), the explicit for of the quadratic form Σ turns out to be

$$\Sigma = \frac{1}{2} \frac{\bar{c}_{\mathrm{v}}}{\theta^+} (\theta - \theta^+)^2 + \frac{1}{2} \bar{\lambda}_{ss} (\mathrm{tr}\,\mathbf{E}_s)^2 - \sum_{b=2}^{N} \bar{\lambda}_{sb} (\mathrm{tr}\,\mathbf{E}_s)(\rho_b - \rho_b^+) / \rho_b^+ \\ + \frac{1}{2} \sum_{b=2}^{N} \sum_{c=2}^{N} \bar{\lambda}_{bc} (\rho_b - \rho_b^+)(\rho_c - \rho_c^+) / \rho_b^+ \rho_c^+ + \mu_{ss}\,\mathrm{tr}(\mathbf{E}_s \mathbf{E}_s) \tag{3.3.36}$$

The fact that Σ is assumed to be positive definite yields

$$\bar{c}_{\mathrm{v}} > 0 \tag{3.3.37}$$

$$\mu_{ss} > 0 \tag{3.3.38}$$

and the $N \times N$ symmetric matrix

$$\begin{bmatrix} \bar{\lambda}_{ss} + \frac{2}{3} \bar{\mu}_{ss} & \bar{\lambda}_{s2} & . & . & . & . & \bar{\lambda}_{sN} \\ \bar{\lambda}_{s2} & \bar{\lambda}_{22} & . & . & . & . & \bar{\lambda}_{2N} \\ . & . & & & & & . \\ . & . & & & & & . \\ . & . & & & & & . \\ \bar{\lambda}_{sN} & \bar{\lambda}_{2N} & . & . & . & . & \bar{\lambda}_{NN} \end{bmatrix}$$

is positive definite.

Frequently, it is physically acceptable to assume that the mixture is *isothermal*. This special case can be justified in cases where the mixture is a perfect conductor. This case is approached when the thermal conductivity κ is large. As equation (3.3.3) illustrates, if κ is allowed to grow without bound, then the temperature must obey $\operatorname{Div}(\operatorname{GRAD}\theta) = 0$. With suitable boundary conditions, the only solution of this equation is $\theta = \theta^{+}$. A *nonconductor* is a mixture for which the constitutive equations are independent of the temperature gradient. In this case, it follows from (3.2.23) and (3.2.24) that $\kappa = \gamma_c = 0$ for $c = 2,\ldots, N$. For a nonconductor, (3.3.3) can be used to eliminate the temperature from the remaining field equations. As in the shifting equilibrium case, one can rewrite the governing constitutive and field equations in terms of a new set of material constants. These constants, which are defined by complicated formulas, are the mixture versions of the isotropic material constants one encounters in the classical theory of elasticity. It is also interesting to point out that our constitutive and field equations contain the degenerate case where there is no relative motion between the constituents. This case can be reached formally by allowing the drag coefficients, ξ_{cb}, to become large. It is a simple exercise to show that in the limit of infinite drag coefficients, equations (3.3.1) and (3.3.2) force the constituent velocities to all be equal.

In closing this section it is useful to note that (3.3.16) can be rearranged to yield

$$\rho_{s_R}\mathbf{K}_s = -(\mathbf{T}_{\mathrm{I}} + \sum_{b=2}^{N} \rho_b^{+}\mu_b \mathbf{I}) + \alpha(\theta - \theta^{+})\mathbf{I}, \tag{3.3.39}$$

where the order term has been dropped. As we shall see later, equation (3.3.39) is frequently used in the applications to eliminate the formal appearance of the solid's chemical potential in favor of the stress $\mathbf{T}_{\mathrm{I}}$.

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4

Porous Elasticity Models With Pore Pressure

In this chapter the model developed in Chapter 3 will be specialized so as to yield the concept of pore pressures. This special case is further specialized to the case of a binary mixture consisting of a fluid and a solid. As a final specialization, the case where each constituent is incompressible is discussed.

4.1 Immiscible Mixtures-Definition of Pore Pressures

In Chapters 1 and 2 examples were discussed which involved the pore pressures of the fluids in the porous solid. In Chapter 1, these quantities were introduced as fundamental physical variables. In Section 7 of Chapter 2 the pore pressures were defined in terms of other parameters associated with the model of a rigid isotropic solid containing incompressible fluids. In this section the pore pressures will be defined in a somewhat more general fashion. For the most part the discussion in this section represents a generalization of a similar discussion in reference 1.

For the b^{th} fluid in a mixture consisting of a solid and N - 1 fluids, the *pore pressure* of the fluid is defined by

$$P_b = \int \gamma_b d\mu_b + \text{constant.} \tag{4.1.1}$$

In order for (4.1.1) to have meaning, the true density $\gamma_b = \rho_b / \varphi_b$ must be a function of the chemical potential μ_b. A trivial functional relationship arises when the b^{th} fluid is incompressible. In this case, γ_b is a constant and (4.1.1) integrates directly. If the results (2.7.14) and (2.7.15) are combined with the definitions (2.7.18) and (2.7.19), one can see that (4.1.1) is implicit in the results of Section 2.7. When the b^{th} fluid is not incompressible, things get somewhat more complicated.

The relationship $\gamma_b = \gamma_b(\mu_b)$ required by (4.1.1) is an extra constitutive equation which must be consistent with our previous constitutive assumptions. Therefore, we must investigate the restrictions that are implied by the assumed existence of the function $\gamma_b = \gamma_b(\mu_b)$. We shall always assume that this function can be inverted and written

$$\mu_b = \tilde{\mu}_b(\gamma_b). \tag{4.1.2}$$

As indicated above, equation (4.1.2) must be consistent with the constitutive assumptions made in Section 3.1. As (3.1.1, (2.7.8) and (3.1.7) show, the chemical potential μ_b is given by

$$\mu_b = \mu_b(\theta, \mathbf{F}_s, \rho_d, \varphi_d). \tag{4.1.3}$$

One case where (4.1.3) and (4.1.2) are consistent is when (4.1.3) degenerates trivially into (4.1.2). If we exclude this possibility, then we must look for a more complicated reconciliation of (4.1.2) and (4.1.3). If (4.1.2) and (4.1.3) are consistent, then it must be true that

$$\gamma_b = \tilde{\mu}_b^{-1} \circ \mu_b(\theta, \mathbf{F}_s, \rho_d, \varphi_d) \tag{4.1.4}$$

or, with the definition (2.1.16),

$$\rho_b / \varphi_b = \tilde{\mu}_b^{-1} \circ \mu_b(\theta, \mathbf{F}_s, \rho_d, \varphi_d). \tag{4.1.5}$$

Equation (4.1.5) can be viewed as a set of N - 1 equations for the determination of ρ_b, $b = 2,...,N$, in terms of $\theta, \mathbf{F}_s, \varphi_2, ..., \varphi_N$. Relationships such as (4.1.5) are not generally consistent with our constitutive assumptions. However, if we assume that the relaxation process governed by (3.1.12) occurs instantaneously, then (3.1.12) is replaced by

$$\sigma_b(\theta, \mathbf{F}_s, \rho_d, \varphi_d) = 0 \tag{4.1.6}$$

Equation (3.4.6) does have the potential of being solved to obtain (4.1.5). If we now restrict the discussion to the linearized model discussed in Sections 3.2 and 3.3, it is possible to construct the restrictions on the constitutive functions which follow from equation (4.1.1).

In the linearized approximation, it is readily established that (4.1.1) and (4.1.2) become

$$P_b = \overset{+}{\gamma}_b \mu_b \tag{4.1.7}$$

and

$$\overset{+}{\rho}_b \mu_b = \Xi_b \left(\frac{\gamma_b - \overset{+}{\gamma}_b}{\overset{+}{\gamma}_b} \right) \tag{4.1.8}$$

respectively, where from (1.1.13),

$$\frac{\gamma_b - \overset{+}{\gamma}_b}{\overset{+}{\gamma}_b} = \frac{\rho_b - \overset{+}{\rho}_b}{\overset{+}{\rho}_b} - \frac{\varphi_b - \overset{+}{\varphi}_b}{\overset{+}{\varphi}_b} \tag{4.1.9}$$

In equation (4.1.7), the assumption has been made that the pore pressure vanishes in the state where the chemical potential vanishes. The quantity Ξ_b in (4.1.8) is a constant material property which is to be determined. The inversion which was assumed in order to reach (4.1.2) implies that $\Xi_b \neq 0$.

Equation (4.1.8) must be consistent with the fact that $\rho_b^+\mu_b$ is given by (3.3.25) and $\varphi_b-\varphi_b^+$ is given by the linearized version of (4.1.6), i.e. by (3.3.21). If (4.1.9) is substituted into (4.1.8) and if (3.3.25) and (3.3.21) are substituted into the result, the resulting formula must hold identically in the variables $tr\mathbf{E}_s, \theta$ and ρ_d, for d =2,...,N. As a result, the following results can easily be obtained

$$\varphi_b^+\bar{\lambda}_{sb}=-\Xi_b\sum_{d=2}^{N}\Phi_{bd}^{-1}\Gamma_{sd} \tag{4.1.10}$$

$$\varphi_b^+\bar{\tau}_b=\Xi_b\sum_{d=2}^{N}\Phi_{bd}^{-1}\Theta_d \tag{4.1.11}$$

and

$$\varphi_b^+\bar{\lambda}_{be}=\Xi_b\left[\varphi_b^+\delta_{be}-\sum_{d=2}^{N}\Phi_{bd}^{-1}\Gamma_{ed}\right] \tag{4.1.12}$$

for fixed b and for e = 2,...,N. In (4.1.12), the quantity δ_{be} denotes the usual Kronecker delta. If every fluid has a pore pressure defined by (4.1.1), then equations (4.1.10) through (4.1.12) represent a system of $2(N-1)+(N-1)^2$ equations which relate the various material constants. If the assumption is made that $\bar{\lambda}_{se}\neq 0$, for e = 2,...,N, it is a straight forward manipulation to derive from these equations the following results:

$$\Xi_b=-\varphi_b^+\bar{\lambda}_{sb}\Big/\sum_{d=2}^{N}\Phi_{bd}^{-1}\Gamma_{sd} \tag{4.1.13}$$

$$\Theta_b=-\sum_{e=2}^{N}\left(\frac{\bar{\tau}_e}{\bar{\lambda}_{se}}\Phi_{be}\left(\sum_{d=2}^{N}\Phi_{ed}^{-1}\Gamma_{sd}\right)\right) \tag{4.1.14}$$

and

$$\Gamma_{bc}=\varphi_b^+\Phi_{bc}+\sum_{e=2}^{N}\left(\frac{\bar{\lambda}_{be}}{\bar{\lambda}_{se}}\Phi_{ce}\left(\sum_{d=2}^{N}\Phi_{ed}^{-1}\Gamma_{sd}\right)\right) \tag{4.1.15}$$

for b,c = 2,..., N. Equations (4.1.14) and (4.1.15) show that the pore pressure definition (4.1.1), when applied to every fluid in the mixture, forces the material constants Θ_b and Γ_{bc}, which appear in (3.3.21) to be determined by the other material constants in the model. One would use equations (4.1.14) and (4.1.15) to eliminate Θ_b and Γ_{bc}, for b,c = 2,..., N, from equation (3.3.21). Equation (4.1.12) enables us to construct explicitly the relationship (4.1.2). In particular, if (4.1.13), is substituted into (4.1.9) and the result is substituted into (4.1.7), the result can be written

$$P_b = K_b \left[\frac{\gamma_b - \gamma_b^+}{\gamma_b^+} \right] \tag{4.1.16}$$

where K_b is the *bulk modulus* for the b^{th} fluid defined by

$$K_b = -\frac{\bar{\lambda}_{sb}}{\sum_{d=2}^{N} \Phi_{bd}^{-1} \Gamma_{sd}} \tag{4.1.17}$$

Because a properly defined bulk modulus is positive, it follows from (4.1.17) that we must require

$$-\frac{\bar{\lambda}_{sb}}{\sum_{d=2}^{N} \Phi_{bd}^{-1} \Gamma_{sd}} > 0 \tag{4.1.18}$$

for $b = 2,...,N$. It is possible to use (4.1.12) to establish that

$$-\frac{\bar{\lambda}_{sb}}{\sum_{d=2}^{N} \Phi_{bd}^{-1} \Gamma_{sd}} = \frac{\bar{\lambda}_{bb}}{\varphi_b^+} \frac{1}{\left(1 - \frac{1}{\varphi_b^+} \sum_{d=2}^{N} \Phi_{bd}^{-1} \Gamma_{bd} \right)} \tag{4.1.19}$$

Equation (4.1.19) provides an alternate formula for the bulk modulus for the b^{th} fluid. It also follows from (4.1.18) and (4.1.19) that

$$\frac{1}{\varphi_b^+} \sum_{d=2}^{N} \Phi_{bd}^{-1} \Gamma_{bd} < 1 \tag{4.1.20}$$

If we utilize the definition (4.1.17), equations (4.1.13), (4.1.14) and (4.1.15) can be written

$$\Xi_b = -\frac{\varphi_b^+ \bar{\lambda}_{sb}}{\sum_{d=2}^{N} \Phi_{bd}^{-1} \Gamma_{sd}} = \frac{\varphi_b^+}{K_b} \tag{4.1.21}$$

$$\Theta_b = -\sum_{e=2}^{N} \left(\frac{\bar{\tau}_e}{\bar{\lambda}_{se}} \Phi_{be} \left(\sum_{d=2}^{N} \Phi_{ed}^{-1} \Gamma_{sd} \right) \right) = \sum_{e=2}^{N} \frac{\bar{\tau}_e \Phi_{be}}{K_e} \tag{4.1.22}$$

and

$$\Gamma_{bc} = \varphi_b^+ \Phi_{bc} + \sum_{e=2}^{N} \left(\frac{\bar{\lambda}_{be}}{\bar{\lambda}_{se}} \Phi_{ce} \left(\sum_{d=2}^{N} \Phi_{ed}^{-1} \Gamma_{sd} \right) \right) = \varphi_b^+ \Phi_{bc} - \sum_{e=2}^{N} \frac{\bar{\lambda}_{be} \Phi_{ec}}{K_e} \tag{4.1.23}$$

These results, allow (3.3.21) to be written

$$\varphi_b - \varphi_b^+ = \frac{\bar{\lambda}_{sb}}{K_b}\operatorname{tr}\mathbf{E}_s + \varphi_b^+(\rho_b - \rho_b^+)/\rho_b^+ - \frac{1}{K_b}\sum_{e=2}^{N}\bar{\lambda}_{be}(\rho_e - \rho_e^+)/\rho_e^+ - \frac{\bar{\tau}_b}{K_b}(\theta - \theta^+) \qquad (4.1.24)$$

Given the result (4.1.7), the chemical potentials can be eliminated from (3.3.25) to obtain

$$\varphi_c^+ P_c = \bar{\tau}_c(\theta - \theta^+) - \bar{\lambda}_{sc}(\operatorname{tr}\mathbf{E}_s) + \sum_{b=2}^{N}\bar{\lambda}_{cb}(\rho_b - \rho_b^+)/\rho_b^+ \qquad (4.1.25)$$

Of course, (4.1.25) also follows from (4.1.24), (4.1.9) and (4.1.16). As in Section 2.7, the *capillary* pressures are defined by P_c - P_2 for c = 3,..., N. With this definition, (4.1.25) can be used to obtain the capillary pressures in terms of $\theta - \theta^+, tr\mathbf{E}_s$ and $(\rho_b - \rho_b^+)/\rho_b^+$.

If equation (4.1.7) is used in (3.3.39), the result is

$$\rho_{s_R}\mathbf{K}_s = -(\mathbf{T}_{\mathrm{I}} + \sum_{b=2}^{N}\varphi_b^+ P_b\mathbf{I}) + \alpha(\theta - \theta^+)\mathbf{I} \qquad (4.1.26)$$

It follows from (4.1.26) and (3.3.24) that

$$\mathbf{T}_{\mathrm{I}} + \sum_{b=2}^{N}\varphi_b^+ P_b\mathbf{I} = \bar{\lambda}_{ss}(\operatorname{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s - \sum_{b=2}^{N}\bar{\lambda}_{sb}((\rho_b - \rho_b^+)/\rho_b^+)\mathbf{I} - (\bar{\tau}_s - \alpha)(\theta - \theta^+)\mathbf{I} \qquad (4.1.27)$$

Equation (3.3.26) can be used to derive an expression for the stress $\mathbf{T}_{\mathrm{I}}$ in terms of the pore pressures. A first step in this derivation involves solving (4.1.25) for the fluid densities in terms of the pore pressures, the temperature and the strain of the solid. Because the quantity Σ, defined by (3.3.36), is positive definite, it is possible to show that the $(N-1)\times(N-1)$ symmetric matrix $\left[\bar{\lambda}_{cb}\right]$ is positive definite and, thus, invertible. Therefore, equation (4.1.25) yields

$$(\rho_b - \rho_b^+)/\rho_b^+ = \sum_{c=2}^{N}\bar{\lambda}_{bc}^{-1}\left[\varphi_c^+ P_c - \bar{\tau}_c(\theta - \theta^+) + \bar{\lambda}_{sc}(\operatorname{tr}\mathbf{E}_s)\right] \qquad (4.1.28)$$

If this result is substituted into (3.3,36) and rearranged, the result turns out to be

$$\begin{aligned}\mathbf{T}_{\mathrm{I}} = {} & (\bar{\lambda}_{ss} - \sum_{b=2}^{N}\sum_{c=2}^{N}\bar{\lambda}_{sb}\bar{\lambda}_{bc}^{-1}\bar{\lambda}_{sc})(\operatorname{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s - \sum_{c=2}^{N}\left(1 + \sum_{b=2}^{N}\bar{\lambda}_{sb}\bar{\lambda}_{bc}^{-1}\right)\varphi_c^+ P_c\mathbf{I} \\ & -(\bar{\tau}_s - \sum_{b=2}^{N}\sum_{c=2}^{N}\bar{\lambda}_{sb}\bar{\lambda}_{bc}^{-1}\bar{\tau}_c - \alpha)(\theta - \theta^+)\mathbf{I}\end{aligned} \qquad (4.1.29)$$

As in classical elasticity, it is possible to solve (4.1.29) for the stress $\mathbf{T}_{\mathrm{I}}$. The result of this inversion is

$$
\begin{aligned}
2\mu_{ss}\mathbf{E}_s = \mathbf{T}_{\mathrm{I}} &- \frac{\bar{\lambda}_{ss} - \sum_{b=2}^{N}\sum_{c=2}^{N}\bar{\lambda}_{sb}\bar{\lambda}_{bc}^{-1}\bar{\lambda}_{sc}}{\bar{\lambda}_{ss} - \sum_{b=2}^{N}\sum_{c=2}^{N}\bar{\lambda}_{sb}\bar{\lambda}_{bc}^{-1}\bar{\lambda}_{sc} + \frac{2}{3}\mu_{ss}}\frac{1}{3}(\operatorname{tr}\mathbf{T}_{\mathrm{I}}) \\
&+ \frac{2\mu_{ss}}{3\left(\bar{\lambda}_{ss} - \sum_{b=2}^{N}\sum_{c=2}^{N}\bar{\lambda}_{sb}\bar{\lambda}_{bc}^{-1}\bar{\lambda}_{sc}\right) + 2\mu_{ss}}\sum_{c=2}^{N}\left(1 + \sum_{b=2}^{N}\bar{\lambda}_{sb}\bar{\lambda}_{bc}^{-1}\right)\varphi_c^{+}P_c\mathbf{I} \\
&+ \frac{2\mu_{ss}}{3\left(\bar{\lambda}_{ss} - \sum_{b=2}^{N}\sum_{c=2}^{N}\bar{\lambda}_{sb}\bar{\lambda}_{bc}^{-1}\bar{\lambda}_{sc}\right) + 2\mu_{ss}}(\bar{\tau}_s - \sum_{b=2}^{N}\sum_{c=2}^{N}\bar{\lambda}_{sb}\bar{\lambda}_{bc}^{-1}\bar{\tau}_c - \alpha)(\theta - \theta^{+})\mathbf{I}
\end{aligned}
\tag{4.1.30}
$$

As we shall see later, it is sometimes convenient to utilize the pore pressures as independent variables in the constitutive equations. Equation (4.1.28) is the key equation which allows for the introduction of pore pressures in place of fluid densities. Equation (4.1.29) is one result of this elimination. Likewise, one can use (4.1.28) to eliminate the fluid densities from the entropy formula (3.3.23). The result of this elimination is

$$
\begin{aligned}
\rho\eta = -\alpha &+ (\bar{c}_{\mathrm{v}} + \theta^{+}\sum_{b=2}^{N}\sum_{c=2}^{N}\bar{\tau}_b\bar{\lambda}_{bc}^{-1}\bar{\tau}_c)(\theta - \theta^{+})/\theta^{+} \\
&+ (\bar{\tau}_s - \sum_{b=2}^{N}\sum_{c=2}^{N}\bar{\tau}_b\bar{\lambda}_{bc}^{-1}\bar{\lambda}_{sc})(\operatorname{tr}\mathbf{E}_s) - \sum_{b=2}^{N}\sum_{c=2}^{N}\bar{\tau}_b\bar{\lambda}_{bc}^{-1}\varphi_c^{+}P_c
\end{aligned}
\tag{4.1.31}
$$

A similar elimination can be made from (4.1.28). The result of this calculation is

$$
(\varphi_b - \varphi_b^{+})/\varphi_b^{+} = (\sum_{c=2}^{N}\bar{\lambda}_{bc}^{-1}\bar{\lambda}_{sc})(\operatorname{tr}\mathbf{E}_s) + \sum_{c=2}^{N}\bar{\lambda}_{bc}^{-1}(\varphi_c^{+}P_c) - \frac{P_b}{K_b} - \sum_{c=2}^{N}\bar{\lambda}_{bc}^{-1}\bar{\tau}_c(\theta - \theta^{+}) \tag{4.1.32}
$$

Equations (4.1.29), (4.1.31) and (4.1.32) reflect a symmetry which is worthy of mention. It is best revealed if we use (4.1.26) to rewrite (4.1.29) as an equation for $\rho_{s_R}\mathbf{K}_s$. The result is

$$
\begin{aligned}
\rho_{s_R}\mathbf{K}_s = &-(\bar{\lambda}_{ss} - \sum_{b=2}^{N}\sum_{c=2}^{N}\bar{\lambda}_{sb}\bar{\lambda}_{bc}^{-1}\bar{\lambda}_{sc})(\operatorname{tr}\mathbf{E}_s)\mathbf{I} - 2\mu_{ss}\mathbf{E}_s \\
&+ \sum_{c=2}^{N}\left(\sum_{b=2}^{N}\bar{\lambda}_{sb}\bar{\lambda}_{bc}^{-1}\right)\varphi_c^{+}P_c\mathbf{I} + (\bar{\tau}_s - \sum_{b=2}^{N}\sum_{c=2}^{N}\bar{\lambda}_{sb}\bar{\lambda}_{bc}^{-1}\bar{\tau}_c)(\theta - \theta^{+})\mathbf{I}
\end{aligned}
\tag{4.1.33}
$$

An examination of (4.1.31), (4.1.32) and (4.1.33) shows that

$$\frac{\partial \varphi_b}{\partial \mathbf{E}_s} = \rho_{s_R} \frac{\partial \mathbf{K}_s}{\partial P_b}, \tag{4.1.34}$$

$$\rho_{s_R} \frac{\partial \mathbf{K}_s}{\partial \theta} = \frac{\partial \rho\eta}{\partial \mathbf{E}_s} \tag{4.1.35}$$

and

$$\frac{\partial \varphi_b}{\partial \theta} = \frac{\partial \rho\eta}{\partial P_b} \tag{4.1.36}$$

These special relationships arise from the definition (4.1.1) and the fact that Ψ_{I}, expressed as a function of $(\theta, \mathbf{E}_s, \rho_b)$, is a thermodynamic potential for the determination of $\rho\eta, \mathbf{K}_s$ and μ_b, for b=2,...,N. It is elementary to show that the function Υ, defined by

$$\Upsilon = \Psi_{\mathrm{I}} - \sum_{b=2}^{N} \rho_b \mu_b \tag{4.1.37}$$

is a thermodynamic potential for $\rho\eta, \mathbf{K}_s$, and ρ_b, for b =2,...,N, in the sense that

$$\rho\eta = -\frac{\partial \Upsilon(\theta, \mathbf{E}_s, \mu_b)}{\partial \theta} \tag{4.1.38}$$

$$\rho_{s_R} \mathbf{K}_s = -\frac{\partial \Upsilon(\theta, \mathbf{E}_s, \mu_b)}{\partial \mathbf{E}_s} \tag{4.1.39}$$

and

$$\rho_c = -\frac{\partial \Upsilon(\theta, \mathbf{E}_s, \mu_b)}{\partial \mu_c} \tag{4.1.40}$$

If we represent the solution of (4.1.1) and (4.1.2) by $\mu_b = \mu_b(P_b)$, then one can define a function $\tilde{\Upsilon}$ by

$$\tilde{\Upsilon}(\theta, \mathbf{E}_s, P_b) = \Upsilon(\theta, \mathbf{E}_s, \mu_b(P_b)) \tag{4.1.41}$$

Given (4.1.38) through (4.1.41) and (4.1.1), it is possible to show that

$$\rho\eta = -\frac{\partial \tilde{\Upsilon}(\theta, \mathbf{E}_s, P_b)}{\partial \theta} \tag{4.1.42}$$

$$\rho_{s_R}\mathbf{K}_s = -\frac{\partial\tilde{\Upsilon}(\theta,\mathbf{E}_s,P_b)}{\partial\mathbf{E}_s} \tag{4.1.43}$$

and

$$\varphi_c = -\frac{\partial\tilde{\Upsilon}(\theta,\mathbf{E}_s,P_b)}{\partial P_c} \tag{4.1.44}$$

The essentials of the argument which produced (4.1.42), (4.1.43) and (4.1.44) can be found in the work of BIOT [Ref. 2, 3, 4].

As mentioned in Section 2.4, the *content* of the b^{th} fluid is defined by

$$m_b = \rho_b\left|\det\mathbf{F}_s\right| \tag{4.1.45}$$

Consistent with the linearized assumptions we have adopted, (4.1.45) can be replaced by

$$(m_b - m_b^+)\big/m_b^+ = (\rho_b - \rho_b^+)\big/\rho_b^+ + \operatorname{tr}\mathbf{E}_s \tag{4.1.46}$$

where the approximation (3.3.11) for $\left|\det\mathbf{F}_s\right|$ has been used. Just as m_b represents the mass of the b^{th} fluid per unit of undeformed solid volume, the volume of the b^{th} fluid per unit of undeformed solid volume is v_b defined by

$$v_b = \varphi_b\left|\det\mathbf{F}_s\right| \tag{4.1.47}$$

The linearized form of (4.1.47) is

$$(v_b - v_b^+)\big/v_b^+ = (\varphi_b - \varphi_b^+)\big/\varphi_b^+ + \operatorname{tr}\mathbf{E}_s \tag{4.1.48}$$

where, for notational convenience, we have not utilized $v_b^+ = \varphi_b^+$.

It is convenient in some applications to use (4.1.46) and (4.1.48) to eliminate fluid densities and volume fractions in favor of the quantities m_b and v_b. If (4.1.46) is used, equation (4.1.25) is replaced by

$$\varphi_c^+ P_c = \overline{\tau}_c(\theta-\theta^+) - (\overline{\lambda}_{sc} + \sum_{b=2}^{N}\overline{\lambda}_{cb})(\operatorname{tr}\mathbf{E}_s) + \sum_{b=2}^{N}\overline{\lambda}_{cb}(m_b - m_b^+)\big/m_b^+ \tag{4.1.49}$$

Likewise, (3.3.23) is replaced by

$$\rho\eta = -\alpha + \frac{\overline{c}_{\text{v}}}{\theta^{+}}(\theta-\theta^{+}) + (\overline{\tau}_s + \sum_{b=2}^{N}\overline{\tau}_b)(\operatorname{tr}\mathbf{E}_s) - \sum_{b=2}^{N}\overline{\tau}_b\,(m_b - m_b^{+})/m_b^{+} \tag{4.1.50}$$

and (3.3.24) is replaced by

$$\rho_{s_R}\mathbf{K}_s = -(\overline{\lambda}_{ss} + \sum_{b=2}^{N}\overline{\lambda}_{sb})(\operatorname{tr}\mathbf{E}_s)\mathbf{I} - 2\mu_{ss}\mathbf{E}_s + \sum_{b=2}^{N}\overline{\lambda}_{sb}((m_b - m_b^{+})/m_b^{+})\mathbf{I} + \overline{\tau}_s(\theta-\theta^{+})\mathbf{I} \tag{4.1.51}$$

A similar rearrangement allows (3.3.26) to be written

$$\begin{aligned}\mathbf{T}_{\text{I}} = {} & (\overline{\lambda}_{ss} + 2\sum_{c=2}^{N}\overline{\lambda}_{sc} + \sum_{b=2}^{N}\sum_{c=2}^{N}\overline{\lambda}_{cb})(\operatorname{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s - \sum_{b=2}^{N}(\overline{\lambda}_{sb} + \sum_{c=2}^{N}\overline{\lambda}_{cb})((m_b - m_b^{+})/m_b^{+})\mathbf{I} \\ & -(\overline{\tau}_s + \sum_{c=2}^{N}\overline{\tau}_c - \alpha)(\theta-\theta^{+})\mathbf{I}\end{aligned} \tag{4.1.52}$$

Likewise, (4.1.24) can be written

$$\frac{\varphi_b - \varphi_b^{+}}{\varphi_b^{+}} = \left(\frac{\overline{\lambda}_{sb} + \sum_{e=2}^{N}\overline{\lambda}_{be}}{\varphi_b^{+}K_b} - 1\right)\operatorname{tr}\mathbf{E}_s + (m_b - m_b^{+})/m_b^{+}) - \frac{1}{\varphi_b^{+}K_b}\sum_{e=2}^{N}\overline{\lambda}_{be}\,(m_e - m_e^{+})/m_e^{+} - \frac{\overline{\tau}_b}{\varphi_b^{+}K_b}(\theta-\theta^{+}) \tag{4.1.53}$$

A more convenient version of (4.1.53) follows by utilizing (4.1.48) to obtain

$$\frac{v_b - v_b^{+}}{v_b^{+}} = \left(\frac{\overline{\lambda}_{sb} + \sum_{e=2}^{N}\overline{\lambda}_{be}}{\varphi_b^{+}K_b}\right)\operatorname{tr}\mathbf{E}_s + (m_b - m_b^{+})/m_b^{+}) - \frac{1}{\varphi_b^{+}K_b}\sum_{e=2}^{N}\overline{\lambda}_{be}\,(m_e - m_e^{+})/m_e^{+} - \frac{\overline{\tau}_b}{\varphi_b^{+}K_b}(\theta-\theta^{+}) \tag{4.1.54}$$

When the independent variables are selected to be $(\theta,\mathbf{E}_s,m_b)$, it is possible to introduce a potential whose derivatives yield (4.1.49) and (4.1.52) [Ref. 3]. The quantity W defined by

$$W(\theta,\mathbf{E}_s,m_b) = \left|\det\mathbf{F}_s\right|\Psi_{\text{I}}(\theta,\mathbf{E}_s,m_b) \tag{4.1.55}$$

is the inner part of the free energy of the mixture *per unit of undeformed solid volume*. If we now use (3.2.15) and (3.3.21), it follows that

$$
\begin{aligned}
W(\theta,\mathbf{E}_s,m_b) &= \alpha(\theta-\theta^+) - \frac{1}{2}\frac{\overline{c}_{\mathrm{v}}}{\theta^+}(\theta-\theta^+)^2 - (\overline{\tau}_s + \sum_{b=2}^{N}\overline{\tau}_b - \alpha)(\theta-\theta^+)(\operatorname{tr}\mathbf{E}_s) \\
&+\sum_{b=2}^{N}\overline{\tau}_b(\theta-\theta^+)\left((m_b-m_b^+)/m_b^+\right) + \frac{1}{2}(\overline{\lambda}_{ss} + 2\sum_{b=2}^{N}\overline{\lambda}_{sb} + \sum_{b=2}^{N}\sum_{c=2}^{N}\overline{\lambda}_{bc})(\operatorname{tr}\mathbf{E}_s)^2 \\
&-\sum_{b=2}^{N}\left(\overline{\lambda}_{sb} + 2\sum_{c=2}^{N}\overline{\lambda}_{cb}\right)(\operatorname{tr}\mathbf{E}_s)\left((m_b-m_b^+)/m_b^+\right) \\
&+\frac{1}{2}\sum_{b=2}^{N}\sum_{c=2}^{N}\overline{\lambda}_{bc}\left((m_b-m_b^+)/m_b^+\right)\left((m_c-m_c^+)/m_c^+\right) + \mu_{ss}\operatorname{tr}(\mathbf{E}_s\mathbf{E}_s)
\end{aligned}
\tag{4.1.56}
$$

It easily follows from (4.1.56) and (4.1.49) that

$$
\varphi_c^+ P_c = \frac{\partial W}{\partial m_c} = \overline{\tau}_c(\theta-\theta^+) - \left(\overline{\lambda}_{sc} + \sum_{b=2}^{N}\overline{\lambda}_{cb}\right)(\operatorname{tr}\mathbf{E}_s) + \sum_{b=2}^{N}\overline{\lambda}_{cb}\left((m_b-m_b^+)/m_b^+\right) \tag{4.1.57}
$$

In a like fashion (4.1.56) and (4.1.52) yield

$$
\begin{aligned}
\mathbf{T}_{\mathrm{I}} = \frac{\partial W}{\partial \mathbf{E}_s} &= (\overline{\lambda}_{ss} + 2\sum_{c=2}^{N}\overline{\lambda}_{sc} + \sum_{b=2}^{N}\sum_{c=2}^{N}\overline{\lambda}_{cb})(\operatorname{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s - \sum_{b=2}^{N}(\overline{\lambda}_{sb} + \sum_{c=2}^{N}\overline{\lambda}_{cb})((m_b-m_b^+)/m_b^+)\mathbf{I} \\
&-(\overline{\tau}_s + \sum_{c=2}^{N}\overline{\tau}_b - \alpha)(\theta-\theta^+)\mathbf{I}
\end{aligned}
\tag{4.1.58}
$$

The relationship of the result (4.1.50) to the quantity $W(\theta,\mathbf{E}_s,m_b)$ is slightly more complicated. First, we introduce the entropy per unit of undeformed solid volume by the definition

$$
\mathrm{H} = \left|\det\mathbf{F}_s\right|\rho\eta \tag{4.1.59}
$$

Given (3.2.15) and (4.1.50), it easily follows that

$$
\mathrm{H} = -\frac{\partial W}{\partial \theta} = -\alpha + \frac{\overline{c}_{\mathrm{v}}}{\theta^+}(\theta-\theta^+) + (\overline{\tau}_s + \sum_{b=2}^{N}\overline{\tau}_b - \alpha)(\operatorname{tr}\mathbf{E}_s) - \sum_{b=2}^{N}\overline{\tau}_b(m_b-m_b^+)/m_b^+ \tag{4.1.60}
$$

4.2 Binary Immiscible Porous Materials With Pore Pressure

The special case of a single fluid contained in a solid is of special importance. Because of the complexity of the governing equations, very few problems have been examined with more than one fluid. For this reason, in this section the case where $N = 2$ will be examined.

For the case where the solid and fluid are compressible, equations (4.1.24), (3.3.22), (3.3.23), (3.3.24) and (4.1.25) reduce to

$$\frac{\varphi_f - \varphi_f^+}{\varphi_f^+} = \frac{\lambda_{sb}}{\varphi_f^+ K_f} \operatorname{tr} \mathbf{E}_s + \left(1 - \frac{\lambda_{ff}}{\varphi_f^+ K_f}\right)((\rho_f - \rho_f^+)/\rho_f^+) - \frac{\bar{\tau}_b}{\varphi_f^+ K_f}(\theta - \theta^+) \tag{4.2.1}$$

$$\begin{aligned}\Psi_{\mathrm{I}} = {} & \alpha(\theta-\theta^+) - \frac{1}{2}\frac{\bar{c}_{\mathrm{v}}}{\theta^+}(\theta-\theta^+)^2 - \bar{\tau}_s(\theta-\theta^+)(\operatorname{tr}\mathbf{E}_s) + \bar{\tau}_f(\theta-\theta^+)(\rho_f-\rho_f^+)/\rho_f^+ \\ & + \frac{1}{2}\bar{\lambda}_{ss}(\operatorname{tr}\mathbf{E}_s)^2 - \bar{\lambda}_{sf}(\operatorname{tr}\mathbf{E}_s)(\rho_f-\rho_f^+)/\rho_f^+ \\ & + \frac{1}{2}\bar{\lambda}_{ff}\left((\rho_f-\rho_f^+)/\rho_f^+\right)^2 + \mu_{ss}\operatorname{tr}(\mathbf{E}_s\mathbf{E}_s),\end{aligned} \tag{4.2.2}$$

$$\rho\eta = -\alpha + \frac{\bar{c}_{\mathrm{v}}}{\theta^+}(\theta-\theta^+) + \bar{\tau}_s(\operatorname{tr}\mathbf{E}_s) - \bar{\tau}_f(\rho_f-\rho_f^+)/\rho_f^+ \tag{4.2.3}$$

$$\begin{aligned}\rho_{s_R}\mathbf{K}_s = {} & -(\mathbf{T}_{\mathrm{I}} + \varphi_f^+ P_f \mathbf{I}) + \alpha(\theta-\theta^+)\mathbf{I} = -\bar{\lambda}_{ss}(\operatorname{tr}\mathbf{E}_s)\mathbf{I} - 2\mu_{ss}\mathbf{E}_s \\ & + \bar{\lambda}_{sf}((\rho_f-\rho_f^+)/\rho_f^+)\mathbf{I} + \bar{\tau}_s(\theta-\theta^+)\mathbf{I}\end{aligned} \tag{4.2.4}$$

and

$$\varphi_f^+ P_f = \bar{\tau}_f(\theta-\theta^+) - \bar{\lambda}_{sf}(\operatorname{tr}\mathbf{E}_s) + \bar{\lambda}_{ff}(\rho_f-\rho_f^+)/\rho_f^+ \tag{4.2.5}$$

where the subscript f denotes the single fluid constituent. From equations (4.1.17), the bulk modulus for the fluid is given by

$$K_f = -\frac{\bar{\lambda}_{sf}\Phi_{ff}}{\Gamma_{sf}} \tag{4.2.6}$$

The inequality (4.1.18) reduces to

$$-\bar{\lambda}_{sf}\frac{\Phi_{ff}}{\Gamma_{sf}} > 0 \tag{4.2.7}$$

Since the coefficient Φ_{ff} must be positive, (4.2.7) shows that

$$\lambda_{sf}/\Gamma_{sf} < 0 \tag{4.2.8}$$

The single fluid versions of (4.1.27), (3.3.26), (4.1.29), (4.1.31) and (4.1.32) take the forms

$$\mathbf{T}_{\mathrm{I}} + \varphi_f^+ P_f \mathbf{I} = \bar{\lambda}_{ss}(\operatorname{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s - \bar{\lambda}_{sf}((\rho_f-\rho_f^+)/\rho_f^+)\mathbf{I} - (\bar{\tau}_s - \alpha)(\theta-\theta^+)\mathbf{I} \tag{4.2.9}$$

$$\mathbf{T}_{\text{I}} = (\bar{\lambda}_{ss} + \bar{\lambda}_{sf})(\operatorname{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s - (\bar{\lambda}_{sf} + \bar{\lambda}_{ff})((\rho_f - \rho_f^+)/\rho_f^+)\mathbf{I} \\ -(\bar{\tau}_s + \bar{\tau}_f - \alpha)(\theta - \theta^+)\mathbf{I} \tag{4.2.10}$$

$$\mathbf{T}_{\text{I}} = (\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2/\bar{\lambda}_{ff})(\operatorname{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s - (1 + \bar{\lambda}_{sf}/\bar{\lambda}_{ff})\varphi_f^+ P_f \mathbf{I} \\ -(\bar{\tau}_s - \bar{\lambda}_{sf}\bar{\tau}_f/\bar{\lambda}_{ff} - \alpha)(\theta - \theta^+)\mathbf{I} \tag{4.2.11}$$

$$\rho\eta = -\alpha + (\bar{c}_{\text{v}} + \theta^+ \bar{\tau}_f^2/\bar{\lambda}_{ff})(\theta - \theta^+)/\theta^+ + (\bar{\tau}_s - \bar{\tau}_f\bar{\lambda}_{sf}/\bar{\lambda}_{ff})(\operatorname{tr}\mathbf{E}_s) - (\bar{\tau}_f/\bar{\lambda}_{ff})\varphi_f^+ P_f \tag{4.2.12}$$

and

$$(\varphi_f - \varphi_f^+)/\varphi_f^+ = (\bar{\lambda}_{sf}/\bar{\lambda}_{ff})(\operatorname{tr}\mathbf{E}_s) + (1/\bar{\lambda}_{ff})\left(1 - \frac{\bar{\lambda}_{ff}}{\varphi_f^+ K_f}\right)\phi_f^+ P_f - (\bar{\tau}_f/\bar{\lambda}_{ff})(\theta - \theta^+) \tag{4.2.13}$$

The field equations which follow from (3.3.33), (3.3.34) and (3.3.35) are

$$\rho_{s_R}\frac{\partial^2 \mathbf{w}_s}{\partial t^2} = (\bar{\lambda}_{ss} + \mu_{ss})\,\text{GRAD}(\text{Div}\,\mathbf{w}_s) + \mu_{ss}\,\text{Div}(\text{GRAD}\,\mathbf{w}_s) \\ + \bar{\lambda}_{sf}\,\text{GRAD}(\text{Div}\,\mathbf{w}_f) + (\alpha_s - \bar{\tau}_s + \gamma)\,\text{GRAD}\,\theta - \xi\left(\frac{\partial \mathbf{w}_s}{\partial t} - \frac{\partial \mathbf{w}_f}{\partial t}\right) \tag{4.2.14}$$

$$\rho_f^+\frac{\partial^2 \mathbf{w}_f}{\partial t^2} = \bar{\lambda}_{sf}\,\text{GRAD}(\text{Div}\,\mathbf{w}_s) + \bar{\lambda}_{ff}\,\text{GRAD}(\text{Div}\,\mathbf{w}_f) + (\alpha_f - \bar{\tau}_f - \gamma)\,\text{GRAD}\,\theta \\ -\xi\left(\frac{\partial \mathbf{w}_f}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t}\right) \tag{4.2.15}$$

and

$$\bar{c}_{\text{v}}\frac{\partial\theta}{\partial t} + \theta^+\left((\bar{\tau}_s - \alpha_s + \varsigma/\theta^+)\,\text{Div}\frac{\partial \mathbf{w}_s}{\partial t} + (\bar{\tau}_f - \alpha_f - \varsigma/\theta^+)\,\text{Div}\frac{\partial \mathbf{w}_f}{\partial t}\right) \\ = \kappa\,\text{Div}(\text{GRAD}\,\theta) \tag{4.2.16}$$

where

$$\xi = \xi_{ff} = -\xi_{sf} \tag{4.2.17}$$

$$\gamma = \gamma_f = -\gamma_s \tag{4.2.18}$$

and

$$\varsigma = \varsigma_f = -\varsigma_s \tag{4.2.19}$$

Given (4.2.17), (4.2.18) and (4.2.19), the inequality (3.2.27) shows that

$$\kappa \geq 0 \tag{4.2.20}$$

and

$$\kappa\xi/\theta^+ \geq \frac{1}{4}(\gamma + \varsigma/\theta^+)^2 \tag{4.2.21}$$

It follows from (4.2.21) that

$$\xi \geq 0 \tag{4.2.22}$$

As mentioned earlier, the coefficient κ is the *thermal conductivity*. The coefficient ξ is called the *drag coefficient*.

4.3 Stability of Equilibrium:Classes of Initial - Boundary Value Problems

In this section some comments will be given about the types of initial-boundary value problems one can formulate with a linearized porous media model of the type being discussed in this Chapter. For simplicity, we shall continue to assume that we are dealing with a porous elastic solid containing one fluid. While much of which is said generalizes to the case of *N* - 1 fluids, it is not necessary to complicate the discussion here.

At several points in the discussion, mention has been made of the inequality (3.3.18). For the case of a solid containing a single fluid, it takes the form

$$\begin{aligned} &\frac{\partial}{\partial t}\int_{\mathscr{V}} \left(\Sigma + \frac{1}{2}\rho_{s_R} \frac{\partial \mathbf{w}_s}{\partial t}\cdot\frac{\partial \mathbf{w}_s}{\partial t} + \frac{1}{2}\rho_f^+ \frac{\partial \mathbf{w}_f}{\partial t}\cdot\frac{\partial \mathbf{w}_f}{\partial t}\right) dv \\ &\leq \oint_{\partial\mathscr{V}} (\mathbf{t} - \frac{\theta-\theta^+}{\theta^+}\mathbf{m})\cdot d\mathbf{s} \end{aligned} \tag{4.3.1}$$

where

$$\mathbf{t} = \mathbf{T}_{\mathrm{I}} \frac{\partial \mathbf{w}_s}{\partial t} - (\varphi_f^+ P_f - \alpha_f(\theta-\theta^+))\left(\frac{\partial \mathbf{w}_f}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t}\right) \tag{4.3.2}$$

Equation (4.3.2) is equation (3.3.17) specialized to the case where $N = 2$ and where the fluid has pore pressure. Of course, equation (4.1.7) has been used to introduce the pore pressure P_f. By use of (3.3.36), in the compressible case Σ is given by

$$\Sigma = \frac{1}{2}\frac{\bar{c}_{\mathrm{v}}}{\theta^+}(\theta-\theta^+)^2 + \frac{1}{2}\bar{\lambda}_{ss}(\operatorname{tr}\mathbf{E}_s)^2 - \bar{\lambda}_{sf}(\operatorname{tr}\mathbf{E}_s)(\rho_f-\rho_f^+)\big/\rho_f^+ + \frac{1}{2}\bar{\lambda}_{ff}\left((\rho_f-\rho_f^+)/\rho_f^+\right)^2 + \mu_{ss}\operatorname{tr}(\mathbf{E}_s\mathbf{E}_s) \tag{4.3.3}$$

The fact that Σ is positive definite yields

$$\bar{c}_{\mathrm{v}} > 0 \tag{4.3.4}$$

$$\bar{\lambda}_{ss} + \frac{2}{3}\mu_{ss} > 0 \tag{4.3.5}$$

$$(\bar{\lambda}_{ss} + \frac{2}{3}\mu_{ss})\bar{\lambda}_{ff} > \bar{\lambda}_{sf}^2 \tag{4.3.6}$$

and

$$\mu_{ss} > 0 \tag{4.3.7}$$

Equations (4.3.5) and (4.3.6) imply that

$$\bar{\lambda}_{ff} > 0 \tag{4.3.8}$$

The essential idea of the stability of equilibrium is to show that the inequality (4.3.1), combined with certain other assumptions, allows one to prove that states that are in equilibrium initially remain so indefinitely. Once this simple physical fact is formally established, it is rather simple to establish a uniqueness theorem for the governing field equations. If the compressible case is considered first, for initial conditions it is required that the initial displacements of the solid and the fluid are such that

$$\mathbf{E}_s(\mathbf{X},0) = \mathbf{0} \tag{4.3.9}$$

$$\frac{\partial \mathbf{w}_s(\mathbf{X},0)}{\partial t} = \mathbf{0} \tag{4.3.10}$$

$$\operatorname{tr}\mathbf{E}_f(\mathbf{X},0) \equiv \operatorname{Div}\mathbf{w}_f(\mathbf{X},0) = 0 \tag{4.3.11}$$

and

$$\frac{\partial \mathbf{w}_f(\mathbf{X},0)}{\partial t} = \mathbf{0} \tag{4.3.12}$$

Therefore, we have assumed that the fluid and the solid have zero initial velocity. In addition, we have assumed that the strain of the solid is initially zero, and, from (3.3.9), that the bulk density of the fluid is initially ρ_f^+. For the temperature, we shall assume that initially

$$\Theta(\mathbf{X},0) = \theta^+ \tag{4.3.13}$$

Next boundary conditions must be prescribed on the surface $\partial\mathscr{V}$. At this point in the discussion, we shall simply say that boundary conditions are prescribed which cause the surface integral in (4.3.1) to vanish. As a result, (4.3.1) reduces to

$$\frac{\partial}{\partial t}\int_{\mathscr{V}} \left(\Sigma + \frac{1}{2}\rho_{s_R} \frac{\partial \mathbf{w}_s}{\partial t}\cdot\frac{\partial \mathbf{w}_s}{\partial t} + \frac{1}{2}\rho_f^+ \frac{\partial \mathbf{w}_f}{\partial t}\cdot\frac{\partial \mathbf{w}_f}{\partial t}\right) dv \le 0 \tag{4.3.14}$$

Given our initial conditions (4.3.9) through (4.3.12), equation (4.3.14) shows that

$$\int_{\mathscr{V}} \left(\Sigma(\mathbf{X},t) + \frac{1}{2}\rho_{s_R} \frac{\partial \mathbf{w}_s(\mathbf{X},t)}{\partial t}\cdot\frac{\partial \mathbf{w}_s(\mathbf{X},t)}{\partial t} + \frac{1}{2}\rho_f^+ \frac{\partial \mathbf{w}_f(\mathbf{X},t)}{\partial t}\cdot\frac{\partial \mathbf{w}_f(\mathbf{X},t)}{\partial t}\right) dv = 0 \tag{4.3.15}$$

for all *t*. If we presume that we are dealing with continuous solutions of the field equations, it follows from (4.3.15) that the integrand must vanish for all $(\mathbf{X},t)$. This fact and the positive definite nature of Σ yield

$$\mathbf{E}_s(\mathbf{X},t) = \mathbf{0} \tag{4.3.16}$$

$$\frac{\partial \mathbf{w}_s(\mathbf{X},t)}{\partial t} = \mathbf{0} \tag{4.3.17}$$

$$\operatorname{tr}\mathbf{E}_f(\mathbf{X},t) \equiv \operatorname{Div}\mathbf{w}_f = 0 \tag{4.3.18}$$

$$\frac{\partial \mathbf{w}_f(\mathbf{X},t)}{\partial t} = \mathbf{0} \tag{4.3.19}$$

and

$$\theta(\mathbf{X},t) = \theta^+ \tag{4.3.20}$$

for all (**X**, *t*). Therefore, with suitable boundary conditions, solutions of the field equations which start out in the equilibrium state defined by (4.3.9) through (4.3.13), remain in that state for future time.

A uniqueness theorem follows from the above results if one considers two possible solutions of the field equations which obey the same initial and boundary conditions. The difference in these two solutions will have zero initial and boundary data. If these initial conditions for the difference solution obey (4.3.9) through (4.3.13), and if the boundary conditions for the difference solution causes the surface integral in (4.3.1) to vanish, then the difference solution must obey (4.3.16) through (4.3.20). It is in the sense of these three equations that the solution of the field equations is unique.

It is not difficult to characterize many cases where the surface integral in (4.3.1) vanishes. If **n** denotes the outward drawn unit normal to the surface $\partial\mathscr{V}$, some of these cases are as follows:

- *i)* The fluid and solid displacements vanish on $\partial\mathscr{V}$,and the temperature equals θ^+.
- *ii)* The stress vector $\mathbf{T}_{\mathrm{I}}\mathbf{n}$ vanishes on $\partial\mathscr{V}$, the fluid and solid displacements are equal and the temperature equals θ^+.
- *iii)* The stress vector $\mathbf{T}_{\mathrm{I}}\mathbf{n}$ vanishes on $\partial\mathscr{V}$, the pore pressure P_f vanishes on $\partial\mathscr{V}$ and the temperature equals θ^+.
- *iv)* The solid displacement vanishes on $\partial\mathscr{V}$, the pore pressure P_f vanishes on $\partial\mathscr{V}$ and the temperature equals θ^+.
- *v)* Cases identical to *i*) and *ii*) except that the temperature condition is replaced by the requirement that $\mathbf{m}\cdot\mathbf{n}$ vanishes on $\partial\mathscr{V}$.
- *vi)* Combinations of *i*) through *v*), where the cases are prescribed on different parts of $\partial\mathscr{V}$.

The reader will recognize in the above list the usual traction and displacement boundary conditions from elasticity. The conditions on the temperature field correspond to cases familiar from conduction heat transfer, i.e. when the boundary is not insulated one prescribes the surface temperature. The conditions on the pore pressure and the relative displacement of the fluid and solid correspond to whether or not the boundary is *pervious* or *impervious* to the flow of fluid across the boundary. If one prescribes the relative displacement to be zero, as in *ii*), then the fluid is not allowed to flow across the boundary. If instead, one prescribes the pore pressure, relative motion is allowed and, thus, fluid can flow across the boundary. In a rough sense the temperature and pore pressure are analogous quantities as far as the prescription of the boundary conditions are concerned. As we shall see later, in a certain special case there is a formal analogy between temperature and pore pressure.

4.4 Incompressible Immiscible Mixtures

In Section 2.7 it was shown how for a mixture consisting of a rigid solid and N - 1 incompressible fluids one could use the constraint of incompressibility to formulate a mathematical model. When the solid is allowed to be a deformable incompressible solid, a similar formulation can be presented [Ref. 5, 6]. However, it is instructive in this Section to take an equivalent approach and obtain the incompressible results as a limit of the results in Section 3.4.

From the results of the Section 4.2, when the mixture is assumed to be in shifting equilibrium and each fluid is assumed to have a pore pressure, the volume fractions are given by (4.1.24). If we make the additional assumption that one of the fluids is incompressible, then by further restrictions on the coefficients in (4.1.24) we can accommodate this special case. Equation (4.1.24) repeated is

$$\varphi_b - \varphi_b^+ = \frac{\bar{\lambda}_{sb}}{K_b}\operatorname{tr}\mathbf{E}_s + \varphi_b^+ (\rho_b - \rho_b^+)\big/\rho_b^+ - \frac{1}{K_b}\sum_{e=2}^{N}\bar{\lambda}_{be}(\rho_e - \rho_e^+)\big/\rho_e^+ - \frac{\bar{\tau}_b}{K_b}(\theta - \theta^+) \tag{4.4.1}$$

If the b^{th} fluid is incompressible, then $\gamma_b = \gamma_b^+$ and equation (4.1.9) reduces to

$$\frac{\rho_b - \rho_b^+}{\rho_b^+} = \frac{\varphi_b - \varphi_b^+}{\varphi_b^+} \tag{4.4.2}$$

In order for equation (4.4.1) to reduce to (4.4.2), it is necessary for

$$\frac{\bar{\lambda}_{sb}}{K_b} \to 0 \tag{4.4.3}$$

$$\frac{\bar{\lambda}_{be}}{K_b} \to 0 \tag{4.4.4}$$

and

$$\frac{\bar{\tau}_b}{K_b} \to 0 \tag{4.4.5}$$

If every fluid is incompressible, then (4.4.3), (4.4.4) and (4.4.5) hold for $b = 2,...,N$. It then follows from (4.1.17), (4.1.23) and (4.1.22) that, in the incompressible limit,

$$\Gamma_{sd} = 0 \tag{4.4.6}$$

$$\Gamma_{dc} = \varphi_b^+ \Phi_{dc} \tag{4.4.7}$$

and

$$\Theta_d = 0 \tag{4.4.8}$$

for $c, d = 2,..., N$. These results reduce (3.3.27) through (3.3.32) to

$$\bar{c}_{\mathrm{v}} = c_{\mathrm{v}} \tag{4.4.9}$$

$$\overline{\tau}_s = \tau_s \tag{4.4.10}$$

$$\overline{\tau}_c = \tau_c \tag{4.4.11}$$

$$\overline{\lambda}_{ss} = \lambda_{ss} \tag{4.4.12}$$

$$\overline{\lambda}_{sc} = \lambda_{sc} \tag{4.4.13}$$

and

$$\overline{\lambda}_{cb} = \lambda_{cb} \tag{4.4.14}$$

As a consequence of (4.4.9) through (4.4.14), in those cases where all of the fluids are incompressible, we need not distinguish between the shifting equilibrium material coefficients and those originally introduced in Section 3.2. This distinction will be ignored without comment throughout the remainder of this chapter.

If the *solid* is also incompressible then we must satisfy the additional condition that γ_s is a constant. Because $\gamma_s = \rho_s / \phi_s$, equations (2.4.9) and (3.2.15) show that

$$\frac{\varphi_s - \varphi_s^+}{\varphi_s^+} = -\operatorname{tr} \mathbf{E}_s \tag{4.4.15}$$

must be true when the solid is incompressible. Equation (4.4.15) also follows from (2.6.11) and (3.2.15). As a result of the relationship (2.6.12), the volume fraction of the solid can be eliminated from (4.4.15) to obtain

$$\sum_{b=2}^{N} (\varphi_b - \varphi_b^+) = \varphi_s^+ \operatorname{tr} \mathbf{E}_s \tag{4.4.16}$$

Equation (4.4.16) is the incompressibility constraint for an incompressible linear elastic solid containing N - 1 incompressible fluids. It is possible that the reader will recognize (4.4.16) as the linearized special case of the constraint (2.6.13). Equation (4.4.16) is an extra relationship between the variables which must be accommodated by the constitutive equations. One method of achieving this accommodation is to allow the pore pressure for the N^{th} fluid to become indeterminant. It follows from (4.1.25) that

$$\phi_N^+ P_N = \tau_N (\theta - \theta^+) - \lambda_{sN} (\operatorname{tr} \mathbf{E}_s) + \sum_{b=2}^{N} \lambda_{Nb} (\rho_b - \rho_b^+) / \rho_b^+ \tag{4.4.17}$$

Since the fluids are incompressible, equation (4.4.2) can be used to replace (4.4.17) by

$$\phi_N^+ P_N = \tau_N(\theta-\theta^+) - \lambda_{sN}(\mathrm{tr}\,\mathbf{E}_s) + \sum_{b=2}^{N} \lambda_{Nb}(\phi_b - \phi_b^+)\big/\phi_b^+ \tag{4.4.18}$$

By appropriate choice of the coefficients in (4.4.18), it can be made to reduce to (4.4.16). The details of this choice are revealed by writing (4.4.18) as

$$\begin{aligned}\phi_N^+ P_N\big/\lambda_{NN} &= (\tau_N\big/\lambda_{NN})(\theta-\theta^+) - (\lambda_{sN}\big/\lambda_{NN})(\mathrm{tr}\,\mathbf{E}_s) \\ &+\sum_{b=2}^{N-1}(\lambda_{Nb}\big/\lambda_{NN})(\varphi_b-\varphi_b^+)\big/\varphi_b^+ + (\varphi_N-\varphi_N^+)\big/\varphi_N^+\end{aligned} \tag{4.4.19}$$

Clearly, if we assume that

$$\frac{1}{\lambda_{NN}} \to 0 \tag{4.4.20}$$

$$\frac{\tau_N}{\lambda_{NN}} \to 0 \tag{4.4.21}$$

$$\frac{\lambda_{Nb}}{\lambda_{NN}} \to \frac{\varphi_b^+}{\varphi_N^+} \tag{4.4.22}$$

for $b = 2,\ldots,N-1$, and

$$\frac{\lambda_{sN}}{\lambda_{NN}} \to \frac{\varphi_s^+}{\varphi_N^+} \tag{4.4.23}$$

where P_N remains bounded, then (4.4.19) reduces to the desired result (4.4.16). Equations (4.4.20), (4.4.21), (4.4.22) and (4.4.23) are additional restrictions on the material constants resulting from the assumption of incompressibility. Unlike (4.4.6), (4.4.7) and (4.4.8), equations (4.4.20), (4.4.21) (4.4.22) and (4.4.23) cause the pore pressure P_N not to appear in the constitutive equations for the model. In short, the pore pressure P_N becomes indeterminate in so far as the constitutive equations are concerned. Of course, this indeterminacy is exactly like the one which appeared in Section 2.7 where the constraint of incompressibility was used during the derivation of the thermodynamic restrictions.

The limits (4.4.20) through (4.4.23) have an impact on the constitutive equations for the pore pressures P_β, for $\beta = 2,\ldots,N-1$. Because of (4.4.20) and (4.4.22), the coefficient $\lambda_{cN} = \lambda_{Nc}$, which appears in (4.1.25) is unbounded in the incompressible limit. However, if one forms the difference $P_\beta - P_N$, for $\beta = 2,\ldots, N$, the result can be seen to have a well defined limit. It follows from (4.1.25) that this difference is given by

$$\varphi_\beta^+(P_\beta - P_N) = (\tau_\beta - \frac{\varphi_\beta^+}{\varphi_N^+}\tau_N)(\theta-\theta^+) - (\lambda_{s\beta} - \frac{\varphi_\beta^+}{\varphi_N^+}\lambda_{sN})(\operatorname{tr}\mathbf{E}_s) + \sum_{\alpha=2}^{N-1}(\lambda_{\beta\alpha} - \frac{\varphi_\beta^+}{\varphi_N^+}\lambda_{N\alpha})(\varphi_\alpha - \varphi_\alpha^+)\big/\varphi_\alpha^+ + (\lambda_{\beta N} - \frac{\varphi_\beta^+}{\varphi_N^+}\lambda_{NN})(\varphi_N - \varphi_N^+)\big/\varphi_N^+, \tag{4.4.24}$$

where (4.4.2) has been used. It follows from (4.4.19) and (4.4.22) that in the incompressible limit the coefficient of $(\varphi_N - \varphi_N^+)\big/\varphi_N^+$ in (4.4.24) vanishes. If the limits of the remaining coefficients in (4.4.24) are written

$$\tau_\beta - \frac{\varphi_\beta^+}{\varphi_N^+}\tau_N \to \tau_\beta^* \tag{4.4.25}$$

$$\lambda_{s\beta} - \frac{\varphi_\beta^+}{\varphi_N^+}\lambda_{sN} \to \lambda_{s\beta}^* \tag{4.4.26}$$

and

$$\lambda_{\beta\alpha} - \frac{\varphi_\beta^+}{\varphi_N^+}\lambda_{N\alpha} \to \lambda_{\beta\alpha}^* \tag{4.4.27}$$

then (4.4.24) becomes

$$\varphi_\beta^+(P_\beta - P_N) = \tau_\beta^*(\theta-\theta^+) - \lambda_{s\beta}^*(\operatorname{tr}\mathbf{E}_s) + \sum_{\alpha=2}^{N-1}\lambda_{\beta\alpha}^*(\varphi_\alpha - \varphi_\alpha^+)\big/\varphi_\alpha^+ \tag{4.4.28}$$

for $\beta = 2,..., N - 1$. Equations (4.4.28) are the constitutive equation for the capillary pressures within the incompressible mixture. A manipulation similar to the one which produced (4.4.24) can be applied to (3.3.24) to obtain

$$\rho_{s_R}\mathbf{K}_s - \varphi_s^+ P_N\mathbf{I} = -(\lambda_{ss} - \frac{\varphi_s^+}{\varphi_N^+}\lambda_{sN})(\operatorname{tr}\mathbf{E}_s)\mathbf{I} - 2\mu_{ss}\mathbf{E}_s + \sum_{\beta=2}^{N-1}(\lambda_{s\beta} - \frac{\varphi_s^+}{\varphi_N^+}\lambda_{N\beta})((\varphi_\beta - \varphi_\beta^+)\big/\varphi_\beta^+)\mathbf{I} + (\lambda_{sN} - \frac{\varphi_s^+}{\varphi_N^+}\lambda_{NN})((\varphi_N - \varphi_N^+)\big/\varphi_N^+)\mathbf{I} + (\tau_s - \frac{\varphi_s^+}{\varphi_N^+}\tau_N)(\theta-\theta^+)\mathbf{I} \tag{4.4.29}$$

As with (4.4.24), the coefficient of $(\varphi_N - \varphi_N^+)\big/\varphi_N^+$ vanishes in the incompressible limit. If the limits of the coefficients in (4.4.29) are written

$$\lambda_{ss} - \frac{\varphi_s^+}{\varphi_N^+}\lambda_{sN} \to \lambda_{ss}^* \tag{4.4.30}$$

$$\lambda_{s\beta} - \frac{\varphi_s^+}{\varphi_N^+}\lambda_{N\beta} \to \lambda_{s\beta}^* \tag{4.4.31}$$

and

$$\tau_s - \frac{\varphi_s^+}{\varphi_N^+}\tau_N \to \tau_s^* \tag{4.4.32}$$

then (4.4.29) becomes

$$\rho_{s_R}\mathbf{K}_s - \varphi_s^+ P_N \mathbf{I} = -\lambda_{ss}^*(\operatorname{tr}\mathbf{E}_s)\mathbf{I} - 2\mu_{ss}\mathbf{E}_s + \sum_{\beta=2}^{N-1}\lambda_{s\beta}^*((\varphi_\beta - \varphi_\beta^+)/\varphi_\beta^+)\mathbf{I} + \tau_s^*(\theta-\theta^+)\mathbf{I}. \tag{4.4.33}$$

Note that (4.4.26) and (4.4.31) provide two apparently different definitions of the coefficients $\lambda_{s\beta}^*$ for β = 2,..., *N* - 1. It is readily shown by use of (4.4.22) and (4.4.23) that the coefficients $\lambda_{s\beta} - (\varphi_\beta^+/\varphi_N^+)\lambda_{sN}$ and $\lambda_{s\beta} - (\varphi_s^+/\varphi_N^+)\lambda_{N\beta}$ have the same incompressible limit.

The incompressible limit of the constitutive equation for the entropy density, equation (3.3.23), is

$$\rho\eta = -\alpha + c_{\text{v}}(\theta-\theta^+)/\theta^+ + \tau_s^* \operatorname{tr}\mathbf{E}_s - \sum_{\beta=2}^{N-1}\tau_\beta^*((\varphi_\beta - \varphi_\beta^+)/\varphi_\beta^+) \tag{4.4.34}$$

Equation (4.4.34) is a consequence of (3.3.23), (4.4.2), (4.4.25), (4.4.32) and the result (4.4.9). Just as equations (4.4.28), (4.4.33) and (4.4.34) have been obtained as a limit of the corresponding compressible result, it is possible to derive an incompressible version of the expression for Ψ_{I}, equation (3.3.22). The result turns out to be

$$\begin{aligned}\Psi_{\text{I}} &= \alpha(\theta-\theta^+) - \frac{1}{2}\frac{c_{\text{v}}}{\theta^+}(\theta-\theta^+)^2 - \tau_s^*(\theta-\theta^+)(\operatorname{tr}\mathbf{E}_s)\\ &+\sum_{\beta=2}^{N-1}\tau_\beta^*(\theta-\theta^+)(\varphi_\beta-\varphi_\beta^+)/\varphi_\beta^+ + \frac{1}{2}\lambda_{ss}^*(\operatorname{tr}\mathbf{E}_s)^2 - \sum_{\beta=2}^{N-1}\lambda_{s\beta}^*(\operatorname{tr}\mathbf{E}_s)(\varphi_\beta-\varphi_\beta^+)/\varphi_\beta^+\\ &+\frac{1}{2}\sum_{\beta=2}^{N-1}\sum_{\alpha=2}^{N-1}\lambda_{\beta\alpha}^*(\varphi_\beta-\varphi_\beta^+)(\varphi_\alpha-\varphi_\alpha^+)/\varphi_\beta^+\varphi_\alpha^+ + \mu_{ss}\operatorname{tr}(\mathbf{E}_s\mathbf{E}_s).\end{aligned} \tag{4.4.35}$$

Given (4.4.35), it is readily established that

$$\rho\eta = -\frac{\partial\Psi_{\text{I}}}{\partial\theta} \tag{4.4.36}$$

$$P_\beta - P_N = \frac{\partial \Psi_{\mathrm{I}}}{\partial \phi_\beta} \tag{4.4.37}$$

and

$$\rho_{s_R}\mathbf{K}_s - \varphi_s^+ P_N \mathbf{I} = -\frac{\partial \Psi_{\mathrm{I}}}{\partial \mathbf{E}_s} \tag{4.4.38}$$

Given (4.4.35) and (4.4.34), it follows that

$$\begin{aligned}\Sigma = \Psi_{\mathrm{I}} + \rho\eta(\theta-\theta^+) &= \frac{1}{2}\frac{c_{\mathrm{v}}}{\theta^+}(\theta-\theta^+)^2 + \frac{1}{2}\lambda_{ss}^*(\operatorname{tr}\mathbf{E}_s)^2 - \sum_{\beta=2}^{N-1}\lambda_{s\beta}^*(\operatorname{tr}\mathbf{E}_s)(\varphi_\beta-\varphi_\beta^+)\big/\varphi_\beta^+ \\ &+ \frac{1}{2}\sum_{\beta=2}^{N-1}\sum_{\alpha=2}^{N-1}\lambda_{\beta\alpha}^*(\varphi_\beta-\varphi_\beta^+)(\varphi_\alpha-\varphi_\alpha^+)\big/\varphi_\beta^+\varphi_\alpha^+ + \mu_{ss}\operatorname{tr}(\mathbf{E}_s\mathbf{E}_s)\end{aligned} \tag{4.4.39}$$

As before, the fact that Σ is assumed to be positive definite yields

$$c_{\mathrm{v}} > 0 \tag{4.4.40}$$

$$\mu_{ss} > 0 \tag{4.4.41}$$

and the $(N-1)\times(N-1)$ symmetric matrix

$$\begin{bmatrix} \lambda_{ss}^* + \frac{2}{3}\mu_{ss}^* & \lambda_{s2}^* & . & . & . & . & \lambda_{s,N-1}^* \\ \lambda_{s2}^* & \lambda_{22}^* & . & . & . & . & \lambda_{2,N-1}^* \\ . & . & & & & & . \\ . & . & & & & & . \\ . & . & & & & & . \\ \lambda_{s,N-1}^* & \lambda_{2,N-1}^* & . & . & . & . & \lambda_{N-1,N-1}^* \end{bmatrix}$$

is positive definite.

It is possible to show that the incompressible versions of (4.1.27), (4.4.23) and (4.1.29) are

$$\begin{aligned}\mathbf{T}_{\mathrm{I}} + P_N\mathbf{I} + \sum_{\beta=2}^{N-1}\varphi_\beta^+(P_\beta-P_N)\mathbf{I} &= \lambda_{ss}^*(\operatorname{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s \\ &- \sum_{\beta=2}^{N-1}\lambda_{s\beta}^*((\varphi_\beta-\varphi_\beta^+)\big/\varphi_\beta^+)\mathbf{I} - (\tau_s^*-\alpha)(\theta-\theta^+)\mathbf{I}\end{aligned} \tag{4.4.42}$$

$$\begin{aligned}\mathbf{T}_{\mathrm{I}} &= -P_N\mathbf{I} + (\lambda^*_{ss} + \sum_{\beta=2}^{N-1}\lambda^*_{s\beta})(\operatorname{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s \\ &\quad -\sum_{\beta=2}^{N-1}(\lambda^*_{s\beta} + \sum_{\alpha=2}^{N-1}\lambda^*_{\alpha\beta})((\varphi_\beta - \varphi^+_\beta)/\varphi^+_\beta)\mathbf{I} - (\tau^*_s + \sum_{\beta=2}^{N-1}\tau^*_\beta - \alpha)(\theta-\theta^+)\mathbf{I}\end{aligned} \tag{4.4.43}$$

and

$$\begin{aligned}\mathbf{T}_{\mathrm{I}} &= -P_N\mathbf{I} + (\lambda^*_{ss} - \sum_{\beta=2}^{N-1}\sum_{\alpha=2}^{N-1}\lambda^*_{s\beta}\lambda^{*-1}_{\beta\alpha}\lambda^*_{s\alpha})(\operatorname{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s \\ &\quad -\sum_{\alpha=2}^{N-1}\left(1+\sum_{\beta=2}^{N-1}\lambda^*_{s\beta}\lambda^{*-1}_{\beta\alpha}\right)\varphi^+_\alpha(P_\alpha - P_N)\mathbf{I} - (\tau^*_s - \sum_{\beta=2}^{N-1}\sum_{\alpha=2}^{N-1}\lambda^*_{s\beta}\lambda^{*-1}_{\beta\alpha}\tau^*_\alpha - \alpha)(\theta-\theta^+)\mathbf{I}\end{aligned} \tag{4.4.44}$$

where the positive definite character of Σ insures that the matrix $\left[\lambda^*_{\beta\alpha}\right]$ has an inverse.

