Hypertrapezoidal Fuzzy Membership Functions

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Abstract

The authors present a method for representing N-dimensional fuzzy membership functions. The proposed method is a generalization of the one-dimensional trapezoidal membership function commonly used in fuzzy systems. The issue of correlation between input variables and a decrease in the rule base size is the motivation for extending the definition of membership functions into more than one domain. The approach outlined in this paper is focused by practical considerations and use of a Bayesian version of fuzzy logic which requires that set membership sum to one. The fuzzy partitioning which stems from the presented method is parameterized by M+1 values, yielding an efficient mechanism for designing complex fuzzy systems.

Introduction

The current state of the art in fuzzy logic system design is the use of fuzzy membership functions which are defined in a single domain, i.e. functions of one variable. These fuzzy membership functions define the degrees of membership that a crisp value has in a fuzzy set. In practice, the membership functions are also either trapezoidal or triangular as in Figure 1. For simple applications this scheme is adequate, easy to design, prototype and adjust.

However, the use of one-dimensional membership functions has proved to be inadequate in practical situations involving complex systems. In particular, previous work in the area of flight mode identification for an "automated pilot advisor" has revealed this inadequacy. Lass [5] describes "as long and tedious" the use of one-dimensional membership functions for maintaining an airplane's state within a particular operating region. He further concludes that fifty-two rules would have been needed to describe one operating mode on a two-dimensional state space. Harral [4] showed that in areas like flight mode identification the real problem for one-dimensional membership functions is the high amount of correlation between the measurable inputs.

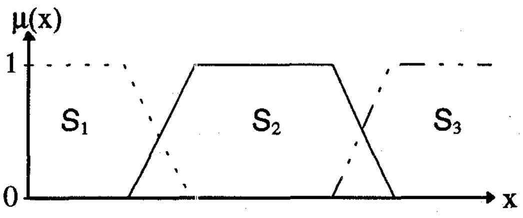

Figure 1. Typical 1D membership functions.

By "correlation" is meant the condition that an aircraft operating mode, say, is represented by irregular, smoothly connected region in a multivariable state-space. The "footprint" of a mode on the x-y plane could look something like Figure 2. One-dimensional membership functions cannot by themselves represent such a relationship. The current practice approximates a smooth representation by composition of two or more single-variable regions. Such a composition is shown in dashed lines on Figure 2. A better approximation would require that each axis be partitioned into more one-dimensional fuzzy sets. However, with additional fuzzy sets come a larger rule base.

The composition of single-variable fuzzy sets into multivariable sets requires a rule base. One-dimensional fuzzy sets are defined on each input space. The typical rules might resemble "If x is HIGH and y is LOW then z is SHORT". These rules can define the system output or an intermediary set used for multilevel rule-bases. The accuracy of the approximation depends on the number of fuzzy sets defined on each input, and the particular connectives used.

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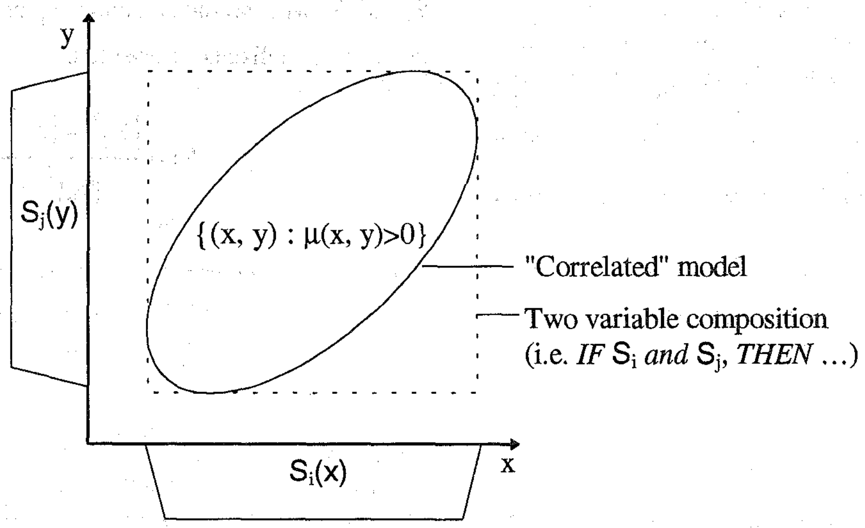

Figure 2. Footprint of fuzzy set when the input variables are correlated.

