PRODUCTS OF REPRESENTATIONS OF THE SYMMETRIC GROUP AND NON-COMMUTATIVE VERSIONS

A Dissertation

by

RIVERA WALTER MOREIRA RODRIGUEZ

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2008

Major Subject: Mathematics

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Approved by:

Chair of Committee,	Marcelo Aguiar
Committee Members,	Frank Sottile
	Huafei Yan
	John Keyser
Head of Department,	Albert Boggess

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ABSTRACT

Products of Representations of the Symmetric Group
and Non-Commutative Versions. (May 2008)

Rivera Walter Moreira Rodriguez, B.S., Universidad de la República;
M.S., Universidad de la República

Chair of Advisory Committee: Dr. Marcelo Aguiar

We construct a new operation among representations of the symmetric group that interpolates between the classical internal and external products, which are defined in terms of tensor product and induction of representations. Following Malvenuto and Reutenauer, we pass from symmetric functions to non-commutative symmetric functions and from there to the algebra of permutations in order to relate the internal and external products to the composition and convolution of linear endomorphisms of the tensor algebra. The new product we construct corresponds to the Heisenberg product of endomorphisms of the tensor algebra. For symmetric functions, the Heisenberg product is given by a construction which combines induction and restriction of representations. For non-commutative symmetric functions, the structure constants of the Heisenberg product are given by an explicit combinatorial rule which extends a well-known result of Garsia, Remmel, Reutenauer, and Solomon for the descent algebra. We describe the dual operation among quasi-symmetric functions in terms of alphabets.

To Walter,

but not because of the kindness.

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Lastly, but first in my heart, I want to thank my wife, María Isabel Cañette, for everything. Her patience rivals that of my advisor.

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1. INTRODUCTION

The space of representations of the symmetric group and the closely related space of symmetric functions are important objects in the field of Algebraic Combinatorics. They have an extremely rich structure. In this work we concentrate on algebra structures in these spaces. Both spaces carry two classical products: the external product and the Kronecker, or internal, product. Our goal is to introduce a new product which interpolates between these classical structures. We call this new operation the Heisenberg product. The reason for the name will be explained in Section 5.

To best understand this new operation we consider not only symmetric functions but other objects, including non-commutative symmetric functions, related by the diagram shown in Figure 1.

The objects in Figure 1 can be divided into three groups, marked with different kind of boxes. We call the first group, marked with square boxes, the *commutative context*. The second group, marked with rounded boxes, is the *non-commutative context*. And finally, the third group has only one element, unframed in Figure 1, which is the space of quasi-symmetric functions. It is important to note that the adjectives "commutative" and "non-commutative" refer only to the external product. The internal product is always non-commutative, as well as the new product we introduce. Each context requires a different and independent construction of the Heisenberg product.

In the commutative context, we start our construction in the category of species. The theory of species was introduced by Joyal [16] in 1980. It provides a general

This dissertation follows the style of *Journal of the American Mathematical Society.*

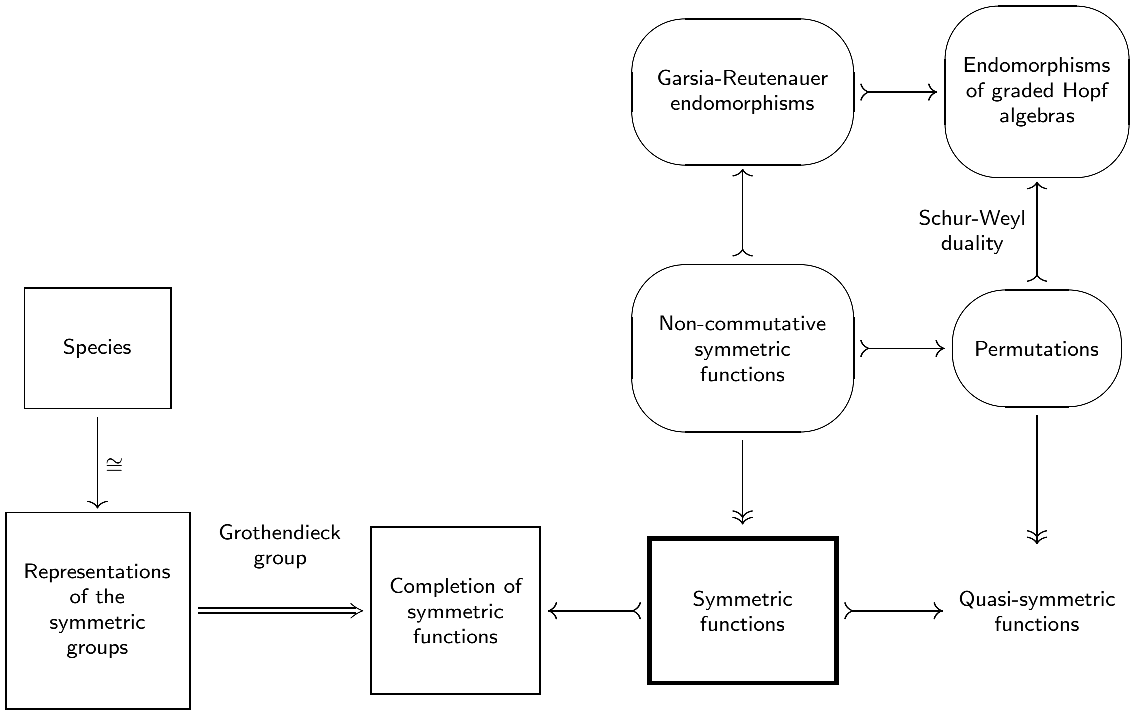

Fig. 1. The spaces where the Heisenberg product is introduced.

context to work with labeled and unlabeled combinatorial structures. Although a species carries essentially the same information as a sequence of representations, over a field of characteristic 0, of the symmetric groups S_n for $n \geq 0$, the language of species is considerably easier than the language of representations, as it is shown in sections 2 and 3.

The equivalence between species and sequences of representations of the symmetric groups is worked out on Section 3. We translate, via this equivalence, the classical and the Heisenberg products to representations of S_n, and we give expressions in terms of restriction and induction of representations.

The relation between representations of S_n and the space of symmetric polynomials in n variables is a classical well-known subject, which involves the Grothendieck group and the Frobenius characteristic. We deal with this passage in Section 4. This is the double arrow in Figure 1. The application of this classical theory to sequences of representations take us to the completion of the space of symmetric functions. This space contains the space of symmetric functions, and the internal and external products restrict to this space. We show that the same holds for the new operation.

The Heisenberg product has a nice combinatorial expression in some basis of symmetric functions, which generalizes combinatorial rules for the external and internal products. These combinatorial formulas are the key to relate the commutative and non-commutative contexts, which are a priori unrelated. That is symbolized in Figure 1 by a boldface rectangle around symmetric functions.

In the non-commutative context, we consider the space of endomorphisms of a graded Hopf algebra. In this space there are several well-known products. One way to look at the space of endomorphisms of a finite-dimensional Hopf algebra H, which emphasizes a relation with the semi-direct product of groups, is to consider the object $H \otimes H^*$. We are interested in an operation which, when defined on $H \otimes H^*$, gives this space the name *Heisenberg double*, and is denoted by $H \# H^*$. This is treated on Section 5.

At this point, our main goal is to restrict the Heisenberg product to the space of non-commutative symmetric functions, which can be embedded into the endomorphisms of the tensor algebra of vector space. We follow two different paths, each of them of particular interest. On one side, we define in Section 5 the notion of Garsia-Reutenauer endomorphisms for an arbitrary Hopf algebra, and we restrict the Heisenberg product to such subspace. When the Hopf algebra is the tensor algebra of a vector space V, then the subspace of Garsia-Reutenauer endomorphisms which are fixed under the action of the group $\mathrm{GL}(V)$ coincide with the non-commutative symmetric functions. We treat this in Section 7. This is the most powerful method to construct the Heisenberg product of non-commutative symmetric functions, since it can be applied to other products and to other subspaces of endomorphisms, as we mention briefly in Section 10. The other path we follow, in Section 6, is to use Schur-Weyl duality to embed the space of permutations into the space of endomorphisms of the tensor algebra. The space of non-commutative symmetric functions can be viewed as a subspace of the space of permutations. We show that the Heisenberg product restricts to such subspace. This construction has the advantage of producing an explicit combinatorial formula for the Heisenberg product of non-commutative symmetric functions. Using this formula it is possible to show that the projection from non-commutative symmetric functions to commutative symmetric functions is a morphism of algebras.

The third group in Figure 1 consists of the space of quasi-symmetric functions. We view this space as the dual of the space of non-commutative symmetric functions. As such, we define in Section 9 a new coproduct of quasi-symmetric functions, dual to the Heisenberg product, which extends the classical internal and external coproducts. Such construction can also be restricted from quasi-symmetric functions to symmetric functions. In this last space we obtain, then, the Heisenberg product and the Heisenberg coproduct, although they are not compatible.

In Section 8 we discuss further connections between the classical and the new structure. In addition, we also study the compatibility of the Heisenberg product with the coproduct in the spaces of commutative and non-commutative symmetric functions. Dually, we consider the compatibility of the Heisenberg coproduct with the usual product in quasi-symmetric functions. This gives new Hopf algebra structures on these spaces.

This thesis is based in a joint work with Marcelo Aguiar and Walter Ferrer [2]. I am grateful to them for allowing me to include such work.

1.1. Terminology

In all the spaces we consider there are at least two well-known products. Although they are closely related by the inclusions, projections, and isomorphisms in Figure 1, unfortunately various different names and notations are used in the literature. Table 1 shows some of the commonly used names.

We call the new product we introduce, in every space, the *Heisenberg* product, and we denote it with the symbol $\#$.

Table 1. Standard terminology and notation for the classical products.

	Species	Representations	Symmetric functions	Non-commutative symmetric functions	Permutations	Endomorphisms of graded Hopf algebras
internal product	Hadamard $(\times)$	Kronecker $(*)$	internal $(*)$	Solomon $(*)$	composition $(\circ)$	composition $(\circ)$
external product	Cauchy $(\cdot)$	induction $(\cdot)$	external $(\cdot)$	external $(\cdot)$	Malvenuto-Reutenauer $(\star)$	convolution $(\star)$

2. SPECIES

2.1. Classical monoidal structures in species

The theory of *species* was introduced by Joyal in [16]. In this section we present the definition, some examples, and the classical monoidal structures in the category of species. We follow the notation and terminology of [1] and [6].

Let $\mathbf{Set}^{\times}$ be the category whose objects are finite sets and whose morphisms are bijections among the sets. Let us fixed a field $\Bbbk$ and let **Vect** be the category of vector spaces over this field.

Definition 2.1.1. A *species* p with values in the category **Vect** is a functor $\mathsf{p} : \mathbf{Set}^{\times} \to \mathbf{Vect}$. A morphism between two species p and q is a natural transformation between the functors p and q.

We denote the category of species and its morphisms by **Sp**. The evaluation of the functor p on a set I is denoted by $\mathsf{p}[I]$, and the evaluation of p on a bijection f is denoted by $\mathsf{p}[f]$. The image of the set $[n] = \{1, 2, \ldots, n\}$ is written, for simplicity, $\mathsf{p}[n]$ instead of $\mathsf{p}\big[[n]\big]$. By definition, $[0] = \emptyset$.

For each n, the vector space $\mathsf{p}[n]$ has a structure of S_n-module. Indeed, given a permutation $\sigma : [n] \to [n]$, the application of the functor p yields a linear map $\mathsf{p}[\sigma] : \mathsf{p}[n] \to \mathsf{p}[n]$. The functoriality of p ensures that the operation of S_n on $\mathsf{p}[n]$ defined by

$$\sigma \cdot x = \mathsf{p}[\sigma](x),$$

for $\sigma \in S_n$ and $x \in [n]$, is a left S_n-module structure on $\mathsf{p}[n]$. Indeed, since σ is a bijection, then $\mathsf{p}[\sigma]$ is an automorphism in the vector space $\mathsf{p}[n]$; and, since p is a functor, $(\tau\sigma) \cdot x = \mathsf{p}[\tau\sigma](x) = \mathsf{p}[\tau]\big(\mathsf{p}[\sigma](x)\big) = \tau \cdot (\sigma \cdot x)$.

Example 2.1.2. We give some examples and we set the notation for some species we will use later.

(a) The *exponential species* e is defined by $\mathsf{e}[I] = \Bbbk\{*\}$ for every finite set I. For $f : I \to J$, the linear map $\mathsf{e}[f] : \mathsf{e}[I] \to \mathsf{e}[J]$ is the unique automorphism of the vector space $\Bbbk\{*\}$ which fixes the element $*$.

(b) The species $\mathsf{1}$ is defined by

$$\mathsf{1}[I] = \begin{cases} \Bbbk\{*\}, & \text{if } I \text{ is empty} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

We define $\mathsf{1}[f]$, for a bijection $f : I \to J$, as the unique linear map between $\mathsf{1}[I]$ and $\mathsf{1}[J]$ which fixes the element $*$ when I and J are empty, or the null map otherwise.

(c) Let $\mathcal{A}$ a functor defined as follows: for a finite set I, the vector space $\mathcal{A}[I]$ has a basis consisting on all the simple graphs with vertex set I. For a bijection $f : I \to J$, the linear map $\mathcal{A}[f]$ is defined on the basis of $\mathcal{A}[I]$ by relabeling the vertices of the graphs according to the bijection f.

(d) The *linear order species* ℓ is defined, for a finite set I, as the vector space $\ell[I]$ spanned by the set of linear orders on the set I. The functor ℓ is defined on morphisms by relabeling the elements, as in Example (c).

To speak of products of species, the proper context is the notion of monoidal categories. The following definition can be found in [17].

Definition 2.1.3. A *monoidal structure* in a category $\mathbf{C}$ is a functor $\square : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ which is associative and has a unit element e, up to natural isomorphisms,

$$\alpha_{a,b,c} : a \square (b \square c) \cong (a \square b) \square c, \quad \lambda_a : e \square a \cong a, \quad \rho_a : a \square e \cong a.$$

Moreover, these isomorphisms must satisfy the commutative diagrams:

$$
\begin{tikzcd}[column sep=small]
 & (a \,\square\, b) \,\square\, (c \,\square\, d) \arrow[dr, "\alpha_{a\square b,c,d}"] & \\
a \,\square\, \big(b \,\square\, (c \,\square\, d)\big) \arrow[ur, "\alpha_{a,b,c\square d}"] \arrow[d, "1\square\alpha_{b,c,d}"'] & & \big((a \,\square\, b) \,\square\, c\big) \,\square\, d \arrow[d, "\alpha_{a,b,c}\square 1"] \\
a \,\square\, \big((b \,\square\, c) \,\square\, d\big) \arrow[rr, "\alpha_{a,b\square c,d}"] & & \big(a \,\square\, (b \,\square\, c)\big) \,\square\, d
\end{tikzcd}
\qquad
\begin{tikzcd}
a \,\square\, (e \,\square\, c) \arrow[rr, "\alpha_{a,e,c}"] \arrow[dr, "1\square\lambda_c"'] & & (a \,\square\, e) \,\square\, c \arrow[dl, "\rho_a\square 1"] \\
 & a \,\square\, c &
\end{tikzcd}
$$

As a basic example of monoidal category, we mention the category of vector spaces over a field $\Bbbk$, with the tensor product $\otimes$. The unit e is the base field $\Bbbk$.

In the category **Sp** of species there are several monoidal structures. The first structure we define is the *sum of species*, which is defined pointwise as the direct sum of vector spaces:

$$(\mathsf{p} \oplus \mathsf{q})[I] = \mathsf{p}[I] \oplus \mathsf{q}[I]. \tag{2.2}$$

The unit for this operation is clearly the species given by $\mathsf{0}[I] = 0$ for every finite set I. We will use this operation in Section 4.1.

In what follows, we use the notation $I = S \sqcup T$ to mean that the set I is decomposed into the disjoint subsets S and T of I.

We concentrate now in two other monoidal structures: the *Cauchy* and *Hadamard* products. These are well-known associative products (see [16, 6]). They are defined, respectively, by

$$(\mathsf{p} \cdot \mathsf{q})[I] = \bigoplus_{I=S\sqcup T} \mathsf{p}[S] \otimes \mathsf{q}[T], \tag{2.3}$$

$$(\mathsf{p} \times \mathsf{q})[I] = \mathsf{p}[I] \otimes \mathsf{q}[I]. \tag{2.4}$$

Note that in the definition of the Cauchy product $(\cdot)$ the direct sum ranges over all the disjoint decompositions of the set I into two subsets.

At the level of morphisms it is easy to see that given two finite sets I and J, and a bijection $f : I \to J$, we obtain a bijection $(S,T) \mapsto \big(f(S), f(T)\big)$ between pairs of sets (S,T) with $I = S \sqcup T$, and sets (S',T') with $J = S' \sqcup T'$. Then, the map $(\mathsf{p} \cdot \mathsf{q})[f]$ is induced by the maps

$$\mathsf{p}[f_{|_S}] \otimes \mathsf{q}[f_{|_T}] : \mathsf{p}[S] \otimes \mathsf{q}[T] \to \mathsf{p}[S'] \otimes \mathsf{q}[T'],$$

where $f_{|_S}$ is the bijection $f_{|_S} : S \to f(S)$. A similar argument applies to the Hadamard product.

The species $\mathsf{1}$ is the unit of $\cdot$ and the species e is the unit of $\times$. The natural maps α, λ, and ρ are induced from the natural linear isomorphisms (associativity and left and right units) of the usual tensor product of vector spaces.

2.2. The Heisenberg product of species

The Heisenberg product of species introduced here is a simple generalization of the Cauchy and Hadamard products, which exhibits a kind of "interpolation" property between these two classical products. The origin of the name for the new product will be explained in Section 5.

Definition 2.2.1. The *Heisenberg product of species* is the functor $\# : \mathbf{Sp} \times \mathbf{Sp} \to \mathbf{Sp}$ given by

$$(\mathsf{p} \# \mathsf{q})[I] = \bigoplus_{I = S \cup T} \mathsf{p}[S] \otimes q[T]. \tag{2.5}$$

The map $(p \# q)[f]$ is defined in the same way as for the Cauchy and Hadamard products. Note that the direct sum runs over *all* the possible decompositions of I into two subsets (compare with the Cauchy product in (2.3)).

This definition contains as extreme cases the classical products, in the following sense. Decompose the expression (2.5) into

$$\begin{aligned}(\mathsf{p} \# \mathsf{q})[I] &= \mathsf{p}[I] \otimes \mathsf{q}[I] + \bigoplus_{\substack{I=S\cup T\\ \emptyset\neq S\cap T\neq I}} \mathsf{p}[S] \otimes \mathsf{q}[T] + \bigoplus_{I=S\sqcup T} \mathsf{p}[S] \otimes \mathsf{q}[T] \\ &= (\mathsf{p} \times \mathsf{q})[I] + \bigoplus_{\substack{I=S\cup T\\ \emptyset\neq S\cap T\neq I}} \mathsf{p}[S] \otimes \mathsf{q}[T] + (\mathsf{p} \cdot \mathsf{q})[I], \qquad (2.6)\end{aligned}$$

and we see that the extreme cases where both subsets S and T are equal to I, or S and T are *disjoint* subsets, give rise to Hadamard and Cauchy products, respectively.

The pairs $(\mathbf{Sp}, \cdot)$ and $(\mathbf{Sp}, \times)$ are monoidal categories. The same is true for the pair $(\mathbf{Sp}, \#)$, and the proof follows the same path as the one for the Cauchy product.

Theorem 2.2.2. *The functor* $\# : \mathbf{Sp} \times \mathbf{Sp} \to \mathbf{Sp}$ *gives a monoidal structure to the category* $\mathbf{Sp}$*, with unit object* $\mathsf{1}$ *defined in* (2.1)*.*

Proof. Let p, q, and r be three species. We prove the associativity property: $\big((\mathsf{p} \# \mathsf{q}) \# \mathsf{r}\big)[I] = \big(\mathsf{p} \# (\mathsf{q} \# \mathsf{r})\big)[I]$ for all finite sets I.

We have

$$\begin{aligned}\big((\mathsf{p} \# \mathsf{q}) \# \mathsf{r}\big)[I] &= \bigoplus_{I=S\cup T} (\mathsf{p} \# \mathsf{q})[S] \otimes \mathsf{r}[T] \\ &= \bigoplus_{I=S\cup T} \bigoplus_{S=U\cup V} \mathsf{p}[U] \otimes \mathsf{q}[V] \otimes \mathsf{r}[T], \qquad (2.7)\\ \big(\mathsf{p} \# (\mathsf{q} \# \mathsf{r})\big)[I] &= \bigoplus_{I=S'\cup T'} \mathsf{p}[S'] \otimes (\mathsf{q} \# \mathsf{r})[T'] \\ &= \bigoplus_{I=S'\cup T'} \bigoplus_{T'=U'\cup V'} \mathsf{p}[S'] \otimes \mathsf{q}[U'] \otimes \mathsf{r}[V']. \qquad (2.8)\end{aligned}$$

The tuples (U, V, T) and (S', U', V') are in bijection by the map $(U, V, T) \mapsto (T, U, V)$, hence equations (2.7) and (2.8) coincide. From Definition (2.5) it is clear that $\mathsf{p} \# \mathsf{1} = \mathsf{1} \# \mathsf{p} = \mathsf{p}$. $\square$

In addition to the relation via interpolation of the three products: Cauchy, Hadamard, and Heisenberg, there is also another relation through the following proposition.

Proposition 2.2.3. *For any species* p *and* q*, we have the isomorphism*

$$(\mathsf{p}\cdot\mathsf{e})\times(\mathsf{q}\cdot\mathsf{e})\cong(\mathsf{p}\#\mathsf{q})\cdot\mathsf{e}, \tag{2.9}$$

where e *is the exponential species, defined in Example 2.1.2(a).*

Proof. To prove Equation (2.9) we evaluate each side on a finite set I. Then, we get

$$\begin{aligned}\big((\mathsf{p}\cdot\mathsf{e})\times(\mathsf{q}\cdot\mathsf{e})\big)[I] &= \Big(\bigoplus_{I=S\sqcup T}\mathsf{p}[S]\otimes\Bbbk\Big)\otimes\Big(\bigoplus_{I=S'\sqcup T'}\mathsf{q}[S']\otimes\Bbbk\Big)\\ &\cong\bigoplus_{S,S'\subseteq I}\mathsf{p}[S]\otimes\mathsf{q}[S'] \qquad (2.10)\\ \big((\mathsf{p}\#\mathsf{q})\cdot\mathsf{e}\big)[I] &= \bigoplus_{I=J\sqcup K}(\mathsf{p}\#\mathsf{q})[J]\otimes\mathsf{e}[K]=\bigoplus_{J\subseteq I}\Big(\bigoplus_{J=S\cup S'}\mathsf{p}[S]\otimes\mathsf{q}[S']\Big)\otimes\Bbbk\\ &\cong\bigoplus_{J\subseteq I}\ \bigoplus_{J=S\sqcup S'}\mathsf{p}[S]\otimes\mathsf{q}[S'], \qquad (2.11)\end{aligned}$$

and clearly (2.10) and (2.11) coincide, and the isomorphisms are natural. □

2.3. Generating series

In this section we assume that every species is *finite-dimensional*, meaning that the vector spaces $\mathsf{p}[n]$ are finite-dimensional for each $n\geq 0$.

Definition 2.3.1. The *generating series* associated to a species p is the formal series

$$F_{\mathsf{p}}(x)=\sum_{n\geq 0}\dim_{\Bbbk}\mathsf{p}[n]\,\frac{x^n}{n!}.$$

Example 2.3.2. The following examples are immediate:

(a) The generating series associated to the exponential species $\mathsf{e}[I]=\Bbbk$ for every finite set I is $F_{\mathsf{e}}(x)=e^x$.

(b) The unit of the Cauchy and Heisenberg products has generating series $F_{\mathsf{1}}(x) = 1$.

(c) The species of linear orders has generating series

$$F_{\ell}(x) = \frac{1}{(1-x)}.$$

The generating series associated to the Cauchy product $\mathsf{p} \cdot \mathsf{q}$ of two species is the usual (Cauchy) product of the power series F_{p} and F_{q}. Similarly, the generating series of $\mathsf{p} \times \mathsf{q}$ is the Hadamard product of the generating series of p and q. In other words, if $F_{\mathsf{p}}(x) = \sum_{n\geq 0} a_n x^n/n!$ and $F_{\mathsf{q}}(x) = \sum_{n\geq 0} b_n x^n/n!$, then

$$F_{\mathsf{p}\cdot\mathsf{q}}(x) = \sum_{n\geq 0}\Bigg(\sum_{i+j=n}\binom{n}{i} a_i b_j\Bigg)\frac{x^n}{n!} \quad \text{and} \quad F_{\mathsf{p}\times\mathsf{q}}(x) = \sum_{n\geq 0} a_n b_n \frac{x^n}{n!}. \tag{2.12}$$

The interpolation property of Equation (2.6) translates into a formula for the generating series of the Heisenberg product of species which contains the cases (2.12).

