REFINED ERROR ESTIMATES FOR MATRIX-VALUED RADIAL BASIS FUNCTIONS

A Dissertation

by

EDWARD J. FUSELIER, JR.

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2006

Major Subject: Mathematics

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ABSTRACT

Refined Error Estimates for Matrix-valued Radial Basis Functions. (May 2006)

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Radial basis functions (RBFs) are probably best known for their applications to scattered data problems. Until the 1990s, RBF theory only involved functions that were scalar-valued. Matrix-valued RBFs were subsequently introduced by Narcowich and Ward in 1994, when they constructed divergence-free vector-valued functions that interpolate data at scattered points. In 2002, Lowitzsch gave the first error estimates for divergence-free interpolants. However, these estimates are only valid when the target function resides in the *native space* of the RBF. In this paper we develop Sobolev-type error estimates for cases where the target function is less smooth than functions in the native space. In the process of doing this, we give an alternate characterization of the native space, derive improved stability estimates for the interpolation matrix, and give divergence-free interpolation and approximation results for band-limited functions. Furthermore, we introduce a new class of matrix-valued RBFs that can be used to produce curl-free interpolants.

To my parents

TABLE OF CONTENTS

CHAPTER		Page
I	INTRODUCTION AND PRELIMINARIES	1
	A. Introduction	1
	B. General Notation	7
	C. The Fourier Transform	7
	D. Sobolev Spaces	8
	1. Scalar-valued Sobolev Spaces	8
	2. Vector-valued Sobolev Spaces	9
	3. Extension and Trace	10
	E. Positive Definite Matrix-Valued Functions	11
II	DIVERGENCE-FREE AND CURL-FREE RBFS	14
	A. Divergence-free Matrix-Valued RBFs	14
	B. Curl-free Matrix-Valued RBFs	14
III	NATIVE SPACES FOR MATRIX-VALUED KERNELS	18
	A. Native Spaces as Reproducing Kernel Hilbert Spaces	18
	B. Alternate Characterizations of Native Spaces	22
	C. Native Spaces as Sobolev Spaces	26
	D. Generalized Interpolation on Native Spaces	28
IV	STABILITY	32
	A. Lower Bounds For $\lambda_{\min}(A_{X,\Phi_{div}})$	32
	B. Lower Bounds For $\lambda_{\min}(A_{X,\Phi_{curl}})$	41
V	BAND-LIMITED INTERPOLATION AND APPROXIMATION	43
	A. Divergence-Free and Curl-Free Approximation	43
	B. Divergence-Free Band-limited Interpolation	44
	C. Curl-Free Band-limited Interpolation	52
VI	ERROR ESTIMATES FOR FUNCTIONS OUTSIDE THE NATIVE SPACE	53
	A. Extensions of Sobolev Spaces to the Native Space	53
	B. Interpolation Error Estimates	58

CHAPTER		Page
VII	CONCLUSIONS AND FUTURE RESEARCH	64
REFERENCES		67
APPENDIX A		71
APPENDIX B		73
VITA		79

LIST OF TABLES

TABLE		Page
I	Popular Examples of RBFs	1
II	Examples of Wendland Functions	12

LIST OF FIGURES

FIGURE Page

1 The Wendland function $\phi_{3,0}$ on $\mathbb{R}^2$. 13

2 The Wendland function $\phi_{3,2}$ on $\mathbb{R}^2$. 13

3 A two-dimensional divergence-free RBF with $\phi = \phi_{3,2}$. 15

4 A two-dimensional curl-free RBF with $\phi = \phi_{3,2}$. 16

CHAPTER I

INTRODUCTION AND PRELIMINARIES

A. Introduction

Radial basis functions (RBFs) are probably best known for their applications to scattered data problems. Suppose you are given a finite set of points $X \subset \mathbb{R}^n$ and data associated with each point and are asked to find a continuous function that fits the data at the points. Given an RBF ϕ one can build an interpolant out of linear combinations of shifts of ϕ, i.e., ϕ generates a "basis" of the approximation space. Also, for RBFs we have $\phi(x) = \phi(\|x\|)$, which leads to the name *radial* basis functions. Such functions do exist, and popular examples include Gaussians, Hardy mulitiquadrics, thin plate splines, and Wendland functions (see Table I and the table on page 12).