The field equations which replace (3.3.33), (3.3.34) and (3.3.35) in the incompressible case turn out to be

$$\begin{aligned}\rho_{s_R}\frac{\partial^2\mathbf{w}_s}{\partial t^2} &= -\varphi^+_s\,\mathrm{GRAD}\,P_N + (\lambda^*_{ss}+\mu_{ss})\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_s) + \mu_{ss}\,\mathrm{Div}(\mathrm{GRAD}\,\mathbf{w}_s) \\ &\quad +\sum_{\beta=2}^{N-1}\lambda^*_{s\beta}\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_\beta) + (\alpha_s - \tau^*_s - \gamma_s)\,\mathrm{GRAD}\,\theta - \sum_{b=2}^{N}\xi_{sb}\left(\frac{\partial\mathbf{w}_b}{\partial t} - \frac{\partial\mathbf{w}_s}{\partial t}\right)\end{aligned} \tag{4.4.45}$$

$$\begin{aligned}\rho^+_\beta\frac{\partial^2\mathbf{w}_\beta}{\partial t^2} &= -\varphi^+_\beta\,\mathrm{GRAD}\,P_N + \lambda^*_{s\beta}\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_s) + \sum_{\beta=2}^{N-1}\lambda^*_{\beta\alpha}\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_\alpha) \\ &\quad +(\alpha_\beta - \tau^*_\beta - \gamma_\beta)\,\mathrm{GRAD}\,\theta - \sum_{b=2}^{N}\xi_{\beta b}\left(\frac{\partial\mathbf{w}_b}{\partial t} - \frac{\partial\mathbf{w}_s}{\partial t}\right)\end{aligned} \tag{4.4.46}$$

for β = 2,…,*N*-1,

$$\rho^+_N\frac{\partial^2\mathbf{w}_N}{\partial t^2} = -\phi^+_N\,\mathrm{GRAD}\,P_N + (\alpha_N - \gamma_N)\,\mathrm{GRAD}\,\theta - \sum_{b=2}^{N}\xi_{Nb}\left(\frac{\partial\mathbf{w}_b}{\partial t} - \frac{\partial\mathbf{w}_s}{\partial t}\right) \tag{4.4.47}$$

and

$$\begin{aligned}&c_{\mathrm{v}}\frac{\partial\theta}{\partial t} + \theta^+(\tau^*_s - \alpha_s - \varsigma_s/\theta^+)\,\mathrm{Div}\frac{\partial\mathbf{w}_s}{\partial t} \\ &\quad +\theta^+\sum_{\beta=2}^{N-1}(\tau^*_\beta - \alpha_\beta - \varsigma_\beta/\theta^+)\,\mathrm{Div}\frac{\partial\mathbf{w}_\beta}{\partial t} - \theta^+(\alpha_N + \varsigma_N/\theta^+)\,\mathrm{Div}\frac{\partial\mathbf{w}_N}{\partial t} = \kappa\,\mathrm{Div}(\mathrm{GRAD}\,\theta)\end{aligned} \tag{4.4.48}$$

In addition, these field equations must be supplemented with the constraint equation (4.4.16) written in terms of displacements. This equation is

$$\varphi_s^+ \operatorname{Div} \mathbf{w}_s + \sum_{b=2}^{N} \varphi_b^+ \operatorname{Div} \mathbf{w}_b = 0 \tag{4.4.49}$$

4.5 Binary Incompressible Immiscible Mixtures

If both the fluid and solid are incompressible, we need to specialize the results of Section 4.4 to the binary case or simply view the incompressible case as the limit of the compressible one. In any case, equations (4.4.35) (4.4.34), (4.4.33) and (4.4.43) take the forms:

$$\begin{aligned}\Psi_{\mathrm{I}} = \alpha(\theta-\theta^+) - \frac{1}{2}\frac{c_{\mathrm{v}}}{\theta^+}(\theta-\theta^+)^2 - \tau_s^*(\theta-\theta^+)(\operatorname{tr}\mathbf{E}_s) \\ + \frac{1}{2}\lambda_{ss}^*(\operatorname{tr}\mathbf{E}_s)^2 + \mu_{ss}\operatorname{tr}(\mathbf{E}_s\mathbf{E}_s)\end{aligned} \tag{4.5.1}$$

$$\rho\eta = -\alpha + c_{\mathrm{v}}(\theta-\theta^+)/\theta^+ + \tau_s^*\operatorname{tr}\mathbf{E}_s \tag{4.5.2}$$

$$\rho_{s_R}\mathbf{K}_s - \varphi_s^+ P_f \mathbf{I} = -\lambda_{ss}^*(\operatorname{tr}\mathbf{E}_s)\mathbf{I} - 2\mu_{ss}\mathbf{E}_s + \tau_s^*(\theta-\theta^+)\mathbf{I} \tag{4.5.3}$$

and

$$\mathbf{T}_{\mathrm{I}} = -P_f\mathbf{I} + \lambda_{ss}^*(\operatorname{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s - (\tau_s^* - \alpha)(\theta-\theta^+)\mathbf{I} \tag{4.5.4}$$

The field equations in this case are special cases of (4.4.45) through (4.4.49). These special cases are

$$\begin{aligned}\rho_{s_R}\frac{\partial^2\mathbf{w}_s}{\partial t^2} = -\varphi_s^+\operatorname{GRAD}P_f + (\lambda_{ss}^* + \mu_{ss})\operatorname{GRAD}(\operatorname{Div}\mathbf{w}_s) + \mu_{ss}\operatorname{Div}(\operatorname{GRAD}\mathbf{w}_s) \\ + (\alpha_s - \tau_s^* + \gamma)\operatorname{GRAD}\theta - \xi\left(\frac{\partial\mathbf{w}_s}{\partial t} - \frac{\partial\mathbf{w}_f}{\partial t}\right)\end{aligned} \tag{4.5.5}$$

$$\rho_f^+\frac{\partial^2\mathbf{w}_f}{\partial t^2} = -\varphi_f^+\operatorname{GRAD}P_f + (\alpha_f - \gamma)\operatorname{GRAD}\theta - \xi\left(\frac{\partial\mathbf{w}_f}{\partial t} - \frac{\partial\mathbf{w}_s}{\partial t}\right) \tag{4.5.6}$$

$$\bar{c}_{\mathrm{v}}\frac{\partial\theta}{\partial t} + \theta^+\left((\tau_s^* - \alpha_s + \varsigma/\theta^+)\operatorname{Div}\frac{\partial\mathbf{w}_s}{\partial t} - (\alpha_f + \varsigma/\theta^+)\operatorname{Div}\frac{\partial\mathbf{w}_f}{\partial t}\right) = \kappa\operatorname{Div}(\operatorname{GRAD}\theta) \tag{4.5.7}$$

and

$$\varphi_s^+ \operatorname{Div} \mathbf{w}_s + \varphi_f^+ \operatorname{Div} \mathbf{w}_f = 0 \tag{4.5.8}$$

In the binary incompressible case, it follows from (4.4.39) that

$$\Sigma = \frac{1}{2}\frac{c_{\mathrm{v}}}{\theta^+}(\theta - \theta^+)^2 + \frac{1}{2}\lambda_{ss}^*(\operatorname{tr}\mathbf{E}_s)^2 + \mu_{ss}\operatorname{tr}(\mathbf{E}_s\mathbf{E}_s) \tag{4.5.9}$$

The result (4.5.9) also follows from (4.5.1), (4.5.2) and the formula, $\Sigma = \Psi_{\mathrm{I}} + \rho\eta(\theta - \theta^+)$. The requirement that Σ be positive definite yields

$$c_{\mathrm{v}} > 0 \tag{4.5.10}$$

$$\lambda_{ss}^* + \frac{2}{3}\mu_{ss} > 0 \tag{4.5.11}$$

and

$$\mu_{ss} > 0 \tag{4.5.12}$$

The arguments in Section 4.3 carry over with minor modification for the incompressible case. Because Σ in (4.5.9) does not depend on $\operatorname{Div}\mathbf{w}_f$, it is not necessary to prescribe the initial condition (4.3.11). However, because the constraint (4.5.8) must be obeyed, (4.3.11) is a consequence of (4.3.9). Likewise, (4.3.18) is not a consequence of the positive definite character of Σ. It is a consequence of the constraint (4.5.8) and (4.3.16).

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5

Models Which Neglect Inertia

In order to reach the results summarized in Sections 4.2 and 4.5, many specializing assumptions were necessary. In this Chapter, additional assumptions which one often finds in the porous elasticity literature are discussed. The two special cases which we shall focus on are the special models one gets when the fluid inertia is neglected, and the special model one gets when both the fluid and solid inertia are neglected. For the moment, we shall not worry about the validity of these approximations.

5.1 Compressible Models

The first special case which shall be considered is the compressible binary porous material defined by equations 4.2.1 through (4.2.22). If the inertia of the fluid constituent is simply neglected, the only formal change is that (4.2.16) is replaced by

$$\xi\left(\frac{\partial \mathbf{w}_f}{\partial t}-\frac{\partial \mathbf{w}_s}{\partial t}\right)=\bar{\lambda}_{sf}\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_s)+\bar{\lambda}_{ff}\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_f)+(\alpha_f-\bar{\tau}_f-\gamma)\,\mathrm{GRAD}\,\theta \quad (5.1.1)$$

Because the highest time derivative of $\mathbf{w}_f$ which now appears in the field equations is one, the number of initial conditions which can be satisfied by the governing field equations is reduced by one. Therefore, the approximate field equations are a *singular perturbation* of those which retain the fluid inertia. Typical of singular perturbation problems, the solutions of the approximate equations cannot be expected to approximate those which retain the fluid inertia in the time neighborhood of the lost initial condition. Usually this fact is described by stating that the approximate equations are not valid within a *boundary layer* of the boundary $t = 0$. It turns out that characteristic times in porous elasticity are rather small. The effect of this fact is that in a nondimensional sense the time boundary layer is usually small. In any case, if we proceed with an investigation of the field equations without fluid inertia, there are certain interesting results which can be obtained.

The first formal manipulation is to eliminate the fluid displacements in favor of the fluid pore pressure. Equations (4.2.11) and (4.2.12) give the stress, $\mathbf{T}_{\mathrm{I}}$, and the entropy, η, in terms of the fluid pore pressure. It is convenient to write (4.2.5) in the form

$$\varphi_f^+ P_f=\bar{\tau}_f(\theta-\theta^+)-\bar{\lambda}_{sf}\,\mathrm{Div}\,\mathbf{w}_s-\bar{\lambda}_{ff}\,\mathrm{Div}\,\mathbf{w}_f \quad (5.1.2)$$

where (3.39), with the order term omitted, (3.2.3) and (3.28) have been used. Given(5.1.2), equation (5.1.1) can be rewritten in the form

$$\xi\left(\frac{\partial \mathbf{w}_f}{\partial t}-\frac{\partial \mathbf{w}_s}{\partial t}\right)=-\varphi_f^{+}\,\text{GRAD}\,P_f+(\alpha_f-\gamma)\,\text{GRAD}\,\theta \tag{5.1.3}$$

Equation (5.1.3) allows for the elimination of the combination $\left(\frac{\partial \mathbf{w}_f}{\partial t}-\frac{\partial \mathbf{w}_s}{\partial t}\right)$ from the constitutive equation for the heat flux **m**. The binary version of (3.2.23), with the order term omitted, is

$$\mathbf{m}=-\kappa\,\text{GRAD}\,\theta-\varsigma\left(\frac{\partial \mathbf{w}_f}{\partial t}-\frac{\partial \mathbf{w}_s}{\partial t}\right) \tag{5.1.4}$$

Therefore, when (5.1.3) is used to eliminate the term $\left(\frac{\partial \mathbf{w}_f}{\partial t}-\frac{\partial \mathbf{w}_s}{\partial t}\right)$ from (5.1.4) the result is

$$\mathbf{m}=-(\kappa+(\varsigma/\xi)(\alpha_f-\gamma))\,\text{GRAD}\,\theta+(\varsigma/\xi)\varphi_f^{+}\,\text{GRAD}\,P_f \tag{5.1.5}$$

Note that in the derivation of (5.1.5) it has been assumed that ξ is nonzero. This assumption, along with the requirement (4.2.22), shows that we have assumed

$$\xi>0 \tag{5.1.6}$$

Next we shall eliminate the displacement of the fluid, $\mathbf{w}_f$, between (5.1.2) and (5.1.3). If we compute the time derivative of (5.1.2) and the divergence of (5.1.3) the results are

$$\varphi_f^{+}\frac{\partial P_f}{\partial t}=\overline{\tau}_f\,\frac{\partial \theta}{\partial t}-\overline{\lambda}_{sf}\,\text{Div}\frac{\partial \mathbf{w}_s}{\partial t}-\overline{\lambda}_{ff}\,\text{Div}\frac{\partial \mathbf{w}_f}{\partial t} \tag{5.1.7}$$

and

$$\xi\left(\text{Div}\frac{\partial \mathbf{w}_f}{\partial t}-\text{Div}\frac{\partial \mathbf{w}_s}{\partial t}\right)=-\varphi_f^{+}\,\text{Div}(\text{GRAD}\,P_f)+(\alpha_f-\gamma)\,\text{Div}(\text{GRAD}\,\theta) \tag{5.1.8}$$

If $\text{Div}\frac{\partial \mathbf{w}_f}{\partial t}$ is eliminated from these two equations, the result is

$$\begin{aligned}(1/\overline{\lambda}_{ff})\varphi_f^{+}\frac{\partial P_f}{\partial t}-(1/\overline{\lambda}_{ff})\overline{\tau}_f\,\frac{\partial \theta}{\partial t}+(1+\overline{\lambda}_{sf}/\overline{\lambda}_{ff})\,\text{Div}\frac{\partial \mathbf{w}_s}{\partial t}\\=(\varphi_f^{+}/\xi)\,\text{Div}(\text{GRAD}\,P_f)-((\alpha_f-\gamma)/\xi)\,\text{Div}(\text{GRAD}\,\theta)\end{aligned} \tag{5.1.9}$$

Equation (5.1.9) replaces the equation of motion for the fluid in the special case where the inertia of the fluid can be neglected. The equation of motion of the solid, (4.2.14), can also be simplified in this case. Equation (5.1.1) can be used to eliminate the term $\xi\left(\frac{\partial \mathbf{w}_f}{\partial t}-\frac{\partial \mathbf{w}_s}{\partial t}\right)$ from (4.2.14), and (5.1.2) can be used to eliminate $\operatorname{Div}\mathbf{w}_f$. The result of this elimination is

$$\begin{aligned}\rho_{s_R}\frac{\partial^2 \mathbf{w}_s}{\partial t^2} &= -\varphi_f^{+}(1+\bar{\lambda}_{sf}/\bar{\lambda}_{ff})\operatorname{GRAD}P_f+(\bar{\lambda}_{ss}-\bar{\lambda}_{sf}^2/\bar{\lambda}_{ff}+\mu_{ss})\operatorname{GRAD}(\operatorname{Div}\mathbf{w}_s)\\ &\quad+\mu_{ss}\operatorname{Div}(\operatorname{GRAD}\mathbf{w}_s)+(\alpha-\bar{\tau}_s+\bar{\tau}_f\,\bar{\lambda}_{sf}/\bar{\lambda}_{ff})\operatorname{GRAD}\theta.\end{aligned} \tag{5.1.10}$$

The energy equation appropriate to this special case follows from (4.2.16). Equation (4.2.16) can be rearranged to the form

$$\begin{aligned}&\bar{c}_{\text{v}}\frac{\partial \theta}{\partial t}+\theta^{+}\left((\bar{\tau}_s-\alpha)\operatorname{Div}\frac{\partial \mathbf{w}_s}{\partial t}+\bar{\tau}_f\operatorname{Div}\frac{\partial \mathbf{w}_f}{\partial t}\right)\\ &\qquad=\operatorname{Div}\left(\kappa\operatorname{GRAD}\theta+\theta^{+}(\varsigma/\theta^{+}+\alpha_f)\left(\frac{\partial \mathbf{w}_s}{\partial t}-\frac{\partial \mathbf{w}_f}{\partial t}\right)\right)\end{aligned} \tag{5.1.11}$$

The desired result can be obtained by use of (5.1.7) to eliminate $\operatorname{Div}(\partial \mathbf{w}_f/\partial t)$ from the left side of (5.1.11) and (5.1.8) to eliminate $\operatorname{Div}\left(\frac{\partial \mathbf{w}_f}{\partial t}-\frac{\partial \mathbf{w}_s}{\partial t}\right)$ from the right side of (5.1.11). The result of this elimination is

$$\begin{aligned}&(\bar{c}_{\text{v}}+\theta^{+}\bar{\tau}_f^2/\bar{\lambda}_{ff})\frac{\partial \theta}{\partial t}-\theta^{+}(1/\bar{\lambda}_{ff})\bar{\tau}_f\varphi_f^{+}\frac{\partial P_f}{\partial t}+\theta^{+}(\bar{\tau}_s-\bar{\tau}_f\,\bar{\lambda}_{sf}/\bar{\lambda}_{ff}-\alpha)\operatorname{Div}\frac{\partial \mathbf{w}_s}{\partial t}\\ &\quad=(\kappa+\theta^{+}(\alpha_f+\varsigma/\theta^{+})(\alpha_f-\gamma)/\xi)\operatorname{Div}(\operatorname{GRAD}\theta)\\ &\qquad-\theta^{+}((\alpha_f+\varsigma/\theta^{+})/\xi)\varphi_f^{+}\operatorname{Div}(\operatorname{GRAD}P_f)\end{aligned} \tag{5.1.12}$$

By rearrangement of the inequality (4.2.21), it is possible to show that

$$\begin{aligned}&(1/\xi)(\kappa+\theta^{+}(\alpha_f+\varsigma/\theta^{+})(\alpha_f-\gamma)/\xi)\\ &\qquad\geq\frac{1}{4}\left\{(\alpha_f+\varsigma/\theta^{+})+(\alpha_f-\gamma)\right\}^2\end{aligned} \tag{5.1.13}$$

The algebra necessary to derive (5.1.13) from (4.2.21) comes from the following sequence of rearrangements

$$\begin{aligned}\frac{1}{4}(\gamma+\varsigma/\theta^{+})^{2} &= \frac{1}{4}\Big((\varsigma/\theta^{+}+\alpha_f)-(\alpha_f-\gamma)\Big)^{2} \\ &= \frac{1}{4}\Big((\varsigma/\theta^{+}+\alpha_f)+(\alpha_f-\gamma)\Big)^{2}-(\varsigma/\theta^{+}+\alpha_f)(\alpha_f-\gamma)\end{aligned} \tag{5.1.14}$$

and the substitution of the result, (5.1.14), into (4.2.21). It follows from (5.1.13) and (5.1.6) that

$$(\kappa+\theta^{+}(\alpha_f+\varsigma/\theta^{+})(\alpha_f-\gamma)/\xi)\geq 0 \tag{5.1.15}$$

Equations (5.1.9), (5.1.10) and (5.1.12) represent the field equations appropriate to the model where fluid inertia has been neglected. These equations, which are intended to yield the solid displacement, $\mathbf{w}_s$, the pore pressure, P_f and the temperature, θ, must be supplemented by boundary and initial conditions. The results of Section 4.3 provide guidance as to the types of initial boundary data which are sufficient to imply uniqueness. If the fluid inertia is neglected, equation (4.3.1) reduces to

$$\frac{\partial}{\partial t}\int_{\mathscr{V}}\left(\Sigma+\frac{1}{2}\rho_{s_R}\frac{\partial \mathbf{w}_s}{\partial t}\cdot\frac{\partial \mathbf{w}_s}{\partial t}\right)dv \leq \oint_{\partial\mathscr{V}}(\mathbf{t}-\frac{\theta-\theta^{+}}{\theta^{+}}\mathbf{m})\cdot d\mathbf{s}. \tag{5.1.16}$$

where

$$\begin{aligned}\Sigma = &\frac{1}{2\theta^{+}}(\overline{c}_{\mathrm{v}}+\theta^{+}\,\overline{\tau}_f^{2}/\overline{\lambda}_{ff})(\theta-\theta^{+})^{2}+\frac{1}{2}(\overline{\lambda}_{ss}-\overline{\lambda}_{sf}^{2}/\overline{\lambda}_{ff})(\mathrm{tr}\,\mathbf{E}_s)^{2} \\ &-(\overline{\tau}_f/\overline{\lambda}_{ff})(\varphi_f^{+}P_f)(\theta-\theta^{+})+\frac{1}{2}(1/\overline{\lambda}_{ff})(\varphi_f^{+}P_f)^{2}+\mu_{ss}\,\mathrm{tr}(\mathbf{E}_s\mathbf{E}_s)\end{aligned} \tag{5.1.17}$$

and

$$\mathbf{t}=\mathbf{T}_{\mathrm{I}}\frac{\partial \mathbf{w}_s}{\partial t}-(\varphi_f^{+}P_f-\alpha_f(\theta-\theta^{+}))\left(\frac{\partial \mathbf{w}_f}{\partial t}-\frac{\partial \mathbf{w}_s}{\partial t}\right) \tag{5.1.18}$$

Equation (5.1.17) follows from (4.3.3) after (5.1.2) and the approximation of (3.3.9)

$$\left(\rho_f-\rho_f^{+}\right)/\rho_f^{+}=-\mathrm{Div}\,\mathbf{w}_f \tag{5.1.19}$$

have been used to eliminate $\left(\rho_f-\rho_f^{+}\right)/\rho_f^{+}$. If the result (5.1.3) is used, (5.1.18) can be replaced by

$$\mathbf{t}=\mathbf{T}_{\mathrm{I}}\frac{\partial \mathbf{w}_s}{\partial t}-(\varphi_f^{+}P_f-\alpha_f(\theta-\theta^{+}))\left(-\frac{\varphi_f^{+}}{\xi}GRADP_f+\frac{(\alpha_f-\gamma)}{\xi}GRAD\theta\right) \tag{5.1.20}$$

An examination of (5.1.20) and (5.1.5) illustrates that, for this approximate theory, unique solutions will result by placing conditions on the stress vector, $\mathbf{T}_{\mathrm{I}}\mathbf{n}$, or the solid displacement, $\mathbf{w}_s$, the pore pressure or its gradient and the temperature or its gradient. The precise nature of these conditions can be deduced from (5.1.17), (5.1.20) and (5.1.5).

If in addition to neglecting the fluid inertia, the assumption is made that the temperature is a constant, then an interesting analogy is obtained between the isothermal poroelasticity of compressible constituents and dynamic thermoelasticity. As mentioned earlier, the isothermal case is obtained is the thermal conductivity κ is assumed to be very large. In the limit as $\kappa \to \infty$, equation (5.1.12) reduces to $\text{Div}(\text{GRAD}\,\theta) = 0$. With appropriate boundary data, the solution to this equation is $\theta = \theta^+$. This result reduces (5.1.9) and (5.1.10) to

$$(1/\bar{\lambda}_{ff})\varphi_f^+ \frac{\partial P_f}{\partial t} + (1 + \bar{\lambda}_{sf}/\bar{\lambda}_{ff})\,\text{Div}\frac{\partial \mathbf{w}_s}{\partial t} = (\varphi_f^+/\xi)\,\text{Div}(\text{GRAD}\,P_f) \tag{5.1.21}$$

and

$$\begin{aligned}\rho_{s_R}\frac{\partial^2 \mathbf{w}_s}{\partial t^2} = -\phi_f^+(1 + \bar{\lambda}_{sf}/\bar{\lambda}_{ff})\,\text{GRAD}\,P_f + (\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2/\bar{\lambda}_{ff} + \mu_{ss})\,\text{GRAD}(\text{Div}\,\mathbf{w}_s)\\ + \mu_{ss}\,\text{Div}(\text{GRAD}\,\mathbf{w}_s)\end{aligned} \tag{5.1.22}$$

An examination of these equations reveals that they are formally identical to the equations of dynamic thermoelasticity [Ref. 1]. In order to establish this analogy, one has to make the following identifications:

Isothermal Poroelasticity	Dynamic Thermoelasticity
$\varphi_f^+ P_f$	The temperature differential
$1/\bar{\lambda}_{ff}$	The specific heat per unit of volume divided by the reference temperature
$\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2/\bar{\lambda}_{ff} + \frac{2}{3}\mu_{ss}$	The isothermal bulk modulus of elasticity
μ_{ss}	The isothermal shear modulus of elasticity
$1/\xi$	The thermal conductivity divided by the reference temperature
$1 + \bar{\lambda}_{sf}/\bar{\lambda}_{ff}$	The stress-temperature modulus

In thermoelasticity, the stress-temperature modulus is usually written as three times the produce of the coefficient of thermal expansion and the isothermal bulk modulus. The analogy between porous elasticity with fluid inertia neglected and dynamic thermoelasticity was first mentioned by Lubinski [Ref. 2]. References 3, 4 and 5 should also be consulted for readers interested in additional information about this analogy.

If the additional approximation is made that the inertia of the solid can also be neglected, then we can obtain yet another set of approximate field equations. This approximation, as we shall see, is frequently adopted as the appropriate one for porous elasticity applications. If the inertia of the solid is neglected, then the field equations (5.1.9) and (5.1.11) are not formally changed. The equation of motion (5.1.10) is replaced by

$$\begin{aligned}(\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff} + \mu_{ss})\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_s) + \mu_{ss}\,\mathrm{Div}(\mathrm{GRAD}\,\mathbf{w}_s) \\ - \varphi_f^+ (1 + \bar{\lambda}_{sf} / \bar{\lambda}_{ff})\,\mathrm{GRAD}\,P_f + (\alpha - \bar{\tau}_s + \bar{\tau}_f\, \bar{\lambda}_{sf} / \bar{\lambda}_{ff})\,\mathrm{GRAD}\,\theta = \mathbf{0}\end{aligned} \tag{5.1.23}$$

Given (4.2.11), a more compact version of (5.1.23) is

$$\mathrm{Div}\,\mathbf{T}_\mathrm{I} = \mathbf{0} \tag{5.1.24}$$

Equation (4.2.11) can be used to eliminate the displacement of the solid from (5.1.9) in favor of the stress $\mathbf{T}_\mathrm{I}$. Some of this elimination has already been performed. It follows from (4.1.30) that

$$\begin{aligned}2\mu_{ss}\mathbf{E}_s = \mathbf{T}_\mathrm{I} - \frac{\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff}}{\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff} + \frac{2}{3}\mu_{ss}} \frac{1}{3}(\mathrm{tr}\,\mathbf{T}_\mathrm{I})\mathbf{I} \\ + \frac{2\mu_{ss}(1 + \bar{\lambda}_{sf} / \bar{\lambda}_{ff})}{3(\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff}) + 2\mu_{ss}} \varphi_f^+ P_f \mathbf{I} + \frac{2\mu_{ss}(\bar{\tau}_s - \bar{\lambda}_{sf}\bar{\tau}_f / \bar{\lambda}_{ff} - \alpha)}{3(\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff}) + 2\mu_{ss}} (\theta - \theta^+)\mathbf{I}\end{aligned} \tag{5.1.25}$$

It follows from (5.1.25) that

$$\begin{aligned}\mathrm{tr}\,\mathbf{E}_s = \mathrm{Div}\,\mathbf{w}_s = \frac{1}{3(\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff}) + 2\mu_{ss}} \mathrm{tr}\,\mathbf{T}_\mathbf{I} + \frac{3(1 + \bar{\lambda}_{sf} / \bar{\lambda}_{ff})}{3(\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff}) + 2\mu_{ss}} \varphi_f^+ P_f \\ + \frac{3(\bar{\tau}_s - \bar{\lambda}_{sf}\bar{\tau}_f / \bar{\lambda}_{ff} - \alpha)}{3(\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff}) + 2\mu_{ss}} (\theta - \theta^+)\end{aligned} \tag{5.1.26}$$

If the time derivative of equation (5.1.26) is used to eliminate the term $\mathrm{Div}\left(\partial \mathbf{w}_s / \partial t\right)$ from (5.1.9) and (5.1.12), the results are

$$\frac{\bar{\lambda}_{ss}+\frac{2}{3}\mu_{ss}+2\bar{\lambda}_{sf}+\bar{\lambda}_{ff}}{\bar{\lambda}_{ff}(\bar{\lambda}_{ss}-\bar{\lambda}_{sf}^2/\bar{\lambda}_{ff}+\frac{2}{3}\mu_{ss})}\varphi_f^+\frac{\partial P_f}{\partial t}$$
$$+\frac{(\bar{\tau}_s-\alpha)(\bar{\lambda}_{ff}+\bar{\lambda}_{sf})-\bar{\tau}_f(\bar{\lambda}_{ss}+\frac{2}{3}\mu_{ss}+\bar{\lambda}_{sf})}{\bar{\lambda}_{ff}(\bar{\lambda}_{ss}-\bar{\lambda}_{sf}^2/\bar{\lambda}_{ff}+\frac{2}{3}\mu_{ss})}\frac{\partial\theta}{\partial t} \tag{5.1.27}$$
$$+\frac{\bar{\lambda}_{ff}+\bar{\lambda}_{sf}}{\bar{\lambda}_{ff}(\bar{\lambda}_{ss}-\bar{\lambda}_{sf}^2/\bar{\lambda}_{ff}+\frac{2}{3}\mu_{ss})}\frac{1}{3}\frac{\partial\,\mathrm{tr}\,\mathbf{T}_{\mathrm{I}}}{\partial t}$$
$$=(\varphi_f^+/\xi)\,\mathrm{Div}(\mathrm{GRAD}\,P_f)-((\alpha_f-\gamma)/\xi)\,\mathrm{Div}(\mathrm{GRAD}\,\theta)$$

and

$$\theta^+\frac{(\bar{\tau}_s-\alpha)(\bar{\lambda}_{ff}+\bar{\lambda}_{sf})-\bar{\tau}_f(\bar{\lambda}_{ss}+\frac{2}{3}\mu_{ss}+\bar{\lambda}_{sf})}{\bar{\lambda}_{ff}(\bar{\lambda}_{ss}-\bar{\lambda}_{sf}^2/\bar{\lambda}_{ff}+\frac{2}{3}\mu_{ss})}\varphi_f^+\frac{\partial P_f}{\partial t}$$
$$+\left(\bar{c}_{\mathrm{v}}+\theta^+\frac{\bar{\tau}_f^2(\bar{\lambda}_{ss}+\frac{2}{3}\mu_{ss})-2(\bar{\tau}_s-\alpha)\bar{\tau}_f\bar{\lambda}_{sf}+(\bar{\tau}_s-\alpha)^2\bar{\lambda}_{ff}}{\bar{\lambda}_{ff}(\bar{\lambda}_{ss}-\bar{\lambda}_{sf}^2/\bar{\lambda}_{ff}+\frac{2}{3}\mu_{ss})}\right)\frac{\partial\theta}{\partial t}$$
$$+\theta^+\frac{(\bar{\tau}_s-\alpha)\bar{\lambda}_{ff}-\bar{\tau}_f\bar{\lambda}_{sf}}{\bar{\lambda}_{ff}(\bar{\lambda}_{ss}-\bar{\lambda}_{sf}^2/\bar{\lambda}_{ff}+\frac{2}{3}\mu_{ss})}\frac{1}{3}\frac{\partial\,\mathrm{tr}\,\mathbf{T}_{\mathrm{I}}}{\partial t} \tag{5.1.28}$$
$$=(\kappa+\theta^+(\alpha_f+\varsigma/\theta^+)((\alpha_f-\gamma)/\xi))\,\mathrm{Div}(\mathrm{GRAD}\,\theta)$$
$$-\theta^+((\alpha_f+\varsigma/\theta^+)/\xi)\varphi_f^+\,\mathrm{Div}(\mathrm{GRAD}\,P_f)$$

In the isothermal case, obtained by allowing the thermal conductivity κ to become unbounded and by imposing sufficient boundary conditions to yield the solution $\Theta(\mathbf{X},t)=\theta^+$, (5.1.27) can be written

$$\frac{\partial(P_f+\frac{B}{3}\mathrm{tr}\,\mathbf{T}_{\mathrm{I}})}{\partial t}=c\,\mathrm{Div}(\mathrm{GRAD}\,P_f) \tag{5.1.29}$$

where c is a positive number called the *compressible consolidation coefficient* defined by

$$c=\frac{1}{\xi}\frac{\bar{\lambda}_{ff}(\bar{\lambda}_{ss}+\frac{2}{3}\mu_{ss})-\bar{\lambda}_{sf}^{2}}{\bar{\lambda}_{ss}+\frac{2}{3}\mu_{ss}+2\bar{\lambda}_{sf}+\bar{\lambda}_{ff}} \tag{5.1.30}$$

and B is defined by

$$B=\frac{1}{\varphi_f^{+}}\frac{\bar{\lambda}_{ff}+\bar{\lambda}_{sf}}{\bar{\lambda}_{ss}+\frac{2}{3}\mu_{ss}+2\bar{\lambda}_{sf}+\bar{\lambda}_{ff}} \tag{5.1.31}$$

The coefficient B is usually called the Skempton coefficient [Ref, 6].

Returning to the case that is not necessarily isothermal, but still making the assumption that the inertia of the fluid and solid can be neglected, we have learned that given a stress field $\mathbf{T}_{\mathrm{I}}=\mathbf{T}_{\mathrm{I}}(\mathbf{X},t)$ which obeys (5.1.24), equations (5.1.27) and (5.1.28) can be utilized to determine the pore pressure and the temperature. Given this information, equation (5.1.25) can be used to calculate the strain field for the solid, $\mathbf{E}_s=\mathbf{E}_s(\mathbf{X},t)$. Unfortunately this strain field does not necessarily correspond to a strain derived from a displacement field $\mathbf{w}_s=\mathbf{w}_s(\mathbf{X},t)$. As can be seen from (3.2.3) and (3.2.8), given $\mathbf{E}_s=\mathbf{E}_s(\mathbf{X},t)$ the displacement field is a solution of the partial differential equation

$$2\mathbf{E}_s(\mathbf{X},t)=\mathrm{GRAD}\,\mathbf{w}_s(\mathbf{X},t)+\left(\mathrm{GRAD}\,\mathbf{w}_s(\mathbf{X},t)\right)^T \tag{5.1.32}$$

As is well known from classical elasticity, the integrability or compatibility condition on this partial differential equation is

$$\begin{aligned}&\mathrm{Div}(\mathrm{GRAD}\,\mathbf{E}_s)+\mathrm{GRAD}(\mathrm{GRAD}(\mathrm{tr}\,\mathbf{E}_s))\\&\qquad-\mathrm{GRAD}(\mathrm{Div}(\mathbf{E}_s))-(\mathrm{GRAD}(\mathrm{Div}(\mathbf{E}_s)))^T=\mathbf{0}\end{aligned} \tag{5.1.33}$$

Equation (5.1.33) is a necessary condition for the existence of a displacement field $\mathbf{w}_s=\mathbf{w}_s(\mathbf{X},t)$. A discussion of (5.1.33) within the context of classical elasticity can be found in any elasticity book. Reference 7 contains an excellent discussion of the compatibility equations of elasticity.

Equation (5.1.33) imposes additional conditions on the stress, pore pressure and temperature fields. The explicit form of these conditions is obtained by and substituting (5.1.25) into (5.1.33). The result turns out to be

$$\begin{aligned}&\operatorname{Div}\operatorname{GRAD}\left(\frac{3(\bar{\lambda}_{ss}-\bar{\lambda}_{sf}^2/\bar{\lambda}_{ff})+2\mu_{ss}}{2(\bar{\lambda}_{ss}-\bar{\lambda}_{sf}^2/\bar{\lambda}_{ff})+2\mu_{ss}}\mathbf{T}_{\mathrm{I}}-\frac{\bar{\lambda}_{ss}-\bar{\lambda}_{sf}^2/\bar{\lambda}_{ff}}{2(\bar{\lambda}_{ss}-\bar{\lambda}_{sf}^2/\bar{\lambda}_{ff})+2\mu_{ss}}(\operatorname{tr}\mathbf{T}_{\mathrm{I}})\mathbf{I}\right)\\&\quad+\operatorname{GRAD}(\operatorname{GRAD}(\operatorname{tr}\mathbf{T}_{\mathrm{I}}))\\&\quad+\frac{\mu_{ss}(1+\bar{\lambda}_{sf}/\bar{\lambda}_{ff})}{(\bar{\lambda}_{ss}-\bar{\lambda}_{sf}^2/\bar{\lambda}_{ff})+\mu_{ss}}\varphi_f^+\left((\operatorname{Div}(\operatorname{GRAD}P_f))\mathbf{I}+\operatorname{GRAD}(\operatorname{GRAD}P_f)\right)\\&\quad+\frac{\mu_{ss}(\bar{\tau}_s-\bar{\lambda}_{sf}\bar{\tau}_f/\bar{\lambda}_{ff}-\alpha)}{(\bar{\lambda}_{ss}-\bar{\lambda}_{sf}^2/\bar{\lambda}_{ff})+\mu_{ss}}\left((\operatorname{Div}(\operatorname{GRAD}\theta))\mathbf{I}+\operatorname{GRAD}(\operatorname{GRAD}\theta)\right)=\mathbf{0}\end{aligned} \tag{5.1.34}$$

where equation (5.1.24) has been used. Equation (5.1.34) is an extra condition which must be obeyed by the stress, the pore pressure and the temperature. The isothermal version of (5.1.34) is equivalent to a result first obtained by Rice & Cleary [Ref. 8].

5.2 Incompressible Models

If the binary fluid solid mixture is incompressible, then the appropriate field and constitutive equations are given in Section 4.5. The constitutive equations (4.5.1) through (4.5.4) remain valid in the case where the fluid inertia is neglected. Equations (5.1.3) and (5.1.5) are also valid with the understanding that the pore pressure, P_f, is the indeterminant pressure. These equations can be used to repeat the calculations of Section 5.1 in the incompressible case. An equivalent approach is to utilize the results of Section 4.4 to calculate the incompressible limits of formulas derived in Section 5.1. We will adopt this approach in this section.

The results we shall need from Section 4.4, specialized to the case of a mixture of a fluid and a solid are (4.4.20) (4.4.21), (4.4.23), (4.4.30, and (4.4.32). These results are

$$\frac{1}{\lambda_{ff}}\to 0 \tag{5.2.1}$$

$$\frac{\tau_f}{\lambda_{ff}}\to 0 \tag{5.2.2}$$

$$\frac{\lambda_{sf}}{\lambda_{ff}}\to\frac{\varphi_s^+}{\varphi_f^+} \tag{5.2.3}$$

$$\lambda_{ss}-\frac{\varphi_s^+}{\varphi_f^+}\lambda_{sf}\to\lambda_{ss}^* \tag{5.2.4}$$

and

$$\tau_s - \frac{\varphi_s^+}{\varphi_f^+}\tau_f \to \tau_s^* \tag{5.2.5}$$

In the manipulations below, we will also need to use the result (2.1.17) in the form

$$\varphi_s^+ + \varphi_f^+ = 1 \tag{5.2.6}$$

A result which follows from (5.2.3) and (5.2.6) is

$$1 + \frac{\lambda_{sf}}{\lambda_{ff}} \to \frac{1}{\varphi_f^+} \tag{5.2.7}$$

It is also true from (5.2.3) and (5.2.4) that

$$\lambda_{ss} - \frac{\lambda_{sf}^2}{\lambda_{ff}} = \lambda_{ss} - \frac{\lambda_{sf}}{\lambda_{ff}}\lambda_{sf} \to \lambda_{ss}^* \tag{5.2.8}$$

and from (5.2.3) and (5.2.5) that

$$\tau_s - \frac{\lambda_{sf}}{\lambda_{ff}}\tau_f \to \tau_s^* \tag{5.2.9}$$

Given (5.2.2) and (5.2.7), equation (5.1.9) reduces, in the incompressible case, to

$$\operatorname{Div}\frac{\partial \mathbf{w}_s}{\partial t} = (\varphi_f^{+2}/\xi)\operatorname{Div}(\operatorname{GRAD} P_f) - \varphi_f^+((\alpha_f - \gamma)/\xi)\operatorname{Div}(\operatorname{GRAD}\theta) \tag{5.2.10}$$

It follows from (5.2.7), (5.2.8) and (5.2.9) that the incompressible limit of (5.1.10) is

$$\begin{aligned}\rho_{s_R}\frac{\partial^2 \mathbf{w}_s}{\partial t^2} = &-\operatorname{GRAD} P_f + (\lambda_{ss}^* + \mu_{ss})\operatorname{GRAD}(\operatorname{Div}\mathbf{w}_s)\\ &+ \mu_{ss}\operatorname{Div}(\operatorname{GRAD}\mathbf{w}_s) - (\tau_s^* - \alpha)\operatorname{GRAD}\theta\end{aligned} \tag{5.2.11}$$

Likewise, it follows from (5.2.2) and (5.2.9) that the incompressible limit of (5.1.12) is

$$\begin{aligned}&c_v\frac{\partial\theta}{\partial t} + \theta^+(\tau_s^* - \alpha)\operatorname{Div}\frac{\partial \mathbf{w}_s}{\partial t}\\ &\quad = (\kappa + \theta^+(\alpha_f + \varsigma/\theta^+)(\alpha_f - \gamma)/\xi)\operatorname{Div}(\operatorname{GRAD}\theta)\\ &\quad - ((\alpha_f + \varsigma/\theta^+)/\xi)\varphi_f^+\operatorname{Div}(\operatorname{GRAD} P_f)\end{aligned} \tag{5.2.12}$$

Equations (5.2.10), (5.2.11) and (5.2.12) are the field equations one would use, along with appropriate initial and boundary data, for the determination of the solid displacement, $\mathbf{w}_s$, the pore pressure, P_f and the temperature, θ, in the case where the fluid and solid are incompressible and the inertia of the fluid is neglected.

If we make the additional assumption, as in Section 5.1, that the inertia of the solid can also be neglected, we can derive incompressible versions of equations (5.1.26) through (5.1.28). (5.1.28). First, consider equation (5.1.26). If we utilize (5.2.8), (5.2.7) and (5.2.9), the incompressible version of (5.1.26) is

$$\operatorname{tr}\mathbf{E}_s = \operatorname{Div}\mathbf{w}_s = \frac{1}{3\lambda_{ss}^* + 2\mu_{ss}}\operatorname{tr}\mathbf{T}_{\mathbf{I}} + \frac{3}{3\lambda_{ss}^* + 2\mu_{ss}}P_f + \frac{3(\tau_s^* - \alpha)}{3\lambda_{ss}^* + 2\mu_{ss}}(\theta - \theta^+) \tag{5.2.13}$$

Likewise, the incompressible version of (5.1.25) is

$$2\mu_{ss}\mathbf{E}_s = \mathbf{T}_{\mathrm{I}} - \frac{\lambda_{ss}^*}{3\lambda_{ss}^* + 2\mu_{ss}}(\operatorname{tr}\mathbf{T}_{\mathrm{I}})\mathbf{I} + \frac{2\mu_{ss}}{3\lambda_{ss}^* + 2\mu_{ss}}P_f\mathbf{I} + \frac{2\mu_{ss}(\tau_s^* - \alpha)}{3\lambda_{ss}^* + 2\mu_{ss}}(\theta - \theta^+)\mathbf{I} \tag{5.2.14}$$

In order to obtain the incompressible version of (5.1.27), we need, first, the incompressible limit of the coefficient

$$\frac{\lambda_{ss} + \frac{2}{3}\mu_{ss} + 2\lambda_{sf} + \lambda_{ff}}{\lambda_{ff}(\lambda_{ss} - \lambda_{sf}^2/\lambda_{ff} + \frac{2}{3}\mu_{ss})}$$

A rearrangement yields

$$\begin{aligned}\frac{\lambda_{ss} + \frac{2}{3}\mu_{ss} + 2\lambda_{sf} + \lambda_{ff}}{\lambda_{ff}(\lambda_{ss} - \lambda_{sf}^2/\lambda_{ff} + \frac{2}{3}\mu_{ss})} &= \frac{\lambda_{ss} - \frac{\lambda_{sf}^2}{\lambda_{ff}} + \frac{2}{3}\mu_{ss} + \lambda_{ff}(1 + \frac{\lambda_{sf}}{\lambda_{ff}})^2}{\lambda_{ff}(\lambda_{ss} - \lambda_{sf}^2/\lambda_{ff} + \frac{2}{3}\mu_{ss})} \\ &= \frac{\frac{1}{\lambda_{ff}}(\lambda_{ss} - (\frac{\lambda_{sf}}{\lambda_{ff}})\lambda_{sf}) + \frac{2}{3}\frac{\mu_{ss}}{\lambda_{ff}} + (1 + \frac{\lambda_{sf}}{\lambda_{ff}})^2}{\lambda_{ss} - \lambda_{sf}^2/\lambda_{ff} + \frac{2}{3}\mu_{ss}}\end{aligned} \tag{5.2.15}$$

As a result,

$$\frac{\lambda_{ss} + \frac{2}{3}\mu_{ss} + 2\lambda_{sf} + \lambda_{ff}}{\lambda_{ff}(\lambda_{ss} - \lambda_{sf}^2/\lambda_{ff} + \frac{2}{3}\mu_{ss})} \to \frac{1}{\varphi_f^{+2}}\frac{1}{\lambda_{ss}^* + \frac{2}{3}\mu_{ss}} \tag{5.2.16}$$

where (5.2.2), (5.2.3), (5.2.7) and (5.2.8) have been used. The coefficient of $\frac{\partial \theta}{\partial t}$ in (5.1.27) can be written

$$\begin{aligned}
\frac{(\tau_s-\alpha)(\lambda_{ff}+\lambda_{sf})-\tau_f(\lambda_{ss}+\frac{2}{3}\mu_{ss}+\lambda_{sf})}{\lambda_{ff}(\lambda_{ss}-\lambda_{sf}^2/\lambda_{ff}+\frac{2}{3}\mu_{ss})} &= \frac{(\tau_s-\alpha)(1+\frac{\lambda_{sf}}{\lambda_{ff}})-\frac{\tau_f}{\lambda_{ff}}(\lambda_{ss}+\frac{2}{3}\mu_{ss}+\lambda_{sf})}{\lambda_{ss}-\lambda_{sf}^2/\lambda_{ff}+\frac{2}{3}\mu_{ss}} \\
&= \frac{(\tau_s-\frac{\lambda_{sf}}{\lambda_{ff}}\tau_f-\alpha)(1+\frac{\lambda_{sf}}{\lambda_{ff}})-\frac{\tau_f}{\lambda_{ff}}(\lambda_{ss}-\lambda_{sf}^2/\lambda_{ff}+\frac{2}{3}\mu_{ss})}{\lambda_{ss}-\lambda_{sf}^2/\lambda_{ff}+\frac{2}{3}\mu_{ss}} \qquad (5.2.17) \\
&\to \frac{1}{\varphi_f^+}\frac{(\tau_s^*-\alpha)}{\lambda_{ss}^*+\frac{2}{3}\mu_{ss}}
\end{aligned}$$

where (5.2.2), (5.2.7) (5.2.8) and (5.2.9) have been used.

The coefficient of $\frac{1}{3}\frac{\partial\,\mathrm{tr}\,\mathbf{T}_\mathrm{I}}{\partial t}$ in (5.1.27) has the incompressible limit

$$\frac{\lambda_{ff}+\lambda_{sf}}{\lambda_{ff}(\lambda_{ss}-\lambda_{sf}^2/\lambda_{ff}+\frac{2}{3}\mu_{ss})} \to \frac{1}{\varphi_f^+}\frac{1}{\lambda_{ss}^*+\frac{2}{3}\mu_{ss}} \qquad (5.2.18)$$

Equations (5.2.16), (5.2.17) and (5.2.18) yield, from (5.1.27),

$$\begin{aligned}
&\frac{1}{\varphi_f^+(\lambda_{ss}^*+\frac{2}{3}\mu_{ss})}\frac{\partial(P_f+\frac{1}{3}\mathrm{tr}\,\mathbf{T}_\mathrm{I})}{\partial t}+\frac{\tau_s^*-\alpha}{\varphi_f^+(\lambda_{ss}^*+\frac{2}{3}\mu_{ss})}\frac{\partial\theta}{\partial t} \qquad (5.2.19) \\
&= (\varphi_f^+/\xi)\,\mathrm{Div}(\mathrm{GRAD}\,P_f)-((\alpha_f-\gamma)/\xi)\,\mathrm{Div}(\mathrm{GRAD}\,\theta)
\end{aligned}$$

Similar arguments yield, for the coefficients of (5.1.28),

$$c_{\text{v}} + \theta^{+} \frac{\tau_f^2(\lambda_{ss} + \frac{2}{3}\mu_{ss}) - 2(\tau_s - \alpha)\tau_f \lambda_{sf} + (\tau_s - \alpha)^2 \lambda_{ff}}{\lambda_{ff}(\lambda_{ss} - \lambda_{sf}^2 / \lambda_{ff} + \frac{2}{3}\mu_{ss})}$$
$$= c_{\text{v}} + \theta^{+} \frac{\tau_f^2(\lambda_{ss} - \lambda_{sf}^2/\lambda_{ff} + \frac{2}{3}\mu_{ss}) + (\tau_s - \frac{\lambda_{sf}}{\lambda_{ff}}\tau_f - \alpha)^2 \lambda_{ff}}{\lambda_{ff}(\lambda_{ss} - \lambda_{sf}^2 / \lambda_{ff} + \frac{2}{3}\mu_{ss})} \tag{5.2.20}$$
$$\rightarrow c_{\text{v}} + \theta^{+} \frac{(\tau_s^* - \alpha)^2}{(\lambda_{ss}^* + \frac{2}{3}\mu_{ss})}$$

and

$$\frac{(\tau_s - \alpha)\lambda_{ff} - \tau_f \lambda_{sf}}{\lambda_{ff}(\lambda_{ss} - \lambda_{sf}^2/\lambda_{ff} + \frac{2}{3}\mu_{ss})} \rightarrow \frac{\tau_s^* - \alpha}{\lambda_{ss}^* + \frac{2}{3}\mu_{ss})} \tag{5.2.21}$$

Therefore, the incompressible version of (5.1.28) is

$$\theta^{+} \frac{\tau_s^* - \alpha}{(\lambda_{ss}^* + \frac{2}{3}\mu_{ss})} \frac{\partial(P_f + \frac{1}{3}\operatorname{tr}\mathbf{T}_{\text{I}})}{\partial t} + \left(c_{\text{v}} + \theta^{+} \frac{(\tau_s^* - \alpha)^2}{(\lambda_{ss}^* + \frac{2}{3}\mu_{ss})} \right) \frac{\partial \theta}{\partial t}$$
$$= (\kappa + \theta^{+}(\alpha_f + \varsigma/\theta^{+})(\alpha_f - \gamma)/\xi)) \operatorname{Div}(\operatorname{GRAD}\theta) \tag{5.2.22}$$
$$-\theta^{+}((\alpha_f + \varsigma/\theta^{+})/\xi)\varphi_f^{+} \operatorname{Div}(\operatorname{GRAD} P_f)$$

In the isothermal limit, i.e. in the case where $\kappa \rightarrow \infty$ in (5.2.22) and sufficient boundary conditions are provided to yield the solution $\Theta(\mathbf{X}, t) = \theta^{+}$, equation (5.2.19) reduces to the incompressible limit of (5.1.29), i.e.

$$\frac{\partial(P_f + \frac{1}{3}\operatorname{tr}\mathbf{T}_{\text{I}})}{\partial t} = c^* \operatorname{Div}(\operatorname{GRAD} P_f) \tag{5.2.23}$$

where c^* is the *incompressible consolidation coefficient*. This coefficient is the incompressible limit of c in (5.1.30) and has the explicit form

$$c^* = \frac{\varphi_f^{+2}}{\xi}(\lambda_{ss}^* + \frac{2}{3}\mu_{ss}) \tag{5.2.24}$$

Notice, in passing, that in the incompressible limit, B, defined by (5.1.31), takes the value

$$B = 1 \tag{5.2.25}$$

Returning to the case that is not necessarily isothermal, but still making the assumption that the inertia of the fluid and solid can be neglected, we have learned that given a stress field $\mathbf{T}_{\mathrm{I}} = \mathbf{T}_{\mathrm{I}}(\mathbf{X},t)$ which obeys (5.1.24), equations (5.2.19) and (5.2.22) can be utilized to determine the pore pressure and the temperature. As in Section 5.1, this information can be used to calculate the strain field for the solid $\mathbf{E}_s(\mathbf{X},t)$. Also, as in Section 5.1, this strain field does not necessarily correspond to a strain derived from a displacement field $\mathbf{w}_s = \mathbf{w}_s(\mathbf{X},t)$. The additional condition one must place on the stress, pore pressure and temperature fields is the incompressible version of (5.1.34). This result is

$$\begin{aligned}
&\operatorname{Div}\operatorname{GRAD}\left(\frac{3\lambda_{ss}^{*}+2\mu_{ss}}{2\lambda_{ss}^{*}+2\mu_{ss}}\mathbf{T}_{\mathrm{I}}-\frac{\lambda_{ss}^{*}}{2\lambda_{ss}^{*}+2\mu_{ss}}(\operatorname{tr}\mathbf{T}_{\mathrm{I}})\mathbf{I}\right)\\
&\quad+\operatorname{GRAD}(\operatorname{GRAD}(\operatorname{tr}\mathbf{T}_{\mathrm{I}}))\\
&\quad+\frac{\mu_{ss}}{\lambda_{ss}^{*}+\mu_{ss}}\Big((\operatorname{Div}(\operatorname{GRAD}P_f))\mathbf{I}+\operatorname{GRAD}(\operatorname{GRAD}P_f)\Big)\\
&\quad+\frac{\mu_{ss}(\tau_s^{*}-\alpha)}{\lambda_{ss}^{*}+\mu_{ss}}\Big((\operatorname{Div}(\operatorname{GRAD}\theta))\mathbf{I}+\operatorname{GRAD}(\operatorname{GRAD}\theta)\Big)=\mathbf{0}
\end{aligned} \tag{5.2.26}$$

For incompressible materials, the case where both the fluid and solid inertias can be neglected, there is another convenient form of the governing partial differential equations. These equations involve the elimination of the pore pressure and the stress in favor of the quantity $\operatorname{Div}\mathbf{w}_s$. This elimination is possible because the pore pressure and the stress enter (5.2.19) and (5.2.22) in the combination $P_f + \frac{1}{3}\operatorname{tr}\mathbf{T}_{\mathrm{I}}$. The manipulations leading to the desired results are somewhat simplified if we, first, solve (5.2.13) for $P_f + \frac{1}{3}\operatorname{tr}\mathbf{T}_{\mathrm{I}}$ to obtain

$$P_f + \frac{1}{3}\operatorname{tr}\mathbf{T}_{\mathrm{I}} = (\lambda_{ss}^{*} + \frac{2}{3}\mu_{ss})\operatorname{Div}\mathbf{w}_s - (\tau_s^{*}-\alpha)(\theta-\theta^{+}) \tag{5.2.27}$$

Given the fact that we are neglecting the inertia of the fluid and the solid, it follows from (5.2.11) that

$$\operatorname{GRAD}P_f = (\lambda_{ss}^{*}+\mu_{ss})\operatorname{GRAD}(\operatorname{Div}\mathbf{w}_s) + \mu_{ss}\operatorname{Div}(\operatorname{GRAD}\mathbf{w}_s) - (\tau_s^{*}-\alpha_s)\operatorname{GRAD}\theta \tag{5.2.28}$$

The divergence of this equation yields

$$\operatorname{Div}(\operatorname{GRAD}P_f) = (\lambda_{ss}^{*}+2\mu_{ss})\operatorname{Div}(\operatorname{GRAD}(\operatorname{Div}\mathbf{w}_s)) - (\tau_s^{*}-\alpha)\operatorname{Div}(\operatorname{GRAD}\theta) \tag{5.2.29}$$

The two results we seek are obtained by utilizing (5.2.27) and (5.2.29) to eliminate $P_f + \frac{1}{3}\operatorname{tr}\mathbf{T}_{\mathrm{I}}$ and $\operatorname{Div}(\operatorname{GRAD} P_f)$, respectively, from (5.2.19) and (5.2.22). The results of this manipulation are

$$\begin{aligned}\frac{\partial \operatorname{Div}\mathbf{w}_s}{\partial t} = \frac{\varphi_f^{+2}}{\xi}(\lambda_{ss}^* + 2\mu_{ss})\operatorname{Div}(\operatorname{GRAD}(\operatorname{Div}\mathbf{w}_s))\\ -\frac{\varphi_f^+}{\xi}(\varphi_f^+(\tau_s^* - \alpha) + (\alpha_f - \gamma))\operatorname{Div}(\operatorname{GRAD}\theta)\end{aligned} \tag{5.2.30}$$

and

$$\begin{aligned}&\theta^+(\tau_s^* - \alpha)\frac{\partial \operatorname{Div}\mathbf{w}_s}{\partial t} + c_{\mathrm{v}}\frac{\partial \theta}{\partial t}\\ &= \left(\kappa + \theta^+\left(\alpha_f + \varsigma/\theta^+\right)\left(\varphi_f^+(\tau_s^* - \alpha) + (\alpha_f - \gamma)/\xi\right)\right)\operatorname{Div}(\operatorname{GRAD}\theta)\\ &\quad -\theta^+(\varphi_f^+/\xi)\left(\alpha_f + \varsigma/\theta^+\right)\left(\lambda_{ss}^* + 2\mu_{ss}\right)\operatorname{Div}(\operatorname{GRAD}(\operatorname{Div}\mathbf{w}_s))\end{aligned} \tag{5.2.31}$$

An interesting conclusion from (5.2.30) is that in the isothermal case, $\operatorname{Div}\mathbf{w}_s$ obeys a classical diffusion equation.

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6

Further Transformations and Material Properties

In spite of the many forms of the governing equations presented in Chapters 4 and 5, there are still further forms which we need to record. These expressions will be given in this Chapter. In addition, an effort will be made to make connections with notations and formalisms one can find in other, very important, references on the subject of porous elasticity. In this chapter, we continue to restrict the discussion to that of a binary mixture of a compressible fluid and a compressible solid with pore pressure. When we discuss the case of an incompressible mixture, we shall follow the technique of Section 5.2 where the incompressible case is derived as a limit of the compressible case.

6.1 Constitutive Equations-Alternate Forms

If we continue to consider a binary mixture of a compressible fluid and a compressible solid with pore pressure, then the following four equations summarize the constitutive equations for the pore pressure, the entropy, the chemical potential for the solid, the inner part of the stress and the volume of the fluid per unit volume of undeformed solid.

$$\varphi_f^+ P_f = \bar{\tau}_f(\theta-\theta^+) - (\bar{\lambda}_{sf}+\bar{\lambda}_{ff})(\operatorname{tr}\mathbf{E}_s) + \bar{\lambda}_{ff}\,(m_b - m_b^+)/m_b^+ \tag{6.1.1}$$

$$\rho\eta = -\alpha + \frac{\bar{c}_v}{\theta^+}(\theta-\theta^+) + (\bar{\tau}_s+\bar{\tau}_f)(\operatorname{tr}\mathbf{E}_s) - \bar{\tau}_f\,(m_f - m_f^+)/m_f^+ \tag{6.1.2}$$

$$\rho_{s_R}\mathbf{K}_s = -(\bar{\lambda}_{ss}+\bar{\lambda}_{sf})(\operatorname{tr}\mathbf{E}_s)\mathbf{I} - 2\mu_{ss}\mathbf{E}_s + \bar{\lambda}_{sf}((m_f - m_f^+)/m_f^+)\mathbf{I} + \bar{\tau}_s(\theta-\theta^+)\mathbf{I} \tag{6.1.3}$$

$$\begin{aligned}\mathbf{T}_{\mathrm{I}} = {} & (\bar{\lambda}_{ss}+2\bar{\lambda}_{sf}+\bar{\lambda}_{ff})(\operatorname{tr}\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s - (\bar{\lambda}_{sf}+\bar{\lambda}_{ff})((m_f - m_f^+)/m_f^+)\mathbf{I} \\ & -(\bar{\tau}_s+\bar{\tau}_f-\alpha)(\theta-\theta^+)\mathbf{I}\end{aligned} \tag{6.1.4}$$

and

$$\frac{\upsilon_f-\upsilon_f^+}{\upsilon_f^+} = \left(\frac{\bar{\lambda}_{sf}+\bar{\lambda}_{ff}}{\varphi_f^+K_f}\right)\operatorname{tr}\mathbf{E}_s + \left(1-\frac{\bar{\lambda}_{ff}}{\varphi_f^+K_f}\right)(m_f - m_f^+)/m_f^+ - \frac{\bar{\tau}_f}{\varphi_f^+K_f}(\theta-\theta^+) \tag{6.1.5}$$

Equation (6.1.1) through (6.1.4) are binary versions of (4.1.49) through (4.1.52), respectively. Equation (6.1.5) is the binary version of equation (4.1.54).

In Section 4.1, we introduced the thermodynamic potential $W(\theta,\mathbf{E}_s,m_b)$ and showed that it was given by equation (4.1.56). The binary mixture version of (4.1.56) is

$$\begin{aligned} W(\theta,\mathbf{E}_s,m_b) &= \alpha(\theta-\theta^+) - \frac{1}{2}\frac{\bar{c}_{\text{v}}}{\theta^+}(\theta-\theta^+)^2 - (\bar{\tau}_s+\bar{\tau}_f-\alpha)(\theta-\theta^+)(\text{tr}\,\mathbf{E}_s) \\ &+\bar{\tau}_f(\theta-\theta^+)(m_f-m_f^+)/m_f^+ + \frac{1}{2}(\bar{\lambda}_{ss}+2\bar{\lambda}_{sf}+\bar{\lambda}_{ff})(\text{tr}\,\mathbf{E}_s)^2 \\ &-\left(\bar{\lambda}_{sf}+2\bar{\lambda}_{ff}\right)(\text{tr}\,\mathbf{E}_s)\left((m_f-m_f^+)/m_f^+\right) \\ &+\frac{1}{2}\bar{\lambda}_{ff}\left((m_f-m_f^+)/m_f^+\right)\left((m_f-m_f^+)/m_f^+\right)+\mu_{ss}\,\text{tr}(\mathbf{E}_s\mathbf{E}_s) \end{aligned} \tag{6.1.6}$$

This result, in turn, yields the binary mixture versions of (4.1.57), (4.1.58) and (4.1.59). These results are

$$\varphi_f^+P_f = \frac{\partial W}{\partial m_f} = \bar{\tau}_f(\theta-\theta^+) - \left(\bar{\lambda}_{sf}+\bar{\lambda}_{ff}\right)(\text{tr}\,\mathbf{E}_s) + \bar{\lambda}_{ff}(m_f-m_f^+)/m_f^+ \tag{6.1.7}$$

$$\begin{aligned} \mathbf{T}_{\text{I}} = \frac{\partial W}{\partial \mathbf{E}_s} &= (\bar{\lambda}_{ss}+2\bar{\lambda}_{sf}+\bar{\lambda}_{ff})(\text{tr}\,\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s - (\bar{\lambda}_{sf}+\bar{\lambda}_{ff})(m_f-m_f^+)/m_f^+\,\mathbf{I} \\ &-(\bar{\tau}_s+\bar{\tau}_f-\alpha)(\theta-\theta^+)\mathbf{I} \end{aligned} \tag{6.1.8}$$

and

$$\text{H} = -\frac{\partial W}{\partial \theta} = -\alpha + \frac{\bar{c}_{\text{v}}}{\theta^+}(\theta-\theta^+) + (\bar{\tau}_s+\bar{\tau}_f-\alpha)(\text{tr}\,\mathbf{E}_s) - \bar{\tau}_f(m_f-m_f^+)/m_f^+ \tag{6.1.9}$$

In the following, the entropy of the mixture per unit of undeformed solid volume, defined by (4.1.59), will be used as the constitutive equation for the entropy.

In several cases in Chapter 4 we wrote the constitutive equations with temperature, the strain of the solid and the fluid pore pressure as the independent variables. The results with this choice of independent variables are as follows:

$$(m_f-m_f^+)/m_f^+ = \frac{1}{\bar{\lambda}_{ff}}\varphi_f^+P_f - \frac{\bar{\tau}_f}{\bar{\lambda}_{ff}}(\theta-\theta^+) + (1+\frac{\bar{\lambda}_{sf}}{\bar{\lambda}_{ff}})(\text{tr}\,\mathbf{E}_s) \tag{6.1.10}$$

$$\text{H} = -\alpha + (\bar{c}_{\text{v}} + \theta^+\frac{\bar{\tau}_f^2}{\bar{\lambda}_{ff}})(\theta-\theta^+)/\theta^+ + (\bar{\tau}_s - \frac{\bar{\lambda}_{sf}}{\bar{\lambda}_{ff}}\bar{\tau}_f - \alpha)(\text{tr}\,\mathbf{E}_s) - \frac{\bar{\tau}_f}{\bar{\lambda}_{ff}}\varphi_f^+P_f \tag{6.1.11}$$

$$\mathbf{T}_{\text{I}} = (\bar{\lambda}_{ss} - \frac{\bar{\lambda}_{sf}^2}{\bar{\lambda}_{ff}})(\text{tr}\,\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s - (1+\frac{\bar{\lambda}_{sf}}{\bar{\lambda}_{ff}})\varphi_f^+P_f\mathbf{I} - (\bar{\tau}_s - \frac{\bar{\lambda}_{sf}}{\bar{\lambda}_{ff}}\tau_c - \alpha)(\theta-\theta^+)\mathbf{I} \tag{6.1.12}$$

and

$$(\upsilon_f - \upsilon_f^+)\big/\upsilon_f^+ = (1+\frac{\bar{\lambda}_{sc}}{\bar{\lambda}_{ff}})(\operatorname{tr}\mathbf{E}_s) + \frac{1}{\bar{\lambda}_{ff}}(1-\frac{\bar{\lambda}_{ff}}{\varphi_f^+ K_f})\varphi_f^+ P_f - \frac{\bar{\tau}_f}{\bar{\lambda}_{ff}}(\theta-\theta^+) \tag{6.1.13}$$

Equation (6.1.10) follows from equation (6.1.7), equations (6.1.11) and (6.1.12) are the result of using (6.1.10) to eliminate the quantity $(m_f - m_f^+)\big/m_f^+$ from (6.1.9) and (6.1.8). Equation (6.1.13) follows by the same elimination applied to (6.1.5).

In Sections 4.1 and 5.1 we found it convenient to eliminate the strain of the solid in favor of the inner part of the stress. Next, we shall record versions of the two sets of constitutive equations above where this elimination has been performed. First, we can solve (6.1.8) to obtain

$$\begin{aligned}2\mu_{ss}\mathbf{E}_s = \mathbf{T}_{\mathrm{I}} &- \frac{\bar{\lambda}_{ss} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ff}}{\bar{\lambda}_{ss} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ff} + \frac{2}{3}\mu_{ss}}\frac{1}{3}(\operatorname{tr}\mathbf{T}_{\mathrm{I}})\mathbf{I} \\ &+ \frac{2\mu_{ss}}{3}\frac{(\bar{\lambda}_{ff} + \bar{\lambda}_{sf})}{\bar{\lambda}_{ss} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ff} + \frac{2}{3}\mu_{ss}}(m_f - m_f^+)\big/m_f^+\,\mathbf{I} \\ &+ \frac{2\mu_{ss}}{3}\frac{(\bar{\tau}_s + \bar{\tau}_f - \alpha)}{\bar{\lambda}_{ss} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ff} + \frac{2}{3}\mu_{ss}}(\theta-\theta^+)\mathbf{I}\end{aligned} \tag{6.1.14}$$

This expression can be used to eliminate $\mathbf{E}_s$ in favor of $\mathbf{T}_{\mathrm{I}}$ in equations (6.1.7), (6.1.8) and (6.1.5). The results turn out to be

$$\begin{aligned}\varphi_f^+ P_f = &-\frac{(\bar{\lambda}_{ff} + \bar{\lambda}_{sf})}{\bar{\lambda}_{ss} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ff} + \frac{2}{3}\mu_{ss}}\frac{1}{3}\operatorname{tr}\mathbf{T}_{\mathrm{I}} + \bar{\lambda}_{ff}\frac{\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2\big/\bar{\lambda}_{ff} + \frac{2}{3}\mu_{ss}}{\bar{\lambda}_{ss} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ff} + \frac{2}{3}\mu_{ss}}(m_b - m_b^+)/m_b^+ \\ &+\frac{\bar{\tau}_f(\bar{\lambda}_{ss} + \bar{\lambda}_{sf} + \frac{2}{3}\mu_{ss}) - (\bar{\lambda}_{ff} + \bar{\lambda}_{sf})(\bar{\tau}_s - \alpha)}{\bar{\lambda}_{ss} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ff} + \frac{2}{3}\mu_{ss}}(\theta-\theta^+)\end{aligned} \tag{6.1.15}$$

$$\mathrm{H} = -\alpha + \left(\overline{c}_{\mathrm{v}} + \theta^{+} \frac{(\overline{\tau}_s + \overline{\tau}_f - \alpha)^2}{\overline{\lambda}_{ss} + 2\overline{\lambda}_{sf} + \overline{\lambda}_{ff} + \frac{2}{3}\mu_{ss}} \right) \frac{(\theta - \theta^{+})}{\theta^{+}} + \frac{(\overline{\tau}_s + \overline{\tau}_f - \alpha)}{\overline{\lambda}_{ss} + 2\overline{\lambda}_{sf} + \overline{\lambda}_{ff} + \frac{2}{3}\mu_{ss}} \frac{1}{3} \operatorname{tr} \mathbf{T}_{\mathrm{I}}$$
$$+ \frac{(\overline{\tau}_s - \alpha)(\overline{\lambda}_{sf} + \overline{\lambda}_{ff}) - \overline{\tau}_f (\overline{\lambda}_{ss} + \overline{\lambda}_{sf} + \frac{2}{3}\mu_{ss})}{\overline{\lambda}_{ss} + 2\overline{\lambda}_{sf} + \overline{\lambda}_{ff} + \frac{2}{3}\mu_{ss}} (m_f - m_f^{+}) \big/ m_f^{+} \tag{6.1.16}$$

and

$$\frac{\upsilon_f - \upsilon_f^{+}}{\upsilon_f^{+}} = \frac{\overline{\lambda}_{sf} + \overline{\lambda}_{ff}}{\varphi_f^{+} K_f} \frac{1}{\overline{\lambda}_{ss} + 2\overline{\lambda}_{sf} + \overline{\lambda}_{ff} + \frac{2}{3}\mu_{ss}} \frac{1}{3} \operatorname{tr} \mathbf{T}_{\mathrm{I}}$$
$$+ \left(\frac{1}{\varphi_f^{+} K_f} \frac{(\overline{\lambda}_{sf} + \overline{\lambda}_{ff})^2}{\overline{\lambda}_{ss} + 2\overline{\lambda}_{sf} + \overline{\lambda}_{ff} + \frac{2}{3}\mu_{ss}} + \left(1 - \frac{\overline{\lambda}_{ff}}{\varphi_f^{+} K_f} \right) \right) (m_f - m_f^{+}) \big/ m_f^{+} \tag{6.1.17}$$
$$+ \frac{1}{\varphi_f^{+} K_f} \frac{(\overline{\lambda}_{sf} + \overline{\lambda}_{ff})(\overline{\tau}_s - \alpha) - \overline{\tau}_f (\overline{\lambda}_{ss} + \overline{\lambda}_{sf} + \frac{2}{3}\mu_{ss})}{\overline{\lambda}_{ss} + 2\overline{\lambda}_{sf} + \overline{\lambda}_{ff} + \frac{2}{3}\mu_{ss}} (\theta - \theta^{+})$$

A similar set of calculations starting with (6.1.10) yields

$$2\mu_{ss} \mathbf{E}_s = \mathbf{T}_{\mathrm{I}} - \frac{\overline{\lambda}_{ss} - \overline{\lambda}_{sf}^2 \big/ \overline{\lambda}_{ff}}{\overline{\lambda}_{ss} - \overline{\lambda}_{sf}^2 \big/ \overline{\lambda}_{ff} + \frac{2}{3}\mu_{ss}} \frac{1}{3} (\operatorname{tr} \mathbf{T}_{\mathrm{I}}) \mathbf{I} + \frac{2\mu_{ss}(1 + \overline{\lambda}_{sf} \big/ \overline{\lambda}_{ff})}{3(\overline{\lambda}_{ss} - \overline{\lambda}_{sf}^2 \big/ \overline{\lambda}_{ff}) + 2\mu_{ss}} \varphi_f^{+} P_f \mathbf{I}$$
$$+ \frac{2\mu_{ss}(\overline{\tau}_s - \overline{\lambda}_{sf} \overline{\tau}_f \big/ \overline{\lambda}_{ff} - \alpha)}{3(\overline{\lambda}_{ss} - \overline{\lambda}_{sf}^2 \big/ \overline{\lambda}_{ff}) + 2\mu_{ss}} (\theta - \theta^{+}) \mathbf{I} \tag{6.1.18}$$

$$(m_f - m_f^{+}) \big/ m_f^{+} = \frac{(1 + \overline{\lambda}_{sf} \big/ \overline{\lambda}_{ff})}{\overline{\lambda}_{ss} - \overline{\lambda}_{sf}^2 \big/ \overline{\lambda}_{ff} + \frac{2}{3}\mu_{ss}} \frac{1}{3} \operatorname{tr} \mathbf{T}_{\mathrm{I}} + \frac{\overline{\lambda}_{ss} + 2\overline{\lambda}_{sf} + \overline{\lambda}_{ff} + \frac{2}{3}\mu_{ss}}{\overline{\lambda}_{ff} (\overline{\lambda}_{ss} - \overline{\lambda}_{sf}^2 \big/ \overline{\lambda}_{ff} + \frac{2}{3}\mu_{ss})} \varphi_f^{+} P_f$$
$$+ \frac{(\overline{\lambda}_{ff} + \overline{\lambda}_{sf})(\tau_s - \alpha) - (\overline{\lambda}_{ss} + \overline{\lambda}_{sf} + \frac{2}{3}\mu_{ss})\tau_f}{\overline{\lambda}_{ff} (\overline{\lambda}_{ss} - \overline{\lambda}_{sf}^2 \big/ \overline{\lambda}_{ff} + \frac{2}{3}\mu_{ss})} (\theta - \theta^{+}) \tag{6.1.19}$$

$$\mathrm{H} = -\alpha + \left(\bar{c}_{\mathrm{v}} + \theta^{+} \frac{\bar{\tau}_f^2((\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff}) + \frac{2}{3}\mu_{ss}) + (\bar{\tau}_s - \frac{\bar{\lambda}_{sf}}{\bar{\lambda}_{ff}}\bar{\tau}_f - \alpha)^2 \bar{\lambda}_{ff}}{\bar{\lambda}_{ff}\left((\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff}) + \frac{2}{3}\mu_{ss} \right)} \right) (\theta - \theta^{+}) / \theta^{+}$$
$$+ \frac{(\bar{\tau}_s - \frac{\bar{\lambda}_{sf}}{\bar{\lambda}_{ff}}\bar{\tau}_f - \alpha)}{(\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff}) + \frac{2}{3}\mu_{ss}} \frac{1}{3}\operatorname{tr}\mathbf{T}_{\mathbf{I}} + \frac{(\bar{\tau}_s - \alpha)(\bar{\lambda}_{ff} + \bar{\lambda}_{sf}) - \bar{\tau}_f \left(\bar{\lambda}_{ss} + \bar{\lambda}_{sf} + \frac{2}{3}\mu_{ss} \right)}{\bar{\lambda}_{ff}\left((\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff}) + \frac{2}{3}\mu_{ss} \right)} \varphi_f^{+} P_f \tag{6.1.20}$$

and

$$(\upsilon_f - \upsilon_f^{+}) / \upsilon_f^{+} = \frac{\left(1 + \bar{\lambda}_{sf} / \bar{\lambda}_{ff}\right)}{\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff} + \frac{2}{3}\mu_{ss}} \frac{1}{3}\operatorname{tr}\mathbf{T}_{\mathbf{I}} + \left(\frac{\bar{\lambda}_{ss} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ff} + \frac{2}{3}\mu_{ss}}{\bar{\lambda}_{ff}((\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff}) + \frac{2}{3}\mu_{ss})} - \frac{1}{\varphi_f^{+} K_f} \right) \varphi_f^{+} P_f$$
$$+ \frac{\left(\bar{\lambda}_{ff} + \bar{\lambda}_{sf}\right)(\bar{\tau}_s - \alpha) - \bar{\tau}_f (\lambda_{ss} + \bar{\lambda}_{sf} + \frac{2}{3}\mu_{ss})}{\bar{\lambda}_{ff}(\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff} + \frac{2}{3}\mu_{ss})} (\theta - \theta^{+}) \tag{6.1.21}$$

Equation (6.1.18) has been given earlier as (5.1.25)

6.2 Connections With Other Formulations

The literature on porous elasticity is quite old and there is a large body of literature available to study. It is important to be able to make connections between the formulation given here, which is built on the formalism of the Theory of Mixtures, with these earlier works. In this section we shall relate the notation we have been using to that found in the pioneering papers of Biot and an important formulation by Rice and Cleary.