For systems with correlation between two variables, x and y, the one-dimensional membership functions describing a set A are of the form $\mu_A(x|y=Y)$ and $\mu_A(y|x=X)$. Correlation of membership can also be approximated by designing conditional membership functions. This is a brute force method of specifying $\mu_A(x|y=Y_k)$ for k=1,2,...K. This requires K one-dimensional fuzzy set definitions for *each* set or mode. In the case that $Y_j < y = Y^* < Y_{j+1}$, an interpolation must be done to approximate $\mu_A(x|y=Y^*)$. This method has been successful [5] but lacks efficiency and is arduous to tune.

Our scheme is a utilitarian multidimensional representation.

The "Sum To One" Design Criteria

Our present work in knowledge-based control [6] admits the alternate fuzzy logic connectives, originally compared by Bellman and Zadeh [1]. These are the usual connectives of the Bayes version of fuzzy logic [7], wherein the membership values sum to unity. That is, for membership functions $\mu_i(x)$,

$$\sum_i \mu_i(x) = 1; \quad \forall x \qquad (1)$$

Membership functions defined in such a manner are referred to as a fuzzy partitioning.

Fuzzy membership functions based on Gaussian probability density functions can easily be extended to N-dimensions. Multidimensional Gaussian membership functions have proved especially useful in the area of clustering [1] and training [8]. However, membership functions based on Gaussian densities generally do not exhibit the desirable property of equation (1). Trapezoidal membership functions, on the other hand, can easily be defined with the design constraint of equation (1).

Hypertrapezoidal Fuzzy Membership Functions

The standard method for defining one-dimensional trapezoidal membership functions is with four points -- a, b, c, and d, as shown in Figure 3. This method, however, is impracticable for defining membership functions on multiple dimensions. Therefore, we propose the hypertrapezoidal membership function as a utilitarian scheme for defining multidimensional membership function.

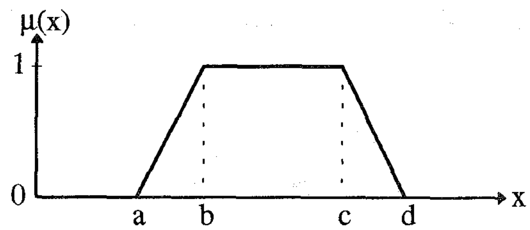

Figure 3. Defining an one-dimensional trapezoidal membership function.

The complete trigonometric derivation of hypertrapezoidal membership functions is straightforward and is not shown here due to space constraints. The derivation consists of four steps: the definition of the prototype points and a "crispness factor", the definition of a relative distance measure, the calculation of a conditional membership function, and the final composition.

Hypertrapezoidal membership functions are defined by prototype points and a crispness factor. In a fuzzy partitioning of an N-dimensional space, let each fuzzy set, S_i, be defined by a prototype point, λ_i. Furthermore, let the partitioning of the space also be parameterized by a crispness factor, σ. The prototype point, λ_i, has a degree of membership in set, S_i, of $\mu_i(\lambda_i)=1$ and a degree of membership in set S_j, of $\mu_j(\lambda_i)=0$ where $j \neq i$.

The crispness factor, $0 \le \sigma \le 1$, determines how much ambiguity exists between the sets of the partitioning. For $\sigma = 1$, no fuzziness exists between the sets and the partitioning is equivalent to a minimum distance classifier. For fuzzy sets, $\sigma < 1$. One way to define the crispness factor is using Figure 4 and equation (2).

$$\sigma = \frac{2\alpha}{d} \tag{2}$$

The crispness factor establishes how much of the space between the prototype points is fuzzy. The prototype points are chosen as ideal representatives of each fuzzy set. Then, the designer's selection of σ specifies the ratio of α and d. See Figure 5 for one-dimensional examples.