Table I. Popular Examples of RBFs

RBF	$\phi(x)$
Gaussians	$e^{-\alpha\|x\|_2^2}$, $\alpha > 0$
Hardy Multiquadrics	$(-1)^{\lceil\beta/2\rceil}(c^2 + \|x\|_2^2)^\beta$, $\beta > 0$, $\beta \notin \mathbb{N}$
Inverse Multiquadrics	$(-1)^{\lceil\beta/2\rceil}(c^2 + \|x\|_2^2)^\beta$, $\beta < 0$, $\beta \notin \mathbb{N}$
Powers	$(-1)^{\lceil\beta/2\rceil}\|x\|_2^\beta$, $\beta > 0$, $\beta \notin 2\mathbb{N}$
Thin Plate Splines	$(-1)^{k+1}\|x\|_2^{2k}\log(\|x\|_2)$, $k \in \mathbb{N}$
Wendland Functions	$\phi_{n,k}$ (see Table II)

Although RBFs were initially studied to solve the interpolation problem, it turns out their applications are much more broad. RBFs can also fit data coming from a

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very large class of continuous linear functionals. In particular, they can interpolate derivative and integral data at any point, and therefore can be used to solve partial differential equations numerically. Furthermore, one can use scalar-valued RBFs to build functions that produce vector-valued interpolants with certain physical properties, such as being divergence-free or curl-free.

Hardy was probably the first to study RBFs for the purpose of scattered data interpolation in the early 1970s. He used the so-called *Hardy multiquadrics* to approximate topographical surfaces [9]. In the late 1970s, Duchon studied the approximation properties of the thin plate spline [5, 6]. Throughout the 1980s, important aspects to the theory were solved, such as the existence and uniqueness of RBF interpolants [17, 18]. With the rise of computational power in the 1990s, RBFs became more popular, and they are now being used for many applications, including computer animation, medical imaging, and fluid dynamics [1, 10, 11, 13, 16].

Until the 1990s, RBF theory only involved functions that were scalar-valued. However, many physical applications involve vector fields that are divergence-free or curl-free, so there was interest in using RBFs to construct vector-valued approximations with similar characteristics. Matrix-valued RBFs were subsequently introduced by Narcowich and Ward in 1994 [22]. They constructed matrix-valued functions that yield divergence-free interpolants at scattered points. Constructing such functions turns out to be fairly simple. If ϕ is a scalar-valued function consider

$$\Phi_{div} := \left(-\Delta I + \nabla\nabla^T\right)\phi,$$

where ∇ is the $n \times 1$ gradient operator and $\Delta = \nabla^T\nabla$ is the Laplacian operator. This is an $n \times n$ matrix-valued function with divergence-free columns. If ϕ is an RBF, then this function can be used to produce divergence-free interpolants. We note that Φ_{div} is *not* a radial function, but because it is usually generated by an RBF ϕ, it is still commonly called a "matrix-valued RBF".

One builds a divergence-free interpolant in the following way. Given a finite point set $X = \{x_j\}_{j=1}^N \subset \mathbb{R}^n$ and data $d_j \in \mathbb{R}^n$ associated with each x_j, we look for coefficient vectors $\{c_j\}_{j=1}^N \subset \mathbb{R}^n$ so that

$$\sum_{j=1}^{N} \Phi_{div}(x_k - x_j)c_j = d_k \quad \forall\, k = 1, \ldots, N.$$

This leads to the matrix equation

$$A_{X,\Phi_{div}} c = d, \tag{1.1}$$

where c and d are $nN \times 1$ vectors whose j^{th} n components are given by c_j and d_j, respectively. Also, $A_{X,\Phi_{div}}$ is an $nN \times nN$ matrix whose $(j,k)^{th}$ $n \times n$ block is given by $\Phi_{div}(x_j - x_k)$. This matrix is symmetric and positive definite, so (1.1) has a unique solution.

In 2002, Lowitzsch [14] gave the first error estimates for the divergence-free interpolants, at least in the case where the data is given by an underlying function with a particular smoothness. She also gave stability estimates for the interpolation process, and used the divergence-free RBFs to successfully model a physical problem described by the Navier-Stokes equation [16].

Matrix-valued RBF theory is quite new, so there is much room for improvement. Much has been discussed about divergence-free functions, but their counterpart, curl-free functions, have not been dealt with yet. In this paper we will address this issue by introducing a class of functions that yield curl-free interpolants. We will see that many of the results we will prove for divergence-free RBFs will carry over to the curl-free case.