Theorem 2.3.3. *Let $F_{\mathsf{p}}(x) = \sum_{n\geq 0} a_n x^n/n!$ and $F_{\mathsf{q}}(x) = \sum_{n\geq 0} b_n x^n/n!$ be the generating series of two species p and q. Then, the generating series of the Heisenberg product $\mathsf{p} \# \mathsf{q}$ is*

$$F_{\mathsf{p}\#\mathsf{q}}(x) = \sum_{n\geq 0}\Bigg(\sum_{\substack{i,j\leq n\\ n\leq i+j}}\binom{n}{n-i,\, n-j,\, i+j-n} a_i b_j\Bigg)\frac{x^n}{n!}.$$

Proof. Recall Definition (2.5) of the Heisenberg product of two species. Let $n \geq 0$, we count the number of terms in the direct sum

$$(\mathsf{p} \# \mathsf{q})[n] = \bigoplus_{[n]=S\cup T} \mathsf{p}[S] \otimes \mathsf{q}[T]$$

by establishing a bijection between the following two sets:

$$\mathcal{A} = \big\{(S,T) \text{ such that } [n] = S \cup T, \text{ and } \# S = i \text{ and } \# T = j\big\}$$

$$\mathcal{B} = \big\{(U,V,W) \text{ such that } [n] = U \sqcup V \sqcup W, \text{ and } \\ \# U + \# W = i \text{ and } \# V + \# W = j\big\}$$

Indeed, just take $U = S \setminus T$, $V = T \setminus S$, and $W = S \cap T$. Clearly, we obtain $\# U = n - i$, $\# V = n - j$, and $\# W = i + j - n$. The multinomial coefficient in the formula of $F_{\mathsf{p}\#\mathsf{q}}$ stands precisely for the number of possible ways to choose the decomposition (U, V, W), while the coefficients a_i and b_j are the dimensions of $\mathsf{p}[S]$ and $\mathsf{q}[T]$, respectively. $\square$

3. REPRESENTATIONS OF THE SYMMETRIC GROUP

3.1. Classical products of representations

The language of species and the more classical language of representations of the symmetric group are essentially the same. In this section we recall the equivalence of these categories and we translate the products defined in Section 2 to the vocabulary of representations.

The translations of the Cauchy and Hadamard products under this equivalence yield two well-known products of representations: the induction product and the Kronecker product. Sometimes they are also called external and internal product, respectively.

Let us fix a field $\Bbbk$ of characteristic zero. Let $\mathrm{Rep}(S_n)$ be the category whose objects are finite-dimensional representations of S_n over the field $\Bbbk$, and whose morphisms are S_n-module homomorphisms. We consider the category

$$\mathbf{R} = \prod_{n \geq 0} \mathrm{Rep}(S_n). \tag{3.1}$$

Recall that if p is a species, each vector space $\mathsf{p}[n]$ is an S_n-module with the action defined by $\sigma \cdot x = \mathsf{p}[\sigma](x)$, for $\sigma \in S_n$ and $x \in \mathsf{p}[n]$. If V is an object in $\mathbf{R}$, we denote its n-th coordinate by V_n, which is a representation of S_n.

Theorem 3.1.1. *The functor $\mathcal{F} : \mathbf{Sp} \to \mathbf{R}$ given by*

$$\mathcal{F}(\mathsf{p}) = \big(\mathsf{p}[0], \mathsf{p}[1], \ldots, \mathsf{p}[n], \ldots\big)$$

is an equivalence of categories.

Proof. Consider the skeleton of the category $\mathbf{Set}^{\times}$ whose objects are the natural intervals $[n]$, and let $\mathbf{C}$ be the category of functors from this skeleton to $\mathbf{Vect}$. The category $\mathbf{C}$ is clearly equivalent to $\mathbf{Sp}$. An object in $\mathbf{C}$ is, then, just a sequence of S_n-modules $(\mathsf{p}[0], \mathsf{p}[1], \dots)$. A map in $\mathbf{C}$ comes from a natural transformation between two species, hence it preserves the action of S_n in each coordinate. Thus, the category $\mathbf{C}$ coincides with $\mathbf{R}$, and the functor $\mathcal{F}$ gives the required equivalence. $\square$

We now give the classical definition of the induction and Kronecker products of representations and in the next section we prove that they correspond under the functor $\mathcal{F}$ to the Cauchy and Hadamard product of species. In these definitions we follow the notation of [22].

Let p and q be non-negative integers. Given permutations $\sigma \in S_p$ and $\tau \in S_q$, let $\sigma \times \tau \in S_{p+q}$ be the permutation

$$(\sigma \times \tau)(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq p, \\ \tau(i-p)+p & \text{if } p+1 \leq i \leq p+q. \end{cases} \tag{3.2}$$

This operation gives an embedding of $S_p \times S_q$ into S_{p+q}, which we call the *standard parabolic embedding*. This is a particular case of Equation (4.1).

Definition 3.1.2. Let V and W be two objects in $\mathbf{R}$. The *induction product* of V and W is the object whose n-th coordinate is the representation of S_n given by

$$(V \cdot W)_n = \bigoplus_{p+q=n} \mathrm{Ind}_{S_p \times S_q}^{S_{p+q}} (V_p \otimes W_q), \tag{3.3}$$

where $S_p \times S_q \hookrightarrow S_{p+q}$ is the standard parabolic embedding. The group $S_p \times S_q$ acts on the space $V_p \otimes W_q$ by $(\sigma, \tau) \cdot (v \otimes w) = (\sigma \cdot v) \otimes (\tau \cdot w)$, for $\sigma \in S_p$, $\tau \in S_q$, and $v \otimes w \in V_p \otimes W_q$.

The *Kronecker product* of V and W is the object whose n-th coordinate is the representation of S_n given by

$$(V * W)_n = V_n \otimes W_n. \tag{3.4}$$

Here the group S_n acts on $V_n \otimes W_n$ diagonally, that is, $\sigma \cdot (v \otimes w) = (\sigma \cdot v) \otimes (\sigma \cdot w)$, for $\sigma \in S_n$ and $v \otimes w \in V_n \otimes W_n$.

Observe that all the operations involved in the definition of both products: induction of representations, direct sum, and tensor product, are functorial. Hence we obtain two functors: $\cdot, * : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$.

It is known that these functors give monoidal structures on $\mathbf{R}$. We will give a proof of this fact for the Heisenberg product in the next section, which contains, as particular cases, the proof for the classical products.

The units for the induction product and for the Kronecker product are the objects defined, respectively, by

$$\mathsf{1} = (\Bbbk, 0, 0, \ldots), \qquad E = (\Bbbk, \Bbbk, \Bbbk, \ldots).$$

where S_n acts trivially in $\Bbbk$.

3.2. The Heisenberg product of representations

To define the Heisenberg product of representations we first need to introduce an embedding similar to the parabolic embedding (3.2).

Let p and q be two non-negative integers, and let n be an integer satisfying $\max(p, q) \leq n \leq p + q$. Define the group

$$S_p \times_n S_q = S_{n-q} \times S_{p+q-n} \times S_{n-p},$$

and consider the embeddings

$$S_p \times_n S_q \hookrightarrow S_n, \quad (\sigma, \rho, \tau) \mapsto \sigma \times \rho \times \tau, \tag{3.5}$$

$$S_p \times_n S_q \hookrightarrow S_p \times S_q, \quad (\sigma, \rho, \tau) \mapsto (\sigma \times \rho, \rho \times \tau). \tag{3.6}$$

In the next definition we use the induction of representations along (3.5) and the restriction of representations along (3.6).

Definition 3.2.1. The *Heisenberg product of representations* is the functor

$$\# : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

defined for V and W in $\mathbf{R}$ as the object whose n-th coordinate is the representation

$$(V \# W)_n = \bigoplus_{\substack{0 \leq p,q \\ \max(p,q) \leq n \leq p+q}} \mathrm{Ind}_{S_p \times_n S_q}^{S_n} \mathrm{Res}_{S_p \times_n S_q}^{S_p \times S_q} (V_p \otimes W_q). \tag{3.7}$$

Note that for each n-th coordinate of $V \# W$ there are finitely many non-negative numbers p and q such that $\max(p, q) \leq n \leq p + q$ and each term of the sum is finite-dimensional, hence the result $V \# W$ is a well-defined object in $\mathbf{R}$.

The interpolation property of the Heisenberg product, analogous to the one in (2.6) for the case of species, takes the following form. In each coordinate n, consider the terms with $p + q = n$. In this case, the embedding (3.6) is the identity and (3.5) is the standard parabolic embedding $S_p \times S_q \hookrightarrow S_{p+q}$. On the other hand, when p and q are both equal to n, the embedding (3.5) is the identity and (3.6) is the diagonal embedding $S_n \hookrightarrow S_n \times S_n$. Thus, we can write

$$\begin{aligned}(V \# W)_n &= \mathrm{Res}_{S_n}^{S_n \times S_n} (V_n \otimes W_n) \oplus \cdots \oplus \bigoplus_{n=p+q} \mathrm{Ind}_{S_p \times S_q}^{S_n} (V_p \otimes W_q) \\ &= (V * W)_n \oplus \cdots \oplus (V \cdot W)_n. \end{aligned} \tag{3.8}$$

The Heisenberg product of two representations of S_p and S_q contains terms of intermediate degrees between $\max(p,q)$ to $p+q$; in this sense it "interpolates" between the Kronecker and induction products. It is a remarkable fact that, as the Kronecker and induction products, the Heisenberg product is associative, and can be lifted to other settings (non-commutative symmetric functions, permutations, and dually, quasi-symmetric functions) as will be shown later in this work, starting in Section 7.

The next theorem proves that the operation defined in (3.7) is the translation of the Heisenberg product in species defined in (2.5). Note that although the functor $\mathcal{F}$ is an equivalence of categories, the language of species is considerably cleaner than the language of representations. The lengthy verifications in Theorem 3.2.2 illustrate this claim.

Note that Theorem 3.2.2 also shows that the unit of the Heisenberg product is the object $\mathsf{1} = (\Bbbk, 0, 0, \ldots)$, image of the species $\mathsf{1}$ under $\mathcal{F}$.

Theorem 3.2.2. *The Heisenberg product of representations makes* $(\mathbf{R}, \#)$ *a monoidal category, and the functor* $\mathcal{F}$ *given in Theorem 3.1.1 preserves the monoidal structures of* $\mathbf{Sp}$ *and* $\mathbf{R}$*:*

$$\mathcal{F}(\mathsf{p} \# \mathsf{q}) = \mathcal{F}(\mathsf{p}) \# \mathcal{F}(\mathsf{p}) \tag{3.9}$$

for any species p *and* q*. The unit of* $(\mathbf{R}, \#)$ *is the image under* $\mathcal{F}$ *of the species* $\mathsf{1}$ (2.1)*:*

$$\mathcal{F}(\mathsf{1}) = (\Bbbk, 0, 0, \ldots)$$

where S_n *acts trivially on* $\Bbbk$*.*

Proof. It is enough to verify that the Heisenberg product of representations defined by (3.7) satisfies Equation (3.9), since $\mathcal{F}$ is already an equivalence of categories and $\mathbf{Sp}$ is a monoidal category with the Heisenberg product.

Fix i, j, and n, three non-negative integers such that $\max(i,j) \leq n \leq i+j$. We claim that we have an isomorphism in $\mathbf{R}$:

$$\bigoplus_{\substack{[n]=S\cup T\\ \# S=i\\ \# T=j}} \mathsf{p}[S]\otimes\mathsf{q}[T] \cong \operatorname{Ind}_{S_i\times_n S_j}^{S_n}\operatorname{Res}_{S_i\times_n S_j}^{S_i\times S_j}\big(\mathsf{p}[i]\otimes\mathsf{q}[j]\big). \tag{3.10}$$

Once this isomorphism is established, taking the direct sum over i and j, we obtain the n-th coordinate of $\mathcal{F}(\mathsf{p}\,\#\,\mathsf{q})$ on the left hand side, and the n-th coordinate of the product $\mathcal{F}(\mathsf{p})\,\#\,\mathcal{F}(\mathsf{q})$ on $\mathbf{R}$ on the right hand side.

The following fact is clear. Let A and B be finite totally ordered sets. Given ordered decompositions $A = A_1\sqcup\cdots\sqcup A_n$ and $B = B_1\sqcup\cdots\sqcup B_n$, with $\# A_i = \# B_i$ for $i = 1,\dots,n$, there is a unique bijection $f : A\to B$ such that $f(A_i) = B_i$ and $f_i = f_{|_{A_i}} : A_i\to B_i$ is increasing, for all $i = 1,\dots,n$. We call f the *canonical* bijection between A and B induced by the ordered partitions.

To define the isomorphism consider the following definitions. Given S and T such that $[n] = S\cup T$, let $S' = S\setminus T$ and $T' = T\setminus S$. If $\# S = i$ and $\# T = j$, then let $f_{S,T} : [n]\to[n]$ be the canonical bijection induced by the following partitions of $[n]$:

$$S'\sqcup(S\cap T)\sqcup T' \quad\text{and}\quad [n-j]\sqcup[n-j+1,i]\sqcup[i+1,n]$$

and let $f_{S'}$, $f_{S\cap T}$, and $f_{T'}$, be the restriction to the corresponding subsets. From the monotonicity conditions for $f_{S,T}$, we get that $f_{S,T}^{-1}$ belongs to $S_i\times_n S_j$.

We consider the standard identification of the induction module $\operatorname{Ind}_H^G(V)$ with the tensor product $\Bbbk G\otimes_{\Bbbk H} V$. Let $u\in\mathsf{p}[S]$ and $v\in\mathsf{q}[T]$, and define the map

$$\begin{aligned} \mathsf{p}[S]\otimes\mathsf{q}[T] &\xrightarrow{\psi} \operatorname{Ind}_{S_i\times_n S_j}^{S_n}\operatorname{Res}_{S_i\times_n S_j}^{S_i\times S_j}\big(\mathsf{p}[i]\otimes\mathsf{q}[j]\big)\\ u\otimes v &\longmapsto f_{S,T}^{-1}\otimes\big(\mathsf{p}[f_{S'}\sqcup f_{S\cap T}](u)\otimes\mathsf{q}[f_{S\cap T}\sqcup f_{T'}](v)\big) \end{aligned} \tag{3.11}$$

and extend it to the direct sum in (3.10).

For a permutation $\sigma \in S_n$, the action of σ in $u \otimes v$ is, according to Theorem 3.1.1,

$$\sigma \cdot (u \otimes v) = \mathsf{p}[\sigma_{|S}](u) \otimes \mathsf{q}[\sigma_{|T}](v). \tag{3.12}$$

Observe that $\sigma \cdot (u \times v) \in \mathsf{p}\big[\sigma(S)\big] \otimes \mathsf{q}\big[\sigma(T)\big]$. The application of the map ψ yields

$$\psi\big(\sigma \cdot (u \otimes v)\big) = f_{\sigma(S),\sigma(T)} \otimes \big(\alpha(u) \otimes \beta(v)\big) \tag{3.13}$$

where $\alpha = \mathsf{p}[f_{\sigma(S')} \sqcup f_{\sigma(S)\cap\sigma(T)}]\mathsf{p}[\sigma_{|S}]$ and $\beta = \mathsf{q}[f_{\sigma(S)\cap\sigma(T)} \sqcup f_{\sigma(T')}]\mathsf{q}[\sigma_{|T}]$. Since we can decompose $\sigma_{|S}$ into $\sigma_{|'_S} \sqcup \sigma_{|S\cap T}$, then by the functoriality of p we get that

$$\alpha = \mathsf{p}\big[(f_{\sigma(S')}\sigma_{|S'}) \sqcup (f_{\sigma(S\cap T)}\sigma_{|S\cap T})\big].$$

Let $\tilde{\sigma}_{S'}$ and $\tilde{\sigma}_{S\cap T}$ be the only bijections such that the following diagrams commute

$$\begin{tikzcd} S' \arrow[r, "f_{S'}"] \arrow[d, "\sigma_{|S'}"'] & {[i]} \arrow[d, "\tilde{\sigma}_{S'}"] \\ \sigma(S') \arrow[r, "f_{\sigma(S')}"] & {[i]} \end{tikzcd} \qquad \begin{tikzcd} S \cap T \arrow[r, "f_{S\cap T}"] \arrow[d, "\sigma_{|S\cap T}"'] & {[n-j+1, i]]} \arrow[d, "\tilde{\sigma}_{S\cap T}"] \\ \sigma(S \cap T) \arrow[r, "f_{\sigma(S\cap T)}"] & {[n-j+1, i]} \end{tikzcd} \tag{3.14}$$

We conclude that α can be rewritten as

$$\alpha = \mathsf{p}\big[(\tilde{\sigma}_{S'} \sqcup \tilde{\sigma}_{S\cap T})(f_{S'} \sqcup f_{S\cap T})\big] = \mathsf{p}[\tilde{\sigma}_{S'} \sqcup \tilde{\sigma}_{S\cap T}]\,\mathsf{p}[f_{S'} \sqcup f_{S\cap T})].$$

We can do the same with respect to β. We deduce, according to the action (3.12), that

$$\alpha(u) \otimes \beta(v) = (\tilde{\sigma}_{S'} \sqcup \tilde{\sigma}_{S\cap T} \sqcup \tilde{\sigma}_{T'}) \cdot \big(\mathsf{p}[f_{S'} \sqcup f_{S\cap T}](u) \otimes \mathsf{q}[f_{S\cap T} \sqcup f_{T'}](v)\big).$$

Note that the permutation $\tilde{\sigma}_{S'} \sqcup \tilde{\sigma}_{S\cap T} \sqcup \tilde{\sigma}_{T'}$ clearly belongs to $S_i \times_n S_j$. Consider Equation (3.13). Since the tensor product of $f^{-1}_{\sigma(S),\sigma(T)}$ with $\alpha(u) \otimes \beta(v)$ is done over this subgroup, then we can move the permutation to the left factor where we get $f^{-1}_{\sigma S,\sigma T}(\tilde{\sigma}_{S'} \sqcup \tilde{\sigma}_{S\cap T} \sqcup \tilde{\sigma}_{T'}) = \sigma f^{-1}_{S,T}$. This equality results again from the diagrams (3.14).

This is precisely the definition of the action of σ on the image of the map ψ.

The map ψ is invertible, since for any element $\sigma \otimes (x \otimes y)$, we decompose $\sigma = \xi(\alpha \times \beta \times \gamma)$, where $\alpha \times \beta \times \gamma \in S_{n-j} \times S_{i+j-n} \times S_{n-i} = S_i \times_n S_j$ and ξ is increasing in the intervals $[n-j]$, $[n-j+1, i]$, and $[i+1, n]$. Define the disjoint sets

$$A = \xi\big([n-j]\big), \quad B = \xi\big([n-j+1, i]\big), \quad C = \xi\big([i+1, n]\big).$$

Then, let $S = A \sqcup B$ and $T = B \sqcup C$. It is straightforward to find $u \otimes v$ in $\mathsf{p}[S] \otimes \mathsf{q}[T]$ such that $\psi(u \otimes v) = \sigma \otimes (x \otimes y)$. Similarly, this process applied to the image of ψ in (3.11) yields back $u \otimes v$. $\square$

Remark 3.2.3. It is clear directly from Definition (2.2) that the sum of species corresponds to the direct sum of S_n-modules in each coordinate, under the functor $\mathcal{F}$:

$$\mathcal{F}(\mathsf{p} \oplus \mathsf{q})_n = \mathcal{F}(\mathsf{p})_n \oplus \mathcal{F}(\mathsf{q})_n. \tag{3.15}$$

3.3. The Grothendieck ring of R

Now we define the Heisenberg product in the Grothendieck group of **R**. This step will help us in Section 4 to construct the Heisenberg product in the space of symmetric functions.

Let $\hat{\mathcal{K}}$ be the Grothendieck group of the category **R**. The group $\hat{\mathcal{K}}$ can be expressed in terms of the Grothendieck groups of the representations of S_n as follows. Let $\mathrm{K}(S_n)$ be the Grothendieck group of the category of finitely generated projective $\Bbbk S_n$-modules. In our situation, in which the field $\Bbbk$ has characteristic 0, any finite-dimensional $\Bbbk S_n$-module is finitely generated projective (and conversely). Then, we have the relation

$$\hat{\mathcal{K}} = \prod_{n \geq 0} \mathrm{K}(S_n).$$

Given an object V of $\mathbf{R}$ (respectively, $V \in \mathrm{Rep}(S_n)$), we denote by $[V]$ the corresponding image in the Grothendieck group $\hat{\mathcal{K}}$ (respectively, $\mathrm{K}(S_n)$).

Our goal is to define an associative product on $\hat{\mathcal{K}}$, induced from the Heisenberg product of representations.

Theorem 3.3.1. *There is an associative product* $\# : \hat{\mathcal{K}} \times \hat{\mathcal{K}} \to \hat{\mathcal{K}}$ *which makes* $\hat{\mathcal{K}}$ *an unital ring and such that*

$$[V] \# [W] = [V \# W]$$

for all $V, W \in \mathbf{R}$.

Proof. The Grothendieck group $\mathrm{K}(S_n)$ can be described explicitly as follows. Let $F^{(n)}$ be the free abelian group generated by the isomorphism classes of representations of S_n, which we denote with the symbol (V), where $V \in \mathrm{Rep}(S_n)$. The Grothendieck group $\mathrm{K}(S_n)$ is the quotient $F^{(n)}/F_0^{(n)}$, where $F_0^{(n)}$ is the subgroup of $F^{(n)}$ generated by the expressions

$$(V \oplus W) - (V) - (W), \tag{3.16}$$

for V and W representations of S_n. For a representation V of S_n, the element $[V]$ is the projection of (V) onto $F^{(n)}/F_0^{(n)}$.

It is clear, due to the functoriality of the operations involved in the definition of the Heisenberg product of representations (3.7), that $V \# V' \cong U \# U'$ whenever $V \cong V'$ and $U \cong U'$, for any representations $U, U' \in \mathrm{Rep}(S_p)$ and $V, V' \in \mathrm{Rep}(S_q)$. Then, the Heisenberg product of representations induces an operation $\# : F^{(p)} \times F^{(q)} \to \hat{\mathcal{K}}$.

We need to check that the operation $\#$ descends to the quotient by $F_0^{(n)}$ in each coordinate, and therefore, this operation can be defined on $\hat{\mathcal{K}}$. To verify this we can use the language of species, which again proves to be more convenient than that of representations. Let p, q, and r be three species. Recalling the definition of sum of species (Equation (2.2)) and Remark 3.2.3 we have

$$\begin{aligned}
\big((\mathsf{p}\oplus\mathsf{q})\,\#\,\mathsf{r}\big)[I] &= \bigoplus_{I=S\cup T}(\mathsf{p}\oplus\mathsf{q})[S]\otimes\mathsf{r}[T] = \bigoplus_{I=S\cup T}\big(\mathsf{p}[S]\oplus\mathsf{q}[S]\big)\otimes\mathsf{r}[T]\\
&\cong \left(\bigoplus_{I=S\cup T}\mathsf{p}[S]\otimes\mathsf{r}[T]\right)\oplus\left(\bigoplus_{I=S\cup T}\mathsf{q}[S]\otimes\mathsf{r}[T]\right)\\
&= \big((\mathsf{p}\,\#\,\mathsf{r})\oplus(\mathsf{q}\,\#\,\mathsf{r})\big)[I],
\end{aligned}$$

and the isomorphism is natural. The same holds for the left multiplication.

The previous discussion shows that the product induced from the Heisenberg product of species is a well defined associative operation:

$$\#:\hat{\mathcal{K}}\times\hat{\mathcal{K}}\to\hat{\mathcal{K}}.$$

The unit of this product is $\big([\Bbbk],0,0,\ldots\big)$, with S_n acting trivially on $\Bbbk$. The associativity and unitality follow from the same properties for representations or species (Theorem 3.2.2). □

We call $\hat{\mathcal{K}}$ the Grothendieck ring with the Heisenberg product. Inside $\hat{\mathcal{K}}$ we have the subgroup $\mathcal{K}$ defined by

$$\mathcal{K}=\bigoplus_{n\geq 0}\mathrm{K}(S_n)\subseteq\hat{\mathcal{K}},$$

which is clearly a subring of $\hat{\mathcal{K}}$ since the definition of the Heisenberg product involves only a finite number of summands. In Section 4 we recall the relation between $\mathcal{K}$ and the space of symmetric functions.

It is convenient to extend the scalars of $\mathcal{K}$ to the field $\Bbbk$. For this, we consider the vector space

$$\mathcal{K}_{\Bbbk}=\mathcal{K}\otimes_{\mathbb{Z}}\Bbbk.$$

Any operation on $\mathcal{K}$ can be trivially extended to $\mathcal{K}_{\Bbbk}$, in particular we can extend the

Heisenberg product as:

$$(v \otimes_{\mathbb{Z}} x) \# (w \otimes_{\mathbb{Z}} y) = (v \# w) \otimes_{\mathbb{Z}} (xy).$$

This extension makes $\mathcal{K}_{\Bbbk}$ a $\Bbbk$-algebra with the Heisenberg product, while $\mathcal{K}$ is just a ring.

4. SYMMETRIC FUNCTIONS

4.1. From species to symmetric functions

Representations of the symmetric group and symmetric functions are closely related. We recall here the connection and we transport the Heisenberg product of representations along this construction, to obtain a product in symmetric functions.

Let us recall the definition of the space of symmetric functions and its relation with the Grothendieck group $\mathcal{K}$, defined in Subsection 3.3.