The second step in the derivation is the definition of an appropriate distance measure relating the distance from the crisp input to two prototype points. This distance measure is a ratio of the distance between two prototype points, and the difference in the distances from the crisp input to the two prototype points. For fuzzy sets S_i and S_j, with prototype points λ_i and λ_j, and a crisp input Λ, that distance measure is

$$\rho_{i|j}(\Lambda) = \frac{|\bar{v}_i|^2 - |\bar{v}_j|^2}{|\bar{v}_{ij}|^2}, \tag{3}$$

where $\bar{v}_{ij}$ is a vector from λ_i to λ_j; $\bar{v}_i$ is a vector from λ_i to Λ; and $\bar{v}_j$ is a vector from λ_j to Λ. This distance measure is used to determine if the crisp input Λ lies completely in fuzzy set i, or completely in fuzzy set j, or in the fuzzy region between the two sets.

The third step in the derivation of hypertrapezoidal membership functions is determining the degree of membership that Λ has in set i, given that set j is the only other set in the partition. Suppose fuzzy sets i and j are the only two sets defined in an N-dimensional space. Using the distance measure of equation (3), that degree of membership is

$$\mu_{i|j}(\Lambda) = \left\{ \begin{array}{ll} 0; & \rho_{i|j}(\Lambda) \ge 1-\sigma \\ 1; & \rho_{i|j}(\Lambda) \le \sigma - 1 \\ \dfrac{\bar{v}_{ij}\cdot\bar{v}_j - \frac{\sigma}{2}|\bar{v}_{ij}|^2}{(1-\sigma)|\bar{v}_{ij}|^2}; & \text{otherwise} \end{array} \right\}. \tag{4}$$

For the first case in equation (4), Λ lies completely in fuzzy set j. For the second case, Λ lies completely in fuzzy set i. The third case is the case of Λ being in the transition from set i to set j.

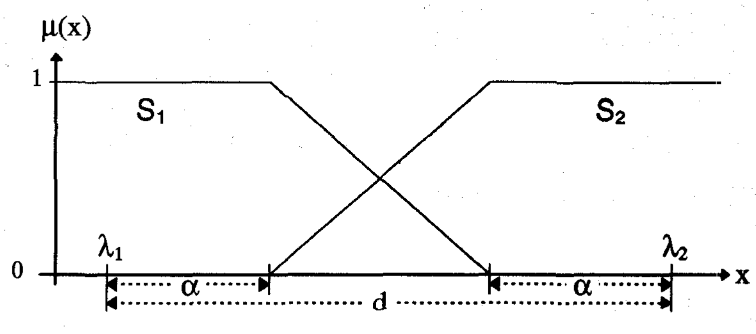

Figure 4. Defining the crispness of a partitioning.

Generally, there will be more than two sets in an input space. Therefore, the final step in the derivation requires combining the membership functions of equation (4) for all j≠i. In this case, the degree of membership of a point Λ in each of the M fuzzy sets can be calculated in one of two ways. The first is based on product/sum inference and is shown in equation (5). The second is based on min/max inference and is shown in equation (6). Both are normalized so that equation (1) is satisfied.

$$\mu_i(\Lambda) = \frac{\prod_{j=1 \neq i}^{M} \mu_{i|j}(\Lambda)}{\sum_{k=1}^{M} \left(\prod_{j=1 \neq k}^{M} \mu_{i|j}(\Lambda) \right)} \quad \text{for } i = 1, 2, \ldots M \tag{5}$$

$$\mu_i(\Lambda) = \frac{\min_j \left(\mu_{i|j}(\Lambda) \right)}{\sum_{k=1}^{M} \left(\min_j \left(\mu_{i|j}(\Lambda) \right) \right)} \quad \text{for } i = 1, 2, \ldots M \tag{6}$$

To summarize, the *design* of hypertrapezoidal membership functions requires two steps –

a) selection of the prototype points, and

b) selection of the crispness factor.

The *computation* of hypertrapezoidal membership functions requires three sets of calculations –

a) the distance measure of equation (3),

b) the conditional membership functions of equation (4), and

c) the composition of equation (5) or (6).

Notice that equations (3) - (6) are general for N dimensions, including N=1. These four equations allow for the use of an N-dimensional membership functions using only M+1 parameters. Additionally, the desirable property of equation (1) is enforced.