Another issue that needs to be resolved is the current error estimates. In order to discuss this further, we need to introduce the idea of the native space. Each scalar RBF ϕ gives rise to a space of functions called the *native space* of ϕ, denoted $\mathcal{N}_\phi$. In the scalar case, these are Hilbert spaces with an inner product $(\cdot,\cdot)_{\mathcal{N}_\phi}$ such that if $f \in \mathcal{N}_\phi$, then $(f, \phi(\cdot - y))_{\mathcal{N}_\phi} = f(y)$. There is an analogue of this in the matrix-valued theory. The error estimates given in [14] are only valid for classes of functions within these native spaces. Native spaces are usually comprised of functions which are very smooth and tend to be small, so such error estimates are quite limited. Our main goal here is to show that the approximation properties of matrix-valued RBFs extend to functions rougher than those in the native space. Finding such estimates for functions outside the native space is sometimes referred to as "escaping" the native space.

Even for scalar-valued RBFs, this is a very recent development. The first "escape" was made by Narowich and Ward in 2002 concerning functions on the n-sphere using *spherical basis functions* (SBFs), which are positive definite functions on the sphere [23]. Results for RBFs on $\mathbb{R}^n$ soon followed [2, 24, 25]. We refer the reader to [19] for a comprehensive overview of these findings. Due to applications to PDEs, it is desirable to obtain error estimates in terms of Sobolev norms. The above findings address this partially, but they only have the appropriate norms on one side of the estimate. However, this issue has recently been completely resolved by Narcowich, Ward, and Wendland for a large class of RBFs, Wendland functions and thin plate splines in particular [26]. Our strategy will be largely based on their approach, and the estimates we present will be of the form

$$\|f - I_X f\|_{H^k(\Omega)} \leq h_{X,\Omega}^{\tau-k} \|f\|_{H^\tau(\Omega)},$$

where $k \leq \tau$ is an integer, $I_X f$ is the RBF interpolant to the target function f on the point set X. Here $h_{X,\Omega}$ represents the *mesh norm*, which we will define later.

This paper is organized as follows. In the rest of this chapter, we introduce notation and state the necessary definitions. The Fourier transform and its inverse are crucial tools in RBF theory, so we give their definitions. Next we discuss Sobolev spaces and some notions concerning Sobolev spaces, such as extensions and trace. Finally we give the definition of matrix-valued positive definite functions and give a brief introduction to RBFs.

In chapter II we will discuss two important classes of matrix-valued RBFs. First we mention divergence-free RBFs. Next we introduce a new class of matrix-valued RBFs, which can be used to produce curl-free interpolants. We finish the chapter by proving that the functions constructed are strictly positive definite.

We will present results on native spaces for matrix-valued kernels in chapter III. While these native spaces were already defined in [14], our presentation here follows the treatment of native spaces for scalar-valued functions found in [30]. We will begin by giving the definition of a *reproducing kernel Hilbert space* (RKHS) and define the native space for a positive definite matrix-valued function. Next we present a uniqueness result, which will allow us to give more useful characterizations native spaces. In particular, we will use the Fourier transform and show that the native space of certain kernels is comprised of functions with a specific smoothness. When the Fourier transform of ϕ has algebraic decay, which is the case with Wendland functions, we will get "sobolev-like" spaces. We end this chapter with a discussion of generalized interpolation on native spaces. This has already dealt with in [14], but it was proved for a different definition of the native space. Our result will show that the two definitions are in fact equivalent.

Perhaps surprisingly, stability plays a crucial role in the escape process. In chapter IV we explore the stability of the interpolation matrix through its spectral condition number. As done in [16] and [22], we do this by estimating the norm of the inverse of $A_{X,\Phi}$. Since the interpolation matrix is symmetric and positive definite, this amounts to bounding its lowest eigenvalue, $\lambda_{min}(A_{X,\Phi})$, from below. The way this is usually done is by finding a matrix-valued function Ψ such that

$$\sum_{j,k=1}^{N} \alpha_j^* \Phi(x_j - x_k)\alpha_k \geq \sum_{j,k=1}^{N} \alpha_j^* \Psi(x_j - x_k)\alpha_k \geq \lambda \|\alpha\|_2,$$

where $\alpha_j \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}^{nN}$ with the j^{th} n elements of α given by α_j. Such a λ is obviously a lower bound for $\lambda_{min}(A_{X,\Phi})$. In [16] and [22], this was done for divergence-free matrix valued functions and λ was found not to depend on N, but only on the dimension n and the minimum separation radius of X, denoted q_X. We will choose a Ψ different than that used in [16] and [22] and obtain slightly improved results.