Consider the ring of polynomials $\mathbb{k}[x_1, \ldots, x_n]$ in n variables. The symmetric group S_n acts on it by permuting the variables. Then, let Λ_n^k be the subring consisting of the homogeneous polynomials of degree k which are invariant under the action of S_n. When $m \geq n$, we can project Λ_m^k onto Λ_n^k via a homomorphism $\rho_{m,n}^k : \Lambda_m^k \to \Lambda_n^k$ which maps the variables x_i with $i \leq n$ to themselves, and the variables x_i with $i > n$ to 0. We define Λ^k as the inverse limit

$$\Lambda^k = \varprojlim_n \Lambda_n^k$$

with respect to the homomorphisms $\rho_{m,n}^k$. The space of *symmetric functions* and its completion, respectively, are defined as (see [18]):

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k, \qquad \hat{\Lambda} = \prod_{k \geq 0} \Lambda^k,$$

The *Frobenius characteristic map* is the linear map

$$\mathrm{ch} : \mathcal{K}_{\mathbb{k}} \to \Lambda, \quad \mathrm{ch}\big([V]\big) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{\mathrm{V}}(\sigma) p_{\mathrm{cycle}(\sigma)},$$

where V is a representation of S_n, the coefficient $\chi_{\mathrm{V}}(\sigma)$ is the character associated to the representation V evaluated at σ, and $p_{\mathrm{cycle}(\sigma)}$ is the power sum (4.5) associated to the cycle-type of σ. The Frobenius characteristic map is an isomorphism [18, Proposition 7.3].

Through the composition of the Grothendieck group of representations of the symmetric group followed by the Frobenius characteristic we obtain an associative product on symmetric functions, which we call *Heisenberg product of symmetric functions*. That is, if $f = \mathrm{ch}(v)$ and $g = \mathrm{ch}(w)$ for $v, w \in \mathcal{K}_{\Bbbk}$, then

$$f \# g = \mathrm{ch}(v \# w).$$

We give more explicit formulae for the Heisenberg product in the space of symmetric functions in Section 4.3.

In summary, we have introduced the Heisenberg product in the objects:

$$\mathbf{Sp} \xrightarrow{\cong} \mathbf{R} \Longrightarrow \hat{\mathcal{K}} \supset \mathcal{K} \xrightarrow{\otimes_{\mathbb{Z}}\Bbbk} \mathcal{K}_{\Bbbk} \xrightarrow{\cong} \Lambda.$$

where the double arrow means the application of the Grothendieck group functor.

4.2. Classical products of symmetric functions

The space of symmetric functions Λ and its completion $\hat{\Lambda}$ are subrings of $\Bbbk[x_1, x_2, \dots]$ and $\Bbbk[\![x_1, x_2, \dots]\!]$, respectively, where the product in the latter spaces is the usual product of polynomials and power series. We call this operation the *external product of symmetric functions.*

With the external product in Λ, the Frobenius characteristic map is an isomorphism when we consider the product in $\mathcal{K}_{\Bbbk}$ which comes from the induction product (3.3) in representations [18].

Observe that the external product of symmetric functions is commutative, though at the level of species and representations we only have an isomorphism $\mathsf{p} \cdot \mathsf{q} \cong \mathsf{q} \cdot \mathsf{p}$.

The *internal product of symmetric functions* is defined usually as the image under the Frobenius characteristic map of the Kronecker product of representations (3.4). More explicitly, if $f = \mathrm{ch}\big([V]\big)$ and $g = \mathrm{ch}\big([W]\big)$, where V and W are representations of S_n, then the internal product of f and g is

$$f * g = \mathrm{ch}\big([V * W]\big).$$

This gives and associative product on Λ. Note, however, that it has no unit in Λ. The unit for the internal product in $\hat{\mathcal{K}}$ is the image of the object $(\Bbbk, \Bbbk, \ldots)$ under the Grothendieck group functor, which does not lie in $\mathcal{K}$.

Let us define some of the well-known linear basis of the space of symmetric functions Λ. We first recall the definition of the objects used to index these bases. A *weak composition* of a non-negative integer n is a finite sequence of non-negative integers $\alpha = (a_1, a_2, \ldots, a_r)$ such that

$$\sum_{i=1}^{n} a_i = n.$$

The numbers a_i are called *parts* of the weak composition α. When all the parts a_i are positive, we say that α is a *composition* of n. A *partition* of n is a composition $\alpha = (a_1, a_2, \ldots, a_r)$ of n whose parts are ordered non-increasingly:

$$a_1 \geq a_2 \geq \cdots \geq a_r.$$

In addition to these basic terms, we define two operators on compositions, which will be used later in relation with the Heisenberg product of symmetric and non-commutative symmetric functions. Given a composition α of n, we denote by $\tilde{\alpha}$ the partition of n resulting from reordering the parts of α in a non-increasing way. If α is a weak composition of n, we let $\hat{\alpha}$ be the composition of n which results from deleting the parts equal to 0 from α.

Let $\alpha = (a_1, \dots, a_r)$ be a weak composition of n and let

$$S_\alpha = S_{a_1} \times \cdots \times S_{a_r}. \tag{4.1}$$

We view S_α as a subgroup of S_n by iterating (3.2). These are called *standard parabolic subgroups* of S_n. Let h_α denote the permutation representation of S_n corresponding to the action by multiplication on the quotient S_n/S_α. By reordering the parts of the composition α we obtain a subgroup conjugate to S_α. Hence, the isomorphism class of h_α does not depend on the order of the parts of α, and thus we will consider the representations h_α for α running over the partitions of n.

If we denote the trivial $\Bbbk S_\alpha$-module by $\mathbf{1}$ (we omit the dependence on α for simplicity), then the representation h_α can also be expressed as

$$h_\alpha = \mathrm{Ind}_{S_\alpha}^{S_n}(\mathbf{1}). \tag{4.2}$$

The symmetric functions associated to the representations h_α are called *complete homogeneous symmetric functions*. We use the same symbol for the representation and for the symmetric function. More explicitly, the symmetric function $h_{(n)}$ can be written

$$h_{(n)}(x) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k}, \tag{4.3}$$

where $(\lambda_1, \dots, \lambda_k)$ runs over all the compositions of n, and the indices $(i_1, i_2, \dots, i_k)$ run over all k-tuples of positive integers. We also define $h_{(0)} = 1$. For a weak composition $\alpha = (a_1, \dots, a_n)$ we have $h_\alpha = h_{(a_1)} \cdots h_{(a_n)}$. The family of functions $\{h_\alpha\}$ with α ranging over the partitions of n form a linear basis of the space Λ^n; and the family of functions $\{h_\alpha\}$, with α a partition, is a basis of Λ.

The external product has a simple expression in the basis of complete homogeneous functions. Given two partitions α and β, denote by $\alpha\beta$ the concatenation and reordering of α and β. For example, if $\alpha = (3, 2, 1, 1)$ and $\beta = (2, 2, 1)$, then $\alpha\beta = (3, 2, 2, 2, 1, 1, 1)$. Then, the external product of h_α and h_β is

$$h_\alpha \cdot h_\beta = h_{\alpha\beta}. \tag{4.4}$$

The internal product also has a nice combinatorial description. We will recall this rule in Remark 4.3.2, as a particular case of the rule for the Heisenberg product in the basis of complete homogeneous symmetric functions.

Another well-known basis for Λ are the symmetric power sums. Let k be a positive integer. We write

$$p_k = \sum_{i \geq 1} x_i^k.$$

Let $\alpha = (a_1, a_2, \dots, a_r)$ be a partition of n. The *symmetric power sum* associated to the partition α is

$$p_\alpha = p_{a_1} p_{a_2} \cdots p_{a_r}. \tag{4.5}$$

The family of functions $\{p_\alpha\}$ is a linear basis of Λ.

The multiplication rules are particularly easy in this basis. The external product of power sums is computed by concatenation: $p_\alpha \cdot p_\beta = p_{\alpha\beta}$. The internal product is computed by

$$p_\alpha * p_\beta = \begin{cases} z(\alpha) p_\alpha, & \text{if } \alpha = \beta; \\ 0, & \text{otherwise}; \end{cases} \tag{4.6}$$

where $z(\alpha)$ is the order of the stabilizer of the conjugacy class of a permutation of cycle-type α. Explicitly, $z(\alpha)$ has the expression

$$z(\alpha) = \prod_r r^{m_r} m_r!, \tag{4.7}$$

where m_r is the number of times r occurs in α. We agree that $z(\alpha) = 1$ when α is the empty partition.

Another important basis of Λ are the Schur functions. We will not work on this basis, but we make some comments in Section 4.6.

4.3. The Heisenberg product of symmetric functions

As we saw in Section 4.1, the Heisenberg product of species corresponds to a product $\# : \hat{\Lambda} \times \hat{\Lambda} \to \hat{\Lambda}$ which restricts to a product $\# : \Lambda \times \Lambda \to \Lambda$. The interpolation property (3.8) translates to an analogous property of symmetric functions:

$$f \# g = f * g + \cdots + f \cdot g$$

for $f, g \in \hat{\Lambda}$. We will give the expression of the Heisenberg product on some of the bases of Λ and we will show an explicit form for this interpolation.

The first basis we consider is the basis of complete homogeneous functions. In order to express the coefficients of the Heisenberg product of two complete homogeneous symmetric functions, we need to define a particular set of plane partitions as follows. Let $\alpha = (a_1, \ldots, a_r) \vDash p$ and $\beta = (b_1, \ldots, b_s) \vDash q$ be two compositions, and n an integer with $\max(p, q) \leq n \leq p + q$. Let $a_0 = n - p$, $b_0 = n - q$, and let $\mathcal{M}^n_{\alpha,\beta}$ be the set of all $(s+1) \times (r+1)$-matrices

$$M = (m_{ij})_{\substack{0 \leq i \leq s \\ 0 \leq j \leq r}}$$

with non-negative integer entries and such that

- the sequence of column sums is $(a_0, a_1, \ldots, a_r)$,
- the sequence of row sums is $(b_0, b_1, \ldots, b_s)$,
- the first entry is $m_{00} = 0$.

We illustrate these conditions as follows:

$$\begin{array}{cccc|c} 0 & m_{01} & \cdots & m_{0r} & n-q \\ m_{10} & m_{11} & \cdots & m_{1r} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{s0} & m_{s1} & \cdots & m_{sr} & b_s \\ \hline n-p & a_1 & \cdots & a_r & \end{array} \tag{4.8}$$

Let $p(M)$ be the partition of n whose parts are the non-zero entries m_{ij} of the matrix M.

The next theorem gives an explicit formula for the Heisenberg product in the linear basis of Λ formed by the complete homogeneous symmetric functions. In addition to be a combinatorial rule useful for computations, this formula allow us to make the connection with the Heisenberg product of non-commutative symmetric functions in Section 8.1.

Theorem 4.3.1. *There is an associative product $\#$ in Λ, interpolating between the internal and external products, which can be expressed in the basis $\{h_\alpha\}$ of complete homogeneous functions as*

$$h_\alpha \# h_\beta = \sum_{n=\max(p,q)}^{p+q} \sum_{M \in \mathcal{M}^n_{\alpha,\beta}} h_{p(M)}, \tag{4.9}$$

where α is a partition of p and β is a partition of q.

For example, using Theorem 4.3.1 we get

$$h_{(2,1)} \# h_{(3)} = h_{(2,1)} + h_{(1,1,1,1)} + h_{(2,1,1)} + h_{(2,2,1)} + h_{(2,1,1,1)} + h_{(3,2,1)}.$$

where the external product is recognized in the last term:

$$h_{(2,1)} \cdot h_{(3)} = h_{(3,2,1)}$$

and the internal product in the first one:

$$h_{(2,1)} * h_{(3)} = h_{(2,1)},$$

since $h_{(3)}$ is the identity for the internal product in Λ^3, together with additional terms of degrees four and five.

Remark 4.3.2. We can deduce from Theorem 4.3.1 a well-known formula for the internal product of h_α and h_β, where α and β are partitions of n, by considering only the term at degree n in the sum (4.9). Indeed, suppose that $p = q = n$. In this case, the top row and leftmost column of (4.8), which add up to $n - p = 0$ and $n - q = 0$, should have entries 0. The remaining part of the matrix is precisely the rule for computing the internal product of symmetric functions in the basis h_α:

$$h_\alpha * h_\beta = \sum_M h_{p(M)}, \tag{4.10}$$

where M ranges over the set of matrices of dimension $n \times n$ with non-negative entries such that its columns add up to α and its rows add up to β. This rule can be found in [26, Exercise 7.84].

On the other hand, consider in Formula (4.9) the case $n = p+q$. In that situation, the only way to fill the matrix (4.8) is by placing zeroes in the entries m_{ij}, with $i \neq 0$ and $j \neq 0$, and by placing the partition α as the top row and the partition β as the leftmost column. Hence, we have the classical formula $h_\alpha \cdot h_\beta = h_{\alpha\beta}$ (4.4).

We show other more interesting examples of the rule for the Heisenberg product in the context of non-commutative symmetric functions in Section 7.2.

The existence of this operation poses the problem of finding an explicit description for its structure constants on the basis of Schur functions. The answer would contain as extreme cases the Littlewood-Richardson rule and (a still unknown) rule for the Kronecker coefficients (see some comments on Section 4.6).

Proof of Theorem 4.3.1. We prove that the following formula holds in the category $\mathbf{R}$:

$$h_\alpha \# h_\beta = \bigoplus_{n=\max(p,q)}^{p+q} \bigoplus_{M \in \mathcal{M}^n_{\alpha,\beta}} h_{p(M)},$$

where the representations h_α are the induced representations defined in (4.2). An application of the Grothendieck group functor and the Frobenius characteristic immediately yields (4.9).

We fix n in the range $\max(p,q) \leq n \leq p+q$. The n-summand of $h_\alpha \# h_\beta$ is, according to (3.7),

$$(h_\alpha \# h_\beta)_n = \mathrm{Ind}^{S_n}_{S_p \times_n S_q} \mathrm{Res}^{S_p \times S_q}_{S_p \times_n S_q} (h_\alpha \otimes h_\beta) = \mathrm{Ind}^{S_n}_{S_p \times_n S_q} \mathrm{Res}^{S_p \times S_q}_{S_p \times_n S_q} \mathrm{Ind}^{S_p \times S_q}_{S_\alpha \times S_\beta}(\mathbf{1}). \quad (4.11)$$

Consider the composition of the first two functors $\mathrm{Res}^{S_p \times S_q}_{S_p \times_n S_q} \mathrm{Ind}^{S_p \times S_q}_{S_\alpha \times S_\beta}$ in the right hand side of (3.7). We use Mackey's formula to interchange them (see [28]), as follows.

Let $\Upsilon \subset S_p \times S_q$ be a set of representatives of the family of double cosets $(S_p \times_n S_q) \setminus (S_p \times S_q) \,/\, (S_\alpha \times S_\beta)$. For each $\upsilon \in \Upsilon$, define

$${}^{\upsilon}(S_\alpha \times S_\beta) = \upsilon^{-1}(S_\alpha \times S_\beta)\upsilon \quad \text{and} \quad S_\alpha \times^{\upsilon}_{n} S_\beta = (S_p \times_n S_q) \cap {}^{\upsilon}(S_\alpha \times S_\beta). \quad (4.12)$$

The following diagram illustrates the relative position of these groups and subgroups

$$\begin{tikzcd}
S_n & & S_p \times S_q & \\
& S_p \times_n S_q \arrow[ul, tail] \arrow[ur, tail] & & S_\alpha \times S_\beta \arrow[ul, tail] \\
& & S_\alpha \times^{\upsilon}_{n} S_\beta \arrow[ul, tail] \arrow[ur, tail] &
\end{tikzcd}$$

In this situation Mackey's formula reads as the equality

$$\mathrm{Res}^{S_p \times S_q}_{S_p \times_n S_q} \mathrm{Ind}^{S_p \times S_q}_{S_\alpha \times S_\beta}(\mathbf{1}) = \bigoplus_{\upsilon \in \Upsilon} \mathrm{Ind}^{S_p \times_n S_q}_{S_\alpha \times^{\upsilon}_{n} S_\beta} \mathrm{Res}^{S_\alpha \times S_\beta}_{S_\alpha \times^{\upsilon}_{n} S_\beta}.$$

Using the transitivity of the induction functor and the property that it commutes with coproducts we deduce that (4.11) can be written as

$$(h_\alpha \# h_\beta)_n = \operatorname{Ind}^{S_n}_{S_p \times_n S_q} \operatorname{Res}^{S_p \times S_q}_{S_p \times_n S_q} (h_\alpha \otimes h_\beta) = \bigoplus_{\upsilon \in \Upsilon} \operatorname{Ind}^{S_n}_{S_\alpha \times^\upsilon_n S_\beta}(\mathbf{1}). \tag{4.13}$$

In Lemma 4.3.3 we construct a bijection $\upsilon \mapsto M_\upsilon$ between Υ and $\mathcal{M}^n_{\alpha,\beta}$ with the property that $S_{p(M_\upsilon)} = S_\alpha \times^\upsilon_n S_\beta$. Then (4.13) becomes

$$(h_\alpha \# h_\beta)_n = \bigoplus_{\upsilon \in \Upsilon} \operatorname{Ind}^{S_n}_{S_\alpha \times^\upsilon_n S_\beta}(\mathbf{1}) = \bigoplus_{\upsilon \in \Upsilon} \operatorname{Ind}^{S_n}_{S_{p(M_\upsilon)}}(\mathbf{1}) = \bigoplus_{M \in \mathcal{M}^n_{\alpha,\beta}} M_{p(M)},$$

proving the theorem. □

Lemma 4.3.3. *In the notations of Theorem 4.3.1, there is a bijection $\upsilon \mapsto M_\upsilon$ between Υ and $\mathcal{M}^n_{\alpha,\beta}$ such that $S_{p(M_\upsilon)} = S_\alpha \times^\upsilon_n S_\beta$.*

Proof. To define the bijection $\Upsilon \to \mathcal{M}^n_{\alpha,\beta}$, we start by splitting the intervals $[1,p]$ and $[1,q]$ as below:

$$\begin{aligned}
E_1 &= [1, a_1], & F_1 &= [1, b_1], \\
E_2 &= [a_1 + 1, a_1 + a_2], & F_2 &= [b_1 + 1, b_1 + b_2], \\
&\ \vdots & &\ \vdots \\
E_k &= [a_1 + \cdots + a_{k-1} + 1, p], & F_s &= [b_1 + \cdots + b_{s-1} + 1, q],
\end{aligned}$$

where $\alpha = (a_1, \ldots, a_k)$ and $\beta = (b_1, \ldots, b_s)$. Given an element $\upsilon = \sigma \times \tau \in S_p \times S_q$ we consider the shuffles $\zeta_\alpha(\sigma) \in \operatorname{Sh}(\alpha)$ and $\zeta_\beta(\tau) \in \operatorname{Sh}(\beta)$ characterized by the equations

$$\sigma = \zeta_\alpha(\sigma) u, \qquad \tau = \zeta_\beta(\tau) v, \tag{4.14}$$

with $u \in S_\alpha$ and $v \in S_\beta$. To simplify the notation, we write $\zeta_\alpha = \zeta_\alpha(\sigma)$ and $\zeta_\beta = \zeta_\beta(\tau)$.

We further split each interval E_i and F_j as below:

$$E_i = E_i' \sqcup E_i'', \qquad F_j = F_j' \sqcup F_j'',$$

such that

$$\begin{aligned} \zeta_\alpha(E_i') &\subseteq [1, n-q], & \zeta_\beta(F_j') &\subseteq [1, p+q-n], \\ \zeta_\alpha(E_i'') &\subseteq [n-q+1, p], & \zeta_\beta(F_j'') &\subseteq [p+q-n+1, q], \end{aligned}$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, s$. Observe that with these definitions we have the decomposition of the interval $[1, n]$ into

$$[1, n-q] = \bigsqcup_{i=1}^{k} \zeta_\alpha(E_i'), \tag{4.15}$$

$$[n-q+1, p] = \bigsqcup_{i=1}^{k} \zeta_\alpha(E_i'') = \bigsqcup_{j=1}^{s} \big(n - q + \zeta_\beta(F_j')\big), \tag{4.16}$$

$$[p+1, n] = \bigsqcup_{j=1}^{s} \big(n - q + \zeta_\beta(F_j'')\big). \tag{4.17}$$

Define the matrix $M_{\sigma\times\tau}$ of dimension $(k+1)\times(s+1)$ whose entries are

$$\begin{aligned} &m_{00} = 0, \\ &m_{i0} = \#\, E_i', && \text{for } i = 1, \ldots, k, \\ &m_{0j} = \#\, F_j'', && \text{for } j = 1, \ldots, s, \\ &m_{ij} = \#\left[\zeta_\alpha(E_i'') \cap \big(n - q + \zeta_\beta(F_j')\big)\right] && \text{otherwise.} \end{aligned}$$

The matrix $M_{\sigma\times\tau}$ belongs to $\mathcal{M}^n_{\alpha,\beta}$. Assume that $i \neq 0$. Since $\zeta_\alpha(E_i'') \subseteq [n-q+$

$1,p] \subseteq \bigsqcup_{j=1}^{s}\big(n-q+\zeta_\beta(F_j'')\big)$, we get

$$\begin{aligned}\sum_{j=0}^{s} m_{ij} &= \#\, E_i' + \sum_{j=1}^{s} \#\left[\zeta_\alpha(E_i'') \cap \big(n-q+\zeta_\beta(F_j')\big)\right] \\ &= \#\, E_i' + \#\Big[\zeta_\alpha(E_i'') \cap \bigsqcup_{j=1}^{s}\big(n-q+\zeta_\beta(F_j')\big)\Big] \\ &= \#\, E_i' + \#\big(\zeta_\alpha(E_i'')\big) = \#\, E_i = a_i.\end{aligned}$$

On the other hand, if $i=0$, then, by (4.17), the sum of m_{0j} for $j=0,\dots,s$, coincides with $\#[p+1,n]=n-p$.

Next we show that the matrix $M_{\sigma\times\tau}$ does not depend on the choice of representative of the coset $(S_p \times_n S_q)v$. Let $x \in S_{n-q}$, $y \in S_{p+q-n}$, and $z \in S_{n-p}$, so that $x \times y \times z \in S_p \times_n S_q$. Consider the representative $v' = \sigma' \times \tau'$ where

$$\sigma' = (x \times y)\sigma \quad \text{and} \quad \tau' = (y \times z)\tau.$$

Let ζ_α' and ζ_β' the shuffles associate to v'. As $\zeta_\alpha(E_i') \subseteq [1,n-q]$, then $(x\times y)\big(\zeta_\alpha(E_i')\big) = x\big(\zeta_\alpha(E_i')\big)$. But we also have $x\big(\zeta_\alpha(E_i')\big) = \zeta_\alpha'(E_i) = \zeta_\alpha'(\tilde{E}_i') \sqcup \zeta_\alpha'(\tilde{E}_i'')$, where $E_i = \tilde{E}_i' \sqcup \tilde{E}_i''$ is the decomposition of E_i corresponding to the shuffle ζ_α', that satisfies $\zeta_\alpha'(\tilde{E}_i') \subseteq [1,n-q]$ and $\zeta_\alpha'(\tilde{E}_i'') \subseteq [n-q+1,p]$. In summary, $x\big(\zeta_\alpha(E_i')\big) \subseteq \zeta_\alpha'(\tilde{E}_i')$. Interchanging the roles of ζ_α and ζ_α' we obtain an equality, which implies that $m_{i0} = \#\, E_i' = \#\, \tilde{E}_i' = m_{i0}'$, where m_{ij}' are the entries of the matrix $M_{v'}$. This proves the equality of the first row of the matrices. The argument for the other rows is similar.

The matrix M_v does not depend on the choice of representative of $v(S_\alpha \times S_\beta)$, since the shuffles satisfying (4.14) are the same for all the elements on this coset. In conclusion, the matrix M_v depends only on the double cosets $(S_p \times_n S_q)v(S_\alpha \times S_\beta)$.

Next we show that the parabolic subgroup $S_{p(M_v)}$ is $S_\alpha \times_n^v S_\beta$. An element of

$S_\alpha \times_n^\upsilon S_\beta$ can be written as $x \times y \times z$ where

$$x \times y = \zeta_\alpha(\sigma_{a_1} \times \cdots \times \sigma_{a_k})\zeta_\alpha^{-1},$$

$$y \times z = \zeta_\beta(\tau_{b_1} \times \cdots \times \tau_{b_s})\zeta_\beta^{-1}.$$

Evaluating at $\zeta_\alpha(E_i')$ we deduce that $\zeta_\alpha\sigma_{a_i}(E_i') = x(E_i')$ and conclude that $\sigma_{a_i}(E_i') = E_i'$. Proceeding in a similar manner with the other decompositions we obtain

$$\sigma_{a_i}(E_i') = E_i', \qquad \tau_{b_j}(F_j') = F_j', \tag{4.18}$$

$$\sigma_{a_i}(E_i'') = E_i'', \qquad \tau_{b_j}(F_j'') = F_j'', \tag{4.19}$$

for all $i = 1, \ldots, k$ and $j = 1, \ldots, s$.

This decomposition can be further refined. Evaluating as above at the subsets $X_{ij} = \zeta_\alpha(E_i'') \cap \zeta_\beta(F_j')$, we obtain the equality

$$\zeta_\alpha\sigma_{a_i}\big(\zeta_\alpha^{-1}(X_{ij})\big) = y(X_{ij}) = \zeta_\beta\tau_{b_j}\big(\zeta_\beta^{-1}(X_{ij})\big).$$

Now, $\zeta_\alpha\sigma_{a_i}\big(\zeta_\alpha^{-1}(X_{ij})\big) \subseteq \zeta_\alpha(E_i'')$ and also $\zeta_\beta\tau_{b_j}\big(\zeta_\beta^{-1}(X_{ij})\big) \subseteq \zeta_\beta(F_j')$. From the above equality we conclude that $\zeta_\alpha\sigma_{a_i}\big(\zeta_\alpha^{-1}(X_{ij})\big) \subseteq \zeta_\alpha(E_i'') \cap \zeta_\beta(F_j')$, and so $\sigma_{a_i}\big(\zeta_\alpha^{-1}(X_{ij})\big) \subseteq \zeta_\alpha^{-1}(X_{ij})$. This inclusion is actually an equality, since both sets have the same cardinality. Therefore, we get the following refinement of (4.18)

$$\sigma_{a_i}(E_i') = E_i', \qquad \sigma_{a_i}\big(\zeta_\alpha^{-1}(X_{ij})\big) = \zeta_\alpha^{-1}(X_{ij}),$$

$$\tau_{b_j}(F_j'') = F_j'', \qquad \tau_{b_j}\big(\zeta_\beta^{-1}(X_{ij})\big) = \zeta_\beta^{-1}(X_{ij}).$$

Note that $\# X_{ij} = m_{ij}$, and thus the previous decomposition shows that $x \times y \times z$ belongs to $S_{p(M)}$.