Examples

The following diagrams are examples of fuzzy

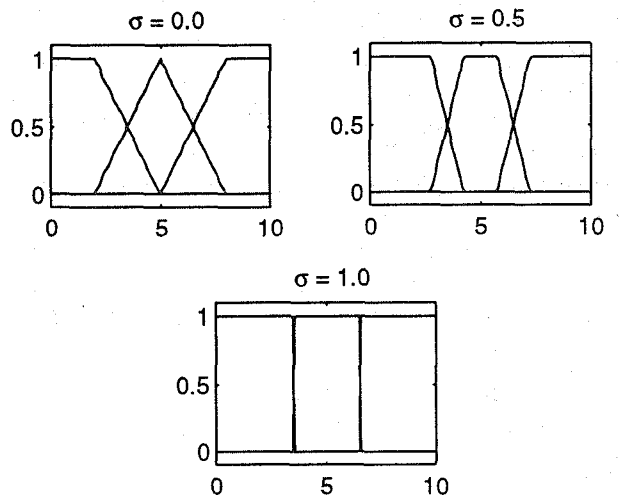

Figure 5. One dimensional examples, for $\lambda = [1 \quad 5 \quad 9]$.

membership functions designed using the described technique. All the examples were made using equation (6). Figure 5 illustrates the use of the described technique for one-dimensional membership functions and the effect of the crispness factor σ on fuzzy sets.

Figure 7 shows an example of three fuzzy sets defined on two domains. The definition of the three sets is accomplished with the following parameters: $\lambda_1 = (9, 1)$, $\lambda_2 = (5, 5)$, $\lambda_3 = (1, 9)$, and $\sigma = 0.5$. A rule base operating on one-dimensional sets could only approximate the correlation represented in the figure.

In the example of Figure 7, a transformation of the axes could also have compensated for the correlation. It is included as a simple example to aid the reader in an intuitive understanding of the design parameters λ_i and σ. Figure 6 shows another example of fuzzy sets defined in a two-dimensional space. In this case, coordinate transformation would not be useful.

Visualization of N-dimensional fuzzy sets defined on more than two domains is not easy. However, hypertrapezoidal membership functions do not require visualization for their design. The results of the application of this technique for problems involving as many as seven inputs will be reported in future publications.

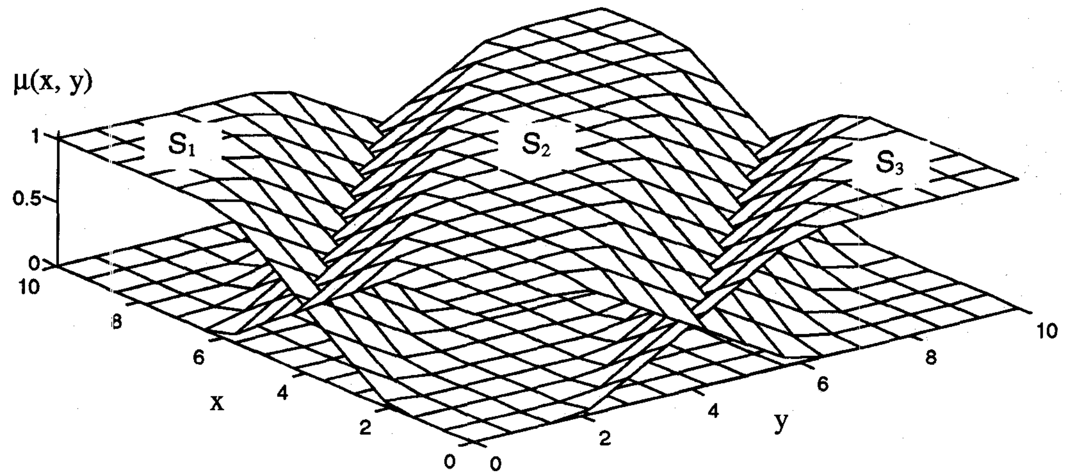

Figure 7. Example of three sets defined on a two-dimensional space.

Conclusion and Future Research

An efficient and simple mechanism for representing and evaluating N-dimensional fuzzy membership functions has been presented. The examples show that fuzzy partitioning of a space can be performed with a small number of parameters. This method of designing N-dimensional membership functions has promise for simplifying the design of complex fuzzy systems. Multidimensional membership functions can account for correlation between the input variables and reduce the number of rules needed in a fuzzy system.

Another significant advantage of the presented approach is in the areas of machine learning and adaptive systems. The small number of parameters needed for hypertrapezoidal membership functions will be valuable for situations requiring training. Both clustering and genetic algorithm techniques could be used to determine good λ_i s and σ from training data.