The first Biot formulation was published in 1941 in Reference 1. This work was an extension and improvement of an earlier formulation of Terzaghi [Ref. 2]. In the notation adopted here, Biot formulated an isothermal porous elasticity model and proposed constitutive equations as follows:

$$\mathbf{E}_s = \frac{1 + \nu_p}{E_p}\mathbf{T}_{\mathrm{I}} - \frac{\nu_p}{E_p}\left(\operatorname{tr}\mathbf{T}_{\mathrm{I}}\right)\mathbf{I} + \frac{1}{3H}P_f\mathbf{I} \tag{6.2.1}$$

and

$$\upsilon_f - \upsilon_f^{+} = \frac{1}{3H}\operatorname{tr}\mathbf{T}_{\mathrm{I}} + \frac{1}{R}P_f \tag{6.2.2}$$

where ν_p, E_p, H and R are material constants. In analogy with classical elasticity, the coefficient ν_p can be identified as the (isothermal) *Poisson ratio at constant pore pressure*. Likewise, E_p is the (isothermal) *Young's modulus at constant pore pressure*. If we compare (6.2.1) and (6.2.2) with isothermal versions of (6.1.18) and (6.1.21), respectively, we can relate the material coefficients introduced by Biot to those used here. The results of this comparison are

$$\nu_P = \frac{\bar{\lambda}_{ss} - \dfrac{\bar{\lambda}_{sf}^2}{\bar{\lambda}_{ff}}}{2(\bar{\lambda}_{ss} - \dfrac{\bar{\lambda}_{sf}^2}{\bar{\lambda}_{ff}} + \mu_{ss})} \tag{6.2.3}$$

$$E_P = \frac{3\mu_{ss}(\bar{\lambda}_{ss} - \dfrac{\bar{\lambda}_{sf}^2}{\bar{\lambda}_{ff}} + \dfrac{2}{3}\mu_{ss})}{\bar{\lambda}_{ss} - \dfrac{\bar{\lambda}_{sf}^2}{\bar{\lambda}_{ff}} + \mu_{ss}} \tag{6.2.4}$$

$$\varphi_f^+ H = \frac{\bar{\lambda}_{ss} - \dfrac{\bar{\lambda}_{sf}^2}{\bar{\lambda}_{ff}} + \dfrac{2}{3}\mu_{ss}}{1 + \dfrac{\bar{\lambda}_{sf}}{\bar{\lambda}_{ff}}} \tag{6.2.5}$$

and

$$\frac{1}{R} = \varphi_f^+ \frac{\bar{\lambda}_{ss} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ff} + \dfrac{2}{3}\mu_{ss}}{\bar{\lambda}_{ff}((\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 / \bar{\lambda}_{ff}) + \dfrac{2}{3}\mu_{ss})} - \frac{1}{K_f} \tag{6.2.6}$$

In an isothermal theory, the only other material property is the drag coefficient ξ. This coefficient is related to the permeability k and the viscosity of the fluid μ_f by the formula

$$\frac{k}{\mu_f} = \frac{\varphi_f^{+2}}{\xi} \Rightarrow \xi = \varphi_f^{+2}\frac{\mu_f}{k} \tag{6.2.7}$$

Equations like (6.2.7) have appeared, earlier, in Section 2.7.

In 1976, Rice and Cleary published an important paper where the Biot model defined by (6.2.1) and (6.2.2) was adopted [Ref. 3]. They redefined certain of the material constants so as to illustrate various points of importance. Rather than the material constants H and R, they introduced constants K_s' and K_s'', where

$$\frac{1}{K_s'} = \frac{1}{K} - \frac{1}{H} \tag{6.2.8}$$

and

$$\frac{\varphi_f^+}{K_s''} = \frac{1}{H} - \frac{1}{R} \tag{6.2.9}$$

The symbol K in (6.2.8) is, in the notation used here, defined by

$$K = \bar{\lambda}_{ss} - \frac{\bar{\lambda}_{sf}^2}{\bar{\lambda}_{ff}} + \frac{2}{3}\mu_{ss} \tag{6.2.10}$$

The symbols H and R are given by (6.2.5) and (6.2.6), respectively. Given the definitions (6.2.8) through (6.2.10) and the results (6.2.5) and (6.2.6), it follows that

$$\frac{1}{K_s'} = \frac{1}{K} - \frac{1}{H} = \frac{1 - \varphi_f^+\left(1 + \frac{\bar{\lambda}_{sf}}{\bar{\lambda}_{ff}}\right)}{\bar{\lambda}_{ss} - \frac{\bar{\lambda}_{sf}^2}{\bar{\lambda}_{ff}} + \frac{2}{3}\mu_{ss}} \tag{6.2.11}$$

and

$$\frac{1}{K_s''} = \frac{1}{\varphi_f^+ H} - \frac{1}{\varphi_f^+ R} = \frac{\bar{\lambda}_{ff}\left(1 + \frac{\bar{\lambda}_{sf}}{\bar{\lambda}_{ff}}\right) - \varphi_f^+\left(\bar{\lambda}_{ss} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ff} + \frac{2}{3}\mu_{ss}\right)}{\bar{\lambda}_{ff}((\bar{\lambda}_{ss} - \bar{\lambda}_{sf}^2 \big/ \bar{\lambda}_{ff}) + \frac{2}{3}\mu_{ss}} - \frac{1}{K_f} \tag{6.2.12}$$

Rice and Cleary wrote the constitutive equations (6.2.1) and (6.2.2) in the forms

$$2G\mathbf{E}_s = \mathbf{T}_{\mathrm{I}} + P_f\mathbf{I} - \frac{\nu_p}{1+\nu_p}\left(\operatorname{tr}\mathbf{T}_{\mathrm{I}} + 3P_f\right)\mathbf{I} + \frac{2G}{3}\left(\frac{1}{H} - \frac{1}{K}\right)P_f\mathbf{I} \tag{6.2.13}$$

and

$$\upsilon_f - \upsilon_f^+ = \frac{1}{3H}(\operatorname{tr}\mathbf{T}_{\mathrm{I}} + 3P_f) - \frac{\varphi_f^+}{K_s''}P_f \tag{6.2.14}$$

where their symbol G is what we have called μ_{ss}. They also wrote their constitutive equations in the form

$$2G\mathbf{E}_s = \mathbf{T}_{\mathrm{I}} - \frac{\nu_p}{1+\nu_p}\left(\operatorname{tr}\mathbf{T}_{\mathrm{I}}\right)\mathbf{I} + \frac{3(\nu_m - \nu_p)}{B(1+\nu_p)(1+\nu_m)} P_f \mathbf{I} \tag{6.2.15}$$

and

$$(m_f - m_f^+)\big/ m_f^+ = \frac{3(\nu_m - \nu_p)}{2GB(1+\nu_m)(1+\nu_p)}\left(\operatorname{tr}\mathbf{T}_{\mathrm{I}} + \frac{3}{B} P_f\right) \tag{6.2.16}$$

where ν_m is the (isothermal) Poisson's ratio at constant m_f. It is defined in terms of the symbols used here by the formula

$$\nu_m = \frac{\bar{\lambda}_{ss} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ff}}{2(\bar{\lambda}_{ss} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ff} + \mu_{ss})} \tag{6.2.17}$$

The symbol B in (6.2.16) is the Skempton coefficient defined by (5.1.31).[1] Equations (6.2.15) and (6.2.16) are easily shown to be the isothermal versions of (6.1.18) and (6.1.19), respectively.

Biot and Willis, wrote their constitutive equations in yet a different form [Ref. 4]. Their formulation adopted a notation used earlier by Biot [Ref. 5]. In any case, they wrote their isothermal constitutive equations as

$$\mathbf{T}_{\mathrm{I}} + \varphi_f^+ P_f \mathbf{I} = A(\operatorname{tr}\mathbf{E}_s) + 2N\mathbf{E}_s - Q((\rho_f - \rho_f^+)\big/\rho_f^+)\mathbf{I} \tag{6.2.18}$$

and

$$-\varphi_f^+ P_f = Q \operatorname{tr}\mathbf{E}_s - R((\rho_f - \rho_f^+)\big/\rho_f^+) \tag{6.2.19}$$

The reader is cautioned that the property R in (6.2.19) is not he same as the R introduced in (6.2.2). In any case, if (6.2.18) and (6.2.19) are compared with the isothermal versions of (4.2.9) and (4.2.5), respectively, we see that

$$A = \bar{\lambda}_{ss} \tag{6.2.20}$$

$$N = \mu_{ss} \tag{6.2.21}$$

$$Q = \bar{\lambda}_{sf} \tag{6.2.22}$$

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[1] Rice and Cleary actually used the symbol ν for what we are calling ν_p and the symbol ν_u for what we are calling ν_m.

and

$$R = \bar{\lambda}_{ff} \tag{6.2.23}$$

A fundamental part of reference 4 is to relate the four constants A, N, Q and R to four constants the authors label μ, κ, δ and γ. The advantage of the constants μ, κ, δ and γ, as explained by the authors, is that they are measurable coefficients. In terms of the notation being used here, the coefficients μ, κ, δ and γ are given by

$$\mu = \mu_{ss} \tag{6.2.24}$$

$$\kappa = \frac{1}{\bar{\lambda}_{ss} - \dfrac{\bar{\lambda}_{sf}^2}{\bar{\lambda}_{ff}} + \dfrac{2}{3}\mu_{ss}} \tag{6.2.25}$$

$$\delta = \frac{1 - \varphi_f^+ \left(1 + \dfrac{\bar{\lambda}_{sf}}{\bar{\lambda}_{ff}}\right)}{\bar{\lambda}_{ss} - \dfrac{\bar{\lambda}_{sf}^2}{\bar{\lambda}_{ff}} + \dfrac{2}{3}\mu_{ss}} \tag{6.2.26}$$

and

$$\gamma = \varphi_f^{+2} \frac{\bar{\lambda}_{ss} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ff} + \dfrac{2}{3}\mu_{ss}}{\bar{\lambda}_{ff}\left(\bar{\lambda}_{ss} - \dfrac{\bar{\lambda}_{sf}^2}{\bar{\lambda}_{ff}} + \dfrac{2}{3}\mu_{ss}\right)} - \varphi_f^+ \frac{1 + \dfrac{\bar{\lambda}_{sf}}{\bar{\lambda}_{ff}}}{\bar{\lambda}_{ss} - \dfrac{\bar{\lambda}_{sf}^2}{\bar{\lambda}_{ff}} + \dfrac{2}{3}\mu_{ss}} \tag{6.2.27}$$

It follows from (6.2.11) and (6.2.26) that

$$\delta = \frac{1}{K_s'} \tag{6.2.28}$$

and

$$\gamma = -\varphi_f^+ \left(\frac{1}{K_s''} + \frac{1}{K_f}\right) \tag{6.2.29}$$

Rice and Cleary have summarized material properties for six rock like porous materials [Ref. 3]. Their Table 1 contains the following numerical values

Table 1 Typical Rock Properties

Property	Ruhr Sandstone	Tennessee Marble	Charcoal Granite	Berea Sandstone	Westerly Granite	Weber Sandstone
μ_{ss} (kbar)	133	240	187	60	160	122
ν_P	.12	.25	.27	.20	.25	.15
ν_m	.31	.27	.30	.33	.34	.29
B	.88	.51	.55	.62	.85	.73
φ_f^+	.02	.02	.02	.19	.01	.06
κ_w (cm/sec)	$2(10)^{-7}$	$(10)^{-10}$	$(10)^{-10}$	$2(10)^{-4}$	$4(10)^{-10}$	$(10)^{-6}$

The quantity κ_w in Table 1 is a permeability measure for water which relates to the permeability in (6.2.7) by the formula

$$\kappa_w = \frac{k}{\mu_w}\gamma_w g = \frac{\varphi_f^{+2}}{\xi}\gamma_w g \tag{6.2.30}$$

where, γ_w is the true density of water, μ_w is the viscosity of water and g is the gravitational constant. Given the information in this table, we can first calculate $\bar{\lambda}_{ff}, \bar{\lambda}_{sf}$ and $\bar{\lambda}_{ss}$ by solving equations (5.131), (6.2.3) and (6.2.17). The result of this elimination is

$$\frac{\bar{\lambda}_{ff}}{2\mu_{ss}} = B^2\varphi_f^{+2}\frac{\left(\dfrac{\nu_m}{1-2\nu_m}+\dfrac{1}{3}\right)^2}{\left(\dfrac{\nu_m}{1-2\nu_m}-\dfrac{\nu_P}{1-2\nu_P}\right)} \tag{6.2.31}$$

$$\frac{\bar{\lambda}_{fs}}{2\mu_{ss}} = B\varphi_f^{+}\left(\frac{\nu_m}{1-2\nu_m}+\frac{1}{3}\right)\left(1-\frac{B\varphi_f^{+}\left(\dfrac{\nu_m}{1-2\nu_m}+\dfrac{1}{3}\right)}{\left(\dfrac{\nu_m}{1-2\nu_m}-\dfrac{\nu_P}{1-2\nu_P}\right)}\right) \tag{6.2.32}$$

and

$$\frac{\bar{\lambda}_{ss}}{2\mu_{ss}} = \left(\frac{\nu_m}{1-2\nu_m}-\frac{\nu_P}{1-2\nu_P}\right)\left(1-\frac{B\varphi_f^{+}\left(\dfrac{\nu_m}{1-2\nu_m}+\dfrac{1}{3}\right)}{\left(\dfrac{\nu_m}{1-2\nu_m}-\dfrac{\nu_P}{1-2\nu_P}\right)}\right)^2+\frac{\nu_P}{1-2\nu_P} \tag{6.2.33}$$

Next, we can calculate the drag coefficient ξ from (6.2.30). In this last calculation we use $\mu_w = .01$ poise and $\gamma_w = 1$ gm/cm^3 for the properties of water. These formulas, and the tabulated values above, yield the following numerical values

Table 2 Calculated Rock Properties-Fluid Mixture Properties

Property	Ruhr Sandstone	Tennessee Marble	Charcoal Granite	Berea Sandstone	Westerly Granite	Weber Sandstone
$\bar{\lambda}_{ff}$ (kbar)	.165	.486	.326	4.44	.075	1.03
$\bar{\lambda}_{sf}$ (kbar)	5.21	4.02	4.13	13.99	3.48	9.91
$\bar{\lambda}_{ss}$ (kbar)	206.41	273.21	271.91	84.05	311.71	147.62
ξ (kbar-sec/cm^2)	$1.9(10)^{-3}$	3.92	3.92	$1.77(10)^{-4}$	.245	$.353(10)^{-3}$

In the remainder of this work, we shall adopt the numerical values above as typical and utilize them for the purposes of illustration. It needs to be stressed that the numerical values for the drag coefficient do presume the fluid is water with the properties given above. The drag coefficients for other fluids can be derived from the above formulas if needed so long as one knows the true density and the viscosity of the fluids. The numerical values adopted from the Rice and Cleary article and their use to calculate the information in Table 2 follows the work of Bowen and Lockett [Ref. 6].

The literature on the poroelastic properties of porous materials is extensive. It is too extensive to be treated properly here. References 7 through 11 might be of interest to the reader. Proelastic properties can also be found in references 12 and 13.

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7

Singular Surfaces and Acceleration Waves

In this chapter we shall look at an elementary application of certain of the partial differential equations introduced in Chapter 6. The objective is to create a better understanding of the physical phenomena characterized by the linear poroelasticity models we have been considering. The application in this chapter is the examination of the propagation of acceleration waves. This information is useful when one tries to understand the properties of the underlying partial differential equations. It is also useful when studying more complicated applications.

7.1 Singular Surfaces

It is possible to gain insight into the equations of linear poroelasticity by examining whether or not they will support solutions with various types of discontinuities such as shock waves and, as we shall define below, acceleration waves. The formalism required to study these kinds of discontinuities can be somewhat complicated. Here, we shall keep the formulism to a minimum and make generous use of reference material. The approach in this section depends heavily on the work of Truesdell and Toupin [Ref. 1] and to a certain extent on the work of Friedlander [Ref. 2].

Our first task is to introduce some of the kinematics of singular surfaces. If $\sum(t)$ is a three dimensional surface, it can be represented by the equation

$$f\left(\mathbf{x},t\right)=0 \tag{7.1.1}$$

then it is true that

$$\mathbf{n}=\operatorname{grad} f\left(\mathbf{x},t\right)\Big/\left\|\operatorname{grad} f\left(\mathbf{x},t\right)\right\| \tag{7.1.2}$$

is the *unit normal* to $\sum(t)$, and

$$u_n=-\frac{\partial f\left(\mathbf{x},t\right)}{\partial t}\Bigg/\left\|\operatorname{grad} f\left(\mathbf{x},t\right)\right\| \tag{7.1.3}$$

is the *normal speed* of $\sum(t)$.

It is convenient to consider a *family of surfaces* defined by

$$f(\mathbf{x},t)=\alpha \tag{7.1.4}$$

where $\alpha=0$ defines the surface $\sum(t)$. Without loss of generality we can regard $\mathbf{n}$ to be directed from the surface $\alpha=0$ towards surfaces with $\alpha>0$. We assume that (7.1.4) can be inverted to yield

$$t=\hat{t}(\mathbf{x},\alpha) \tag{7.1.5}$$

Given (7.1.5), a function φ of $(\mathbf{x},t)$ can be replaced by a function $\hat{\varphi}$ of $(\mathbf{x},\alpha)$ by the rule

$$\hat{\varphi}(\mathbf{x},\alpha)=\varphi(\mathbf{x},\hat{t}(\mathbf{x},\alpha)) \tag{7.1.6}$$

It follows from (7.1.6) that

$$\operatorname{grad}\hat{\varphi}(\mathbf{x},\alpha)=\operatorname{grad}\varphi(\mathbf{x},t)+\frac{\partial\varphi(\mathbf{x},t)}{\partial t}\operatorname{grad}\hat{t}(\mathbf{x},\alpha) \tag{7.1.7}$$

Likewise it follows from (7.1.4) and (7.1.5) that

$$\operatorname{grad}\hat{t}(\mathbf{x},\alpha)=-\operatorname{grad} f(\mathbf{x},t)\Big/\frac{\partial f(\mathbf{x},t)}{\partial t} \tag{7.1.8}$$

By formulas like (7.1.2) and (7.1.3), (7.1.8) can be written

$$\operatorname{grad}\hat{t}(\mathbf{x},\alpha)=\frac{\mathbf{n}_{(\alpha)}}{u_{n(\alpha)}} \tag{7.1.9}$$

where $\mathbf{n}_{(\alpha)}$ is the unit normal to the surface (7.1.4) and $u_{n(\alpha)}$ is its normal speed. The special surface $\alpha=0$ has its normal and normal speed denoted by $\mathbf{n}$ and u_n, respectively. Given (7.1.9), equation (7.1.7) becomes

$$\operatorname{grad}\hat{\varphi}(\mathbf{x},\alpha)=\operatorname{grad}\varphi(\mathbf{x},t)+\frac{\mathbf{n}_{(\alpha)}}{u_{n(\alpha)}}\frac{\partial\varphi(\mathbf{x},t)}{\partial t} \tag{7.1.10}$$

The *displacement derivative* of $\varphi(\mathbf{x},t)$, following a particular surface $\alpha=0$, is defined by

$$\frac{\delta\varphi(\mathbf{x},t)}{\delta t}=\left(\operatorname{grad}\hat{\varphi}(\mathbf{x},\alpha)\Big|_{\alpha=0}\right)\cdot\mathbf{n}u_n \tag{7.1.11}$$

Physically, the displacement derivative represents the time rate of change of the quantity $\varphi(\mathbf{x},t)$ seen by an observer moving with speed u_n along a curve whose tangent is $\mathbf{n}$. This curve is called the *normal trajectory*. It follows from (7.1.10) that the displacement derivative can be written

$$\frac{\delta\varphi(\mathbf{x},t)}{\delta t}=\frac{\partial\varphi(\mathbf{x},t)}{\partial t}+\operatorname{grad}\varphi(\mathbf{x},t)\cdot\mathbf{n}u_n \tag{7.1.12}$$

If we continue to single out a particular moving surface, $\alpha=0$, at the time t, this surface can be regarded as the common boundary between two regions $\mathscr{V}^+$ and $\mathscr{V}^-$. Let $\varphi(\mathbf{x},t)$, for each t, be continuous in $\mathscr{V}^+$ and $\mathscr{V}^-$ and approach limits φ^+ and φ^- as $\mathbf{x}$ approaches a point $\mathbf{x}_0$ on the surface while remaining in $\mathscr{V}^+$ and $\mathscr{V}^-$, respectively. The *jump* of φ across the surface $\alpha=0$ is defined by

$$[\![\varphi]\!]=\varphi^- -\varphi^+ \tag{7.1.13}$$

where, formally,

$$\varphi^- =\lim_{\alpha\uparrow 0^-}\hat{\varphi}(\mathbf{x},\alpha) \tag{7.1.14}$$

and

$$\varphi^+ =\lim_{\alpha\downarrow 0^+}\hat{\varphi}(\mathbf{x},\alpha) \tag{7.1.15}$$

The jump $[\![\varphi]\!]$ is a function of position on the surface as well as the time. As a matter of convention, we regard the unit normal $\mathbf{n}$ as directed into $\mathscr{V}^+$.

Because differentiation at constant α commutes with the jump operation, the jump of (7.1.10) yields

$$\operatorname{grad}[\![\varphi]\!]=[\![\operatorname{grad}\varphi(\mathbf{x},t)]\!]+[\![\frac{\partial\varphi(\mathbf{x},t)}{\partial t}]\!]\frac{\mathbf{n}}{u_n} \tag{7.1.16}$$

The projection of (7.1.16) tangential to the surface is a result known as the geometrical condition of compatibility [Ref. 1, Sec. 174]. If, as a special case, the jump of $[\![\varphi]\!]$ is zero, (7.1.16) yields

$$[\![\operatorname{grad}\varphi(\mathbf{x},t)]\!]=-[\![\frac{\partial\varphi(\mathbf{x},t)}{\partial t}]\!]\frac{\mathbf{n}}{u_n} \tag{7.1.17}$$

Equation (7.1.17) is known as *Maxwell's theorem* [Ref. 1, Sec. 175]. It yields the interesting kinematic result that the jump $[\![\operatorname{grad}\varphi(\mathbf{x},t)]\!]$ is necessarily normal to the surface in the case where $[\![\varphi]\!]$ is zero. Returning to the case where $[\![\varphi]\!]\neq 0$, it follows from (7.1.11) that

$$[\![\frac{\delta\varphi(\mathbf{x},t)}{\delta t}]\!]=\frac{\delta[\![\varphi(\mathbf{x},t)]\!]}{\delta t} \qquad (7.1.18)$$

and, from (7.1.12) and (7.1.18),

$$\frac{\delta[\![\varphi(\mathbf{x},t)]\!]}{\delta t}=[\![\frac{\partial\varphi(\mathbf{x},t)}{\partial t}]\!]+[\![\operatorname{grad}\varphi(\mathbf{x},t)]\!]\cdot\mathbf{n}u_n \qquad (7.1.19)$$

Equation (7.1.19) is known as the *kinematic condition of compatibility* [Ref. 1, Sec. 180]. In the special case where $[\![\varphi]\!]$ is zero, (7.1.19) yields

$$[\![\frac{\partial\varphi(\mathbf{x},t)}{\partial t}]\!]=-[\![\operatorname{grad}\varphi(\mathbf{x},t)]\!]\cdot\mathbf{n}u_n \qquad (7.1.20)$$

which also follows from (7.1.17).

7.2 Acceleration Waves

Next, we shall utilize the above kinematic results to study various types of singular surfaces propagating in porous materials defined by equations (4.2.14) through (4.2.22). In particular, we are interested in a binary mixture with pore pressure consisting of a compressible isotropic linear elastic solid and a compressible linear elastic fluid. As in Chapter 4, the mixture is assumed to have pore pressure. In particular, the porous material is governed by the following partial differential equations:

$$\begin{aligned}\rho_{s_R}\frac{\partial^2\mathbf{w}_s}{\partial t^2}&=(\bar{\lambda}_{ss}+\mu_{ss})\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_s)+\mu_{ss}\,\mathrm{Div}(\mathrm{GRAD}\,\mathbf{w}_s)\\&\quad+\bar{\lambda}_{sf}\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_f)+(\alpha_s-\bar{\tau}_s+\gamma)\,\mathrm{GRAD}\,\theta-\xi\left(\frac{\partial\mathbf{w}_s}{\partial t}-\frac{\partial\mathbf{w}_f}{\partial t}\right)\end{aligned} \qquad (7.2.1)$$

$$\begin{aligned}\rho_f^+\frac{\partial^2\mathbf{w}_f}{\partial t^2}&=\bar{\lambda}_{sf}\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_s)+\bar{\lambda}_{ff}\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_f)+(\alpha_f-\bar{\tau}_f-\gamma)\,\mathrm{GRAD}\,\theta\\&\quad-\xi\left(\frac{\partial\mathbf{w}_f}{\partial t}-\frac{\partial\mathbf{w}_s}{\partial t}\right)\end{aligned} \qquad (7.2.2)$$

and

$$\bar{c}_v \frac{\partial \theta}{\partial t} + \theta^+ \left((\bar{\tau}_s - \alpha_s + \varsigma/\theta^+) \operatorname{Div} \frac{\partial \mathbf{w}_s}{\partial t} + (\bar{\tau}_f - \alpha_f - \varsigma/\theta^+) \operatorname{Div} \frac{\partial \mathbf{w}_f}{\partial t} \right) = \kappa \operatorname{Div}(\operatorname{GRAD} \theta) \tag{7.2.3}$$

where

$$\kappa \geq 0 \tag{7.2.4}$$

and

$$\kappa\xi/\theta^+ \geq \frac{1}{4}(\gamma + \varsigma/\theta^+)^2 \tag{7.2.5}$$

As indicated in Chapter 4, it follows from (7.2.5) that

$$\xi \geq 0 \tag{7.2.6}$$

Also as indicated in Chapter 4, we shall require that the inequalities (4.3.4) through (4.3.8) be satisfied. These formulas, repeated here, are

$$\bar{c}_v > 0 \tag{7.2.7}$$

$$\bar{\lambda}_{ss} + \frac{2}{3}\mu_{ss} > 0 \tag{7.2.8}$$

$$(\bar{\lambda}_{ss} + \frac{2}{3}\mu_{ss})\bar{\lambda}_{ff} > \bar{\lambda}_{sf}^2 \tag{7.2.9}$$

$$\mu_{ss} > 0 \tag{7.2.10}$$

and

$$\bar{\lambda}_{ff} > 0 \tag{7.2.11}$$

In addition to the field equations (7.2.1), (7.2.2) and (7.2.3), there are jump balance equations which hold across the surface of discontinuity. We have not developed these equations in this textbook. A careful development of these equations at this point would take us too deeply into the details of Continuum Mechanics. A quick and somewhat inelegant way of deriving the jump balance equations involves the integration of (7.2.1), (7.2.2) and (7.2.3) over a volume that moves with and contains the singular surface. The results of this calculation turn out to be[1]

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[1] The basic material that would allow for the derivation of (7.2.12) through (7.2.14) from the foundations of Continuum Mechanics can be found reference 1, pages 525-529. Reference 4 also contains sufficient material to be helpful in understanding the jump balance equations. A careful examination of the analysis leading to equations

$$\rho_{s_R} u_n [\![\frac{\partial \mathbf{w}_s}{\partial t}]\!] + [\![\bar{\lambda}_{ss} \left(\operatorname{Div} \mathbf{w}_s\right) \mathbf{I} + \mu_{ss} \left(\operatorname{GRAD} \mathbf{w}_s + \left(\operatorname{GRAD} \mathbf{w}_s\right)^T \right) + \bar{\lambda}_{sf} \left(\operatorname{Div} \mathbf{w}_f\right) \mathbf{I}]\!] \mathbf{n} + (\alpha_s - \bar{\tau}_s + \gamma) [\![\theta]\!] \mathbf{n} = \mathbf{0} \tag{7.2.12}$$

$$\rho_f^+ u_n [\![\frac{\partial \mathbf{w}_f}{\partial t}]\!] + [\![\bar{\lambda}_{sf} \left(\operatorname{Div} \mathbf{w}_s\right) + \bar{\lambda}_{ff} \left(\operatorname{Div} \mathbf{w}_f\right)]\!] \mathbf{n} + (\alpha_f - \bar{\tau}_f - \gamma) [\![\theta]\!] \mathbf{n} = \mathbf{0} \tag{7.2.13}$$

and

$$u_n [\![\bar{c}_v \theta + \theta^+ \left(\bar{\tau}_s - \alpha_s\right) \operatorname{Div} \mathbf{w}_s + \theta^+ \left(\bar{\tau}_f - \alpha_f\right) \operatorname{Div} \mathbf{w}_f]\!] = [\![-\kappa \operatorname{GRAD} \theta - \varsigma \left(\frac{\partial \mathbf{w}_f}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t} \right)]\!] \cdot \mathbf{n} \tag{7.2.14}$$

By definition, an *acceleration wave* in the porous material defined by (7.2.1) through (7.2.6) is a moving singular surface across which[2]

$$[\![\mathbf{w}_f]\!] = [\![\mathbf{w}_s]\!] = \mathbf{0} \tag{7.2.15}$$

$$[\![\frac{\partial \mathbf{w}_f}{\partial t}]\!] = [\![\frac{\partial \mathbf{w}_s}{\partial t}]\!] = \mathbf{0} \tag{7.2.16}$$

$$[\![\theta]\!] = 0 \tag{7.2.17}$$

but $[\![\frac{\partial^2 \mathbf{w}_f}{\partial t^2}]\!]$, $[\![\frac{\partial^2 \mathbf{w}_s}{\partial t^2}]\!]$ and $[\![\frac{\partial \theta}{\partial t}]\!]$ are not necessarily zero. If we select φ in (7.1.17) to be $\mathbf{w}_f, \mathbf{w}_s$, θ, $\frac{\partial \mathbf{w}_f}{\partial t}$ and $\frac{\partial \mathbf{w}_s}{\partial t}$, respectfully, then equations (7.2.15), (7.2.16) and (7.2.17) yield

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(7.2.1) through (7.2.14) will reveal that the jump balance equations <u>do</u> <u>include</u> terms arising from the momentum supplies introduced in equation (2.2.2). When the constitutive equations for the momentum supplies depend upon density gradients, deformation gradients, temperature gradients and other gradients, one cannot automatically assume they are sufficiently smooth to yield zero contribution to the jump balance of linear momentum equation. Indeed, they do contribute a terms to the jump balance of linear momentum. These terms are in (7.2.12) and (7.2.13). The contribution from the momentum supplies has been treated incorrectly in some of the historical literature. Examples can be found in the articles written by Bowen [Ref. 5, 6].

[2] If we had included a proof of the results (7.2.12), (7.2.13) and (7.2.14), it would have been seen that we had already assumed (7.2.15).

$$[\![\text{GRAD}\,\mathbf{w}_f]\!] = \mathbf{0} \tag{7.2.18}$$

$$[\![\text{GRAD}\,\mathbf{w}_s]\!] = \mathbf{0} \tag{7.2.19}$$

$$[\![\text{GRAD}\left(\frac{\partial \mathbf{w}_f}{\partial t}\right)]\!] = -[\![\frac{\partial^2 \mathbf{w}_f}{\partial t^2}]\!] \otimes \frac{\mathbf{n}}{u_n} \tag{7.2.20}$$

$$[\![\text{GRAD}\left(\frac{\partial \mathbf{w}_s}{\partial t}\right)]\!] = -[\![\frac{\partial^2 \mathbf{w}_s}{\partial t^2}]\!] \otimes \frac{\mathbf{n}}{u_n} \tag{7.2.21}$$

and

$$[\![\text{GRAD}\,\theta]\!] = -[\![\frac{\partial \theta}{\partial t}]\!] \frac{\mathbf{n}}{u_n} \tag{7.2.22}$$

Because of (7.2.18), we can take φ in (7.1.20) to be $\text{grad}\,\mathbf{w}_f$ and the result can be written

$$u_n^2 [\![\text{GRAD}\left(\text{GRAD}\,\mathbf{w}_f\right)]\!] = [\![\frac{\partial^2 \mathbf{w}_f}{\partial t^2}]\!] \otimes \mathbf{n} \otimes \mathbf{n} \tag{7.2.23}$$

after (7.2.21) is used. An identical calculation yields

$$u_n^2 [\![\text{GRAD}\left(\text{GRAD}\,\mathbf{w}_s\right)]\!] = [\![\frac{\partial^2 \mathbf{w}_s}{\partial t^2}]\!] \otimes \mathbf{n} \otimes \mathbf{n} \tag{7.2.24}$$

It follows from (7.2.23) that

$$u_n^2 [\![\text{GRAD}\left(\text{Div}\,\mathbf{w}_f\right)]\!] = \left([\![\frac{\partial^2 \mathbf{w}_f}{\partial t^2}]\!] \cdot \mathbf{n} \right) \mathbf{n} \tag{7.2.25}$$

Likewise, it follows from (7.2.24) that

$$u_n^2 [\![\text{GRAD}\left(\text{Div}\,\mathbf{w}_s\right)]\!] = \left([\![\frac{\partial^2 \mathbf{w}_s}{\partial t^2}]\!] \cdot \mathbf{n} \right) \mathbf{n} \tag{7.2.26}$$

and

$$u_n^2 [\![\text{Div}\left(\text{GRAD}\,\mathbf{w}_s\right)]\!] = [\![\frac{\partial^2 \mathbf{w}_s}{\partial t^2}]\!] \tag{7.2.27}$$

Given the definition of an acceleration wave, equations (7.2.12) and (7.2.13) are identically satisfied. Equation (7.2.14), which represents the jump balance of energy across the acceleration wave, reduces to

$$\kappa[\![\text{GRAD}\,\theta]\!]\cdot\mathbf{n}=0 \tag{7.2.28}$$

Equation (7.2.28) provides a point of departure for two important special cases. The first case is that of a *conductor*, where we rule out the case $\kappa=0$. In this case, (7.2.4) is replaced by

$$\kappa>0 \tag{7.2.29}$$

and (7.2.28) forces

$$[\![\text{GRAD}\,\theta]\!]\cdot\mathbf{n}=0 \tag{7.2.30}$$

which, in turn, from (7.1.20) yields

$$[\![\frac{\partial\theta}{\partial t}]\!]=0 \tag{7.2.31}$$

Finally, this last result and (7.1.17) yield

$$[\![\text{GRAD}\,\theta]\!]=\mathbf{0} \tag{7.2.32}$$

The second case we shall examine is for a *nonconductor*, i.e., when $\kappa=0$. In this case, which we shall examine below, we cannot utilize the results (7.2.31) and (7.2.32).

Consider first, the case of a conductor. On the assumption that an acceleration wave, as we have defined it, exists, the balance equations (7.2.1), (7.2.2) and (7.2.3) can be used to computer necessary conditions that must be satisfied. If we form the jump of these equations across the acceleration wave, it follows from (7.2.16), (7.2.25), (7.2.26), (7.2.27) and (7.2.32) that (7.2.1) reduces to

$$\rho_{s_R}[\![\frac{\partial^2\mathbf{w}_s}{\partial t^2}]\!]=(\bar{\lambda}_{ss}+\mu_{ss})\left([\![\frac{\partial^2\mathbf{w}_s}{\partial t^2}]\!]\cdot\mathbf{n}\right)\frac{\mathbf{n}}{u_n^2}+\mu_{ss}u_n^2[\![\frac{\partial^2\mathbf{w}_s}{\partial t^2}]\!]\frac{1}{u_n^2}+\bar{\lambda}_{sf}\left([\![\frac{\partial^2\mathbf{w}_f}{\partial t^2}]\!]\cdot\mathbf{n}\right)\frac{\mathbf{n}}{u_n^2} \tag{7.2.33}$$

A similar calculation yields from (7.2.2)

$$\rho_f^+[\![\frac{\partial^2\mathbf{w}_f}{\partial t^2}]\!]=\bar{\lambda}_{sf}\left([\![\frac{\partial^2\mathbf{w}_s}{\partial t^2}]\!]\cdot\mathbf{n}\right)\frac{\mathbf{n}}{u_n^2}+\bar{\lambda}_{ff}\left([\![\frac{\partial^2\mathbf{w}_f}{\partial t^2}]\!]\cdot\mathbf{n}\right)\frac{\mathbf{n}}{u_n^2} \tag{7.2.34}$$

An entirely similar calculation yields, from (7.2.3),

$$\left[\mathrm{Div}(\mathrm{GRAD}\,\theta)\right] = -\frac{\theta^+}{\kappa}\left((\overline{\tau}_s - \alpha_s + \varsigma/\theta^+)\left[\frac{\partial^2 \mathbf{w}_s}{\partial t^2}\right]\cdot\frac{\mathbf{n}}{u_n} + (\overline{\tau}_f - \alpha_f - \varsigma/\theta^+)\left[\frac{\partial^2 \mathbf{w}_f}{\partial t^2}\right]\cdot\frac{\mathbf{n}}{u_n}\right) \quad (7.2.35)$$

Equations (7.2.33) and (7.2.34), that do not involve the jumps in derivatives of the temperature, can be rearranged to yield

$$(\overline{\lambda}_{ss} + \mu_{ss})(\mathbf{a}_s\cdot\mathbf{n})\mathbf{n} + \mu_{ss}\mathbf{a}_s + \overline{\lambda}_{sf}(\mathbf{a}_f\cdot\mathbf{n})\mathbf{n} = \rho_{s_R}u_n^2\mathbf{a}_s \quad (7.2.36)$$

and

$$\overline{\lambda}_{sf}(\mathbf{a}_s\cdot\mathbf{n})\mathbf{n} + \overline{\lambda}_{ff}(\mathbf{a}_f\cdot\mathbf{n})\mathbf{n} = \rho_f^+u_n^2\mathbf{a}_f \quad (7.2.37)$$

where

$$\mathbf{a}_f = \left[\frac{\partial^2 \mathbf{w}_f}{\partial t^2}\right] \quad (7.2.38)$$

and

$$\mathbf{a}_s = \left[\frac{\partial^2 \mathbf{w}_s}{\partial t^2}\right] \quad (7.2.39)$$

are the respective *amplitudes* of the acceleration wave. For a given direction $\mathbf{n}$, equations (7.2.36) and (7.2.37) determine the possible squared speeds and the direction of the two amplitudes, $\mathbf{a}_f$ and $\mathbf{a}_s$.

The analysis of (7.2.36) and (7.2.37) is straightforward. If we form the cross product of each equation with the unit normal $\mathbf{n}$, the two equations yield

$$\left(u_n^2 - \frac{\mu_{ss}}{\rho_{s_R}}\right)\mathbf{a}_s\times\mathbf{n} = \mathbf{0} \quad (7.2.40)$$

and

$$\mathbf{a}_f\times\mathbf{n} = \mathbf{0} \quad (7.2.41)$$

Equation (7.2.41) shows that the amplitude for the fluid is necessarily parallel to the unit normal. In other words the acceleration wave in the fluid is *longitudinal.* Equation (7.2.40) shows that when $u_n^2 = \dfrac{\mu_{ss}}{\rho_{s_R}}$, the amplitude of the acceleration wave in the solid has components perpendicular to the unit normal. This *transverse wave* has multiplicity of two since the plane perpendicular to the unit normal is two dimensional. In the following, we shall designate the squared speed for the transverse wave by $u_{n(3)}^2$. Equation (7.2.40) shows that

$$u_{n(3)}^2 = \frac{\mu_{ss}}{\rho_{s_R}} \tag{7.2.42}$$

Equation (7.2.10) shows that the squared wave speed $u_{n(3)}^2$ is, indeed, positive.

If we next project (7.2.36) and (7.2.37) in the direction of the unit normal, the results are

$$(\bar{\lambda}_{ss} + 2\mu_{ss} - \rho_{s_R} u_n^2)(\mathbf{a}_s \cdot \mathbf{n}) + \bar{\lambda}_{sf}(\mathbf{a}_f \cdot \mathbf{n}) = 0 \tag{7.2.43}$$

and

$$\bar{\lambda}_{sf}(\mathbf{a}_s \cdot \mathbf{n}) + \left(\bar{\lambda}_{ff} - \rho_f^+ u_n^2\right)(\mathbf{a}_f \cdot \mathbf{n}) = 0 \tag{7.2.44}$$

It is useful to rewrite (7.2.43) and (7.2.44) as the matrix equation

s

$$\left(\mathbf{Q} - u_n^2 \mathbf{M}\right)\mathbf{a} = \mathbf{0} \tag{7.2.45}$$

where

$$\mathbf{Q} = \begin{bmatrix} \bar{\lambda}_{ff} & \bar{\lambda}_{sf} \\ \bar{\lambda}_{sf} & \bar{\lambda}_{ss} + 2\mu_{ss} \end{bmatrix} \tag{7.2.46}$$

$$\mathbf{M} = \begin{bmatrix} \rho_f^+ & 0 \\ 0 & \rho_{s_R} \end{bmatrix} \tag{7.2.47}$$

and

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_f \cdot \mathbf{n} \\ \mathbf{a}_s \cdot \mathbf{n} \end{bmatrix} \tag{7.2.48}$$

Equation (7.2.45) shows that in order that the normal components of the two amplitudes be nonzero, the normal wave speeds must be solutions of

$$\det\left(\mathbf{Q} - u_n^2 \mathbf{M}\right) = 0 \tag{7.2.49}$$

If the determinant (7.2.49), which is a quadratic in the squared normal speed, is expanded and solved for these squared speeds, the results are

$$\left.\begin{matrix} u_{n(1)}^2 \\ u_{n(2)}^2 \end{matrix}\right\} = \frac{1}{2}\left(\frac{\bar{\lambda}_{ff}}{\rho_f^+} + \frac{\bar{\lambda}_{ss} + 2\mu_{ss}}{\rho_{s_R}} \pm \sqrt{\left(\frac{\bar{\lambda}_{ff}}{\rho_f^+} - \frac{\bar{\lambda}_{ss} + 2\mu_{ss}}{\rho_{s_R}} \right)^2 + \frac{4\bar{\lambda}_{sf}^2}{\rho_f^+ \rho_{s_R}}} \right) \tag{7.2.50}$$

Because of the inequality (7.2.9), the symmetric matrix $\mathbf{Q}$ is positive definite. The diagonal matrix $\mathbf{M}$ is also positive definite. Therefore, the two squared wave speeds given by (7.2.50) are the positive eigenvalues of the matrix $\mathbf{Q}$ with respect to the inner product induced by $\mathbf{M}$. The matrix $\mathbf{a}$, defined by (7.2.48), is the eigenvector for the eigenvalue problem (7.2.45).

The numerical values in Table 2 of Section 6.2 can be used to calculate the wave speeds from equations (7.2.50) and (7.2.42). In order to proceed with these calculations, we need numerical values for the true densities of the solid and the fluid. As with the tables in Section 6.2, we shall take the fluid to be water and use $\gamma_f = \gamma_w = 1$ gm/cm^3. The true densities for the solids are taken from Farmer [Ref. 7]. These values and the resulting acceleration wave speeds are shown in the table below.

Table 1 Calculated Acceleration Wave Speeds

Property	Ruhr Sandstone	Tennessee Marble	Charcoal Granite	Berea Sandstone	Westerly Granite	Weber Sandstone
γ_{s_R} (g/cm^3)	2.6	2.7	2.7	2.6	2.7	2.6
$u_{n(1)}$ (m/sec)	4340	5346	4955	3210	4805	4059
$u_{n(2)}$ (m/sec)	728	1522	1220	1313	740	1124
$u_{n(3)}$ (m/sec)	2285	3012	2658	1688	2369	2234

As reflected in this table, in the following we shall always assume that the eigenvalues $u_{n(1)}^2$ and $u_{n(2)}^2$ are distinct.

Since the three wave speeds $u_{n(1)}^2$, $u_{n(2)}^2$ and $u_{n(3)}^2$ are constant, a wave of a given initial shape retains that shape as the wave propagates, i.e., the waves form a family of parallel surfaces. For example, an initially plane wave remains plane. As an illustration of the effect of diffusion on the growth and decay of acceleration, we shall next investigate the growth and decay of *plane* acceleration waves. As a consequence of this assumption the unit normal, $\mathbf{n}$, is a constant. This calculation requires that we utilize (7.1.19) with the choice $\varphi = \dfrac{\partial^2 \mathbf{w}_f}{\partial t^2}$. Therefore,

$$\frac{\delta}{\delta t}\left[\frac{\partial^2 \mathbf{w}_f}{\partial t^2}\right] = \left[\frac{\partial^3 \mathbf{w}_f}{\partial t^3}\right] + \left[\mathrm{GRAD}\frac{\partial^2 \mathbf{w}_f}{\partial t^2}\right]\cdot \mathbf{n} u_n \tag{7.2.51}$$

A entirely similar calculation with $\varphi = \text{GRAD}\dfrac{\partial \mathbf{w}_f}{\partial t}$ yields

$$\frac{\delta}{\delta t}\left[\!\left[\text{GRAD}\frac{\partial \mathbf{w}_f}{\partial t}\right]\!\right] = \left[\!\left[\text{GRAD}\frac{\partial^2 \mathbf{w}_f}{\partial t^2}\right]\!\right] + \left[\!\left[\text{GRAD}\left(\text{GRAD}\frac{\partial \mathbf{w}_f}{\partial t}\right)\right]\!\right]\cdot \mathbf{n}u_n \tag{7.2.52}$$

The term $\left[\!\left[\text{GRAD}\dfrac{\partial \mathbf{w}_f}{\partial t}\right]\!\right]$ in (7.2.52) can be eliminated in favor of $\left[\!\left[\dfrac{\partial^2 \mathbf{w}_f}{\partial t^2}\right]\!\right]$ if we use (7.2.20). The result of this elimination can be used to eliminate the term $\left[\!\left[\text{GRAD}\dfrac{\partial^2 \mathbf{w}_f}{\partial t^2}\right]\!\right]$ from (7.2.51). The first step in this elimination yields

$$2\frac{\delta}{\delta t}\left[\!\left[\frac{\partial^2 \mathbf{w}_f}{\partial t^2}\right]\!\right] + \left(\left[\!\left[\frac{\partial^2 \mathbf{w}_f}{\partial t^2}\right]\!\right] \otimes \frac{\delta}{\delta t}\left(\frac{\mathbf{n}}{u_n}\right)\right)\mathbf{n}u_n = \left[\!\left[\frac{\partial^3 \mathbf{w}_f}{\partial t^3}\right]\!\right] - u_n^2\left(\left[\!\left[\text{GRAD}\left(\text{GRAD}\frac{\partial \mathbf{w}_f}{\partial t}\right)\right]\!\right]\right)(\mathbf{n},\mathbf{n}) \tag{7.2.53}$$

An expansion of the second term yields,

$$\begin{aligned} 2\frac{\delta}{\delta t}\left[\!\left[\frac{\partial^2 \mathbf{w}_f}{\partial t^2}\right]\!\right] &+ \left(\left[\!\left[\frac{\partial^2 \mathbf{w}_f}{\partial t^2}\right]\!\right]\frac{\delta \mathbf{n}}{\delta t}\cdot \mathbf{n} - \frac{1}{u_n}\left[\!\left[\frac{\partial^2 \mathbf{w}_f}{\partial t^2}\right]\!\right]\frac{\delta u_n}{\delta t}\right) \\ &= \left[\!\left[\frac{\partial^3 \mathbf{w}_f}{\partial t^3}\right]\!\right] - u_n^2\left(\left[\!\left[\text{GRAD}\left(\text{GRAD}\frac{\partial \mathbf{w}_f}{\partial t}\right)\right]\!\right]\right)(\mathbf{n},\mathbf{n}) \end{aligned} \tag{7.2.54}$$

Because $\mathbf{n}\cdot\mathbf{n} = 1$, $\dfrac{\delta \mathbf{n}}{\delta t}\cdot \mathbf{n} = 0$. Therefore, (7.2.54) reduces to

$$\begin{aligned} 2\frac{\delta}{\delta t}\left[\!\left[\frac{\partial^2 \mathbf{w}_f}{\partial t^2}\right]\!\right] &- \frac{1}{u_n}\left[\!\left[\frac{\partial^2 \mathbf{w}_f}{\partial t^2}\right]\!\right]\frac{\delta u_n}{\delta t} \\ &= \left[\!\left[\frac{\partial^3 \mathbf{w}_f}{\partial t^3}\right]\!\right] - u_n^2\left(\left[\!\left[\text{GRAD}\left(\text{GRAD}\frac{\partial \mathbf{w}_f}{\partial t}\right)\right]\!\right]\right)(\mathbf{n},\mathbf{n}) \end{aligned} \tag{7.2.55}$$

In our case, since $\mathbf{n}$ and u_n are constants, (7.2.54) and (7.2.55) both reduce to

$$2\frac{\delta}{\delta t}\left[\!\left[\frac{\partial^2 \mathbf{w}_f}{\partial t^2}\right]\!\right] = \left[\!\left[\frac{\partial^3 \mathbf{w}_f}{\partial t^3}\right]\!\right] - u_n^2\left(\left[\!\left[\text{GRAD}\left(\text{GRAD}\frac{\partial \mathbf{w}_f}{\partial t}\right)\right]\!\right]\right)(\mathbf{n},\mathbf{n}) \tag{7.2.56}$$

An identical calculation yields

$$2\frac{\delta}{\delta t}[\![\frac{\partial^2 \mathbf{w}_s}{\partial t^2}]\!] = [\![\frac{\partial^3 \mathbf{w}_s}{\partial t^3}]\!] - u_n^2\left([\![\text{GRAD}\left(\text{GRAD}\frac{\partial \mathbf{w}_s}{\partial t}\right)]\!]\right)(\mathbf{n},\mathbf{n}) \tag{7.2.57}$$

In equations (7.2.56) and (7.2.57) the last terms are, in Cartesian components,

$$\left([\![\text{GRAD}\left(\text{GRAD}\frac{\partial \mathbf{w}_f}{\partial t}\right)]\!]\right)(\mathbf{n},\mathbf{n}) = [\![\frac{\partial^3 \mathbf{w}_f}{\partial X_k \partial X_j \partial t}]\!] n_k n_j \tag{7.2.58}$$

and

$$\left([\![\text{GRAD}\left(\text{GRAD}\frac{\partial \mathbf{w}_s}{\partial t}\right)]\!]\right)(\mathbf{n},\mathbf{n}) = [\![\frac{\partial^3 \mathbf{w}_s}{\partial X_k \partial X_j \partial t}]\!] n_k n_j \tag{7.2.59}$$

respectively.

Equations (7.2.56) and (7.2.57) provide ordinary differential equations for the amplitudes (7.2.38) and (7.2.39) after we have evaluated their right hand sides. The time derivatives of (7.2.2) and (7.2.1) yield the jumps

$$\begin{aligned}\rho_f^+[\![\frac{\partial^3 \mathbf{w}_f}{\partial t^3}]\!] &= \bar{\lambda}_{sf}[\![\text{GRAD}(\text{Div}\frac{\partial \mathbf{w}_s}{\partial t})]\!] + \bar{\lambda}_{ff}[\![\text{GRAD}(\text{Div}\frac{\partial \mathbf{w}_f}{\partial t})]\!] + (\alpha_f - \bar{\tau}_f - \gamma)[\![\text{GRAD}\frac{\partial \theta}{\partial t}]\!] \\ &\quad -\xi\left([\![\frac{\partial^2 \mathbf{w}_f}{\partial t^2}]\!] - [\![\frac{\partial^2 \mathbf{w}_s}{\partial t^2}]\!]\right)\end{aligned} \tag{7.2.60}$$

and

$$\begin{aligned}\rho_{s_R}[\![\frac{\partial^3 \mathbf{w}_s}{\partial t^3}]\!] &= (\bar{\lambda}_{ss} + \mu_{ss})[\![\text{GRAD}(\text{Div}\frac{\partial \mathbf{w}_s}{\partial t})]\!] + \mu_{ss}[\![\text{Div}(\text{GRAD}\frac{\partial \mathbf{w}_s}{\partial t})]\!] \\ &\quad + \bar{\lambda}_{sf}[\![\text{GRAD}(\text{Div}\frac{\partial \mathbf{w}_f}{\partial t})]\!] + (\alpha_s - \bar{\tau}_s + \gamma)[\![\text{GRAD}\frac{\partial \theta}{\partial t}]\!] - \xi\left([\![\frac{\partial^2 \mathbf{w}_s}{\partial t^2}]\!] - [\![\frac{\partial^2 \mathbf{w}_f}{\partial t^2}]\!]\right)\end{aligned} \tag{7.2.61}$$

The next formal step is to substitute (7.2.60) and (7.2.61) into (7.2.56) and (7.2.57), respectively. The results turn out to be

$$\begin{aligned}2\rho_f^+\frac{\delta \mathbf{a}_f}{\delta t} &= \bar{\lambda}_{sf}[\![\text{GRAD}(\text{Div}\frac{\partial \mathbf{w}_s}{\partial t})]\!] + \bar{\lambda}_{ff}[\![\text{GRAD}(\text{Div}\frac{\partial \mathbf{w}_f}{\partial t})]\!] \\ &\quad - \rho_f^+ u_n^2\left([\![\text{GRAD}\left(\text{GRAD}\frac{\partial \mathbf{w}_f}{\partial t}\right)]\!]\right)(\mathbf{n},\mathbf{n}) \\ &\quad - \xi(\mathbf{a}_f - \mathbf{a}_s) + (\alpha_f - \bar{\tau}_f - \gamma)[\![\text{GRAD}\frac{\partial \theta}{\partial t}]\!]\end{aligned} \tag{7.2.62}$$

and

$$
\begin{aligned}
2\rho_{s_R}\frac{\delta \mathbf{a}_s}{\delta t} = {} & (\bar{\lambda}_{ss}+\mu_{ss})\left[\!\left[\operatorname{GRAD}(\operatorname{Div}\frac{\partial \mathbf{w}_s}{\partial t})\right]\!\right] + \mu_{ss}\left[\!\left[\operatorname{Div}(\operatorname{GRAD}\frac{\partial \mathbf{w}_s}{\partial t})\right]\!\right] \\
& + \bar{\lambda}_{sf}\left[\!\left[\operatorname{GRAD}(\operatorname{Div}\frac{\partial \mathbf{w}_f}{\partial t})\right]\!\right] - \rho_{s_R}u_n^2\left(\left[\!\left[\operatorname{GRAD}\left(\operatorname{GRAD}\frac{\partial \mathbf{w}_s}{\partial t}\right)\right]\!\right]\right)(\mathbf{n},\mathbf{n}) \\
& - \xi\left(\mathbf{a}_s - \mathbf{a}_f\right) \\
& + (\alpha_s - \bar{\tau}_s + \gamma)\left[\!\left[\operatorname{GRAD}\frac{\partial \theta}{\partial t}\right]\!\right]
\end{aligned}
\tag{7.2.63}
$$

These two complicated looking equations are simplified somewhat if we use the kinematic formulas

$$
\left[\!\left[\operatorname{GRAD}(\operatorname{Div}\frac{\partial \mathbf{w}_f}{\partial t})\right]\!\right] = \left(\left(\left[\!\left[\operatorname{GRAD}\left(\operatorname{GRAD}\frac{\partial \mathbf{w}_f}{\partial t}\right)\right]\!\right]\right)(\mathbf{n},\mathbf{n})\right)\cdot\mathbf{n} \tag{7.2.64}
$$

$$
\left[\!\left[\operatorname{GRAD}(\operatorname{Div}\frac{\partial \mathbf{w}_s}{\partial t})\right]\!\right] = \left(\left(\left[\!\left[\operatorname{GRAD}\left(\operatorname{GRAD}\frac{\partial \mathbf{w}_s}{\partial t}\right)\right]\!\right]\right)(\mathbf{n},\mathbf{n})\right)\cdot\mathbf{n} \tag{7.2.65}
$$

and

$$
\left[\!\left[\operatorname{Div}(\operatorname{GRAD}\frac{\partial \mathbf{w}_s}{\partial t})\right]\!\right] = \left(\left[\!\left[\operatorname{GRAD}\left(\operatorname{GRAD}\frac{\partial \mathbf{w}_s}{\partial t}\right)\right]\!\right]\right)(\mathbf{n},\mathbf{n}) \tag{7.2.66}
$$

These formulas can be derived from (7.1.16). Rather than include their derivations here, it is convenient to point out that they are special cases of Equation (176.8) of Ref. 1. The reduction of Equation (178.8) of Ref. 1 to the results (7.2.64) through (7.2.66), does make use of (7.2.15), (7.2.16), the plane wave assumption, the fact that the speeds are constants, and the fact that all of the amplitudes are assumed not to vary tangent to the wave front. It simplifies the notation somewhat if we introduce the symbols

$$
\mathbf{d}_f = \left(\left[\!\left[\operatorname{GRAD}\left(\operatorname{GRAD}\frac{\partial \mathbf{w}_f}{\partial t}\right)\right]\!\right]\right)(\mathbf{n},\mathbf{n}) \tag{7.2.67}
$$

and

$$
\mathbf{d}_s = \left(\left[\!\left[\operatorname{GRAD}\left(\operatorname{GRAD}\frac{\partial \mathbf{w}_f}{\partial t}\right)\right]\!\right]\right)(\mathbf{n},\mathbf{n}) \tag{7.2.68}
$$

These symbols along with (7.2.64) through (7.2.66) allow (7.2.62) and (7.2.63) to be written

$$2\rho_f^+ \frac{\delta \mathbf{a}_f}{\delta t} = \bar{\lambda}_{ff}\left(\mathbf{d}_f \cdot \mathbf{n}\right)\mathbf{n} + \bar{\lambda}_{sf}\left(\mathbf{d}_s \cdot \mathbf{n}\right)\mathbf{n} - \rho_f^+ u_n^2 \mathbf{d}_f \\ -\xi\left(\mathbf{a}_f - \mathbf{a}_s\right) + (\alpha_f - \bar{\tau}_f - \gamma)\left[\!\left[\mathrm{GRAD}\frac{\partial \theta}{\partial t}\right]\!\right] \tag{7.2.69}$$

and

$$2\rho_{s_R} \frac{\delta \mathbf{a}_s}{\delta t} = \bar{\lambda}_{sf}\left(\mathbf{d}_f \cdot \mathbf{n}\right)\mathbf{n} + (\bar{\lambda}_{ss} + \mu_{ss})\left(\mathbf{d}_s \cdot \mathbf{n}\right)\mathbf{n} + \mu_{ss}\mathbf{d}_s - \rho_{s_R} u_n^2 \mathbf{d}_s \\ -\xi\left(\mathbf{a}_s - \mathbf{a}_f\right) + (\alpha_s - \bar{\tau}_s + \gamma)\left[\!\left[\mathrm{GRAD}\frac{\partial \theta}{\partial t}\right]\!\right] \tag{7.2.70}$$

The term $\left[\!\left[\mathrm{GRAD}\frac{\partial \theta}{\partial t}\right]\!\right]$, which appears in both (7.2.69) and (7.2.70), can be related to the two amplitudes if we utilize (7.2.35) and certain of our kinematic formulas. Equation (7.2.35), rewritten in terms of the amplitudes (7.2.38) and (7.2.39) is

$$\left[\!\left[\mathrm{Div}(\mathrm{GRAD}\,\theta)\right]\!\right] = -\frac{\theta^+}{\kappa}\left((\bar{\tau}_s - \alpha_s + \varsigma/\theta^+)\mathbf{a}_s \cdot \frac{\mathbf{n}}{u_n} + (\bar{\tau}_f - \alpha_f - \varsigma/\theta^+)\mathbf{a}_f \cdot \frac{\mathbf{n}}{u_n}\right) \tag{7.2.71}$$

The kinematic formula which links $\left[\!\left[\mathrm{GRAD}\frac{\partial \theta}{\partial t}\right]\!\right]$ to $\left[\!\left[\mathrm{Div}(\mathrm{GRAD}\,\theta)\right]\!\right]$ is

$$\left[\!\left[\mathrm{GRAD}\frac{\partial \theta}{\partial t}\right]\!\right] = -u_n\left[\!\left[\mathrm{Div}(\mathrm{GRAD}\,\theta)\right]\!\right]\mathbf{n} \tag{7.2.72}$$

The derivation of this equation is similar to the derivation of some of the other kinematic formulas used thus far. One begins with (7.1.20) and makes the choice $\varphi = \mathrm{GRAD}\,\theta$. The result can be written

$$\left[\!\left[\mathrm{GRAD}\frac{\partial \theta}{\partial t}\right]\!\right] = -\left[\!\left[\mathrm{GRAD}\left(\mathrm{GRAD}\,\theta\right)\right]\!\right]\mathbf{n}u_n \tag{7.2.73}$$

Given (7.2.32) and the choice $\varphi = \mathrm{GRAD}\,\theta$, it follows from (7.1.17) and (7.1.20) that

$$\left[\!\left[\mathrm{GRAD}\left(\mathrm{GRAD}\,\theta\right)\right]\!\right] = \left(\mathbf{n}\cdot\left[\!\left[\mathrm{GRAD}\left(\mathrm{GRAD}\,\theta\right)\right]\!\right]\mathbf{n}\right)\mathbf{n}\otimes\mathbf{n} \tag{7.2.74}$$

This result yields

$$\left[\!\left[\mathrm{GRAD}\left(\mathrm{GRAD}\,\theta\right)\right]\!\right]\mathbf{n} = \left(\mathbf{n}\cdot\left[\!\left[\mathrm{GRAD}\left(\mathrm{GRAD}\,\theta\right)\right]\!\right]\mathbf{n}\right)\mathbf{n} \tag{7.2.75}$$

Also, since $\text{Div}(\text{GRAD}\,\theta) = \text{tr}\left(\text{GRAD}\left(\text{GRAD}\,\theta\right)\right)$, it follows from (7.2.74) that

$$\left[\text{Div}(\text{GRAD}\,\theta)\right] = \text{tr}\left(\left[\text{GRAD}\left(\text{GRAD}\,\theta\right)\right]\right) = \mathbf{n}\cdot\left[\text{GRAD}\left(\text{GRAD}\,\theta\right)\right]\mathbf{n} \tag{7.2.76}$$

Equations (7.2.75) and (7.2.76) allow (7.2.73) to be written in the form (7.2.72). With the result (7.2.72), it follow from (7.2.71) that

$$\left[\text{GRAD}\frac{\partial\theta}{\partial t}\right] = \frac{\theta^+}{\kappa}\left((\overline{\tau}_s - \alpha_s + \varsigma/\theta^+)\mathbf{a}_s\cdot\mathbf{n} + (\overline{\tau}_f - \alpha_f - \varsigma/\theta^+)\mathbf{a}_f\cdot\mathbf{n}\right)\mathbf{n} \tag{7.2.77}$$

When (7.2.77) is substituted into (7.2.62) and (7.2.63) the results are

$$\begin{aligned}2\rho_f^+\frac{\delta\mathbf{a}_f}{\delta t} &= \overline{\lambda}_{ff}\left(\mathbf{d}_f\cdot\mathbf{n}\right)\mathbf{n} + \overline{\lambda}_{sf}\left(\mathbf{d}_s\cdot\mathbf{n}\right)\mathbf{n} - \rho_f^+u_n^2\mathbf{d}_f - \xi\left(\mathbf{a}_f - \mathbf{a}_s\right)\\ &\quad -\frac{\theta^+}{\kappa}(\overline{\tau}_f - \alpha_f + \gamma)(\overline{\tau}_f - \alpha_f - \varsigma/\theta^+)\left(\mathbf{a}_f\cdot\mathbf{n}\right)\mathbf{n}\\ &\quad -\frac{\theta^+}{\kappa}(\overline{\tau}_f - \alpha_f + \gamma)(\overline{\tau}_s - \alpha_s + \varsigma/\theta^+)\left(\mathbf{a}_s\cdot\mathbf{n}\right)\mathbf{n}\end{aligned} \tag{7.2.78}$$

and

$$\begin{aligned}2\rho_{s_R}\frac{\delta\mathbf{a}_s}{\delta t} &= \overline{\lambda}_{sf}\left(\mathbf{d}_f\cdot\mathbf{n}\right)\mathbf{n} + (\overline{\lambda}_{ss} + \mu_{ss})\left(\mathbf{d}_s\cdot\mathbf{n}\right)\mathbf{n} + \mu_{ss}\mathbf{d}_s - \rho_{s_R}u_n^2\mathbf{d}_s - \xi\left(\mathbf{a}_s - \mathbf{a}_f\right)\\ &\quad -\frac{\theta^+}{\kappa}(\overline{\tau}_s - \alpha_s - \gamma)(\overline{\tau}_f - \alpha_f - \varsigma/\theta^+)\left(\mathbf{a}_f\cdot\mathbf{n}\right)\mathbf{n}\\ &\quad -\frac{\theta^+}{\kappa}(\overline{\tau}_s - \alpha_s - \gamma)(\overline{\tau}_s - \alpha_s + \varsigma/\theta^+)\left(\mathbf{a}_s\cdot\mathbf{n}\right)\mathbf{n}\end{aligned} \tag{7.2.79}$$

Equations (7.2.78) and (7.2.79) are used to determine the magnitudes of the amplitudes $\mathbf{a}_f$ and $\mathbf{a}_s$. The directions of these amplitudes are determined by the propagation condition (7.2.36) and (7.2.37). Utilizing these directions, the approach to the solutions involves projecting (7.2.78) and (7.2.79) in such a fashion that the jumps $\mathbf{d}_f$ and $\mathbf{d}_s$ are eliminated. This approach is best illustrated by first studying the transverse acceleration wave. In this case, the normal speed squared is given by (7.2.42) and the amplitude of the acceleration wave in the fluid obeys (7.2.41). Given these results, the projection of (7.2.78) into the transverse plane is obtained by forming its cross product with the unit normal, $\mathbf{n}$. The result of this calculation is

$$\mathbf{n}\times\mathbf{d}_f = \frac{\xi}{\rho_f^+u_{n(3)}^2}\mathbf{n}\times\mathbf{a}_s \tag{7.2.80}$$

Likewise, the cross product of (7.2.79) with the unit normal yields

$$\frac{\delta \mathbf{n}\times\mathbf{a}_s}{\delta t} = -\frac{\xi}{2\rho_{s_R}}\mathbf{n}\times\mathbf{a}_s \tag{7.2.81}$$

Equation (7.2.81) yields the amplitude of the transverse wave as a function of time to be

$$\mathbf{n}\times\mathbf{a}_s\left(t\right) = \mathbf{n}\times\mathbf{a}_s\left(0\right)e^{-\frac{\xi}{2\rho_{s_R}}t} \tag{7.2.82}$$

Because of the thermodynamic result (7.2.6), the amplitude of the transverse wave is either constant or it decays exponentially according to the formula (7.2.82). Given (7.2.82), the tangential projection $\mathbf{n}\times\mathbf{d}_f$ is, from (7.2.80) given by

$$\mathbf{n}\times\mathbf{d}_f = \frac{\xi}{\rho_f^+ u_{n(3)}^2}\mathbf{n}\times\mathbf{a}_s\left(0\right)e^{-\frac{\xi}{2\rho_{s_R}}t} \tag{7.2.83}$$

For either of the longitudinal waves, we can project (7.2.78) and (7.2.79) in the normal direction and obtain

$$\begin{aligned} 2\rho_f^+\frac{\delta \mathbf{a}_f\cdot\mathbf{n}}{\delta t} &= \bar{\lambda}_{ff}\mathbf{d}_f\cdot\mathbf{n} + \bar{\lambda}_{sf}\mathbf{d}_s\cdot\mathbf{n} - \rho_f^+ u_n^2\mathbf{d}_f\cdot\mathbf{n} - \xi\left(\mathbf{a}_f\cdot\mathbf{n} - \mathbf{a}_s\cdot\mathbf{n}\right)\\ &\quad -\frac{\theta^+}{\kappa}(\bar{\tau}_f - \alpha_f + \gamma)(\bar{\tau}_f - \alpha_f - \varsigma/\theta^+)\mathbf{a}_f\cdot\mathbf{n}\\ &\quad -\frac{\theta^+}{\kappa}(\bar{\tau}_f - \alpha_f + \gamma)(\bar{\tau}_s - \alpha_s + \varsigma/\theta^+)\mathbf{a}_s\cdot\mathbf{n} \end{aligned} \tag{7.2.84}$$

and

$$\begin{aligned} 2\rho_{s_R}\frac{\delta \mathbf{a}_s\cdot\mathbf{n}}{\delta t} &= \bar{\lambda}_{sf}\mathbf{d}_f\cdot\mathbf{n} + (\bar{\lambda}_{ss} + 2\mu_{ss})\mathbf{d}_s\cdot\mathbf{n} - \rho_{s_R}u_n^2\mathbf{d}_s\cdot\mathbf{n} - \xi\left(\mathbf{a}_s\cdot\mathbf{n} - \mathbf{a}_f\cdot\mathbf{n}\right)\\ &\quad -\frac{\theta^+}{\kappa}(\bar{\tau}_s - \alpha_s - \gamma)(\bar{\tau}_f - \alpha_f - \varsigma/\theta^+)\left(\mathbf{a}_f\cdot\mathbf{n}\right)\\ &\quad -\frac{\theta^+}{\kappa}(\bar{\tau}_s - \alpha_s - \gamma)(\bar{\tau}_s - \alpha_s + \varsigma/\theta^+)\left(\mathbf{a}_s\cdot\mathbf{n}\right) \end{aligned} \tag{7.2.85}$$

These two equations are best analyzed by writing them as the single matrix equation

$$2\mathbf{M}\frac{\delta \mathbf{a}}{\delta t} = \left(\mathbf{Q} - u_n^2\mathbf{M}\right)\mathbf{d} - \xi\mathbf{E}\mathbf{a} - \frac{\theta^+}{\kappa}\left(\mathbf{A} - \mathbf{B}\right)\left(\mathbf{A}^T - \mathbf{D}^T\right)\mathbf{a} \tag{7.2.86}$$

where $\mathbf{a}, \mathbf{M}$ and $\mathbf{Q}$ are given be (7.2.48), (7.2.47) and (7.2.45), respectively. In addition, the matrices $\mathbf{E}$ and $\mathbf{H}$ are given by

$$\mathbf{E} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{7.2.87}$$

and

$$\mathbf{A} = \begin{bmatrix} \overline{\tau}_f - \alpha_f \\ \overline{\tau}_s - \alpha_s \end{bmatrix} \tag{7.2.88}$$

$$\mathbf{B} = \gamma \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{7.2.89}$$

$$\mathbf{D} = \frac{\varsigma}{\theta^+} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{7.2.90}$$

From (7.2.45), the column matrix $\mathbf{a}$ is the eigenvector of $\mathbf{Q}$ associated with the eigenvalue u^2 with respect to the inner product induced by $\mathbf{M}$. Because we have assumed these eigenvalues are distinct, we can write each eigenvector $\mathbf{a}$ in the form

$$\mathbf{a} = a\mathbf{r} \tag{7.2.91}$$

where a is a scalar and $\mathbf{r}$ is a normalized column matrix with the properties

$$\left(\mathbf{Q} - u^2\mathbf{M}\right)\mathbf{r} = \mathbf{0} \tag{7.2.92}$$

and

$$\mathbf{r}^T\mathbf{M}\mathbf{r} = 1 \tag{7.2.93}$$

The matrix equation (7.2.86) can be projected in the direction $\mathbf{r}$ to yield

$$2\frac{\delta a}{\delta t} = \mathbf{r}^T\left(\mathbf{Q} - u_n^2\mathbf{M}\right)\mathbf{d} - \left(\xi\mathbf{r}^T\mathbf{E}\mathbf{r} + \frac{\theta^+}{\kappa}\mathbf{r}^T\left(\mathbf{A} - \mathbf{B}\right)\left(\mathbf{A}^T - \mathbf{D}^T\right)\mathbf{r}\right)a \tag{7.2.94}$$

after (7.2.93) is utilized. The symmetry of the matrices $\mathbf{Q}$ and $\mathbf{M}$ allow the term $\mathbf{r}^T\left(\mathbf{Q} - u_n^2\mathbf{M}\right)\mathbf{d}$ to be written $\mathbf{d}^T\left(\mathbf{Q} - u_n^2\mathbf{M}\right)\mathbf{r}$ which by (7.2.92) is zero. Therefore, the magnitude of the acceleration wave with the speed u_n is determined by solving the ordinary differential equation

$$2\frac{\delta a}{\delta t}=-\left(\xi\mathbf{r}^T\mathbf{E}\mathbf{r}+\frac{\theta^+}{\kappa}\mathbf{r}^T\left(\mathbf{A}-\mathbf{B}\right)\left(\mathbf{A}^T-\mathbf{D}^T\right)\mathbf{r}\right)a \tag{7.2.95}$$

The behavior of the solution depends upon the sign of the sign of the term $\xi\mathbf{r}^T\mathbf{E}\mathbf{r}+\frac{\theta^+}{\kappa}\mathbf{r}^T\left(\mathbf{A}-\mathbf{B}\right)\left(\mathbf{A}^T-\mathbf{D}^T\right)\mathbf{r}$. This coefficient, as we shall see, cannot be negative. As a result, the amplitude of the longitudinal acceleration wave is either a constant of decays exponentially.