The map $\upsilon \mapsto M_\upsilon$ is invertible, since from the entries of the matrix M_υ we can recover the shuffles ζ_α and ζ_β, which are in the same double coset as υ. $\square$

4.4. The Heisenberg product in terms of the classical products

The Heisenberg product of symmetric functions can be expressed in terms of the classical products and the coproduct of symmetric functions. This fact was brought to our attention by A. Zelevinsky.

We remark that this relation is specific to symmetric functions and does not hold true for the Heisenberg product of non-commutative symmetric functions or of permutations. These are discussed in Sections 6.2 and 7.2 (see also remark after Theorem 7.2.3).

The space Λ of symmetric functions has a coproduct which is dual to the external product with respect to the pairing

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu},$$

where s_λ is the Schur function associated to the partition λ (see Section 4.6). In the basis of complete homogeneous symmetric functions h_α, this coproduct has the expression

$$\Delta(h_a) = \sum_{i+j=a} h_i \otimes h_j, \qquad \Delta(h_{(a_1,\dots,a_n)}) = \Delta(h_{a_1}) \cdots \Delta(h_{a_n}). \tag{4.20}$$

Theorem 4.4.1. *The Heisenberg product of two symmetric functions f and g can be written in terms of the classical products as*

$$f \# g = \sum f_1 \cdot (f_2 * g_1) \cdot g_2, \tag{4.21}$$

where we use Sweedler notation: $\Delta(f) = \sum f_1 \otimes f_2$ and $\Delta(g) = \sum g_1 \otimes g_2$.

Proof. Let p and q be non-negative integers. Let $\alpha = (a_1, \dots, a_r)$ and $\beta = (b_1, \dots, b_s)$ be partitions of p and q, respectively. We prove Identity (4.21) by computing $h_\alpha \# h_\beta$ using Formula (4.9).

Note that $\Delta(h_\alpha)$ can be computed, from Equation (4.20), as

$$\Delta(h_\alpha) = \sum h_{(a_1^{(1)},\ldots,a_r^{(1)})} \otimes h_{(a_1^{(2)},\ldots,a_r^{(2)})} \tag{4.22}$$

where $a_i^{(k)} \geq 0$ and $a_i^{(1)} + a_i^{(2)} = a_i$ for $k = 1, 2$ and $i = 1, \ldots, r$. Recall the convention that $h_{(0)} = 1$. Let

$$\alpha_1 = (a_1^{(1)}, \ldots, a_r^{(1)}), \qquad \beta_1 = (b_1^{(1)}, \ldots, b_s^{(1)}),$$
$$\alpha_2 = (a_1^{(2)}, \ldots, a_r^{(2)}), \qquad \beta_2 = (b_1^{(2)}, \ldots, b_s^{(2)}),$$

so that $h_{\alpha_1} \otimes h_{\alpha_2}$ is a term of $\Delta(h_\alpha)$ and $h_{\beta_1} \otimes h_{\beta_2}$ is a term of $\Delta(h_\beta)$.

Now we compute $h_{\alpha_1} \cdot (h_{\alpha_2} * h_{\beta_1}) \cdot h_{\beta_2}$ by using formulas (4.10) and (4.4). We can assume that α_2 and β_1 are weak compositions of the same integer, otherwise the product $h_{\alpha_1} * h_{\alpha_2}$ is 0. Then

$$h_{\alpha_1} \cdot (h_{\alpha_2} * h_{\beta_1}) \cdot h_{\beta_2} = \sum_M h_{\alpha_1} \cdot h_{p(M)} \cdot h_{\beta_2} = \sum_M h_{\alpha_1 p(M) \beta_2}, \tag{4.23}$$

where $M = (m_{ij})$ ranges over the matrices $r \times s$ which fill the diagram

$$\begin{array}{ccc|c} m_{11} & \cdots & m_{1r} & b_1^{(1)} \\ \vdots & \ddots & \vdots & \vdots \\ m_{s1} & \cdots & m_{sr} & b_s^{(1)} \\ \hline a_1^{(2)} & \cdots & a_r^{(2)} & \end{array}$$

The index $\alpha_1 p(M) \beta_2$ of the functions h in (4.23) can be identified with $p(M')$ where

M' is the matrix

$$\begin{array}{cccc|c}
0 & a_1^{(1)} & \cdots & a_r^{(1)} & \\
b_1^{(2)} & m_{11} & \cdots & m_{1r} & b_1^{(1)} + b_1^{(2)} = b_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
b_s^{(2)} & m_{s1} & \cdots & m_{sr} & b_s^{(1)} + b_s^{(2)} = b_s \\
\hline
 & a_1^{(1)} + a_1^{(2)} & \cdots & a_r^{(1)} + a_r^{(2)} & \\
 & = a_1 & & = a_r &
\end{array} \tag{4.24}$$

It is clear that M' fits into Diagram (4.8) and, thus, $h_{\alpha_1 p(M) \beta_2} = h_{p(M')}$ is a term of the sum (4.9).

Conversely, given a matrix in $\mathcal{M}^n_{\alpha,\beta}$ for some n satisfying $\max(p,q) \leq n \leq p+q$, we can recover the weak compositions α_1, α_2, β_1, and β_2, from Diagram (4.24). This shows that both expressions in (4.21) coincide. $\square$

4.5. The Heisenberg product of symmetric power sums

The Heisenberg product also has a nice combinatorial expression in the basis of Λ formed by the symmetric power sums (4.5).

Theorem 4.5.1. *The Heisenberg product in the basis of power sums can be expressed as*

$$p_\lambda \# p_\mu = \sum_{\substack{\alpha\delta=\lambda \\ \beta\delta=\mu}} z(\delta)\, p_{\alpha\delta\beta}, \tag{4.25}$$

where $z(\delta)$ is the order of the stabilizer of the conjugacy class of a permutation of cycle-type δ (recall Equation (4.7)).

Proof. The coproduct on the basis of power sums is determined by requiring the functions p_n, with n a non-negative integer, to be primitive elements: $\Delta(p_n) = 1 \otimes$

$p_n + p_n \otimes 1$. More explicitly,

$$\Delta(p_\lambda) = \sum_{\alpha\beta=\lambda} p_\alpha \otimes p_\beta.$$

Then, formula (4.21) reads

$$p_\lambda \# p_\mu = \sum_{\substack{\alpha_1\alpha_2=\lambda \\ \beta_1\beta_2=\mu}} p_{\alpha_1} \cdot (p_{\alpha_2} * p_{\beta_1}) \cdot p_{\beta_2}.$$

But $p_{\alpha_2} * p_{\beta_1} = z(\alpha_2)\delta_{\alpha_2\beta_1}$. Since the external product of power sums is done by concatenating the partitions, we obtain the result of the theorem. □

As a particular case, assume that λ and μ are partitions of n. Note that there is a term in degree n only when $\lambda = \mu$, otherwise δ would never be the empty partition and the degree of $p_{\alpha\delta\beta}$ would be strictly greater than n. Therefore, the only term in degree n is

$$\begin{cases} z(\lambda)\, p_\lambda, & \text{if } \lambda = \mu; \\ 0, & \text{otherwise}; \end{cases}$$

which is the expression of the internal product in the basis of power sums (4.6).

On the other hand for any partitions λ and μ, when δ is the empty partition, we obtain the term of largest degree, namely $p_{\alpha\beta}$, since $z(\delta) = 1$ in this case. This gives the external product $p_\lambda \cdot p_\mu = p_{\lambda\mu}$.

Note that the coefficients of Formula (4.25) in the basis of power sums are not necessarily the numbers $z(\delta)$. Indeed, the partition λ may be decomposed, in general, in more than one way as $\lambda = \alpha\delta$, since the operation of concatenation of partitions involves a reordering of the final result. For example, let (1^n) be the partitions with n parts equal to 1. Then,

$$p_{(1^u)} \# p_{(1^v)} = \sum_{n=\max(u,v)}^{u+v} \binom{u}{n-v}\binom{v}{n-u}(u+v-n)!\, p_{(1^n)}. \tag{4.26}$$

In this case, the partitions of Formula (4.25) are $\alpha = (1^{n-v})$, $\beta = (1^{n-u})$, and $\delta = (1^{u+v-n})$. The number of possible decompositions of (1^u) into two partitions of length $n-v$ and $u+v-n$ is $\binom{u}{n-v}$, and the same argument for (1^v) yields the second binomial coefficient. The remaining factor of the coefficient is $z(\delta) = z\big((1^{u+v-n})\big) = (u+v-n)!$, according to Formula (4.7).

From the explicit expression (4.3) for the complete homogeneous symmetric functions, it is clear that $h_{(1^u)} = p_{(1^u)}$. Hence, Formula (4.26) can also be deduced from Theorem (4.3.1). We use this method in Example 4 of Section 7, in the context of non-commutative symmetric functions.

4.6. The Heisenberg product of Schur functions

Let λ be a partition of n. The Schur function s_λ is the Frobenius characteristic of the irreducible representation of the symmetric group indexed by the partition λ. The Schur functions s_λ, with λ varying over all the partitions, form another linear basis of Λ.

The external product of Schur functions has a well-known combinatorial expression via the Littlewood-Richardson rule (see [18, Proposition I.9.2] or [26, Section A1.3]).

Question. For the internal product of Schur functions, on the other hand, a combinatorial rule is yet unknown. This problem has been the object of intense study. There are a lot of partial results for particular cases of partitions λ and μ (see, for example, [27, 5]).

Given that the Heisenberg product contains the classical products, a combinatorial rule for this product would yield a rule for the internal product, as well as containing as a special case the Littlewood-Richardson rule. Since the Schur functions correspond to the irreducible representations of the symmetric group, then the coefficients of $s_\lambda \# s_\mu$ in the basis of Schur functions are non-negative. Indeed, since the product can be expressed as a representation (3.7), it can be expanded into the irreducible representations with the adequate multiplicities. Thus, it makes sense to look for a combinatorial rule for the coefficients of the Heisenberg product which will contain the combinatorial rule for the Kronecker product. We do not tackle this problem in this work.

We show some examples of Heisenberg products of Schur functions. They are computed using the combinatorial rule (4.9) in the basis of complete homogeneous functions, and using a linear change of basis. The summands are grouped in rows according to their degrees. In the following example, the degree 3 terms (in the first row) constitute the internal product $s_{(2,1)} * s_{(2,1)}$, and the degree 6 terms constitute the external product $s_{(2,1)} \cdot s_{(2,1)}$.

$$\begin{aligned} s_{(2,1)} \# s_{(2,1)} = {} & s_{(2,1)} + s_{(3)} + s_{(1,1,1)} \\ & + 6s_{(3,1)} + 2s_{(1,1,1,1)} + 6s_{(2,1,1)} + 4s_{(2,2)} + 2s_{(4)} \\ & + s_{(5)} + 4s_{(2,1,1,1)} + 5s_{(2,2,1)} + 4s_{(4,1)} + 5s_{(3,2)} + 6s_{(3,1,1)} + s_{(1,1,1,1,1)} \\ & + s_{(2,2,1,1)} + s_{(3,3)} + 2s_{(3,2,1)} + s_{(4,1,1)} + s_{(3,1,1,1)} + s_{(2,2,2)} + s_{(4,2)}, \end{aligned}$$

In the next example, the partitions indexing the Schur functions are of different integers, and hence there is no internal product involved:

$$\begin{aligned} s_{(2,1)} \# s_{(4)} = {} & s_{(1,3)} + s_{(2,1,1)} + s_{(2,2)} \\ & + s_{(5)} + 2s_{(2,2,1)} + s_{(2,1,1,1)} + 3s_{(4,1)} + 3s_{(3,2)} + 3s_{(3,1,1)} \\ & + s_{(3,3)} + 3s_{(5,1)} + 2s_{(3,2,1)} + 3s_{(4,1,1)} + s_{(3,1,1,1)} + s_{(6)} + 3s_{(2,4)} \\ & + s_{(3,1)} + s_{(5,2)} + s_{(4,2,1)} + s_{(5,1,1)}. \end{aligned}$$

5. ENDOMORPHISMS OF HOPF ALGEBRAS

5.1. The algebra of endomorphisms of a Hopf algebra

In this part we start looking at the spaces marked with an oval box in Diagram 1. The theory in this context is independent of the one developed in the previous sections. In Section 8 we make the connection between both parts.

Let $(H, m, \Delta, \iota, \varepsilon, S)$ be an arbitrary Hopf algebra, where $m : H \otimes H \to H$ is the product, $\Delta : H \to H \otimes H$ is the coproduct, $\iota : \Bbbk \to H$ is the unit, $\varepsilon : H \to \Bbbk$ is the counit, and $S : H \to H$ is the antipode. We consider the space $\operatorname{End}(H)$ of linear endomorphisms of H, consisting of the linear maps from H to itself. Observe that we do not require the maps in $\operatorname{End}(H)$ to preserve the Hopf algebra structure of H. The space $\operatorname{End}(H)$ carries several associative products. Let $f, g \in \operatorname{End}(H)$. The *composition* $g \circ f$ and *convolution* $f \star g$ of endomorphisms are respectively defined by the diagrams

$$\begin{tikzcd} & H \arrow[dr, "g"] & \\ H \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & H \end{tikzcd} \qquad \begin{tikzcd} H \otimes H \arrow[r, "f\otimes g"] & H \otimes H \arrow[d, "m"] \\ H \arrow[u, "\Delta"] \arrow[r, "f\star g"'] & H \end{tikzcd} \tag{5.1}$$

The unit of the composition product is the identity function, and the unit of the convolution product is the map $\iota\varepsilon$. We do not show the usual proofs of associativity here, but we do it for the Heisenberg product in Section 5.2. Observe that the composition of f and g is written $g \circ f$, which has the factors in the opposite order of $f \star g$.

These products are often defined in a different setting [21] as follows. Given a Hopf algebra H, let H^* be its linear dual with the algebra structure given by the product

$$(f * g)(h) = \sum f(h_1)g(h_2),$$

for $f, g \in H^*$, $h \in H$, and where we use Sweedler's notation for the coproduct: $\Delta(h) = \sum h_1 \otimes h_2$.

Consider the space $H^* \otimes H$. There is a canonical inclusion $H^* \otimes H \hookrightarrow \mathrm{End}(H)$ sending the element $f \otimes h$ to the endomorphism $x \mapsto f(x)h$. When H is finite-dimensional this inclusion is an isomorphism. Note, however, that we do not make any finite-dimensional assumption of H for what follows.

With this notation, the composition and convolution products restricted to the space $H^* \otimes H$ are defined, respectively, by

$$(g \otimes \ell) \circ (f \otimes k) = g(k)(f \otimes \ell), \qquad (f \otimes k) \star (g \otimes \ell) = (f * g) \otimes k\ell.$$

Observe that neither product of endomorphisms is commutative.

5.2. The Heisenberg product of endomorphisms

The Heisenberg product is an extension of the diagrams in (5.1).

Definition 5.2.1. Let $f, g \in \mathrm{End}(H)$. The *Heisenberg product of endomorphisms* $f \# g$ is defined by the diagram

$$\begin{tikzcd}
 & H^{\otimes 3} \arrow[r, "\text{cyclic}"] & H^{\otimes 3} \arrow[dr, "1\otimes m"] & \\
H^{\otimes 2} \arrow[ur, "\Delta\otimes 1"] & & & H^{\otimes 2} \arrow[d, "1\otimes g"] \\
H^{\otimes 2} \arrow[u, "f\otimes 1"] & & & H^{\otimes 2} \arrow[dl, "m"] \\
 & H \arrow[ul, "\Delta"] \arrow[r, "f\#g"'] & H &
\end{tikzcd} \tag{5.2}$$

where the map cyclic : $H^{\otimes 3} \to H^{\otimes 3}$ is $x \otimes y \otimes z \mapsto y \otimes z \otimes x$. The unit of the Heisenberg product is the map $\iota\varepsilon$.

The Heisenberg product can be expressed in the setting described in Section 5.1 as follows. The algebra H acts on the left on H^* by translation as

$$(h \cdot f)(k) = f(kh).$$

Then, the Heisenberg product is the operation on $H^* \otimes H$ defined by

$$(a \otimes h) \# (b \otimes k) = \sum a(h_1 \cdot b) \otimes h_2 k. \tag{5.3}$$

The space $H^* \otimes H$ with the Heisenberg product is usually called *Heisenberg double*. From now on, we use the expression (5.2) for the Heisenberg product in the space $\mathrm{End}(H)$.

We consider some subspaces of $\mathrm{End}(H)$ for a particular kind of Hopf algebra H. This will allow us to make the connection to permutations in Section 6.

Assume that H is a graded connected Hopf algebra, that is: $H = \bigoplus_{n \geq 0} H_n$ with $H_0 \cong \Bbbk$. Moreover, assume that m and Δ are degree-preserving maps. We are interested in the subspace of $\mathrm{End}(H)$ consisting of linear endomorphisms of H which preserve the grading and are zero except on finitely many components:

$$\mathrm{end}(H) = \bigoplus_{n \geq 0} \mathrm{End}(H_n).$$

The following proposition shows that $\mathrm{end}(H)$ is a subalgebra of $\mathrm{End}(H)$ with respect to the Heisenberg product.

Proposition 5.2.2. *Let H be a graded connected Hopf algebra. The composition, convolution, and Heisenberg products of* $\mathrm{End}(H)$ *restrict to* $\mathrm{end}(H)$. *Moreover, if*

$f \in \mathrm{End}(H_p)$ and $g \in \mathrm{End}(H_q)$ then

$$f \# g \in \bigoplus_{n=\max(p,q)}^{p+q} \mathrm{End}(H_n) \tag{5.4}$$

and the top and bottom components of $f \# g$ are

$$(f \# g)_{p+q} = f \star g \quad \textit{and, if } p = q, \quad (f \# g)_p = g \circ f. \tag{5.5}$$

Proof. Let $h \in H_n$. The coproduct of h is

$$\Delta(h) = \sum_{a+b=n} h_a \otimes h_b,$$

with $h_a \in H_a$ and $h_b \in H_b$. We evaluate Diagram (5.2) at h to get an explicit form for the Heisenberg product of f and g:

$$(f \# g)(h) = \sum_{a+b=n} f(h_a)_2 g\big(h_b f(h_a)_1\big). \tag{5.6}$$

Suppose that f and g belong to $\mathrm{end}(H)$. The computation of the degree of every term in the sum yields

$$\begin{aligned}
\deg\big[f(h_a)_2 g\big(h_b f(h_a)_1\big)\big] &= \deg\big(f(h_a)_2\big) + \deg\big[g\big(h_b f(h_a)_1\big)\big] \\
&= \deg\big(f(h_a)_2\big) + \deg\big(h_b f(h_a)_1\big) \\
&= \deg\big(f(h_a)_2\big) + \deg(h_b) + \deg\big(f(h_a)_1\big) \\
&= \deg\big(f(h_a)\big) + \deg(h_b) \\
&= a + b = n,
\end{aligned}$$

proving that $f \# g$ is in $\mathrm{end}(H)$.

We can refine the previous analysis as follows. Assume that $f \in \mathrm{End}(H_p)$ and

$g \in \mathrm{End}(H_q)$. Then, Expression (5.6) is zero unless

$$a = p \quad \text{and} \quad b + \deg\big(f(h_a)_1\big) = q. \tag{5.7}$$

Adding these two equations we get that $n = a + b \leq p + q$. On the other hand, $p = a \leq a + b = n$ and $q = b + \deg\big(f(h_a)_1\big) \leq b + a = n$, hence $\max(p, q) \leq n$. This proves (5.4).

If we set $n = p + q$ in (5.7) then we get $\deg\big(f(h_a)_1\big) = 0$, and (5.6) reduces to convolution diagram in (5.1). If we set $n = p = q$, then $\deg(h_b) = \deg\big(f(h_a)_2\big) = 0$, and (5.6) reduces to $g\big(f(h)\big) = (g \circ f)(h)$, which is the composition product. □

Thus, the Heisenberg product interpolates between the composition and convolution products. The analogous interpolation property at all other non-commutative levels (permutations and non-commutative symmetric functions) is a consequence of this general result.

5.3. Garsia-Reutenauer endomorphisms

In this section we show that the Heisenberg product of endomorphisms of a Hopf algebra H can be restricted to the subspace of *Garsia-Reutenauer endomorphisms.* These endomorphisms are characterized in terms of their action on products of primitive elements of H. When H is the tensor algebra of a vector space, a result of Garsia and Reutenauer relates this subspace with the space of non-commutative symmetric functions via Schur-Weyl duality (Lemma 6.1.1 and Theorem 7.2.1).

Definition 5.3.1. Let H be an arbitrary Hopf algebra. If $h_1, \ldots, h_n \in H$, define

$$G(h_1, \ldots, h_n) = \mathrm{Span}(h_{\sigma(1)} \cdots h_{\sigma(n)} \mid \sigma \in S_n),$$

that is, $G(h_1, \ldots, h_n)$ is the linear span of the products of the elements $h_{\sigma(1)}, \ldots, h_{\sigma(n)}$

with σ varying over all the permutations in S_n.

Let $\mathrm{Prim}(H)$ be the subspace of primitive elements of H, that is, those elements $h \in H$ such that

$$\Delta(h) = 1 \otimes h + h \otimes 1.$$

The following lemma lists some basic properties of the subspaces $G(h_1, \dots, h_n)$ which we will need in the next theorem. For a concise expression of the coproduct in the next lemma we consider the set of (p,q)-*shuffles*, defined for non-negative integers p and q as the set $\mathrm{Sh}(p,q)$ consisting of all the permutations $\xi \in S_{p+q}$ which satisfy

$$\xi(1) < \cdots < \xi(p) \quad \text{and} \quad \xi(p+1) < \cdots < \xi(p+q). \tag{5.8}$$

In other words, the permutations in $\mathrm{Sh}(p,q)$ are those permutations of S_{p+q} which have no descents except perhaps at position p.

Lemma 5.3.2. *For any $h_1, \dots, h_n \in H$ we have:*

(i) If $a \in G(h_1, \dots, h_k)$ and $b \in G(h_{k+1}, \dots, h_n)$, then $ab \in G(h_1, \dots h_n)$.

(ii) If $a \in G(h_1, \dots, h_n)$ and $h_1, \dots, h_n \in \mathrm{Prim}(H)$, then

$$\Delta(a) = \sum_{\substack{k+\ell=n \\ \xi \in \mathrm{Sh}(k,\ell)}} a_\xi^{(1)} \otimes a_\xi^{(2)},$$

where $a_\xi^{(1)} \in G(h_{\xi(1)}, \dots, h_{\xi(k)})$ and $a_\xi^{(2)} \in G(h_{\xi(k+1)}, \dots, h_{\xi(n)})$.

Proof. Let $a = h_{\sigma(1)} \cdots h_{\sigma(k)} \in G(h_1, \dots, h_k)$ for some permutation $\sigma \in S_k$, and $b = h_{\tau(1)} \cdots h_{\tau(k)} \in G(h_{k+1}, \dots, h_n)$ for some permutation $\tau \in S_{n-k}$. Then, the product ab is the product of the elements h_i sorted via the permutation $\sigma \times \tau \in S_n$, which belongs to $G(h_1, \dots, h_n)$. By bilinearity of the product we conclude (i).

To prove (ii), note that if $h_1, \ldots, h_n$ are primitive elements of H, then we have

$$\begin{aligned}\Delta(h_1 \cdots h_n) = \Delta(h_1) \cdots \Delta(h_n) &= (1 \otimes h_1 + h_1 \otimes 1) \cdots (1 \otimes h_n + h_n \otimes 1)\\ &= \sum_{\substack{k+\ell=n\\ \xi \in \mathrm{Sh}(k,\ell)}} h_{\xi(1)} \cdots h_{\xi(k)} \otimes h_{\xi(k+1)} \cdots h_{\xi(n)}, \qquad (5.9)\end{aligned}$$

where the k left terms of the tensor in the sum come from the election of k second terms of $1 \otimes h_i + h_i \otimes 1$, and similarly with ℓ and the first terms of $1 \otimes h_i + h_i \otimes 1$.

If we compute $\Delta(h_{\sigma(1)} \cdots h_{\sigma(n)})$ instead, for a permutation $\sigma \in S_n$, we get as first factors in the tensors in (5.9) the expression $h_{\sigma(\xi(1))} \cdots h_{\sigma(\xi(k))}$, which clearly belongs to $G(h_1, \ldots, h_k)$. The same argument applies to the second factors of the terms in (5.9). $\square$

Definition 5.3.3. Let H be an arbitrary Hopf algebra. The space $\Sigma(H)$ of *Garsia-Reutenauer endomorphisms* of H consists of those endomorphisms $T \in \mathrm{End}(H)$ which leave each space $G(h_1, \ldots, h_n)$ invariant, for any finite set of primitive elements $h_1, \ldots, h_n$ of H. Thus,

$$\Sigma(H) = \big\{T \in \mathrm{End}(H) \;\big|\; T\big(G(h_1, \ldots, h_n)\big) \subseteq G(h_1, \ldots, h_n) \\ \text{for all } h_1, \ldots, h_n \in \mathrm{Prim}(H)\big\}.$$

The following theorem is the key result of this section for the interpretation of the Heisenberg product in the space of non-commutative symmetric functions.