We will discuss band-limited functions in chapter V. In the scalar theory, the final escape of the native space in [26] was made by using the approximation properties of band-limited functions, which are functions in L_2 whose Fourier transforms are compactly supported. These functions are analytic, and their smoothness puts them in most native spaces. We will show that band-limited functions can simultaneously approximate *and* interpolate both functions in the native space and rougher functions, enabling one to eventually use a triangle inequality to escape the native space.

In this chapter VI we present the main result of the paper, which is to show that interpolants rising from matrix-valued RBFs can approximate functions that are more rough than those in the native space. We begin the chapter with a discussion on extending Sobolev functions from a bounded domain $\Omega \subset \mathbb{R}^n$ to the native space. Once a function is extended to the native space, best approximation properties of interpolants can be used to help estimate the error. The error estimates we give are in terms of the *mesh norm*. Given a compact set $\Omega \subset \mathbb{R}^n$ and a finite set $X \subset \Omega$, the mesh norm is given by

$$h_{X,\Omega} := \sup_{x\in\Omega} \inf_{x_j\in X} \|x - x_j\|_2.$$

As stated before, the norms involved in the estimates will be Sobolev norms.

Finally, we end with a brief summary in chapter VII. We also include some possible problems for further study.

B. General Notation

We will use the usual multi-index notation: Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of nonnegative integers and define $|\alpha| := \sum_j \alpha_j$. We will use $\|x\|_2$ to denote the standard euclidean norm of $x \in \mathbb{R}^n$. If f is a matrix-valued function or distribution, we write f^* for the conjugate transpose of f, i.e., $f^* = \bar{f}^T$. We define the *ceiling function* $\lceil x \rceil$ to be the function that returns the integer k such that $k - 1 < x \leq k$, and the *floor function* $\lfloor x \rfloor$ to be the functions that returns the integer k such that $k \leq x < k + 1$. Also, we let $(x)_+ = x$ if $x \geq 0$ and 0 otherwise.

Let $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$. We will say $f \in C^k(\Omega)$ if f is k-times continuously differentiable. L_p spaces are defined in the usual way: we say $f \in L_p(\Omega)$ if $\int_\Omega |f|^p dx$ is finite. In the case $\Omega = \mathbb{R}^n$, we define $C^k := C^k(\mathbb{R}^n)$ and $L_p := L_p(\mathbb{R}^n)$. Also, if f is vector-valued, we say that $f \in C^k(\Omega)$ or $f \in L_p(\Omega)$ if each of its components are in $C^k(\Omega)$ or $L_p(\Omega)$, respectively. This should cause no confusion.

C. The Fourier Transform

The Fourier transform plays an important role in the theory of RBFs. We will use the following convention for the *Fourier transform* of a function or tempered distribution:

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix^T\xi} d\xi,$$

and let the *inverse Fourier transform* of a function or tempered distribution be defined by

$$\check{f}(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\xi) e^{i\xi^T x} d\xi.$$

If f is a matrix-valued function, we will take $\widehat{f}$ to be the matrix of Fourier transforms of each component of f.

D. Sobolev Spaces

1. Scalar-valued Sobolev Spaces

The error estimates will deal with vector-valued functions whose components reside in Sobolev spaces, which we define now. Let Ω be an open domain in $\mathbb{R}^n$, $1 \leq p < \infty$, and k be a non-negative integer. Suppose u is locally integrable and that the distributional derivatives $D^\alpha u$ exist for all $|\alpha| \leq k$. Then we define the Sobolev norm to be

$$\|u\|_{W_p^k(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}.$$

For the case $p = \infty$ we have

$$\|u\|_{W_\infty^k(\Omega)} := \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

We define the Sobolev spaces to be

$$W_p^k(\Omega) := \left\{ u \in L_{loc}^1(\Omega) : \|u\|_{W_p^k(\Omega)} < \infty \right\}.$$