The proof that the term $\xi\mathbf{r}^T\mathbf{E}\mathbf{r}+\frac{\theta^+}{\kappa}\mathbf{r}^T\left(\mathbf{A}-\mathbf{B}\right)\left(\mathbf{A}^T-\mathbf{D}^T\right)\mathbf{r}$ cannot be negative is straight forward but somewhat complicated. The first step is to note the identity

$$\begin{aligned}\left(\mathbf{A}-\mathbf{B}\right)\left(\mathbf{A}^T-\mathbf{D}^T\right)&=\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)-\frac{1}{2}\left(\mathbf{B}-\mathbf{D}\right)\right)\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)+\frac{1}{2}\left(\mathbf{B}-\mathbf{D}\right)\right)^T\\&=\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)^T-\frac{1}{4}\left(\mathbf{B}-\mathbf{D}\right)\left(\mathbf{B}-\mathbf{D}\right)^T\\&\quad+\frac{1}{2}\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)\left(\mathbf{B}-\mathbf{D}\right)^T-\frac{1}{2}\left(\mathbf{B}-\mathbf{D}\right)\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)^T\end{aligned} \tag{7.2.96}$$

It follows from this expression that

$$\begin{aligned}\mathbf{r}^T\left(\mathbf{A}-\mathbf{B}\right)\left(\mathbf{A}^T-\mathbf{D}^T\right)\mathbf{r}&=\mathbf{r}^T\left(\begin{array}{l}\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)^T-\frac{1}{4}\left(\mathbf{B}-\mathbf{D}\right)\left(\mathbf{B}-\mathbf{D}\right)^T\\+\frac{1}{2}\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)\left(\mathbf{B}-\mathbf{D}\right)^T-\frac{1}{2}\left(\mathbf{B}-\mathbf{D}\right)\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)^T\end{array}\right)\mathbf{r}\\&=\mathbf{r}^T\left(\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)^T\right)\mathbf{r}-\frac{1}{4}\mathbf{r}^T\left(\mathbf{B}-\mathbf{D}\right)\left(\mathbf{B}-\mathbf{D}\right)^T\mathbf{r}\\&\quad+\frac{1}{2}\mathbf{r}^T\left(\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)\left(\mathbf{B}-\mathbf{D}\right)^T-\left(\mathbf{B}-\mathbf{D}\right)\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)^T\right)\mathbf{r}\\&=\left(\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)^T\mathbf{r}\right)^2-\frac{1}{4}\mathbf{r}^T\left(\mathbf{B}-\mathbf{D}\right)\left(\mathbf{B}-\mathbf{D}\right)^T\mathbf{r}\end{aligned} \tag{7.2.97}$$

The definitions (7.2.89) and (7.2.90) show that

$$\left(\mathbf{B}-\mathbf{D}\right)\left(\mathbf{B}-\mathbf{D}\right)^T=\left(\gamma+\frac{\varsigma}{\theta^+}\right)^2\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1&1\end{bmatrix}=\left(\gamma+\frac{\varsigma}{\theta^+}\right)^2\mathbf{E} \tag{7.2.98}$$

where the definition (7.2.87) has been used. Equations (7.2.97) and (7.2.98) show that

$$\xi\mathbf{r}^T\mathbf{E}\mathbf{r}+\frac{\theta^+}{\kappa}\mathbf{r}^T\left(\mathbf{A}-\mathbf{B}\right)\left(\mathbf{A}^T-\mathbf{D}^T\right)\mathbf{r}=\left(\xi-\frac{1}{4}\frac{\theta^+}{\kappa}\left(\gamma+\frac{\varsigma}{\theta^+}\right)^2\right)\mathbf{r}^T\mathbf{E}\mathbf{r}+\frac{\theta^+}{\kappa}\left(\left(\mathbf{A}-\frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)^T\mathbf{r}\right)^2 \quad (7.2.99)$$

Because we are working with the assumption that $\kappa>0$, equation (7.2.5) tells us that $\xi-\frac{1}{4}\frac{\theta^+}{\kappa}\left(\gamma+\frac{\varsigma}{\theta^+}\right)^2\geq 0$. Also, the definition (7.2.87) shows that $\mathbf{r}^T\mathbf{E}\mathbf{r}=\left(r_g-r_s\right)^2\geq 0$. Because the last term in (7.2.99) is non-negative, we have the result

$$\xi\mathbf{r}^T\mathbf{E}\mathbf{r}+\frac{\theta^+}{\kappa}\mathbf{r}^T\left(\mathbf{A}-\mathbf{B}\right)\left(\mathbf{A}^T-\mathbf{D}^T\right)\mathbf{r}\geq 0 \quad (7.2.100)$$

Equation (7.2.100) gives the result stated above, i.e., that the amplitude of each longitudinal acceleration wave cannot increase.

There are further simplifications the formula (7.2.99) that are useful. If we look at the two cases $\mathbf{r}=\mathbf{r}_1$ and $\mathbf{r}=\mathbf{r}_2$, it is possible to show that

$$\mathbf{r}_1^T\mathbf{E}\mathbf{r}_1=\operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{E}\right)\frac{u_{n(1)}^2-u_0^2}{u_{n(1)}^2-u_{n(2)}^2}=\left(\frac{1}{\rho_f^+}+\frac{1}{\rho_{s_R}}\right)\frac{u_{n(1)}^2-u_0^2}{u_{n(1)}^2-u_{n(2)}^2} \quad (7.2.101)$$

and

$$\mathbf{r}_2^T\mathbf{E}\mathbf{r}_2=\operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{E}\right)\frac{u_0^2-u_{n(2)}^2}{u_{n(1)}^2-u_{n(2)}^2}=\left(\frac{1}{\rho_f^+}+\frac{1}{\rho_{s_R}}\right)\frac{u_0^2-u_{n(2)}^2}{u_{n(1)}^2-u_{n(2)}^2} \quad (7.2.102)$$

where

$$u_0^2=\frac{\operatorname{tr}\left(\operatorname{adj}\left(\mathbf{Q}\right)\mathbf{E}\right)}{\operatorname{tr}\left(\operatorname{adj}\left(\mathbf{M}\right)\mathbf{E}\right)}=\frac{\bar{\lambda}_{ff}+2\bar{\lambda}_{sf}+\bar{\lambda}_{ss}+2\mu_{ss}}{\rho_f^++\rho_{s_R}} \quad (7.2.103)$$

The derivation of (7.2.101) and (7.2.102) involves a rather straight forward use of the spectral representation of $\mathbf{Q}$. Given the eigenvalue problem (7.2.45), it is possible to show that the spectral representation of $\mathbf{Q}$ is

$$\mathbf{Q}=u_{n(1)}^2\mathbf{M}\mathbf{r}_1^T\mathbf{r}_1\mathbf{M}+u_{n(2)}^2\mathbf{M}\mathbf{r}_2^T\mathbf{r}_2\mathbf{M} \quad (7.2.104)$$

where

$$\mathbf{M} = \mathbf{M}\mathbf{r}_1^T\mathbf{r}_1\mathbf{M} + \mathbf{M}\mathbf{r}_2^T\mathbf{r}_2\mathbf{M} \tag{7.2.105}$$

These two equations yield

$$\mathbf{M}\mathbf{r}_1^T\mathbf{r}_1\mathbf{M} = \frac{\mathbf{Q} - u_{n(2)}^2\mathbf{M}}{u_{n(1)}^2 - u_{n(2)}^2} \tag{7.2.106}$$

and

$$\mathbf{M}\mathbf{r}_2^T\mathbf{r}_2\mathbf{M} = \frac{\mathbf{Q} - u_{n(1)}^2\mathbf{M}}{u_{n(2)}^2 - u_{n(1)}^2} \tag{7.2.107}$$

Given (7.2.106), it follows that

$$\begin{aligned} \mathbf{r}_1^T\mathbf{E}\mathbf{r}_1 &= \operatorname{tr}\left(\mathbf{r}_1\mathbf{r}_1^T\mathbf{E}\right) = \operatorname{tr}\left(\mathbf{M}\mathbf{r}_1\mathbf{r}_1^T\mathbf{M}\mathbf{M}^{-1}\mathbf{E}\mathbf{M}^{-1}\right) \\ &= \operatorname{tr}\left(\left(\frac{\mathbf{Q} - u_{n(2)}^2\mathbf{M}}{u_{n(1)}^2 - u_{n(2)}^2}\right)\mathbf{M}^{-1}\mathbf{E}\mathbf{M}^{-1}\right) = \frac{\operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{Q}\mathbf{M}^{-1}\mathbf{E}\right) - u_{n(2)}^2\operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{E}\right)}{u_{n(1)}^2 - u_{n(2)}^2} \end{aligned} \tag{7.2.108}$$

Because the matrix $\mathbf{Q}$ obeys its own characteristic equation, it is true that

$$\left(\mathbf{Q}\mathbf{M}^{-1}\right)^2 - \operatorname{tr}\left(\mathbf{Q}\mathbf{M}^{-1}\right)\mathbf{Q}\mathbf{M}^{-1} + \det\left(\mathbf{Q}\mathbf{M}^{-1}\right)\mathbf{I} = \mathbf{0} \tag{7.2.109}$$

This equation can be rearranged into the form

$$\mathbf{M}^{-1}\mathbf{Q}\mathbf{M}^{-1} - \operatorname{tr}\left(\mathbf{Q}\mathbf{M}^{-1}\right)\mathbf{M}^{-1} + \det\left(\mathbf{Q}\mathbf{M}^{-1}\right)\mathbf{Q}^{-1} = \mathbf{0} \tag{7.2.110}$$

Therefore,

$$\begin{aligned} \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{Q}\mathbf{M}^{-1}\mathbf{E}\right) &= \operatorname{tr}\left(\mathbf{Q}\mathbf{M}^{-1}\right)\operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{E}\right) - \det\left(\mathbf{Q}\mathbf{M}^{-1}\right)\operatorname{tr}\left(\mathbf{Q}^{-1}\mathbf{E}\right) \\ &= \operatorname{tr}\left(\mathbf{Q}\mathbf{M}^{-1}\right)\operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{E}\right) - \det\mathbf{M}^{-1}\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right) \end{aligned} \tag{7.2.111}$$

where $\operatorname{adj}\mathbf{Q}$ denotes the adjoint of the matrix $\mathbf{Q}$. Because $\operatorname{tr}\left(\mathbf{Q}\mathbf{M}^{-1}\right) = u_{n(1)}^2 + u_{n(2)}^2$ and $\det\left(\mathbf{Q}\mathbf{M}^{-1}\right) = u_{n(1)}^2 u_{n(2)}^2$, (7.2.111) can be rewritten

$$\operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{Q}\mathbf{M}^{-1}\mathbf{E}\right) = \left(u_{n(1)}^2 + u_{n(2)}^2\right)\operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{E}\right) - \det\mathbf{M}^{-1}\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right) \tag{7.2.112}$$

Equation (7.2.112) allows (7.2.108) to be written

$$\mathbf{r}_1^T \mathbf{E} \mathbf{r}_1 = \frac{\operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{Q}\mathbf{M}^{-1}\mathbf{E}\right) - u_{n(2)}^2 \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{E}\right)}{u_{n(1)}^2 - u_{n(2)}^2} = \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{E}\right) \frac{u_{n(1)}^2 - \det \mathbf{M}^{-1} \operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right) \Big/ \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{E}\right)}{u_{n(1)}^2 - u_{n(2)}^2}$$
$$= \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{E}\right) \frac{u_{n(1)}^2 - \operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right) \Big/ \operatorname{tr}\left(\left(\operatorname{adj}\mathbf{M}\right)\mathbf{E}\right)}{u_{n(1)}^2 - u_{n(2)}^2} = \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{E}\right) \frac{u_{n(1)}^2 - u_0^2}{u_{n(1)}^2 - u_{n(2)}^2} \tag{7.2.113}$$

where the definition (7.2.103) has been used. The derivation of (7.2.102) follows by an identical argument.

Clearly the quantity u_0^2 has the physical dimensions of velocity squared. One can show from (7.2.1) through (7.2.3) that u_0^2 is the squared longitudinal wave speed in the limiting case where $\xi \to \infty$ and $\kappa \to \infty$. The case $\xi \to \infty$ corresponds to the case where there is no diffusion, i.e., the drag is so great that the two constituents move at the same velocity. The case $\kappa \to \infty$ corresponds, with appropriate boundary conditions, to the case where the temperature is a constant. For this reason, u_0 is called the *frozen isothermal wave speed*. This speed first appeared in the porous media article by Biot [Ref. 8]. If we adopt the numerical values shown in Table 2 of Section 6.2 and the true density values shown in Table 1 of this Section, the following values for the frozen isothermal wave speed are obtained.

Table 2 Calculated Frozen Isothermal Wave Speeds

Property	Ruhr Sandstone	Tennessee Marble	Charcoal Granite	Berea Sandstone	Westerly Granite	Weber Sandstone
γ_{s_R} (g/cm^3)	2.6	2.7	2.7	2.6	2.7	2.6
u_0 (m/sec)	4337	5345	4954	3209	4802	4059

This table and Table 1 of this Section show that the frozen isothermal wave speeds are close to the acceleration wave speeds $u_{n(1)}$. It is a theoretical result that one can establish from the spectral representation (7.2.104) that

$$u_{n(1)}^2 \geq u_0^2 \geq u_{n(2)}^2 \tag{7.2.114}$$

Tables 1 and 2 show that, to the numerical approximation used here, $u_0 = u_{n(1)}$, for Weber Sandstone.

A similar calculation allows the term $\frac{\theta^+}{\kappa}\left(\left(\mathbf{A} - \frac{1}{2}\left(\mathbf{B}+\mathbf{D}\right)\right)^T \mathbf{r}\right)^2$ in (7.2.99) to be written

$$\frac{\theta^+}{\kappa}\left(\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T\mathbf{r}_1\right)^2=\frac{\theta^+}{\kappa}\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T\mathbf{r}_1\mathbf{r}_1^T\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)$$
$$=\frac{\bar{c}_{\text{v}}}{\kappa}\frac{\left(p_{n(1)}^2-u_{n(1)}^2\right)\left(u_{n(1)}^2-p_{n(2)}^2\right)}{u_{n(1)}^2-u_{n(2)}^2} \tag{7.2.115}$$

and

$$\frac{\theta^+}{\kappa}\left(\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T\mathbf{r}_2\right)^2=\frac{\theta^+}{\kappa}\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T\mathbf{r}_2\mathbf{r}_2^T\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)$$
$$=\frac{\bar{c}_{\text{v}}}{\kappa}\frac{\left(p_{n(1)}^2-u_{n(2)}^2\right)\left(p_{n(2)}^2-u_{n(2)}^2\right)}{u_{n(1)}^2-u_{n(2)}^2} \tag{7.2.116}$$

for the two cases $\mathbf{r}=\mathbf{r}_1$ and $\mathbf{r}=\mathbf{r}_2$, respectively. The quantities $p_{n(1)}^2$ and $p_{n(2)}^2$ are the two roots of the equation

$$\det\left(\mathbf{Q}+\frac{\theta}{c_v}\left(A-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)\left(A-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T-p_n^2\mathbf{M}\right)=0 \tag{7.2.117}$$

These quantities are positive because the symmetric matrix $\mathbf{Q}+\frac{\theta}{c_v}\left(A-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)\left(A-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T$ is positive definite. It is possible to use the spectral form of this matrix to establish that

$$p_{n(1)}^2\geq u_{n(1)}^2\geq p_{n(2)}^2\geq u_{n(2)}^2 \tag{7.2.118}$$

If we adopt the formulas (7.2.101) and (7.2.115), it follows from (7.2.99) that

$$\xi\mathbf{r}_1^T\mathbf{E}\mathbf{r}_1+\frac{\theta^+}{\kappa}\mathbf{r}_1^T(\mathbf{A}-\mathbf{B})\left(\mathbf{A}^T-\mathbf{D}^T\right)\mathbf{r}_1$$
$$=\left(\xi-\frac{1}{4}\frac{\theta^+}{\kappa}\left(\gamma+\frac{\varsigma}{\theta^+}\right)^2\right)\left(\frac{1}{\rho_f^+}+\frac{1}{\rho_{s_R}}\right)\frac{u_{n(1)}^2-u_0^2}{u_{n(1)}^2-u_{n(2)}^2}+\frac{\bar{c}_{\text{v}}}{\kappa}\frac{\left(p_{n(1)}^2-u_{n(1)}^2\right)\left(u_{n(1)}^2-p_{n(2)}^2\right)}{u_{n(1)}^2-u_{n(2)}^2} \tag{7.2.119}$$

for the choice $\mathbf{r}=\mathbf{r}_1$. By an entirely similar calculation,

$$\xi \mathbf{r}_2^T \mathbf{E} \mathbf{r}_2 + \frac{\theta^+}{\kappa} \mathbf{r}_2^T \left(\mathbf{A} - \mathbf{B}\right)\left(\mathbf{A}^T - \mathbf{D}^T\right)\mathbf{r}_2 = \left(\xi - \frac{1}{4}\frac{\theta^+}{\kappa}\left(\gamma + \frac{\varsigma}{\theta^+}\right)^2\right)\left(\frac{1}{\rho_f^+} + \frac{1}{\rho_{s_R}}\right)\frac{u_0^2 - u_{n(2)}^2}{u_{n(1)}^2 - u_{n(2)}^2} + \frac{\overline{c}_v}{\kappa}\frac{\left(p_{n(1)}^2 - u_{n(2)}^2\right)\left(p_{n(2)}^2 - u_{n(2)}^2\right)}{u_{n(1)}^2 - u_{n(2)}^2} \quad (7.2.120)$$

for the choice $\mathbf{r} = \mathbf{r}_2$.

Equations (7.2.119) and (7.2.120) have the advantage of allowing the calculation of the coefficient in the amplitude ordinary differential equation, equation (7.2.95), without explicit knowledge of the eigenvectors $\mathbf{r}_1$ and $\mathbf{r}_2$. In the next chapter we shall investigate the propagation of plane harmonic waves in the poroelastic mixture adopted in this chapter. We shall see the expressions (7.2.119) and (7.2.120) arise in one of the limiting cases examined for these types of waves.

In closing this chapter, it is useful to briefly examine a case omitted in the derivation of the above results. This case is the one ruled out by the assumption (7.2.29), i.e. $\kappa = 0$. If we allow for this possibility, then (7.2.28) is satisfied without necessarily having $\left[\!\left[\text{GRAD}\,\theta\right]\!\right] \cdot \mathbf{n} = 0$. In this case, the inequality (7.2.5) forces

$$\gamma = -\frac{\varsigma}{\theta^+} \quad (7.2.121)$$

or, from the definitions (7.2.89) and (7.2.90),

$$\mathbf{B} = \mathbf{D} \quad (7.2.122)$$

Logically, if one reaches the result $\kappa = 0$ with the argument that the constitutive equations do not have a dependence on the temperature gradient, one could conclude from (3.2.24) that γ and, from (7.2.121), ς are zero. For our purposes here, the result (7.2.121) is a sufficient specialization.

Rather than repeat the calculations we carried out the case for a conductor, it is convenient to simply state the key results. It can be shown from (7.2.3), with $\kappa = 0$, that

$$u_n^2 \overline{c}_v \left[\!\left[\text{GRAD}\,\theta\right]\!\right] = -\theta^+ \left(\overline{\tau}_f - \alpha_f - \gamma\right)\left(\mathbf{a}_f \cdot \mathbf{n}\right)\mathbf{n} - \theta^+ \left(\overline{\tau}_s - \alpha_s + \gamma\right)\left(\mathbf{a}_s \cdot \mathbf{n}\right)\mathbf{n} \quad (7.2.123)$$

Another jump that arises in the derivation of the differential equation for the amplitude is $\left[\!\left[\text{GRAD}\frac{\partial \theta}{\partial t}\right]\!\right]$. This quantity is also determined by the energy equation and can be shown to take the form

$$\overline{c}_v \left[\!\left[\text{GRAD}\frac{\partial \theta}{\partial t}\right]\!\right] = -\theta^+ \left(\overline{\tau}_f - \alpha_f - \gamma\right)\left(\mathbf{d}_f \cdot \mathbf{n}\right)\mathbf{n} - \theta^+ \left(\overline{\tau}_s - \alpha_s + \gamma\right)\left(\mathbf{d}_s \cdot \mathbf{n}\right)\mathbf{n} \quad (7.2.124)$$

The results for the nonconductor shall be given without proof. One can formally adopt (7.2.36) through (7.2.50) with the following changes:

a) Replace $\bar{\lambda}_{ff}$ with $\bar{\lambda}_{ff} + \dfrac{\theta^+}{\bar{c}_v}\left(\bar{\tau}_f - \alpha_f - \gamma\right)^2$.

b) Replace $\bar{\lambda}_{sf}$ with $\bar{\lambda}_{sf} + \dfrac{\theta^+}{\bar{c}_v}\left(\bar{\tau}_f - \alpha_f - \gamma\right)\left(\bar{\tau}_s - \alpha_s + \gamma\right)$

c) Replace $\bar{\lambda}_{ss}$ with $\bar{\lambda}_{ss} + \dfrac{\theta^+}{\bar{c}_v}\left(\bar{\tau}_s - \alpha_s + \gamma\right)^2$.

The amplitude equation for transverse waves, equation (7.2.82), can be adopted without formal change. For the longitudinal waves, the amplitude is given by

$$2\frac{\delta a}{\delta t} = -\left(\xi \mathbf{r}^T \mathbf{E}\mathbf{r}\right)a \tag{7.2.125}$$

rather than (7.2.95). However, in this case the eigenvectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are calculated from (7.2.92) with $\mathbf{Q}$ given by

$$\mathbf{Q} = \begin{bmatrix} \bar{\lambda}_{ff} + \dfrac{\theta^+}{\bar{c}_v}\left(\bar{\tau}_f - \alpha_f - \gamma\right)^2 & \bar{\lambda}_{sf} + \dfrac{\theta^+}{\bar{c}_v}\left(\bar{\tau}_f - \alpha_f - \gamma\right)\left(\bar{\tau}_s - \alpha_s + \gamma\right) \\ \bar{\lambda}_{sf} + \dfrac{\theta^+}{\bar{c}_v}\left(\bar{\tau}_f - \alpha_f - \gamma\right)\left(\bar{\tau}_s - \alpha_s + \gamma\right) & \bar{\lambda}_{ss} + \dfrac{\theta^+}{\bar{c}_v}\left(\bar{\tau}_s - \alpha_s + \gamma\right)^2 + 2\mu_{ss} \end{bmatrix} \tag{7.2.126}$$

rather than by (7.2.46).

The nonconductor case just discussed also provides a framework for the study of the growth and decay of shock waves in porous materials defined by (7.2.1) through (7.2.3). By definition, a *shock wave* in the porous material defined by (7.2.1) through (7.2.6) is a moving singular surface across which

$$[\mathbf{w}_f] = [\mathbf{w}_s] = \mathbf{0} \tag{7.2.127}$$

but $[\dfrac{\partial^2 \mathbf{w}_f}{\partial t^2}]$, $[\dfrac{\partial^2 \mathbf{w}_s}{\partial t^2}]$ and $[\dfrac{\partial \theta}{\partial t}]$ are not necessarily zero. The amplitudes of the shock wave are defined by

$$\mathbf{a}_f = [\frac{\partial \mathbf{w}_f}{\partial t}] \tag{7.2.128}$$

and

$$\mathbf{a}_s = [\frac{\partial \mathbf{w}_s}{\partial t}] \tag{7.2.129}$$

The jump balance equations (7.2.12) through (7.2.14), with $\kappa = 0$, yield the results

$$\rho_{s_R} u_n^2 \mathbf{a}_s - \left(\bar{\lambda}_{ss} + \mu_{ss}\right)(\mathbf{a}_s \cdot \mathbf{n})\mathbf{n} - \mu_{ss}\mathbf{a}_s - \bar{\lambda}_{sf}\left(\mathbf{a}_f \cdot \mathbf{n}\right)\mathbf{n} + (\alpha_s - \bar{\tau}_s + \gamma)u_n[\theta]\mathbf{n} = \mathbf{0} \tag{7.2.130}$$

$$\rho_f^+ u_n^2 \mathbf{a}_f - \bar{\lambda}_{sf}(\mathbf{a}_s \cdot \mathbf{n})\mathbf{n} - \bar{\lambda}_{ff}\left(\mathbf{a}_f \cdot \mathbf{n}\right)\mathbf{n} + (\alpha_f - \bar{\tau}_f - \gamma)u_n[\theta]\mathbf{n} = \mathbf{0} \tag{7.2.131}$$

and

$$u_n \bar{c}_v[\theta] + \theta^+\left(\alpha_s - \bar{\tau}_s + \gamma\right)\mathbf{a}_s \cdot \mathbf{n} + \theta^+\left(\alpha_f - \bar{\tau}_f - \gamma\right)\mathbf{a}_f \cdot \mathbf{n} = 0 \tag{7.2.132}$$

Equation (7.2.132) gives the jump in the temperature across the shock wave in terms of the two shock amplitudes. If this result is substituted into (7.2.130) and (7.2.131) the result is

$$\begin{aligned}&\left(\bar{\lambda}_{sf} + \frac{\theta^+}{\bar{c}_v}\left(\alpha_s - \bar{\tau}_s + \gamma\right)\left(\alpha_f - \bar{\tau}_f - \gamma\right)\right)\left(\mathbf{a}_f \cdot \mathbf{n}\right)\mathbf{n}\\&+\left(\bar{\lambda}_{ss} + \frac{\theta^+}{\bar{c}_v}\left(\alpha_s - \bar{\tau}_s + \gamma\right)\left(\alpha_s - \bar{\tau}_s + \gamma\right) + \mu_{ss}\right)\left(\mathbf{a}_s \cdot \mathbf{n}\right)\mathbf{n} + \mu_{ss}\mathbf{a}_s = \rho_{s_R} u_n^2 \mathbf{a}_s\end{aligned} \tag{7.2.133}$$

$$\begin{aligned}&\left(\bar{\lambda}_{ff} + \frac{\theta^+}{\bar{c}_v}\left(\alpha_f - \bar{\tau}_f - \gamma\right)^2\right)\left(\mathbf{a}_f \cdot \mathbf{n}\right)\mathbf{n} + \left(\bar{\lambda}_{sf} + \frac{\theta^+}{\bar{c}_v}\left(\alpha_s - \bar{\tau}_s + \gamma\right)\left(\alpha_f - \bar{\tau}_f - \gamma\right)\right)\left(\mathbf{a}_s \cdot \mathbf{n}\right)\mathbf{n}\\&= \rho_f^+ u_n^2 \mathbf{a}_f\end{aligned} \tag{7.2.134}$$

These equations are formally identical to our previous results for acceleration waves (7.2.36) and (7.2.37) *providing* that we replace the coefficients according to the rules a), b) and c) above. The fact that shock waves in a linear theory have the same propagation as that for acceleration waves is typical. Sometimes such shock waves are called *weak shocks*. The growth and decay of these weak shocks in the longitudinal are given by equations formally identical to (7.2.125).

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8

Plane Harmonic Waves

In this chapter we shall build upon the results in Chapter 7 and look at a more interesting kind of wave propagation problem. The problem adopts one dimensional versions of the governing partial differential equations studied in Chapter 7 and examines a certain type of wave solution. The solution is that of a plane harmonic wave. The one dimensional assumption is made as a matter of convenience and can be avoided as was done in Chapter 6 by the assumption of a plane wave. For simplicity here, we shall just adopt the one dimensional formulation. The interest in studying harmonic waves is that it represents a nontrivial solution of the governing equations that carries a lot of interesting physical information about the features of the model. It is a solution that lends itself to analytical analysis as opposed to many other interesting problems that require numerical solutions. To a significant extent, the results in this chapter are a special case of the work of Bowen [Ref. 1] and Bowen and Reinicke [Ref. 2]. The earlier works of Biot [Ref. 3] and of Craine and Johnson [Ref. 4] are also important to the understanding of the material in this Chapter. This chapter is also influenced by the formalism used by Chadwick [Ref. 5] when he studied plane harmonic waves in thermoelasticity.

8.1 One Dimensional Governing Equations

If we again adopt the governing partial differential equations (7.2.1) through (7.2.3) and the restrictions on the properties (7.2.4) through (7.2.11), the one dimensional case corresponds to the assumptions

$$\mathbf{w}_s = \left(w_s \left(X, t \right), 0, 0 \right) \tag{8.1.1}$$

$$\mathbf{w}_f = \left(w_f \left(X, t \right), 0, 0 \right) \tag{8.1.2}$$

and

$$\theta = \theta\left(X, t \right) \tag{8.1.3}$$

These specializations reduce the governing partial differential equations (7.2.1), (7.2.2) and (7.2.3) to

$$\rho_{s_R} \frac{\partial^2 w_s}{\partial t^2} = (\bar{\lambda}_{ss} + 2\mu_{ss}) \frac{\partial^2 w_s}{\partial X^2} + \bar{\lambda}_{sf} \frac{\partial^2 w_f}{\partial X^2} + (\alpha_s - \bar{\tau}_s + \gamma) \frac{\partial \theta}{\partial X} - \xi \left(\frac{\partial w_s}{\partial t} - \frac{\partial w_f}{\partial t} \right) \tag{8.1.4}$$

$$\rho_f^+ \frac{\partial^2 w_f}{\partial t^2} = \bar{\lambda}_{sf} \frac{\partial^2 w_s}{\partial X^2} + \bar{\lambda}_{ff} \frac{\partial^2 w_f}{\partial X^2} + (\alpha_f - \bar{\tau}_f - \gamma) \frac{\partial \theta}{\partial X} - \xi \left(\frac{\partial w_f}{\partial t} - \frac{\partial w_s}{\partial t} \right) \quad (8.1.5)$$

and

$$\bar{c}_{\text{v}} \frac{\partial \theta}{\partial t} + \theta^+ \left((\bar{\tau}_s - \alpha_s + \varsigma / \theta^+) \frac{\partial^2 w_s}{\partial X \partial t} + (\bar{\tau}_f - \alpha_f - \varsigma / \theta^+) \frac{\partial^2 w_f}{\partial X \partial t} \right) = \kappa \frac{\partial^2 \theta}{\partial X^2} \quad (8.1.6)$$

The manipulation of these equations is facilitated if we write them in matrix form as follows:

$$\mathbf{M} \frac{\partial^2 \mathbf{w}}{\partial t^2} = \mathbf{Q} \frac{\partial^2 \mathbf{w}}{\partial X^2} - \left(\mathbf{A} - \mathbf{B} \right) \frac{\partial \theta}{\partial X} - \xi \mathbf{E} \frac{\partial \mathbf{w}}{\partial t} \quad (8.1.7)$$

and

$$\bar{c}_{\text{v}} \frac{\partial \theta}{\partial t} + \theta^+ \left(\mathbf{A}^T - \mathbf{D}^T \right) \frac{\partial^2 \mathbf{w}}{\partial X \partial t} = \kappa \frac{\partial^2 \theta}{\partial X^2} \quad (8.1.8)$$

where

$$\mathbf{w}(X,t) = \begin{bmatrix} w_f(X,t) \\ w_s(X,t) \end{bmatrix} \quad (8.1.9)$$

and, as in Chapter 7,

$$\mathbf{M} = \begin{bmatrix} \rho_f^+ & 0 \\ 0 & \rho_{s_R} \end{bmatrix} \quad (8.1.10)$$

$$\mathbf{Q} = \begin{bmatrix} \bar{\lambda}_{ff} & \bar{\lambda}_{sf} \\ \bar{\lambda}_{sf} & \bar{\lambda}_{ss} + 2\mu_{ss} \end{bmatrix} \quad (8.1.11)$$

$$\mathbf{E} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (8.1.12)$$

and

$$\mathbf{A} = \begin{bmatrix} \bar{\tau}_f - \alpha_f \\ \bar{\tau}_s - \alpha_s \end{bmatrix} \quad (8.1.13)$$

$$\mathbf{B} = \gamma \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{8.1.14}$$

$$\mathbf{D} = \frac{\varsigma}{\theta^+} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{8.1.15}$$

It is perhaps useful to note in passing that one can derive from (8.1.7) and (8.1.8) a single uncoupled partial differential equation for each of the unknowns. In the case $\kappa > 0$, the result is three uncoupled partial differential equations of the form

$$Lw_f = Lw_s = L\theta = 0 \tag{8.1.16}$$

where L is the sixth order differential operator defined by

$$\begin{aligned} L =& \frac{\partial}{\partial t}\left[\left(\frac{\partial^2}{\partial t^2} - p_{n(1)}^2 \frac{\partial^2}{\partial X^2}\right)\left(\frac{\partial^2}{\partial t^2} - p_{n(2)}^2 \frac{\partial^2}{\partial X^2}\right) + \omega_\xi \frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t^2} - p_0^2 \frac{\partial^2}{\partial X^2}\right)\right] \\ &- \frac{\kappa}{\bar{c}_v}\frac{\partial^2}{\partial X^2}\left[\left(\frac{\partial^2}{\partial t^2} - u_{n(1)}^2 \frac{\partial^2}{\partial X^2}\right)\left(\frac{\partial^2}{\partial t^2} - u_{n(2)}^2 \frac{\partial^2}{\partial X^2}\right) + \omega_0 \frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t^2} - u_0^2 \frac{\partial^2}{\partial X^2}\right)\right] \end{aligned} \tag{8.1.17}$$

where ω_0 and ω_1 are characteristic frequencies defined by

$$\omega_0 = \left(\xi - \frac{\theta^+}{4\kappa}\left(\gamma + \frac{\varsigma}{\theta^+}\right)^2\right)\frac{\operatorname{tr}\left((\operatorname{adj}\mathbf{M})\mathbf{E}\right)}{\det \mathbf{M}} = \left(\xi - \frac{\theta^+}{4\kappa}\left(\gamma + \frac{\varsigma}{\theta^+}\right)^2\right)\left(\frac{1}{\rho_R} + \frac{1}{\rho_f^+}\right) \tag{8.1.18}$$

and

$$\omega_\xi = \xi \frac{\operatorname{tr}\left((\operatorname{adj}\mathbf{M})\mathbf{E}\right)}{\det \mathbf{M}} = \xi\left(\frac{1}{\rho_R} + \frac{1}{\rho_f^+}\right) \tag{8.1.19}$$

Note, in passing, that the inequalities (7.2.5) and (7.2.6) insure that the characteristic times ω_0 and ω_ξ are non-negative. Also note that the definition (8.1.18) assumes a strengthened version of (7.2.4) in the form

$$\kappa > 0 \tag{8.1.20}$$

The quantities $u_{n(1)}^2$ and $u_{n(2)}^2$ in (8.1.17) are given by (7.2.50) and u_0^2 is given by (7.2.103). The quantities $p_{n(1)}^2$ and $p_{n(2)}^2$, as in equation (7.2.117), are the roots of

$$\det\left(\mathbf{Q}+\frac{\theta^{+}}{2c_{\mathrm{v}}^{+}}\left(\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^{T}\right)-p^{2}\mathbf{M}\right)=0 \tag{8.1.21}$$

and p_0^2 is given by

$$p_0^2=\frac{\operatorname{tr}\left(\left(\mathbf{Q}+\frac{\theta^{+}}{2c_{\mathrm{v}}^{+}}\left(\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^{T}\right)\right)\mathbf{E}\right)}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{M}\right)\mathbf{E}\right)} \tag{8.1.22}$$

We shall say more about these roots in Section 8.2.

8.2 Plane Progressive Waves: Dispersion Relation

A plane progressive wave is a solution of (8.1.7) and (8.1.8) of the form

$$\mathbf{w}(X,t)=\mathbf{a}e^{i(kx-\omega t)} \tag{8.2.1}$$

and

$$\theta(X,t)-\theta^{+}=be^{i(kx-\omega t)} \tag{8.2.2}$$

where the 2×1 matrix $\mathbf{a}$ and the scalar b are the amplitudes of the wave. The quantity k is the *wave number* and ω is the *frequency*. The point if view is that the frequency is given and we seek the relationship $k(\omega)$ forced by requiring (8.2.1) and (8.2.2) to be solutions of the partial differential equations (8.1.7) and (8.1.8). The relationship $k(\omega)$ is called the *dispersion relation*. Given this relationship, the *phase velocity* and the *attenuation coefficient* are defined by

$$q=\frac{\omega}{\mathscr{R}(k)} \tag{8.2.3}$$

and

$$s=\mathscr{I}(k) \tag{8.2.4}$$

where $\mathscr{R}(k)$ denotes the real part of $k(\omega)$ and $\mathscr{I}(k)$ its imaginary part.

If (8.2.1) and (8.2.2) are substituted into (8.1.7) and (8.1.8), respectively, the result is

$$\left(k^{2}\mathbf{Q}-\omega^{2}\mathbf{M}-i\omega\xi\mathbf{E}\right)\mathbf{a}+ik\left(\mathbf{A}-\mathbf{B}\right)b=0 \tag{8.2.5}$$

and

$$\theta^{+}\left(\mathbf{A}^{T}-\mathbf{D}^{T}\right)\omega k\mathbf{a}+\left(\kappa k^{2}-i\omega\bar{c}_{\mathrm{v}}\right)b=0 \tag{8.2.6}$$

In order that equations (8.2.5) and (8.2.6) yield nonzero amplitudes, the 3×3 determinant of coefficients must vanish, i.e.,[1]

$$\begin{aligned}&\left(\kappa k^2 - i\omega\bar{c}_{\mathrm{v}}\right)\det\left(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\right)\\&\quad -\theta^{+} i\omega k^2\left(\mathbf{A}^T-\mathbf{D}^T\right)\operatorname{adj}\left(\left(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\right)\right)\left(\mathbf{A}-\mathbf{B}\right)=0\end{aligned} \tag{8.2.7}$$

This expression can be written

$$\begin{aligned}&i\omega\left(\det\left(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\right)+\frac{\theta^{+}}{c_{\mathrm{v}}}k^2\left(\mathbf{A}^T-\mathbf{D}^T\right)\operatorname{adj}\left(\left(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\right)\right)\left(\mathbf{A}-\mathbf{B}\right)\right)\\&\quad -\frac{\kappa}{c_{\mathrm{v}}}k^2\det\left(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\right)=0\end{aligned} \tag{8.2.8}$$

Next, we shall show that (8.2.8) can be written

$$\begin{aligned}&i\omega\left(\left(p_{n(1)}^2k^2-\omega^2\right)\left(p_{n(2)}^2k^2-\omega^2\right)-i\omega\omega_{\xi}\left(p_0^2k^2-\omega^2\right)\right)\\&\quad -\frac{\kappa}{\bar{c}_{\mathrm{v}}}k^2\left(\left(u_{n(1)}^2k^2-\omega^2\right)\left(u_{n(2)}^2k^2-\omega^2\right)-i\omega\omega_0\left(u_0^2k^2-\omega^2\right)\right)=0\end{aligned} \tag{8.2.9}$$

The first formula we need is

$$\begin{aligned}&\det\left(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\right)\\&\quad =\det\mathbf{M}\left(\left(u_{n(1)}^2k^2-\omega^2\right)\left(u_{n(2)}^2k^2-\omega^2\right)-i\omega\xi\frac{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{M}\right)\mathbf{E}\right)}{\det\mathbf{M}}\left(u_0^2k^2-\omega^2\right)\right)\end{aligned} \tag{8.2.10}$$

This formula follows from the elementary formula for the determinant of the sum of 2×2 matrices, $\det\left(\mathbf{A}+\mathbf{B}\right)=\det\mathbf{A}+\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{A}\right)\mathbf{B}\right)+\det\mathbf{B}$ and the formulas

$$\det\left(k^2\mathbf{Q}-\omega^2\mathbf{M}\right)=\det\mathbf{M}\left(k^2u_1^2-\omega^2\right)\left(k^2u_2^2-\omega^2\right) \tag{8.2.11}$$

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[1] For the 3×3 matrix $\mathbf{F}=\begin{bmatrix}A_{11} & A_{12} & B_1\\ A_{21} & A_{22} & B_2\\ C_1 & C_2 & D\end{bmatrix}$, it is not difficult to show that $\det\mathbf{F}=\mathbf{D}\det\mathbf{A}-\mathbf{C}\left(\operatorname{adj}\mathbf{A}\right)\mathbf{B}$, where $\mathbf{A}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}B_1\\ B_2\end{bmatrix}$ and $\mathbf{C}=\begin{bmatrix}C_1 & C_2\end{bmatrix}$.

$$u_0^2 = \frac{\mathrm{tr}\left(\mathrm{adj}(\mathbf{Q})\mathbf{E}\right)}{\mathrm{tr}\left(\mathrm{adj}(\mathbf{M})\mathbf{E}\right)} \tag{8.2.12}$$

and

$$\det \mathbf{E} = 0 \tag{8.2.13}$$

Equation (8.2.11) follows from (7.2.92), equation (8.2.12) is the definition (7.2.103) repeated, and (8.2.13) follows from the definition (7.2.87).

The next formula we need involves a rather complicated rearrangement of the term $\left(\mathbf{A}^T - \mathbf{D}^T\right)\mathrm{adj}\left(\left(k^2\mathbf{Q} - \omega^2\mathbf{M} - i\omega\xi\mathbf{E}\right)\right)\left(\mathbf{A} - \mathbf{B}\right)$. First, because the matrix $\mathrm{adj}\left(\left(k^2\mathbf{Q} - \omega^2\mathbf{M} - i\omega\xi\mathbf{E}\right)\right)$ is symmetric, we can use (7.2.96) and write

$$\begin{aligned}
&\left(\mathbf{A}^T - \mathbf{D}^T\right)\mathrm{adj}\left(\left(k^2\mathbf{Q} - \omega^2\mathbf{M} - i\omega\xi\mathbf{E}\right)\right)\left(\mathbf{A} - \mathbf{B}\right) \\
&\quad = \mathrm{tr}\left(\left(\mathrm{adj}\left(k^2\mathbf{Q} - \omega^2\mathbf{M} - i\omega\xi\mathbf{E}\right)\right)\left(\mathbf{A} - \mathbf{B}\right)\left(\mathbf{A}^T - \mathbf{D}^T\right)\right) \\
&\quad = \mathrm{tr}\left(\left(\mathrm{adj}\left(k^2\mathbf{Q} - \omega^2\mathbf{M} - i\omega\xi\mathbf{E}\right)\right)\left(\left(\mathbf{A} - \frac{1}{2}(\mathbf{B}+\mathbf{D})\right)\left(\mathbf{A} - \frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T - \frac{1}{4}(\mathbf{B}-\mathbf{D})(\mathbf{B}-\mathbf{D})^T\right)\right) \\
&\quad = \mathrm{tr}\left(\left(\mathrm{adj}\left(k^2\mathbf{Q} - \omega^2\mathbf{M} - i\omega\xi\mathbf{E}\right)\right)\left(\left(\mathbf{A} - \frac{1}{2}(\mathbf{B}+\mathbf{D})\right)\left(\mathbf{A} - \frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T\right)\right) \\
&\quad - \frac{1}{4}\mathrm{tr}\left(\left(\mathrm{adj}\left(k^2\mathbf{Q} - \omega^2\mathbf{M} - i\omega\xi\mathbf{E}\right)\right)\left((\mathbf{B}-\mathbf{D})(\mathbf{B}-\mathbf{D})^T\right)\right)
\end{aligned} \tag{8.2.14}$$

Equation (7.2.98), repeated here, tells us that

$$\left(\mathbf{B} - \mathbf{D}\right)\left(\mathbf{B} - \mathbf{D}\right)^T = \left(\gamma + \frac{\varsigma}{\theta^+}\right)^2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} = \left(\gamma + \frac{\varsigma}{\theta^+}\right)^2 \mathbf{E} \tag{8.2.15}$$

This identity along with the definition (8.2.12) allows the last term in (8.2.14) to be written

$$
\begin{aligned}
-\frac{1}{4}\operatorname{tr}\Big(\big(\operatorname{adj}\big(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\big)\big)\big((\mathbf{B}-\mathbf{D})(\mathbf{B}-\mathbf{D})^T\big)\Big) &= -\frac{1}{4}\left(\gamma+\frac{\varsigma}{\theta^+}\right)^2\operatorname{tr}\Big(\big(\operatorname{adj}\big(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\big)\big)\mathbf{E}\Big)\\
&= -\frac{1}{4}\left(\gamma+\frac{\varsigma}{\theta^+}\right)^2\operatorname{tr}\Big(\big(\operatorname{adj}\big(k^2\mathbf{Q}-\omega^2\mathbf{M}\big)\big)\mathbf{E}\Big)\\
&= -\frac{1}{4}\left(\gamma+\frac{\varsigma}{\theta^+}\right)^2\operatorname{tr}\big(\operatorname{adj}\mathbf{M}\big)\big(u_0^2k^2-\omega^2\big)
\end{aligned}
\tag{8.2.16}
$$

Given (8.2.14) and (8.2.16), it follows that

$$
\begin{aligned}
&\det\big(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\big)+\frac{\theta^+}{c_{\text{v}}}k^2\left(\mathbf{A}^T-\mathbf{D}^T\right)\operatorname{adj}\Big(\big(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\big)\Big)\big(\mathbf{A}-\mathbf{B}\big)\\
&=\det\big(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\big)\\
&\qquad+\frac{\theta^+}{c_{\text{v}}}k^2\operatorname{tr}\left(\big(\operatorname{adj}\big(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\big)\big)\left(\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T\right)\right)\\
&\qquad-\frac{\theta^+}{4c_{\text{v}}}k^2\operatorname{tr}\Big(\big(\operatorname{adj}\big(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\big)\big)\big((\mathbf{B}-\mathbf{D})(\mathbf{B}-\mathbf{D})^T\big)\Big)\\
&=\det\left(k^2\left(\mathbf{Q}+\frac{\theta^+}{c_{\text{v}}}\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T\right)-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\right)\\
&\qquad-\frac{\theta^+}{4c_{\text{v}}}k^2\left(\gamma+\frac{\varsigma}{\theta^+}\right)^2\operatorname{tr}\big(\operatorname{adj}\mathbf{M}\big)\big(u_0^2k^2-\omega^2\big)
\end{aligned}
\tag{8.2.17}
$$

This formula utilized, again, the identity $\det(\mathbf{A}+\mathbf{B})=\det\mathbf{A}+\operatorname{tr}\big((\operatorname{adj}\mathbf{A})\mathbf{B}\big)+\det\mathbf{B}$ except this time to combine terms. This formula also made use of the result
$\det\left(\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T\right)=0$.

Given (8.2.17), the result (8.2.8) can be written

$$
\begin{aligned}
&i\omega\det\left(k^2\left(\mathbf{Q}+\frac{\theta^+}{c_{\text{v}}}\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T\right)-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\right)\\
&\qquad-\frac{\kappa}{c_{\text{v}}}k^2\left(\det\big(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\big)+i\omega\frac{\theta^+}{4\kappa}\left(\gamma+\frac{\varsigma}{\theta^+}\right)^2\operatorname{tr}\big(\operatorname{adj}\mathbf{M}\big)\big(u_0^2k^2-\omega^2\big)\right)=0
\end{aligned}
\tag{8.2.18}
$$

Given (8.2.10), the last factor in (8.2.18) can be written

$$\det\left(k^2\mathbf{Q}-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\right)+i\omega\frac{\theta^+}{4\kappa}\left(\gamma+\frac{\varsigma}{\theta^+}\right)^2\operatorname{tr}\left(\operatorname{adj}\mathbf{M}\right)\left(u_0^2k^2-\omega^2\right)$$
$$=\det\mathbf{M}\left(\left(u_{n(1)}^2k^2-\omega^2\right)\left(u_{n(2)}^2k^2-\omega^2\right)-i\omega\omega_0\left(u_0^2k^2-\omega^2\right)\right) \tag{8.2.19}$$

where the definition of the characteristic frequency ω_0, given by (8.1.18), has been used. Equation (8.2.18) reduces to the result (8.2.9) if we use the identity

$$\det\left(k^2\left(\mathbf{Q}+\frac{\theta^+}{c_\text{v}}\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T\right)-\omega^2\mathbf{M}-i\omega\xi\mathbf{E}\right)$$
$$=\det\mathbf{M}\left(\left(p_{n(1)}^2k^2-\omega^2\right)\left(p_{n(2)}^2k^2-\omega^2\right)-i\omega\omega_\xi\left(p_0^2k^2-\omega^2\right)\right) \tag{8.2.20}$$

Equation (8.2.20) is derived by the same calculation that produced (8.2.10) with the exception that now the symmetric matrix $\mathbf{Q}+\frac{\theta^+}{c_\text{v}}\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T$ is in the formula. The definition (8.1.19) has also been used.

A few words or explanation need to be given about the parameters $p_{n(1)}^2$, $p_{n(2)}^2$ and p_0^2 defined by (8.1.21) and (8.1.22). Because the symmetric matrix $\mathbf{Q}+\frac{\theta^+}{c_\text{v}}\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)\left(\mathbf{A}-\frac{1}{2}(\mathbf{B}+\mathbf{D})\right)^T$ is also positive definite, the roots of (8.1.21) are positive. Thus, $p_{n(1)}$ and $p_{n(2)}$, like $u_{n(1)}$ and $u_{n(2)}$, are characteristic speeds. The same is true for p_0. The same kind of argument that yielded (7.2.114) can be used to establish

$$p_{n(1)}^2\geq p_0^2\geq p_{n(2)}^2 \tag{8.2.21}$$

As was mentioned in Section 7.2 in equation (7.2.118), another set of inequalities on can establish are

$$p_{n(1)}^2\geq u_{n(1)}^2\geq p_{n(2)}^2\geq u_{n(2)}^2 \tag{8.2.22}$$

and

$$p_0^2\geq u_0^2 \tag{8.2.23}$$

These inequalities follow from the definitions (7.2.50), (7.2.51), (8.1.21) and (8.1.22). It is useful to note in passing that the definition (8.1.22) of p_0^2 can be replaced by

$$p_0^2 = \frac{\operatorname{tr}\left(\left(\mathbf{Q} + \frac{\theta^+}{c_\text{v}^+}\mathbf{A}\mathbf{A}^T\right)\mathbf{E}\right)}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{M}\right)\mathbf{E}\right)} \tag{8.2.24}$$

The dispersion relation (8.2.9) gives the wave number $k(\omega)$ in terms of the frequency, ω, six characteristic wave speeds squared, $p_{n(1)}^2, p_{n(2)}^2, p_{n(2)}^2, u_{n(1)}^2, u_{n(2)}^2$ and u_0^2, two characteristic frequencies ω_0 and ω_ξ and the dimensional ratio $\dfrac{\kappa}{\overline{c}_\text{v}}$. It is useful to isolate a third characteristic frequency ω_κ defined by

$$\frac{1}{\omega_\kappa} = \frac{\kappa}{\overline{c}_\text{v} u_{n(2)}^2} \tag{8.2.25}$$

The velocity squared $u_{n(2)}^2$ in (8.2.25) can be replaced by any of the other characteristic velocities. Given the definition (8.2.25), the dispersion relation (8.2.9) takes the form

$$\begin{aligned} u_{n(2)}^2 k^2 \left(\left(u_{n(1)}^2 k^2 - \omega^2\right)\left(u_{n(2)}^2 k^2 - \omega^2\right) - i\omega\omega_0\left(u_0^2 k^2 - \omega^2\right)\right) \\ -i\omega\omega_\kappa\left(\left(p_{n(1)}^2 k^2 - \omega^2\right)\left(p_{n(2)}^2 k^2 - \omega^2\right) - i\omega\omega_\xi\left(p_0^2 k^2 - \omega^2\right)\right) = 0 \end{aligned} \tag{8.2.26}$$

It is this third order polynomial in the wave number squared. This equation determines the wave number $k(\omega)$ in terms of the frequency, ω, the six characteristic wave speeds squared, $p_{n(1)}^2, p_{n(2)}^2, p_0^2, u_{n(1)}^2, u_{n(2)}^2$ and u_0^2 and the three characteristic frequencies ω_0, ω_ξ and ω_κ.

8.3 High Frequency Approximation

It is possible to gain insight into the behavior of the plane harmonic waves as defined by (8.2.1) and (8.2.2) by examining high and low frequency approximations to the dispersion relation. In the high frequency case, we look for approximations of the form

$$\frac{k^2}{\omega^2} = \alpha_0 + \alpha_1 \frac{1}{\omega} + \alpha_2 \frac{1}{\omega^2} + O\left(\frac{1}{\omega^3}\right) \tag{8.3.1}$$

where the coefficients α_0, α_1 and α_3 are to be determined. The calculation of these coefficients can be tedious if done by hand.[2] Essentially what one does is to write the dispersion relationship (8.2.26) in the form

---

[2] The calculations of the coefficients α_0, α_1 and α_3 are greatly facilitated by use of a symbolic manipulator such as that found in Matlab or in Maple.

$$
\begin{aligned}
& i\frac{\omega_\kappa}{\omega}\left(\left(p_{n(1)}^2\frac{k^2}{\omega^2}-1\right)\left(p_{n(2)}^2\frac{k^2}{\omega^2}-1\right)-i\frac{\omega_\xi}{\omega}\left(p_0^2\frac{k^2}{\omega^2}-1\right)\right) \\
& \qquad -u_{n(2)}^2\frac{k^2}{\omega^2}\left(\left(u_{n(1)}^2\frac{k^2}{\omega^2}-1\right)\left(u_{n(2)}^2\frac{k^2}{\omega^2}-1\right)-i\frac{\omega_0}{\omega}\left(u_0^2\frac{k^2}{\omega^2}-1\right)\right)=0
\end{aligned}
\tag{8.3.2}
$$

and then utilize the approximation (8.3.1) to derive the equations satisfied by α_0,α_1 and α_3. The results of this calculation turn out to be

$$
\begin{aligned}
\frac{k_{(1)}^2}{\omega^2} &= \frac{1}{u_{n(1)}^2}+\frac{i}{u_{n(1)}^2}\left(\frac{\left(p_{n(1)}^2-u_{n(1)}^2\right)\left(u_{n(1)}^2-p_{n(2)}^2\right)}{u_{n(2)}^2\left(u_{n(1)}^2-u_{n(2)}^2\right)}\right)\frac{\omega_\kappa}{\omega}+\frac{i}{u_{n(1)}^2}\left(\frac{u_{n(1)}^2-u_0^2}{u_{n(1)}^2-u_{n(2)}^2}\right)\frac{\omega_0}{\omega} \\
&+\frac{\left(u_{n(1)}^2-u_0^2\right)\left(u_0^2-u_{n(2)}^2\right)}{\left(u_{n(1)}^2-u_{n(2)}^2\right)^3}\left(\frac{\omega_0}{\omega}\right)^2 \\
&-\frac{\left(\begin{aligned}&-\left(p_{n(2)}^2+p_{n(1)}^2\right)u_{n(1)}^2u_0^2-\left(p_{n(1)}^2p_{n(2)}^2\right)\left(u_{n(1)}^2+u_{n(2)}^2-2u_0^2\right)\\&+2\left(p_{n(2)}^2+p_{n(1)}^2\right)u_{n(1)}^2u_{n(2)}^2-\left(p_{n(1)}^2+p_{n(2)}^2\right)u_{n(2)}^2u_0^2+u_{n(1)}^6+2u_{n(1)}^2u_{n(2)}^2u_0^2-3u_{n(1)}^4u_{n(2)}^2\end{aligned}\right)}{u_{n(2)}^2\left(u_{n(1)}^2-u_{n(2)}^2\right)^3}\left(\frac{\omega_0\omega_\kappa}{\omega^2}\right) \\
&-\left(\frac{p_0^2-u_{n(1)}^2}{u_{n(2)}^2\left(u_{n(1)}^2-u_{n(2)}^2\right)}\right)\left(\frac{\omega_\xi\omega_\kappa}{\omega^2}\right) \\
&-\frac{\left(2\left(p_{n(2)}^2-u_{n(1)}^2\right)\left(p_{n(1)}^2-u_{n(1)}^2\right)\left(p_{n(1)}^2p_{n(2)}^2-\left(p_{n(1)}^2+p_{n(2)}^2\right)u_{n(2)}^2+2u_{n(1)}^2u_{n(2)}^2-u_{n(1)}^4\right)\right)}{u_{n(2)}^4\left(u_{n(1)}^2-u_{n(2)}^2\right)^3}\left(\frac{\omega_\kappa^2}{\omega^2}\right)+O\left(\frac{1}{\omega^3}\right)
\end{aligned}
\tag{8.3.3}
$$

$$\frac{k_{(2)}^2}{\omega^2} = \frac{1}{u_{n(2)}^2} + \frac{i}{u_{n(2)}^2}\left(\frac{\left(p_{n(1)}^2 - u_{n(2)}^2\right)\left(p_{n(2)}^2 - u_{n(2)}^2\right)}{u_{n(2)}^2\left(u_{n(1)}^2 - u_{n(2)}^2\right)}\right)\frac{\omega_\kappa}{\omega} + \frac{i}{u_{n(2)}^2}\left(\frac{u_0^2 - u_{n(2)}^2}{u_{n(1)}^2 - u_{n(2)}^2}\right)\frac{\omega_0}{\omega}$$

$$-\frac{\left(u_{n(1)}^2 - u_0^2\right)\left(u_0^2 - u_{n(2)}^2\right)}{\left(u_{n(1)}^2 - u_{n(2)}^2\right)^3}\left(\frac{\omega_0}{\omega}\right)^2$$

$$+\frac{\begin{pmatrix} -\left(p_{n(2)}^2 + p_{n(1)}^2\right)u_{n(1)}^2 u_0^2 - \left(p_{n(1)}^2 p_{n(2)}^2\right)\left(u_{n(1)}^2 + u_{n(2)}^2 - 2u_0^2\right) \\ +2\left(p_{n(2)}^2 + p_{n(1)}^2\right)u_{n(1)}^2 u_{n(2)}^2 - \left(p_{n(1)}^2 + p_{n(2)}^2\right)u_{n(2)}^2 u_0^2 + u_{n(2)}^6 + 2u_{n(1)}^2 u_{n(2)}^2 u_0^2 - 3u_{n(1)}^2 u_{n(2)}^4 \end{pmatrix}}{u_{n(2)}^2\left(u_{n(1)}^2 - u_{n(2)}^2\right)^3}\left(\frac{\omega_0\omega_\kappa}{\omega^2}\right)$$

$$+\left(\frac{p_0^2 - u_{n(2)}^2}{u_{n(2)}^2\left(u_{n(1)}^2 - u_{n(2)}^2\right)}\right)\left(\frac{\omega_\xi\omega_\kappa}{\omega^2}\right)$$

$$+\frac{\left(2\left(p_{n(2)}^2 - u_{n(2)}^2\right)\left(p_{n(1)}^2 - u_{n(2)}^2\right)\left(p_{n(1)}^2 p_{n(2)}^2 - \left(p_{n(1)}^2 + p_{n(2)}^2\right)u_{n(1)}^2 + 2u_{n(1)}^2 u_{n(2)}^2 - u_{n(2)}^4\right)\right)}{u_{n(2)}^4\left(u_{n(1)}^2 - u_{n(2)}^2\right)^3}\left(\frac{\omega_\kappa^2}{\omega^2}\right) + O\left(\frac{1}{\omega^3}\right) \tag{8.3.4}$$

and

$$\frac{k_{(3)}^2}{\omega^2} = \frac{i}{u_{n(2)}^2}\frac{\omega_\kappa}{\omega} + \left(\frac{\omega_\kappa\omega_0}{u_{n(2)}^2} - \frac{\omega_\kappa\omega_\xi}{u_{n(2)}^2} + \frac{\left(p_{n(1)}^2 - u_{n(1)}^2\right) + \left(p_{n(2)}^2 - u_{n(2)}^2\right)}{u_{n(2)}^2}\frac{\omega_\kappa^2}{u_{n(2)}^2}\right)\frac{1}{\omega^2} + O\left(\frac{1}{\omega^3}\right) \tag{8.3.5}$$

for the three squared wave numbers. For simplicity, we shall confine our discussion to the positive roots of these squared wave numbers. The negative roots simply correspond to waves propagating in the negative x direction. Each of these three roots will be referred to as a *mode of propagation*. The derivations of (8.3.3) and (8.3.4) have adopted the inequality $u_{n(1)}^2 > u_{n(2)}^2$ as a strengthened version of (8.2.22).

It follows from (8.3.3) that the phase velocity and the attenuation coefficient for the first mode are

$$q_{(1)} = u_{n(1)} \left(\begin{aligned} & 1 - u_{n(1)}^2 \frac{\left(u_{n(1)}^2 - u_0^2\right)\left(u_0^2 - u_{n(2)}^2\right)}{2\left(u_{n(1)}^2 - u_{n(2)}^2\right)^3} \left(\frac{\omega_0}{\omega}\right)^2 \\ & + u_{n(1)}^2 \frac{\left(\begin{aligned} & -\left(p_{n(2)}^2 + p_{n(1)}^2\right) u_{n(1)}^2 u_0^2 - \left(p_{n(1)}^2 p_{n(2)}^2\right)\left(u_{n(1)}^2 + u_{n(2)}^2 - 2u_0^2\right) \\ & + 2\left(p_{n(2)}^2 + p_{n(1)}^2\right) u_{n(1)}^2 u_{n(2)}^2 - \left(p_{n(1)}^2 + p_{n(2)}^2\right) u_{n(2)}^2 u_0^2 + u_{n(1)}^6 + 2u_{n(1)}^2 u_{n(2)}^2 u_0^2 - 3u_{n(1)}^4 u_{n(2)}^2 \end{aligned} \right)}{2u_{n(2)}^2 \left(u_{n(1)}^2 - u_{n(2)}^2\right)^3} \left(\frac{\omega_0 \omega_\kappa}{\omega^2}\right) \\ & + u_{n(1)}^2 \left(\frac{p_0^2 - u_{n(1)}^2}{2u_{n(2)}^2 \left(u_{n(1)}^2 - u_{n(2)}^2\right)} \right) \left(\frac{\omega_\xi \omega_\kappa}{\omega^2}\right) \\ & + u_{n(1)}^2 \frac{\left(2\left(p_{n(2)}^2 - u_{n(1)}^2\right)\left(p_{n(1)}^2 - u_{n(1)}^2\right)\left(p_{n(1)}^2 p_{n(2)}^2 - \left(p_{n(1)}^2 + p_{n(2)}^2\right) u_{n(2)}^2 + 2u_{n(1)}^2 u_{n(2)}^2 - u_{n(1)}^4\right)\right)}{2u_{n(2)}^4 \left(u_{n(1)}^2 - u_{n(2)}^2\right)^3} \left(\frac{\omega_\kappa^2}{\omega^2}\right) \end{aligned} \right) + O\left(1/\omega^3\right) \tag{8.3.6}$$

and

$$s_{(1)} = \frac{1}{2u_{n(1)}} \left(\frac{\left(p_{n(1)}^2 - u_{n(1)}^2\right)\left(u_{n(1)}^2 - p_{n(2)}^2\right)}{u_{n(2)}^2 \left(u_{n(1)}^2 - u_{n(2)}^2\right)} \omega_\kappa + \frac{\left(u_{n(1)}^2 - u_0^2\right)}{\left(u_{n(1)}^2 - u_{n(2)}^2\right)} \omega_0 \right) + O\left(1/\omega\right) \tag{8.3.7}$$

Likewise, it follows from (8.3.4) that for the second mode

$$q_{(2)}=u_{n(2)}\left(\begin{array}{l}1+u_{n(2)}^2\dfrac{\left(u_{n(1)}^2-u_0^2\right)\left(u_0^2-u_{n(2)}^2\right)}{2\left(u_{n(1)}^2-u_{n(2)}^2\right)^3}\left(\dfrac{\omega_0}{\omega}\right)^2\\ -\dfrac{\left(\begin{array}{l}-\left(p_{n(2)}^2+p_{n(1)}^2\right)u_{n(1)}^2u_0^2-\left(p_{n(1)}^2p_{n(2)}^2\right)\left(u_{n(1)}^2+u_{n(2)}^2-2u_0^2\right)\\ +2\left(p_{n(2)}^2+p_{n(1)}^2\right)u_{n(1)}^2u_{n(2)}^2-\left(p_{n(1)}^2+p_{n(2)}^2\right)u_{n(2)}^2u_0^2+u_{n(2)}^6+2u_{n(1)}^2u_{n(2)}^2u_0^2-3u_{n(1)}^2u_{n(2)}^4\end{array}\right)}{2\left(u_{n(1)}^2-u_{n(2)}^2\right)^3}\left(\dfrac{\omega_0\omega_\kappa}{\omega^2}\right)\\ -\left(\dfrac{p_0^2-u_{n(2)}^2}{2\left(u_{n(1)}^2-u_{n(2)}^2\right)}\right)\left(\dfrac{\omega_\xi\omega_\kappa}{\omega^2}\right)\\ -\dfrac{\left(2\left(p_{n(2)}^2-u_{n(2)}^2\right)\left(p_{n(1)}^2-u_{n(2)}^2\right)\left(p_{n(1)}^2p_{n(2)}^2-\left(p_{n(1)}^2+p_{n(2)}^2\right)u_{n(1)}^2+2u_{n(1)}^2u_{n(2)}^2-u_{n(2)}^4\right)\right)}{2u_{n(2)}^2\left(u_{n(1)}^2-u_{n(2)}^2\right)^3}\left(\dfrac{\omega_\kappa^2}{\omega^2}\right)\end{array}\right)$$
$$+O\left(1/\omega^3\right) \qquad (8.3.8)$$

and

$$s_{(2)}=\frac{1}{2u_{n(2)}}\left(\frac{\left(p_{n(1)}^2-u_{n(2)}^2\right)\left(p_{n(2)}^2-u_{n(2)}^2\right)}{u_{n(2)}^2\left(u_{n(1)}^2-u_{n(2)}^2\right)}\omega_\kappa+\frac{\left(u_0^2-u_{n(2)}^2\right)}{\left(u_{n(1)}^2-u_{n(2)}^2\right)}\omega_0\right)+O\left(1/\omega\right) \qquad (8.3.9)$$

Equations (8.3.6) through (8.3.9) show that the in the high frequency limit, the two phase velocities approach constants and the corresponding attenuation coefficients are positive numbers. The two phase velocities in this limit correspond to the acceleration wave speeds $u_{n(1)}$ and $u_{n(2)}$ defined by equation (7.2.50). The attenuation coefficients are positive because of the inequalities (7.2.114) and (8.2.22) along with the thermodynamic results (7.2.4) through (7.2.6). These latter results tell us that ω_κ and ω_0 are positive numbers. If the attenuation coefficient (8.3.7) is compared to the one in (7.2.119), we see that the term in the bracket of (8.3.7) is the same factor in (7.2.119). A similar correspondence exists between (8.3.9) and (7.2.120). The high frequency correspondence acceleration wave speeds and phase velocities and attenuation coefficients is a standard result. The usual explanation is that the acceleration wave calculation yields a short time or, equivalently, high frequency solution in the neighborhood of the boundary represented by the acceleration wave.