Theorem 5.3.4. *If H is a Hopf algebra, the space $\Sigma(H)$ of Garsia-Reutenauer endomorphisms is a subalgebra of* $\mathrm{End}(H)$ *with respect to the Heisenberg product.*

Proof. Given a primitive element h, we have $\mu\varepsilon(h) = 0$, hence the unit of the Heisenberg product is in $\Sigma(H)$.

Take two endomorphisms f and g in $\Sigma(H)$, and let $h_1, \ldots, h_n \in \mathrm{Prim}(H)$. Then, we have by definition (5.2)

$$(f \# g)(h_1 \cdots h_n) = \sum_{\substack{k+\ell=n \\ \xi \in \mathrm{Sh}(k,\ell)}} \big(f(h_{\xi(1)} \cdots h_{\xi(k)})\big)_2 g\Big(h_{\xi(k+1)} \cdots h_{\xi(n)} \big(f(h_{\xi(1)} \cdots h_{\xi(k)})\big)_1\Big). \tag{5.10}$$

As $f(h_{\xi(1)} \cdots h_{\xi(k)}) \in G(h_{\xi(1)} \cdots h_{\xi(k)})$, it follows from part *(ii)* of Lemma 5.3.2 that

$$\Delta\big(f(h_{\xi(1)} \cdots h_{\xi(k)})\big) = \sum_{\substack{r+s=k \\ \eta \in \mathrm{Sh}(r,s)}} a_\eta^{(1)} \otimes a_\eta^{(2)},$$

with $a_\eta^{(1)} \in G(h_{\xi\eta(1)}, \ldots, h_{\xi\eta(r)})$ and $a_\eta^{(2)} \in G(h_{\xi\eta(r+1)}, \ldots, h_{\xi\eta(k)})$. Hence, we rewrite Equation (5.10) as

$$(f \# g)(h_1 \cdots h_n) = \sum_{\substack{k+\ell=n \\ \xi \in \mathrm{Sh}(k,\ell)}} a_\eta^{(2)}\, g(h_{\xi(k+1)} \cdots h_{\xi(n)}\, a_\eta^{(1)}).$$

But the argument of g belongs to $G(h_{\xi(k+1)}, \ldots, h_{\xi(n)}, h_{\xi\eta(1)}, \ldots, h_{\xi\eta(r)})$. Using that $g \in \Sigma(H)$ and using part *(i)* of Lemma 5.3.2 we obtain that

$$\begin{aligned}(f \# g)(h_1 \cdots h_n) &\in G(h_{\xi\eta(r+1)}, \ldots, h_{\xi\eta(k)}, h_{\xi(k+1)}, \ldots, h_{\xi(n)}, h_{\xi\eta(1)}, \ldots, h_{\xi\eta(r)}) \\ &\subseteq G(h_1, \ldots, h_n),\end{aligned}$$

proving that $f \# g \in \Sigma(H)$. $\square$

6. PERMUTATIONS

6.1. From endomorphisms to permutations

In order to translate the Heisenberg product from endomorphisms of Hopf algebras to permutations we specialize the construction of Section 5.2 and we use the Schur-Weyl duality theorem.

Let $\Bbbk$ be a field of characteristic zero. Consider

$$T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$$

be the tensor algebra of a vector space V over the field $\Bbbk$. It is a graded connected Hopf algebra with product defined by concatenation

$$(v_1 \otimes \cdots \otimes v_k)(w_1 \otimes \cdots \otimes w_\ell) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell,$$

and with coproduct uniquely determined by

$$v \mapsto 1 \otimes v + v \otimes 1 \quad \text{for } v \in V. \tag{6.1}$$

In other words, the elements of the vector space V are primitives. By definition of the product, we can write $v_1 \otimes \cdots \otimes v_n = v_1 \cdots v_n$, hence we will omit the tensors when writing elements of $T(V)$. Then, by Proposition 5.2.2 we get the Heisenberg product in $\mathrm{End}\big(T(V)\big)$ and in $\mathrm{end}\big(T(V)\big)$.

The general linear group $\mathrm{GL}(V)$ acts on V and hence on each $V^{\otimes n}$ diagonally. Schur-Weyl duality [24, 11] states that the only endomorphisms of $T(V)$ which commute with the action of $\mathrm{GL}(V)$ are linear combinations of permutations. Let

$$\mathbb{S} = \bigoplus_{n\geq 0} \Bbbk S_n$$

be the direct sum of all symmetric group algebras. The product in $\mathcal{S}$ is determined, in the basis of permutations, by $\sigma \circ \tau$ (usual composition of permutations) when σ and τ belong to the same homogeneous component of $\mathcal{S}$, and is 0 in any other case. In this section, to simplify the notation, we use juxtaposition for the composition of permutations, that is, we write $\sigma\tau$ for the composition $\sigma \circ \tau$.

Lemma 6.1.1 (Schur-Weyl duality)**.** *Let V a infinite dimensional vector space over the field $\Bbbk$. Let Ψ be the map*

$$\Psi : \mathcal{S} \to \mathrm{end}_{\mathrm{GL}(V)}\big(T(V)\big),$$

defined by sending $\sigma \in S_n$ to the endomorphism $\Psi(\sigma)$ of $T(V)$, which in degree n is given by the right action of σ on $V^{\otimes n}$:

$$v_1 \cdots v_n \xmapsto{\Psi(\sigma)} v_{\sigma(1)} \cdots v_{\sigma(n)}$$

and is 0 in the other homogeneous components. Then, ψ is an isomorphism of vector spaces.

6.2. The Heisenberg product of permutations

Malvenuto and Reutenauer [20] deduce from Lemma 6.1.1 that $\mathcal{S}$ is closed under the convolution product. The same argument gives us:

Theorem 6.2.1. *The space $\mathcal{S}$ is closed under the Heisenberg product of endomorphisms.*

Proof. We need to prove that the maps involved in Definition 5.2.1, for the particular case $H = T(V)$, commute with the action of $\mathrm{GL}(V)$.

Let us consider first the map $\Delta : T(V) \to T(V) \otimes T(V)$. We need to check that $\Delta(a \cdot x) = a \cdot \Delta(x)$ for all $a \in \mathrm{GL}(V)$ and $x \in T(V)$. Since Δ is a multiplicative map it is enough to take $x \in V$, and since the elements of V are primitive, we have

$$\begin{aligned}\Delta(a \cdot x) &= 1 \otimes (a \cdot x) + (a \cdot x) \otimes 1 \\ &= a \cdot (1 \otimes x + x \otimes 1) = a \cdot \Delta(x).\end{aligned}$$

We used that $\mathrm{GL}(V)$ acts diagonally on $T(V) \otimes T(V)$ and that $a \cdot 1 = 1$, since $\mathrm{GL}(V)$ acts trivially on $T^0(V)$. This proves that the coproduct commutes with the action of $\mathrm{GL}(V)$.

Next we consider the product $m : T(V) \otimes T(V) \to T(V)$. Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ vectors in V. We obtain

$$\begin{aligned}\big(a \cdot (v_1 \cdots v_n)\big)\big(a \cdot (w_1 \cdots w_n)\big) &= \big((a \cdot v_1) \cdots (a \cdot v_n)\big)\big((a \cdot w_1) \cdots (a \cdot w_m)\big) \\ &= a \cdot (v_1 \cdots v_n w_1 \cdots w_m),\end{aligned}$$

using again the diagonal action of $\mathrm{GL}(V)$ on the tensors.

The map cyclic : $T(V)^{\otimes 3} \to T(V)^{\otimes 3}$ clearly commutes with the diagonal action since it is just a permutation of the tensors. The remaining maps in Definition 5.2.1 are tensors of maps already commuting with the action of $\mathrm{GL}(V)$. In conclusion, the composition

$$f \# g = m(1 \otimes g)(1 \otimes m)\,\mathrm{cyclic}(\Delta \otimes 1)(f \otimes 1)\Delta$$

commutes with the action of $\mathrm{GL}(V)$. The result follows from the application of Lemma 6.1.1. $\square$

This conceptual argument is important because it can be applied to other dualities than Schur-Weyl's, i.e., to centralizer algebras of groups (or even Hopf algebras) acting on the tensor algebra other than the general linear group. It can also be applied to other products of endomorphisms, a remarkable case being that of *Drinfeld* product. We mention some preliminary results about the Drinfeld product in

Section 10.

Next we show an explicit formula for the Heisenberg product of two permutations. Consider the set of shuffles, $\mathrm{Sh}(p,q)$, defined in (5.8). We denote by $\beta_{p,q}$ the shuffle of maximal length in $\mathrm{Sh}(p,q)$, namely, the permutation

$$\beta_{p,q} = \begin{pmatrix} 1 & 2 & \cdots & p & p+1 & p+2 & \cdots & p+q \\ q+1 & q+2 & \cdots & q+p & 1 & 2 & \cdots & q \end{pmatrix}.$$

The identity in S_n is denoted by Id_n.

Theorem 6.2.2. *Let $\sigma \in S_p$ and $\tau \in S_q$. Then, the Heisenberg product of permutations can be expressed as*

$$\sigma \# \tau = \sum_{n=\max(p,q)}^{p+q} \sum_{\substack{\xi \in \mathrm{Sh}(p,n-p) \\ \eta \in \mathrm{Sh}(p+q-n,n-q)}} \xi\big((\sigma\eta) \times \mathrm{Id}_{n-p}\big)\beta_{2n-p-q,p+q-n}(\mathrm{Id}_{n-q} \times \tau). \tag{6.2}$$

This formula includes, as always, the Malvenuto-Reutenauer product of permutations as defined by Malvenuto-Reutenauer [19, 20] and studied in detail in [4], when $n = p+q$,

$$\sigma \star \tau = \sum_{\xi \in \mathrm{Sh}(p,q)} \xi(\sigma \times \tau).$$

When $n = p = q$, the sets of shuffles in (6.2) contain just the identity, and the maximal shuffle $\beta_{0,n}$ is also the identity. Hence, (6.2) reduces to the composition product $\sigma\tau$.

Proof. First we give an expression for the coproduct defined in (6.1). Since the elements $v_i \in V$ are primitive elements, Formula (5.9) gives:

$$\Delta(v_1 \cdots v_n) = \sum_{\substack{p+q=n \\ \xi \in \mathrm{Sh}(p,q)}} v_{\xi(1)} \cdots v_{\xi(p)} \otimes v_{\xi(p+1)} \cdots v_{\xi(n)}. \tag{6.3}$$

Then, using (5.2) for the particular Hopf algebra $H = T(V)$ and for the endomorphisms $\Psi(\sigma)$ and $\Psi(\tau)$ induced by the permutations $\sigma \in S_p$ and $\tau \in S_q$, respectively, we obtain

$$\begin{aligned}
&\big(\Psi(\sigma) \# \Psi(\tau)\big)(v_1 \cdots v_n) \\
&\qquad = \sum_{\substack{r+s=n \\ \xi\in \mathrm{Sh}(r,s)}} \big(\Psi(\sigma)(v_{\xi(1)} \cdots v_{\xi(r)})\big)_2 \; \Psi(\tau)\Big(v_{\xi_{r+1}} \cdots v_{\xi_n} \, \big(\Psi(\sigma)(v_{\xi(1)} \cdots v_{\xi(r)})\big)_1\Big).
\end{aligned}$$

The only non-zero terms occur when $r = p$, since $\Psi(\sigma)$ and $\Psi(\tau)$ are degree preserving endomorphisms. Hence

$$\begin{aligned}
&\big(\Psi(\sigma) \# \Psi(\tau)\big)(v_1 \cdots v_n) \\
&= \sum_{\xi\in \mathrm{Sh}(p,n-p)} \big(\Psi(\sigma)(v_{\xi(1)} \cdots v_{\xi(p)})\big)_2 \; \Psi(\tau)\Big(v_{\xi(p+1)} \cdots v_{\xi(n)} \, \big(\Psi(\sigma)(v_{\xi(1)} \cdots v_{\xi(p)})\big)_1\Big) \\
&= \sum_{\substack{\xi\in \mathrm{Sh}(p,n-p) \\ u+v=p \\ \eta\in \mathrm{Sh}(u,v)}} v_{\xi\sigma\eta(u+1)} \cdots v_{\xi\sigma\eta(p)} \Psi(\tau)\big(v_{\xi(p+1)} \cdots v_{\xi(n)} v_{\xi\sigma\eta(1)} \cdots v_{\xi\sigma\eta(u)}\big) \\
&= \sum_{\substack{\xi\in \mathrm{Sh}(p,n-p) \\ \eta\in \mathrm{Sh}(p+q-n,n-q)}} v_{\xi\sigma\eta(p+q-n+1)} \cdots v_{\xi\sigma\eta(p)} v_{\xi\tau(p+1)} \cdots v_{\xi\tau(n)} v_{\xi\sigma\eta\tau(1)} \cdots v_{\xi\sigma\eta\tau(p+q-n)} \\
&= \sum_{\substack{\xi\in \mathrm{Sh}(p,n-p) \\ \eta\in \mathrm{Sh}(p+q-n,n-q)}} \Psi\Big[\xi\big((\sigma\eta) \times \mathrm{Id}_{n-p}\big)\beta_{2n-p-q,p+q-n}(\mathrm{Id}_{n-q} \times \tau)\Big](v_1 \cdots v_n),
\end{aligned}$$

which proves the theorem. $\square$

For instance, writing the permutations in word format:

$$\begin{aligned}
12 \# 132 = {} & 132 + 231 + 321 \\
& + 1234 + 1243 + 1324 + 2134 + 2143 + 2314 \\
& + 3124 + 3142 + 3214 + 4123 + 4132 + 4213 \\
& + 12354 + 13254 + 14253 + 15243 + 23154 \\
& + 24153 + 25143 + 34152 + 35142 + 45132,
\end{aligned}$$

where the degree of the permutations in the result varies from $\max(2, 3) = 3$ to $2 + 3 = 5$. There is no permutation corresponding to the composition of 12 and 132 since the operation $12 \circ 132$ in $\mathbb{S}$ is 0.

7. NON-COMMUTATIVE SYMMETRIC FUNCTIONS

7.1. Classical products of non-commutative symmetric functions

In this section we define the space of descents of permutations and two classical products.

The *descent set* of a permutation $\sigma \in S_n$ is the subset of $[n-1]$ defined by

$$\mathrm{Des}(\sigma) = \big\{ i \in [n-1] \mid \sigma(i) > \sigma(i+1) \big\}. \tag{7.1}$$

Given $J \subseteq [n-1]$, define $\mathcal{B}_J$ as the set of permutations $\sigma \in S_n$ with $\mathrm{Des}(\sigma) \subseteq J$, and consider the following elements of $\Bbbk S_n$:

$$X_J = \sum_{\sigma \in \mathcal{B}_J} \sigma. \tag{7.2}$$

It is convenient to index the elements X_J by compositions of n by means of the bijection

$$(a_1, a_2, \ldots, a_r) \longleftrightarrow \{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_{r-1}\}. \tag{7.3}$$

For instance, if $n = 9$, then $X_{(1,2,4,2)} = X_{\{1,3,7\}}$.

Let Σ_n be the subspace of $\Bbbk S_n$ linearly spanned by the elements X_α as α runs over all compositions of n and define

$$\Sigma = \bigoplus_{n \geq 0} \Sigma_n.$$

A fundamental result of Solomon [25] states that Σ_n is a subalgebra of the symmetric group algebra $\Bbbk S_n$ with the composition product of permutations. This is *Solomon's descent algebra*. We denote the composition product in this space with the symbol $*$. The rule for multiplying two elements in the linear basis of elements

X_I is similar to the rule for multiplying complete homogeneous functions in Λ (Theorem 4.3.1). We will deduce it as a particular case of the rule for the Heisenberg product in Section 7.3.

It is also well-known that Σ is closed under the external product of permutations [14, 15, 20]. The notation for the external product in this space is $X_I \cdot X_J$. In fact, we have

$$X_{(a_1,\ldots,a_r)} \cdot X_{(b_1,\ldots,b_s)} = X_{(a_1,\ldots,a_r,b_1,\ldots,b_s)}. \tag{7.4}$$

The space Σ with the external product is the algebra of *non-commutative symmetric functions.*

There is another basis of Σ which we will use in one of the theorems of this section. Given a set $J \subseteq [n-1]$, define $\mathcal{D}_J = \{\sigma \in S_n \mid \mathrm{Des}(\sigma) = J\}$ (compare with the definition of $\mathcal{B}_J$). Then, the elements

$$Y_J = \sum_{\sigma \in \mathcal{D}_J} \sigma \tag{7.5}$$

when J runs over the subsets of $[n-1]$ are a basis of the space Σ_n.

7.2. The Heisenberg product of non-commutative symmetric functions

An important result of Garsia and Reutenauer characterizes the elements of $\mathcal{S}$ that belong to Σ in terms of their action on the tensor algebra.

Let V be an infinite dimensional vector space over a field of characteristic zero. Recall Definition 5.3.3 of the space of Garsia-Reutenauer endomorphisms, $\Sigma(H)$, of an arbitrary Hopf algebra H. The next theorem states that the elements of Σ are precisely those Garsia-Reutenauer endomorphisms of the Hopf algebra $T(V)$ (in the sense of Definition 5.3.3) which commute with the action of $\mathrm{GL}(V)$.

Theorem 7.2.1 (Garsia and Reutenauer [13]). *Let V be an infinite-dimensional* *vector space. We have*

$$\Psi(\Sigma) = \Sigma(T(V)) \cap \mathrm{end}_{\mathrm{GL}(V)}(T(V)),$$

where $\Psi : \mathcal{S} \to \mathrm{end}_{\mathrm{GL}(V)}\big(T(V)\big)$ *is the isomorphism of Lemma 6.1.1 (Schur-Weyl duality) and* $\Sigma(T(V))$ *is the space of Garsia-Reutenauer endomorphisms of* $T(V)$*.*

The following theorem introduces the Heisenberg product in Σ as the restriction of the Heisenberg product of permutations. In view of the interpolation property of the Heisenberg product, we obtain the the classical products as particular cases.

Theorem 7.2.2. *The space* $\Sigma \subseteq \mathcal{S}$ *is closed under the Heisenberg product.*

Proof. This a straightforward result from Theorem 7.2.1, from the general result about Garsia-Reutenauer endomorphisms (Theorem 5.3.4), and from Schur-Weyl duality (Lemma 6.1.1). $\square$

The proof of this theorem is valid in other situations where there are generalizations of Schur-Weyl duality (see Section 10 for more details).

In this work we present two more proofs for the fact that the Heisenberg product restricts to the space Σ. The first proof gives a combinatorial rule for the Heisenberg product in the linear basis $\{X_I\}$. Such rule contains the classical rule of Solomon, which we will mention in the examples in Section 7.3. The second proof is a bijective proof which generalizes a proof of Schocker [23] for the composition product.

Yet another proof of Theorem 7.2.2, which we do not show here, can be done by extending the Heisenberg product to the Coxeter complex of the symmetric group (that is, the faces of the permutahedron). This makes a connection with recent work of Brown, Mahajan, Schocker, and others on this aspect of the theory of descent algebras [9, 3, 23].

The structure coefficients of the Heisenberg product on the basis (X_α) are expressed in terms of the matrices $\mathcal{M}^n_{\alpha,\beta}$ defined in Section 4.3. Recall that α and β are compositions, so that the order of the entries is significative. For a matrix $M \in \mathcal{M}^n_{\alpha,\beta}$, denote by $c(M)$ the composition of n whose parts are the non-zero entries of M, read from left to right and from top to bottom.

Theorem 7.2.3. *Let $\alpha \vDash p$ and $\beta \vDash q$ be two compositions. Then*

$$X_\alpha \# X_\beta = \sum_{n=\max(p,q)}^{p+q} \sum_{M \in \mathcal{M}^n_{\alpha,\beta}} X_{c(M)}. \tag{7.6}$$

In particular, we obtain that Σ is closed under the Heisenberg product.

Observe that even though this formula is similar to the one in symmetric functions (4.9), the occurrence of the compositions as indices of the basis makes the connection between the Heisenberg product and the external and Solomon products considerably harder than in the commutative context. In particular, a formula such as (4.21) expressing the Heisenberg product in terms of the external and internal product, does not longer hold.

We present several applications of Formula (7.6) in the examples in Section 7.3.

This proof is important because it allows us to make the connection with the Heisenberg product of representations of the symmetric group. This point is taken up in Section 8.1

Proof of Theorem 7.2.3. Let us take a fixed integer n between $\max(p, q)$ and $p + q$. To the compositions $\alpha = (a_1, \ldots, a_r)$ and $\beta = (b_1, \ldots, b_s)$ we associate the following sets:

$$\begin{aligned}
E_0^n &= [p+1, n], & F_0^n &= [1, n-q], \\
E_1^n &= [1, a_1], & F_1^n &= n-q+[1, b_1], \\
E_2^n &= [a_1+1, a_1+a_2], & F_2^n &= n-q+[b_1+1, b_1+b_2], \\
&\vdots & &\vdots \\
E_r^n &= [a_1+\cdots+a_{r-1}+1, p], & F_s^n &= n-q+[b_1+\cdots+b_{s-1}+1, q].
\end{aligned}$$

Observe that the family of intervals $\{E_j^n\}_{j\in\{0,\dots,r\}}$ and $\{F_i^n\}_{i\in\{0,\dots,s\}}$ are partitions of $[1,n]$. It is also clear that $\sigma \in \mathcal{B}_\alpha$ if and only if $\sigma \times \mathrm{Id}_{n-p}$ is increasing in E_j^n for all $j \in \{0,\dots,r\}$. Similarly, $\tau \in \mathcal{B}_\beta$ if and only if $\mathrm{Id}_{n-q} \times \tau$ is increasing in F_i^n for all $i \in \{0,\dots,s\}$. Observe, also, that $\# E_j^n$ is the j-th coordinate of the weak composition $(n-p, a_1, \dots, a_r)$, and $\# F_i^n$ is the i-th coordinate of $(n-q, b_1, \dots, b_s)$.

Given $\eta \in \mathrm{Sh}(p+q-n, n-q)$ and $\tau \in \mathcal{B}_\beta$, call $\varphi_{\eta,\tau} = (\eta \times \mathrm{Id}_{n-p})\beta_0(\mathrm{Id}_{n-q} \times \tau)$ and define the matrix

$$M_{\eta,\tau} = \left\{\#(F_i^n \cap \varphi_{\eta,\tau}^{-1} E_j^n)\right\}_{\substack{0\le i\le s,\\ 0\le j\le r}}$$

where we have abbreviated $\beta_{2n-p-q,p+q-n} = \beta_0$. In this situation $M_{\eta,\tau} \in \mathcal{M}_{\alpha,\beta}^n$. Indeed, if we call $m_{ij} = \#(F_i^n \cap \varphi_{\eta,\tau}^{-1} E_j^n)$, for $i=j=0$ we have that

$$\varphi_{\eta,\tau}[1, n-q] = \eta[p+q-n+1, p] \subseteq [1,p],$$

which shows that the intersection $F_0^n \cap \varphi_{\eta,\tau}^{-1} E_0^n$ is empty, and then $m_{00} = 0$. The sum $m_{0j} + \cdots + m_{sj}$ equals the number of elements of E_j^n, which is, as noted before, the j-th entry of the composition $(n-p, a_1, \dots, a_r)$. The same argument applies to the sum of the rows. In this manner, sending $\tau \mapsto M_{\eta,\tau}$ we define a map $\mathcal{B}_\beta \to \mathcal{M}_{\alpha,\beta}^n$.

Take $M = \{m_{ij}\} \in \mathcal{M}^n_{\alpha,\beta}$ and let $\mathcal{B}^{\eta,n}_{\beta}(M)$ the corresponding fiber of this map:

$$\mathcal{B}^{\eta,n}_{\beta}(M) = \big\{\tau \in \mathcal{B}_\beta \;\big|\; \#(F^n_i \cap \varphi^{-1}_{\eta,\tau} E^n_j) = m_{ij} \text{ for all } j \in \{0,\dots,r\},\, i \in \{0,\dots,s\}\big\}.$$

Therefore, we have a partition of $\mathcal{B}_\beta = \bigcup_{M\in\mathcal{M}^n_{\alpha,\beta}} \mathcal{B}^{\eta,n}_{\beta}(M)$.