It is also possible to have Sobolev spaces of fractional order. Let $1 \leq p < \infty$, k be a non-negative integer, and $0 < t < 1$. We define the Sobolev space $W_p^{k+t}(\Omega)$ to be all u such that the following norm is finite:

$$\|u\|_{W_p^{k+t}(\Omega)} := \left(\|u\|^p_{W_p^k(\Omega)} + \sum_{|\alpha|=k} \int_\omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+pt}} dxdy \right)^{1/p}.$$

In the special case $p = 2$, we define $H^\tau(\Omega) := W_2^\tau(\Omega)$. It is well-known that $H^\tau(\Omega)$ is a Hilbert space, and that in the case of $\Omega = \mathbb{R}^n$, we may use the Fourier transform to characterize $H^\tau(\mathbb{R}^n)$:

$$H^\tau(\mathbb{R}^n) := \left\{ u \in L_2(\mathbb{R}^n) : \widehat{u}(\cdot) \left(1 + \|\cdot\|_2^2\right)^{\tau/2} \in L_2(\mathbb{R}^n) \right\}.$$

The inner product in $H^\tau(\mathbb{R}^n)$ is given by

$$\langle g, f\rangle_{H^\tau(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + \|\xi\|_2^2)^\tau \overline{\widehat{f}(\xi)} \widehat{g}(\xi) d\xi.$$

2. Vector-valued Sobolev Spaces

Let $u : \Omega \to \mathbb{R}^n$, with u_j denoting the jth coordinate of u. If $u_j \in W_p^k(\Omega)$ for all $j = 1, \ldots, n$, then we say $u \in (W_p^k(\Omega))^n$. We impose the following norms on $u \in (W_p^k(\Omega))^n$ for $1 \le p < \infty$:

$$|u|_{(W_p^k(\Omega))^n} := \left(\sum_{j=1}^n |u_j|^p_{W_p^k(\Omega)} \right)^{1/p}, \ \|u\|_{(W_p^k(\Omega))^n} := \left(\sum_{j=1}^n \|u_j\|^p_{W_p^k(\Omega)} \right)^{1/p}.$$

For $p = \infty$ we have:

$$|u|_{(W_\infty^k(\Omega))^n} := \max_{1 \le j \le n} |u_j|_{W_\infty^k(\Omega)}, \ \|u\|_{(W_\infty^k(\Omega))^n} := \max_{1 \le j \le n} \|u_j\|_{W_\infty^k(\Omega)}.$$

Note that $(H^\tau(\Omega))^n$ is a Hilbert space, and in the special case $\Omega = \mathbb{R}^n$ the inner product can be defined by

$$\langle g, f\rangle_{(H^\tau(\mathbb{R}^n))^n} = \int_{\mathbb{R}^n} (1 + \|\xi\|_2^2)^\tau \widehat{f}(\xi)^* \widehat{g}(\xi) d\xi.$$

When the context is clear, we will use the notation $H^\tau(\Omega) = (H^\tau(\Omega))^n$. This should cause no confusion.

We will also be interested in spaces that are divergence-free or curl-free. A function is *divergence-free* if and only if $\nabla \cdot f = 0$. For $\tau \geq 0$, we define space

$$H^\tau_{div}(\Omega) := \{f \in H^\tau(\Omega) : \nabla \cdot f = 0\}.$$

This is a closed subspace of $H^\tau(\Omega)$.

We would like to define a similar space for curl-free functions. In the case $n = 2$, a function f is *curl-free* if and only if $\partial f_2/\partial x - \partial f_1/\partial y = 0$. When $n = 3$, a function f is curl-free if and only if $\nabla \times f = 0$. We will use the sloppy notation $\nabla \times f$ to represent the curl of a vector field if $n = 2$. Thus for $n = 2$ or 3 we define

$$H^\tau_{curl}(\Omega) := \{f \in H^\tau(\Omega) : \nabla \times f = 0\}.$$

This is also a closed subspace of $H^\tau(\Omega)$. When $n > 3$, there is no simple analogue for curl involving a nice differential operator. However, using differential forms and Poincaré's Lemma we see that a vector-valued function on a manifold has no rotation if and only if it is the differential of a scalar valued function. Therefore for general n we will say a function $f \in H^\tau(\mathbb{R}^n)$ is *curl-free* on $\mathbb{R}^n$ if and only if there is a scalar-valued function in $H^{\tau+1}(\mathbb{R}^n)/\mathbb{R}$ such that $\nabla\phi = f$.