The high frequency approximations for the phase velocity and the attenuation coefficient for the third mode turn out to be

$$q_{(3)} = u_{n(2)}\sqrt{\frac{2\omega}{\omega_\kappa}}\left(1-\frac{u_{n(2)}^2}{2\omega_\kappa}\left(\frac{\omega_\kappa\omega_0}{u_{n(2)}^2}-\frac{\omega_\kappa\omega_\xi}{u_{n(2)}^2}+\frac{\left(p_{n(1)}^2-u_{n(1)}^2\right)+\left(p_{n(2)}^2-u_{n(2)}^2\right)}{u_{n(2)}^2}\frac{\omega_\kappa^2}{u_{n(2)}^2}\right)\frac{1}{\omega}\right)+O\left(\frac{1}{\omega^{3/2}}\right) \quad (8.3.10)$$

and

$$s_{(3)} = \frac{1}{u_{n(2)}}\sqrt{\frac{\omega\omega_\kappa}{2}}\left(1-\frac{u_{n(2)}^2}{2\omega_\kappa}\left(\frac{\omega_\kappa\omega_0}{u_{n(2)}^2}-\frac{\omega_\kappa\omega_\xi}{u_{n(2)}^2}+\frac{\left(p_{n(1)}^2-u_{n(1)}^2\right)+\left(p_{n(2)}^2-u_{n(2)}^2\right)}{u_{n(2)}^2}\frac{\omega_\kappa^2}{u_{n(2)}^2}\right)\frac{1}{\omega}\right)+O\left(\frac{1}{\omega^{3/2}}\right) \quad (8.3.11)$$

The fact that the phase velocity depends upon the frequency, even in the high frequency limit, illustrates that the third mode is dispersive.

8.4 Low Frequency Approximation

In the low frequency case, we look for approximations of the form

$$k^2 = \alpha_0 + \alpha_1\omega + \alpha_2\omega^2 + \alpha_3\omega^3 + O\left(\omega^4\right) \quad (8.4.1)$$

where the coefficients $\alpha_0, \alpha_1, \alpha_2$ and α_3 are to be determined. Like the high frequency case, the calculation details are tedious. They calculations begin with the substitution of (8.4.1) into (8.2.26). The next step is to solve for the unknown coefficients. The results for the three wave numbers squared turn out to be[3]

$$k_{(1)}^2 = \frac{1}{p_0^2}\omega^2 + \left(\frac{\omega_0}{\omega_\kappa\omega_\xi}u_{n(2)}^2\left(p_0^2-u_0^2\right)+\frac{1}{\omega_\xi}\left(p_{n(1)}^2-p_0^2\right)\left(p_0^2-p_{n(2)}^2\right)\right)\frac{i}{p_0^6}\omega^3 + O\left(\omega^4\right) \quad (8.4.2)$$

---

[3] It is important to note that the result (8.4.2) assumes that ω_κ and ω_ξ are nonzero. The case where ω_κ is zero, as shown in the definition (8.2.25), corresponds to an infinite conductivity κ. This case, as equation (8.1.8) shows, forces the temperature to obey $\frac{\partial^2\theta}{\partial X^2}=0$. With suitable boundary conditions, we can conclude that the case $\omega_\kappa = 0$ corresponds to isothermal poroelasticity. In any case, for the case $\omega_\kappa = 0$, the results (8.4.2) through (8.4.3) are replaced by

$$k_{(1)}^2 = \frac{1}{u_0^2}\omega^2 + \left(u_{n(1)}^2-u_0^2\right)\left(u_0^2-u_{n(2)}^2\right)\frac{i}{\omega_\xi u_0^6}\omega^3 + O\left(\omega^4\right)$$

and

$$k_{(2)}^2 = i\frac{u_0^2}{u_{n(1)}^2u_{n(2)}^2}\omega_\xi\omega + \frac{\left(u_0^2-u_{n(2)}^2\right)u_{n(1)}^2+u_0^2u_{n(2)}^2}{u_0^2u_{n(1)}^2u_{n(2)}^2}\omega^2 + i\frac{\left(u_{n(1)}^2-u_0^2\right)\left(u_0^2-u_{n(2)}^2\right)}{\omega_\xi u_0^6}\omega^3 + O\left(\omega^4\right),$$

Respectively.

$$k_{(2)}^2 = \frac{i}{2u_{n(1)}^2 u_{n(2)}^4}\left(\omega_0 u_0^2 u_{n(2)}^2 + \omega_\kappa p_{n(1)}^2 p_{n(2)}^2 - \sqrt{\left(\omega_0 u_0^2 u_{n(2)}^2 + \omega_\kappa p_{n(1)}^2 p_{n(2)}^2\right)^2 - 4\omega_\xi \omega_\kappa u_{n(1)}^2 u_{n(2)}^4 p_0^2}\right)\omega + O\left(\omega^2\right) \tag{8.4.3}$$

and

$$k_{(3)}^2 = \frac{i}{2u_{n(1)}^2 u_{n(2)}^4}\left(\omega_0 u_0^2 u_{n(2)}^2 + \omega_\kappa p_{n(1)}^2 p_{n(2)}^2 + \sqrt{\left(\omega_0 u_0^2 u_{n(2)}^2 + \omega_\kappa p_{n(1)}^2 p_{n(2)}^2\right)^2 - 4\omega_\xi \omega_\kappa u_{n(1)}^2 u_{n(2)}^4 p_0^2}\right)\omega + O\left(\omega^2\right) \tag{8.4.4}$$

where, in order to avoid long and uninteresting formulas, we have truncated the expansions for the second and third modes at terms linear in the frequency ω. The first mode has been taken to higher order terms because it is nondispersive in the low frequency limit. It readily follows from (8.4.2) that

$$q_{(1)} = p_0 + O\left(\omega^2\right) \tag{8.4.5}$$

and the corresponding attenuation coefficient is

$$s_{(1)} = \frac{\omega^2}{2p_0}\left(\frac{\omega_0}{\omega_\kappa \omega_\xi}\frac{u_{n(2)}^2\left(p_0^2 - u_0^2\right)}{p_0^4} + \frac{1}{\omega_\xi}\frac{\left(p_{n(1)}^2 - p_0^2\right)\left(p_0^2 - p_{n(2)}^2\right)}{p_0^4}\right) + O\left(\omega^3\right) \tag{8.4.6}$$

The corresponding quantities for the second and third mode are

$$q_{(2)} = \frac{2u_{n(1)}u_{n(2)}^2\sqrt{\omega}}{\sqrt{\left(\omega_0 u_0^2 u_{n(2)}^2 + \omega_\kappa p_{n(1)}^2 p_{n(2)}^2 - \sqrt{\left(\omega_0 u_0^2 u_{n(2)}^2 + \omega_\kappa p_{n(1)}^2 p_{n(2)}^2\right)^2 - 4\omega_\xi \omega_\kappa u_{n(1)}^2 u_{n(2)}^4 p_0^2}\right)}} + O\left(\omega\right) \tag{8.4.7}$$

$$s_{(2)} = \frac{\sqrt{\omega}}{2u_{n(1)}u_{n(2)}^2}\sqrt{\left(\omega_0 u_0^2 u_{n(2)}^2 + \omega_\kappa p_{n(1)}^2 p_{n(2)}^2 - \sqrt{\left(\omega_0 u_0^2 u_{n(2)}^2 + \omega_\kappa p_{n(1)}^2 p_{n(2)}^2\right)^2 - 4\omega_\xi \omega_\kappa u_{n(1)}^2 u_{n(2)}^4 p_0^2}\right)} + O\left(\omega\right) \tag{8.4.8}$$

$$q_{(3)} = \frac{2u_{n(1)}u_{n(2)}^2\sqrt{\omega}}{\sqrt{\left(\omega_0 u_0^2 u_{n(2)}^2 + \omega_\kappa p_{n(1)}^2 p_{n(2)}^2 + \sqrt{\left(\omega_0 u_0^2 u_{n(2)}^2 + \omega_\kappa p_{n(1)}^2 p_{n(2)}^2\right)^2 - 4\omega_\xi \omega_\kappa u_{n(1)}^2 u_{n(2)}^4 p_0^2}\right)}} + O\left(\omega\right) \tag{8.4.9}$$

and

$$s_{(3)} = \frac{\sqrt{\omega}}{2u_{n(1)}u_{n(2)}^2}\sqrt{\left(\omega_0 u_0^2 u_{n(2)}^2 + \omega_\kappa p_{n(1)}^2 p_{n(2)}^2 + \sqrt{\left(\omega_0 u_0^2 u_{n(2)}^2 + \omega_\kappa p_{n(1)}^2 p_{n(2)}^2\right)^2 - 4\omega_\xi \omega_\kappa u_{n(1)}^2 u_{n(2)}^4 p_0^2}\right)} + O(\omega) \tag{8.4.10}$$

8.5 High and Low Frequency Approximation for Nonconductors

The results summarized in Section 8.3 and 8.4 assume the result (8.1.20), i.e., the thermal conductivity is positive. In this Section, we shall briefly record the results for the dispersion relation in the case of a nonconductor, i.e., when

$$\kappa = 0 \tag{8.5.1}$$

In terms of the characteristic frequencies, the nonconductor case corresponds to

$$\frac{1}{\omega_\kappa} = 0 \tag{8.5.2}$$

and

$$\omega_0 = \omega_\xi \tag{8.5.3}$$

The result (8.5.3) follows from the definitions (8.1.18) and (8.1.19) along with the thermodynamic result (7.2.121). It follows from (8.2.26) that in this limiting case the dispersion relation $k(\omega)$ is a solution of

$$\left(p_{n(1)}^2 k^2 - \omega^2\right)\left(p_{n(2)}^2 k^2 - \omega^2\right) - i\omega\omega_\xi\left(p_0^2 k^2 - \omega^2\right) = 0 \tag{8.5.4}$$

The high and low frequency approximations to this quadratic in k^2 are as follows:

High Frequency

$$\frac{k_{(1)}^2}{\omega^2} = \frac{1}{p_{n(1)}^2} + \frac{i}{p_{n(1)}^2}\left(\frac{p_{n(1)}^2 - p_0^2}{p_{n(1)}^2 - p_{n(2)}^2}\right)\frac{\omega_0}{\omega} + \frac{\left(p_{n(1)}^2 - p_0^2\right)\left(p_0^2 - p_{n(2)}^2\right)}{\left(p_{n(1)}^2 - p_{n(2)}^2\right)^3}\left(\frac{\omega_0}{\omega}\right)^2 + O\left(\frac{1}{\omega^3}\right) \tag{8.5.5}$$

$$\frac{k_{(2)}^2}{\omega^2} = \frac{1}{p_{n(2)}^2} + \frac{i}{p_{n(2)}^2}\left(\frac{p_0^2 - p_{n(2)}^2}{p_{n(1)}^2 - p_{n(2)}^2}\right)\frac{\omega_0}{\omega} - \frac{\left(p_{n(1)}^2 - p_0^2\right)\left(p_0^2 - p_{n(2)}^2\right)}{\left(p_{n(1)}^2 - p_{n(2)}^2\right)^3}\left(\frac{\omega_0}{\omega}\right)^2 + O\left(\frac{1}{\omega^3}\right) \tag{8.5.6}$$

Low Frequency

$$k_{(1)}^2 = \frac{1}{p_0^2}\omega^2 + \left(p_{n(1)}^2 - p_0^2\right)\left(p_0^2 - p_{n(2)}^2\right)\frac{i}{\omega_\xi p_0^6}\omega^3 + O\left(\omega^4\right) \tag{8.5.7}$$

$$k_{(2)}^2 = i\frac{p_0^2}{p_{n(1)}^2 p_{n(2)}^2}\omega_\xi \omega + \frac{\left(p_0^2 - p_{n(2)}^2\right)p_{n(1)}^2 + p_0^2 p_{n(2)}^2}{p_0^2 p_{n(1)}^2 p_{n(2)}^2}\omega^2 + i\frac{\left(p_{n(1)}^2 - p_0^2\right)\left(p_0^2 - p_{n(2)}^2\right)}{\omega_\xi p_0^6}\omega^3 + O\left(\omega^4\right) \tag{8.5.8}$$

Equations (8.5.5), (8.5.6) and (8.5.7) are nondispersive modes in the high and low frequency range respectively. The phase velocities are $p_{n(1)}$, $p_{n(2)}$ and p_0, respectively. It is possible to confirm that the attenuation coefficients in these cases are positive.

8.6 Phase Velocities

The results in Sections 8.3 through 8.5 allow us to identify the phase velocities associated with the various high and low frequency approximations. In this way, we gain physical insight into the meaning of the six constants $u_0^2, u_{n(1)}^2, u_{n(2)}^2, p_0^2, p_{n(1)}^2$ and $p_{n(2)}^2$ that appear in all of our formulas. The following table shows when these constants correspond to phase velocities for non dispersive modes of plane progressive waves.

Table of Phase Velocities

	High Frequency Approximation	Low Frequency Approximation
$0<\omega_\kappa,\omega_\xi,\omega_0<\infty$	$u_{n(1)}$ and $u_{n(2)}$	p_0
$1/\omega_\kappa = 0, 0<\omega_\xi,\omega_0<\infty$	$p_{n(1)}$ and $p_{n(2)}$	p_0
$\omega_\kappa = 0, 0<\omega_\xi,\omega_0<\infty$	$u_{n(1)}$ and $u_{n(2)}$	u_0

8.7 Numerical Example

It is interesting to examine the solutions to (8.2.26) for a hypothetical porous elastic material defined by specific numerical values. In this example, we shall make the choices

$$\begin{aligned} p_{n(1)} &= 4500 \text{ m/sec} \\ u_{n(1)} &= 4000 \text{ m/sec} \\ p_0 &= 3500 \text{ m/sec} \\ u_0 &= 3000 \text{ m/sec} \\ p_{n(2)} &= 2500 \text{ m/sec} \\ u_{n(2)} &= 2000 \text{ m/sec} \end{aligned} \tag{8.7.1}$$

These numbers have been selected such that the inequalities (7.2.114), (8.2.21), (8.2.22) and (8.2.23) are obeyed. In addition, the numbers selected are in the range of values displayed in Tables 1 and 2 of Chapter 7.

We also need typical numbers for the three characteristic frequencies ω_0, ω_ξ and ω_κ. It follows from the definitions (8.1.18) and (8.1.19) that

$$\omega_\xi \geq \omega_0 \tag{8.7.2}$$

Consistent with this inequality, in our numerical example, we shall initially make the choices

$$\begin{aligned} \omega_0 &= 10(10)^7 \text{ rad/sec} \\ \omega_\xi &= 15(10)^7 \text{ rad/sec} \\ \omega_\kappa &= 40(10)^7 \text{ rad/sec} \end{aligned} \tag{8.7.3}$$

These numbers reflect the relationship (8.7.2) and are of the order of magnitude that one would get for the materials shown in Tables 1 and 2 of Section 6.2. We shall also allow the characteristic frequency ω_κ to be zero (infinite conductivity κ) and infinite (zero conductivity κ).

We shall first look at plots of the phase velocity for the hypothetical poroelastic material defined by (8.7.1) and (8.7.3). The three modes displayed in the figure below correspond to the three modes ordered as in Section 8.3. The resulting plot is

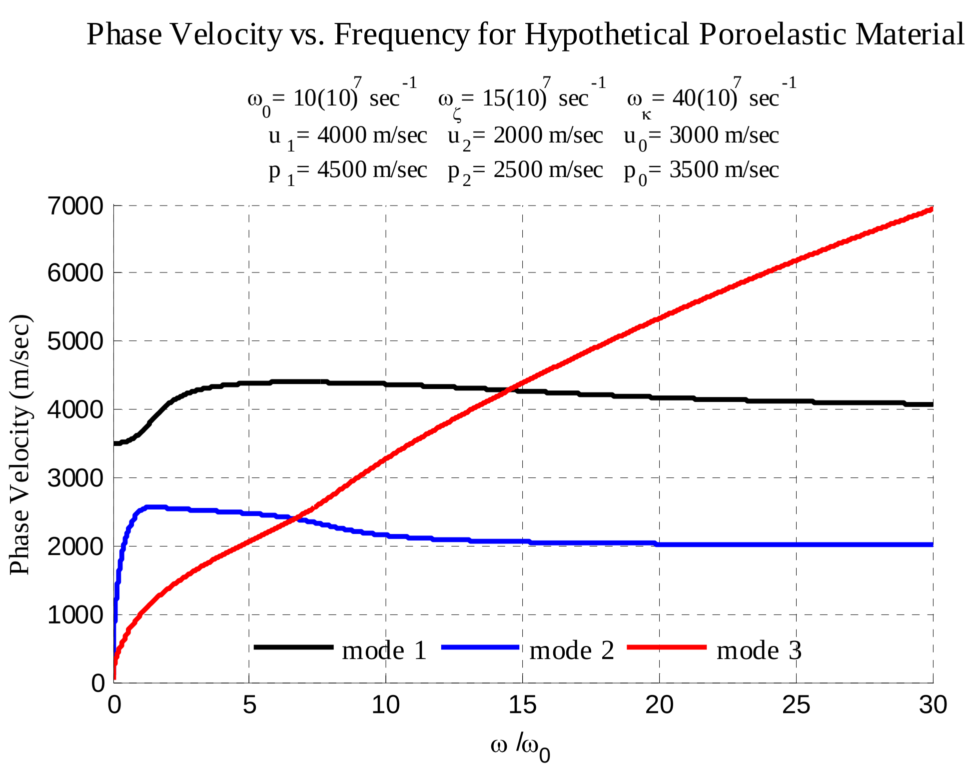

Figure 1

As the approximate results in Sections 8.3 and 8.4 illustrated, Figure 1 shows that the first mode, for low frequency, has a phase velocity p_0 which grows, for high frequencies, to the phase velocity u_1. The second mode is dispersive for low frequencies but approaches the nondispersive phase velocity u_2 for high frequencies.

The attenuation coefficients for these three modes plot as follows

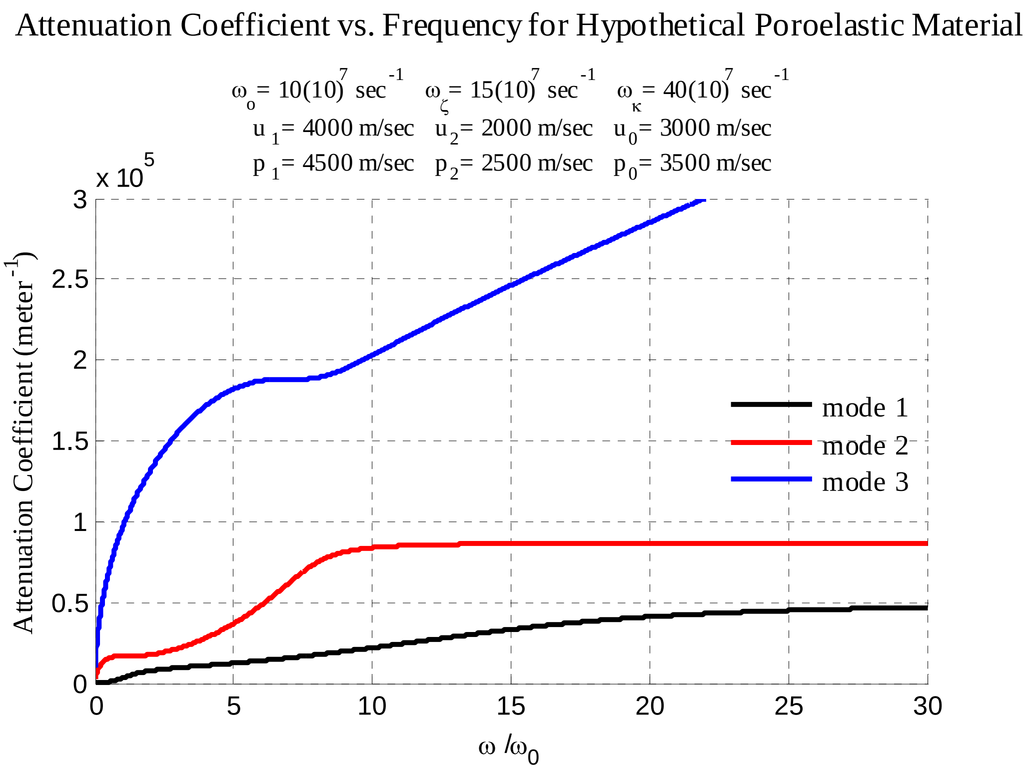

Figure 2

Figure 2 illustrates the analytical results contained in (8.3.7) and (8.3.9) that the attenuation coefficients for modes 1 and 2 approach constants for large frequencies.

The two limiting cases mentioned in Sections 8.4 and 8.5, namely, the cases of infinite and zero thermal conductivity, yield the following two results for the phase velocities.

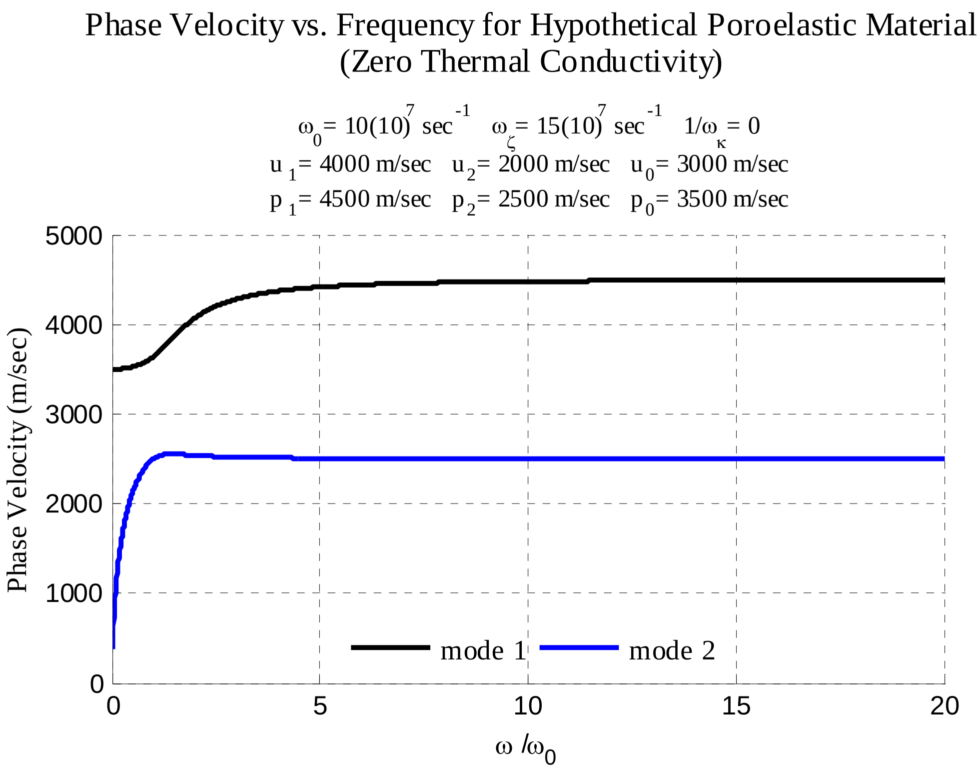

Figure 3

and

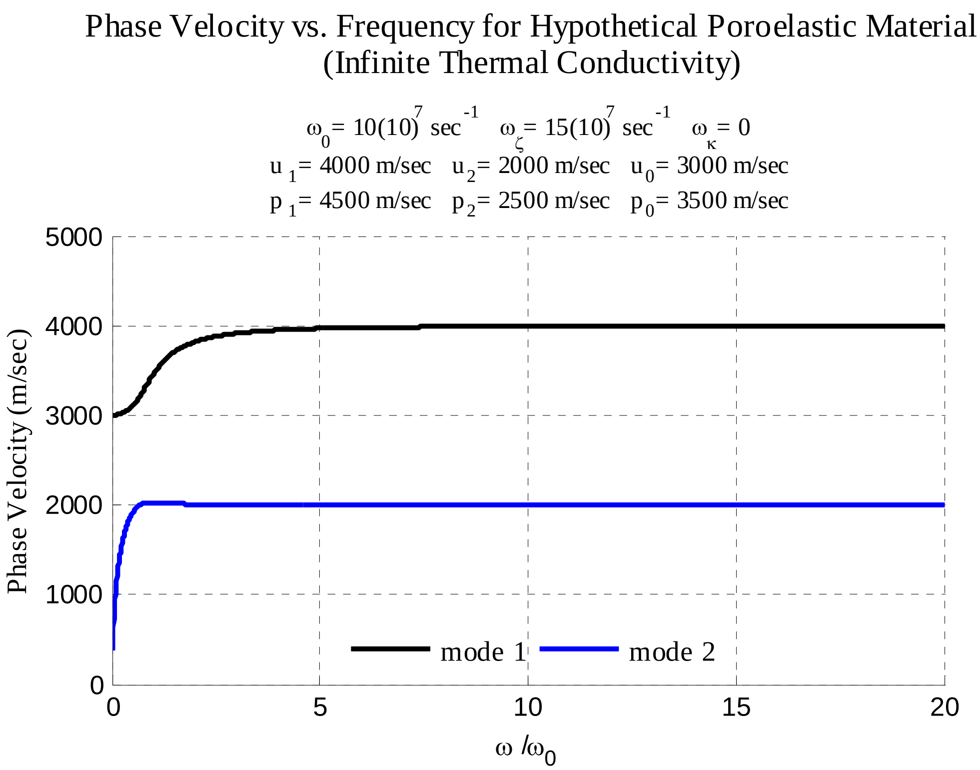

Figure 4

These curves reflect the phase velocity results summarized in Section 8.6.

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9

Boundary Initial Value Problems: Inertia Neglected

In Chapter 5, we examined the field and constitutive equations in the special case where it was acceptable to neglect the inertia of the fluid and, later, the inertia of the solid. In this chapter, we shall adopt the field equations where both of these inertias are neglected and, in addition, we assume the porous elastic material is isothermal. Given these assumptions, we shall formulate rather simple boundary initial value problems and indicate how they might be solved. One of the historically important problems is one first solved by Biot [Ref. 1]. We shall generate this solution and some of its generalizations. The formalism for this chapter is drawn, in part, from the formulation given by Bowen [Ref. 2].

9.1 Governing Partial Differential Equations

We begin this discussion with the constitutive equations and field equations given in Section 4.2. Equations (4.2.1) through (4.2.5, repeated here, are

$$\frac{\varphi_f-\varphi_f^+}{\varphi_f^+}=\frac{\lambda_{sb}}{\varphi_f^+K_f}\operatorname{tr}\mathbf{E}_s+\left(1-\frac{\lambda_{ff}}{\varphi_f^+K_f}\right)((\rho_f-\rho_f^+)\big/\rho_f^+)-\frac{\overline{\tau}_b}{\varphi_f^+K_f}(\theta-\theta^+) \tag{9.1.1}$$

$$\begin{aligned}\Psi_{\mathrm{I}}=\alpha(\theta-\theta^+)-\frac{1}{2}\frac{\overline{c}_{\mathrm{v}}}{\theta^+}(\theta-\theta^+)^2-\overline{\tau}_s(\theta-\theta^+)(\operatorname{tr}\mathbf{E}_s)+\overline{\tau}_f(\theta-\theta^+)(\rho_f-\rho_f^+)\big/\rho_f^+\\+\frac{1}{2}\overline{\lambda}_{ss}(\operatorname{tr}\mathbf{E}_s)^2-\overline{\lambda}_{sf}(\operatorname{tr}\mathbf{E}_s)(\rho_f-\rho_f^+)\big/\rho_f^+\\+\frac{1}{2}\overline{\lambda}_{ff}\left((\rho_f-\rho_f^+)\big/\rho_f^+\right)^2+\mu_{ss}\operatorname{tr}(\mathbf{E}_s\mathbf{E}_s),\end{aligned} \tag{9.1.2}$$

$$\rho\eta=-\alpha+\frac{\overline{c}_{\mathrm{v}}}{\theta^+}(\theta-\theta^+)+\overline{\tau}_s(\operatorname{tr}\mathbf{E}_s)-\overline{\tau}_f(\rho_f-\rho_f^+)\big/\rho_f^+ \tag{9.1.3}$$

$$\begin{aligned}\rho_{s_R}\mathbf{K}_s=-(\mathbf{T}_{\mathrm{I}}+\varphi_f^+P_f\mathbf{I})+\alpha(\theta-\theta^+)\mathbf{I}=-\overline{\lambda}_{ss}(\operatorname{tr}\mathbf{E}_s)\mathbf{I}-2\mu_{ss}\mathbf{E}_s\\+\overline{\lambda}_{sf}((\rho_f-\rho_f^+)\big/\rho_f^+)\mathbf{I}+\overline{\tau}_s(\theta-\theta^+)\mathbf{I}\end{aligned} \tag{9.1.4}$$

and

$$\varphi_f^+P_f=\overline{\tau}_f(\theta-\theta^+)-\overline{\lambda}_{sf}(\operatorname{tr}\mathbf{E}_s)+\overline{\lambda}_{ff}(\rho_f-\rho_f^+)\big/\rho_f^+ \tag{9.1.5}$$

Equation (4.2.9) is

$$\mathbf{T}_{\mathrm{I}} + \varphi_f^+ P_f \mathbf{I} = \bar{\lambda}_{ss}(\mathrm{tr}\,\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s - \bar{\lambda}_{sf}\left((\rho_f - \rho_f^+)/\rho_f^+\right)\mathbf{I} - (\bar{\tau}_s - \alpha)(\theta - \theta^+)\mathbf{I} \tag{9.1.6}$$

The field equations which follow from these constitutive equations are equations (4.2.14), (4.2.15) and (4.2.16). These equations, repeated here, are

$$\begin{aligned}\rho_{s_R}\frac{\partial^2 \mathbf{w}_s}{\partial t^2} &= (\bar{\lambda}_{ss} + \mu_{ss})\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_s) + \mu_{ss}\,\mathrm{Div}(\mathrm{GRAD}\,\mathbf{w}_s) \\ &+ \bar{\lambda}_{sf}\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_f) + (\alpha_s - \bar{\tau}_s + \gamma)\,\mathrm{GRAD}\,\theta - \xi\left(\frac{\partial \mathbf{w}_s}{\partial t} - \frac{\partial \mathbf{w}_f}{\partial t}\right)\end{aligned} \tag{9.1.7}$$

$$\begin{aligned}\rho_f^+\frac{\partial^2 \mathbf{w}_f}{\partial t^2} &= \bar{\lambda}_{sf}\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_s) + \bar{\lambda}_{ff}\,\mathrm{GRAD}(\mathrm{Div}\,\mathbf{w}_f) + (\alpha_f - \bar{\tau}_f - \gamma)\,\mathrm{GRAD}\,\theta \\ &-\xi\left(\frac{\partial \mathbf{w}_f}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t}\right)\end{aligned} \tag{9.1.8}$$

and

$$\begin{aligned}\bar{c}_{\mathrm{v}}\frac{\partial \theta}{\partial t} + \theta^+\left((\bar{\tau}_s - \alpha_s + \varsigma/\theta^+)\,\mathrm{Div}\frac{\partial \mathbf{w}_s}{\partial t} + (\bar{\tau}_f - \alpha_f - \varsigma/\theta^+)\,\mathrm{Div}\frac{\partial \mathbf{w}_f}{\partial t}\right) \\ = \kappa\,\mathrm{Div}(\mathrm{GRAD}\,\theta)\end{aligned} \tag{9.1.9}$$

The material constants κ, ξ, γ and ς obey (4.2.20), (4.2.21) and (4.2.22). These equations, repeated here, are

$$\kappa \geq 0 \tag{9.1.10}$$

and

$$\kappa\xi/\theta^+ \geq \frac{1}{4}(\gamma + \varsigma/\theta^+)^2 \tag{9.1.11}$$

It follows from (9.1.10) and (9.1.11) that

$$\xi \geq 0 \tag{9.1.12}$$

As has been mentioned, the coefficient κ is the *thermal conductivity*. The coefficient ξ is called the *drag coefficient*.

The special case of the above equations we wish to study in this chapter are those that neglect the inertia of the fluid and the solid and for which the temperature is a constant. The isothermal assumption can be formally obtained from (9.1.9) by allowing $\kappa \to \infty$. In this limit, the temperature is a solution of

$$\text{Div}(\text{GRAD}\,\theta) = 0 \tag{9.1.13}$$

With suitable boundary conditions, the solution of this partial differential equation is

$$\theta = \Theta\left(\mathbf{X}, t\right) = \theta^{+} \tag{9.1.14}$$

Given (9.1.14) and our assumption to neglect the inertia of the fluid and the solid, the two equations of motion (9.1.7) and (9.1.8) reduce to

$$(\bar{\lambda}_{ss} + \mu_{ss})\text{GRAD}(\text{Div}\,\mathbf{w}_s) + \mu_{ss}\,\text{Div}(\text{GRAD}\,\mathbf{w}_s) + \bar{\lambda}_{sf}\,\text{GRAD}(\text{Div}\,\mathbf{w}_f) - \xi\left(\frac{\partial \mathbf{w}_s}{\partial t} - \frac{\partial \mathbf{w}_f}{\partial t}\right) = \mathbf{0} \tag{9.1.15}$$

$$\bar{\lambda}_{sf}\,\text{GRAD}(\text{Div}\,\mathbf{w}_s) + \bar{\lambda}_{ff}\,\text{GRAD}(\text{Div}\,\mathbf{w}_f) - \xi\left(\frac{\partial \mathbf{w}_f}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t}\right) = \mathbf{0} \tag{9.1.16}$$

The constitutive equations appropriate to this isothermal special case are special cases of (9.1.5) and (9.1.6). We shall write these special cases as

$$\varphi_f^{+} P_f = -\bar{\lambda}_{sf}\,\text{tr}\,\mathbf{E}_s - \bar{\lambda}_{ff}\,\text{tr}\,\mathbf{E}_f \tag{9.1.17}$$

and

$$\mathbf{T}_{\text{I}} + \varphi_f^{+} P_f \mathbf{I} = \bar{\lambda}_{ss}(\text{tr}\,\mathbf{E}_s)\mathbf{I} + 2\mu_{ss}\mathbf{E}_s + \bar{\lambda}_{sf}\left(\text{tr}\,\mathbf{E}_f\right)\mathbf{I} \tag{9.1.18}$$

where (3.39) and the identity

$$\text{tr}\,\mathbf{E}_f = \text{Div}\,\mathbf{w}_f \tag{9.1.19}$$

have been used.

9.2 Some Properties of the Space Part Of The Operator

It is helpful when we talk about solutions to boundary initial value problems based upon the partial differential equations (9.1.15) and (9.1.16) to introduce certain properties of the space part of the operator that defines these equations. In particular, we can define a formal linear operator L by the equation

$$L\mathbf{w} = -\begin{bmatrix} \bar{\lambda}_{sf} \text{ GRAD}(\text{Div}\,\mathbf{w}_s) + \bar{\lambda}_{ff} \text{ GRAD}(\text{Div}\,\mathbf{w}_f) \\ (\bar{\lambda}_{ss} + \mu_{ss})\text{GRAD}(\text{Div}\,\mathbf{w}_s) + \mu_{ss} \text{ Div}(\text{GRAD}\,\mathbf{w}_s) + \bar{\lambda}_{sf} \text{ GRAD}(\text{Div}\,\mathbf{w}_f) \end{bmatrix} \tag{9.2.1}$$

where

$$\mathbf{w} = \begin{bmatrix} \mathbf{w}_f \\ \mathbf{w}_s \end{bmatrix} \tag{9.2.2}$$

The operator L is defined on the set of twice differentiable functions defined on an open subset of a spatial region $\mathscr{V}$. The definitions (9.2.1) and (9.2.2) adopt a mixed vector-matrix notation. If it aids ones thinking, the vectors in these two equations can be thought of as column matrices. This view allows (9.2.2), for example, to be seen as a column matrix of dimension 6×1. The way this notation has been adopted will be more or less obvious in the manipulation below.

If we define the inner produce by

$$\langle \mathbf{w}, \mathbf{v} \rangle = \int_{\mathscr{V}} \left(\mathbf{w}_f \cdot \mathbf{v}_f + \mathbf{w}_s \cdot \mathbf{v}_s \right) d\upsilon \tag{9.2.3}$$

Then it is possible to show from (9.2.1) that

$$\begin{aligned} \langle \mathbf{w}, L\mathbf{v} \rangle = & \oint_{\partial\mathscr{V}} \begin{pmatrix} \left(-\bar{\lambda}_{ff} \text{ Div}\,\mathbf{v}_f - \bar{\lambda}_{sf} \text{ Div}\,\mathbf{v}_s \right) \mathbf{w}_f \\ + \left(-\bar{\lambda}_{sf} \left(\text{Div}\,\mathbf{v}_f \right) \mathbf{I} - \bar{\lambda}_{ss} \left(\text{Div}\,\mathbf{v}_s \right) \mathbf{I} - \mu_{ss} \left(\text{GRAD}\,\mathbf{v}_s + \left(\text{GRAD}\,\mathbf{v}_s \right)^T \right) \right) \mathbf{w}_s \end{pmatrix} \cdot d\mathbf{s} \\ & + \int_{\mathscr{V}} \begin{pmatrix} \bar{\lambda}_{ff} \text{ Div}\,\mathbf{w}_f \text{ Div}\,\mathbf{v}_f + \bar{\lambda}_{sf} \left(\text{Div}\,\mathbf{w}_f \text{ Div}\,\mathbf{v}_s + \text{Div}\,\mathbf{w}_s \text{ Div}\,\mathbf{v}_f \right) + \bar{\lambda}_{ss} \text{ Div}\,\mathbf{w}_s \text{ Div}\,\mathbf{v}_s \\ + \frac{1}{2} \mu_{ss} \text{ tr} \left(\left(\text{GRAD}\,\mathbf{w}_s + \left(\text{GRAD}\,\mathbf{w}_s \right)^T \right) \left(\text{GRAD}\,\mathbf{v}_s + \left(\text{GRAD}\,\mathbf{v}_s \right)^T \right) \right) \end{pmatrix} \end{aligned} \tag{9.2.4}$$

Equation (9.2.4) is the first Green's identity for the operator L. A more convenient form of this equation is

$$\begin{aligned} \langle \mathbf{w}, L\mathbf{v} \rangle = & \oint_{\partial\mathscr{V}} \left(\varphi_f^+ P_f \left(\mathbf{v} \right) \mathbf{w}_f - \left(\mathbf{T}_I \left(\mathbf{v} \right) + \varphi_f^+ P_f \left(\mathbf{v} \right) \mathbf{I} \right) \mathbf{w}_s \right) \cdot d\mathbf{s} \\ & + \int_{\mathscr{V}} \begin{pmatrix} \bar{\lambda}_{ff} \text{ Div}\,\mathbf{w}_f \text{ Div}\,\mathbf{v}_f + \bar{\lambda}_{sf} \left(\text{Div}\,\mathbf{w}_f \text{ Div}\,\mathbf{v}_s + \text{Div}\,\mathbf{w}_s \text{ Div}\,\mathbf{v}_f \right) + \bar{\lambda}_{ss} \text{ Div}\,\mathbf{w}_s \text{ Div}\,\mathbf{v}_s \\ + \frac{1}{2} \mu_{ss} \text{ tr} \left(\left(\text{GRAD}\,\mathbf{w}_s + \left(\text{GRAD}\,\mathbf{w}_s \right)^T \right) \left(\text{GRAD}\,\mathbf{v}_s + \left(\text{GRAD}\,\mathbf{v}_s \right)^T \right) \right) \end{pmatrix} \end{aligned} \tag{9.2.5}$$

where, from (9.1.17), $P_f\left(\mathbf{v}\right)$ is the pore pressure calculated from the pair $\mathbf{v}=\begin{bmatrix}\mathbf{v}_f\\ \mathbf{v}_s\end{bmatrix}$ and , from (9.1.18), $\mathbf{T}_I\left(\mathbf{v}\right)$ is the stress calculated from the same pair. It follows from (9.2.5) that the second Green's identity for the operator L is

$$\left\langle \mathbf{w}, L\mathbf{v}\right\rangle - \left\langle L\mathbf{w},\mathbf{v}\right\rangle = \oint_{\partial\mathscr{V}}\begin{pmatrix}\varphi_f^+\left(P_f\left(\mathbf{v}\right)\mathbf{w}_f - P_f\left(\mathbf{w}\right)\mathbf{v}_f\right)\\ -\left(\mathbf{T}_I\left(\mathbf{v}\right)+\varphi_f^+P_f\left(\mathbf{v}\right)\mathbf{I}\right)\mathbf{w}_s + \left(\mathbf{T}_I\left(\mathbf{w}\right)+\varphi_f^+P_f\left(\mathbf{w}\right)\mathbf{I}\right)\mathbf{v}_s\end{pmatrix}\cdot d\mathbf{s} \qquad (9.2.6)$$

or, equivalently,

$$\left\langle \mathbf{w}, L\mathbf{v}\right\rangle - \left\langle L\mathbf{w},\mathbf{v}\right\rangle = \oint_{\partial\mathscr{V}}\begin{pmatrix}\varphi_f^+\left(P_f\left(\mathbf{v}\right)\left(\mathbf{w}_f-\mathbf{w}_s\right) - P_f\left(\mathbf{w}\right)\left(\mathbf{v}_f-\mathbf{v}_s\right)\right)\\ -\mathbf{T}_I\left(\mathbf{v}\right)\mathbf{w}_s + \mathbf{T}_I\left(\mathbf{w}\right)\mathbf{v}_s\end{pmatrix}\cdot d\mathbf{s} \qquad (9.2.7)$$

Equation (9.2.7) allow us to characterize those circumstances where the operator L is self adjoint, i.e., when

$$\left\langle \mathbf{w}, L\mathbf{v}\right\rangle = \left\langle L\mathbf{w},\mathbf{v}\right\rangle \qquad (9.2.8)$$

The following fives cases yield this result:

i) The fluid and solid displacements vanish on $\partial\mathscr{V}$.

ii) The stress vector $\mathbf{T}_I\mathbf{n}$ vanishes on $\partial\mathscr{V}$ and the fluid and solid displacements are equal.

iii) The stress vector $\mathbf{T}_I\mathbf{n}$ vanishes on $\partial\mathscr{V}$ and the pore pressure P_f vanishes on $\partial\mathscr{V}$.

iv) The solid displacement vanishes on $\partial\mathscr{V}$ and the pore pressure P_f vanishes on $\partial\mathscr{V}$.

v) Combinations of *i*) through *iv*), where the cases are prescribed on different parts of $\partial\mathscr{V}$.

where $\mathbf{n}$ is the outward drawn unit normal to the surface $\partial\mathscr{V}$.

From a formal stand point, each of these five choices defines a different operator. The operator is defined formally by (9.2.1) and by its domain, i.e. by the set of functions in its domain. Each of the above five choices represent possible different choices of the domain of L.

As explained in Section 4.3, the above list contains the usual traction and displacement boundary conditions from elasticity. The conditions on the pore pressure and the relative displacement of the fluid and solid correspond to whether or not the boundary is *pervious* or *impervious* to the flow of fluid across the boundary. If one prescribes the relative displacement to be zero, as in *ii*), then the fluid is not allowed to flow across the boundary. If instead, one prescribes the pore pressure, relative motion is allowed and, thus, fluid can flow across the boundary.

Given boundary conditions taken from the list *i)* through *v)* above, it follows from (9.2.5) that

$$\langle \mathbf{w}, L\mathbf{w} \rangle = \int_{\mathscr{V}} \begin{pmatrix} \bar{\lambda}_{ff} \left(\operatorname{Div} \mathbf{w}_f \right)^2 + 2\bar{\lambda}_{sf} \operatorname{Div} \mathbf{w}_f \operatorname{Div} \mathbf{w}_s + \bar{\lambda}_{ss} \left(\operatorname{Div} \mathbf{w}_s \right)^2 \\ + \frac{1}{2} \mu_{ss} \operatorname{tr}\left(\left(\operatorname{GRAD} \mathbf{w}_s + \left(\operatorname{GRAD} \mathbf{w}_s \right)^T \right)^2 \right) \end{pmatrix} \tag{9.2.9}$$

The inequalities (4.3.5), (4.2.6) and (4.3.7) allow us to conclude that for the operators defined by the categories of boundary conditions *i)* through *v)*, the operator L is positive semi definite, i.e.,

$$\langle \mathbf{w}, L\mathbf{w} \rangle \geq 0 \tag{9.2.10}$$

As usual, an eigenvalue of the self adjoint positive semi definite operator L is a nonzero function $\mathbf{u}$ that obeys

$$L\mathbf{u} = \lambda \mathbf{u} \tag{9.2.11}$$

The quantity λ is the eigenvalue associated with the eigenfunction $\mathbf{u}$. Because the operator is self adjoint, the eigenvalues are necessarily real numbers. Because the operator is positive semi definite, the eigenvalues are nonnegative real numbers. It is also a standard result that when one has two different eigenvalues, the corresponding eigenvectors are orthogonal with respect to the inner product defined by (9.2.3).

9.3 Boundary Initial Value Problems

Consider next, a boundary initial value problem for the partial differential equations (9.1.15) and (9.1.16) as follows:

Boundary Conditions:

$$\mathbf{T}_I(\mathbf{w})\mathbf{n} = -\mathbf{s}(\mathbf{X}, t) \quad \text{for} \quad (\mathbf{X}, t) \in \partial\mathscr{V}_1 \times (0, \infty) \tag{9.3.1}$$

$$P_f(\mathbf{w}) = r(\mathbf{X}, t) \quad \text{for} \quad (\mathbf{X}, t) \in \partial\mathscr{V}_1 \times (0, \infty) \tag{9.3.2}$$

$$\mathbf{w} = \mathbf{k}(\mathbf{X}, t) \quad \text{for} \quad (\mathbf{X}, t) \in \partial\mathscr{V}_2 \times (0, \infty) \tag{9.3.3}$$

Initial Condition:

$$\mathbf{w}(\mathbf{X}, 0) = \mathbf{f}(\mathbf{X}) \quad \text{for} \quad \mathbf{X} \in \mathscr{V} \tag{9.3.4}$$

where the boundary $\partial\mathscr{V}$ of the volume $\mathscr{V}$ consists of two parts $\partial\mathscr{V}_1$ and $\partial\mathscr{V}_2$. On the part $\partial\mathscr{V}_1$ the stress vector $\mathbf{T}_I\mathbf{n}$ and the pore pressure P_f are prescribed by the functions $\mathbf{s}(\mathbf{X},t)$ and $r(\mathbf{X},t)$, respectively. On the remainder, $\partial\mathscr{V}_2$, the displacement is prescribed by the function $\mathbf{k}(\mathbf{X},t)$. The initial condition is given by the function $\mathbf{f}(\mathbf{X})$. The four functions $\mathbf{s}(\mathbf{X},t)$, $r(\mathbf{X},t)$, $\mathbf{k}(\mathbf{X},t)$ and $\mathbf{f}(\mathbf{X})$ represent the data for this problem.

Other categories of boundary initial value problems can be constructed where, instead of prescribing the pore pressure, one prescribes the relative displacement $\mathbf{w}_f-\mathbf{w}_s$ on the boundary $\partial\mathscr{V}_1$. An example of this kind of boundary condition would be one where the boundary $\partial\mathscr{V}_1$ was impervious to the flow of the fluid. For the moment, we shall focus our discussion on the boundary initial value problem defined by (9.3.1) through (9.3.4).

There are conditions on the data that are required by the governing partial differential equations. It is easily established by summing (9.1.15) and (9.1.16) and making use of (9.1.17) and (9.1.18) that

$$\operatorname{Div}\mathbf{T}_I(\mathbf{X},t)=0 \tag{9.3.5}$$

This result, when forced to hold at $t=0$ shows that the initial condition (9.3.4) is not fully arbitrary. The stress calculated from the initial displacement $\mathbf{w}(\mathbf{X},0)=\mathbf{f}(\mathbf{X})$ from the constitutive equations (9.1.17) and (9.1.18) must obey (9.3.5). From a physical standpoint, this restriction on the initial data simply states that the porous elastic material is initially in a state that obeys the governing balance of linear momentum equations.

For certain boundary conditions, the result (9.3.5) implies a restriction on the surface traction $\mathbf{s}(\mathbf{X},t)$ in (9.3.1). If (9.3.5) is integrated over the region $\mathscr{V}$ we obtain the overall equilibrium result

$$\oint_{\partial\mathscr{V}}\mathbf{T}_I d\mathbf{s}=\mathbf{0} \tag{9.3.6}$$

For boundary initial value problems like above, but in the special case where $\partial\mathscr{V}_1=\partial\mathscr{V}$, the result (9.3.6) and with the boundary condition (9.3.1) force the requirement

$$\oint_{\partial\mathscr{V}}\mathbf{s}(\mathbf{X},t)ds=\mathbf{0} \tag{9.3.7}$$

on the data. The condition (9.3.7) is no more than the requirement that the data be prescribed such that. for all t. the porous elastic material is in overall elastic equilibrium.

9.4 A One Dimensional Example

It is instructive to illustrate some of the features of problems governed by (9.1.15) and (9.1.16) by working a one dimensional example. The example is motivated by one worked by Biot in a fundamental paper published in 1941 [Ref. 2]. The particular problem is one where the two displacements are given by the one dimensional representations

$$\mathbf{w}_f = \left(w_f\left(X,t\right),0,0\right) \tag{9.4.1}$$

and

$$\mathbf{w}_s = \left(w_s\left(X,t\right),0,0\right) \tag{9.4.2}$$

In this special case, we can write the partial differential equations (9.1.15) and (9.1.16) as

$$\xi\mathbf{E}\frac{\partial\mathbf{w}}{\partial t} - \mathbf{Q}\frac{\partial^2\mathbf{w}}{\partial X^2} = \mathbf{0} \tag{9.4.3}$$

where,

$$\mathbf{w} = \mathbf{w}\left(X,t\right) = \begin{bmatrix} w_f\left(X,t\right) \\ w_s\left(X,t\right) \end{bmatrix} \tag{9.4.4}$$

$$\mathbf{Q} = \begin{bmatrix} \bar{\lambda}_{ff} & \bar{\lambda}_{sf} \\ \bar{\lambda}_{sf} & \bar{\lambda}_{ss} + 2\mu_{ss} \end{bmatrix} \tag{9.4.5}$$

and

$$\mathbf{E} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{9.4.6}$$

Of course, the definitions (9.4.4) through (9.4.6) have appeared in earlier chapters. The symbol $\mathbf{Q}$ was introduced in equations (7.2.46) and (8.1.11). The symbol $\mathbf{E}$ was first introduced in equation (7.2.87) and, again, in (8.1.12). The specialization of (9.1.15) and (9.1.16) to the one dimensional equation is a special case of the derivation of (8.1.7) and (8.1.8) of Chapter 8. The matrix of displacements (9.4.4) was also introduced in Chapter 8. It should be evident that it is the one dimensional counterpart of equation (9.2.2) above.

For this one dimensional problem, the constitutive equation (9.1.17) and (9.1.18) can be written in the matrix form

$$\begin{bmatrix} -\varphi_f^+ P_f \\ T_I + \varphi_f^+ P_f \end{bmatrix} = \mathbf{Q}\frac{\partial\mathbf{w}}{\partial X} \tag{9.4.7}$$

The boundary conditions we shall first apply to this one dimensional problem are one dimensional versions of (9.3.1) through (9.3.4). We are interested in solving a problem in the interval $X \in (0,h)$ where the boundary conditions and initial condition are

$$T_I(0,t) = -s(t) \tag{9.4.8}$$

$$P_f(0,t) = r(t) \tag{9.4.9}$$

$$\mathbf{w}(h,t) = \mathbf{k}(t) \tag{9.4.10}$$

and

$$\mathbf{w}(X,0) = \mathbf{f}(X) \tag{9.4.11}$$

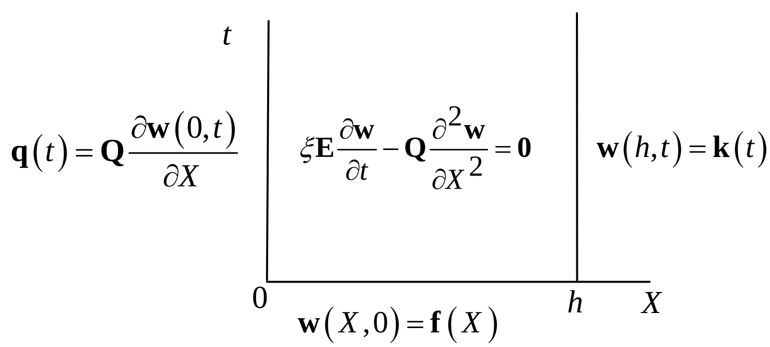
If we adopt the symbol $\mathbf{q}(t)$ defined by

$$\mathbf{q}(t) = \begin{bmatrix} -\varphi_f^+ r(t) \\ -s(t) + \varphi_f^+ r(t) \end{bmatrix} \tag{9.4.12}$$

then the two boundary conditions (9.4.8) and (9.4.9) can be combined with (9.4.7) to write the boundary condition at $X = 0$ as

$$\mathbf{q}(t) = \mathbf{Q}\frac{\partial \mathbf{w}(0,t)}{\partial X} \tag{9.4.13}$$

It is instructive to display the partial differential equation, the boundary conditions and the initial condition on the following schematic



The one dimensional form of (9.3.5) is

$$\frac{\partial T_I}{\partial X} = 0 \tag{9.4.14}$$

Thus, the stress cannot depend upon X. This fact can be represented in an equivalent way by multiplying (9.4.3) by $\operatorname{adj}\mathbf{E}$ to obtain

$$(\operatorname{adj}\mathbf{E})\mathbf{Q}\frac{\partial^2 \mathbf{w}}{\partial X^2} = \mathbf{0} \tag{9.4.15}$$

This simple formula can be integrated twice to yield

$$(\operatorname{adj}\mathbf{E})\mathbf{Q}\mathbf{w}(X,t) = (\operatorname{adj}\mathbf{E})\mathbf{Q}\mathbf{k}(t) + (\operatorname{adj}\mathbf{E})\mathbf{q}(t)(X-h) \tag{9.4.16}$$

after the boundary conditions (9.4.10) and (9.4.13) are used. If this formula is evaluated at $t = 0$, we see that the functions $\mathbf{f}(X)$, $\mathbf{k}(t)$ and $\mathbf{q}(t)$ are linked by the formula

$$(\operatorname{adj}\mathbf{E})\mathbf{Q}\mathbf{f}(X) = (\operatorname{adj}\mathbf{E})\mathbf{Q}\mathbf{k}(0) + (\operatorname{adj}\mathbf{E})\mathbf{q}(0)(X-h) \tag{9.4.17}$$

Equation is the one dimensional version of the data compatibility issue mentioned in Section 9.3. In the following, we shall assume that (9.4.17) is obeyed by our choices of the three functions $\mathbf{f}(X)$, $\mathbf{k}(t)$ and $\mathbf{q}(t)$.

In order to exploit eigenfunction techniques to solve this boundary initial value problem, we need to change the dependent variable in a manner sufficient to produce homogeneous boundary conditions. Because the problem is one dimensional, it is not difficult to see that the function $\mathbf{v}(X,t)$ defined by

$$\mathbf{v}(X,t) = \mathbf{w}(X,t) - \mathbf{Q}^{-1}\mathbf{q}(t)(X-h) - \mathbf{k}(t) \tag{9.4.18}$$

obeys the partial differential equation

$$\xi\mathbf{E}\frac{\partial \mathbf{v}}{\partial t} = \mathbf{Q}\frac{\partial^2 \mathbf{v}}{\partial X^2} + \mathbf{p}(t) \tag{9.4.19}$$

where

$$\mathbf{p}(t) = -\xi\mathbf{E}\mathbf{Q}^{-1}\dot{\mathbf{q}}(t)(X-h) - \xi\mathbf{E}\dot{\mathbf{k}}(t) \tag{9.4.20}$$

The boundary conditions and the initial value that the function $\mathbf{v}(X,t)$ must obey follow from (9.4.18) and the corresponding conditions on $\mathbf{w}(X,t)$. These results are

$$\frac{\partial \mathbf{v}(0,t)}{\partial X} = \mathbf{0} \tag{9.4.21}$$

$$\mathbf{v}(h,t) = \mathbf{0} \tag{9.4.22}$$

and

$$\mathbf{v}(X,0) = \mathbf{f}(X) - \mathbf{Q}^{-1}\mathbf{q}(0)(X-h) - \mathbf{k}(0) \tag{9.4.23}$$

The gain with the transformation (9.4.18) is that the function $\mathbf{v}(X,t)$ obeys homogeneous boundary conditions. The trade off is that the partial differential equation obeyed by $\mathbf{v}(X,t)$ is inhomogeneous and its initial condition is more complicated than that required of $\mathbf{w}(X,t)$.

It is useful at this point to note that the definition (9.4.18) and the result (9.4.16) combine to yield

$$(\text{adj}\mathbf{E})\mathbf{Q}\mathbf{v}(X,t) = (\text{adj}\mathbf{E})\mathbf{Q}\left(\mathbf{w}(X,t) - \mathbf{Q}^{-1}\mathbf{q}(t)(X-h) - \mathbf{k}(t)\right) = \mathbf{0} \tag{9.4.24}$$

Of course, this result also follows from (9.4.19) and the two boundary conditions, (9.4.21) and (9.4.22).

The solution technique we shall utilize is to represent the solution as an expansion in the eigenfunctions of the space part of the operator in (9.4.19). As explained in Section 9.2, the eigenfunctions are nonzero functions $\mathbf{u}(X)$ that obey

$$-\mathbf{Q}\frac{d^2\mathbf{u}}{dX^2} = \lambda\mathbf{u} \quad \text{on } (0,h) \tag{9.4.25}$$

and the boundary conditions

$$\frac{d\mathbf{u}(0)}{dX} = 0 \qquad \text{and} \quad \mathbf{u}(h) = \mathbf{0} \tag{9.4.26}$$

It is elementary to show that solutions of (9.4.25) that obey (9.4.26) are of the form

$$\mathbf{u}_n = \mathbf{b}_n \cos\frac{(2n-1)\pi X}{2h} \qquad \text{for} \quad n = 1,2,\ldots,\infty \tag{9.4.27}$$

where, for each n,

$$\left(\mathbf{Q} - \frac{4h^2}{(2n-1)^2}\lambda_n \mathbf{I}\right)\mathbf{b}_n = \mathbf{0} \tag{9.4.28}$$

The symbol $\mathbf{I}$ in (9.4.28) denotes the 2×2 identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{9.4.29}$$

Equation (9.4.28), which arises from the substitution of (9.4.27) into (9.4.25), shows that the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n, ...$ and the column matrices $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n, ...$ are determined by the algebraic eigenvalue problem for the 2×2 symmetric matrix $\mathbf{Q}$. If $\alpha_{(1)}$ and $\alpha_{(2)}$ are the eigenvalues $\mathbf{Q}$, then they are the two roots of

$$\det(\mathbf{Q} - \alpha \mathbf{I}) = 0 \tag{9.4.30}$$

Given these two roots, it follows from (9.4.28) that the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n, ...$ are

$$\lambda_{n(1)} = \frac{(2n-1)^2 \pi^2}{4h^2}\alpha_{(1)} \tag{9.4.31}$$

and

$$\lambda_{n(2)} = \frac{(2n-1)^2 \pi^2}{4h^2}\alpha_{(2)} \tag{9.4.32}$$

The form of (9.4.31) and (9.4.32) show that the column matrices $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n, ...$ actually do not depend upon the index n. They are the two nonzero column matrices that obey

$$\left(\mathbf{Q} - \alpha_{(1)}\mathbf{I}\right)\mathbf{b}_{(1)} = \mathbf{0} \tag{9.4.33}$$

and

$$\left(\mathbf{Q} - \alpha_{(2)}\mathbf{I}\right)\mathbf{b}_{(2)} = \mathbf{0} \tag{9.4.34}$$

Given this construction, the solution of our boundary initial value problem for $\mathbf{v}(X,t)$ will be of the form

$$\mathbf{v}(X,t) = \sum_{n=1}^{\infty} T_{n(1)}(t)\mathbf{b}_{(1)} \cos\frac{(2n-1)\pi X}{2h} + \sum_{n=1}^{\infty} T_{n(2)}(t)\mathbf{b}_{(2)} \cos\frac{(2n-1)\pi X}{2h} \tag{9.4.35}$$

It simplifies the analysis somewhat if we rewrite (9.4.35) as

$$\mathbf{v}(X,t)=\sum_{n=1}^{\infty}\mathbf{T}_n(t)\cos\frac{(2n-1)\pi X}{2h} \tag{9.4.36}$$

where $\mathbf{T}_n(t)$ is a column matrix whose representation with respect to the basis of eigenvectors of $\mathbf{Q}$ is

$$\mathbf{T}_n(t)=T_{n(1)}(t)\mathbf{b}_{(1)}+T_{n(2)}(t)\mathbf{b}_{(2)} \tag{9.4.37}$$

As constructed, the proposed solution (9.4.36) obeys the boundary conditions (9.4.21) and (9.4.22). Our next step is to determine the time dependence of the solution. If (9.4.36) is substituted into (9.4.19) the result is

$$\sum_{n=1}^{\infty}\left(\xi\mathbf{E}\frac{d\mathbf{T}_n(t)}{dt}+\frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\mathbf{T}_n(t)\right)\cos\frac{(2n-1)\pi X}{2h}=\mathbf{p}(t) \tag{9.4.38}$$

The orthogonality of the functions $\left\{\cos\frac{(2n-1)\pi X}{2h}\right\}$ allow us to derive from (9.4.38) the result

$$\xi\mathbf{E}\frac{d\mathbf{T}_n(t)}{dt}+\frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\mathbf{T}_n(t)=\mathbf{p}_n(t) \tag{9.4.39}$$

where

$$\mathbf{p}_n(t)=\frac{2}{h}\xi\mathbf{E}\mathbf{Q}^{-1}\dot{\mathbf{q}}(t)\int_0^h(h-X)\cos\frac{(2n-1)\pi X}{2h}dX-\frac{2}{h}\xi\mathbf{E}\dot{\mathbf{k}}(t)\int_0^h\cos\frac{(2n-1)\pi X}{2h}dX \tag{9.4.40}$$

After the elementary integrals in (9.4.40) are evaluated, we obtain

$$\mathbf{p}_n(t)=\frac{8h}{(2n-1)^2\pi^2}\xi\mathbf{E}\mathbf{Q}^{-1}\dot{\mathbf{q}}(t)+\frac{4}{(2n-1)\pi}(-1)^n\xi\mathbf{E}\dot{\mathbf{k}}(t) \tag{9.4.41}$$

The solution of (9.4.39) requires an initial condition. This condition comes from (9.4.23) and (9.4.36). These two equations yield

$$\sum_{n=1}^{\infty}\mathbf{T}_n(0)\cos\frac{(2n-1)\pi X}{2h}=\mathbf{f}(X)-\mathbf{Q}^{-1}\mathbf{q}(0)(X-h)-\mathbf{k}(0) \tag{9.4.42}$$

The orthogonality of the functions $\left\{\cos\frac{(2n-1)\pi X}{2h}\right\}$ can be used again to derive from (9.4.42) the result

$$\begin{aligned}\mathbf{T}_n(0) &= \frac{2}{h}\int_0^h \mathbf{f}(X)\cos\frac{(2n-1)\pi X}{2h}dX \\ &+\frac{2}{h}\mathbf{Q}^{-1}\mathbf{q}(0)\int_0^h (h-X)\cos\frac{(2n-1)\pi X}{2h}dX - \frac{2}{h}\mathbf{k}(0)\int_0^h \cos\frac{(2n-1)\pi X}{2h}dX\end{aligned} \tag{9.4.43}$$

The integrals in the second and third terms of (9.4.43) can be evaluated to yield the result

$$\mathbf{T}_n(0) = \mathbf{f}_n + \frac{8h}{(2n-1)^2\pi^2}\mathbf{Q}^{-1}\mathbf{q}(0) + \frac{4}{(2n-1)\pi}(-1)^n\,\mathbf{k}(0) \tag{9.4.44}$$

where, we have introduced the symbol $\mathbf{f}_n$ defined by

$$\mathbf{f}_n = \frac{2}{h}\int_0^h \mathbf{f}(X)\cos\frac{(2n-1)\pi X}{2h}dX \tag{9.4.45}$$

In the manipulations that follow, it is convenient to simply refer to the initial condition as $\mathbf{T}_n(0)$ and to utilize the explicit formula (9.4.44) at the end of the calculation.

In the way of summary, at this point in the solution we must find the solution of the matrix ordinary differential equation (9.4.39), repeated here,

$$\xi\mathbf{E}\frac{d\mathbf{T}_n(t)}{dt} + \frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\mathbf{T}_n(t) = \mathbf{p}_n(t) \tag{9.4.46}$$

where $\mathbf{p}_n(t)$ is given by (9.4.41), repeated here,

$$\mathbf{p}_n(t) = \frac{8h}{(2n-1)^2\pi^2}\xi\mathbf{E}\mathbf{Q}^{-1}\dot{\mathbf{q}}(t) + \frac{4}{(2n-1)\pi}(-1)^n\,\xi\mathbf{E}\dot{\mathbf{k}}(t) \tag{9.4.47}$$

subject to the initial condition (9.4.44). Because the 2×2 matrix $\mathbf{E}$ defined by (9.4.6) is singular, solutions of (9.4.46) necessarily reflect a certain indeterminacy. If we multiply (9.4.46) by $\operatorname{adj}\mathbf{E}$ and use the formula

$$(\operatorname{adj}\mathbf{E})\mathbf{E} = (\det\mathbf{E})\mathbf{I} = \mathbf{0} \tag{9.4.48}$$

equation (9.4.46) yields

$$\left(\operatorname{adj}\mathbf{E}\right)\mathbf{Q}\mathbf{T}_n\left(t\right)=\mathbf{0} \tag{9.4.49}$$

after one uses the definition (9.4.47). This result also follows from (9.4.36) and (9.4.24).