For $\xi \in \mathrm{Sh}(p, n-p)$ and $\eta \in \mathrm{Sh}(p+q-n, n-q)$, let us denote $g^n_{\xi,\eta}(\sigma,\tau) = \xi\big((\sigma\eta)\times \mathrm{Id}_{n-p}\big)\beta_0(\mathrm{Id}_{n-q}\times\tau)$, the n-term in the sum (6.2). The function $g^n_{\xi,\eta}$ is bilinear, and we can write

$$X_\alpha \# X_\beta = \sum_n \sum_{\xi,\eta} g^n_{\xi,\eta}(X_\alpha, X_\beta).$$

From now on as n is fixed we will omit it in the notations of the sets and the functions. Next we show that

$$\sum_{\xi,\eta} g_{\xi,\eta}(X_\alpha, X_\beta) = \sum_{M\in\mathcal{M}_{\alpha,\beta}} X_{c(M)}.$$

For this, we write

$$\begin{aligned}
\sum_{\xi,\eta} g_{\xi,\eta}(X_\alpha, X_\beta) &= \sum_{\xi,\eta} g_{\xi,\eta}\Bigg(\sum_{\sigma\in\mathcal{B}_\alpha}\sigma,\ \sum_{M\in\mathcal{M}_{\alpha,\beta}}\ \sum_{\tau\in\mathcal{B}^{\eta}_{\beta}(M)}\tau\Bigg)\\
&= \sum_{M\in\mathcal{M}_{\alpha,\beta}}\ \sum_{\xi,\eta}\ \sum_{\sigma\in\mathcal{B}_\alpha}\ \sum_{\tau\in\mathcal{B}^{\eta}_{\beta}(M)} g_{\xi,\eta}(\sigma,\tau). \qquad (7.7)
\end{aligned}$$

If we denote by $S_{\alpha,\beta}(M)$ the set of elements (ξ,η,σ,τ) such that $\xi \in \mathrm{Sh}(p, n-p)$, $\eta \in \mathrm{Sh}(p+q-n, n-q)$, $\sigma \in \mathcal{B}_\alpha$ and $\tau \in \mathcal{B}^{\eta}_{\alpha}(M)$; then the map $\psi : S_{\alpha,\beta}(M) \to \mathcal{B}_{c(M)}$ given by $\psi(\xi,\eta,\sigma,\tau) = g_{\xi,\eta}(\sigma,\tau)$ is a bijection. We prove this fact in Lemma 7.2.5. In this situation, if we group together the last three sums of (7.7) we obtain

$$\sum_{\xi,\eta} g_{\xi,\eta}(X_\alpha, X_\beta) = \sum_{M\in\mathcal{M}_{\alpha,\beta}} X_{c(M)},$$

which concludes the proof of the theorem. $\square$

In the following two lemmas we assume the notations of the previous theorem.

Lemma 7.2.4. *For $\eta \in \mathrm{Sh}(p+q-n, n-q)$, $\tau \in \mathcal{B}_\beta$ and for all $i = 0, \ldots, s$ and $j = 0, \ldots, r$, the sets*

$$F_i \cap \varphi_{\eta,\tau}^{-1} E_j$$

are disjoint intervals. Moreover, in each of these intervals the function $\varphi_{\eta,\tau}$ is increasing and has image either contained in $[1, p]$ or contained in $[p+1, n]$.

Proof. As η and τ are fixed throughout this lemma, we write $\varphi = \varphi_{\eta,\tau}$. Let $x, y \in F_i \cap \varphi^{-1} E_j$ with $x < y$. Consider z such that $x < z < y$. Therefore, $x, y \in F_i$ and, since F_i is an interval, we conclude that $z \in F_i$.

On the other hand $\varphi(x), \varphi(y) \in E_j$. Since $\tau \in \mathcal{B}_\beta$, then $\mathrm{Id} \times \tau$ is increasing in F_i:

$$(\mathrm{Id} \times \tau)(x) < (\mathrm{Id} \times \tau)(z) < (\mathrm{Id} \times \tau)(y). \tag{7.8}$$

In order to prove that $\varphi(z)$ also belongs to E_j, we consider the following cases:

1. Assume that $j = 0$. Then, $\varphi(x), \varphi(y) \in E_0 = [p+1, n]$. Since $(\eta \times \mathrm{Id})$ is the identity on that interval, this implies that $\beta_0(\mathrm{Id} \times \tau)(x)$ and $\beta_0(\mathrm{Id} \times \tau)(y)$ are in $[p+1, n]$. But $\beta_0^{-1}[p+1, n] = [n-q+1, 2n-p-q]$ and β_0 is increasing in that set. Therefore, the three terms in (7.8) belong to $[n-q+1, 2n-p-q]$ and, applying $(\eta \times \mathrm{Id})\beta_0$, which is increasing on this set, we obtain that $\varphi(x) < \varphi(z) < \varphi(y)$.

2. Assume that $j > 0$. Consider the cases:

   (a) Assume that $i = 0$. In this case we have $x, z, y \in F_0 = [1, n-q]$. Then, applying $\mathrm{Id} \times \tau|_{F_0} = \mathrm{Id}$ we continue in the same set. The permutation β_0 sends increasingly $[1, n-q]$ into $[p+q-n+1, p]$. In this last interval, η is also increasing. Thus, the inequality (7.8) implies that $\varphi(x) < \varphi(z) < \varphi(y)$.

   (b) Assume that $i > 0$. We have that $x, y, z \in F_j \subset [n-q+1, n]$. Applying

$\mathrm{Id}\times\tau$ we have that the terms of (7.8) are also in $[n-q+1, n]$. If $(\mathrm{Id}\times\tau)(x) \in [n-q+1, 2n-p-q]$, then $\beta_0(\mathrm{Id}\times\tau)(x) \in [p+1, n]$ and $\varphi(x) \in [p+1, n] = E_0$, which contradicts the assumption $j > 0$. Therefore, the terms in (7.8) belong to $[2n-p-q+1, n]$. The permutation β_0 maps increasingly this interval into $[1, p+q-n]$, and η is also increasing in that image. Thus, we conclude that $\varphi(x) < \varphi(z) < \varphi(y)$.

In all the cases we obtain that $\varphi(x) < \varphi(z) < \varphi(y)$, and since $\varphi(x)$ and $\varphi(y)$ belong to the interval E_j, we deduce that $\varphi(z) \in E_j$. This proves that $F_i \cap \varphi^{-1}E_j$ is an interval.

Notice that along the way we also proved that φ is increasing in the intervals $F_i \cap \varphi^{-1}E_j$ as well as the assertions concerning the images.

The fact that the intervals $F_i \cap \varphi^{-1}E_j$ are disjoint follows immediately from the fact that the sets E_j, for $j = 0, \ldots, r$, and the sets F_i, for $i = 0, \ldots, s$, are disjoint. This finishes the proof. $\square$

Lemma 7.2.5. *For $M \in \mathcal{M}_{\alpha,\beta}$, the map $\psi : S_{\alpha,\beta}(M) \to \mathcal{B}_{c(M)}$, which sends $(\xi, \eta, \sigma, \tau)$ to $g_{\xi,\eta}(\sigma, \tau)$, is a bijection.*

Proof. For the matrix $M = \{m_{ij}\}$, denote by s_{ij} the sum of the entries $m_{k\ell}$ of M for $(k, \ell) \leq (i, j)$ with respect to the lexicographical order of pairs. We define $R_{00} = [1, s_{00}]$ and $R_{ij} = [s_{k\ell}, s_{ij}]$ where s_{ij} covers $s_{k\ell}$. Observe that some of the intervals R_{ij} may be empty. Also note that $\# R_{ij} = m_{ij}$.

The sequence $(R_{00}, R_{01}, \ldots, R_{sr})$ is a weak composition whose nonzero parts form a partition of the interval $[n]$ and $\gamma \in \mathcal{B}_{c(M)}$ if and only if γ is increasing in R_{ij} for all $i \in \{0, \ldots, s\}$ and $j \in \{0, \ldots, r\}$.

Since $M \in \mathcal{M}_{\alpha,\beta}$ and therefore, $\sum_j \#(R_{ij}) = \sum_j m_{ij} = \# F_i$, it follows that

$$F_i = \bigcup_j R_{ij}. \tag{7.9}$$

Moreover, if $\eta \in \mathrm{Sh}(p+q-n, n-q)$ and $\tau \in \mathcal{B}_\beta^\eta$, then $F_i \cap \varphi_{\eta,\tau}^{-1} E_j = R_{ij}$. This can be seen from the fact both sets are intervals with the same cardinal and from the following relation:

$$\bigcup_j (F_i \cap \varphi_{\eta,\tau}^{-1} E_j) = F_i = \bigcup_j R_{ij}.$$

In particular, we deduce that $\varphi_{\eta,\tau}$ is increasing in R_{ij}.

Given $(\xi, \eta, \sigma, \tau) \in S_{\alpha,\beta}(M)$, we will show that $g_{\xi,\eta}(\sigma,\tau) \in \mathcal{B}_{c(M)}$. To prove this, since $\varphi_{\eta,\tau}|_{R_{ij}}$ is increasing and $\varphi_{\eta,\tau} R_{ij} \subseteq E_j$, we observe that

$$(\sigma \times \mathrm{Id})(\eta \times \mathrm{Id})\beta_0(\mathrm{Id} \times \tau)_{|R_{ij}}$$

is also increasing. According to Lemma 7.2.4, the images of R_{ij} under the previous permutation are in $[1,p]$ or $[p+1,n]$, where ξ is increasing. Therefore, left multiplying by ξ we deduce that $g_{\xi,\eta}(\sigma,\tau)$ is increasing in R_{ij}, which proves that it belongs to $\mathcal{B}_{c(M)}$.

We prove now that ψ is bijective. Given $\gamma \in \mathcal{B}_{w(M)}$, we show that there exists a unique quadruple $(\xi, \eta, \sigma, \tau) \in S_{\alpha,\beta}(M)$ such that $\psi(\xi, \eta, \sigma, \tau) = \gamma$.

Assume there exists such a quadruple. Using the fact that $E_j = \bigcup_i \varphi_{\eta,\tau} R_{ij}$, we deduce that

$$\xi(\sigma \times \mathrm{Id})E_j = \gamma\big(\bigcup_i R_{ij}\big). \tag{7.10}$$

This proves the uniqueness of the permutation $\xi(\sigma \times \mathrm{Id})$, in other words, it is the only permutation which maps E_j increasingly into the set on the right side; and this implies the uniqueness of ξ and σ. Therefore, we have that $(\eta \times \mathrm{Id})\beta_0(\mathrm{Id} \times \tau) = (\sigma \times \mathrm{Id})^{-1}\xi^{-1}\gamma$. Thus, η is characterized by the image of $[1, n-q]$ under the permutation on the right, which is $\eta[p+q-n+1,p]$. The uniqueness of τ follows immediately.

Given $\gamma \in \mathcal{B}_{c(M)}$, to construct (ξ,η,σ,τ) we note that

$$\#(E_j) = \sum_i m_{ij} = \#\Big(\bigcup_i R_{ij}\Big) = \#\Big(\gamma\Big(\bigcup_i R_{ij}\Big)\Big), \tag{7.11}$$

and, thus, we can construct a permutation μ such that (7.10) is verified, increasingly mapping E_j into $\gamma\big(\bigcup_i R_{ij}\big)$. This permutation can be written as $\mu = \xi(\sigma \times \mu')$ with $\xi \in \mathrm{Sh}(p, n-p)$, $\sigma \in S_p$ and $\mu' \in S_{n-p}$. Since μ is increasing on $E_0 = [p+1, n]$ we conclude that $\mu' = \mathrm{Id}_{n-p}$, and from the monotony on E_j with $j > 0$ we deduce that $\sigma \in \mathcal{B}_\alpha$. In the same way as before, we construct η by mapping the interval $[1, n-q]$ and for this, we will show that

$$(\sigma \times \mathrm{Id})^{-1}\xi^{-1}\gamma \text{ is increasing in } F_i \text{ for all } i. \tag{7.12}$$

In particular, for $i = 0$, we obtain the desired property to define η. We then consider $\beta_0^{-1}(\eta \times \mathrm{Id})^{-1}(\sigma \times \mathrm{Id})^{-1}\gamma$, which equals $\mathrm{Id} \times \tau$ for some $\tau \in S_p$. Using (7.12) for $i > 0$ we conclude that $\tau \in \mathcal{B}_\beta$; and it follows from (7.11) that the constructed τ belongs to $\mathcal{B}^\eta_\beta(M)$.

It remains to prove (7.12). Take $x_1, x_2 \in F_i$ with $x_1 < x_2$. Then, $x_1 \in R_{ij_1}$ and $x_2 \in R_{ij_2}$ for some $j_1 \le j_2$. Assume that $j_1 = j_2$, then $\gamma(x_1) < \gamma(x_2)$. In this case, we have $\gamma(x_1) = \xi(\sigma \times \mathrm{Id})(e_1)$ and $\gamma(x_2) = \xi(\sigma \times \mathrm{Id})(e_2)$ with $e_1, e_2 \in E_j$. Since σ is increasing in E_j we obtain that $e_1 < e_2$ as desired.

On the other hand, if $j_1 < j_2$, then $e_1 \in E_{j_1}$ and $e_2 \in E_{j_2}$ and the conclusion follows easily as all the elements of E_{j_1} are smaller than those of E_{j_2}. □

Now we present a bijective proof of the part of Theorem 7.2.2 that guarantees the stability of Σ under the Heisenberg product. Our method applied to the composition product is similar to the one presented in [23].

First we fix some basic notations that will be used in the rest of this section. If $\sigma, \nu \in S_n$, then we denote

$$\nu \triangleright \sigma = \sigma^{-1}\nu\sigma,$$

that is, $\nu \triangleright \sigma$ is the result of right conjugating nu by σ. The set of transpositions is stable under conjugation. The transposition permuting the elements $1 \leq i < j \leq n$ is written as α_{ij} and α_{ii+1} is abbreviated as α_i. The transpositions α_i are called elementary transpositions.

Definition 7.2.6. Given σ and τ in S_n, we say that σ, τ are *descent related* and write $\sigma \sim \tau$ if there exists a non-elementary transposition α such that:

(i) $\sigma = \tau\alpha$;

(ii) $\alpha \triangleright \tau^{-1}$ is elementary.

In other words, for some $i, j, k = 1, \ldots, n$ with j and k non-consecutive, $\sigma = \alpha_i\tau$ and $\sigma = \tau\alpha_{jk}$.

Hence, τ and σ are descent related if τ differs from σ only in that two consecutive values that do not appear in consecutive positions have been swapped. For instance, $(3214) \sim (4213) \sim (4312)$.

It is well-known that a equivalence class under the transitive and reflexive closure of the relation $\sim$ consists of those permutations with the same descent set, as defined in (7.1) (see, for example, [7]).

Theorem 7.2.7 (Theorem 7.2.2)**.** *The subspace $\Sigma \subseteq \mathcal{S}$ is stable under the Heisenberg product.*

Proof. To simplify the writing along this proof we set the following notation for the

Heisenberg product of $\sigma \in S_p$ and $\tau \in S_q$:

$$\sigma \# \tau = \sum_{n=\max(p,q)}^{p+q} \sum_{\substack{\xi \in \mathrm{Sh}(p,n-p) \\ \eta \in \mathrm{Sh}(p+q-n,n-q)}} f^n_{\sigma,\tau}(\xi,\eta)$$

where $f^n_{\sigma,\tau}(\xi,\eta) = \xi\big((\sigma\eta) \times \mathrm{Id}_{n-p}\big)\beta_{2n-p-q,p+q-n}(\mathrm{Id}_{n-q} \times \tau)$. In what follows we will omit the subindices from the permutation $\beta_{2n-p-q,p+q-n}$ and from the identities, writing just β and Id, respectively.

If (σ,τ) are as above and $\gamma \in S_n$, observe that there exists at most *one* pair of shuffles (ξ,η) such that $f^n_{\sigma,\tau}(\xi,\eta) = \gamma$. Indeed, if $f^n_{\sigma,\tau}(\xi,\eta) = f^n_{\sigma,\tau}(\xi',\eta')$, canceling the rightmost factors in

$$\xi(\sigma\eta \times \mathrm{Id})\beta(\mathrm{Id} \times \tau) = \xi'(\sigma\eta' \times \mathrm{Id})\beta(\mathrm{Id} \times \tau).$$

we deduce the equality $\xi(\sigma\eta \times \mathrm{Id}) = \xi'(\sigma\eta' \times \mathrm{Id})$. The map $\mathrm{Sh}(p,q) \times S_p \times S_q \to S_n$ given by $(\zeta,\nu,\delta) \mapsto \zeta(\nu \times \delta)$ is a bijection by definition of $\mathrm{Sh}(p,q)$, and we conclude that $\xi = \xi'$ and $\eta = \eta'$.

Let us consider the linear basis $\{\mathcal{D}_J\}$ of Σ_n defined in (7.5). If $I \subset [p-1]$ and $J \subset [q-1]$, we write

$$Y_I \# Y_J = \sum_{n=\max(p,q)}^{p+q} \sum_{\rho \in S_n} \# D^\rho_{I,J} \cdot \rho,$$

where the sets $D^\rho_{I,J}$ are defined as

$$D^\rho_{I,J} = \big\{(\sigma,\tau) \in \mathcal{D}_I \times \mathcal{D}_J \mid \rho = f^n_{\sigma,\tau}(\xi,\eta) \text{ for some } \\ \xi \in \mathrm{Sh}(p,n-p) \text{ and } \eta \in \mathrm{Sh}(p+q-n,n-q)\big\}$$

We prove that if $\rho \sim \rho'$, then there exists a bijection between $D^\rho_{I,J}$ and $D^{\rho'}_{I,J}$. Hence, the number $\# D^\rho_{I,J}$ only depends on the descent class of ρ. If we call $d^U_{I,J} = \# D^\rho_{I,J}$, where $U = \mathrm{Des}(\rho)$, we obtain the formula

$$Y_I \# Y_J = \sum_{n=\max(p,q)}^{p+q} \sum_{U \subset [n-1]} d_{I,J}^U Y_U$$

that proves the theorem.

Assume that $\rho, \rho' \in S_n$ are descent related, that is, $\rho \sim \rho'$. This means that for a non-elementary transposition α and an elementary transposition α', we have that $\rho' = \rho\alpha$ and $\rho' = \alpha'\rho$.

In order to construct the bijection $\psi_{\rho\rho'} : D_{I,J}^{\rho} \to D_{I,J}^{\rho'}$ we "move" step by step the elementary transposition α' from left to right, and in accordance with the result of these successive movements, we define the values of $\psi_{\rho\rho'}(\sigma, \tau)$.

Write

$$\rho' = \alpha'\xi(\sigma\eta \times \mathrm{Id})\beta(\mathrm{Id} \times \tau), \tag{7.13}$$

then we have the following excluding cases:

(A) Assume that $\alpha'\xi \in \mathrm{Sh}(p, n-p)$, then define

$$\psi_{\rho\rho'}(\sigma, \tau) = (\sigma, \tau).$$

(B) Assume that $\alpha'\xi \notin \mathrm{Sh}(p, n-p)$. In accordance with Lemma 7.2.9, we have $\alpha' \triangleright \xi = \alpha_i$, where α_i is an elementary transposition different from α_p; i.e., $\alpha'\xi = \xi\alpha_i$. Write $\rho' = \xi\alpha_i(\sigma\eta \times \mathrm{Id})\beta(\mathrm{Id} \times \tau)$, and consider the following disjoint cases:

(B_1) Assume $p < i$. In this case $\rho' = \xi(\sigma\eta \times \mathrm{Id})\alpha_i\beta(\mathrm{Id} \times \tau)$. It follows from Lemma 7.2.10 that $\alpha_i \triangleright \beta$ is non elementary only when $\alpha_i = \alpha_{p+q-n}$. As $p + q - n \le p < i$ this cannot happen. Then, we can write $\rho' = \xi(\sigma\eta \times \mathrm{Id})\beta\alpha_\ell(\mathrm{Id}\times\tau)$ for some elementary $\alpha_\ell = \alpha_{\beta^{-1}(i)}$. In our hypothesis it follows that $\ell \ge n - q$. Otherwise we commute α_ℓ and $\mathrm{Id} \times \tau$ in order to obtain the equality $\rho' = \rho\alpha_\ell$ that implies $\alpha_\ell = \alpha$. This contradicts the fact that α is non elementary. Moreover, it cannot happen that $\ell = n - q$, since that implies $i = p$ which contradicts the assumption $i < p$. Hence, we can write

$$\rho' = \xi(\sigma\eta \times \mathrm{Id})\beta(\mathrm{Id} \times \alpha_\ell\tau) \tag{7.14}$$

and $\mathrm{Des}(\alpha_\ell\tau) = \mathrm{Des}(\tau)$. Indeed, if $\mathrm{Des}(\alpha_\ell\tau) \neq \mathrm{Des}(\tau)$, Lemma 7.2.8 would imply that $\alpha = \alpha_\ell \rhd \tau$ is an elementary transposition. In this situation define

$$\psi_{\rho\rho'}(\sigma, \tau) = (\sigma, \alpha_\ell\tau)$$

(B_2) Assume $i + 1 < p$ and write

$$\rho' = \xi(\alpha_i\sigma\eta \times \mathrm{Id})\beta(\mathrm{Id} \times \tau). \tag{7.15}$$

Consider the following disjoint cases:

(B_{21}) If $\mathrm{Des}(\alpha_i\sigma) = \mathrm{Des}(\sigma)$ we define

$$\psi_{\rho\rho'}(\sigma, \tau) = (\alpha_i\sigma, \tau).$$

(B_{22}) If $\mathrm{Des}(\alpha_i\sigma) \neq \mathrm{Des}(\sigma)$, we apply Lemma 7.2.8 in order to conclude that $\alpha_i \rhd \sigma$ is an elementary transposition α_m. Write

$$\rho' = \xi\big(\sigma(\alpha_m\eta) \times \mathrm{Id}\big)\beta(\mathrm{Id} \times \tau) \tag{7.16}$$

and consider the following alternatives.

(B_{221}) Assume that the permutation $\alpha_m\eta \in \mathrm{Sh}(p + q - n, n - q)$. In this case we define

$$\psi_{\rho\rho'}(\sigma, \tau) = (\sigma, \tau).$$

(B_{222}) Assume that the permutation $\alpha_m\eta \notin \mathrm{Sh}(p+q-n, n-q)$. In this case $\alpha_m \triangleright \eta$ is an elementary transposition that we call α_t, and by Lemma 7.2.9 we know that $\alpha_t \neq \alpha_{p+q-n}$. Then, using Lemma 7.2.10, we conclude that $\alpha_t \triangleright \beta$ is elementary. Define $\alpha_r = \alpha_t \triangleright \beta$. Then $r \geq n-q$, otherwise we prove as before that α is elementary. Moreover, if $r = n-q$ then $t = p$, and this would imply that $\alpha_i = \alpha_p$, which contradicts our assumptions. Hence, writing

$$\rho' = \xi(\sigma\eta \times \mathrm{Id})\beta\big(\mathrm{Id} \times (\alpha_r\tau)\big), \tag{7.17}$$

we have that $\mathrm{Des}(\alpha_r\tau) = \mathrm{Des}(\tau)$. If $\mathrm{Des}(\alpha_r\tau) \neq \mathrm{Des}(\tau)$, applying as before Lemma 7.2.8, we conclude that α is elementary. In this situation we define

$$\psi_{\rho\rho'}(\sigma, \tau) = (\sigma, \alpha_r\tau).$$

Observe that equations (7.13) to (7.17), together with the corresponding assumptions, show that $\psi_{\rho\rho'}(\sigma, \tau)$ belongs to $D^{\rho'}_{I,J}$.

The last step of the proof of the theorem, is the verification that $\psi_{\rho\rho'}$ is bijective. We prove that $(\psi_{\rho'\rho} \circ \psi_{\rho\rho'})(\sigma, \tau) = (\sigma, \tau)$ for all $(\sigma, \tau) \in D^{\rho}_{I,J}$. This will follow from the fact that, if for some pair of transpositions α, α' with α non elementary and α' elementary, we have that

$$\rho' = \rho\alpha = \alpha'\rho$$

then we also have that

$$\rho = \rho'\alpha = \alpha'\rho'.$$

For example, assume that we are in the alternative (B_1), where

$$\rho' = \alpha'\xi(\sigma\eta \times \mathrm{Id})\beta(\mathrm{Id} \times \tau)$$

and where $\alpha'\xi \notin \mathrm{Sh}(p, n-p)$ and $\alpha'\xi = \xi\alpha_i$, with $p < i$. By successive conjugations we move α' to the far right and produce α_ℓ that verifies:

$$\rho' = \xi(\sigma\eta \times \mathrm{Id})\beta\big(\mathrm{Id} \times (\alpha_\ell\tau)\big).$$

Recall that in this case we defined $\psi_{\rho\rho'}(\sigma, \tau) = (\sigma, \alpha_\ell\tau)$.

In order to compute $\psi_{\rho'\rho}(\sigma, \alpha_\ell\tau)$ we write

$$\rho = \alpha'\rho' = \alpha'\xi(\sigma\eta \times \mathrm{Id})\beta(\mathrm{Id} \times \alpha_\ell\tau)$$

and perform the necessary conjugations. Clearly we are again in the situation (B_1), and when we "move" α' we obtain again α_ℓ. Thus, $\psi_{\rho'\rho}(\sigma, \alpha_\ell\tau) = (\sigma, \alpha_\ell\alpha_\ell\tau) = (\sigma, \tau)$. For the other cases the argument is similar. □

The following lemmas were used in the previous theorem.

Lemma 7.2.8. *Assume that $\sigma, \alpha_i \in S_n$ and that α_i is an elementary transposition. The following assertions are equivalent:*

(a) $\mathrm{Des}(\alpha_i\sigma) = \mathrm{Des}(\sigma)$,

(b) $\alpha_i \triangleright \sigma$ *is a non-elementary transposition.*

Proof. The fact that *(b)* implies *(a)* follows from the result mentioned before: the equivalence classes associated to the relation $\sim$ are the sets of permutation with the same descent set.