Another important extension of the hypertrapezoidal membership function is motivated by the work of Foster and Khambhampati[3] in the area of multidimensional Gaussian membership functions. Instead of a single point in space defining the center of a Gaussian membership function, they used a vector in space to define the "top ridge" of an elongated Gaussian fuzzy set. This allows for more variety in the shapes of the designed membership functions. For hypertrapezoids, this would involve replacing the λ_i s with vectors $\bar{\lambda}_i$.

Future work will also compare and contrast the use of N-dimensional membership functions with one-dimensional membership functions. First, a classical

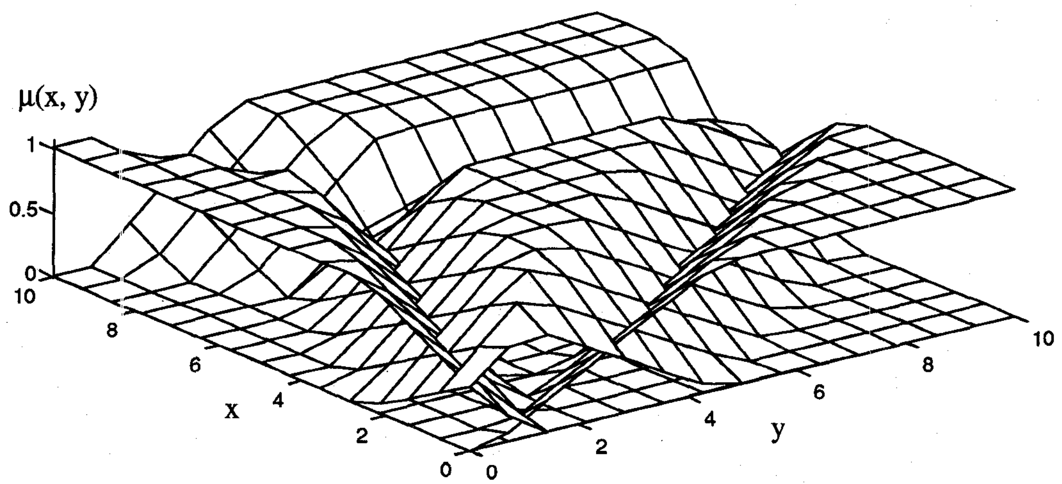

Figure 6. Example of four sets defined on a two dimensional space.

fuzzy logic control problem (an inverted pendulum or truck backer-upper) will be used for comparison. Work is also now underway to implement this new technique for flight mode analysis in a pilot advisory system. It is believed that the real advantage of this technique will be in complex systems like the pilot advisory system.

References

[1] R. E. Bellman and L. A. Zadeh, "Decision-making in a Fuzzy Environment," *Management Science,* vol. 17, #4, December 1970, pp. 141-164.

[2] H. R. Berenji and P. S. Khedkar, "Clustering in Product Space for Fuzzy Inference," *Second IEEE International Conference on Fuzzy Systems,* San Fransisco, CA, March 1993, pp. 1402 - 1407.

[3] G. T. Foster and C. Khambhampati, "Optimal Set Placement and Multi-dimensional Fuzzy Sets for Fuzzy Logic Controllers," *Proceedings of the International Conference on Control'94,* March 1994, pp. 658 - 663.

[4] V. Q. Harral, "Genetic Algorithms for Tuning Fuzzy Membership Functions in Flight Control Software," M.S. Thesis, Department of Electrical Engineering, Texas A&M University, College Station, Texas, 1995.

[5] S. L. Lass, "An Object-oriented, Knowledge-based Aircraft Approach Controller,", M.S. Thesis, Department of Electrical Engineering, Texas A&M University, College Station, Texas, 1995.

[6] J. H. Painter, "Knowledge-based Control," Proceedings of the 1992 IEEE Symposium on Computer Aided Control System Design, Napa, CA, March 1992, pp. 138-147.

[7] J. H. Painter, "Fuzzy Decision and Control, The Bayes Context" *Proceedings of the 32nd IEEE Conference on Decision and Control,* San Antonio, Texas, December 15-17, 1993.

[8] L. X. Wang, *Adaptive Fuzzy Systems and Control.* Englewood, NJ: PTR Prentice Hall, 1994.