There are various ways to solve (9.4.46). Probably the most direct is to calculate its Laplace transform and use standard methods to solve and invert the result. The Laplace transform of (9.4.46) is

$$\left(s\xi\mathbf{E}+\frac{\left(2n-1\right)^2\pi^2}{4h^2}\mathbf{Q}\right)\overline{\mathbf{T}}_n\left(s\right)=\xi\mathbf{E}\mathbf{T}_n\left(0\right)+\overline{\mathbf{p}}_n\left(s\right) \tag{9.4.50}$$

where $\overline{\mathbf{T}}_n\left(s\right)$ is the Laplace transform of $\mathbf{T}_n\left(t\right)$ and $\overline{\mathbf{p}}_n\left(s\right)$ is the Laplace transform of $\mathbf{p}_n\left(t\right)$. Equation (9.4.50) yields

$$\overline{\mathbf{T}}_n\left(s\right)=\xi\left(s\xi\mathbf{E}+\frac{\left(2n-1\right)^2\pi^2}{4h^2}\mathbf{Q}\right)^{-1}\mathbf{E}\mathbf{T}_n\left(0\right)+\left(s\xi\mathbf{E}+\frac{\left(2n-1\right)^2\pi^2}{4h^2}\mathbf{Q}\right)^{-1}\overline{\mathbf{p}}_n\left(s\right) \tag{9.4.51}$$

The inverse matrix that appears in (9.4.51) can be written

$$\left(s\xi\mathbf{E}+\frac{\left(2n-1\right)^2\pi^2}{4h^2}\mathbf{Q}\right)^{-1}=\frac{\operatorname{adj}\left(s\xi\mathbf{E}+\frac{\left(2n-1\right)^2\pi^2}{4h^2}\mathbf{Q}\right)}{\det\left(s\xi\mathbf{E}+\frac{\left(2n-1\right)^2\pi^2}{4h^2}\mathbf{Q}\right)} \tag{9.4.52}$$

Because we are dealing with 2×2 matrices, it is true that

$$\operatorname{adj}\left(s\xi\mathbf{E}+\frac{\left(2n-1\right)^2\pi^2}{4h^2}\mathbf{Q}\right)=s\xi\operatorname{adj}\mathbf{E}+\frac{\left(2n-1\right)^2\pi^2}{4h^2}\operatorname{adj}\mathbf{Q} \tag{9.4.53}$$

It is also true that

$$\det\left(s\xi\mathbf{E}+\frac{\left(2n-1\right)^2\pi^2}{4h^2}\mathbf{Q}\right)=s\xi\frac{\left(2n-1\right)^2\pi^2}{4h^2}\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)+\frac{\left(2n-1\right)^4\pi^4}{16h^4}\det\mathbf{Q} \tag{9.4.54}$$

where $\det\mathbf{E}=\mathbf{0}$ has been used. Given (9.4.53) and (9.4.54), we can write (9.4.52) as

$$\left(s\xi\mathbf{E}+\frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\right)^{-1}=\frac{s\xi\,\mathrm{adj}\mathbf{E}+\frac{(2n-1)^2\pi^2}{4h^2}\mathrm{adj}\mathbf{Q}}{s\xi\frac{(2n-1)^2\pi^2}{4h^2}\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)+\frac{(2n-1)^4\pi^4}{16h^4}\det\mathbf{Q}}$$

$$=\frac{1}{\xi\frac{(2n-1)^2\pi^2}{4h^2}\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\frac{s\xi\,\mathrm{adj}\mathbf{E}+\frac{(2n-1)^2\pi^2}{4h^2}\mathrm{adj}\mathbf{Q}}{\left(s+\frac{(2n-1)^2\pi^2}{4h^2}\frac{\det\mathbf{Q}}{\xi\,\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\right)}$$

$$=\frac{1}{\frac{(2n-1)^2\pi^2}{4h^2}\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\mathrm{adj}\mathbf{E}$$
$$+\left(\frac{1}{\xi\,\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\left(\mathrm{adj}\mathbf{Q}-\frac{\det\mathbf{Q}}{\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\mathrm{adj}\mathbf{E}\right)\right)\frac{1}{\left(s+\frac{(2n-1)^2\pi^2}{4h^2}\frac{\det\mathbf{Q}}{\xi\,\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\right)}$$

$$=\frac{1}{\frac{(2n-1)^2\pi^2}{4h^2}\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\mathrm{adj}\mathbf{E}$$
$$+\left(\frac{1}{\xi\,\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\left(\mathrm{adj}\mathbf{Q}-\frac{\det\mathbf{Q}}{\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\mathrm{adj}\mathbf{E}\right)\right)\frac{1}{\left(s+\frac{(2n-1)^2\pi^2}{4h^2}c\right)} \tag{9.4.55}$$

where c is the consolidation coefficient defined by equation (5.1.30), repeated here as

$$c=\frac{1}{\xi}\frac{\bar{\lambda}_{ff}(\bar{\lambda}_{ss}+\frac{2}{3}\mu_{ss})-\bar{\lambda}_{sf}^2}{\bar{\lambda}_{ss}+\frac{2}{3}\mu_{ss}+2\bar{\lambda}_{sf}+\bar{\lambda}_{ff}}=\frac{\det\mathbf{Q}}{\xi\,\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)} \tag{9.4.56}$$

If this long formula is substituted into (9.4.51), we must invert

$$
\begin{aligned}
\overline{\mathbf{T}}_n(s) &= \xi \left(\begin{array}{l} \dfrac{1}{\dfrac{(2n-1)^2\pi^2}{4h^2}\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\operatorname{adj}\mathbf{E} \\ +\left(\dfrac{1}{\xi\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}-\dfrac{\det\mathbf{Q}}{\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\operatorname{adj}\mathbf{E}\right)\right)\dfrac{1}{\left(s+\dfrac{(2n-1)^2\pi^2}{4h^2}c\right)} \end{array} \right)\mathbf{E}\mathbf{T}_n(0) \\
&+ \left(\begin{array}{l} \dfrac{1}{\dfrac{(2n-1)^2\pi^2}{4h^2}\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\operatorname{adj}\mathbf{E} \\ +\left(\dfrac{1}{\xi\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}-\dfrac{\det\mathbf{Q}}{\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\operatorname{adj}\mathbf{E}\right)\right)\dfrac{1}{\left(s+\dfrac{(2n-1)^2\pi^2}{4h^2}c\right)} \end{array} \right)\overline{\mathbf{p}}_n(s) \\
&= \frac{1}{\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\frac{(\operatorname{adj}\mathbf{Q})\mathbf{E}}{\left(s+\dfrac{(2n-1)^2\pi^2}{4h^2}c\right)}\mathbf{T}_n(0) \\
&+ \frac{1}{\xi\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\frac{\operatorname{adj}\mathbf{Q}}{\left(s+\dfrac{(2n-1)^2\pi^2}{4h^2}c\right)}\overline{\mathbf{p}}_n(s)
\end{aligned} \tag{9.4.57}
$$

where (9.4.48) has been used. We have also used $(\operatorname{adj}\mathbf{E})\overline{\mathbf{p}}_n(s)=\mathbf{0}$ which follows from (9.4.47) and (9.4.48). The inverse Laplace transform of (9.4.57) is[1]

$$
\begin{aligned}
\mathbf{T}_n(t) &= \frac{1}{\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}(\operatorname{adj}\mathbf{Q})\mathbf{E}\mathbf{T}_n(0)e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct} \\
&+ \frac{1}{\xi\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\operatorname{adj}\mathbf{Q}\int_0^t \mathbf{p}_n(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau
\end{aligned} \tag{9.4.58}
$$

Next, we shall utilize (9.4.44) and (9.4.47) to write (9.4.58) as

---

[1] While we do not need the formula here, it is useful to note that the identity $(\operatorname{adj}\mathbf{Q})\mathbf{E}+(\operatorname{adj}\mathbf{E})\mathbf{Q}=\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)\mathbf{I}$ and the formula (9.4.49) allow the solution (9.4.58) to be written

$$
\mathbf{T}_n(t)=\mathbf{T}_n(0)e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct}+\frac{1}{\xi\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\operatorname{adj}\mathbf{Q}\int_0^t \mathbf{p}_n(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau
$$

This form of the solution illustrates that the initial condition is, in fact, satisfied.

$$
\begin{aligned}
\mathbf{T}_n(t) &= \frac{1}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\begin{pmatrix} \mathbf{f}_n + \dfrac{8h}{(2n-1)^2\pi^2}\mathbf{Q}^{-1}\mathbf{q}(0) \\ +\dfrac{4}{(2n-1)\pi}(-1)^n\mathbf{k}(0)\end{pmatrix} e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct} \\
&\quad + \frac{1}{\xi\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\operatorname{adj}\mathbf{Q}\int_0^t \begin{pmatrix} \dfrac{8h}{(2n-1)^2\pi^2}\xi\mathbf{E}\mathbf{Q}^{-1}\dot{\mathbf{q}}(\tau) \\ +\dfrac{4}{(2n-1)\pi}(-1)^n\xi\mathbf{E}\dot{\mathbf{k}}(\tau)\end{pmatrix} e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau \\
&= \frac{1}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\begin{pmatrix} \mathbf{f}_n + \dfrac{8h}{(2n-1)^2\pi^2}\mathbf{Q}^{-1}\mathbf{q}(0) \\ +\dfrac{4}{(2n-1)\pi}(-1)^n\mathbf{k}(0)\end{pmatrix} e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct} \\
&\quad + \frac{1}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\mathbf{Q}^{-1}\frac{8h}{(2n-1)^2\pi^2}\int_0^t \dot{\mathbf{q}}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau \\
&\quad + \frac{1}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\frac{4}{(2n-1)\pi}(-1)^n\int_0^t \dot{\mathbf{k}}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau \\
&= \frac{1}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\mathbf{f}_n e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct} \\
&\quad + \frac{8h}{(2n-1)^2\pi^2}\frac{1}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\mathbf{Q}^{-1}\begin{pmatrix} \mathbf{q}(0)e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct} \\ +\displaystyle\int_0^t \dot{\mathbf{q}}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau\end{pmatrix} \\
&\quad + \frac{4}{(2n-1)\pi}(-1)^n\frac{1}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\begin{pmatrix} \mathbf{k}(0)e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct} \\ +\displaystyle\int_0^t \dot{\mathbf{k}}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau\end{pmatrix}
\end{aligned} \tag{9.4.59}
$$

If we utilize integration by parts to write

$$
\begin{aligned}
&\mathbf{q}(0)e^{-\frac{(2n-1)^2\pi^2}{4h^2}\frac{\det\mathbf{Q}}{\xi\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}t} + \int_0^t \dot{\mathbf{q}}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau \\
&\qquad = \mathbf{q}(t) - \frac{(2n-1)^2\pi^2}{4h^2}\frac{\det\mathbf{Q}}{\xi\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\int_0^t \mathbf{q}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau
\end{aligned} \tag{9.4.60}
$$

and

$$\begin{aligned}&\mathbf{k}(0)e^{-\frac{(2n-1)^2\pi^2}{4h^2}\frac{\det\mathbf{Q}}{\xi\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}t}+\int_0^t\dot{\mathbf{k}}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}\frac{\det\mathbf{Q}}{\xi\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}(t-\tau)}d\tau\\&=\mathbf{k}(t)-\frac{(2n-1)^2\pi^2}{4h^2}\frac{\det\mathbf{Q}}{\xi\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}\int_0^t\mathbf{k}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}\frac{\det\mathbf{Q}}{\xi\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}(t-\tau)}d\tau\end{aligned}\tag{9.4.61}$$

the solution (9.4.59) can be written

$$\begin{aligned}\mathbf{T}_n(t)&=\frac{1}{\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}(\operatorname{adj}\mathbf{Q})\mathbf{E}\mathbf{f}_n e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct}\\&+\frac{8h}{(2n-1)^2\pi^2}\frac{1}{\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}(\operatorname{adj}\mathbf{Q})\mathbf{E}\mathbf{Q}^{-1}\left(\begin{aligned}&\mathbf{q}(t)\\&-\frac{(2n-1)^2\pi^2}{4h^2}\frac{\det\mathbf{Q}}{\xi\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}\int_0^t\mathbf{q}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau\end{aligned}\right)\\&+\frac{4}{(2n-1)\pi}(-1)^n\frac{1}{\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}(\operatorname{adj}\mathbf{Q})\mathbf{E}\left(\begin{aligned}&\mathbf{k}(t)\\&-\frac{(2n-1)^2\pi^2}{4h^2}\frac{\det\mathbf{Q}}{\xi\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}\int_0^t\mathbf{k}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau\end{aligned}\right)\end{aligned}\tag{9.4.62}$$

If this result is combined with (9.4.36), the solution for $\mathbf{v}(X,t)$ can be written

$$\mathbf{v}(X,t)=\sum_{n=1}^{\infty}\mathbf{T}_n(t)\cos\frac{(2n-1)\pi X}{2h}$$

$$=\sum_{n=1}^{\infty}\left(\begin{array}{l}\frac{1}{\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}(\operatorname{adj}\mathbf{Q})\mathbf{E}\mathbf{f}_n e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct}\\ +\frac{8h}{(2n-1)^2\pi^2}\frac{1}{\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}(\operatorname{adj}\mathbf{Q})\mathbf{E}\mathbf{Q}^{-1}\\ \times\left(\begin{array}{l}\mathbf{q}(t)\\ -\frac{(2n-1)^2\pi^2}{4h^2}\frac{\det\mathbf{Q}}{\xi\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}\int_0^t\mathbf{q}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau\end{array}\right)\\ +\frac{4}{(2n-1)\pi}(-1)^n\frac{1}{\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}(\operatorname{adj}\mathbf{Q})\mathbf{E}\\ \times\left(\begin{array}{l}\mathbf{k}(t)\\ -\frac{(2n-1)^2\pi^2}{4h^2}\frac{\det\mathbf{Q}}{\xi\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}\int_0^t\mathbf{k}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau\end{array}\right)\end{array}\right)\cos\frac{(2n-1)\pi X}{2h} \tag{9.4.63}$$

Because

$$\frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi X}{2h}=h-X \tag{9.4.64}$$

and

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)}(-1)^n\right)\cos\frac{(2n-1)\pi X}{2h}=-1 \tag{9.4.65}$$

(9.4.63) reduces to

$$
\begin{aligned}
\mathbf{v}(X,t) &= \sum_{n=1}^{\infty} \mathbf{T}_n(t)\cos\frac{(2n-1)\pi X}{2h} \\
&= \sum_{n=1}^{\infty} \left(\begin{aligned}
&\frac{1}{\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}(\operatorname{adj}\mathbf{Q})\mathbf{E}\mathbf{f}_n e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct} \\
&-\frac{2}{h}\frac{1}{\xi\left(\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)\right)^2}(\operatorname{adj}\mathbf{Q})\mathbf{E}(\operatorname{adj}\mathbf{Q})\left(\int_0^t \mathbf{q}(\tau) e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)} d\tau\right) \\
&-(-1)^n \frac{(2n-1)\pi}{h^2}\frac{\det\mathbf{Q}}{\xi\left(\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)\right)^2}(\operatorname{adj}\mathbf{Q})\mathbf{E}\left(\int_0^t \mathbf{k}(\tau) e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)} d\tau\right)
\end{aligned} \right)\cos\frac{(2n-1)\pi X}{2h} \\
&+\frac{1}{\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}(\operatorname{adj}\mathbf{Q})\mathbf{E}\mathbf{Q}^{-1}\mathbf{q}(t)(h-X) \\
&-\frac{1}{\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}(\operatorname{adj}\mathbf{Q})\mathbf{E}\left(\mathbf{k}(t)\right)
\end{aligned}
\tag{9.4.66}
$$

This result combines with (9.4.18) to yield

$$
\begin{aligned}
&\mathbf{w}(X,t)=\mathbf{v}(X,t)+\mathbf{Q}^{-1}\mathbf{q}(t)(X-h)+\mathbf{k}(t)\\
&=\sum_{n=1}^{\infty}\left(\begin{array}{l}
\dfrac{1}{\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}(\mathrm{adj}\mathbf{Q})\mathbf{E}\mathbf{f}_n e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct}\\
-\dfrac{2}{h}\dfrac{1}{\xi\left(\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)\right)^2}(\mathrm{adj}\mathbf{Q})\mathbf{E}(\mathrm{adj}\mathbf{Q})\left(\displaystyle\int_0^t \mathbf{q}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau\right)\\
-(-1)^n\dfrac{(2n-1)\pi}{h^2}\dfrac{\det\mathbf{Q}}{\xi\left(\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)\right)^2}(\mathrm{adj}\mathbf{Q})\mathbf{E}\left(\displaystyle\int_0^t \mathbf{k}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau\right)
\end{array}\right)\cos\frac{(2n-1)\pi X}{2h}\\
&+\left(\frac{1}{\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}(\mathrm{adj}\mathbf{Q})\mathbf{E}-\mathbf{I}\right)\mathbf{Q}^{-1}\mathbf{q}(t)(h-X)-\left(\frac{1}{\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}(\mathrm{adj}\mathbf{Q})\mathbf{E}-\mathbf{I}\right)\mathbf{k}(t)\\
&=\frac{1}{\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}(\mathrm{adj}\mathbf{Q})\mathbf{E}\sum_{n=1}^{\infty}\mathbf{f}_n e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct}\cos\frac{(2n-1)\pi X}{2h}\\
&-\frac{2}{h}\frac{\det\mathbf{Q}}{\xi\left(\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)\right)^2}(\mathrm{adj}\mathbf{Q})\mathbf{E}\sum_{n=1}^{\infty}\left(\begin{array}{l}
\mathbf{Q}^{-1}\left(\displaystyle\int_0^t \mathbf{q}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau\right)\\
(-1)^n\dfrac{(2n-1)\pi}{2h}\left(\displaystyle\int_0^t \mathbf{k}(\tau)e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau\right)
\end{array}\right)\cos\frac{(2n-1)\pi X}{2h}\\
&+\frac{\mathrm{adj}\mathbf{E}}{\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\mathbf{q}(t)(X-h)+\frac{(\mathrm{adj}\mathbf{E})\mathbf{Q}}{\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\mathbf{k}(t)
\end{aligned}
\tag{9.4.67}
$$

where the algebraic identity

$$
(\mathrm{adj}\mathbf{Q})\mathbf{E}+(\mathrm{adj}\mathbf{E})\mathbf{Q}=\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)\mathbf{I} \tag{9.4.68}
$$

has been used to simplify the last two terms.

Equation (9.4.67) is the analytical solution to the boundary initial value problem we have posed. Because of the relationship (9.4.17) and various identities one can write among the 2×2 matrices that appear in the answer, one can write the solution in various other equivalent forms.

It is straight forward to show that the solution (9.4.67) obeys the boundary and initial conditions (9.4.8) through (9.4.11). Also, we note in passing that the solution (9.4.67) yields the following formula for the initial velocity.

$$
\frac{\partial\mathbf{w}(X,0)}{\partial t}=\frac{\det\mathbf{Q}}{\xi\,\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\frac{\partial^2\mathbf{f}(X)}{\partial X^2}+\frac{\mathrm{adj}\mathbf{E}}{\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\dot{\mathbf{q}}(0)(X-h)+\frac{\mathrm{adj}\mathbf{E}}{\mathrm{tr}\left((\mathrm{adj}\mathbf{Q})\mathbf{E}\right)}\mathbf{Q}\dot{\mathbf{k}}(0) \tag{9.4.69}
$$

This result, which can be shown to be consistent with (9.4.16), will be useful in Section 9.5

9.5 Biot Problem

The original problem solved by Biot is the special case of the solution (9.4.67) corresponding to the choices [Ref. 1]

$$\mathbf{q}(t) = P_0 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{9.5.1}$$

$$\mathbf{k}(t) = \mathbf{0} \tag{9.5.2}$$

and

$$\mathbf{f}(X) = \frac{P_0}{\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}(h - X)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{9.5.3}$$

It follows from (9.4.12) that the choice (9.5.1) corresponds to the assumption that the pore pressure at the point $x = 0$ is zero and the stress T_I at $x = 0$ is the pressure P_0. As explained earlier, the choice $P_f = 0$ at a boundary means that the boundary is pervious to the flow of the fluid out of and into the solid. The choice (9.5.2) corresponds to the assumption that the displacement vanishes at $x = h$.

The initial condition (9.5.3) arises from the physical assumption that the initial stress $T_I(0)$ equals the imposed pressure P_0 and that the change in fluid content at $t = 0$ is zero. The change in content is given by (4.1.46). In terms of the notation we are using, the assumption that at $t = 0$ the change in fluid content is zero reduces (4.1.46) to

$$-\frac{\partial w_f(X,0)}{\partial X} + \frac{\partial w_s(X,0)}{\partial X} = 0 \tag{9.5.4}$$

It follows from the one dimensional forms of (9.1.17) and (9.1.18) that

$$T_I(0) = -P_0 = \left(\bar{\lambda}_{ff} + \bar{\lambda}_{sf}\right)\frac{\partial w_f(X,0)}{\partial X} + \left(\bar{\lambda}_{ss} + 2\mu_{ss} + \bar{\lambda}_{sf}\right)\frac{\partial w_s(X,0)}{\partial X} \tag{9.5.5}$$

When we use (9.5.4), equation (9.5.5) simplifies to

$$-P_0 = \left(\bar{\lambda}_{ff} + 2\bar{\lambda}_{sf} + \bar{\lambda}_{ss} + 2\mu_{ss}\right)\frac{\partial w_f(X,0)}{\partial X} = \operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)\frac{\partial w_f(X,0)}{\partial X} \tag{9.5.6}$$

We can integrate (9.5.6) and obtain

$$w_f\left(X,0\right)=-\frac{P_0}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(X-h\right) \tag{9.5.7}$$

where we have selected the constant of integration such that $w_f\left(X,0\right)$ vanishes at $X=h$. This choice makes the initial displacement at $X=h$ agree with the displacement at $X=h$ initially. Given (9.5.7), we obtain from (9.5.4)

$$w_s\left(X,0\right)=-\frac{P_0}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(X-h\right) \tag{9.5.8}$$

Equations (9.5.7) and (9.5.8) give the specified initial condition (9.5.3). We mentioned in Sections 3 and 4 that the initial condition and the boundary conditions must obey a compatibility condition. For one dimensional problems, we derived equation (9.4.17), repeated here ,

$$\left(\operatorname{adj}\mathbf{E}\right)\mathbf{Qf}\left(X\right)=\left(\operatorname{adj}\mathbf{E}\right)\mathbf{Qk}\left(0\right)+\left(\operatorname{adj}\mathbf{E}\right)\mathbf{q}\left(0\right)\left(X-h\right) \tag{9.5.9}$$

It is easily established that this result is obeyed for our choices (9.5.1) through (9.5.3). In fact, the initial condition (9.5.3) was constructed such that it obeys (9.5.9).

Given (9.5.1) through (9.5.3), the solution (9.4.67) reduces to

$$\begin{aligned}\mathbf{w}\left(X,t\right)=&\frac{2P_0c}{h}\frac{1}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{EQ}^{-1}\begin{bmatrix}0\\1\end{bmatrix}\sum_{n=1}^{\infty}\left(\int_0^t e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau\right)\cos\frac{\left(2n-1\right)\pi X}{2h}\\&+\frac{P_0}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{E}\right)\begin{bmatrix}0\\-1\end{bmatrix}\left(X-h\right)\end{aligned} \tag{9.5.10}$$

The integral that appears in (9.5.10) can be evaluated to yield the final form of the answer

$$\begin{aligned}\mathbf{w}\left(X,t\right)=&\frac{8P_0h}{\pi^2}\frac{1}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{EQ}^{-1}\begin{bmatrix}0\\1\end{bmatrix}\\&\times\sum_{n=1}^{\infty}\left(\frac{1}{\left(2n-1\right)^2}\left(1-e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct}\right)\right)\cos\frac{\left(2n-1\right)\pi X}{2h}\\&+\frac{P_0}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\begin{bmatrix}1\\1\end{bmatrix}\left(h-X\right)\end{aligned} \tag{9.5.11}$$

where the simple formula $\left(\operatorname{adj}\mathbf{E}\right)\begin{bmatrix}0\\-1\end{bmatrix}=-\begin{bmatrix}1\\1\end{bmatrix}$ has also been used.[2]

It is instructive to plot the solution (9.5.11) for a typical poroelastic solid. If we adopt the material properties for Ruhr Sandstone as tabulated in Tables 1 and 2 of Chapter 6, then we shall use the following numerical values:

$$\mathbf{Q}=\begin{bmatrix}\bar{\lambda}_{ff} & \bar{\lambda}_{sf}\\ \bar{\lambda}_{sf} & \bar{\lambda}_{ss}+2\mu_{ss}\end{bmatrix}=\begin{bmatrix}.165 & 5.21\\ 5.21 & 206.41+2\left(133\right)\end{bmatrix}=\begin{bmatrix}.165 & 5.21\\ 5.21 & 472.41\end{bmatrix}\text{ kbar} \tag{9.5.12}$$

$$\varphi_f^{+}=.02 \tag{9.5.13}$$

and

$$\xi=1.9\left(10\right)^{-3}\text{ kbar-sec/cm}^2 \tag{9.5.14}$$

For purposes of illustration, we shall take

$$P_0=10^2\text{ kbar} \tag{9.5.15}$$

We shall plot the solution (9.5.11) by first writing it the equivalent dimensionless form

$$\begin{aligned}\frac{\mathbf{w}\left(X,t\right)}{h}=&\frac{8}{\pi^2}\frac{1}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}/P_0\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}/P_0\right)\mathbf{E}\left(\mathbf{Q}/P_0\right)^{-1}\begin{bmatrix}0\\1\end{bmatrix}\\ &\times\sum_{n=1}^{\infty}\left(\frac{1}{\left(2n-1\right)^2}\left(1-e^{-\frac{\left(2n-1\right)^2\pi^2}{4}\left(\frac{tc}{h^2}\right)}\right)\right)\cos\left(\frac{\left(2n-1\right)\pi}{2}\frac{X}{h}\right)\\ &+\frac{1}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}/P_0\right)\mathbf{E}\right)}\begin{bmatrix}1\\1\end{bmatrix}\left(1-\frac{X}{h}\right)\end{aligned} \tag{9.5.16}$$

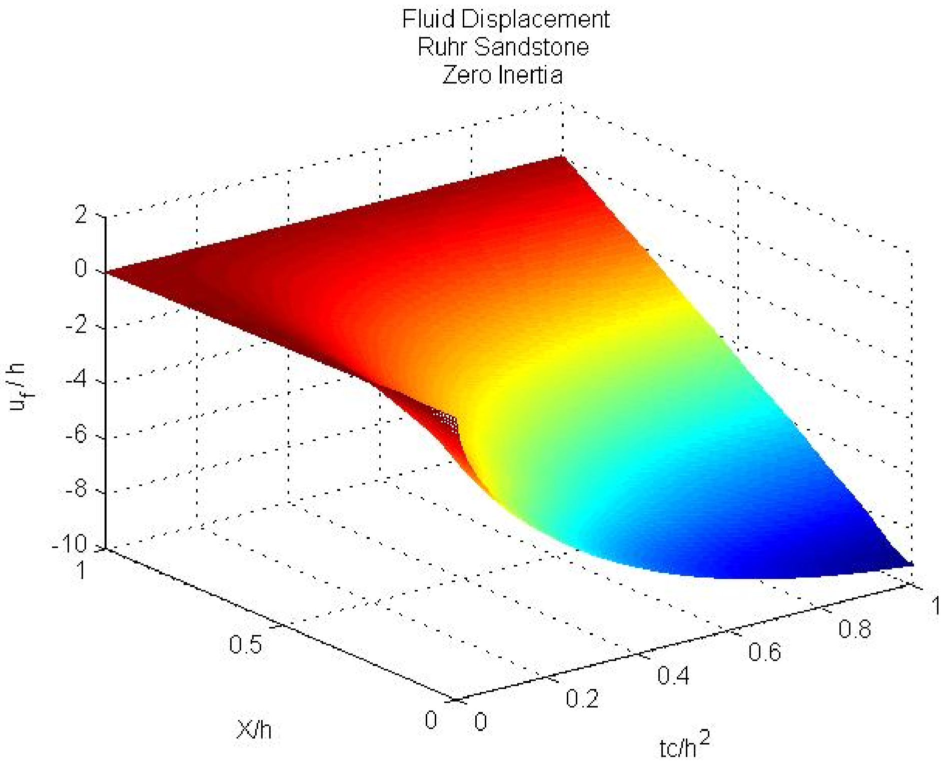
---

[2] It is easily shown from (9.5.11) that as $t\to\infty$, the displacement $\mathbf{w}\left(X,t\right)$ approaches

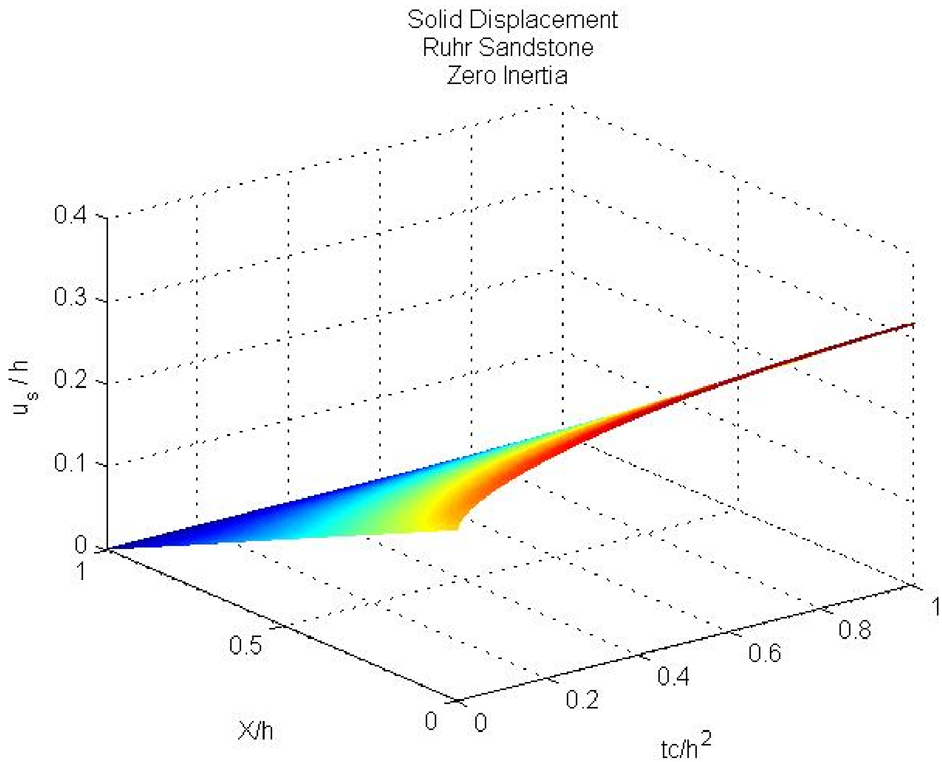
$$\mathbf{w}_{\infty}=P_0\left(h-X\right)\mathbf{Q}^{-1}\begin{bmatrix}0\\1\end{bmatrix}$$

and forming a plot of $\dfrac{\mathbf{w}(X,t)}{h}$ verses the dimensionless distance $\dfrac{X}{h}$ and the dimensionless time $\dfrac{tc}{h^2}$.

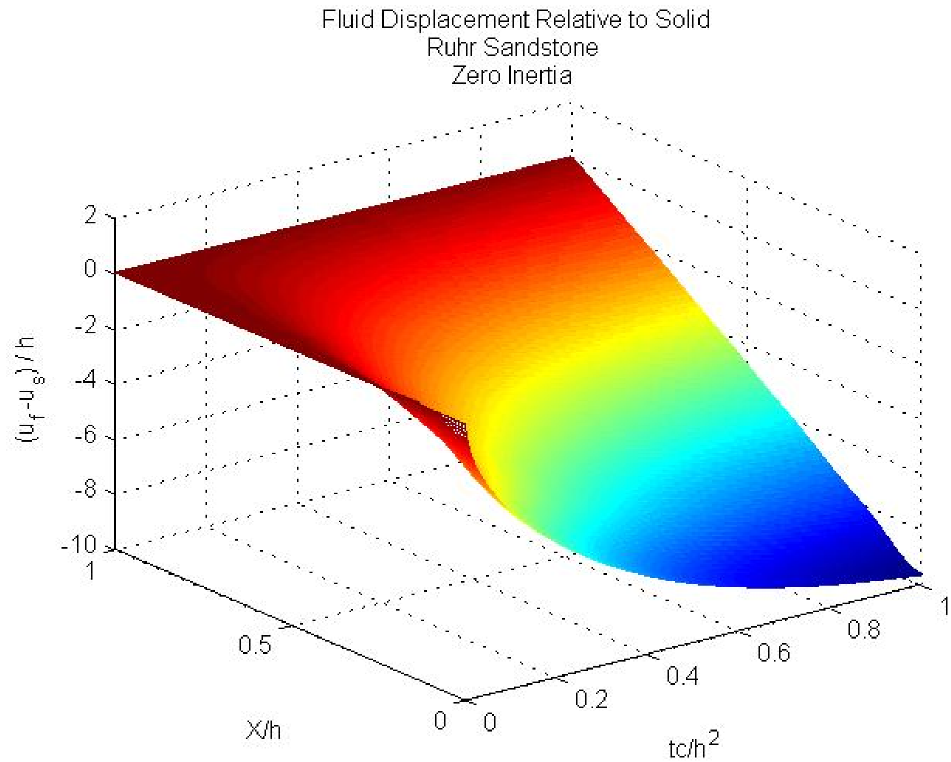
The resulting plots of the fluid and solid displacements are



and



An examination of these two figures shows that roughly speaking the displacement of the fluid is negative and that of the solid is positive. Thus, the fluid tends to move out of the solid. This observation is somewhat easier to see if we plot the displacement of the fluid relative to that of the solid. The figure in this case is



Except very close to $\frac{tc}{h^2}=0$, this figure shows that the relative displacement of the fluid is negative.

The plots we have displayed do not need numerical values of h and c. However, it is useful to note that (9.5.12), (9.5.14) and the definition (9.4.56) yield

$$c = .0055\ \text{m}^2/\text{sec} \tag{9.5.17}$$

for Ruhr Sandstone. Thus, a dimensionless time in the range $0<\frac{tc}{h^2}<1$ corresponds to actual times of $0<t<505$ hours for a poroelastic solid of length $h=100$ m. This observation reveals the slow nature of the diffusion process governed by the poroelasticity equations. The characteristic time $\frac{h^2}{c}$ is typically large resulting in slow displacement changes over time.

In Section 9.5, the formula (9.4.69) for the initial velocity was given. For the example being examined in this section, (9.4.69) yields

$$\frac{\partial \mathbf{w}(X,0)}{\partial t} = \mathbf{0} \tag{9.5.18}$$

While it is a little difficult to see, the result (9.5.18) is shown in the above figures.

9.6 Modified Biot Problem: Time Dependent External Pressure

It is interesting to modify the problem discussed in Section 9.5 by allowing the externally applied pressure to be time dependent. In order to illustrate this point, we replace (9.5.1) through (9.5.3) by

$$\mathbf{q}(t) = P_0 \cos\omega t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{9.6.1}$$

$$\mathbf{k}(t) = \mathbf{0} \tag{9.6.2}$$

and

$$\mathbf{f}(X) = \frac{P_0 \cos\omega t}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}(h - X)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{9.6.3}$$

where ω is a prescribed external frequency. The choice (9.6.1) corresponds to the assumption that the pore pressure at the point $x = 0$ is zero and the stress T_I at $x = 0$ is the pressure $P_0 \cos\omega t$. The choice (9.6.2) retains the assumption used in Section 9.5 that vanishes at $x = h$. The compatibility requirement between the initial condition and the boundary conditions that we have mentioned is satisfied by the choices (9.6.1) through (9.6.3).

Given (9.6.1) through (9.6.3), the solution (9.4.67) reduces to

$$\begin{aligned}\mathbf{w}(X,t) = {} & \frac{2P_0 c}{h}\frac{1}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\mathbf{Q}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sum_{n=1}^{\infty}\left(\int_0^t \cos\omega\tau e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-\tau)}d\tau\right)\cos\frac{(2n-1)\pi X}{2h} \\ & + \frac{P_0\cos\omega t}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{E}\right)\begin{bmatrix} 0 \\ -1 \end{bmatrix}(X - h)\end{aligned} \tag{9.6.4}$$

After the integral in (9.6.4) is evaluated, the solution (9.6.4) becomes

$$\mathbf{w}(X,t)=\frac{2P_0c}{h}\frac{1}{\mathrm{tr}((\mathrm{adj}\mathbf{Q})\mathbf{E})}(\mathrm{adj}\mathbf{Q})\mathbf{E}\mathbf{Q}^{-1}\begin{bmatrix}0\\1\end{bmatrix}\sum_{n=1}^{\infty}\left(\begin{array}{l}\dfrac{1}{\left(\omega^2+\dfrac{(2n-1)^4\pi^4}{16h^4}c^2\right)}\\ \times\left(\begin{array}{l}\dfrac{(2n-1)^2\pi^2}{4h^2}c\left(\cos\omega t-e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct}\right)\\ +\omega\sin\omega t\end{array}\right)\end{array}\right)\cos\frac{(2n-1)\pi X}{2h}$$

$$+\frac{P_0\cos\omega t}{\mathrm{tr}((\mathrm{adj}\mathbf{Q})\mathbf{E})}\begin{bmatrix}1\\1\end{bmatrix}(h-X) \qquad (9.6.5)$$

where, again, the simple formula $(\mathrm{adj}\mathbf{E})\begin{bmatrix}0\\-1\end{bmatrix}=-\begin{bmatrix}1\\1\end{bmatrix}$ has also been used. This solution, unlike the solution (9.5.11), displays a long time oscillation with the imposed frequency ω. A more compact version of the solution (9.6.5) is

$$\mathbf{w}(X,t)=\frac{8hP_0}{\pi^2}\frac{1}{\mathrm{tr}((\mathrm{adj}\mathbf{Q})\mathbf{E})}(\mathrm{adj}\mathbf{Q})\mathbf{E}\mathbf{Q}^{-1}\begin{bmatrix}0\\1\end{bmatrix}\sum_{n=1}^{\infty}\frac{\cos\varphi_n}{(2n-1)^2}\left(\cos(\omega t-\varphi_n)-\cos\varphi_n e^{-\frac{(2n-1)^2\pi^2}{4h^2}ct}\right)\cos\frac{(2n-1)\pi X}{2h}$$

$$+\frac{P_0\cos\omega t}{\mathrm{tr}((\mathrm{adj}\mathbf{Q})\mathbf{E})}\begin{bmatrix}1\\1\end{bmatrix}(h-X) \qquad (9.6.6)$$

where φ_n is the angle defined by

$$\cos\varphi_n=\frac{\dfrac{(2n-1)^2\pi^2}{4h^2}c}{\sqrt{\omega^2+\dfrac{(2n-1)^4\pi^4}{16h^4}c^2}} \qquad (9.6.7)$$

and

$$\sin\varphi_n=\frac{\omega}{\sqrt{\omega^2+\dfrac{(2n-1)^4\pi^4}{16h^4}c^2}} \qquad (9.6.8)$$

We shall repeat the plots of Section 9.5 but for the solution (9.6.5) or, equivalently, (9.6.6). We shall again adopt the numerical values for Ruhr Sandstone displayed in equations (9.5.12) through (9.5.15). We shall need to prescribe the external frequency ω. However, we shall first display the solution (9.6.5) in dimensionless form as follows:

$$\frac{\mathbf{w}(X,t)}{h}=\frac{2}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}/P_0\right)\mathbf{E}\right)}\left(\operatorname{adj}\mathbf{Q}/P_0\right)\mathbf{E}\left(\mathbf{Q}/P_0\right)^{-1}\begin{bmatrix}0\\1\end{bmatrix}\sum_{n=1}^{\infty}\left(\begin{array}{l}\dfrac{1}{\left(\left(\dfrac{\omega h^2}{c}\right)^2+\dfrac{(2n-1)^4\pi^4}{16}\right)}\\ \times\left(\begin{array}{l}\dfrac{(2n-1)^2\pi^2}{4}\left(\begin{array}{l}\cos\left(\left(\dfrac{\omega h^2}{c}\right)\left(\dfrac{tc}{h^2}\right)\right)\\ -e^{-\frac{(2n-1)^2\pi^2}{4}\left(\frac{tc}{h^2}\right)}\end{array}\right)\\ +\dfrac{\omega h^2}{c}\sin\left(\left(\dfrac{\omega h^2}{c}\right)\left(\dfrac{tc}{h^2}\right)\right)\end{array}\right)\end{array}\right)\cos\left(\frac{(2n-1)\pi}{2}\frac{X}{h}\right)$$

$$+\frac{\cos\left(\left(\dfrac{\omega h^2}{c}\right)\left(\dfrac{tc}{h^2}\right)\right)}{\operatorname{tr}\left(\left(\operatorname{adj}\mathbf{Q}/P_0\right)\mathbf{E}\right)}\begin{bmatrix}1\\1\end{bmatrix}\left(1-\frac{X}{h}\right)$$

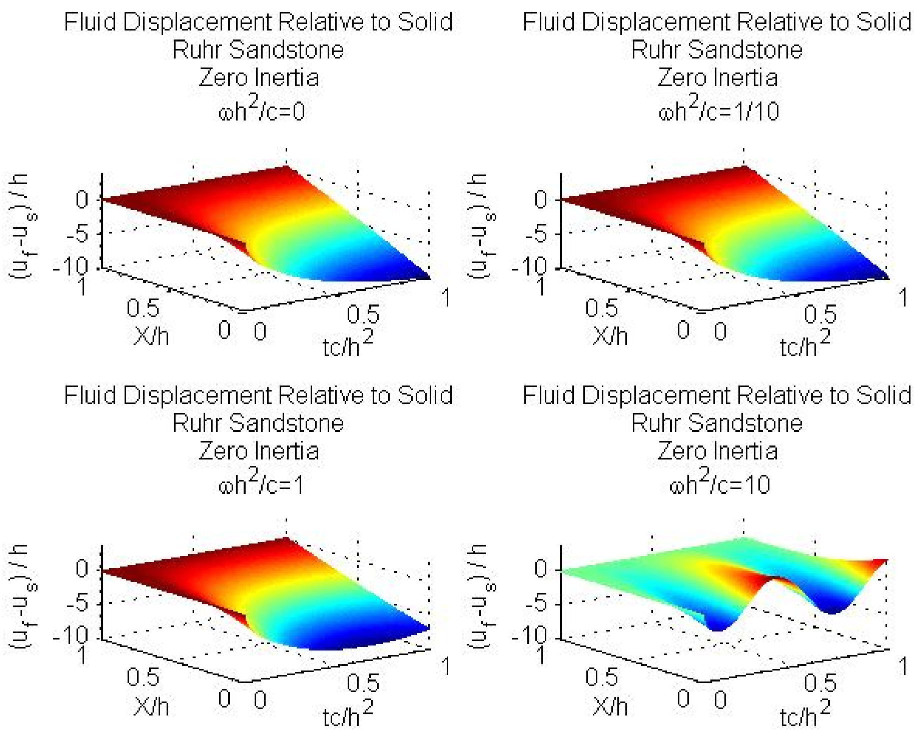
(9.6.9)

The form of this equation suggests plots of $\dfrac{\mathbf{w}(X,t)}{h}$ verses the dimensionless time $\dfrac{tc}{h^2}$ and the dimensionless distance $\dfrac{X}{h}$ for various values of the dimensionless ratio $\dfrac{\omega h^2}{c}$. This last ratio, written,

$$\frac{\omega h^2}{c}=\frac{h^2/c}{1/\omega} \tag{9.6.10}$$

is the ratio of the characteristic time of diffusion to the characteristic time of the oscillation of the external load. As we pointed out in Section 9.5, the characteristic time of diffusion is typically large. Because it is externally controlled, the characteristic time of the load oscillation can be small or large relative to the characteristic time of diffusion. We shall form plots of $\dfrac{\mathbf{w}(X,t)}{h}$ verses the dimensionless distance $\dfrac{X}{h}$ and the dimensionless time $\dfrac{tc}{h^2}$ for various values of the ratio $\dfrac{\omega h^2}{c}$.

The resulting plots of the fluid and solid displacements for three different choices of the ratio (9.6.10) are



This series of figures suggests that for small values of the dimensionless frequency $\frac{\omega h^2}{c}$, the solution is not unlike that of the zero frequency case. However, when this ratio is large, as illustrated in the last figure, the solution is fundamentally different as one would expect.

As with the example in Section 9.5, the formula (9.4.69) yields

$$\frac{\partial \mathbf{w}(X,0)}{\partial t} = \mathbf{0} \tag{9.6.11}$$

for the initial conditions (9.6.1) through (9.6.3).

9.7 The Use of Green's Functions

When one encounters inhomogeneous boundary conditions as in Section 9.3, it is often advantageous to formulate the solution in terms of Green's Functions. The representation of the solution to the boundary initial value problem stated in Section 9.3 in terms of Green's functions is given in Ref. 2. Basically, one defines the Green's function as the 6×6 matrix of functions. We shall use the notation $\mathbf{G}(\mathbf{X},t;\mathbf{X}_0,t_0)$. As the notation suggests, the Green's function depends upon pairs of points and pairs of times. It is helpful, when dealing with the notation, to represent the elements of the 6×6 matrix by six 6×1 column matrices that we can write

$$\mathbf{G}(\mathbf{X},t;\mathbf{X}_0,t_0)=\begin{bmatrix}\mathbf{g}_{f(1)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{f(2)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{f(3)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{f(4)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{f(5)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{f(6)}(\mathbf{X},t;\mathbf{X}_0,t_0)\\ \mathbf{g}_{s(1)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{s(2)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{s(3)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{s(4)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{s(5)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{s(6)}(\mathbf{X},t;\mathbf{X}_0,t_0)\end{bmatrix} \tag{9.7.1}$$

where, for example, $\mathbf{g}_{f(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)$, is a three dimensional vector that we have represented by a 3×1 column vector.

The Green's function is the 6×6 matrix of functions introduced in (9.7.1) defined such that, for every pair $(\mathbf{X}_0,t_0)\in\mathscr{V}\times(-\infty,\infty)$, it obeys the boundary initial value problem

$$\mathscr{D}\mathbf{G}(\mathbf{X},t;\mathbf{X}_0,t_0)=\delta(t-t_0)\delta(\mathbf{X}-\mathbf{X}_0)\mathbf{I}\qquad\text{for}\qquad(\mathbf{X},t)\in\mathscr{V}\times(-\infty,\infty) \tag{9.7.2}$$

$$\mathbf{T}_I(\mathbf{G})\mathbf{n}=\mathbf{0}\qquad\text{for}\qquad(\mathbf{X},t)\in\partial\mathscr{V}_1\times(-\infty,\infty) \tag{9.7.3}$$

$$P_f(\mathbf{G})=0\qquad\text{for}\qquad(\mathbf{X},t)\in\partial\mathscr{V}_1\times(-\infty,\infty) \tag{9.7.4}$$

$$\mathbf{G}=0\quad\text{for}\quad(\mathbf{X},t)\in\partial\mathscr{V}_2\times(-\infty,\infty) \tag{9.7.5}$$

and

$$\mathbf{G}(\mathbf{X},t,\mathbf{X}_0,t_0)=\mathbf{0}\qquad\text{for}\qquad\mathbf{X}\in\mathscr{V}\text{ and }t<t_0 \tag{9.7.6}$$

where $\mathscr{D}$ is the formal linear operator defined by

$$\begin{aligned}\mathscr{D}\mathbf{w}&=\begin{bmatrix}\xi\left(\dfrac{\partial\mathbf{w}_f}{\partial t}-\dfrac{\partial\mathbf{w}_s}{\partial t}\right)-\bar{\lambda}_{sf}\,\text{GRAD}(\text{Div}\,\mathbf{w}_s)-\bar{\lambda}_{ff}\,\text{GRAD}(\text{Div}\,\mathbf{w}_f)\\ -\xi\left(\dfrac{\partial\mathbf{w}_f}{\partial t}-\dfrac{\partial\mathbf{w}_s}{\partial t}\right)-(\bar{\lambda}_{ss}+\mu_{ss})\text{GRAD}(\text{Div}\,\mathbf{w}_s)-\mu_{ss}\,\text{Div}(\text{GRAD}\,\mathbf{w}_s)-\bar{\lambda}_{sf}\,\text{GRAD}(\text{Div}\,\mathbf{w}_f)\end{bmatrix}\\ &=\begin{bmatrix}\xi\left(\dfrac{\partial\mathbf{w}_f}{\partial t}-\dfrac{\partial\mathbf{w}_s}{\partial t}\right)\\ -\xi\left(\dfrac{\partial\mathbf{w}_f}{\partial t}-\dfrac{\partial\mathbf{w}_s}{\partial t}\right)\end{bmatrix}+L\,\mathbf{w}\end{aligned} \tag{9.7.7}$$

The notation in (9.7.7) is the same as used in Section 9.2. In particular, the quantity $\mathbf{w}$ is defined by (9.2.2) and can be thought of as a column matrix of dimension 6×1. Given this interpretation, the symbol $\mathscr{L}\mathbf{G}$ is calculated by the rule (9.7.7) applied to each column of the 6×6 matrix $\mathbf{G}$. The symbol $\mathbf{I}$ in (9.7.2) represents the 6×6 identity matrix. The symbol $\delta(t-t_0)$ represents the one dimensional delta function, and the symbol $\delta(\mathbf{X}-\mathbf{X}_0)$ denotes the three dimensional delta function.

While it is questionable whether or not our notation should be exploited excessively, there is a benefit of going the next step and writing

$$\begin{bmatrix} \left(\dfrac{\partial \mathbf{w}_f}{\partial t}-\dfrac{\partial \mathbf{w}_s}{\partial t}\right) \\ -\left(\dfrac{\partial \mathbf{w}_f}{\partial t}-\dfrac{\partial \mathbf{w}_s}{\partial t}\right) \end{bmatrix} = \mathbf{E}\frac{\partial \mathbf{w}}{\partial t} \tag{9.7.8}$$

where $\mathbf{E}$ is the symmetric 6×6 matrix

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \tag{9.7.9}$$

This formalism allows the operator $\mathscr{L}$ to be defined by

$$\mathscr{D}\mathbf{w} = \xi\mathbf{E}\frac{\partial \mathbf{w}}{\partial t} + L\,\mathbf{w} \tag{9.7.10}$$

rather than by (9.7.7).

Given the definition of the Green's function $\mathbf{G}(\mathbf{X},t;\mathbf{X}_0,t_0)$, the theory of Green's functions can be used to write the solution of the boundary initial value problem defined by (9.3.1) through (9.3.4) in terms of its initial and boundary data. The details of this formalism can be found in Ref. 2, and the result is

$$\begin{aligned}\mathbf{w}(\mathbf{X},t)=\xi\int_{\mathscr{V}}\mathbf{G}(\mathbf{X},t;\mathbf{X}_0,0)\mathbf{Ef}(\mathbf{X}_0)d\upsilon_0-\int_0^t\int_{\partial\mathscr{V}_1}\left(\sum_{\alpha=1}^{6}\mathbf{e}_\alpha\left(\mathbf{g}_{s(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)\cdot\mathbf{s}(\mathbf{X}_0,t_0)\right)\right)d\upsilon_0dt_0\\ -\varphi_f^+\int_0^t\int_{\partial\mathscr{V}_1}\left(\sum_{\alpha=1}^{6}\mathbf{e}_\alpha\left(\left(\mathbf{g}_{f(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)-\mathbf{g}_{s(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)\right)\cdot\mathbf{n}r(\mathbf{X}_0,t_0)\right)\right)d\upsilon_0dt_0\\ -\int_0^t\int_{\partial\mathscr{V}_2}\left(\sum_{\alpha=1}^{6}\mathbf{e}_\alpha\left(\mathbf{T}_I\left(\mathbf{g}_{(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)\right)\mathbf{k}_s(\mathbf{X}_0,t_0)\right)\right)\cdot d\mathbf{s}_0dt_0\\ +\varphi_f^+\int_0^t\int_{\partial\mathscr{V}_2}\left(\sum_{\alpha=1}^{6}\mathbf{e}_\alpha\left(P_f\left(\mathbf{g}_{(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)\right)\right)\left(\mathbf{k}_f(\mathbf{X}_0,t_0)-\mathbf{k}_s(\mathbf{X}_0,t_0)\right)\right)\cdot d\mathbf{s}_0dt_0\end{aligned} \tag{9.7.11}$$

where $\mathbf{g}_{(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)$ is the column matrix defined by

$$\mathbf{g}_{(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)=\begin{bmatrix}\mathbf{g}_{f(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)\\ \mathbf{g}_{s(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)\end{bmatrix}, \tag{9.7.12}$$

$\mathbf{k}_f(\mathbf{X}_0,t_0)$ and $\mathbf{k}_s(\mathbf{X}_0,t_0)$ are obtained from the boundary condition (9.3.3) expressed as the column matrix

$$\mathbf{k}(\mathbf{X},t)=\begin{bmatrix}\mathbf{k}_f(\mathbf{X},t)\\ \mathbf{k}_s(\mathbf{X},t)\end{bmatrix} \tag{9.7.13}$$

and $\mathbf{e}_\alpha$, for $\alpha=1,2,...,6$, is the 6×1 column matrix with entries equal to zero except in the α position where the value is 1

The techniques that can be used to actually calculate the Green's function are complicated. One method is, however, straight forward. The method is to represent the solution of the initial boundary value problem (9.7.2) through (9.7.6) by an expansion in the eigenfunctions of the space part of the operator $\mathscr{L}$, i.e., in terms of the eigenfunctions of the operator L. These eigenfunctions, as defined in Section 9.2 allow us to seek solutions for the Green's function of the form

$$\mathbf{G}(\mathbf{X},t;\mathbf{X}_0,t_0)=\sum_{n=1}^{\infty}T_n(t;t_0)\mathbf{u}_n(\mathbf{X})\mathbf{u}_n(\mathbf{X}_0)^T \tag{9.7.14}$$

The solution in terms of eigenfunctions of L will also be complicated. It is helpful at this point in the discussion to illustrate this kind of solution for the class of one dimensional problems discussed in Section 9.4. In this case, the initial boundary value problem we are discussing is defined by (9.4.3) and (9.4.8) through (9.4.11). The Green's function is the 2×2 matrix of functions

$$\mathbf{G}\left(X,t;X_0,t_0\right)=\begin{bmatrix} g_{f(1)}\left(X,t;X_0,t_0\right) & g_{f(2)}\left(X,t;X_0,t_0\right) \\ g_{s(1)}\left(X,t;X_0,t_0\right) & g_{s(2)}\left(X,t;X_0,t_0\right) \end{bmatrix} \tag{9.7.15}$$

that obeys the one dimensional forms of (9.7.2) through (9.7.6). Therefore, the Green's function is the solution of

$$\xi\mathbf{E}\frac{\partial\mathbf{G}}{\partial t}-\mathbf{Q}\frac{\partial^2\mathbf{G}}{\partial X^2}=\delta\left(t-t_0\right)\delta\left(X-X_0\right) \qquad \text{for} \qquad \left(X,t\right)\in\left(0,h\right)\times\left(-\infty,\infty\right) \tag{9.7.16}$$

$$\frac{\partial\mathbf{G}\left(0,t;X_0,t_0\right)}{\partial X}=\mathbf{0} \tag{9.7.17}$$

$$\mathbf{G}\left(h,t;X_0,t_0\right)=0 \tag{9.7.18}$$

and

$$\mathbf{G}\left(x,t;x_0,t_0\right)=\mathbf{0} \qquad \text{for} \quad t<t_0 \tag{9.7.19}$$

As in Section (9.4), the eigenfunctions of the space part of the one dimensional operator are defined by (9.4.25) and (9.4.26) are given by (9.4.27) where the eigenvalues are given by (9.4.31) and (9.4.32). In the notation adopted in Section 9.4, the representation of the solution (9.7.14) becomes

$$\mathbf{G}\left(X,t;X_0,t_0\right)=\sum_{n=1}^{\infty}\begin{pmatrix} T_{n(11)}\left(t;t_0\right)\mathbf{b}_{(1)}\mathbf{b}_{(1)}^T+T_{n(12)}\left(t;t_0\right)\mathbf{b}_{(1)}\mathbf{b}_{(2)}^T \\ +T_{n(21)}\left(t;t_0\right)\mathbf{b}_{(2)}\mathbf{b}_{(1)}^T+T_{n(22)}\left(t;t_0\right)\mathbf{b}_{(2)}\mathbf{b}_{(2)}^T \end{pmatrix}\cos\frac{\left(2n-1\right)\pi X}{2h}\cos\frac{\left(2n-1\right)\pi X_0}{2h} \tag{9.7.20}$$

Just as when we expressed (9.4.35) in the form (9.4.36), we can simplify the notation somewhat if we write (9.7.20) as

$$\mathbf{G}\left(X,t;X_0,t_0\right)=\sum_{n=1}^{\infty}\mathbf{K}_n\left(t;t_0\right)\cos\frac{\left(2n-1\right)\pi X}{2h}\cos\frac{\left(2n-1\right)\pi X_0}{2h} \tag{9.7.21}$$

where, for each n, $\mathbf{K}_n\left(t;t_0\right)$ is a 2×2 matrix whose representation with respect to the basis of eigenvectors of $\mathbf{Q}$ is

$$\begin{aligned}\mathbf{K}_n\left(t;t_0\right)=T_{n(11)}\left(t;t_0\right)\mathbf{b}_{(1)}\mathbf{b}_{(1)}^T+T_{n(12)}\left(t;t_0\right)\mathbf{b}_{(1)}\mathbf{b}_{(2)}^T \\ +T_{n(21)}\left(t;t_0\right)\mathbf{b}_{(2)}\mathbf{b}_{(1)}^T+T_{n(22)}\left(t;t_0\right)\mathbf{b}_{(2)}\mathbf{b}_{(2)}^T\end{aligned} \tag{9.7.22}$$

If (9.7.21) is substituted into (9.7.16), it is easily shown that $\mathbf{K}_n\left(t;t_0\right)$ must obey

$$\xi \mathbf{E} \frac{d\mathbf{K}_n(t;t_0)}{dt} + \frac{(2n-1)^2 \pi^2}{4h^2} \mathbf{Q}\mathbf{K}_n(t;t_0) = \frac{2}{h}\delta(t-t_0) \tag{9.7.23}$$

It follows from (9.7.19) and (9.7.21) that the matrix $\mathbf{K}_n(t;t_0)$ must obey

$$\mathbf{K}_n(t;t_0) = \mathbf{0} \qquad \text{for} \quad t < t_0 \tag{9.7.24}$$

The same technique used to solve (9.4.46) can be applied to finding the solution of (9.7.23). The result is

$$\mathbf{K}_n(t;t_0) = H(t-t_0)\frac{2c}{h}\frac{(\operatorname{adj}\mathbf{Q})\mathbf{E}\mathbf{Q}^{-1}}{\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})} e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-t_0)} + \delta(t-t_0)\frac{8h}{\pi^2}\frac{\operatorname{adj}\mathbf{E}}{\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}\frac{1}{(2n-1)^2} \tag{9.7.25}$$

where $H(t-t_0)$ is the Heaviside step function and, as above, $\delta(t-t_0)$ is the delta function. Given (9.7.25), equation (9.7.21) can be written[3]

$$\begin{aligned}\mathbf{G}(X,t;X_0,t_0) &= H(t-t_0)\frac{2c}{h}\frac{(\operatorname{adj}\mathbf{Q})\mathbf{E}\mathbf{Q}^{-1}}{\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}\sum_{n=1}^{\infty} e^{-\frac{(2n-1)^2\pi^2}{4h^2}c(t-t_0)} \cos\frac{(2n-1)\pi X}{2h}\cos\frac{(2n-1)\pi X_0}{2h} \\ &+\delta(t-t_0)\frac{8h}{\pi^2}\frac{\operatorname{adj}\mathbf{E}}{\operatorname{tr}((\operatorname{adj}\mathbf{Q})\mathbf{E})}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi X}{2h}\cos\frac{(2n-1)\pi X_0}{2h}\end{aligned} \tag{9.7.26}$$

For the one dimensional problem we are discussing, the solution (9.7.11) can be written

$$\begin{aligned}\mathbf{w}(X,t) &= \xi\int_0^h \mathbf{G}(X,t;X_0,0)\mathbf{E}\mathbf{f}(X_0)\,dX_0 - \int_0^t \mathbf{G}(X,t;0,t_0)\mathbf{q}(t_0)\,dt_0 \\ &-\int_0^t \frac{\partial \mathbf{G}(X,t;h,t_0)}{\partial X_0}\mathbf{Q}\mathbf{k}(t_0)\,dt_0\end{aligned} \tag{9.7.27}$$

where $\mathbf{q}(t)$ is defined by (9.4.12). If (9.7.26) were to be substituted into (9.7.27), a long and somewhat tedious calculation will again yield the solution (9.4.67).

References

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[3] If convenient, the identity

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi X}{2h}\cos\frac{(2n-1)\pi X_0}{2h} = \begin{cases} h - X_0 & \text{for} \quad X < X_0 \\ h - X & \text{for} \quad X > X_0 \end{cases}$$

can be used to simplify further (9.7.26)

1. BIOT, M. A., General Theory of Three Dimensional Consolidation, *J. Appl. Phys*, **12**, 155-164, 1941.
2. BOWEN, R. M., Green's Functions for Consolidation Problems, *Lett. Appl. Engng. Sci*., **19**, 455-466, 1981.

10

Boundary Initial Value Problems: Inertia Included

In this chapter we shall discuss the solution of a certain class of porous elasticity problems but, unlike Chapter 9, retain the inertia terms. The formalism for this chapter is drawn, in part, from the formulation given by Bowen and Lockett [Ref. 1].

10.1 Governing Partial Differential Equations

We shall adopt the isothermal constitutive equations listed in Section 9.1 except that we shall restore the neglected inertia terms. As in Section 9.1, we shall make assumptions sufficient to treat the temperature as a constant. Thus, the constitutive equations and field equations we shall utilize are as follows:

$$\frac{\varphi_f - \varphi_f^+}{\varphi_f^+} = \frac{\lambda_{sb}}{\varphi_f^+ K_f} \operatorname{tr} \mathbf{E}_s + \left(1 - \frac{\lambda_{ff}}{\varphi_f^+ K_f}\right)((\rho_f - \rho_f^+)/\rho_f^+) \tag{10.1.1}$$

$$\begin{aligned}\Psi_{\mathrm{I}} = \frac{1}{2}\bar{\lambda}_{ss}(\operatorname{tr}\mathbf{E}_s)^2 - \bar{\lambda}_{sf}(\operatorname{tr}\mathbf{E}_s)(\rho_f - \rho_f^+)/\rho_f^+ \\ + \frac{1}{2}\bar{\lambda}_{ff}\left((\rho_f - \rho_f^+)/\rho_f^+\right)^2 + \mu_{ss}\operatorname{tr}(\mathbf{E}_s\mathbf{E}_s),\end{aligned} \tag{10.1.2}$$

$$\rho\eta = -\alpha + \bar{\tau}_s(\operatorname{tr}\mathbf{E}_s) - \bar{\tau}_f(\rho_f - \rho_f^+)/\rho_f^+ \tag{10.1.3}$$

$$\begin{aligned}\rho_{s_R}\mathbf{K}_s = -(\mathbf{T}_{\mathrm{I}} + \varphi_f^+ P_f \mathbf{I}) = -\bar{\lambda}_{ss}(\operatorname{tr}\mathbf{E}_s)\mathbf{I} - 2\mu_{ss}\mathbf{E}_s \\ + \bar{\lambda}_{sf}((\rho_f - \rho_f^+)/\rho_f^+)\mathbf{I}\end{aligned} \tag{10.1.4}$$

and

$$\varphi_f^+ P_f = -\bar{\lambda}_{sf}(\operatorname{tr}\mathbf{E}_s) + \bar{\lambda}_{ff}(\rho_f - \rho_f^+)/\rho_f^+ \tag{10.1.5}$$

The field equations which follow from these constitutive equations are the isothermal versions of equations (4.2.14) and (4.2.15). These equations, repeated here, are

$$\rho_{s_R} \frac{\partial^2 \mathbf{w}_s}{\partial t^2} = (\bar{\lambda}_{ss} + \mu_{ss})\,\text{GRAD}(\text{Div}\,\mathbf{w}_s) + \mu_{ss}\,\text{Div}(\text{GRAD}\,\mathbf{w}_s) + \bar{\lambda}_{sf}\,\text{GRAD}(\text{Div}\,\mathbf{w}_f) - \xi\left(\frac{\partial \mathbf{w}_s}{\partial t} - \frac{\partial \mathbf{w}_f}{\partial t}\right) \tag{10.1.6}$$

and

$$\rho_f^+ \frac{\partial^2 \mathbf{w}_f}{\partial t^2} = \bar{\lambda}_{sf}\,\text{GRAD}(\text{Div}\,\mathbf{w}_s) + \bar{\lambda}_{ff}\,\text{GRAD}(\text{Div}\,\mathbf{w}_f) - \xi\left(\frac{\partial \mathbf{w}_f}{\partial t} - \frac{\partial \mathbf{w}_s}{\partial t}\right) \tag{10.1.7}$$

Given the form of (10.1.6) and (10.1.7), the results in Section 9.2 remain valid when the inertia terms are restored.

10.2 Boundary Initial Value Problems

As a generalization of the boundary initial value problem considered in Chapter 9, we shall consider the following:

Boundary Conditions:

$$\mathbf{T}_I(\mathbf{w})\mathbf{n} = -\mathbf{s}(\mathbf{X},t) \quad \text{for} \quad (\mathbf{X},t) \in \partial\mathscr{V}_1 \times (0,\infty) \tag{10.2.1}$$

$$P_f(\mathbf{w}) = r(\mathbf{X},t) \quad \text{for} \quad (\mathbf{X},t) \in \partial\mathscr{V}_1 \times (0,\infty) \tag{10.2.2}$$

$$\mathbf{w} = \mathbf{k}(\mathbf{X},t) \quad \text{for} \quad (\mathbf{X},t) \in \partial\mathscr{V}_2 \times (0,\infty) \tag{10.2.3}$$

Initial Conditions:

$$\mathbf{w}(\mathbf{X},0) = \mathbf{f}(\mathbf{X}) \quad \text{for} \quad \mathbf{X} \in \mathscr{V} \tag{10.2.4}$$

$$\frac{\partial \mathbf{w}(\mathbf{X},0)}{\partial t} = \mathbf{g}(\mathbf{X}) \quad \text{for} \quad \mathbf{X} \in \mathscr{V} \tag{10.2.5}$$

where, as in Section 9.3, the boundary $\partial\mathscr{V}$ of the volume $\mathscr{V}$ consists of two parts $\partial\mathscr{V}_1$ and $\partial\mathscr{V}_2$. On the part $\partial\mathscr{V}_1$ the stress vector $\mathbf{T}_I\mathbf{n}$ and the pore pressure P_f are prescribed by the functions $\mathbf{s}(\mathbf{X},t)$ and $r(\mathbf{X},t)$, respectively. On the remainder, $\partial\mathscr{V}_2$, the displacement is prescribed by the function $\mathbf{k}(\mathbf{X},t)$. The initial displacement given by the function $\mathbf{f}(\mathbf{X})$, and the initial velocity is given by the function $\mathbf{g}(\mathbf{X})$. The five functions $\mathbf{s}(\mathbf{X},t)$, $r(\mathbf{X},t)$, $\mathbf{k}(\mathbf{X},t)$, $\mathbf{f}(\mathbf{X})$ and $\mathbf{g}(\mathbf{X})$ represent the data for this problem.

It is interesting to note that when inertia is restored, we do not have the data compatibility that originated with (9.3.5). As illustrated in the examples worked in Sections 9.4 through 9.6, this compatibility condition links the data at $t = 0$.

10.3 A One Dimensional Example

If we make the same simplification as adopted in Section 9.4, the resulting one dimensional problem is to find the solution of

$$\mathbf{M}\frac{\partial^2 \mathbf{w}(X,t)}{\partial t^2} + \xi \mathbf{E}\frac{\partial \mathbf{w}(X,t)}{\partial t} = \mathbf{Q}\frac{\partial^2 \mathbf{w}(X,t)}{\partial X^2} \tag{10.3.1}$$

where,

$$\mathbf{w} = \mathbf{w}(X,t) = \begin{bmatrix} w_f(X,t) \\ w_s(X,t) \end{bmatrix} \tag{10.3.2}$$

$$\mathbf{M} = \begin{bmatrix} \rho_f^+ & 0 \\ 0 & \rho_{s_R} \end{bmatrix} \tag{10.3.3}$$

$$\mathbf{Q} = \begin{bmatrix} \bar{\lambda}_{ff} & \bar{\lambda}_{sf} \\ \bar{\lambda}_{sf} & \bar{\lambda}_{ss} + 2\mu_{ss} \end{bmatrix} \tag{10.3.4}$$

and

$$\mathbf{E} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{10.3.5}$$

As mentioned in Chapter 9, the above definitions have appeared in earlier chapters. Equation (10.3.1) is the isothermal version of (8.1.7).

For this one dimensional problem, the constitutive equation (10.1.4) and (10.1.5) can again be written in the matrix form

$$\begin{bmatrix} -\varphi_f^+ P_f \\ T_I + \varphi_f^+ P_f \end{bmatrix} = \mathbf{Q}\frac{\partial \mathbf{w}}{\partial X} \tag{10.3.6}$$

The one dimensional form of the boundary initial value problem introduced in Section 9.2 is easily seen to be

$$\mathbf{q}(t) \equiv \begin{bmatrix} -\varphi_f^+ r(t) \\ -s(t) + \varphi_f^+ r(t) \end{bmatrix} = \mathbf{Q}\frac{\partial \mathbf{w}(0,t)}{\partial X} \qquad \text{for} \qquad t \in (0,\infty) \tag{10.3.7}$$

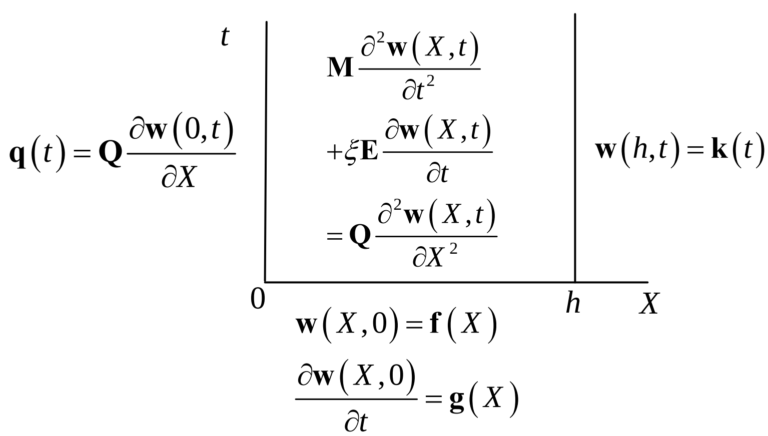
$$\mathbf{w}(h,t) = \mathbf{k}(t) \qquad \text{for} \qquad t \in (0,\infty) \tag{10.3.8}$$

$$\mathbf{w}(X,0) = \mathbf{f}(X) \qquad \text{for} \qquad X \in (0,h) \tag{10.3.9}$$

and

$$\frac{\partial \mathbf{w}(X,0)}{\partial t} = \mathbf{g}(X) \qquad \text{for} \qquad X \in (0,h) \tag{10.3.10}$$

We can display the partial differential equation, the boundary conditions and the initial condition on the following schematic



As in Section 9.4, in order to exploit eigenfunction techniques to solve this boundary initial value problem, we need to change the dependent variable in a manner sufficient to produce homogeneous boundary conditions. The transformation that achieves this objective is again achieved by defining a function $\mathbf{v}(X,t)$ by

$$\mathbf{v}(X,t) = \mathbf{w}(X,t) - \mathbf{Q}^{-1}\mathbf{q}(t)(X-h) - \mathbf{k}(t) \tag{10.3.11}$$

The partial differential equation that $\mathbf{v}(X,t)$ obeys follows from (10.3.1) and (10.3.11). This result is

$$\mathbf{M}\frac{\partial^2 \mathbf{v}}{\partial t^2} + \xi \mathbf{E}\frac{\partial \mathbf{v}}{\partial t} = \mathbf{Q}\frac{\partial^2 \mathbf{v}}{\partial X^2} + \mathbf{p}(t) \tag{10.3.12}$$

where

$$\mathbf{p}(X,t) = -\mathbf{M}\left(\mathbf{Q}^{-1}\ddot{\mathbf{q}}(t)(X-h)+\ddot{\mathbf{k}}(t)\right)-\xi\mathbf{E}\left(\mathbf{Q}^{-1}\dot{\mathbf{q}}(t)(X-h)+\dot{\mathbf{k}}(t)\right) \tag{10.3.13}$$

The boundary conditions and the initial value that the function $\mathbf{v}(X,t)$ must obey follow from (10.3.11) and the corresponding conditions on $\mathbf{w}(X,t)$. These results are

$$\frac{\partial \mathbf{v}(0,t)}{\partial X} = \mathbf{0} \tag{10.3.14}$$

$$\mathbf{v}(h,t) = \mathbf{0} \tag{10.3.15}$$

$$\mathbf{v}(X,0) = \mathbf{f}(X) - \mathbf{Q}^{-1}\mathbf{q}(0)(X-h) - \mathbf{k}(0) \tag{10.3.16}$$

and

$$\frac{\partial \mathbf{v}(X,0)}{\partial t} = \mathbf{g}(X) - \mathbf{Q}^{-1}\dot{\mathbf{q}}(0)(X-h) - \dot{\mathbf{k}}(0) \tag{10.3.17}$$

As in Section 9.4, we again seek a solution of our boundary initial value problem for $\mathbf{v}(X,t)$ in the form

$$\mathbf{v}(X,t) = \sum_{n=1}^{\infty} \mathbf{T}_n(t)\cos\frac{(2n-1)\pi X}{2h} \tag{10.3.18}$$

where $\mathbf{T}_n(t)$ is a 2×1 column matrix that determines the time dependence of $\mathbf{v}(X,t)$. As constructed, the proposed solution (10.3.18) obeys the boundary conditions (10.3.14) and (10.3.15) . Our next step is to determine the time dependence of the solution. If (10.3.18) is substituted into (10.3.12) the result is

$$\sum_{n=1}^{\infty}\left(\mathbf{M}\frac{d^2\mathbf{T}_n(t)}{dt^2} + \xi\mathbf{E}\frac{d\mathbf{T}_n(t)}{dt} + \frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\mathbf{T}_n(t)\right)\cos\frac{(2n-1)\pi X}{2h} = \mathbf{p}(X,t) \tag{10.3.19}$$

The orthogonality of the functions $\left\{\cos\frac{(2n-1)\pi X}{2h}\right\}$ allow us to derive from (10.3.19) the result

$$\mathbf{M}\frac{d^2\mathbf{T}_n(t)}{dt^2} + \xi\mathbf{E}\frac{d\mathbf{T}_n(t)}{dt} + \frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\mathbf{T}_n(t) = \mathbf{p}_n(t) \tag{10.3.20}$$

where, from (10.3.13),

$$
\begin{aligned}
\mathbf{p}_n(t) &= \frac{2}{h}\int_0^h \mathbf{p}(X,t)\cos\frac{(2n-1)\pi X}{2h}\,dX \\
&= \frac{2}{h}\int_0^h \Big(-\mathbf{M}\big(\mathbf{Q}^{-1}\ddot{\mathbf{q}}(t)(X-h)+\ddot{\mathbf{k}}(t)\big)-\xi\mathbf{E}\big(\mathbf{Q}^{-1}\dot{\mathbf{q}}(t)(X-h)+\dot{\mathbf{k}}(t)\big)\Big)\cos\frac{(2n-1)\pi X}{2h}\,dX \\
&= \big(\mathbf{M}\mathbf{Q}^{-1}\ddot{\mathbf{q}}(t)+\xi\mathbf{E}\mathbf{Q}^{-1}\dot{\mathbf{q}}(t)\big)\frac{2}{h}\int_0^h (h-X)\cos\frac{(2n-1)\pi X}{2h}\,dX \\
&\quad -\big(\mathbf{M}\ddot{\mathbf{k}}(t)+\xi\mathbf{E}\dot{\mathbf{k}}(t)\big)\frac{2}{h}\int_0^h \cos\frac{(2n-1)\pi X}{2h}\,dX \\
&= \big(\mathbf{M}\mathbf{Q}^{-1}\ddot{\mathbf{q}}(t)+\xi\mathbf{E}\mathbf{Q}^{-1}\dot{\mathbf{q}}(t)\big)\frac{8h}{\pi^2}\frac{1}{(2n-1)^2}+\frac{4}{(2n-1)}(-1)^n\big(\mathbf{M}\ddot{\mathbf{k}}(t)+\xi\mathbf{E}\dot{\mathbf{k}}(t)\big)
\end{aligned}
\tag{10.3.21}
$$

after the elementary integrals are evaluated.