Conversely, assume that $\mathrm{Des}(\alpha_i\sigma) = \mathrm{Des}(\sigma)$ and that $\alpha_i \triangleright \sigma$ is the elementary transposition α_j. Then, $\sigma^{-1}\alpha_i\sigma = \alpha_j$ and $\mathrm{Des}(\alpha_i\sigma) = \mathrm{Des}(\sigma)$. From the first equality we deduce that $\sigma(j) = i$ and $\sigma(j+1) = i+1$ or $\sigma(j) = i+1$ and $\sigma(j+1) = i$. In the first case $j \notin \mathrm{Des}(\sigma)$ but as $\alpha_i\sigma(j) = i+1$ and $\alpha_i\sigma(j+1) = i$, $j \in \mathrm{Des}(\alpha_i\sigma)$. In the other case one verifies that $j \in \mathrm{Des}(\sigma)$ and $j \notin \mathrm{Des}(\alpha_i\sigma)$. $\square$

Lemma 7.2.9. *Assume that $p + q = n$ and that $\xi \in \mathrm{Sh}(p, q)$. If $\alpha_i\xi \notin \mathrm{Sh}(p, q)$ for an elementary transposition α_i, then $\alpha_i \triangleright \xi$ is an elementary transposition. Moreover $\alpha_i \triangleright \xi \neq \alpha_p$.*

Proof. If $\alpha_i\xi \notin \mathrm{Sh}(p, q)$, then $\mathrm{Des}(\alpha_i\xi) \neq \mathrm{Des}(\xi)$ and applying Lemma 7.2.8 we conclude that $\alpha_i \triangleright \xi$ is an elementary transposition. The last assertion of the lemma can be proved as follows. If $\alpha_i \triangleright \xi = \alpha_p$, then $\xi^{-1}(i) = p$ and $\xi^{-1}(i+1) = p+1$ or $\xi^{-1}(i) = p+1$ and $\xi^{-1}(i+1) = p$. In the first case $\xi(p) = i$ and $\xi(p+1) = p+1$ and then, ξ does not have a descent at p and this implies that $\xi = \mathrm{Id}$. Then, $\alpha_i\xi = \alpha_p \in \mathrm{Sh}(p, q)$ and this is a contradiction. In the second case, $\xi(p) = i$ and $\xi(p+1) = i$ and this implies that $i = p$, which means that $\xi = \alpha_p$. Again, this is impossible because then we would have $\alpha_i\xi = \mathrm{Id} \in \mathrm{Sh}(p, q)$, contradicting the hypothesis. $\square$

Lemma 7.2.10. *Assume that $p + q = n$, and let α_i an elementary transposition. Then, $\alpha_i \triangleright \beta_{p,q}$ is non elementary if and only if $\alpha_i = \alpha_q$.*

Proof. Clearly $\alpha_q \triangleright \beta_{p,q} = \alpha_{p+q,1}$ which is not elementary. Conversely, if $\alpha_i \triangleright \beta_{p,q}$ is not elementary, it follows from Lemma 7.2.8 that $\mathrm{Des}(\alpha_i\beta_{p,q}) = \{p\}$. A direct inspection shows that the composition $\alpha_i\beta_{p,q}$ does not introduce additional descents only when $\alpha_i = \alpha_q$. $\square$

7.3. Applications

We present some examples of the combinatorial rule for the Heisenberg product, including the rules for the classical products and some interesting identities.

1. *Internal product.* Consider the case $p = q = n$, and let $\alpha = (a_1, \ldots, a_r)$ and $\beta = (b_1, \ldots, b_s)$ compositions of n. As observed in Section 7.2, the sum of terms of degree n in $X_\alpha \# X_\beta$ is the Solomon's product $X_\alpha * X_\beta$. In this case, an element of $\mathcal{M}^n_{\alpha,\beta}$ has the shape described below:

$$\begin{array}{cccc|c} 0 & m_{01} & \cdots & m_{0r} & 0 \\ m_{10} & m_{11} & \cdots & m_{1r} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{s0} & m_{s1} & \cdots & m_{sr} & b_s \\ \hline 0 & a_1 & \cdots & a_r & \end{array}$$

and this implies that the first row and column have to be zero. Then $\mathcal{M}^n_{\alpha,\beta}$ can be identified with the set of matrices M such that its columns add up to α and its rows add up to β. Denoting such set by $\mathcal{M}$, We get

$$X_\alpha * X_\beta = \sum_{M \in \mathcal{M}} X_{c(M)}.$$

This is Solomon's rule for the internal product in the space Σ, which is a well-known formula given in [12, 19].

2. *External product.* Assume p and q are arbitrary non-negative integers. Let $\alpha = (a_1, \ldots, a_r)$ a composition of p and $\beta = (b_1, \ldots, b_s)$ a composition of q. Consider $n = p + q$. The sum of the terms of degree n in $X_\alpha \# X_\beta$ is the convolution product $X_\alpha \cdot X_\beta$. We have $n - p = q$ and $n - q = p$, and the set $\mathcal{M}^n_{\alpha,\beta}$ is the set of matrices with non-negative integral entries which can fill the

black dots in the following diagram:

$$\begin{array}{cccc|c} 0 & \bullet & \cdots & \bullet & p \\ \bullet & \bullet & \cdots & \bullet & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bullet & \bullet & \cdots & \bullet & b_s \\ \hline q & a_1 & \cdots & a_r & \end{array}$$

where the rightmost column and the lowest row indicate as usual the values of the sum of the corresponding rows and columns, respectively. The only way to fill this matrix is by placing $(a_1, \ldots, a_r)$ in the top row, $(b_1, \ldots, b_s)$ in the leftmost column, and zeroes in the rest of the matrix. Call M the corresponding matrix:

$$M = \begin{pmatrix} 0 & a_1 & \cdots & a_r \\ b_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_s & 0 & \cdots & 0 \end{pmatrix}.$$

Then $\mathcal{M}^n_{\alpha,\beta} = \{M\}$, the weak composition $c(M)$ is $\alpha\beta$ (concatenation of both compositions), and

$$X_\alpha \cdot X_\beta = X_{(a_1,\ldots,a_r,b_1,\ldots,b_s)},$$

which is Formula (7.4).

3. *Product of identities.* Consider the compositions $\alpha = (p)$ and $\beta = (q)$. Let n an integer satisfying $\max(p, q) \le n \le p+q$. Then, the matrix below is the only way to fill in the associated framework for the Heisenberg product of $X_{(p)}$ and

$X_{(q)}$:

$$\begin{array}{cc|c} 0 & n-q & n-q \\ n-p & p+q-n & q \\ \hline n-p & p & \end{array} .$$

Then, we have that

$$X_{(p)} \# X_{(q)} = \sum_{n=\max(p,q)}^{p+q} X_{(n-q,n-p,p+q-n)}. \tag{7.18}$$

Note that some of the entries of $(n-q, n-p, p+q-n)$ can be 0, in which case we just remove such entry, according to the definition of $c(M)$ in Section 7.2.

Under the bijection (7.3), the subset that corresponds to the composition (p) is the empty set. As the only permutation with empty descent set is the identity, we get

$$X_{(p)} = \mathrm{Id}_p,$$

hence Formula (7.18) is actually the Heisenberg product of the identities of S_p and S_q.

4. *Products* $X_{(1^p)} \# X_{(1^q)}$. For a non-negative integer p, we denote by (1^p) the composition of p given by $(1, \ldots, 1)$, that is, all its parts are 1. In order to compute $X_{(1^p)} \# X_{(1^q)}$ we need to fill in all possible ways the following matrix of size $(q+1) \times (p+1)$:

$$\begin{array}{cccc|c} 0 & \bullet & \cdots & \bullet & n-q \\ \bullet & \bullet & \cdots & \bullet & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bullet & \bullet & \cdots & \bullet & 1 \\ \hline n-p & 1 & \cdots & 1 & \end{array}$$

In the first row, we can only put 1's and 0's, exactly $n-q$ ones and the rest of zeroes. Similarly for the first column with $n-p$ ones and the rest of zeroes. All in all we have exactly

$$\binom{p}{n-q} \times \binom{q}{n-p}$$

possibilities as choices for the first row and column. Observe that if one of the elements of the first row is one, all the elements of the corresponding column have to be zero and something similar happens with the first column. Hence, in order to fill out the remaining spaces of the framework, we eliminate the rows and columns whose "headings" have been filled with a one and all we have to consider are

$$\big(q-(n-p)\big) \times \big(p-(n-q)\big) = (p+q-n) \times (p+q-n)$$

matrices, that must have exactly one 1 en each row and column. Thus, we have to consider all the possible permutation square matrices of size $p+q-n$, which are as many as $(p+q-n)!$. Then, the total number of possible matrices to fill in the above framework is

$$\binom{p}{n-q}\binom{q}{n-p}(p+q-n)!.$$

Moreover, it is clear that all the matrices M that can be fitted into the above framework consists only of 1's. Moreover, as in each of the matrices there are exactly n ones, we have $c(M) = (1^n)$.

Hence, we obtain the following formula:

$$X_{(1^p)} \# X_{(1^q)} = \sum_{n=\max(p,q)}^{p+q} \binom{p}{n-q}\binom{q}{n-p}(p+q-n)!\, X_{(1^n)}.$$

Compare this formula with Formula (4.26) for the Heisenberg product of symmetric power functions. This same argument could have been used to prove Formula (4.26), by invoking Theorem 4.3.1.

5. *Powers of* $X_{(1)}$. The Heisenberg product of $X_{(1)}$ by itself n times gives:

$$X_{(1)}^{\#(n)} = \sum_{k=1}^{n} S(n,k) X_{(1^k)}, \tag{7.19}$$

where $S(n,k)$ are the Stirling numbers of second kind (we used the symbol $\#(n)$ as exponent to emphasize that the power is with respect to the Heisenberg product). Equation (7.19) can be easily proved by induction on n. For $n = 1$ the result is trivial. Assume that (7.19) is true for $n-1$ and compute

$$X_{(1)}^{\#(n)} = X_{(1)}^{\#(n-1)} \# X_{(1)} = \sum_{k=1}^{n-1} S(n-1,k) X_{(1^k)} \# X_{(1)}. \tag{7.20}$$

The product $X_{(1^k)} \# X_{(1)}$ only has terms of degree k and $k+1$. They are found by filling the following $2 \times (k+1)$ matrices

$$\begin{array}{ccccc|c} 0 & 1 & 0 & \cdots & 1 & k-1 \\ 0 & 0 & 1 & \cdots & 0 & 1 \\ \hline 0 & 1 & 1 & \cdots & 1 & \end{array} \quad , \qquad \begin{array}{cccc|c} 0 & 1 & \cdots & 1 & k \\ 1 & 0 & \cdots & 0 & 1 \\ \hline 1 & 1 & \cdots & 1 & \end{array} .$$

There is k ways to fill the first matrix and all the possibilities yield the composition $(1,\dots,1)$ of k. And there is only one way to fill the second matrix, which produces the composition $(1,\dots,1)$ of $k+1$. Substituting in (7.20) and reindexing the sums we obtain

$$X_{(1)}^{\#(n)} = \sum_{k=1}^{n} \big[kS(n-1,k) + S(n-1,k-1) \big] X_{(1^k)} = \sum_{k=1}^{n} S(n,k) X_{(1^k)},$$

using a well-known recurrence formula for the Stirling numbers. This proves Equation (7.19).

8. HOPF ALGEBRAS WITH THE HEISENBERG PRODUCT

8.1. From non-commutative to commutative symmetric functions

In the previous sections we constructed the following commutative diagram of algebras with respect to the Heisenberg product:

$$\begin{tikzcd}
\Sigma\big(T(V)\big) \arrow[r, tail] & \mathrm{end}\big(T(V)\big) \\
\Sigma \arrow[u, hook] \arrow[r, tail] & \mathcal{S} \arrow[u]
\end{tikzcd}$$

The algebra $(\Sigma, \cdot)$ of non-commutative symmetric functions and the algebra $(\mathcal{S}, \star)$ of Malvenuto-Reutenauer are non-commutative. We want to extend the previous diagram to the commutative algebra $(\Lambda, \cdot)$ of symmetric functions, where we already defined the Heisenberg structure in Section 4.

In this section we define a surjective linear map $\Sigma \twoheadrightarrow \Lambda$ and we show that it preserves the Heisenberg structures in Λ and Σ, as constructed in Section 4.3 and Section 7.2, respectively.

Let $n \geq 0$ and define the linear map $\pi_n : \Sigma_n \to \Lambda_n$ by mapping the basis $\{X_\alpha\}$ of Σ_n onto the the basis of complete homogeneous symmetric functions $\{h_\alpha\}$:

$$\pi_n(X_\alpha) = h_{\tilde{\alpha}}, \tag{8.1}$$

where α is a composition of n and $\tilde{\alpha}$ is the partition of n obtained by reordering the entries of α. Let us denote by $\pi : \Sigma \to \Lambda$ the map induced in the direct sum.

It is known that the map π is a morphism of algebras with respect to the external and internal products. We prove next that the same is true with respect to the Heisenberg product. This connection is what motivates the name "Heisenberg" for the product of species, representations and symmetric functions.

Theorem 8.1.1. *For any pair of compositions α and β it holds*

$$\pi(X_\alpha \# X_\beta) = h_{\tilde{\alpha}} \# h_{\tilde{\beta}}.$$

Proof. Comparing equations (4.9) and (7.6), it is enough to construct a bijection $\psi : \mathcal{M}^n_{\alpha,\beta} \to \mathcal{M}^n_{\tilde{\alpha},\tilde{\beta}}$, for a fixed n, such that

$$p\big(\psi(M)\big) = \widetilde{c(M)} \tag{8.2}$$

for all $M \in \mathcal{M}^n_{\alpha,\beta}$. Recall that $\widetilde{c(M)}$ is the reordering of the parts of the composition $c(M)$ to make it a partition.

Let σ and τ be two permutations which reorder the compositions $\alpha = (a_1, \dots, a_r)$ and $\beta = (b_1, \dots, b_s)$, respectively, into partitions. This means that

$$\tilde{\alpha} = (a_{\sigma(1)}, \dots, a_{\sigma(r)}), \quad \tilde{\beta} = (b_{\tau(1)}, \dots, b_{\tau(r)}),$$

are partitions. Define $\psi(M)$ as the matrix obtained from M by permuting its columns with the permutation $\mathrm{Id}_1 \times \sigma$ and its rows with the permutation $\mathrm{Id}_1 \times \tau$. The result belongs to $\mathcal{M}^n_{\tilde{\alpha},\tilde{\beta}}$. By using the inverse permutations $\mathrm{Id}_1 \times \sigma^{-1}$ and $\mathrm{Id}_1 \times \tau^{-1}$ we see that this construction is a bijection. Moreover, since the entries of M and $\psi(M)$ are the same, we get Equation (8.2). $\square$

From this theorem we obtain following the diagram of algebras with the Heisenberg product:

$$\begin{tikzcd}
\Sigma\big(T(V)\big) \arrow[r, tail] & \mathrm{end}\big(T(V)\big) \\
\Sigma \arrow[u, hook] \arrow[r, tail] \arrow[d, two heads, "\pi"'] & \mathcal{S} \arrow[u] \\
\Lambda &
\end{tikzcd}$$

We will see in the next sections that the spaces Σ and Λ are actually Hopf algebras with the Heisenberg product.

8.2. Hopf structures on non-commutative and commutative symmetric functions

The space Σ of non-commutative symmetric functions has a structure of coalgebra given by the coproduct

$$\Delta(X_{(a_1,\ldots,a_r)}) = \sum_{\substack{b_i+c_i=a_i \\ 0\leq b_i,c_i}} X_{(b_1,\ldots,b_r)^{\frown}} \otimes X_{(c_1,\ldots,c_r)^{\frown}}, \tag{8.3}$$

where $^{\frown}$ indicates that parts equal to zero are omitted.

Next we proved that Δ is compatible with the Heisenberg product in Σ.

Theorem 8.2.1. *The space* $(\Sigma, \#, \Delta)$ *is a cocommutative Hopf algebra.*

Proof. It is enough to prove that Δ is a morphism of algebras with respect to the Heisenberg product. Let α and β compositions of p and q, respectively. We use Formula (7.6) to compute

$$\Delta(X_\alpha \# X_\beta) = \sum_n \sum_{M\in\mathcal{M}^n_{\alpha,\beta}} \Delta(X_{c(M)}) = \sum_{n=p\vee q}^{p+q} \sum_{M\in\mathcal{M}^n_{\alpha,\beta}} \sum_{\gamma+\gamma'=c(M)} X_\gamma \otimes X_{\gamma'}. \tag{8.4}$$

On the other hand,

$$\begin{aligned}\Delta(X_\alpha) \# \Delta(X_\beta) &= \Bigg(\sum_{\gamma+\gamma'=\alpha} X_\gamma \otimes X_{\gamma'}\Bigg) \# \Bigg(\sum_{\delta+\delta'=\beta} X_\delta \otimes X_{\delta'}\Bigg) \\ &= \sum_{\substack{\gamma+\gamma'=\alpha \\ \delta+\delta'=\beta}} (X_\gamma \# X_\delta) \otimes (X_{\gamma'} \# X_{\delta'}) \\ &= \sum_{\substack{\gamma+\gamma'=\alpha \\ \delta+\delta'=\beta}} \sum_{n,n'} \sum_{\substack{M\in\mathcal{M}^n_{\delta,\gamma} \\ M'\in\mathcal{M}^{n'}_{\delta',\gamma'}}} X_{c(M)} \otimes X_{c(M')}.\end{aligned} \tag{8.5}$$

We show that the sums (8.4) and (8.5) are the same as follows: take an octuple of indices corresponding to the sum (8.5): $(\gamma, \gamma', \delta, \delta', n, n', M, M')$ and construct the quadruple $\big(n+n', M+M', c(M), c(M')\big)$. Denote by $\mathrm{col}(M)$ the vector whose entries are the sum of the columns of the matrix M, and similarly, $\mathrm{row}(M)$ to the sum of rows. Since

$$\mathrm{col}(M+M') = \mathrm{col}(M) + \mathrm{col}(M') = \big(n - |\gamma|\big)\gamma + \big(n' - |\gamma'|\big)\gamma' = (n+n'-p)\alpha,$$

where $|\zeta|$ is the sum of the parts of a composition ζ, and similarly with $\mathrm{row}(M+M') = (n+n'-q)\beta$, we see that $M+M' \in \mathcal{M}^n_{\alpha,\beta}$. As $c(M)+c(M') = c(M+M')$, if we set

$$(\widetilde{n}, \widetilde{M}, \widetilde{\gamma}, \widetilde{\gamma'}) = \big(n+n', M+M', c(M), c(M')\big)$$

it is clear that $(\widetilde{n}, \widetilde{M}, \widetilde{\gamma}, \widetilde{\gamma'})$ is a quadruple of indices appearing in the sum (8.4) and that the corresponding summands of (8.4) and (8.5) are the same.

Moreover, it is clear that the above correspondence between the indices of the sums is bijective. $\square$

Remark 8.2.2. The coproduct defined in (8.3) is the restriction to Σ of a coproduct in $\mathcal{S}$ which has the expression, for $\sigma \in S_n$,

$$\Delta(\sigma) = \sum_{p=0}^{n} \sigma_p \otimes \sigma'_{n-p},$$

where $\sigma_p \in S_p$ and $\sigma'_{n-p} \in S_{n-p}$ are the only permutations such that $\sigma = (\sigma_p \times \sigma'_{n-p})\xi^{-1}$ with $\xi \in \mathrm{Sh}(p, n-p)$.

This coproduct is compatible with the convolution product in $\mathcal{S}$, making $(\mathcal{S}, \star, \Delta)$ a graded connected Hopf algebra. However, Δ is not compatible with the composition product in $\mathcal{S}$ [19], although they become compatible in the restriction to Σ and in the projection to Λ. In particular, this means that the Heisenberg product is not compatible with the coproduct in $\mathcal{S}$.

The space Λ of symmetric functions also has a coproduct, that we already used to relate the Heisenberg product to the external and internal products (4.20). Recall from (4.22) that, in the linear basis of functions $\{h_\alpha\}$, the coproduct is expressed as

$$\Delta(h_{(a_1,\dots,a_r)}) = \sum_{\substack{b_i+c_i=a_i \\ 0 \le b_i, c_i}} h_{(b_1,\dots,b_r)} \otimes h_{(c_1,\dots,c_r)}. \tag{8.6}$$

Then, we have the following theorem.

Theorem 8.2.3. *The space* $(\Lambda, \#, \Delta)$ *is a cocommutative Hopf algebra, and the map* $\pi : \Sigma \to \Lambda$ *defined in* (8.1) *is a morphism of Hopf algebras.*

Proof. By Theorem 8.1.1 we know that π is multiplicative. Recall that π send X_α into $h_{\tilde{\alpha}}$, which coincides with h_α. By equations (8.6) and (8.3) it is immediate that π also preserves the comultiplication:

$$\Delta\big(\pi(X_\alpha)\big) = \Delta(h_\alpha) = \sum h_{\alpha_1} \otimes h_{\alpha_2} = \sum \pi(X_{\alpha_1}) \otimes \pi(X_{\alpha_2}) = (\pi \otimes \pi)\big(\Delta(X_\alpha)\big).$$

And, since $\pi(X_\alpha \# X_\beta) = \pi(X_\alpha) \# \pi(X_\beta)$, the compatibility of the coproduct and the Heisenberg product in the space Σ induces the compatibility Δ and $\#$ in the space Λ. $\square$

8.3. Isomorphisms between Heisenberg and classical structures

The Heisenberg product is related to the internal and external products not only through interpolation, but also through certain isomorphisms. This statement must be qualified as follows. The external and Heisenberg products are isomorphic, but the isomorphism is not degree-preserving. The Heisenberg and internal products are also isomorphic, once they are extended to the completion with respect to the grading. Both results hold for symmetric functions and for non-commutative symmetric functions, as discussed next.

Theorem 8.3.1. *The map $\psi : (\Sigma, \cdot, \Delta) \to (\Sigma, \#, \Delta)$ given by*

$$\psi(X_{(a_1,\ldots,a_r)}) = X_{(a_1)} \# \cdots \# X_{(a_r)} \tag{8.7}$$

is an isomorphism of Hopf algebras (which does not preserve gradings).

Proof. Since the Heisenberg product has the external product as the only term in the upper degree, the matrix of the linear map ψ in the basis (X_α) is triangular with 1 in the diagonal. Hence, it is invertible. The map ψ is also multiplicative, since the external product in the basis (X_α) is the concatenation of the compositions.

It remains to prove that ψ is comultiplicative, that is,

$$\Delta\big(\psi(X_\alpha) \# \psi(X_\beta)\big) = (\psi \otimes \psi)\Delta(X_\alpha \cdot X_\beta). \tag{8.8}$$

Since it was already proved that ψ is multiplicative, it is enough to prove (8.8) on the algebra generators $X_{(a)}$ for a non-negative integer a. For the right hand side we have

$$\Delta(X_{(a_1)} \cdot X_{(a_2)}) = \Delta(X_{(a_1,a_2)}) = \sum_{\substack{a+b=a_1 \\ a'+b'=a_2}} X_{(a,a')^\frown} \otimes X_{(b,b')^\frown}.$$

Applying the map $\psi \otimes \psi$ and using formula (7.6) to compute $\psi(X_{(a,a')}) = X_{(a)} \# X_{(a')}$ and $\psi(X_{(b,b')}) = X_{(b)} \# X_{(b')}$ (note that we assume $\psi(X_{(0)})$ to be the identity) we get

$$(\psi \otimes \psi)\Delta(X_{(a_1,a_2)^\frown}) = \sum_{\substack{a+b=a_1 \\ a'+b'=a_2}} \sum_{n,m} X_{(n-a',n-a,a+a'-n)^\frown} \otimes X_{(m-b',m-b,b+b'-m)^\frown}. \tag{8.9}$$

On the other hand, taking into account that $\psi(X_\alpha) = X_\alpha$ for partitions with only one part, the left hand side is

$$\Delta\big(X_{(a_1)} \# X_{(a_2)}\big) = \sum_k \sum_{\substack{c_1+c_1'=k-a_2 \\ c_2+c_2'=k-a_1 \\ c_3+c_3'=a_1+a_2-k}} X_{(c_1,c_2,c_3)^\frown} \otimes X_{(c_1',c_2',c_3')^\frown}, \tag{8.10}$$

By collecting together the terms in (8.9) with $n+m=k$ and interchanging the sums, it is easy to see that (8.9) and (8.10) are the same expression. □

Corollary 8.3.2. *The map $(\Lambda, \cdot, \Delta) \to (\Lambda, \#, \Delta)$ given by*

$$h_{(a_1,\dots,a_r)} \mapsto h_{(a_1)} \# \cdots \# h_{(a_r)} \tag{8.11}$$

is an isomorphism of Hopf algebras (which does not preserve gradings).

The Heisenberg and internal products are also isomorphic at the level of $\hat{\Lambda}$.

Theorem 8.3.3. *The map $(\hat{\Lambda}, \#) \to (\hat{\Lambda}, *)$ given by*

$$f \mapsto f \cdot \sum_{n \geq 0} h_{(n)} \tag{8.12}$$

is an isomorphism of algebras.

Proof. This isomorphism follows from isomorphism (2.9) in the category of species.

Note that the species e corresponds to the object $(\mathbf{1}_0, \mathbf{1}_1, \dots)$ in the category $\mathbf{R}$, where $\mathbf{1}_n$ is the trivial S_n-modules. Applying the Kronecker group construction and then the Frobenius map ch we obtain that e corresponds to the element $\sum_{n\geq 0} h_{(n)}$ in $\hat{\Lambda}$. $\square$

Remark 8.3.4. The isomorphism $(\Sigma, \cdot) \cong (\Sigma, \#)$ of Theorem 8.3.1 does *not* extend to an isomorphism between $(\hat{\Sigma}, \cdot)$ and $(\hat{\Sigma}, \#)$. Indeed, consider the element

$$X_{(1)} + X_{(1,1)} + X_{(1,1,1)} + \cdots$$

of $\hat{\Sigma}$. This element would map to

$$X_{(1)} + X_{(1)} \# X_{(1)} + X_{(1)} \# X_{(1)} \# X_{(1)} + \cdots .$$

Each of the terms in this infinite sum contributes a term of degree 1 (namely, $X_{(1)}$); therefore, this infinite sum is not a well-defined element of $\hat{\Sigma}$.