The solution of (10.3.21) requires an initial condition. These conditions follow from (10.3.11), (10.3.16) and (10.3.17). These three equations yield

$$
\sum_{n=1}^{\infty}\mathbf{T}_n(0)\cos\frac{(2n-1)\pi X}{2h}=\mathbf{f}(X)-\mathbf{Q}^{-1}\mathbf{q}(0)(X-h)-\mathbf{k}(0) \tag{10.3.22}
$$

and

$$
\sum_{n=1}^{\infty}\frac{d\mathbf{T}_n(0)}{dt}\cos\frac{(2n-1)\pi X}{2h}=\mathbf{g}(X)-\mathbf{Q}^{-1}\dot{\mathbf{q}}(0)(X-h)-\dot{\mathbf{k}}(0) \tag{10.3.23}
$$

The orthogonality of the functions $\left\{\cos\frac{(2n-1)\pi X}{2h}\right\}$ can be used again to derive from (10.3.22) and (10.3.23) the results

$$
\begin{aligned}
\mathbf{T}_n(0) &= \frac{2}{h}\int_0^h \mathbf{f}(X)\cos\frac{(2n-1)\pi X}{2h}\,dX \\
&\quad +\frac{2}{h}\mathbf{Q}^{-1}\mathbf{q}(0)\int_0^h (h-X)\cos\frac{(2n-1)\pi X}{2h}\,dX-\frac{2}{h}\mathbf{k}(0)\int_0^h\cos\frac{(2n-1)\pi X}{2h}\,dX
\end{aligned}
\tag{10.3.24}
$$

and

$$
\begin{aligned}
\frac{d\mathbf{T}_n(0)}{dt} &= \frac{2}{h}\int_0^h \mathbf{g}(X)\cos\frac{(2n-1)\pi X}{2h}\,dX \\
&\quad +\frac{2}{h}\mathbf{Q}^{-1}\dot{\mathbf{q}}(0)\int_0^h (h-X)\cos\frac{(2n-1)\pi X}{2h}\,dX-\frac{2}{h}\dot{\mathbf{k}}(0)\int_0^h\cos\frac{(2n-1)\pi X}{2h}\,dX
\end{aligned}
\tag{10.3.25}
$$

The integrals in the second and third terms of (10.3.24) can be evaluated to yield the result

$$\mathbf{T}_n(0) = \mathbf{f}_n + \frac{8h}{(2n-1)^2\pi^2}\mathbf{Q}^{-1}\mathbf{q}(0) + \frac{4}{(2n-1)\pi}(-1)^n\mathbf{k}(0) \tag{10.3.26}$$

where, we have introduced the symbol $\mathbf{f}_n$ defined by

$$\mathbf{f}_n = \frac{2}{h}\int_0^h \mathbf{f}(X)\cos\frac{(2n-1)\pi X}{2h}dX \tag{10.3.27}$$

Likewise, the integrals in the second and third terms of (10.3.25) can be evaluated to yield

$$\frac{d\mathbf{T}_n(0)}{dt} = \mathbf{g}_n + \frac{8h}{(2n-1)^2\pi^2}\mathbf{Q}^{-1}\dot{\mathbf{q}}(0) + \frac{4}{(2n-1)\pi}(-1)^n\dot{\mathbf{k}}(0) \tag{10.3.28}$$

where, we have introduced the symbol $\mathbf{g}_n$ defined by

$$\mathbf{g}_n = \frac{2}{h}\int_0^h \mathbf{g}(X)\cos\frac{(2n-1)\pi X}{2h}dX \tag{10.3.29}$$

Following the approach in Section 9.4, we shall utilize the Laplace transform method to solve (10.3.20). The Laplace transform of this equation yields

$$\left(s^2\mathbf{M} + s\xi\mathbf{E} + \frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\right)\overline{\mathbf{T}}_n(s) = (s\mathbf{M} + \xi\mathbf{E})\mathbf{T}_n(0) + \mathbf{Q}\frac{d\mathbf{T}_n(0)}{dt} + \overline{\mathbf{p}}_n(s) \tag{10.3.30}$$

It is convenient at this point to define the 2×2 matrix

$$\overline{\mathbf{K}}_n(s) = \frac{2}{h}\left(s^2\mathbf{M} + s\xi\mathbf{E} + \frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\right)^{-1} \tag{10.3.31}$$

Because we are dealing with 2×2 matrices it is true that

$$\overline{\mathbf{K}}_n(s) = \frac{2}{h}\frac{s^2\,\mathrm{adj}\,\mathbf{M} + s\xi\,\mathrm{adj}\,\mathbf{E} + \frac{(2n-1)^2\pi^2}{4h^2}\mathrm{adj}\,\mathbf{Q}}{\det\left(s^2\mathbf{M} + s\xi\mathbf{E} + \frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\right)} \tag{10.3.32}$$

Therefore, we need to calculate the inverse Laplace transform of

$$\overline{\mathbf{T}}_n(s)=\frac{h}{2}\overline{\mathbf{K}}_n(s)\left(\left(s\mathbf{M}+\xi\mathbf{E}\right)\mathbf{T}_n(0)+\mathbf{M}\frac{d\mathbf{T}_n(0)}{dt}+\overline{\mathbf{p}}_n(s)\right) \tag{10.3.33}$$

If we now use (10.3.21), (10.3.26) and (10.3.28), equation (10.3.33) reduces to

$$\overline{\mathbf{T}}_n(s)=\frac{h}{2}\overline{\mathbf{K}}_n(s)\begin{pmatrix}\left(s\mathbf{M}+\xi\mathbf{E}\right)\mathbf{f}_n+\mathbf{M}\mathbf{g}_n+\dfrac{8h}{(2n-1)^2\pi^2}\left(\mathbf{M}s^2+\xi s\mathbf{E}\right)\mathbf{Q}^{-1}\overline{\mathbf{q}}(s)\\ +\dfrac{4}{(2n-1)\pi}(-1)^n\left(\mathbf{M}s^2+\xi s\mathbf{E}\right)\overline{\mathbf{k}}(s)(\ \ )\end{pmatrix} \tag{10.3.34}$$

This expression can be simplified further by use of (10.3.31) written in the form

$$\frac{h}{2}\overline{\mathbf{K}}_n(s)\left(s^2\mathbf{M}+s\xi\mathbf{E}\right)=\mathbf{I}-\frac{h}{2}\frac{(2n-1)^2\pi^2}{4h^2}\overline{\mathbf{K}}_n(s)\mathbf{Q} \tag{10.3.35}$$

The result of this substitution is

$$\begin{aligned}\overline{\mathbf{T}}_n(s)=&\frac{h}{2}\overline{\mathbf{K}}_n(s)(s\mathbf{M}+\xi\mathbf{E})\mathbf{f}_n+\frac{h}{2}\overline{\mathbf{K}}_n(s)\mathbf{M}\mathbf{g}_n-\overline{\mathbf{K}}_n(s)\overline{\mathbf{q}}(s)-\frac{(2n-1)\pi}{2h}(-1)^n\overline{\mathbf{K}}_n(s)\mathbf{Q}\overline{\mathbf{k}}(s)\\&+\frac{8h}{(2n-1)^2\pi^2}\mathbf{Q}^{-1}\overline{\mathbf{q}}(s)+\frac{4}{(2n-1)\pi}(-1)^n\overline{\mathbf{k}}(s)\end{aligned} \tag{10.3.36}$$

If $\mathbf{K}_n(t)$ is the inverse Laplace transform of $\overline{\mathbf{K}}_n(s)$, then the inverse of (10.3.36) is

$$\begin{aligned}\mathbf{T}_n(t)=&\frac{h}{2}\left(\frac{\partial\mathbf{K}_n(t)}{\partial t}\mathbf{M}+\xi\mathbf{K}_n(t)\mathbf{E}\right)\mathbf{f}_n+\frac{h}{2}\mathbf{K}_n(t)\mathbf{M}\mathbf{g}_n-\int_0^t\mathbf{K}_n(t-\tau)\mathbf{q}(\tau)d\tau\\&-\frac{(2n-1)\pi}{2h}(-1)^n\int_0^t\mathbf{K}_n(t-\tau)\mathbf{Q}\mathbf{k}(\tau)d\tau+\frac{8h}{(2n-1)^2\pi^2}\mathbf{Q}^{-1}\mathbf{q}(t)+\frac{4}{(2n-1)\pi}(-1)^n\mathbf{k}(t)\end{aligned} \tag{10.3.37}$$

Among the elementary properties of the Laplace transform that were used to derive (10.3.37) from (10.3.36) is the formula

$$\lim_{s\to\infty}\left(s\overline{\mathbf{K}}_n(s)\right)=\mathbf{K}_n(0)=\mathbf{0} \tag{10.3.38}$$

The first part of (10.3.38) is the initial value theorem for Laplace transforms. The fact that $\lim_{s\to\infty}\left(s\overline{\mathbf{K}}_n(s)\right)=\mathbf{0}$ follows from the definition (10.3.32). Another result that follows from(10.3.32) and the initial value theorem is

$$\lim_{s\to\infty}\left(s^2\bar{\mathbf{K}}_n(s)\right)=\lim_{s\to\infty}\left(s\left(s\bar{\mathbf{K}}_n(s)\right)\right)=\frac{\partial\mathbf{K}_n(0)}{\partial t}=\frac{2}{h}\mathbf{M}^{-1} \tag{10.3.39}$$

Given the formula (10.3.37), we can use (10.3.18) and (10.3.11) to obtain the following solution for $\mathbf{w}(X,t)$

$$\begin{aligned}\mathbf{w}(X,t)&=\sum_{n=1}^{\infty}\mathbf{T}_n(t)\cos\frac{(2n-1)\pi X}{2h}+\mathbf{Q}^{-1}\mathbf{q}(t)(X-h)+\mathbf{k}(t)\\&=\frac{h}{2}\sum_{n=1}^{\infty}\mathbf{K}_n(t)\mathbf{M}\mathbf{g}_n\cos\frac{(2n-1)\pi X}{2h}+\frac{h}{2}\sum_{n=1}^{\infty}\left(\frac{\partial\mathbf{K}_n(t)}{\partial t}\mathbf{M}+\xi\mathbf{K}_n(t)\mathbf{E}\right)\mathbf{f}_n\cos\frac{(2n-1)\pi X}{2h}\\&-\sum_{n=1}^{\infty}\left(\int_0^t\mathbf{K}_n(t-\tau)\mathbf{q}(\tau)d\tau\right)\cos\frac{(2n-1)\pi X}{2h}\\&-\frac{\pi}{2h}\sum_{n=1}^{\infty}\left((2n-1)(-1)^n\int_0^t\mathbf{K}_n(t-\tau)\mathbf{Q}\mathbf{k}(\tau)d\tau\right)\cos\frac{(2n-1)\pi X}{2h}\end{aligned} \tag{10.3.40}$$

In order to utilize the solution (10.3.40), we must use (10.3.32) to invert

$$\bar{\mathbf{K}}_n(s)=\frac{2}{h}\frac{s^2\operatorname{adj}\mathbf{M}+s\xi\operatorname{adj}\mathbf{E}+\dfrac{(2n-1)^2\pi^2}{4h^2}\operatorname{adj}\mathbf{Q}}{\det\left(s^2\mathbf{M}+s\xi\mathbf{E}+\dfrac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\right)} \tag{10.3.41}$$

Ultimately, the properties of this inverse depend upon the zeros of the fourth order polynomial

$$\det\left(\beta_n^2\mathbf{M}+\beta_n\xi\mathbf{E}+\frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\right)=0 \tag{10.3.42}$$

We shall examine the nature of these zeros in the next Section.

10.4 Properties of the Roots of $\det\left(\beta_n^2\mathbf{M}+\beta_n\xi\mathbf{E}+\frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\right)=0$

The fourth order polynomial (10.3.42) will have either four real roots, two real and one complex conjugate pair or two complex conjugate pairs. If we expand the determinant in (10.3.42), the polynomial is

$$
\begin{aligned}
&\left(\det\mathbf{M}\right)\beta_n^4+\xi\,\mathrm{tr}\left(\left(\mathrm{adj}\mathbf{M}\right)\mathbf{E}\right)\beta_n^3\\
&\quad+\frac{\left(2n-1\right)^2\pi^2}{4h^2}\mathrm{tr}\left(\left(\mathrm{adj}\mathbf{M}\right)\mathbf{Q}\right)\beta_n^2+\xi\frac{\left(2n-1\right)^2\pi^2}{4h^2}\mathrm{tr}\left(\left(\mathrm{adj}\mathbf{Q}\right)\mathbf{E}\right)\beta_n+\frac{\left(2n-1\right)^4\pi^4}{16h^4}\det\mathbf{Q}=0
\end{aligned}
\tag{10.4.1}
$$

In parts of this section, it will be convenient to utilize the definitions (7.2.50), (7.2.103) and (8.1.19) to write the polynomial (10.4.1) in the form

$$
\begin{aligned}
&\beta_n^4+\omega_\xi\beta_n^3+\frac{\left(2n-1\right)^2\pi^2}{4h^2}\left(u_{n(1)}^2+u_{n(2)}^2\right)\beta_n^2\\
&\quad+\frac{\left(2n-1\right)^2\pi^2}{4h^2}\omega_\xi u_0^2\beta_n+\frac{\left(2n-1\right)^4\pi^4}{16h^4}u_{n(1)}^2u_{n(2)}^2=0
\end{aligned}
\tag{10.4.2}
$$

Equation (10.4.2) is only valid in the case where $\det\mathbf{M}\neq 0$. The case $\mathbf{M}=\mathbf{0}$ formally corresponds to the inertia free approximation discussed in Chapter 9 and elsewhere in these Lectures.

Because $\xi\geq 0$, and because the symmetric matrix $\mathbf{Q}$ defined by (7.2.46) is positive definite and the symmetric matrix $\mathbf{E}$ defined by (7.2.87) is positive semi definite, it is a theorem that the polynomial (10.3.42) has no roots with positive real parts [Ref. 2, page 246]. In our case, unless we wish to study the case $\mathbf{M}=\mathbf{0}$ mentioned above, the symmetric matrix $\mathbf{M}$ is also positive definite.

It is interesting to characterize the case where the real parts of the roots of (10.4.2) are zero. In the special case $\xi=0$, it readily follows from (10.4.2) that the four roots are the complex numbers

$$
\beta_n\Big|_{\xi=0}=\begin{cases}\pm i\dfrac{\left(2n-1\right)\pi}{2h}u_{n(1)}\\[2ex] \pm i\dfrac{\left(2n-1\right)\pi}{2h}u_{n(2)}\end{cases}
\tag{10.4.3}
$$

Conversely, if one forces the roots of (10.4.2) to have zero real parts, i.e., to be of the form

$$
\beta_n=i\alpha
\tag{10.4.4}
$$

where α is real, it is readily established from (10.4.2) that

$$\alpha = \begin{cases} \pm\dfrac{(2n-1)\pi}{2h} u_{n(1)} \\ \pm\dfrac{(2n-1)\pi}{2h} u_{n(2)} \end{cases} \tag{10.4.5}$$

and

$$\omega_\xi \left(\alpha^2 - \frac{(2n-1)^2 \pi^2}{4h^2} u_0^2 \right) = 0 \tag{10.4.6}$$

Equation (10.4.6) is obeyed when $\xi = 0$ or when u_0 equals $u_{n(1)}$ or $u_{n(2)}$. As was pointed out in Section 7.2, it is a theoretical result that

$$u_{n(1)}^2 \geq u_0^2 \geq u_{n(2)}^2 \tag{10.4.7}$$

The numerical values in Table 1 of Section 6.2 illustrate that for these example materials u_0 and $u_{n(1)}$ are close in their values. To the accuracy of the numerical approximation used in this table, $u_0 = u_{n(1)}$ for Weber Sandstone. The summary of the discussion just completed is that when the real part of the roots of (10.4.8) are zero, then either $\xi = 0$ or u_0 equals either $u_{n(1)}$ or $u_{n(2)}$. In the following, we shall always make the assumption that

$$\xi > 0 \tag{10.4.9}$$

We shall not, however, rule out the case where u_0 equals $u_{n(2)}$

Given what we know about the roots of (10.4.2), in those cases where the real parts of the roots are less than zero, we can use the final value theorem from Laplace transform theory and conclude from (10.3.41) that

$$\lim_{t\to\infty} \mathbf{K}_n(t) = \lim_{s\to 0}\left(s\bar{\mathbf{K}}_n(s)\right) = \mathbf{0} \tag{10.4.10}$$

$$\lim_{t\to\infty} \frac{\partial \mathbf{K}_n(t)}{\partial t} = \lim_{s\to 0}\left(s\left(s\bar{\mathbf{K}}_n(s)\right)\right) = \lim_{s\to 0}\left(s^2\bar{\mathbf{K}}_n(s)\right) = \mathbf{0} \tag{10.4.11}$$

and

$$\lim_{t\to\infty} \int_0^t \mathbf{K}_n(t-\tau)\,d\tau = \lim_{s\to 0} \bar{\mathbf{K}}_n(s) = \frac{8h}{\pi^2} \frac{1}{(2n-1)^2} \mathbf{Q}^{-1} \tag{10.4.12}$$

These results are available without our actually computing the inverse of (10.3.41) and, from (10.4.9) and the above argument, they hold when u_0 does not equal either $u_{n(1)}$ or $u_{n(2)}$

Analytical solutions to fourth order polynomials such as (10.4.2) are difficult to interpret. Because of this fact, it is convenient to look at certain analytical approximations. The first is based upon the observation just made, namely that u_0 and $u_{n(1)}$ are close. We shall measure their closeness by a dimensionless parameter defined by

$$\varepsilon = \frac{u_1^2 - u_0^2}{u_0^2} \tag{10.4.13}$$

and look for roots of the form of an asymptotic expansion

$$\beta_n = a_0 + a_1\varepsilon + O\left(\varepsilon^2\right) \tag{10.4.14}$$

While algebraically tedious, it is possible to derive from (10.4.2) the following approximations for the four roots[1,2]

---

[1] The derivation of the solutions (10.4.15) and (10.4.16) is facilitated if one uses a symbolic manipulator such as found in Maple or Matlab.

[2] Equations (10.4.15) and (10.4.16) display a scaling that can also be established from (10.4.2), namely, that the roots of (10.4.2) will always be expressible as function of the dimensionless ratios of the form

$$\beta_n = \omega_\xi f_n\left(\frac{u_0}{h\omega_\xi}, \frac{u_{n(1)}}{h\omega_\xi}, \frac{u_{n(2)}}{h\omega_\xi}, \frac{(2n-1)\pi}{2}\right)$$

$$
\left.\begin{matrix}\beta_{n(1)}\\ \beta_{n(2)}\end{matrix}\right\}=-\frac{1}{2}\omega_\xi\pm\frac{1}{2}\sqrt{\omega_\xi^2-\frac{(2n-1)^2\pi^2}{h^2}u_{n(2)}^2}
$$

$$
+\left(\frac{\omega_\xi\left(-\frac{1}{2}\omega_\xi\pm\frac{1}{2}\sqrt{\omega_\xi^2-\frac{(2n-1)^2\pi^2}{h^2}u_{n(2)}^2}\right)u_0^2\frac{(2n-1)^2\pi^2}{4h^2}}{\left(\begin{matrix}\omega_\xi\left(u_0^2+u_{n(2)}^2\right)\frac{(2n-1)^2\pi^2}{4h^2}+2\left(-\frac{1}{2}\omega_\xi\pm\frac{1}{2}\sqrt{\omega_\xi^2-\frac{(2n-1)^2\pi^2}{h^2}u_{n(2)}^2}\right)\left(u_0^2-u_{n(2)}^2\right)\frac{(2n-1)^2\pi^2}{4h^2}\\ +\omega_\xi^2\left(-\frac{1}{2}\omega_\xi\pm\frac{1}{2}\sqrt{\omega_\xi^2-\frac{(2n-1)^2\pi^2}{h^2}u_{n(2)}^2}\right)\end{matrix}\right)}\right)\varepsilon
$$

$$
+O\left(\varepsilon^2\right) \tag{10.4.15}
$$

and

$$
\left.\begin{matrix}\beta_{n(3)}\\ \beta_{n(4)}\end{matrix}\right\}=-\frac{1}{2}\frac{(2n-1)^2\pi^2}{4h^2}\left(u_0^2-u_{n(2)}^2\right)\frac{\omega_\xi}{\left(\omega_\xi^2+\frac{(2n-1)^2\pi^2}{4h^2}\frac{\left(u_0^2-u_{n(2)}^2\right)^2}{u_0^2}\right)}\varepsilon
$$

$$
\pm i\frac{(2n-1)\pi}{2h}u_0\pm i\frac{1}{2}u_0\frac{(2n-1)^3\pi^3}{8h^3}\frac{\left(u_0^2-u_{n(2)}^2\right)^2}{\left(\omega_\xi^2u_0^2+\frac{(2n-1)^2\pi^2}{4h^2}\left(u_0^2-u_{n(2)}^2\right)^2\right)}\varepsilon \tag{10.4.16}
$$

$$
+O\left(\varepsilon^2\right)
$$

These approximate expressions obey the theoretical conclusions summarized above. Namely, since we have assumed (10.4.9), the real part of the roots are negative unless u_0 equals either $u_{n(1)}$ or $u_{n(2)}$. Equation (10.4.15) shows that the two roots $\beta_{n(1)}$ and $\beta_{n(2)}$ are real for all n such that

$$
\omega_\xi^2-\frac{(2n-1)^2\pi^2}{h^2}u_{n(2)}^2\geq 0 \tag{10.4.17}
$$

It follows from (10.4.17) that the n's that cause $\beta_{n(1)}$ and $\beta_{n(2)}$ to be real obey

$$n \le \frac{1}{2}\left(1+\frac{\omega_\xi h}{\pi u_{n(2)}}\right) \tag{10.4.18}$$

As we shall see next, the largest integer allowed by (10.4.18) for six materials introduced in Chapter 6 is large.

Because we are going to display numerical values for the roots of (10.4.2), it is helpful to repeat some of the numerical results contained in earlier Chapters. From Table 2 of Section 6.2 and Tables 1 and 2 of Section 7.2

Table 1 Rock Properties-Fluid Mixture Properties

Property	Ruhr Sandstone	Tennessee Marble	Charcoal Granite	Berea Sandstone	Westerly Granite	Weber Sandstone
γ_{s_R} (g/cm^3)	2.6	2.7	2.7	2.6	2.7	2.6
φ_f^+	.02	.02	.02	.19	.01	.06
$\bar{\lambda}_{ff}$ (kbar)	.165	.486	.326	4.44	.075	1.03
$\bar{\lambda}_{sf}$ (kbar)	5.21	4.02	4.13	13.990	3.48	9.91
$\bar{\lambda}_{ss}$ (kbar)	206.41	273.21	271.91	84.05	311.71	147.62
μ_{ss} (kbar)	133	240	187	60	160	122
$u_{n(1)}$ (m/sec)	4340	5346	4955	3210	4805	4059
$u_{n(2)}$ (m/sec)	728	1522	1220	1313	740	1124
$u_{n(3)}$ (m/sec)	2285	3012	2658	1688	2369	2234
u_0 (m/sec)	4337	5345	4954	3209	4802	4059
ξ (kbar-sec/cm^2)	$1.9(10)^{-3}$	3.92	3.92	$1.77(10)^{-4}$	.245	$.353(10)^{-3}$
ω_ξ (1/sec)	$9.58(10)^7$	$1.98(10)^{11}$	$1.98(10)^{11}$	$1.02(10)^6$	$2.46(10)^{11}$	$6.06(10)^7$
c (m^2/sec)	.005550	.000011	.000008	1.698365	.000022	.208487

where numerical values for ω_ξ and the consolidation coefficient, c, have been added to the table.

If we utilize values from Table 1, we can calculate the factor $\frac{1}{2}\left(1+\frac{\omega_\xi h}{\pi u_{n(2)}}\right)$ in equation (10.4.18). The results are

Table 2 The factor $\frac{1}{2}\left(1+\frac{\omega_{\xi}h}{\pi u_{n(2)}}\right)$ for $h=100$ m

	Ruhr Sandstone	Tennessee Marble	Charcoal Granite	Berea Sandstone	Westerly Granite	Weber Sandstone
$\frac{1}{2}\left(1+\frac{\omega_{\xi}h}{\pi u_{n(2)}}\right)$	2,091,883	2,065,612,193	2,576,551,743.	12,310	528,983,521	85,833

The numbers show that ,unless we adopt an h that is especially small, the integer n that would cause the roots cause $\beta_{n(1)}$ and $\beta_{n(2)}$ to be complex is large. The smallest n arises for Berea Sandstone and, in that case, it would take $12,000$ terms in our solution (10.3.40) before the two real roots would need to be replaced by a complex conjugate pair.

In the following, we shall only consider the case where (10.4.2) has two real roots and a single pair of complex conjugate roots. When we need actual numbers for these roots, we shall take them from the appendices to this Chapter. These appendices provide the first fifty roots of (10.4.2) for the six poroelastic materials whose properties are given in the above table. For our purposes here, it is useful to display, from the information in the appendices, the first ten roots for Ruhr Sandstone. These values are

Table 3 The First Ten Roots of (10.4.2) for Ruhr Sandstone for $h=100$ m

n	First Root: $\beta_{n(1)}$	Second Root: $\beta_{n(2)}$	Third Root: $\beta_{n(3)}$	Fourth Root: $\beta_{n(4)}$
1	-0.000001	-95745682.888539	68.12334i	-68.12334i
2	-0.000012	-95745682.888527	204.37002i	-204.37002i
3	-0.000034	-95745682.888504	-0.000001+340.6167i	-0.000001-340.6167i
4	-0.000067	-95745682.88847	-0.000002+476.86338i	-0.000002-476.86338i
5	-0.000111	-95745682.888423	-0.000003+613.11006i	-0.000003-613.11006i
6	-0.000166	-95745682.888366	-0.000004+749.35674i	-0.000004-749.35674i
7	-0.000231	-95745682.888297	-0.000006+885.60342i	-0.000006-885.60342i
8	-0.000308	-95745682.888216	-0.000008+1021.8501i	-0.000008-1021.8501i
9	-0.000396	-95745682.888124	-0.00001+1158.09678i	-0.00001-1158.09678i
10	-0.000494	-95745682.88802	-0.000013+1294.34346i	-0.000013-1294.34346i

Another useful approximation to the roots of (10.4.2) is when we look for solutions of the form

$$\beta_n = a_0 + a_1\varepsilon_n + a_2\varepsilon_n^2 + O\left(\varepsilon_n^3\right) \tag{10.4.19}$$

where ε_n is a dimensionless parameter defined by

$$\varepsilon_n = \frac{(2n-1)\pi}{2h}\frac{u_0}{\omega_\xi} \tag{10.4.20}$$

Unlike the approximations (10.4.15) and (10.4.16), because ε_n increases with n, the approximation given in the following are not useful for all n. However, when we make the choice $h = 100$ m, ε_n is small for rather large n for the example materials shown in the above table. The approximations to the roots of the form (10.4.19) turn out to be

$$\left.\begin{matrix}\beta_{n(1)} \\ \beta_{n(2)}\end{matrix}\right\} = \begin{cases} -\dfrac{u_{n(1)}^2 u_{n(2)}^2}{u_0^4}\omega_\xi \varepsilon_n^2 + O\left(\varepsilon_n^4\right) \\ -\omega_\xi + \dfrac{u_{n(1)}^2 + u_{n(2)}^2 - u_0^2}{u_0^2}\omega_\xi \varepsilon_n^2 + O\left(\varepsilon_n^4\right) \end{cases} \tag{10.4.21}$$

and

$$\left.\begin{matrix}\beta_{n(3)} \\ \beta_{n(4)}\end{matrix}\right\} = \begin{cases} -\dfrac{\left(u_{n(1)}^2 - u_0^2\right)\left(u_0^2 - u_{n(2)}^2\right)}{2u_0^4}\omega_\xi \varepsilon_n^2 + i\omega_\xi \varepsilon_n + O\left(\varepsilon_n^3\right) \\ -\dfrac{\left(u_{n(1)}^2 - u_0^2\right)\left(u_0^2 - u_{n(2)}^2\right)}{2u_0^4}\omega_\xi \varepsilon_n^2 - i\omega_\xi \varepsilon_n + O\left(\varepsilon_n^3\right) \end{cases} \tag{10.4.22}$$

The two sets of approximations, (10.4.15) and (10.4.16) and (10.4.21) and (10.4.22), are self consistent in the sense that if we assume $\varepsilon_n = \dfrac{(2n-1)\pi}{2h}\dfrac{u_0}{\omega_\xi}$ is small in (10.4.15) and (10.4.16) and, likewise, assume $\varepsilon = \dfrac{u_1^2 - u_0^2}{u_0^2}$ is small in (10.4.21) and (10.4.22) the resulting two sets of new approximations yield the same results. Also, if (10.4.21) and (10.4.22) are utilized to calculate the first ten roots for Ruhr Sandstone, the results are excellent approximations to the numbers in Table 3 above.

10.5 Inversion of $\bar{\mathbf{K}}_n(s)$

Given what we have learned about the roots of (10.3.42), we shall always assume a factorization of the form

$$\det\left(\beta_n^2\mathbf{M} + \beta_n\xi\mathbf{E} + \frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\right) = \det\mathbf{M}\left(\beta_n + \alpha_n\right)\left(\beta_n + \gamma_n\right)\left(\left(\beta_n + \zeta_n\right)^2 + \omega_n^2\right) \tag{10.5.1}$$

which allows to identify the four roots by[3]

$$\left.\begin{matrix}\beta_{n(1)}\\ \beta_{n(2)}\\ \beta_{n(3)}\\ \beta_{n(4)}\end{matrix}\right\} = \begin{cases}-\alpha_n\\ -\gamma_n\\ -\varsigma_n + i\omega_n\\ -\varsigma_n - i\omega_n\end{cases} \tag{10.5.2}$$

Given (10.5.1), it follows from (10.3.41) that we must compute the inverse Laplace transform of

$$\bar{\mathbf{K}}_n(s) = \frac{2}{h}\frac{s^2\,\mathrm{adj}\mathbf{M} + s\xi\,\mathrm{adj}\mathbf{E} + \dfrac{(2n-1)^2\pi^2}{4h^2}\mathrm{adj}\mathbf{Q}}{\det\mathbf{M}(s+\alpha_n)(s+\gamma_n)\left((s+\varsigma_n)^2+\omega_n^2\right)} \tag{10.5.3}$$

As the ratio of two polynomials in s, the inverse of (10.5.3) can be shown to be

$$\mathbf{K}_n(t) = \frac{2}{h}\begin{pmatrix} e^{-\alpha_n t}\mathbf{A}_n + e^{-\gamma_n t}\mathbf{B}_n - e^{-\varsigma_n t}\cos\omega_n t\left(\mathbf{A}_n + \mathbf{B}_n\right) \\ +e^{-\varsigma_n t}\sin\omega_n t\left(\dfrac{\alpha_n-\varsigma_n}{\omega_n}\mathbf{A}_n + \dfrac{\gamma_n-\varsigma_n}{\omega_n}\mathbf{B}_n + \dfrac{1}{\omega_n}\mathbf{M}^{-1}\right)\end{pmatrix} \tag{10.5.4}$$

where

[3] The coefficients of the polynomial (10.4.2) and its roots (10.5.2) can be shown to be related by the formulas

$$\begin{aligned}
\omega_\xi &= \beta_{n(1)} + \beta_{n(2)} + \beta_{n(3)} + \beta_{n(4)} \\
&= -\left(\alpha_n + \gamma_n + 2\varsigma_n\right) \\
\frac{(2n-1)^2\pi^2}{4h^2}\left(u_{n(1)}^2 + u_{n(2)}^2\right) &= \beta_{n(1)}\beta_{n(2)} + \left(\beta_{n(1)} + \beta_{n(2)}\right)\left(\beta_{n(3)} + \beta_{n(4)}\right) + \beta_{n(3)}\beta_{n(4)} \\
&= \alpha_n\gamma_n + 2\varsigma_n\left(\alpha_n + \gamma_n\right) + \varsigma_n^2 + \omega_n^2 \\
\frac{(2n-1)^2\pi^2}{4h^2}\omega_\xi u_0^2 &= \left(\beta_{n(1)} + \beta_{n(2)}\right)\beta_{n(3)}\beta_{n(4)} + \left(\beta_{n(3)} + \beta_{n(4)}\right)\beta_{n(1)}\beta_{n(2)} \\
&= -\left(\left(\alpha_n + \gamma_n\right)\left(\varsigma_n^2 + \omega_n^2\right) + 2\varsigma_n\alpha_n\gamma_n\right) \\
\frac{(2n-1)^4\pi^4}{16h^4}u_{n(1)}^2u_{n(2)}^2 &= \beta_{n(1)}\beta_{n(2)}\beta_{n(3)}\beta_{n(4)} \\
&= \alpha_n\gamma_n\left(\varsigma_n^2 + \omega_n^2\right)
\end{aligned}$$

$$\mathbf{A}_n = \frac{1}{\det \mathbf{M}(\gamma_n - \alpha_n)\left((\alpha_n - \varsigma_n)^2 + \omega_n^2\right)}\left(\alpha_n^2 \operatorname{adj}\mathbf{M} - \alpha_n \xi \operatorname{adj}\mathbf{E} + \frac{(2n-1)^2 \pi^2}{4h^2}\operatorname{adj}\mathbf{Q}\right) \quad (10.5.5)$$

and

$$\mathbf{B}_n = \frac{1}{\det \mathbf{M}(\alpha_n - \gamma_n)\left((\gamma_n - \varsigma_n)^2 + \omega_n^2\right)}\left(\gamma_n^2 \operatorname{adj}\mathbf{M} - \gamma_n \xi \operatorname{adj}\mathbf{E} + \frac{(2n-1)^2 \pi^2}{4h^2}\operatorname{adj}\mathbf{Q}\right) \quad (10.5.6)$$

Given (10.5.4), the solution (10.3.40) is complete. It is helpful to note in passing that (10.5.4) obeys the conditions

$$\mathbf{K}_n(0) = \mathbf{0} \quad (10.5.7)$$

$$\dot{\mathbf{K}}_n(0) = \frac{2}{h}\mathbf{M}^{-1} \quad (10.5.8)$$

and, if $\varsigma_n \neq 0$,

$$\lim_{t\to\infty}\mathbf{K}_n(t) = \mathbf{0} \quad (10.5.9)$$

The results (10.5.7), (10.5.8) and (10.5.9) were stated previously in equation (10.3.38), (10.3.39) and (10.4.10), respectively.

10.6 The Use of Green's Functions

As in Section 9.7 there are some advantages when dealing with boundary initial value problems with inhomogeneous boundary conditions to formulate the solution in terms of Green's Functions. The formulation is similar to that given in Section 9.7 except that the boundary value problem is the one formulated in Section 10.2 and the underlying differential operator is the one defined by (10.1.6) and (10.1.7).

As in Section 9.7, we shall use the notation $\mathbf{G}(\mathbf{X},t;\mathbf{X}_0,t_0)$ for the 6×6 matrix of functions that constitute the Green's function. As the notation suggests, the Green's function still depends upon pairs of points and pairs of times. It is helpful, when we need to display the components of this matrix to write

$$\mathbf{G}(\mathbf{X},t;\mathbf{X}_0,t_0) = \begin{bmatrix} \mathbf{g}_{f(1)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{f(2)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{f(3)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{f(4)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{f(5)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{f(6)}(\mathbf{X},t;\mathbf{X}_0,t_0) \\ \mathbf{g}_{s(1)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{s(2)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{s(3)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{s(4)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{s(5)}(\mathbf{X},t;\mathbf{X}_0,t_0) & \mathbf{g}_{s(6)}(\mathbf{X},t;\mathbf{X}_0,t_0) \end{bmatrix}$$

(10.6.1)

where, again each entry is a three dimensional vector that we have represented by a 3×1 column vector.

The Green's function is the 6×6 matrix of functions introduced in (10.6.1) defined such that, for every pair $(\mathbf{X}_0, t_0)\in\mathscr{V}\times(-\infty,\infty)$, it obeys the boundary initial value problem

$$\mathscr{D}\mathbf{G}(\mathbf{X},t;\mathbf{X}_0,t_0)=\delta(t-t_0)\,\delta(\mathbf{X}-\mathbf{X}_0)\,\mathbf{I} \qquad \text{for} \qquad (\mathbf{X},t)\in\mathscr{V}\times(-\infty,\infty) \tag{10.6.2}$$

$$\mathbf{T}_I(\mathbf{G})\mathbf{n}=\mathbf{0} \qquad \text{for} \qquad (\mathbf{X},t)\in\partial\mathscr{V}_1\times(-\infty,\infty) \tag{10.6.3}$$

$$P_f(\mathbf{G})=0 \qquad \text{for} \qquad (\mathbf{X},t)\in\partial\mathscr{V}_1\times(-\infty,\infty) \tag{10.6.4}$$

$$\mathbf{G}=0 \quad \text{for} \quad (\mathbf{X},t)\in\partial\mathscr{V}_2\times(-\infty,\infty) \tag{10.6.5}$$

and

$$\mathbf{G}(\mathbf{X},t,\mathbf{X}_0,t_0)=\mathbf{0} \qquad \text{for} \qquad \mathbf{X}\in\mathscr{V} \text{ and } t<t_0 \tag{10.6.6}$$

where $\mathscr{D}$ is now formal linear operator defined by

$$\begin{aligned}\mathscr{D}\mathbf{w}&=\begin{bmatrix}\rho_f^{+}\dfrac{\partial^2\mathbf{w}_f}{\partial t^2}+\xi\left(\dfrac{\partial\mathbf{w}_f}{\partial t}-\dfrac{\partial\mathbf{w}_s}{\partial t}\right)-\bar{\lambda}_{sf}\,\text{GRAD}(\text{Div}\,\mathbf{w}_s)-\bar{\lambda}_{ff}\,\text{GRAD}(\text{Div}\,\mathbf{w}_f)\\ \rho_{s_R}\dfrac{\partial^2\mathbf{w}_s}{\partial t^2}-\xi\left(\dfrac{\partial\mathbf{w}_f}{\partial t}-\dfrac{\partial\mathbf{w}_s}{\partial t}\right)-(\bar{\lambda}_{ss}+\mu_{ss})\,\text{GRAD}(\text{Div}\,\mathbf{w}_s)-\mu_{ss}\,\text{Div}(\text{GRAD}\,\mathbf{w}_s)-\bar{\lambda}_{sf}\,\text{GRAD}(\text{Div}\,\mathbf{w}_f)\end{bmatrix}\\ &=\begin{bmatrix}\rho_f^{+}\dfrac{\partial^2\mathbf{w}_f}{\partial t^2}+\xi\left(\dfrac{\partial\mathbf{w}_f}{\partial t}-\dfrac{\partial\mathbf{w}_s}{\partial t}\right)\\ \rho_{s_R}\dfrac{\partial^2\mathbf{w}_s}{\partial t^2}-\xi\left(\dfrac{\partial\mathbf{w}_f}{\partial t}-\dfrac{\partial\mathbf{w}_s}{\partial t}\right)\end{bmatrix}+L\,\mathbf{w}\end{aligned} \tag{10.6.7}$$

As in Section 9.7, we can write

$$\begin{bmatrix}\left(\dfrac{\partial\mathbf{w}_f}{\partial t}-\dfrac{\partial\mathbf{w}_s}{\partial t}\right)\\ -\left(\dfrac{\partial\mathbf{w}_f}{\partial t}-\dfrac{\partial\mathbf{w}_s}{\partial t}\right)\end{bmatrix}=\mathbf{E}\frac{\partial\mathbf{w}}{\partial t} \tag{10.6.8}$$

where $\mathbf{E}$ is the symmetric 6×6 matrix

$$\mathbf{E}=\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \tag{10.6.9}$$

In addition, we can write

$$\begin{bmatrix} \rho_f^+ \dfrac{\partial^2 \mathbf{w}_f}{\partial t^2} \\ \rho_{s_R} \dfrac{\partial^2 \mathbf{w}_s}{\partial t^2} \end{bmatrix} = \mathbf{M}\frac{\partial^2 \mathbf{w}}{\partial t^2} \tag{10.6.10}$$

where $\mathbf{M}$ is the symmetric 6×6 matrix

$$\mathbf{M}=\begin{bmatrix} \rho_f^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_f^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_f^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{s_R} & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_{s_R} & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_{s_R} \end{bmatrix} \tag{10.6.11}$$

This formalism allows the operator $\mathscr{L}$ to be defined by

$$\mathscr{D}\mathbf{w} = \mathbf{M}\frac{\partial^2 \mathbf{w}}{\partial t^2} + \xi \mathbf{E}\frac{\partial \mathbf{w}}{\partial t} + L\,\mathbf{w} \tag{10.6.12}$$

rather than by (10.6.7).

Given the definition of the Green's function $\mathbf{G}(\mathbf{X},t;\mathbf{X}_0,t_0)$, the theory of Green's functions can be used to write the solution of the boundary initial value problem defined by (10.2.1) through (10.2.5) in terms of its initial and boundary data. The result is

$$
\begin{aligned}
\mathbf{w}(\mathbf{X},t) = & \int_{\mathscr{V}} \mathbf{G}(\mathbf{X},t;\mathbf{X}_0,0)\mathbf{Mg}(\mathbf{X}_0)d\upsilon_0 + \frac{\partial}{\partial t}\int_{\mathscr{V}} \mathbf{G}(\mathbf{X},t;\mathbf{X}_0,0)\mathbf{Mf}(\mathbf{X}_0)d\upsilon_0 \\
& \xi\int_{\mathscr{V}} \mathbf{G}(\mathbf{X},t;\mathbf{X}_0,0)\mathbf{Ef}(\mathbf{X}_0)d\upsilon_0 - \int_0^t\int_{\partial\mathscr{V}_1}\left(\sum_{\alpha=1}^{6}\mathbf{e}_\alpha\left(\mathbf{g}_{s(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)\cdot\mathbf{s}(\mathbf{X}_0,t_0)\right)\right)d\upsilon_0 dt_0 \\
& -\varphi_f^+\int_0^t\int_{\partial\mathscr{V}_1}\left(\sum_{\alpha=1}^{6}\mathbf{e}_\alpha\left(\left(\mathbf{g}_{f(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)-\mathbf{g}_{s(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)\right)\cdot\mathbf{n}r(\mathbf{X}_0,t_0)\right)\right)d\upsilon_0 dt_0 \qquad (10.6.13) \\
& -\int_0^t\int_{\partial\mathscr{V}_2}\left(\sum_{\alpha=1}^{6}\mathbf{e}_\alpha\left(\mathbf{T}_I\left(\mathbf{g}_{(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)\right)\mathbf{k}_s(\mathbf{X}_0,t_0)\right)\right)\cdot d\mathbf{s}_0 dt_0 \\
& +\varphi_f^+\int_0^t\int_{\partial\mathscr{V}_2}\left(\sum_{\alpha=1}^{6}\mathbf{e}_\alpha\left(P_f\left(\mathbf{g}_{(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)\right)\right)\left(\mathbf{k}_f(\mathbf{X}_0,t_0)-\mathbf{k}_s(\mathbf{X}_0,t_0)\right)\right)\cdot d\mathbf{s}_0 dt_0
\end{aligned}
$$

where $\mathbf{g}_{(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0)$ is the column matrix defined by

$$\mathbf{g}_{(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0) = \begin{bmatrix} \mathbf{g}_{f(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0) \\ \mathbf{g}_{s(\alpha)}(\mathbf{X},t;\mathbf{X}_0,t_0) \end{bmatrix}, \qquad (10.6.14)$$

$\mathbf{k}_f(\mathbf{X}_0,t_0)$ and $\mathbf{k}_s(\mathbf{X}_0,t_0)$ are obtained from the boundary condition (10.2.3) expressed as the column matrix

$$\mathbf{k}(\mathbf{X},t) = \begin{bmatrix} \mathbf{k}_f(\mathbf{X},t) \\ \mathbf{k}_s(\mathbf{X},t) \end{bmatrix} \qquad (10.6.15)$$

and $\mathbf{e}_\alpha$, for $\alpha = 1,2,...,6$, is the 6×1 column matrix with entries equal to zero except in the α position where the value is 1

The eigenfunction expansion technique mentioned in Section 9.7 also works in the case where the inertia terms are present. The eigenfunctions, as defined in Section 9.2 allow us to seek solutions for the Green's function of the form.

$$\mathbf{G}(\mathbf{X},t;\mathbf{X}_0,t_0) = \sum_{n=1}^{\infty} T_n(t;t_0)\mathbf{u}_n(\mathbf{X})\mathbf{u}_n(\mathbf{X}_0)^T \qquad (10.6.16)$$

If we apply this formalism to the one dimensional problem considered in Section 10.3, the Green's function is the 2×2 matrix of functions

$$\mathbf{G}(X,t;X_0,t_0) = \begin{bmatrix} g_{f(1)}(X,t;X_0,t_0) & g_{f(2)}(X,t;X_0,t_0) \\ g_{s(1)}(X,t;X_0,t_0) & g_{s(2)}(X,t;X_0,t_0) \end{bmatrix} \qquad (10.6.17)$$

that obeys the one dimensional forms of (10.6.2) through (10.6.6). Therefore, the Green's function is the solution of

$$\mathbf{M}\frac{\partial^2 \mathbf{G}}{\partial t^2}+\xi\mathbf{E}\frac{\partial \mathbf{G}}{\partial t}-\mathbf{Q}\frac{\partial^2 \mathbf{G}}{\partial X^2}=\delta(t-t_0)\delta(X-X_0) \qquad \text{for} \qquad (X,t)\in(0,h)\times(-\infty,\infty) \tag{10.6.18}$$

$$\frac{\partial \mathbf{G}(0,t;X_0,t_0)}{\partial X}=\mathbf{0} \tag{10.6.19}$$

$$\mathbf{G}(h,t;X_0,t_0)=0 \tag{10.6.20}$$

and

$$\mathbf{G}(x,t;x_0,t_0)=\mathbf{0} \qquad \text{for} \quad t<t_0 \tag{10.6.21}$$

As in Section (9.4), the eigenfunctions of the space part of the one dimensional operator are defined by (9.4.25) and (9.4.26) are given by (9.4.27) where the eigenvalues are given by (9.4.31) and (9.4.32). In the notation adopted in Section 9.7, the representation of the solution (10.6.16) becomes

$$\mathbf{G}(X,t;X_0,t_0)=\sum_{n=1}^{\infty}\mathbf{K}_n(t;t_0)\cos\frac{(2n-1)\pi X}{2h}\cos\frac{(2n-1)\pi X_0}{2h} \tag{10.6.22}$$

where, for each n, $\mathbf{K}_n(t;t_0)$ is a 2×2 matrix whose representation with respect to the basis of eigenvectors of $\mathbf{Q}$ is

$$\begin{aligned}\mathbf{K}_n(t;t_0)=&T_{n(11)}(t;t_0)\mathbf{b}_{(1)}\mathbf{b}_{(1)}^T+T_{n(12)}(t;t_0)\mathbf{b}_{(1)}\mathbf{b}_{(2)}^T\\&+T_{n(21)}(t;t_0)\mathbf{b}_{(2)}\mathbf{b}_{(1)}^T+T_{n(22)}(t;t_0)\mathbf{b}_{(2)}\mathbf{b}_{(2)}^T\end{aligned} \tag{10.6.23}$$

If (10.6.22) is substituted into (10.6.18), it is easily shown that $\mathbf{K}_n(t;t_0)$ must obey

$$\mathbf{M}\frac{d^2\mathbf{K}_n(t;t_0)}{dt^2}+\xi\mathbf{E}\frac{d\mathbf{K}_n(t;t_0)}{dt}+\frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\mathbf{K}_n(t;t_0)=\frac{2}{h}\delta(t-t_0) \tag{10.6.24}$$

It follows from (10.6.21) and (10.6.22) that the matrix $\mathbf{K}_n(t;t_0)$ must obey

$$\mathbf{K}_n(t;t_0)=\mathbf{0} \qquad \text{for} \quad t<t_0 \tag{10.6.25}$$

The same Laplace transform techniques that were used to solve (10.3.20) can be used to establish that

$$\mathbf{K}_n\left(t;t_0\right)=\frac{2}{h}\mathscr{L}^{-1}\left(e^{-st_0}\frac{s^2\operatorname{adj}\mathbf{M}+s\xi\operatorname{adj}\mathbf{E}+\dfrac{\left(2n-1\right)^2\pi^2}{4h^2}\operatorname{adj}\mathbf{Q}}{\det\left(s^2\mathbf{M}+s\xi\mathbf{E}+\dfrac{\left(2n-1\right)^2\pi^2}{4h^2}\mathbf{Q}\right)}\right) \tag{10.6.26}$$

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transform operator. A comparison of (10.6.26) with (10.3.41) shows that

$$\mathbf{K}_n\left(t;t_0\right)=H\left(t-t_0\right)\mathbf{K}_n\left(t-t_0\right) \tag{10.6.27}$$

where $H\left(t-t_0\right)$ is the Heaviside step function that was introduced in Section 9.7 and $\mathbf{K}_n\left(t\right)$ is defined by (10.5.4). Thus, without explicit mention, our manipulations in Section 10.3 through 10.5 were, in effect, manipulations equivalent to finding the Green's function for our boundary initial value problem. The results in Section 10.3 are equivalent to those here. The one dimensional form of (10.6.13) is

$$\begin{aligned}\mathbf{w}\left(X,t\right)=&\int_0^h\mathbf{G}\left(X,t;X_0,0\right)\mathbf{Mg}\left(X_0\right)dX_0+\frac{\partial}{\partial t}\int_0^h\mathbf{G}\left(X,t;X_0,0\right)\mathbf{Mf}\left(X_0\right)dX_0\\&+\xi\int_0^h\mathbf{G}\left(X,t;X_0,0\right)\mathbf{Ef}\left(X_0\right)dX_0-\int_0^t\mathbf{G}\left(X,t;0,t_0\right)\mathbf{q}\left(t_0\right)dt_0\\&-\int_0^t\frac{\partial\mathbf{G}\left(X,t;h,t_0\right)}{\partial X_0}\mathbf{Qk}\left(t_0\right)dt_0\end{aligned} \tag{10.6.28}$$

where $\mathbf{q}\left(t\right)$ is defined by (10.3.7). If (10.6.22) were to be substituted into (10.6.28) one would again derive (10.3.40).

10.7 Biot Problem with Inertia

In this section we shall resolve the problem introduced in Section 9.5 except that this time we shall include the inertia terms for the fluid and the solid. As in Section 9.5, the boundary and initial conditions are

$$\mathbf{q}\left(t\right)=P_0\begin{bmatrix}0\\-1\end{bmatrix} \tag{10.7.1}$$

$$\mathbf{k}\left(t\right)=\mathbf{0} \tag{10.7.2}$$

and

$$\mathbf{f}(X)=\frac{P_0}{\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}(h-X)\begin{bmatrix}1\\1\end{bmatrix} \tag{10.7.3}$$

The boundary initial value problem for the model with inertia requires that the initial velocity be prescribed. We shall make the choice

$$\mathbf{g}(X)=\mathbf{0} \tag{10.7.4}$$

The physical meaning of the boundary conditions (10.7.1) and (10.7.2) and the initial condition (10.7.3) are explained in Section 9.5. The second initial condition (10.7.4) forces this solution with inertia to obey the same initial condition, i.e., zero velocity, that is automatically obeyed by the inertia free solution when the other boundary and initial conditions (10.7.1) through (10.7.3) are obeyed. The proof of this assertion is given in Section 9.5 where equation (9.5.19) was established.

Given (10.7.1) through (10.7.4), the solution (10.3.40) reduces to

$$\begin{aligned}\mathbf{w}(X,t)=&\frac{h}{2}\sum_{n=1}^{\infty}\left(\frac{\partial\mathbf{K}_n(t)}{\partial t}\mathbf{M}+\xi\mathbf{K}_n(t)\mathbf{E}\right)\mathbf{f}_n\cos\frac{(2n-1)\pi X}{2h}\\&-\sum_{n=1}^{\infty}\left(\int_0^t\mathbf{K}_n(t-\tau)\mathbf{q}(\tau)\,d\tau\right)\cos\frac{(2n-1)\pi X}{2h}\end{aligned} \tag{10.7.5}$$

It follows from (10.3.27) and (10.7.3) that

$$\begin{aligned}\mathbf{f}_n&=\frac{2}{h}\int_0^h\mathbf{f}(X)\cos\frac{(2n-1)\pi X}{2h}dX=\frac{P_0}{\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\begin{bmatrix}1\\1\end{bmatrix}\frac{2}{h}\int_0^h(h-X)\cos\frac{(2n-1)\pi X}{2h}dX\\&=\frac{P_0}{\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\begin{bmatrix}1\\1\end{bmatrix}\frac{8h}{(2n-1)^2\pi^2}\end{aligned} \tag{10.7.6}$$

This result along with (10.7.1) reduce the answer (10.7.5) to

$$\begin{aligned}\mathbf{w}(X,t)=&\frac{4h^2P_0}{\pi^2\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\frac{\partial\mathbf{K}_n(t)}{\partial t}\mathbf{M}\begin{bmatrix}1\\1\end{bmatrix}\cos\frac{(2n-1)\pi X}{2h}\\&+P_0\sum_{n=1}^{\infty}\left(\int_0^t\mathbf{K}_n(t-\tau)\,d\tau\right)\begin{bmatrix}0\\1\end{bmatrix}\cos\frac{(2n-1)\pi X}{2h}\end{aligned} \tag{10.7.7}$$

where $\mathbf{E}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ has been used. The analytical solution is essentially complete when one substitutes (10.5.4) into (10.7.7). It follows from (10.5.4) that

$$\frac{h}{2}\frac{\partial \mathbf{K}_n(t)}{\partial t} = -\alpha_n e^{-\alpha_n t}\mathbf{A}_n - \gamma_n e^{-\gamma_n t}\mathbf{B}_n + \left(\varsigma_n e^{-\varsigma_n t}\cos\omega_n t + \omega_n e^{-\varsigma_n t}\sin\omega_n t\right)\left(\mathbf{A}_n + \mathbf{B}_n\right)$$
$$+\left(-\varsigma_n e^{-\varsigma_n t}\sin\omega_n t + \omega_n e^{-\varsigma_n t}\cos\omega_n t\right)\left(\frac{\alpha_n - \varsigma_n}{\omega_n}\mathbf{A}_n + \frac{\gamma_n - \varsigma_n}{\omega_n}\mathbf{B}_n + \frac{1}{\omega_n}\mathbf{M}^{-1}\right) \quad (10.7.8)$$

and

$$\frac{h}{2}\int_0^t \mathbf{K}_n(t-\tau)\,d\tau = \frac{1}{\alpha_n}\left(1-e^{-\alpha_n t}\right)\mathbf{A}_n + \frac{1}{\gamma_n}\left(1-e^{-\gamma_n t}\right)\mathbf{B}_n$$
$$-\frac{1}{\omega_n^2+\varsigma_n^2}\left(\xi_n\left(1-e^{-\varsigma_n t}\cos\omega_n t\right) + \omega_n e^{-\varsigma_n t}\sin\omega_n t\right)\left(\mathbf{A}_n + \mathbf{B}_n\right) \quad (10.7.9)$$
$$+\frac{1}{\omega_n^2+\varsigma_n^2}\left(\omega_n\left(1-e^{-\varsigma_n t}\cos\omega_n t\right) - \xi_n e^{-\varsigma_n t}\sin\omega_n t\right)\left(\begin{array}{l}\dfrac{\alpha_n - \varsigma_n}{\omega_n}\mathbf{A}_n \\ +\dfrac{\gamma_n - \varsigma_n}{\omega_n}\mathbf{B}_n + \dfrac{1}{\omega_n}\mathbf{M}^{-1}\end{array}\right)$$

In Section 9.5, it was pointed out that the long time solution of the inertia free Biot problem, i.e. the solution (9.5.12) is $\mathbf{w}_\infty = P_0(h-X)\mathbf{Q}^{-1}\begin{bmatrix}0\\1\end{bmatrix}$. It turns out that the solution (10.7.7) yields the same result. To see this, we note from (10.7.7) that

$$\mathbf{w}_\infty = \lim_{t\to\infty}\mathbf{w}(X,t) = \frac{4h^2P_0}{\pi^2\,\mathrm{tr}\left((\mathrm{adj}\,\mathbf{Q})\mathbf{E}\right)}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left(\lim_{t\to\infty}\frac{\partial\mathbf{K}_n(t)}{\partial t}\right)\mathbf{M}\begin{bmatrix}1\\1\end{bmatrix}\cos\frac{(2n-1)\pi X}{2h}$$
$$+P_0\sum_{n=1}^{\infty}\left(\lim_{t\to\infty}\int_0^t\mathbf{K}_n(t-\tau)\,d\tau\right)\begin{bmatrix}0\\1\end{bmatrix}\cos\frac{(2n-1)\pi X}{2h} \quad (10.7.10)$$

It follows from equations (10.4.11) and (10.4.12) that (10.7.10) reduces to

$$\mathbf{w}_\infty = \lim_{t\to\infty}\mathbf{w}(X,t) = P_0\frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi X}{2h}\mathbf{Q}^{-1}\begin{bmatrix}0\\1\end{bmatrix} \quad (10.7.11)$$

Equation (9.4.64) allows us to write the last result as

$$\mathbf{w}_\infty = \lim_{t\to\infty}\mathbf{w}(X,t) = P_0(h-X)\mathbf{Q}^{-1}\begin{bmatrix}0\\1\end{bmatrix} \quad (10.7.12)$$

which is identical to the result for the inertia free case. It is important to keep in mind that the key results (10.4.11) and (10.4.12) assume we are not dealing with a case where u_0 equals either $u_{n(1)}$ or $u_{n(2)}$. In such a case, the real part of the roots $\beta_{n(3)} = -\varsigma_n + i\omega_n$ and $\beta_{n(4)} = -\varsigma_n - i\omega_n$ is zero. As (10.7.8) and (10.7.9) show, if $\varsigma_n = 0$ it is no longer true that (10.4.10) through (10.4.12) hold.

As in the inertia free solution in Section 9.5, it is useful to plot the solution (10.7.7). We shall again adopt the material properties for Ruhr Sandstone as tabulated in Tables 1 and 2 of Chapter 6, and repeated in Table 1 of this chapter. The numerical values we shall adopt are again

$$\mathbf{Q} = \begin{bmatrix} \bar{\lambda}_{ff} & \bar{\lambda}_{sf} \\ \bar{\lambda}_{sf} & \bar{\lambda}_{ss} + 2\mu_{ss} \end{bmatrix} = \begin{bmatrix} .165 & 5.21 \\ 5.21 & 206.41 + 2(133) \end{bmatrix} = \begin{bmatrix} .165 & 5.21 \\ 5.21 & 472.41 \end{bmatrix} \text{ kbar} \tag{10.7.13}$$

$$\varphi_f^+ = .02 \tag{10.7.14}$$

and

$$\xi = 1.9(10)^{-3} \text{ kbar-sec/cm}^2 \tag{10.7.15}$$

Also, it follows from the numbers in Table 1 that

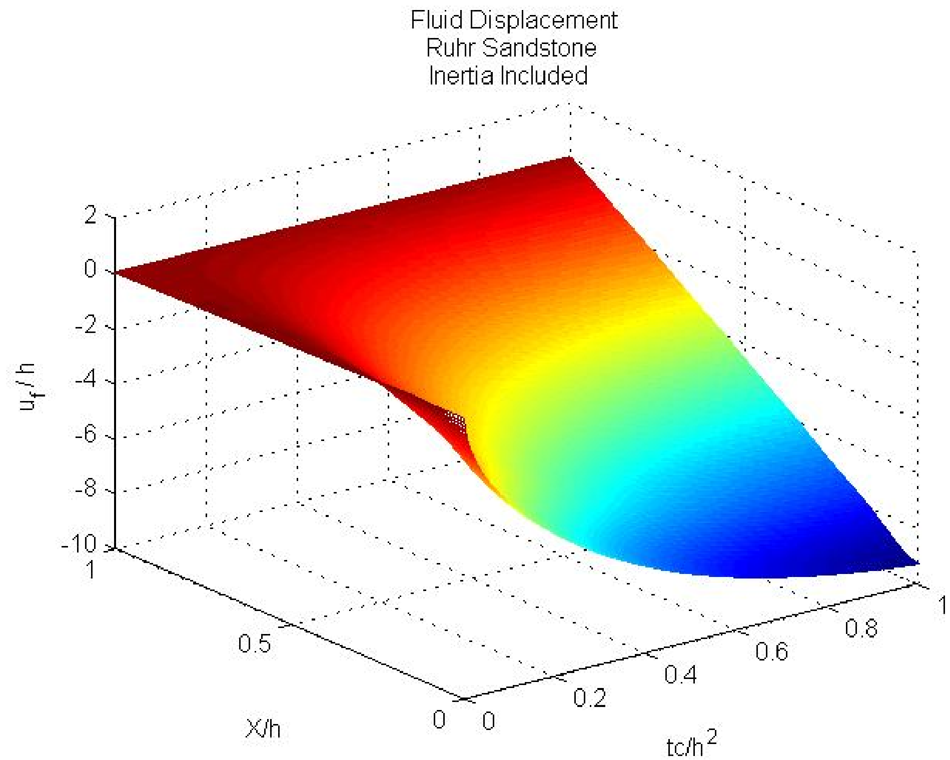
$$\mathbf{M} = \begin{bmatrix} \varphi_f^+ \gamma_f & 0 \\ 0 & (1-\varphi_f^+)\gamma_{s_R} \end{bmatrix} = \begin{bmatrix} .02 & 0 \\ 0 & .98(2.6) \end{bmatrix} \text{ g/cm}^3 \tag{10.7.16}$$

We shall again take

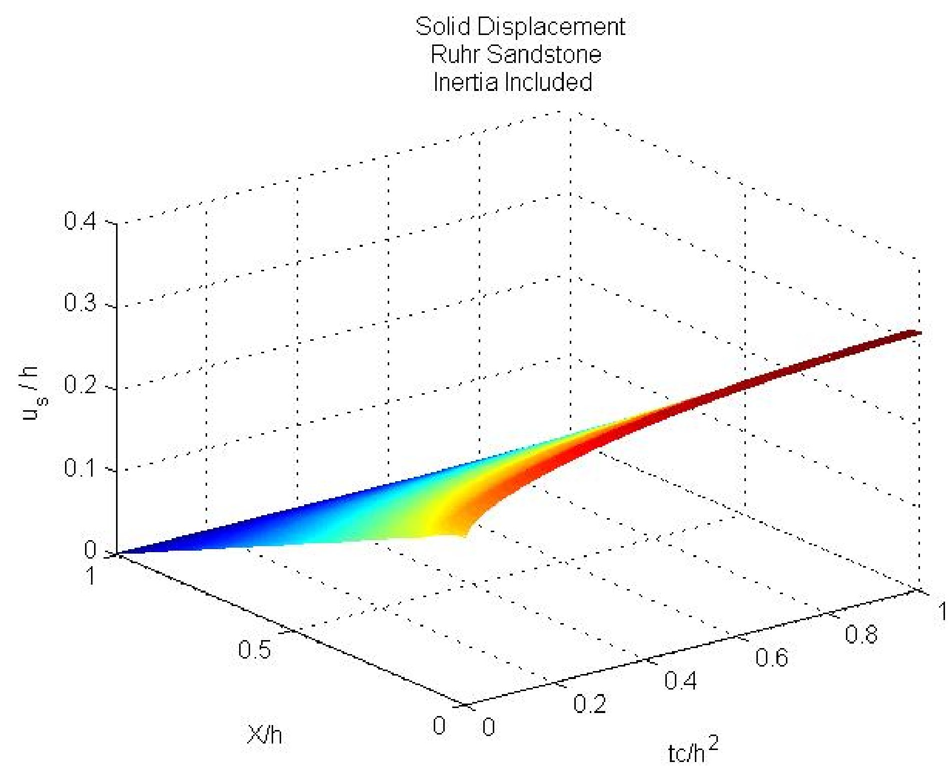
$$P_0 = 10^2 \text{ kbar} \tag{10.7.17}$$

As in Section 9.5, we shall plot the solution (10.7.7) by plotting the dimensionless displacement $\dfrac{\mathbf{w}(X,t)}{h}$ verses the dimensionless distance $\dfrac{X}{h}$ and the dimensionless time $\dfrac{tc}{h^2}$. Unlike the inertia free solution (9.5.17), the solution (10.7.7) is not naturally represented in terms of these dimensionless groups. Each of the roots (10.5.2) has dimensions of inverse time and could be used to normalize the time. We have made the choice $\dfrac{tc}{h^2}$ in order to allow comparisons to be made with the inertia free solution (9.5.17).

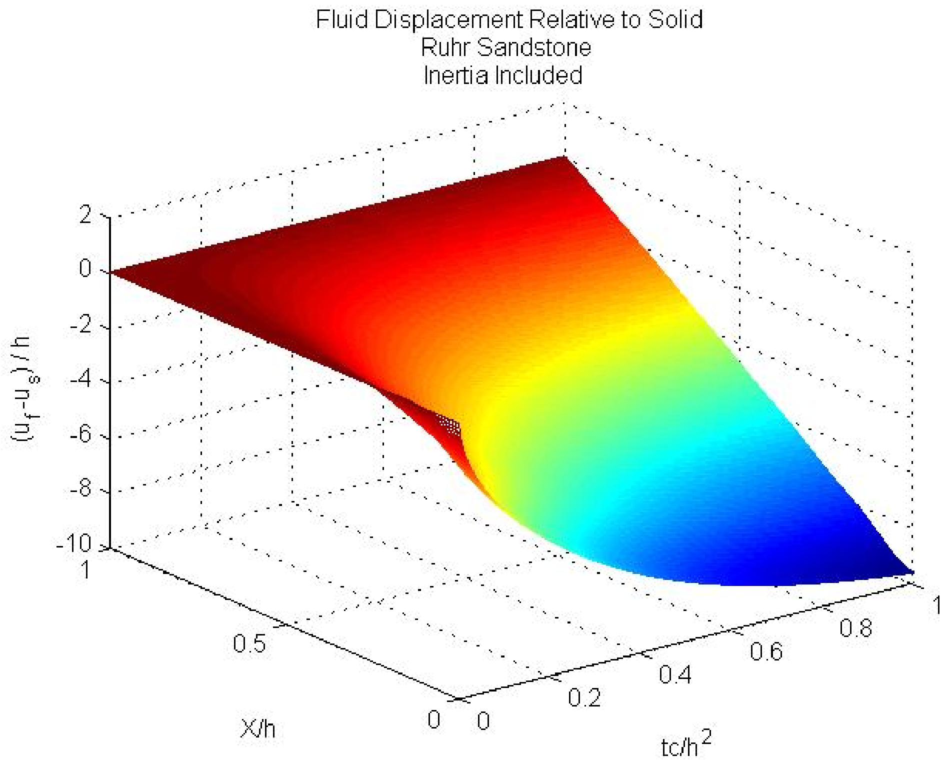
The resulting plots of the fluid and solid displacements are


Fluid Displacement
Ruhr Sandstone
Inertia Included
u_f / h
X/h
tc/h^2

and


Solid Displacement
Ruhr Sandstone
Inertia Included
u_s / h
X/h
tc/h^2

The displacement of the fluid relative to that of the solid is



The evident conclusion from these three figures is that they appear virtually identical to the corresponding three figures in Section 9.5. In effect, the problem we have posed for the poroelastic solid with inertia produces results that, at least to the scale of observation in the above three figures, are indistinguishable from the corresponding problem without inertia.

One issue that masks the differences in the two models is the initial condition (10.7.4) which we forced in the inertia case but was a theorem in the inertia free case. The partial differential equations which define the model without inertia are (9.1.15) and (9.1.16). Because they are first order in time, they represent singular perturbations of the equations used in this chapter, equations (10.1.6) and (10.1.7). As singular perturbations, they will not satisfy the same number of initial conditions as will the more general equations. The problem just solved does not exploit this difference because we forced the initial condition (10.7.4). A more significant display of the differences would arise if we selected a different initial velocity. As equation (9.4.68) displays, the model used in Section 9.5 is such that the initial condition is determined by the data, i.e., the values of the initial displacement, the imposed load and the displacement at $x = h$ determines the initial velocity. For the equations with inertia, the initial velocity is a part of the data.

In the next section, we shall consider the more general problem where the external pressure oscillates with a prescribed frequency. We considered this problem in Section 9.6 for the case of zero inertia.

10.8 Biot Problem with Inertia: Time Dependent External Pressure

As in Section 9.6, we adopt the boundary and initial conditions

$$\mathbf{q}(t) = P_0 \cos\omega t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{10.8.1}$$

$$\mathbf{k}(t) = \mathbf{0} \tag{10.8.2}$$

and

$$\mathbf{f}(X) = \frac{P_0 \cos\omega t}{\operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}(h - X)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{10.8.3}$$

where ω is a prescribed external frequency. We shall continue to adopt (10.7.4) as the initial condition. Thus, we shall take

$$\mathbf{g}(X) = \mathbf{0} \tag{10.8.4}$$

Given (10.8.1) through (10.8.4), the solution (10.3.40) reduces to

$$\begin{aligned} \mathbf{w}(X,t) = {} & \frac{h}{2}\sum_{n=1}^{\infty}\left(\frac{\partial \mathbf{K}_n(t)}{\partial t}\mathbf{M} + \xi \mathbf{K}_n(t)\mathbf{E}\right)\mathbf{f}_n \cos\frac{(2n-1)\pi X}{2h} \\ & -\sum_{n=1}^{\infty}\left(\int_0^t \mathbf{K}_n(t-\tau)\mathbf{q}(\tau)\,d\tau\right)\cos\frac{(2n-1)\pi X}{2h} \end{aligned} \tag{10.8.5}$$

where $\mathbf{q}(t)$ is given by (10.8.1) and $\mathbf{f}_n$ is again given by (10.7.6). If these two results are used, (10.8.5) reduces to

$$\begin{aligned} \mathbf{w}(X,t) = {} & \frac{4h^2 P_0}{\pi^2 \operatorname{tr}\left((\operatorname{adj}\mathbf{Q})\mathbf{E}\right)}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left(\frac{\partial \mathbf{K}_n(t)}{\partial t}\mathbf{M}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos\frac{(2n-1)\pi X}{2h} \\ & + P_0\sum_{n=1}^{\infty}\left(\int_0^t \mathbf{K}_n(t-\tau)\cos\omega\tau\,d\tau\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos\frac{(2n-1)\pi X}{2h} \end{aligned}$$

Given the formula (10.5.4), a series of straight forward, but lengthy, integrations yields

$$\frac{h}{2}\int_0^t \mathbf{K}_n(t-\tau)\cos\omega\tau d\tau = \left(\frac{1}{\alpha_n^2+\omega^2}\left(\alpha_n\left(\cos\omega t - e^{-\alpha_n t}\right)+\omega\sin\omega t\right)\right)\mathbf{A}_n$$
$$+\left(\frac{1}{\gamma_n^2+\omega^2}\left(\gamma_n\left(\cos\omega t - e^{-\gamma_n t}\right)+\omega\sin\omega t\right)\right)\mathbf{B}_n$$
$$-\frac{1}{\sqrt{\left(\omega_n^2+\zeta_n^2-\omega^2\right)^2+\left(2\omega\zeta_n\right)^2}}\begin{pmatrix}\sqrt{\xi_n^2+\omega^2}\sin\left(\omega t+\psi_n\right)\\ +\sqrt{\omega_n^2+\zeta_n^2}e^{-\zeta_n t}\sin\left(\omega_n t-\varphi_n\right)\end{pmatrix}\left(\mathbf{A}_n+\mathbf{B}_n\right)$$
$$+\frac{1}{\sqrt{\left(\omega_n^2+\zeta_n^2-\omega^2\right)^2+\left(2\omega\zeta_n\right)^2}}\begin{pmatrix}\omega_n\cos\left(\omega t-\sigma_n\right)\\ -\sqrt{\omega_n^2+\zeta_n^2}e^{-\zeta_n t}\cos\left(\omega_n t-\phi_n\right)\end{pmatrix}\begin{pmatrix}\dfrac{\alpha_n-\xi_n}{\omega_n}\mathbf{A}_n\\ +\dfrac{\gamma_n-\xi_n}{\omega_n}\mathbf{B}_n\\ +\dfrac{1}{\omega_n}\mathbf{M}^{-1}\end{pmatrix}$$

where

$$\sin\varphi_n = \frac{\zeta_n\left(\omega^2+\omega_n^2+\zeta_n^2\right)}{\sqrt{\omega_n^2+\zeta_n^2}\sqrt{\left(\omega_n^2+\zeta_n^2-\omega^2\right)^2+\left(2\omega\zeta_n\right)^2}} \tag{10.8.8}$$

$$\cos\varphi_n = \frac{\omega_n\left(\omega_n^2+\zeta_n^2-\omega^2\right)}{\sqrt{\omega_n^2+\zeta_n^2}\sqrt{\left(\omega_n^2+\zeta_n^2-\omega^2\right)^2+\left(2\omega\zeta_n\right)^2}} \tag{10.8.9}$$

$$\sin\psi_n = \frac{\zeta_n\left(\omega^2+\omega_n^2+\zeta_n^2\right)}{\sqrt{\xi_n^2+\omega^2}\sqrt{\left(\omega_n^2+\zeta_n^2-\omega^2\right)^2+\left(2\omega\zeta_n\right)^2}} \tag{10.8.10}$$

$$\cos\psi_n = \frac{\omega\left(\omega^2-\omega_n^2+\zeta_n^2\right)}{\sqrt{\xi_n^2+\omega^2}\sqrt{\left(\omega_n^2+\zeta_n^2-\omega^2\right)^2+\left(2\omega\zeta_n\right)^2}} \tag{10.8.11}$$

$$\sin\sigma_n = \frac{2\omega\zeta_n}{\sqrt{\left(\omega_n^2+\zeta_n^2-\omega^2\right)^2+\left(2\omega\zeta_n\right)^2}} \tag{10.8.12}$$

and

$$\cos\sigma_n = \frac{\omega_n^2+\zeta_n^2-\omega^2}{\sqrt{\left(\omega_n^2+\zeta_n^2-\omega^2\right)^2+\left(2\omega\zeta_n\right)^2}} \tag{10.8.13}$$

This solution displays a transient solution, $\mathbf{w}_t$, defined by

$$\begin{aligned}\mathbf{w}_t\left(X,t\right) &= \frac{4h^2P_0}{\pi^2\,\mathrm{tr}\left(\left(\mathrm{adj}\mathbf{Q}\right)\mathbf{E}\right)}\sum_{n=1}^{\infty}\frac{1}{\left(2n-1\right)^2}\left(\frac{\partial\mathbf{K}_n\left(t\right)}{\partial t}\mathbf{M}\right)\begin{bmatrix}1\\1\end{bmatrix}\cos\frac{\left(2n-1\right)\pi X}{2h}\\ &+P_0\sum_{n=1}^{\infty}\left(\int_0^t\mathbf{K}_n\left(t-\tau\right)\cos\omega\tau d\tau\Big|\right)_{\text{Transient}}\begin{bmatrix}0\\1\end{bmatrix}\cos\frac{\left(2n-1\right)\pi X}{2h}\end{aligned} \tag{10.8.14}$$

where $\dfrac{\partial\mathbf{K}_n\left(t\right)}{\partial t}$ is given by (10.7.8) and $\left(\int_0^t\mathbf{K}_n\left(t-\tau\right)\cos\omega\tau d\tau\Big|\right)_{\text{Transient}}$, from (10.8.7) is given by

$$\begin{aligned}\frac{h}{2}\int_0^t\mathbf{K}_n\left(t-\tau\right)\cos\omega\tau d\tau\Big|_{\text{Transient}} &= e^{-\alpha_n t}\left(-\frac{\alpha_n}{\alpha_n^2+\omega^2}\right)\mathbf{A}_n+e^{-\gamma_n t}\left(-\frac{\gamma_n}{\gamma_n^2+\omega^2}\right)\mathbf{B}_n\\ &-e^{-\zeta_n t}\frac{1}{\sqrt{\left(\omega_n^2+\zeta_n^2-\omega^2\right)^2+\left(2\omega\zeta_n\right)^2}}\left(\sqrt{\omega_n^2+\zeta_n^2}\sin\left(\omega_n t-\varphi_n\right)\right)\left(\mathbf{A}_n+\mathbf{B}_n\right)\\ &+e^{-\zeta_n t}\frac{1}{\sqrt{\left(\omega_n^2+\zeta_n^2-\omega^2\right)^2+\left(2\omega\zeta_n\right)^2}}\left(-\sqrt{\omega_n^2+\zeta_n^2}\cos\left(\omega_n t-\phi_n\right)\right)\begin{pmatrix}\dfrac{\alpha_n-\xi_n}{\omega_n}\mathbf{A}_n\\ +\dfrac{\gamma_n-\xi_n}{\omega_n}\mathbf{B}_n\\ +\dfrac{1}{\omega_n}\mathbf{M}^{-1}\end{pmatrix}\end{aligned} \tag{10.8.15}$$

Likewise, the steady state solution is given by

$$\mathbf{w}_s\left(X,t\right) = P_0\sum_{n=1}^{\infty}\left(\int_0^t\mathbf{K}_n\left(t-\tau\right)\cos\omega\tau d\tau\Big|_{\text{Steady}}\right)\begin{bmatrix}0\\1\end{bmatrix}\cos\frac{\left(2n-1\right)\pi X}{2h} \tag{10.8.16}$$

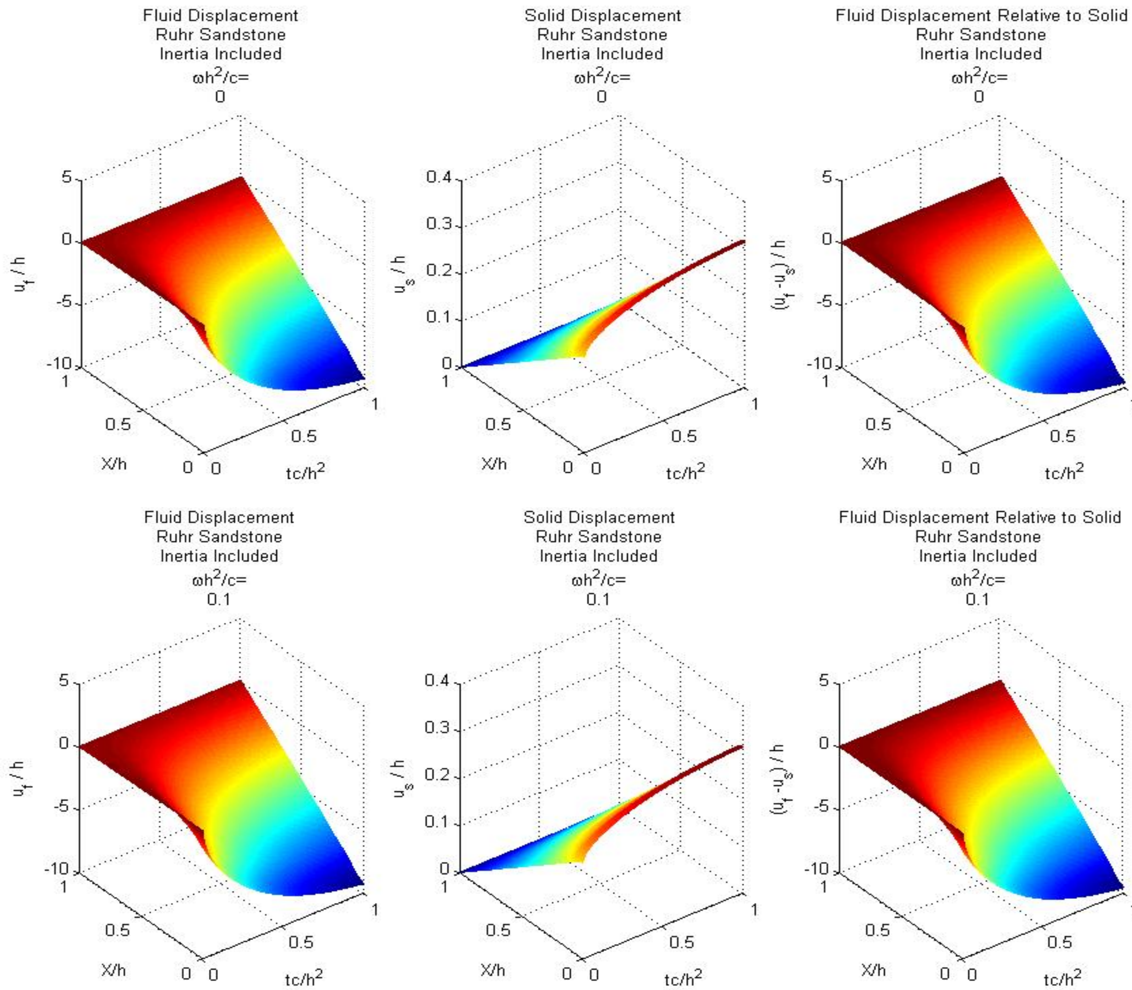
where, from (10.8.7),

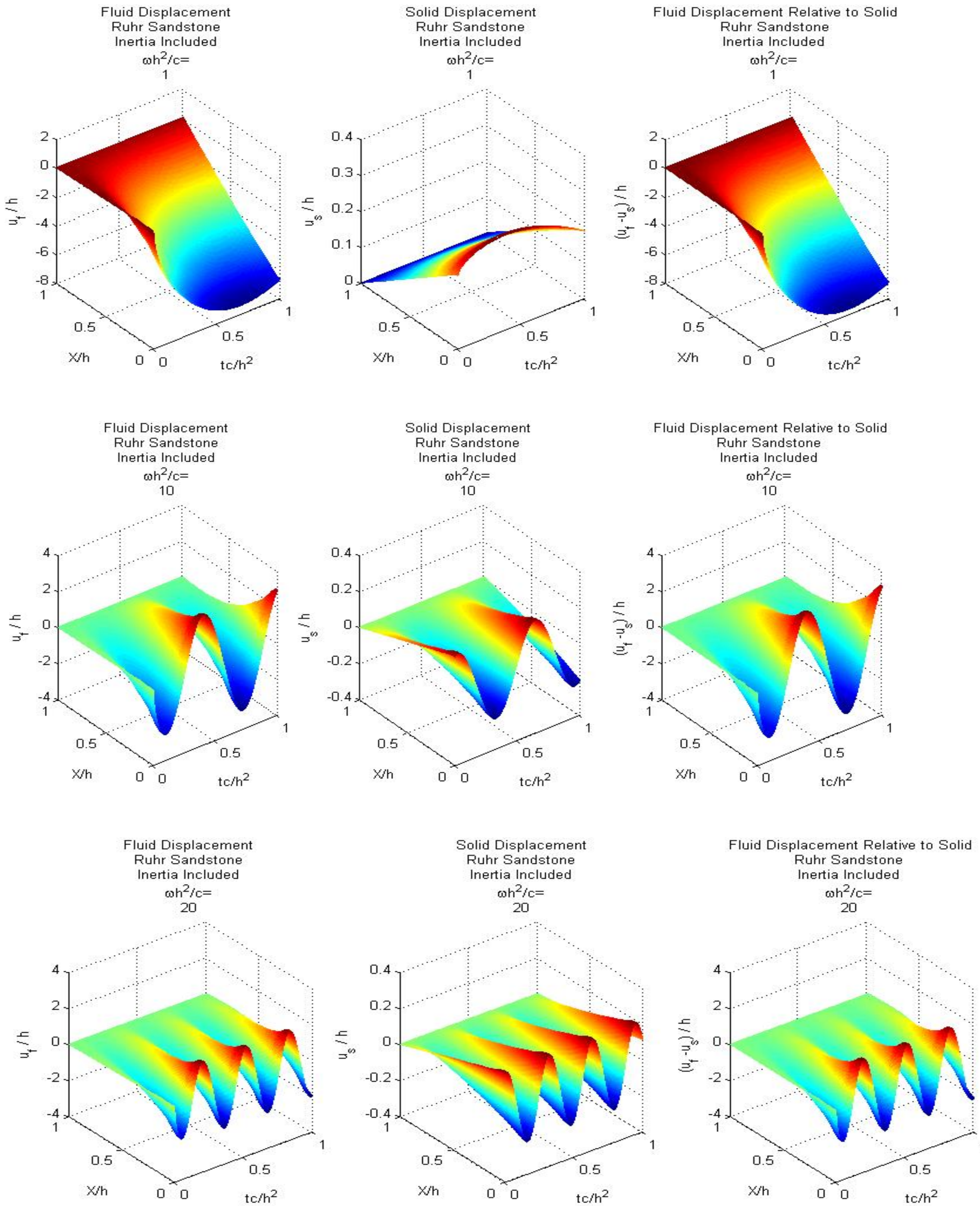
$$
\begin{aligned}
\int_0^t \mathbf{K}_n(t-\tau)\cos\omega\tau d\tau\Big|_{\text{Steady}} &= \frac{1}{\alpha_n^2+\omega^2}\left(\alpha_n\cos\omega t+\omega\sin\omega t\right)\mathbf{A}_n \\
&+\frac{1}{\gamma_n^2+\omega^2}\left(\gamma_n\cos\omega t+\omega\sin\omega t\right)\mathbf{B}_n \\
&-\frac{\sqrt{\xi_n^2+\omega^2}}{\sqrt{\left(\omega_n^2+\zeta_n^2-\omega^2\right)^2+\left(2\omega\zeta_n\right)^2}}\sin\left(\omega t+\psi_n\right)\left(\mathbf{A}_n+\mathbf{B}_n\right) \\
&+\frac{\omega_n}{\sqrt{\left(\omega_n^2+\zeta_n^2-\omega^2\right)^2+\left(2\omega\zeta_n\right)^2}}\cos\left(\omega t-\sigma_n\right)\begin{pmatrix} \dfrac{\alpha_n-\xi_n}{\omega_n}\mathbf{A}_n \\ +\dfrac{\gamma_n-\xi_n}{\omega_n}\mathbf{B}_n \\ +\dfrac{1}{\omega_n}\mathbf{M}^{-1} \end{pmatrix}
\end{aligned}
$$

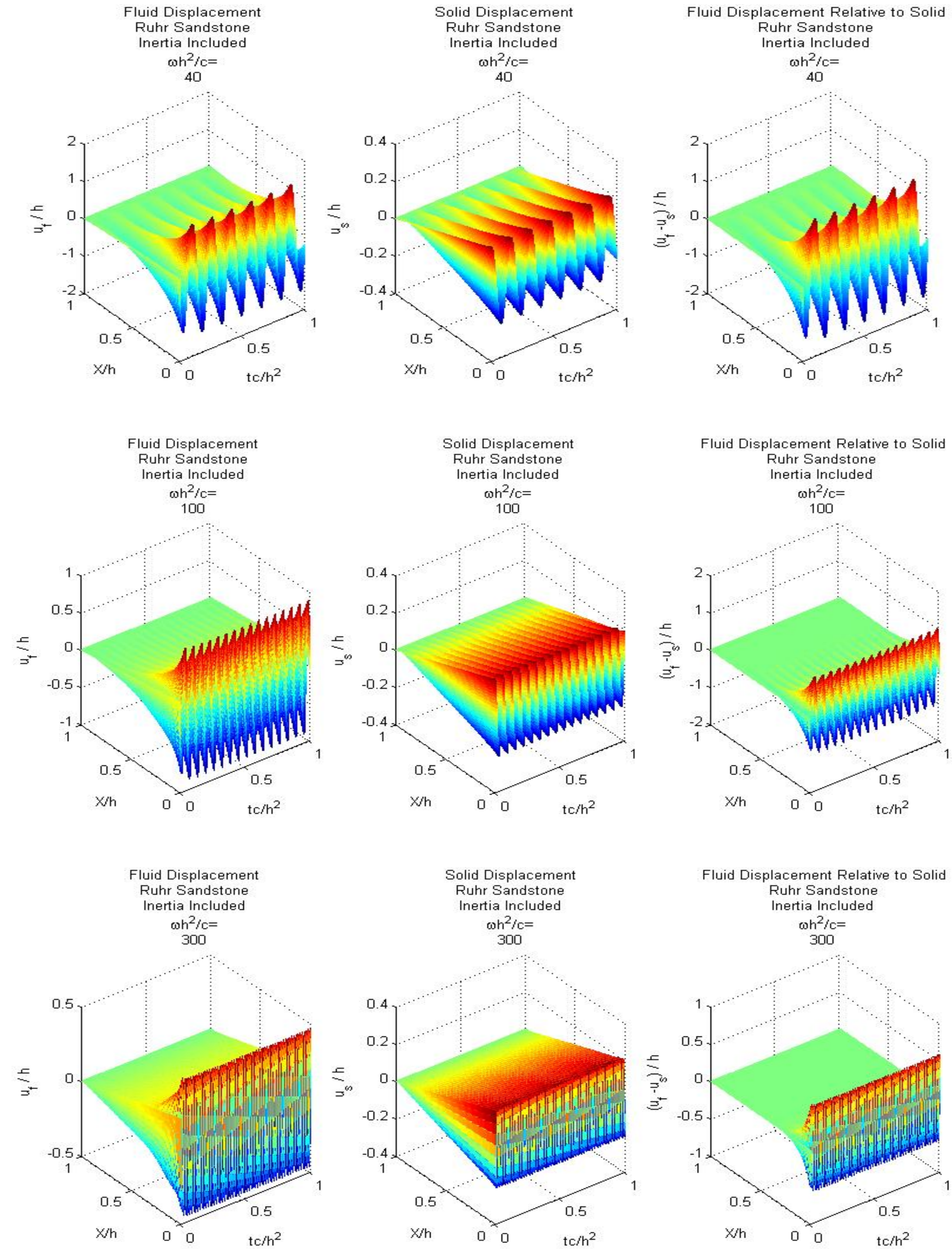
Next, we shall present a series of plots that attempt to show the similarities and the differences produced by the solution (10.8.6) and the inertia free solution of the same problem, equation (9.65). As we did in Section 10.7, we shall present the results in dimensionless forms using $\frac{\mathbf{w}(X,t)}{h}$ as the dimensionless distance, $\frac{tc}{h^2}$ as the dimensionless time and $\frac{X}{h}$ as the dimensionless distance. As explained in Section 10.7, the choice of the dimensionless time is somewhat arbitrary given the many possible choices. The physical problem simply does not naturally scale with this choice. It is done here so that we can make comparisons of the results with inertia free results in Section 9.6. A similar issue arises with our choice of the ratio $\frac{\omega h^2}{c}$ as the dimensionless frequency. Its choice is convenient when we compare our results to those in Section 9.6. Because the consolidation coefficient c for the poroelastic materials listed in Table 1 are small and because we are typically dealing with large values of the length h the dimensionless frequencies can be large for modest sized imposed frequencies ω. For example, if we select ω to be the frequency associated with the choice $n=1$ for the third and fourth roots for Ruhr Sandstone shown in Table 3 above, then $\omega = 68.12334$ yields a dimensionless frequency, $\frac{\omega h^2}{c}$, of approximately $1,228(10)^5$.

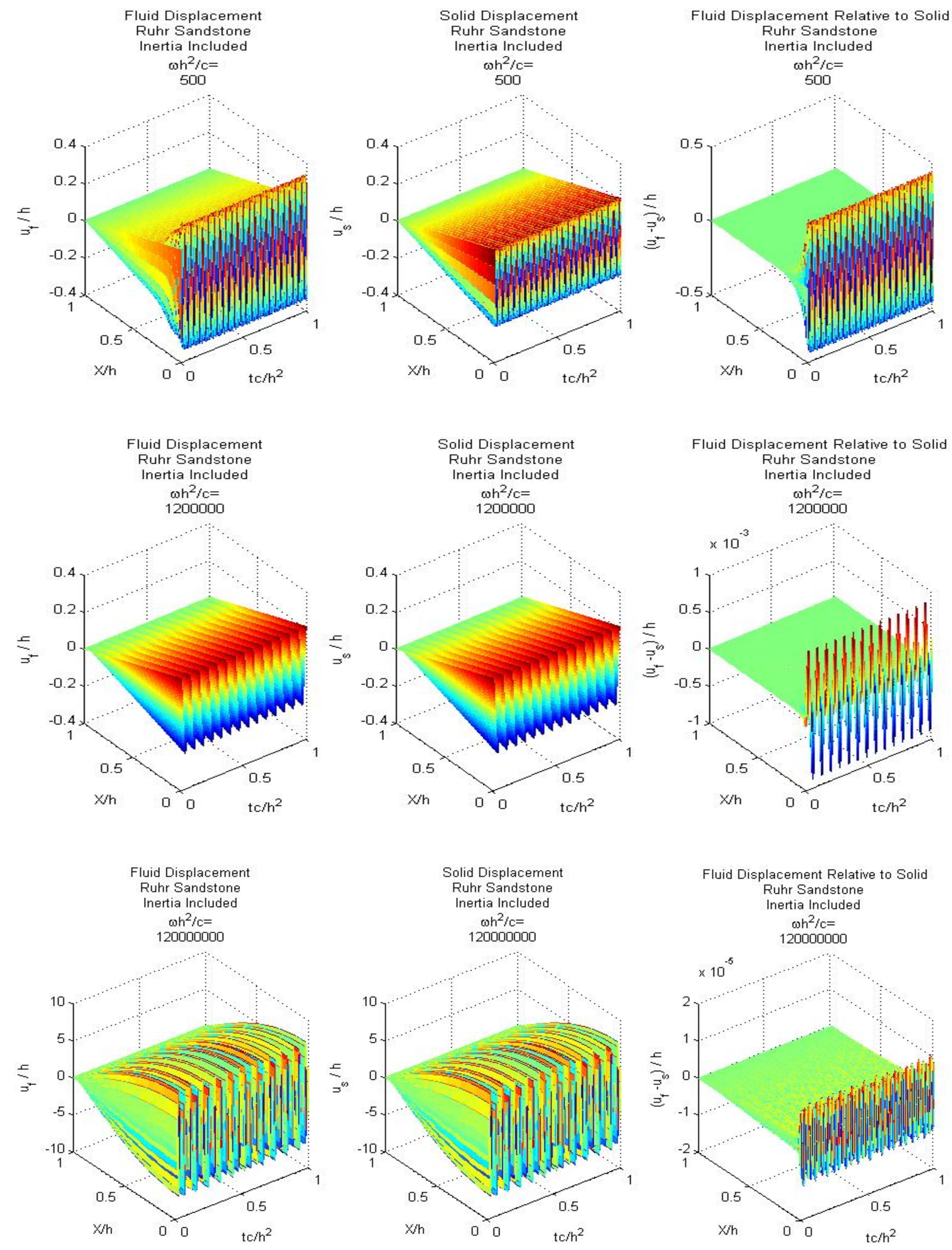
As in Section 9.6, we shall form plots of $\frac{\mathbf{w}(X,t)}{h}$ verses the dimensionless distance $\frac{X}{h}$ and the dimensionless time $\frac{tc}{h^2}$ for various values of the ratio $\frac{\omega h^2}{c}$. The range of values selected for the dimensionless frequencies will be large enough to have actual frequencies near the smaller natural frequencies for Ruhr Sandstone that show in Table 3.

The resulting plots of the fluid and solid displacements for several different choices of the ratio $\frac{\omega h^2}{c}$ are



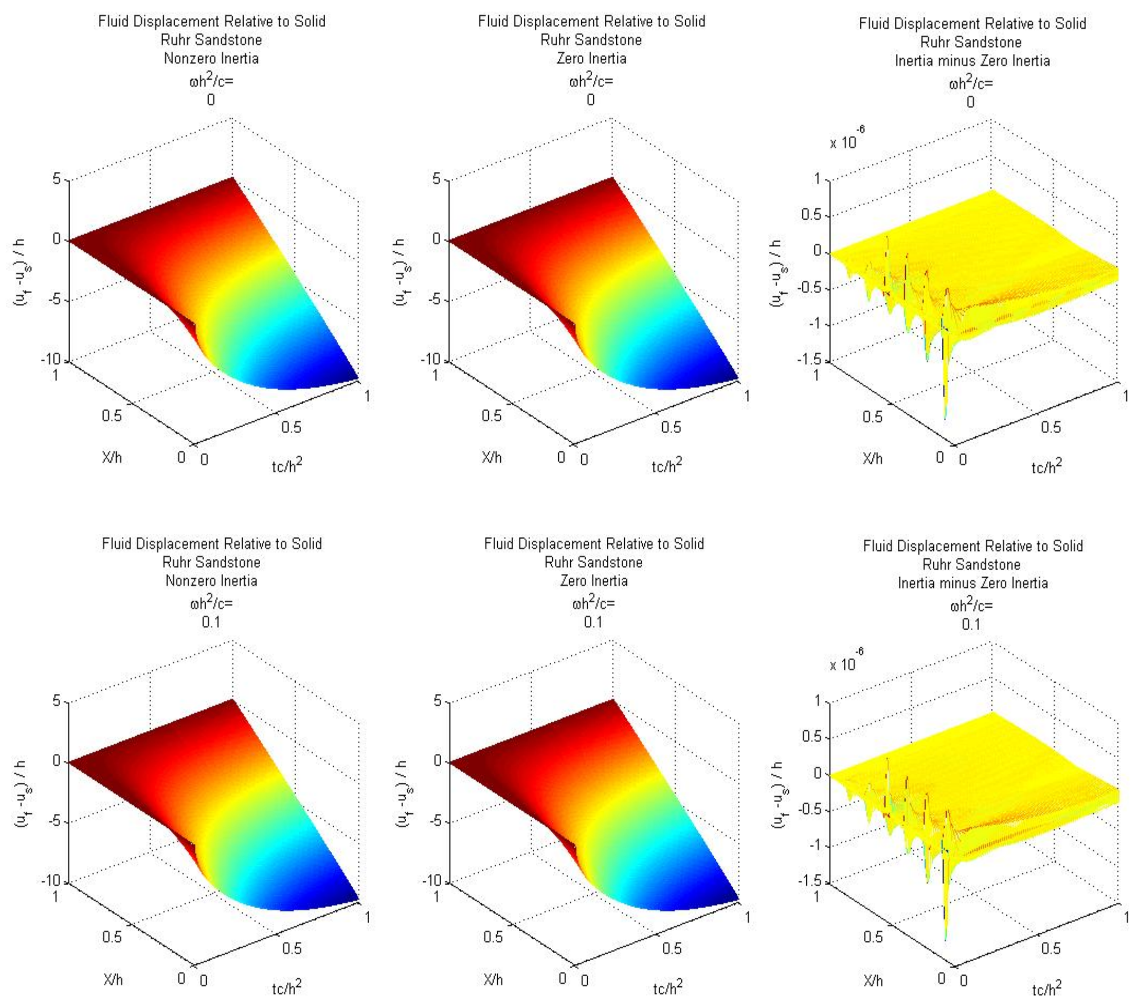

Fluid Displacement
Ruhr Sandstone
Inertia Included
ωh2/c=
1
Solid Displacement
Ruhr Sandstone
Inertia Included
ωh2/c=
1
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia Included
ωh2/c=
1
Fluid Displacement
Ruhr Sandstone
Inertia Included
ωh2/c=
10
Solid Displacement
Ruhr Sandstone
Inertia Included
ωh2/c=
10
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia Included
ωh2/c=
10
Fluid Displacement
Ruhr Sandstone
Inertia Included
ωh2/c=
20
Solid Displacement
Ruhr Sandstone
Inertia Included
ωh2/c=
20
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia Included
ωh2/c=
20
uf / h
us / h
(uf -us) / h
X/h
tc/h2

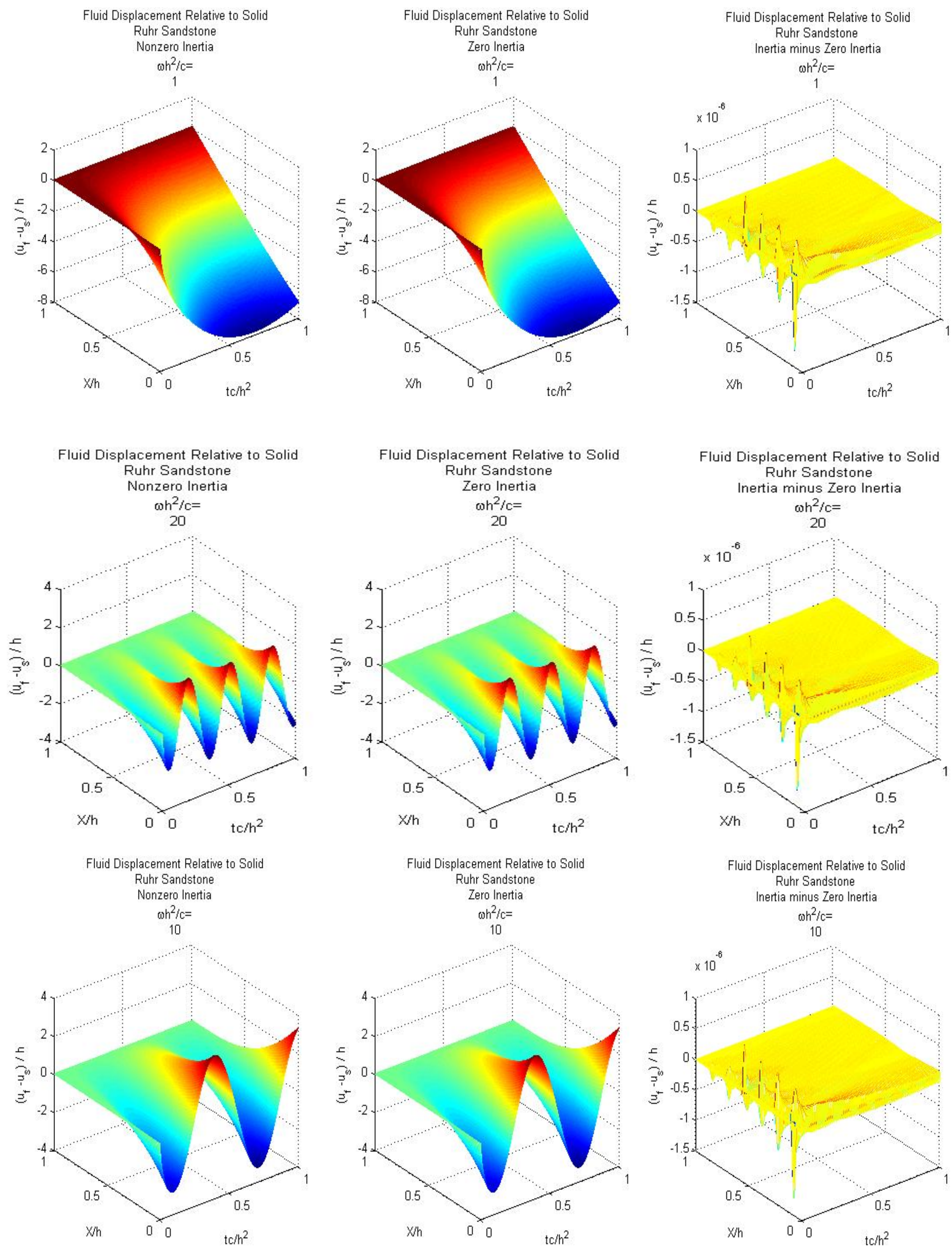

Fluid Displacement
Ruhr Sandstone
Inertia Included
ωh2/c=
40
u_f / h
X/h
tc/h2
Solid Displacement
Ruhr Sandstone
Inertia Included
ωh2/c=
40
u_s / h
X/h
tc/h2
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia Included
ωh2/c=
40
(u_f - u_s) / h
X/h
tc/h2
Fluid Displacement
Ruhr Sandstone
Inertia Included
ωh2/c=
100
u_f / h
X/h
tc/h2
Solid Displacement
Ruhr Sandstone
Inertia Included
ωh2/c=
100
u_s / h
X/h
tc/h2
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia Included
ωh2/c=
100
(u_f - u_s) / h
X/h
tc/h2
Fluid Displacement
Ruhr Sandstone
Inertia Included
ωh2/c=
300
u_f / h
X/h
tc/h2
Solid Displacement
Ruhr Sandstone
Inertia Included
ωh2/c=
300
u_s / h
X/h
tc/h2
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia Included
ωh2/c=
300
(u_f - u_s) / h
X/h
tc/h2

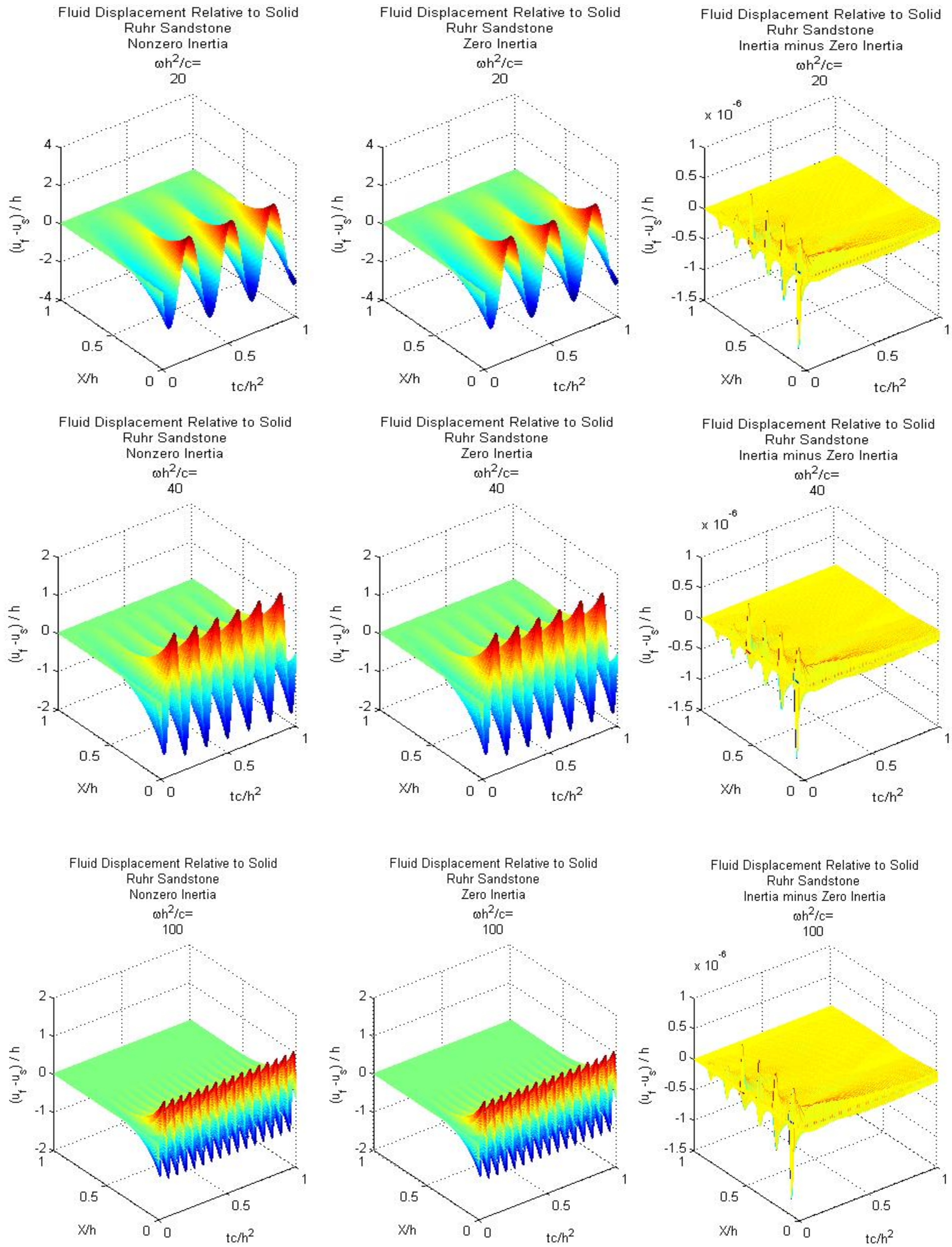


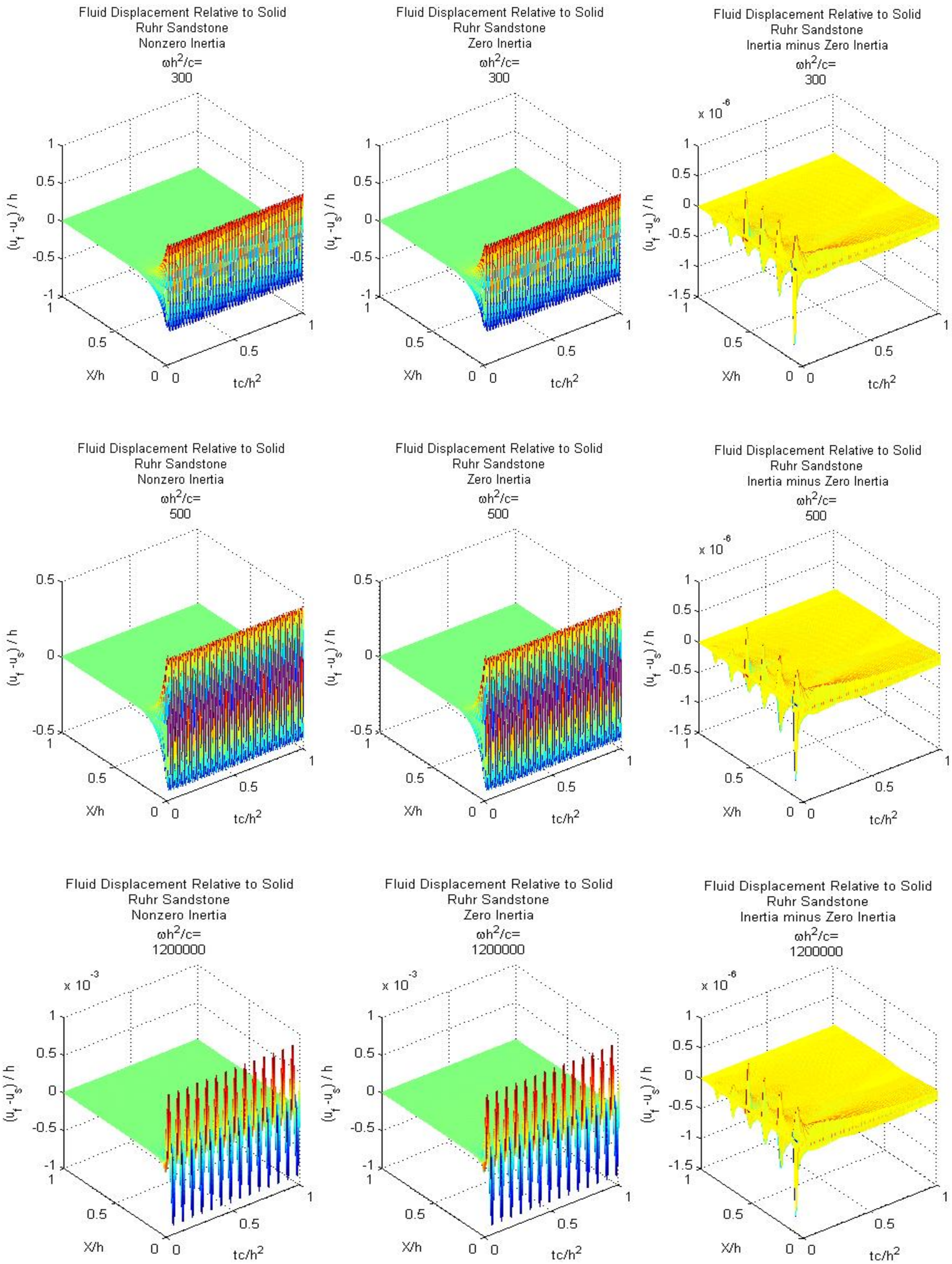
This series of figures suggests that for small values of the dimensionless frequency $\frac{\omega h^2}{c}$, the solution is not unlike that of the zero frequency cases shown in Section 9.6 when the inertia is zero and in Section 10.8 when the imposed frequency is zero. However as the imposed frequency increases to near the first natural frequency, $\omega = 68.12334$, the two displacements become large. At frequencies this large, the imposed frequency is near one of the natural frequencies. This typical resonance phenomena shows a circumstance where the inertia terms in the governing partial differential equations significantly alter the solution.

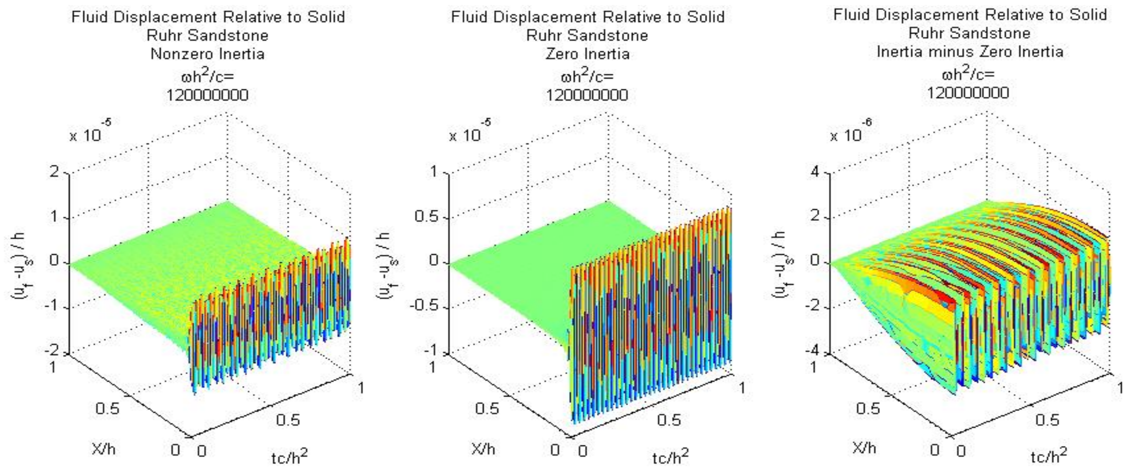
In reference 2, Bowen and Lockett look at the zero mass case as an approximation of the case with inertia included. They establish analytical expressions which, when satisfied, justify neglecting the inertia. They also show analytically the large displacements illustrated in the figures above. We shall examine some of these results later. It is interesting to numerically illustrate the effects of inertia by constructing plots of the solutions with inertia minus the same solutions without inertia. These difference plots illustrate differences that are not apparent in the figures above as well as those in Sections 9.6 and 10.7. Because of the physical significance of the relative displacements between the fluid and the solid, we shall plot these displacements with inertia, without inertia and then for the difference between these two plots. We shall display these plots for various external frequencies. The resulting plots are the following:




Fluid Displacement Relative to Solid
Ruhr Sandstone
Nonzero Inertia
ωh²/c=
1
Fluid Displacement Relative to Solid
Ruhr Sandstone
Zero Inertia
ωh²/c=
1
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia minus Zero Inertia
ωh²/c=
1
x 10^-6
Fluid Displacement Relative to Solid
Ruhr Sandstone
Nonzero Inertia
ωh²/c=
20
Fluid Displacement Relative to Solid
Ruhr Sandstone
Zero Inertia
ωh²/c=
20
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia minus Zero Inertia
ωh²/c=
20
x 10^-6
Fluid Displacement Relative to Solid
Ruhr Sandstone
Nonzero Inertia
ωh²/c=
10
Fluid Displacement Relative to Solid
Ruhr Sandstone
Zero Inertia
ωh²/c=
10
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia minus Zero Inertia
ωh²/c=
10
x 10^-6
(u_f -u_s) / h
X/h
tc/h²


Fluid Displacement Relative to Solid
Ruhr Sandstone
Nonzero Inertia
ωh²/c=
20
Fluid Displacement Relative to Solid
Ruhr Sandstone
Zero Inertia
ωh²/c=
20
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia minus Zero Inertia
ωh²/c=
20
x 10⁻⁶
Fluid Displacement Relative to Solid
Ruhr Sandstone
Nonzero Inertia
ωh²/c=
40
Fluid Displacement Relative to Solid
Ruhr Sandstone
Zero Inertia
ωh²/c=
40
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia minus Zero Inertia
ωh²/c=
40
x 10⁻⁶
Fluid Displacement Relative to Solid
Ruhr Sandstone
Nonzero Inertia
ωh²/c=
100
Fluid Displacement Relative to Solid
Ruhr Sandstone
Zero Inertia
ωh²/c=
100
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia minus Zero Inertia
ωh²/c=
100
x 10⁻⁶
(u_f -u_s) / h
X/h
tc/h²


Fluid Displacement Relative to Solid
Ruhr Sandstone
Nonzero Inertia
ωh²/c=
300
Fluid Displacement Relative to Solid
Ruhr Sandstone
Zero Inertia
ωh²/c=
300
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia minus Zero Inertia
ωh²/c=
300
x 10⁻⁶
Fluid Displacement Relative to Solid
Ruhr Sandstone
Nonzero Inertia
ωh²/c=
500
Fluid Displacement Relative to Solid
Ruhr Sandstone
Zero Inertia
ωh²/c=
500
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia minus Zero Inertia
ωh²/c=
500
x 10⁻⁶
Fluid Displacement Relative to Solid
Ruhr Sandstone
Nonzero Inertia
ωh²/c=
1200000
x 10⁻³
Fluid Displacement Relative to Solid
Ruhr Sandstone
Zero Inertia
ωh²/c=
1200000
x 10⁻³
Fluid Displacement Relative to Solid
Ruhr Sandstone
Inertia minus Zero Inertia
ωh²/c=
1200000
x 10⁻⁶
(u_f - u_s) / h
X/h
tc/h²



References

1. BOWEN, R. M., AND R. R. LOCKETT, Inertial Effects in Poroelasticity, *J. Appl. Mech.*, **50**, 334-342, 1983.
2. BELLMAN, R., Introduction to Matrix Analysis, *McGraw-Hill*, New York, 1960.

Appendix A

Roots of $\det\left(\beta_n^2\mathbf{M}+\beta_n\xi\mathbf{E}+\frac{(2n-1)^2\pi^2}{4h^2}\mathbf{Q}\right)=0$

Ruhr Sandstone

n	First Root: $\beta_{n(1)} = -\alpha_n$	Second Root: $\beta_{n(2)} = -\gamma_n$	Third Root: $\beta_{n(3)} = -\varsigma_n + i\omega_n$	Fourth Root: $\beta_{n(4)} = -\varsigma_n - i\omega_n$
1	-0.000001	-95745682.888539	68.12334i	-68.12334i
2	-0.000012	-95745682.888527	204.37002i	-204.37002i
3	-0.000034	-95745682.888504	-0.000001+340.6167i	-0.000001-340.6167i
4	-0.000067	-95745682.88847	-0.000002+476.86338i	-0.000002-476.86338i
5	-0.000111	-95745682.888423	-0.000003+613.11006i	-0.000003-613.11006i
6	-0.000166	-95745682.888366	-0.000004+749.35674i	-0.000004-749.35674i
7	-0.000231	-95745682.888297	-0.000006+885.60342i	-0.000006-885.60342i
8	-0.000308	-95745682.888216	-0.000008+1021.8501i	-0.000008-1021.8501i
9	-0.000396	-95745682.888124	-0.00001+1158.09678i	-0.00001-1158.09678i
10	-0.000494	-95745682.88802	-0.000013+1294.34346i	-0.000013-1294.34346i
11	-0.000604	-95745682.887905	-0.000016+1430.59014i	-0.000016-1430.59014i
12	-0.000724	-95745682.887778	-0.000019+1566.83682i	-0.000019-1566.83682i
13	-0.000856	-95745682.88764	-0.000022+1703.0835i	-0.000022-1703.0835i
14	-0.000998	-95745682.88749	-0.000026+1839.33018i	-0.000026-1839.33018i
15	-0.001152	-95745682.887329	-0.00003+1975.57686i	-0.00003-1975.57686i
16	-0.001316	-95745682.887156	-0.000034+2111.82354i	-0.000034-2111.82354i
17	-0.001491	-95745682.886972	-0.000038+2248.07022i	-0.000038-2248.07022i
18	-0.001678	-95745682.886776	-0.000043+2384.3169i	-0.000043-2384.3169i
19	-0.001875	-95745682.886569	-0.000048+2520.56358i	-0.000048-2520.56358i
20	-0.002083	-95745682.88635	-0.000054+2656.81026i	-0.000054-2656.81026i
21	-0.002302	-95745682.88612	-0.000059+2793.05694i	-0.000059-2793.05694i
22	-0.002532	-95745682.885878	-0.000065+2929.30362i	-0.000065-2929.30362i
23	-0.002773	-95745682.885624	-0.000071+3065.5503i	-0.000071-3065.5503i
24	-0.003025	-95745682.885359	-0.000078+3201.79698i	-0.000078-3201.79698i
25	-0.003288	-95745682.885083	-0.000085+3338.04366i	-0.000085-3338.04366i
26	-0.003562	-95745682.884795	-0.000092+3474.29034i	-0.000092-3474.29034i
27	-0.003847	-95745682.884495	-0.000099+3610.53702i	Appendix A
28	-0.004143	-95745682.884184	-0.000106+3746.7837i	-0.000106-3746.7837i
29	-0.00445	-95745682.883862	-0.000114+3883.03038i	-0.000114-3883.03038i
30	-0.004767	-95745682.883528	-0.000123+4019.27706i	-0.000123-4019.27706i
31	-0.005096	-95745682.883182	-0.000131+4155.52374i	-0.000131-4155.52374i
32	-0.005436	-95745682.882825	-0.00014+4291.77042i	-0.00014-4291.77042i
33	-0.005786	-95745682.882456	-0.000149+4428.0171i	-0.000149-4428.0171i
34	-0.006148	-95745682.882076	-0.000158+4564.26378i	-0.000158-4564.26378i
35	-0.00652	-95745682.881685	-0.000168+4700.51046i	-0.000168-4700.51046i
36	-0.006904	-95745682.881281	-0.000177+4836.75714i	-0.000177-4836.75714i
37	-0.007298	-95745682.880867	-0.000188+4973.00382i	-0.000188-4973.00382i
38	-0.007704	-95745682.88044	-0.000198+5109.2505i	-0.000198-5109.2505i
39	-0.00812	-95745682.880003	-0.000209+5245.49718i	-0.000209-5245.49718i
40	-0.008547	-95745682.879554	-0.00022+5381.74386i	-0.00022-5381.74386i
41	-0.008986	-95745682.879093	-0.000231+5517.99054i	-0.000231-5517.99054i
42	-0.009435	-95745682.87862	-0.000242+5654.23722i	-0.000242-5654.23722i
43	-0.009895	-95745682.878137	-0.000254+5790.4839i	-0.000254-5790.4839i
44	-0.010366	-95745682.877641	-0.000266+5926.73058i	-0.000266-5926.73058i
45	-0.010848	-95745682.877134	-0.000279+6062.97726i	-0.000279-6062.97726i
46	-0.011341	-95745682.876616	-0.000291+6199.22394i	-0.000291-6199.22394i
47	-0.011845	-95745682.876086	-0.000304+6335.47062i	-0.000304-6335.47062i
48	-0.01236	-95745682.875545	-0.000318+6471.7173i	-0.000318-6471.7173i
49	-0.012886	-95745682.874992	-0.000331+6607.96398i	-0.000331-6607.96398i
50	-0.013423	-95745682.874427	-0.000345+6744.21066i	-0.000345-6744.21066i

Tennessee Marble

n	First Root: $\beta_{n(1)} = -\alpha_n$	Second Root: $\beta_{n(2)} = -\gamma_n$	Third Root: $\beta_{n(3)} = -\varsigma_n + i\omega_n$	Fourth Root: $\beta_{n(4)} = -\varsigma_n - i\omega_n$
1	0	-197481481481.481	-83.964i	83.964i
2	-0.00000003	-197481481481.481	-251.8919i	251.8919i
3	-0.00000007	-197481481481.481	-419.8198i	419.8198i
4	-0.00000014	-197481481481.481	-587.7478i	587.7478i
5	-0.00000023	-197481481481.481	-755.6757i	755.6757i
6	-0.00000035	-197481481481.481	-923.6036i	923.6036i
7	-0.00000049	-197481481481.481	-1091.5315i	1091.5315i
8	-0.00000065	-197481481481.481	-1259.4595i	1259.4595i
9	-0.00000084	-197481481481.481	-1427.3874i	1427.3874i
10	-0.00000104	-197481481481.481	-1595.3153i	1595.3153i
11	-0.00000128	-197481481481.481	-1763.2433i	1763.2433i
12	-0.00000153	-197481481481.481	-1931.1712i	1931.1712i
13	-0.00000181	-197481481481.481	-2099.0991i	2099.0991i
14	-0.00000211	-197481481481.481	-2267.027i	2267.027i
15	-0.00000243	-197481481481.481	-2434.955i	2434.955i
16	-0.00000278	-197481481481.481	-2602.8829i	2602.8829i
17	-0.00000315	-197481481481.481	-2770.8108i	2770.8108i
18	-0.00000354	-197481481481.481	-2938.7388i	2938.7388i
19	-0.00000396	-197481481481.481	-3106.6667i	3106.6667i
20	-0.0000044	-197481481481.481	-3274.5946i	3274.5946i
21	-0.00000486	-197481481481.481	-3442.5226i	3442.5226i
22	-0.00000535	-197481481481.481	-3610.4505i	3610.4505i
23	-0.00000586	-197481481481.481	-3778.3784i	3778.3784i
24	-0.00000639	-197481481481.481	-3946.3063i	3946.3063i
25	-0.00000695	-197481481481.481	-4114.2343i	4114.2343i
26	-0.00000753	-197481481481.481	-4282.1622i	4282.1622i
27	-0.00000813	-197481481481.481	-4450.0901i	4450.0901i
28	-0.00000875	-197481481481.481	-4618.0181i	4618.0181i
29	-0.0000094	-197481481481.481	-4785.946i	4785.946i
30	-0.00001007	-197481481481.481	-4953.8739i	4953.8739i
31	-0.00001077	-197481481481.481	-5121.8018i	5121.8018i
32	-0.00001149	-197481481481.481	-5289.7298i	5289.7298i
33	-0.00001223	-197481481481.481	-5457.6577i	5457.6577i
34	-0.00001299	-197481481481.481	-5625.5856i	5625.5856i
35	-0.00001378	-197481481481.481	-5793.5136i	5793.5136i
36	-0.00001459	-197481481481.481	-5961.4415i	5961.4415i
37	-0.00001542	-197481481481.481	-6129.3694i	6129.3694i
38	-0.00001628	-197481481481.481	-6297.2974i	6297.2974i
39	-0.00001716	-197481481481.481	-6465.2253i	6465.2253i
40	-0.00001806	-197481481481.481	-6633.1532i	6633.1532i
41	-0.00001899	-197481481481.481	-6801.0811i	6801.0811i
42	-0.00001994	-197481481481.481	-6969.0091i	6969.0091i
43	-0.00002091	-197481481481.481	-7136.937i	7136.937i
44	-0.0000219	-197481481481.481	-7304.8649i	7304.8649i
45	-0.00002292	-197481481481.481	-7472.7929i	7472.7929i
46	-0.00002396	-197481481481.481	-7640.7208i	7640.7208i
47	-0.00002503	-197481481481.481	-7808.6487i	7808.6487i
48	-0.00002612	-197481481481.481	-7976.5766i	7976.5766i
49	-0.00002723	-197481481481.481	-8144.5046i	8144.5046i
50	-0.00002836	-197481481481.481	-8312.4325i	8312.4325i

Charcoal Granite

n	First Root: $\beta_{n(1)} = -\alpha_n$	Second Root: $\beta_{n(2)} = -\gamma_n$	Third Root: $\beta_{n(3)} = -\varsigma_n + i\omega_n$	Fourth Root: $\beta_{n(4)} = -\varsigma_n - i\omega_n$
1	0	-197481481481.481	-77.8296i	77.8296i
2	-0.00000002	-197481481481.481	-233.4887i	233.4887i
3	-0.00000005	-197481481481.481	-389.1478i	389.1478i
4	-0.00000009	-197481481481.481	-544.807i	544.807i
5	-0.00000015	-197481481481.481	-700.4661i	700.4661i
6	-0.00000022	-197481481481.481	-856.1253i	856.1253i
7	-0.00000031	-197481481481.481	-1011.7844i	1011.7844i
8	-0.00000042	-197481481481.481	-1167.4435i	1167.4435i
9	-0.00000054	-197481481481.481	-1323.1027i	1323.1027i
10	-0.00000067	-197481481481.481	-1478.7618i	1478.7618i
11	-0.00000082	-197481481481.481	-1634.421i	1634.421i
12	-0.00000098	-197481481481.481	-1790.0801i	1790.0801i
13	-0.00000116	-197481481481.481	-1945.7392i	1945.7392i
14	-0.00000136	-197481481481.481	-2101.3984i	2101.3984i
15	-0.00000156	-197481481481.481	-2257.0575i	2257.0575i
16	-0.00000179	-197481481481.481	-2412.7167i	2412.7167i
17	-0.00000202	-197481481481.481	-2568.3758i	2568.3758i
18	-0.00000228	-197481481481.481	-2724.0349i	2724.0349i
19	-0.00000255	-197481481481.481	-2879.6941i	2879.6941i
20	-0.00000283	-197481481481.481	-3035.3532i	3035.3532i
21	-0.00000313	-197481481481.481	-3191.0124i	3191.0124i
22	-0.00000344	-197481481481.481	-3346.6715i	3346.6715i
23	-0.00000377	-197481481481.481	-3502.3306i	3502.3306i
24	-0.00000411	-197481481481.481	-3657.9898i	3657.9898i
25	-0.00000446	-197481481481.481	-3813.6489i	3813.6489i
26	-0.00000484	-197481481481.481	-3969.3081i	3969.3081i
27	-0.00000522	-197481481481.481	-4124.9672i	4124.9672i
28	-0.00000562	-197481481481.481	-4280.6263i	4280.6263i
29	-0.00000604	-197481481481.481	-4436.2855i	4436.2855i
30	-0.00000647	-197481481481.481	-4591.9446i	4591.9446i
31	-0.00000692	-197481481481.481	-4747.6038i	4747.6038i
32	-0.00000738	-197481481481.481	-4903.2629i	4903.2629i
33	-0.00000786	-197481481481.481	-5058.922i	5058.922i
34	-0.00000835	-197481481481.481	-5214.5812i	5214.5812i
35	-0.00000885	-197481481481.481	-5370.2403i	5370.2403i
36	-0.00000937	-197481481481.481	-5525.8995i	5525.8995i
37	-0.00000991	-197481481481.481	-5681.5586i	5681.5586i
38	-0.00001046	-197481481481.481	-5837.2177i	5837.2177i
39	-0.00001102	-197481481481.481	-5992.8769i	5992.8769i
40	-0.0000116	-197481481481.481	-6148.536i	6148.536i
41	-0.0000122	-197481481481.481	-6304.1952i	6304.1952i
42	-0.00001281	-197481481481.481	-6459.8543i	6459.8543i
43	-0.00001343	-197481481481.481	-6615.5134i	6615.5134i
44	-0.00001407	-197481481481.481	-6771.1726i	6771.1726i
45	-0.00001473	-197481481481.481	-6926.8317i	6926.8317i
46	-0.0000154	-197481481481.481	-7082.4909i	7082.4909i
47	-0.00001608	-197481481481.481	-7238.15i	7238.15i
48	-0.00001678	-197481481481.481	-7393.8091i	7393.8091i
49	-0.00001749	-197481481481.481	-7549.4683i	7549.4683i
50	-0.00001822	-197481481481.481	-7705.1274i	7705.1274i

Berea Sandstone

n	First Root: $\beta_{n(1)} = -\alpha_n$	Second Root: $\beta_{n(2)} = -\gamma_n$	Third Root: $\beta_{n(3)} = -\varsigma_n + i\omega_n$	Fourth Root: $\beta_{n(4)} = -\varsigma_n - i\omega_n$
1	-0.00041905	-1015624.531	-50.4107i	50.4107i
2	-0.00377149	-1015624.5276	-151.232i	151.232i
3	-0.01047637	-1015624.5209	-252.0534i	252.0534i
4	-0.02053369	-1015624.5108	-352.8747i	352.8747i
5	-0.03394345	-1015624.4974	-453.6961i	453.6961i
6	-0.05070564	-1015624.4806	-554.5174i	554.5174i
7	-0.07082028	-1015624.4605	-0.0001-655.3388i	-0.0001+655.3388i
8	-0.09428736	-1015624.437	-0.0001-756.1601i	-0.0001+756.1601i
9	-0.12110688	-1015624.4101	-0.0001-856.9815i	-0.0001+856.9815i
10	-0.15127884	-1015624.3799	-0.0001-957.8028i	-0.0001+957.8028i
11	-0.18480324	-1015624.3463	-0.0002-1058.6241i	-0.0002+1058.6241i
12	-0.22168009	-1015624.3093	-0.0002-1159.4455i	-0.0002+1159.4455i
13	-0.26190938	-1015624.269	-0.0002-1260.2668i	-0.0002+1260.2668i
14	-0.30549111	-1015624.2254	-0.0003-1361.0882i	-0.0003+1361.0882i
15	-0.35242529	-1015624.1783	-0.0003-1461.9095i	-0.0003+1461.9095i
16	-0.40271191	-1015624.128	-0.0004-1562.7309i	-0.0004+1562.7309i
17	-0.45635099	-1015624.0742	-0.0004-1663.5522i	-0.0004+1663.5522i
18	-0.51334251	-1015624.0171	-0.0005-1764.3736i	-0.0005+1764.3736i
19	-0.57368648	-1015623.9567	-0.0005-1865.1949i	-0.0005+1865.1949i
20	-0.6373829	-1015623.8929	-0.0006-1966.0163i	-0.0006+1966.0163i
21	-0.70443177	-1015623.8257	-0.0006-2066.8376i	-0.0006+2066.8376i
22	-0.77483309	-1015623.7552	-0.0007-2167.659i	-0.0007+2167.659i
23	-0.84858687	-1015623.6813	-0.0008-2268.4803i	-0.0008+2268.4803i
24	-0.92569311	-1015623.604	-0.0008-2369.3017i	-0.0008+2369.3017i
25	-1.0061518	-1015623.5234	-0.0009-2470.123i	-0.0009+2470.123i
26	-1.08996295	-1015623.4395	-0.001-2570.9444i	-0.001+2570.9444i
27	-1.17712657	-1015623.3521	-0.0011-2671.7657i	-0.0011+2671.7657i
28	-1.26764264	-1015623.2615	-0.0012-2772.5871i	-0.0012+2772.5871i
29	-1.36151118	-1015623.1674	-0.0012-2873.4084i	-0.0012+2873.4084i
30	-1.45873219	-1015623.07	-0.0013-2974.2297i	-0.0013+2974.2297i
31	-1.55930566	-1015622.9693	-0.0014-3075.0511i	-0.0014+3075.0511i
32	-1.66323161	-1015622.8652	-0.0015-3175.8724i	-0.0015+3175.8724i
33	-1.77051002	-1015622.7577	-0.0016-3276.6938i	-0.0016+3276.6938i
34	-1.88114091	-1015622.6469	-0.0017-3377.5151i	-0.0017+3377.5151i
35	-1.99512428	-1015622.5327	-0.0018-3478.3365i	-0.0018+3478.3365i
36	-2.11246013	-1015622.4151	-0.0019-3579.1578i	-0.0019+3579.1578i
37	-2.23314845	-1015622.2942	-0.002-3679.9792i	-0.002+3679.9792i
38	-2.35718926	-1015622.1699	-0.0021-3780.8005i	-0.0021+3780.8005i
39	-2.48458256	-1015622.0423	-0.0023-3881.6219i	-0.0023+3881.6219i
40	-2.61532834	-1015621.9113	-0.0024-3982.4432i	-0.0024+3982.4432i
41	-2.74942662	-1015621.777	-0.0025-4083.2646i	-0.0025+4083.2646i
42	-2.88687739	-1015621.6393	-0.0026-4184.0859i	-0.0026+4184.0859i
43	-3.02768065	-1015621.4982	-0.0027-4284.9073i	-0.0027+4284.9073i
44	-3.17183642	-1015621.3538	-0.0029-4385.7286i	-0.0029+4385.7286i
45	-3.31934468	-1015621.206	-0.003-4486.55i	-0.003+4486.55i
46	-3.47020545	-1015621.0549	-0.0032-4587.3713i	-0.0032+4587.3713i
47	-3.62441873	-1015620.9004	-0.0033-4688.1927i	-0.0033+4688.1927i
48	-3.78198452	-1015620.7426	-0.0034-4789.014i	-0.0034+4789.014i
49	-3.94290283	-1015620.5813	-0.0036-4889.8354i	-0.0036+4889.8354i
50	-4.10717365	-1015620.4168	-0.0037-4990.6567i	-0.0037+4990.6567i

Westerly Granite

n	First Root: $\beta_{n(1)} = -\alpha_n$	Second Root: $\beta_{n(2)} = -\gamma_n$	Third Root: $\beta_{n(3)} = -\varsigma_n + i\omega_n$	Fourth Root: $\beta_{n(4)} = -\varsigma_n - i\omega_n$
1	-0.00000001	-24591657313.8795	-75.434i	75.434i
2	-0.00000005	-24591657313.8795	-226.302i	226.302i
3	-0.00000014	-24591657313.8795	-377.17i	377.17i
4	-0.00000027	-24591657313.8795	-528.038i	528.038i
5	-0.00000045	-24591657313.8795	-678.906i	678.906i
6	-0.00000067	-24591657313.8795	-829.774i	829.774i
7	-0.00000093	-24591657313.8795	-980.642i	980.642i
8	-0.00000124	-24591657313.8795	-1131.51i	1131.51i
9	-0.00000159	-24591657313.8795	-1282.378i	1282.378i
10	-0.00000199	-24591657313.8795	-1433.246i	1433.246i
11	-0.00000243	-24591657313.8795	-1584.114i	1584.114i
12	-0.00000291	-24591657313.8795	-1734.982i	1734.982i
13	-0.00000344	-24591657313.8795	-1885.85i	1885.85i
14	-0.00000401	-24591657313.8795	-2036.718i	2036.718i
15	-0.00000462	-24591657313.8795	-2187.586i	2187.586i
16	-0.00000528	-24591657313.8795	-2338.454i	2338.454i
17	-0.00000599	-24591657313.8795	-2489.3221i	2489.3221i
18	-0.00000674	-24591657313.8795	-2640.1901i	2640.1901i
19	-0.00000753	-24591657313.8795	-2791.0581i	2791.0581i
20	-0.00000836	-24591657313.8795	-2941.9261i	2941.9261i
21	-0.00000924	-24591657313.8795	-3092.7941i	3092.7941i
22	-0.00001017	-24591657313.8795	-3243.6621i	3243.6621i
23	-0.00001114	-24591657313.8795	-3394.5301i	3394.5301i
24	-0.00001215	-24591657313.8795	-3545.3981i	3545.3981i
25	-0.0000132	-24591657313.8795	-3696.2661i	3696.2661i
26	-0.0000143	-24591657313.8795	-3847.1341i	3847.1341i
27	-0.00001545	-24591657313.8795	-3998.0021i	3998.0021i
28	-0.00001663	-24591657313.8795	-4148.8701i	4148.8701i
29	-0.00001787	-24591657313.8795	-4299.7381i	4299.7381i
30	-0.00001914	-24591657313.8795	-4450.6061i	4450.6061i
31	-0.00002046	-24591657313.8795	-4601.4741i	4601.4741i
32	-0.00002183	-24591657313.8795	-4752.3421i	4752.3421i
33	-0.00002323	-24591657313.8795	-4903.2101i	4903.2101i
34	-0.00002468	-24591657313.8795	-5054.0781i	5054.0781i
35	-0.00002618	-24591657313.8795	-5204.9461i	5204.9461i
36	-0.00002772	-24591657313.8795	-5355.8141i	5355.8141i
37	-0.0000293	-24591657313.8795	-5506.6821i	5506.6821i
38	-0.00003093	-24591657313.8795	-5657.5501i	5657.5501i
39	-0.0000326	-24591657313.8795	-5808.4181i	5808.4181i
40	-0.00003432	-24591657313.8795	-5959.2861i	5959.2861i
41	-0.00003608	-24591657313.8795	-6110.1541i	6110.1541i
42	-0.00003788	-24591657313.8795	-6261.0221i	6261.0221i
43	-0.00003973	-24591657313.8795	-6411.8901i	6411.8901i
44	-0.00004162	-24591657313.8795	-6562.7581i	6562.7581i
45	-0.00004356	-24591657313.8795	-6713.6261i	6713.6261i
46	-0.00004554	-24591657313.8795	-6864.4941i	6864.4941i
47	-0.00004756	-24591657313.8795	-7015.3621i	7015.3621i
48	-0.00004963	-24591657313.8795	-7166.2302i	7166.2302i
49	-0.00005174	-24591657313.8795	-7317.0982i	7317.0982i
50	-0.00005389	-24591657313.8795	-7467.9662i	7467.9662i

Weber Sandstone

n	First Root: $\beta_{n(1)} = -\alpha_n$	Second Root: $\beta_{n(2)} = -\gamma_n$	Third Root: $\beta_{n(3)} = -\varsigma_n + i\omega_n$	Fourth Root: $\beta_{n(4)} = -\varsigma_n - i\omega_n$
1	-0.00005144	-6061920.3491	-63.7532i	63.7532i
2	-0.00046298	-6061920.3487	-191.2597i	191.2597i
3	-0.00128605	-6061920.3479	-318.7661i	318.7661i
4	-0.00252066	-6061920.3466	-446.2726i	446.2726i
5	-0.00416681	-6061920.345	-573.779i	573.779i
6	-0.0062245	-6061920.3429	-701.2855i	701.2855i
7	-0.00869372	-6061920.3404	-828.7919i	828.7919i
8	-0.01157448	-6061920.3375	-956.2984i	956.2984i
9	-0.01486677	-6061920.3342	-1083.8048i	1083.8048i
10	-0.01857061	-6061920.3305	-1211.3113i	1211.3113i
11	-0.02268598	-6061920.3264	-1338.8177i	1338.8177i
12	-0.02721288	-6061920.3218	-1466.3242i	1466.3242i
13	-0.03215133	-6061920.3169	-0.0001-1593.8306i	-0.0001+1593.8306i
14	-0.03750131	-6061920.3115	-0.0001-1721.3371i	-0.0001+1721.3371i
15	-0.04326283	-6061920.3057	-0.0001-1848.8435i	-0.0001+1848.8435i
16	-0.04943588	-6061920.2995	-0.0001-1976.35i	-0.0001+1976.35i
17	-0.05602047	-6061920.2929	-0.0001-2103.8564i	-0.0001+2103.8564i
18	-0.0630166	-6061920.2859	-0.0001-2231.3629i	-0.0001+2231.3629i
19	-0.07042427	-6061920.2785	-0.0001-2358.8693i	-0.0001+2358.8693i
20	-0.07824347	-6061920.2706	-0.0001-2486.3758i	-0.0001+2486.3758i
21	-0.08647421	-6061920.2624	-0.0002-2613.8822i	-0.0002+2613.8822i
22	-0.09511649	-6061920.2537	-0.0002-2741.3887i	-0.0002+2741.3887i
23	-0.10417031	-6061920.2446	-0.0002-2868.8951i	-0.0002+2868.8951i
24	-0.11363566	-6061920.2351	-0.0002-2996.4016i	-0.0002+2996.4016i
25	-0.12351254	-6061920.2252	-0.0002-3123.908i	-0.0002+3123.908i
26	-0.13380097	-6061920.2149	-0.0002-3251.4145i	-0.0002+3251.4145i
27	-0.14450093	-6061920.2041	-0.0003-3378.9209i	-0.0003+3378.9209i
28	-0.15561243	-6061920.193	-0.0003-3506.4274i	-0.0003+3506.4274i
29	-0.16713547	-6061920.1814	-0.0003-3633.9338i	-0.0003+3633.9338i
30	-0.17907004	-6061920.1694	-0.0003-3761.4403i	-0.0003+3761.4403i
31	-0.19141615	-6061920.157	-0.0004-3888.9467i	-0.0004+3888.9467i
32	-0.2041738	-6061920.1442	-0.0004-4016.4532i	-0.0004+4016.4532i
33	-0.21734299	-6061920.131	-0.0004-4143.9596i	-0.0004+4143.9596i
34	-0.23092371	-6061920.1174	-0.0004-4271.4661i	-0.0004+4271.4661i
35	-0.24491597	-6061920.1033	-0.0004-4398.9725i	-0.0004+4398.9725i
36	-0.25931976	-6061920.0889	-0.0005-4526.479i	-0.0005+4526.479i
37	-0.2741351	-6061920.074	-0.0005-4653.9854i	-0.0005+4653.9854i
38	-0.28936197	-6061920.0587	-0.0005-4781.4919i	-0.0005+4781.4919i
39	-0.30500037	-6061920.043	-0.0006-4908.9983i	-0.0006+4908.9983i
40	-0.32105032	-6061920.0269	-0.0006-5036.5048i	-0.0006+5036.5048i
41	-0.3375118	-6061920.0104	-0.0006-5164.0112i	-0.0006+5164.0112i
42	-0.35438482	-6061919.9935	-0.0006-5291.5177i	-0.0006+5291.5177i
43	-0.37166938	-6061919.9761	-0.0007-5419.0241i	-0.0007+5419.0241i
44	-0.38936547	-6061919.9584	-0.0007-5546.5306i	-0.0007+5546.5306i
45	-0.4074731	-6061919.9402	-0.0007-5674.037i	-0.0007+5674.037i
46	-0.42599227	-6061919.9216	-0.0008-5801.5435i	-0.0008+5801.5435i
47	-0.44492297	-6061919.9026	-0.0008-5929.0499i	-0.0008+5929.0499i
48	-0.46426521	-6061919.8832	-0.0009-6056.5564i	-0.0009+6056.5564i
49	-0.48401899	-6061919.8634	-0.0009-6184.0628i	-0.0009+6184.0628i
50	-0.50418431	-6061919.8431	-0.0009-6311.5693i	-0.0009+6311.5693i

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