Similarly, the isomorphism $(\hat{\Lambda}, \#) \cong (\hat{\Lambda}, *)$ of Theorem 8.3.3 does not restrict to an isomorphism between $(\Lambda, \#)$ and $(\Lambda, *)$. Indeed, the element $1 \in \Lambda$ maps to $\sum_{n\geq 0} h_{(n)}$ which is in $\hat{\Lambda}$ but not in Λ.

Moreover, a natural question is whether a similar isomorphism to (8.12) holds in $\hat{\Sigma} = \prod_{n\geq 0} \Sigma_n$, that is, whether

$$\varphi : f \mapsto f \cdot \sum_{n\geq 0} X_{(n)} \quad \text{or} \quad \psi : f \mapsto \sum_{n\geq 0} X_{(n)} \cdot f$$

are isomorphisms between $(\hat{\Sigma}, \#)$ and $(\hat{\Sigma}, *)$. The answer is no, and the following counterexample shows that the maps are not multiplicative. Compute $X_{(3)} \# X_{(3)}$ using the rule (7.6):

$$X_{(3)} \# X_{(3)} = X_{(3)} + X_{(1,1,2)} + X_{(2,2,1)} + X_{(3,3)}.$$

Then, an application of φ yields

$$\varphi(X_{(3)} \# X_{(3)}) = \sum_{n\geq 0} X_{(3,n)^\frown} + \sum_{n\geq 0} X_{(1,1,2,n)^\frown} + \sum_{n\geq 0} X_{(2,2,1,n)^\frown} + \sum_{n\geq 0} X_{(3,3,n)^\frown}. \quad (8.13)$$

On the other hand, computing $\varphi(X_{(3)}) \circ \varphi(X_{(3)})$ using Solomon's rule, gives

$$\begin{aligned}\varphi(X_{(3)}) * \varphi(X_{(3)}) &= \sum_{n,m} X_{(3,n)^\frown} * X_{(3,m)^\frown} = \sum_{n} X_{(3,n)^\frown} * X_{(3,n)^\frown} \\ &= \sum_{n\geq 0} X_{(3,n)^\frown} + X_{(2,1,1,n)^\frown} + X_{(1,2,2,n)^\frown} + X_{(3,3,n)^\frown}. \quad (8.14)\end{aligned}$$

We can see that (8.13) and (8.14) are different since, for example, the term $X_{(2,1,1,n)^\frown}$ appears in (8.14) but there is no term in (8.13) whose index is a composition starting with $2, 1, 1$. A similar argument shows that ψ is not multiplicative, either.

9. QUASI-SYMMETRIC FUNCTIONS

9.1. Classical coproducts of quasi-symmetric functions

To summarize the previous sections, we constructed the Heisenberg product in Σ and Λ so that they are Hopf algebras. In this section we consider the quasi-symmetric functions $\mathcal{Q}$, dual to the non-commutative symmetric functions, and we introduce the Heisenberg coproduct by dualization. The space $\mathcal{Q}$ will fill the following commutative diagram of Hopf algebras, where Λ and Σ have the external product and the coproduct defined in Section 8.2, and $\mathcal{S}$ has the Malvenuto-Reutenauer product:

$$\begin{tikzcd} \Sigma \arrow[r, tail] \arrow[d, two heads] & \mathcal{S} \arrow[d, two heads, "F"] \\ \Lambda \arrow[r, tail, "i"] & \mathcal{Q} \end{tikzcd}$$

The diagram is self dual with respect to the antidiagonal. The maps F and i are the dual of the inclusion of Σ in $\mathcal{S}$ and the projection of Σ onto Λ, respectively. The map F is described in detail in [4] and we will not use it here. Note that the space $\mathcal{S}$ is a Hopf algebra only with respect to the Malvenuto-Reutenauer product (as noted in Remark 8.2.2).

Let $\mathbf{X} = \{x_1, x_2, \ldots\}$ be a countable set, totally ordered by $x_1 < x_2 < \cdots$. We say that $\mathbf{X}$ is an *alphabet*. Let $\Bbbk[[\mathbf{X}]]$ be the algebra of formal power series on $\mathbf{X}$ and $\mathcal{Q} = \mathcal{Q}(\mathbf{X})$ the subspace linearly spanned by the elements

$$M_\alpha = \sum_{i_1 < \cdots < i_r} x_{i_1}^{a_1} \cdots x_{i_r}^{a_r} \tag{9.1}$$

as $\alpha = (a_1, \ldots, a_r)$ runs over all compositions of n, for $n \geq 0$. The space $\mathcal{Q}$ is a graded subalgebra of $\Bbbk[[\mathbf{X}]]$ known as the algebra of *quasi-symmetric functions* [15].

In other words, an element $f \in \Bbbk[\![\mathbf{X}]\!]$ is a quasi-symmetric function if the coefficients of $x_1^{k_1} \cdots x_n^{k_n}$ and of $y_1^{k_1} \cdots y_n^{k_n}$ coincide whenever $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, and for any positive integers $k_1, \ldots, k_n$. It is clear that any symmetric function is quasi-symmetric, hence we have the inclusion of algebras $i : \Lambda \hookrightarrow \mathcal{Q}$. In [20] it is proved that this map is the dual of the projection $\pi : \Sigma \twoheadrightarrow \Lambda$ defined in Section 8.1.

The duality between Σ and $\mathcal{Q}$ is realized by the following pairing between the homogeneous components of degree n:

$$\langle M_\alpha, X_\beta \rangle = \delta_{\alpha,\beta}. \tag{9.2}$$

It is known [14, 15, 20] that this pairing identifies the product of quasi-symmetric functions with the coproduct (8.3) of Σ

$$\langle fg, u \rangle = \langle f \otimes g, \Delta(u) \rangle,$$

for $f, g \in \mathcal{Q}$ and $u \in \Sigma$.

The algebra $\mathcal{Q}$ also carries two coproducts Δ_i and Δ_e which are defined via evaluation of quasi-symmetric functions on alphabets. To express this evaluation we first define some operations on alphabets.

Definition 9.1.1. Let $\mathbf{X}$ and $\mathbf{Y}$ be two alphabets. The sum $\mathbf{X} + \mathbf{Y}$ is the disjoint union $\mathbf{X} \sqcup \mathbf{Y}$ together with a total order that extend the order in $\mathbf{X}$ and $\mathbf{Y}$ satisfying

$$x < y$$

for $x \in \mathbf{X}$ and $y \in \mathbf{Y}$.

The product $\mathbf{X} \times \mathbf{Y}$ is the cartesian product of $\mathbf{X}$ and $\mathbf{Y}$ together with the reverse lexicographic order:

$$(x, y) \leq (x', y') \qquad \text{means} \qquad y < y' \text{ or } (y = y' \text{ and } x < x').$$

The coproducts internal and external coproducts in $\mathcal{Q}$ are defined by the formulas

$$\Delta_i\big(f(\mathbf{X})\big) = f(\mathbf{X} \times \mathbf{Y}) \quad \text{and} \quad \Delta_e\big(f(\mathbf{X})\big) = f(\mathbf{X} + \mathbf{Y}),$$

where we identify $\mathcal{Q}(\mathbf{X}, \mathbf{Y}) \cong \mathcal{Q}(\mathbf{X}) \otimes \mathcal{Q}(\mathbf{X})$ to obtain maps $\Delta_i, \Delta_e : \mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q}$. This is usually called separation of variables.

The coproducts Δ_i and Δ_e in $\mathcal{Q}$ are dual through the pairing (9.2) to the internal and external product in Σ, respectively. In other words,

$$\langle \Delta_* f, u \otimes v \rangle = \langle f, uv \rangle, \quad \langle \Delta_. f, u \otimes v \rangle = \langle f, u \cdot v \rangle,$$

for any $f, g \in \mathcal{Q}$ and $u, v \in \Sigma$. Here we set $\langle f \otimes g, u \otimes v \rangle = \langle f, u \rangle \langle g, v \rangle$.

9.2. The Heisenberg coproduct of quasi-symmetric functions

Let $\Delta_\#$ be the coproduct of $\mathcal{Q}$ dual to the Heisenberg product of Σ, that is, $\Delta_\#$ satisfies

$$\langle \Delta_\# f, u \otimes v \rangle = \langle f, u \# v \rangle,$$

for all $f \in \mathcal{Q}$ and $u, v \in \Sigma$. We call $\Delta_\#$ the Heisenberg coproduct. Since the Heisenberg product is a sum of terms of various degrees (5.4), the Heisenberg coproduct is a finite sum of the form

$$\Delta_\#(f) = \sum_i f_i \otimes f_i'$$

with $0 \leq \deg(f_i)$ and $\deg(f_i') \leq \deg(f) \leq \deg(f_i) + \deg(f_i')$. The terms corresponding to $\deg(f) = \deg(f_i) = \deg(f_i')$ and to $\deg(f) = \deg(f_i) + \deg(f_i')$ are the coproducts $\Delta_i(f)$ and $\Delta_e(f)$, respectively.

We now give an expression of the Heisenberg coproduct as an evaluation over an alphabet. Let $\mathbf{1}+\mathbf{X}$ denote the alphabet $\mathbf{X}$ together with a new variable x_0 smaller than all the others and with the property $x_0^k = x_0$ for any natural k. Let

$$(\mathbf{1}+\mathbf{X}) \times (\mathbf{1}+\mathbf{Y}) - \mathbf{1}$$

be the Cartesian product of the alphabets $\mathbf{1}+\mathbf{X}$ and $\mathbf{1}+\mathbf{X}$ with reverse lexicographic ordering and with the variable (x_0, y_0) removed. We can suggestively denote $(\mathbf{1}+\mathbf{X}) \times (\mathbf{1}+\mathbf{Y}) - \mathbf{1}$ by $\mathbf{X}+\mathbf{Y}+\mathbf{XY}$, although the order is given properly by the former expression.

The following result was obtained in conversation with Arun Ram.

Theorem 9.2.1. *For any $f \in \mathcal{Q}$,*

$$\Delta_{\#}\big(f(\mathbf{X})\big) = f(\mathbf{X}+\mathbf{Y}+\mathbf{XY}).$$

Proof. We have to show that, with respect to the pairing (9.2),

$$\langle M_\gamma(\mathbf{X}+\mathbf{Y}+\mathbf{XY}), X_\alpha \otimes X_\beta\rangle = \langle M_\gamma, X_\alpha \# X_\beta\rangle \tag{9.3}$$

for all γ, α and β compositions of n, p and q, respectively. Let us fix a composition γ of n and let k the length of γ. Denote the set of indices of $M_\gamma(\mathbf{X}+\mathbf{Y}+\mathbf{XY})$ by

$$\mathcal{Y} = \big\{\big((i_1, j_1), \dots, (i_k, j_k)\big) \bigm| (i_1, j_1) < \cdots < (i_k, j_k)\big\}.$$

Consider the set $\mathcal{A}_{\alpha,\beta} = \big\{M \in \mathcal{M}^n_{\alpha,\beta} \mid w(M) = \gamma\big\}$ and define the map

$$\psi : \mathcal{Y} \to \bigcup_{\alpha,\beta} \mathcal{A}_{\alpha,\beta}$$

as follows: given $(i_1, j_1) < \cdots < (i_k, j_k)$, let $\widetilde{M} = (\widetilde{m}_{ij})$ be a matrix of zeroes big enough to set $\widetilde{m}_{j_\ell i_\ell} = \gamma_\ell$ (as usual in these proofs, we start the indices in 0). Then, remove all zero rows and columns, except those with index 0; let us call M to the result. Since $(0,0)$ is not a possible index, we have $m_{00} = 0$. Thus, $M \in \mathcal{M}^n_{\alpha,\beta}$ where α is the composition obtained by adding all the rows of M but the first, and analogously with β and the rows of M.

The map ψ is surjective, since, given some $M \in \mathcal{A}_{\alpha,\beta}$, we can build a sequence of indices in $\mathcal{Y}$ by reading the nonzero entries of M, say m_{uv}, and considering the pairs (v,u) lexicographically ordered. Therefore, we can write

$$M_\gamma(\mathbf{X}+\mathbf{Y}+\mathbf{XY}) = \sum_{q\in\mathcal{Y}} (xy)^\gamma_q = \sum_{\alpha,\beta} \sum_{M\in\mathcal{A}_{\alpha,\beta}} \sum_{q\in\psi^{-1}(M)} (xy)^\gamma_q$$

where $(xy)^\gamma_q$ denotes the monomial $(x_{i_1}y_{i_1})^{\gamma_1}\cdots(x_{i_k}y_{i_k})^{\gamma_k}$ and q is the tuple of indices $\big((i_1,j_1),\ldots,(i_k,j_k)\big)$. Collecting together the x's and y's establishes a bijection between the terms of the last sum indexed over $\psi^{-1}(M)$ and the terms of $M_\alpha(\mathbf{X})M_\beta(\mathbf{Y})$. Indeed, take a term from this product, given by indices $i_{r_1} < \cdots < i_{r_k}$ and $j_{s_1} < \cdots < j_{s_\ell}$, and build the pairs (j_{s_u}, i_{r_v}) such that $m_{v,u} \neq 0$. We also have to consider the pairs $(0, i_{r_v})$ and $(j_{s_u}, 0)$ according to nonzero entries in the first row and column of M. Ordering these indices it is clear that they belong to $\psi^{-1}(M)$ and this is the inverse process of grouping x's and y's.

Then, we can write

$$M_\gamma(\mathbf{X}+\mathbf{Y}+\mathbf{XY}) = \sum_{\alpha,\beta} \sum_{M\in\mathcal{A}_{\alpha,\beta}} M_\alpha(\mathbf{X})M_\beta(\mathbf{Y}) = \sum_{\alpha,\beta} \#\mathcal{A}_{\alpha,\beta} M_\alpha(\mathbf{X})M_\beta(\mathbf{Y}).$$

which obviously implies Equation (9.3). $\square$

We can express the dual of the isomorphism in Theorem 8.3.1 in term of alphabets in the full dual of Σ, which is $\hat{\mathcal{Q}} = \prod_{n\geq 0} \mathcal{Q}_n$. The pairing $\langle , \rangle : \Sigma \times \hat{\mathcal{Q}} \to \Bbbk$ is defined by

$$\langle f, g\rangle = \sum_n \langle f_n, g_n\rangle_n$$

where f_n and g_n are the restrictions of f and g to the homogeneous components of degree n, and $\langle , \rangle_n$ is the pairing defined in (9.2).

For this, given an alphabet $\mathbf{X}$ we define its exponential, $\mathbf{e}(\mathbf{X})$, by

$$\mathbf{e}(\mathbf{X}) = \mathbf{X} + \mathbf{X}^{(2)} + \mathbf{X}^{(3)} + \cdots$$

where the *divided power* $\mathbf{X}^{(n)}$ is the set

$$\mathbf{X}^{(n)} = \big\{(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \in \mathbf{X}^n \;\big|\; x_{i_1} < x_{i_2} < \cdots < x_{i_n}\big\}. \tag{9.4}$$

We endow $\mathbf{e}(\mathbf{X})$ with the reverse lexicographic order. With this notations, the following equation holds:

$$\mathbf{e}(\mathbf{X} + \mathbf{Y}) = \big(1 + \mathbf{e}(\mathbf{X})\big)\big(1 + \mathbf{e}(\mathbf{Y})\big) - 1$$

where the equality is considered as ordered sets. Indeed, denote by $(x)_k$ the monomial $x_{i_1} \cdots x_{i_k}$ with $i_1 < \cdots < i_k$. Then, given $(x)_k(y)_\ell < (x')_{k'}(y')_{\ell'}$ in $\mathbf{e}(\mathbf{X} + \mathbf{Y})$, it is immediate to see that either $(y)_\ell < (y')_{\ell'}$ or $(y)_\ell = (y')_{\ell'}$ and $(x)_k < (x')_{k'}$, which is the definition of the order in the left hand side. Clearly, the same argument applies in the other direction.

Theorem 9.2.2. *The dual of the isomorphism ψ from $(\Sigma, \cdot, \Delta)$ to $(\Sigma, \#, \Delta)$ of Theorem 8.3.1 with respect to the pairing $\langle , \rangle$, is the isomorphism ψ^* from $(\hat{\mathcal{Q}}, \cdot, \Delta_{\#})$ to $(\hat{\mathcal{Q}}, \cdot, \Delta)$ given by*

$$\psi^*(f) = f\big(\mathbf{e}(\mathbf{X})\big).$$

Proof. We have to show that $\big\langle \psi(X_\gamma), f \big\rangle = \big\langle X_\gamma, \psi^*(f) \big\rangle$. Observe that, from the definition of the pairing, it is enough to prove this equation on each grade. Moreover, it is enough to prove it for the generators of the algebra $(\Sigma, \cdot)$ since, for g and g'

generators

$$\begin{aligned}
\big\langle \psi(g\cdot g'), f\big\rangle &= \big\langle \psi(g) \# \psi(g'), f\big\rangle \\
&= \big\langle \psi(g)\otimes\psi(g'), f(\mathbf{X}+\mathbf{Y}+\mathbf{XY})\big\rangle \\
&= \sum_i \big\langle \psi(g), f_i(\mathbf{X})\big\rangle \, \big\langle \psi(g'), f_i'(\mathbf{Y})\big\rangle \\
&= \sum_i \big\langle g, f_i\big(\mathbf{e}(\mathbf{X})\big)\big\rangle \, \big\langle g', f_i'\big(\mathbf{e}(\mathbf{Y})\big)\big\rangle \\
&= \big\langle g\otimes g', f\big(\mathbf{e}(\mathbf{X})+\mathbf{e}(\mathbf{Y})+\mathbf{e}(\mathbf{X})\,\mathbf{e}(\mathbf{Y})\big)\big\rangle \\
&= \big\langle g\otimes g', f\big(\mathbf{e}(\mathbf{X}+\mathbf{Y})\big)\big\rangle \\
&= \big\langle g\otimes g', \Delta\big(f(\mathbf{e}(\mathbf{X}))\big)\big\rangle \\
&= \big\langle g\cdot g', f\big(\mathbf{e}(\mathbf{X})\big)\big\rangle.
\end{aligned}$$

Thus, it is enough to prove the duality for the set of generators given by $X_{(n)}$ for $n \geq 0$, and for $f = M_\alpha$ where α is a composition of n. In this case we have $\psi(X_{(n)}) = X_{(n)}$ and the equation $\big\langle X_{(n)}, M_\alpha\big\rangle = \big\langle X_{(n)}, M_\alpha\big(\mathbf{e}(\mathbf{X})\big)\big\rangle = \delta_{(n),\alpha}$ is immediately verified. □

9.3. The antipode of symmetric functions

Endowed with the coproduct $\Delta_\#$, the algebra $\hat{\mathfrak{Q}}$ is a connected Hopf algebra, in duality with the graded connected Hopf algebra $(\Sigma, \#, \Delta)$. The Heisenberg coproduct restricts to the subalgebra $\hat{\Lambda}$ of $\hat{\mathfrak{Q}}$, and then the completion of the space of symmetric functions becomes a Hopf algebra with the usual product and the Heisenberg coproduct. In this section we express the antipode of this Hopf algebra in terms of alphabets.

First, define the evaluation of quasi-symmetric functions on the the opposite of an alphabet $\mathbf{X}$ by the equation

$$M_\alpha(-\mathbf{X}) = (-1)^r \sum_{i_1 \geq \cdots \geq i_r} x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}, \tag{9.5}$$

for any composition $\alpha = (a_1, \ldots, a_r)$ (compare with the definition of M_α in (9.1)). Second, define the alphabet

$$\mathbf{X}^* = \mathbf{X} + \mathbf{X}^2 + \mathbf{X}^3 + \cdots \tag{9.6}$$

as the disjoint union of the Cartesian powers $\mathbf{X}^n$ under reverse lexicographic order. For instance $(x_3, x_1, x_2) < (x_2, x_2) < (x_1, x_3, x_2)$.

Theorem 9.3.1. *The antipode of the Hopf algebra $\hat{\Lambda}$ endowed with the Heisenberg coproduct is*

$$S_\#(f) = f\big((-\mathbf{X})^*\big).$$

Proof. By Theorem 9.2.1, it is enough to prove that $M_{(a)}\big(\mathbf{X} + (-\mathbf{X})^* + \mathbf{X}(-\mathbf{X})^*\big) = 0$ for any alphabet $\mathbf{X}$ and for any positive integer a. Once this is established, we use the multiplicativity and the infinite linearity of both the antipode and the evaluation on alphabets to deduce the result.

By selecting variables from each of the three alphabets $\mathbf{X}$, $(-\mathbf{X})^*$, and $\mathbf{X}(-\mathbf{X})^*$, we can write

$$\begin{aligned} &M_{(a)}\big(\mathbf{X} + (-\mathbf{X})^* + \mathbf{X}(-\mathbf{X})^*\big) \\ &\qquad = \sum x_i^a + \sum_r (-1)^r \sum (x_{i_1} \cdots x_{i_r})^a + \sum_r (-1)^r \sum x_j^a (x_{i_1} \cdots x_{i_r})^a. \end{aligned}$$

It is easy to see that the first sum cancel with the terms with $r = 1$ of the second sum, while the remaining terms of the second sum cancel with the last sum. This concludes the proof. $\square$

Question. An interesting question is whether the result of Theorem 9.2.1 can be generalized to give an expression of the antipode of the Hopf algebra $\hat{\mathcal{Q}}$ with the Heisenberg coproduct.

10. CONCLUSIONS AND FURTHER DIRECTIONS

In summary, we constructed a new product in the category of species (commutative context) and in the space of endomorphisms of a Hopf algebra (non-commutative context), which interpolates between two well-known classical products. The construction is essentially different in both contexts. However, we showed that they coincide when specialized and restricted to the space of symmetric functions. Apart from the spaces of species and endomorphisms, we were able to define the new product in several intermediate spaces.

As a final note, we mention three possible ways to extend and to apply the tools we developed in this work.

Consider the space $H^* \otimes H$ where H is a cocommutative Hopf algebra, as we did in Section 5. This space has, in addition to the Heisenberg product we considered, another product which is called the *Drinfeld product*. The space is denoted with the symbol $H^* \bowtie H$ and is called the *Drinfeld double* [21]. The definition is similar to the definition (5.3) of the Heisenberg product, but instead of considering the action of H on H^* by translation, we consider the action of H on H^* by conjugation:

$$(h \cdot f)(k) = \sum f\big(S(h_1)(k)h_2\big),$$

where $\Delta(h) = \sum h_1 \otimes h_2$ and S is the antipode of H. Then, the Drinfeld product is defined by

$$(f \otimes h) \bowtie (g \otimes k) = \sum f(h_1 \cdot g) \otimes h_2 k. \tag{10.1}$$

The Drinfeld product can also be defined in the space $\mathrm{End}(H)$ as

$$(f \bowtie g)(h) = \sum f(h_1)_3 g\Big(S\big(f(h_1)_1\big)h_2 f(h_1)_2\Big), \tag{10.2}$$

and Equation (10.1) results from applying the canonical embedding $H^* \otimes H \hookrightarrow \mathrm{End}(H)$.

Since the operations involved in the definition of the Drinfeld product commute with the action of the group $\mathrm{GL}(V)$ (recall Section 6.2), we can apply Schur-Weyl duality (Lemma 6.1.1). Similarly, we can apply the general considerations on the Garsia-Reutenauer endomorphisms, and we get the following theorem:

Theorem 10.0.2. *The Drinfeld product of endomorphisms defined by Equation* (10.2) *restricts to the space of permutations and to the space of non-commutative symmetric functions.*

The work still to be done is to carry the combinatorial proofs, as we did with the Heisenberg product, to obtain an explicit combinatorial formula for the Drinfeld product in the basis X_α of Σ. This would allow us to immediately project this product to the space of symmetric functions.

Another interesting question with respect to the Drinfeld product is whether there is a commutative analogue at the level of species and representations, and hence in Λ, which coincides with the projection from Σ.

A different direction to explore consists in using other versions of the Schur-Weyl duality to perform the constructions of sections 6 and 7. For example, let q be a non-degenerate symmetric bilinear form on a finite-dimensional vector space V over a field of characteristic zero, and consider the orthogonal group $\mathrm{O}(V,q)$ acting on $V^{\otimes n}$. Then, Schur-Weyl duality gives the isomorphism

$$\mathcal{B}_n \cong \mathrm{End}_{\mathrm{O}(V,q)}(V^{\otimes n}),$$

where $\mathcal{B}_n$ is the Brauer algebra [8, 10, 11], and the isomorphism comes from extending the action of S_n to an action of the Brauer algebra on $V^{\otimes n}$. Then, we can define the

Heisenberg product on the Brauer algebra with the technique of Section 6. Similarly, an interesting question is whether the application of the techniques of Section 7, namely Theorem 7.2.1 for the orthogonal group, yields an object which is bigger than the space of non-commutative symmetric functions.

Finally, we want to stress the interest of studying the Heisenberg product in the basis of Schur functions, as a way of "continuously deforming" results from the usual product to the Kronecker product of symmetric functions.

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VITA

Name: Rivera Walter Moreira Rodriguez

Address: Department of Mathematics
c/o Dr. Marcelo Aguiar
Texas A&M University M.S. 3368
College Station, TX 77843

Email address: wmoreira@math.tamu.edu

Education: B.S., Mathematics, Universidad de la República, 1998
M.S., Mathematics, Universidad de la República, 2000