3. Extension and Trace

Let $\Omega \subset \mathbb{R}^n$, and let $f \in W^\tau_p(\Omega)$. Two concepts we will need are that of extending f to $W^\tau_p(\mathbb{R}^n)$ and extending f to the boundary of Ω, denoted by $\partial\Omega$. If Ω has a Lipschitz boundary and satisfies an interior cone condition, there is a continuous extension operator $\mathfrak{E} : W_p^\tau(\Omega) \to W_p^\tau(\mathbb{R}^n)$ such that $\mathfrak{E}f|_\Omega = f$ and

$$\|\mathfrak{E}f\|_{W_p^\tau(\mathbb{R}^n)} \leq C\|f\|_{W_p^\tau(\Omega)}.$$

Also, the same operator works for *all* Sobolev spaces $W_p^\tau(\Omega)$. The operator was produced by Stein for integer τ in [27] and extended to fractional τ later (for a proof of the fractional case, see [4]). We will refer to this operator as *Stein's operator* When f is vector-valued, we will let $\mathfrak{E}f$ denote Stein's operator acting on each component of f.

Another important idea is that of a *trace*, which one gets by extending a Sobolev function to the boundary of its domain. It is well-known that when $\partial\Omega$ is Lipschitz, the trace exists and is continuous in the following way

$$\|f|_{\partial\Omega}\|_{W_p^{\tau-1/p}(\partial\Omega)} \leq C\|f\|_{W_p^\tau(\Omega)}.$$

These notions will be especially important in chapter VI.

E. Positive Definite Matrix-Valued Functions

An $m \times m$ matrix-valued function Φ is *positive definite* on $\mathbb{R}^n$ if given any finite, distinct set of points $X := \{x_1, \ldots, x_N\} \subset \mathbb{R}^n$ we have

$$\sum_{j,k} \alpha_j^T \Phi(x_j - x_k)\alpha_k \geq 0$$

for all $\alpha_1, \ldots, \alpha_N$ in $\mathbb{R}^m$. If the inequality is strict when $\alpha_i \neq 0$ for some i, then we say the Φ is *strictly positive definite* (SPD). This is equivalent to saying that the $mN \times mN$ matrix whose $(j,k)^{th}$ $m \times m$ block is given by $\Phi(x_j - x_k)$ is positive definite, and hence invertible. We will denote this matrix by $A_{X,\Phi}$. In this paper we will only concentrate on the cases $m = 1$ and $m = n$, where n is the dimension of the domain.

When $m = 1$, we get the special case of scalar-valued positive definite functions. A positive definite function ϕ that depends only on the length of its argument, i.e., $\phi(x) = \phi(\|x\|_2)$, is called a *Radial Basis function* (RBF). Some popular examples are given in Table 1. Wendland functions are particularly important because they are piecewise polynomials with compact support, and are hence easy to compute. We will let $\phi_{n,k}$ denote the Wendland function that is $2k$ times continuously differentiable and is positive definite on $\mathbb{R}^n$. Table II lists several of these functions. Examples of Wendland functions are graphed in Figures 1 and 2.

Table II. Examples of Wendland Functions

Space dimension	Function	Smoothness
$n = 1$	$\phi_{1,0}(x) = (1 - \|x\|_2)_+$	C^0
	$\phi_{1,1}(x) = (1 - \|x\|_2)_+^3(3\|x\|_2 + 1)$	C^2
	$\phi_{1,2}(x) = (1 - \|x\|_2)_+^5(8\|x\|_2^2 + 5\|x\|_2 + 1)$	C^4
$n \leq 3$	$\phi_{3,0}(x) = (1 - \|x\|_2)_+^2$	C^0
	$\phi_{3,1}(x) = (1 - \|x\|_2)_+^4(4\|x\|_2 + 1)$	C^2
	$\phi_{3,2}(x) = (1 - \|x\|_2)_+^6(35\|x\|_2^2 + 18\|x\|_2 + 3)$	C^4
$n \leq 5$	$\phi_{5,0}(x) = (1 - \|x\|_3)_+^3$	C^0
	$\phi_{5,1}(x) = (1 - \|x\|_2)_+^5(5\|x\|_2 + 1)$	C^2
	$\phi_{5,2}(x) = (1 - \|x\|_2)_+^7(16\|x\|_2^2 + 7\|x\|_2 + 1)$	C^4

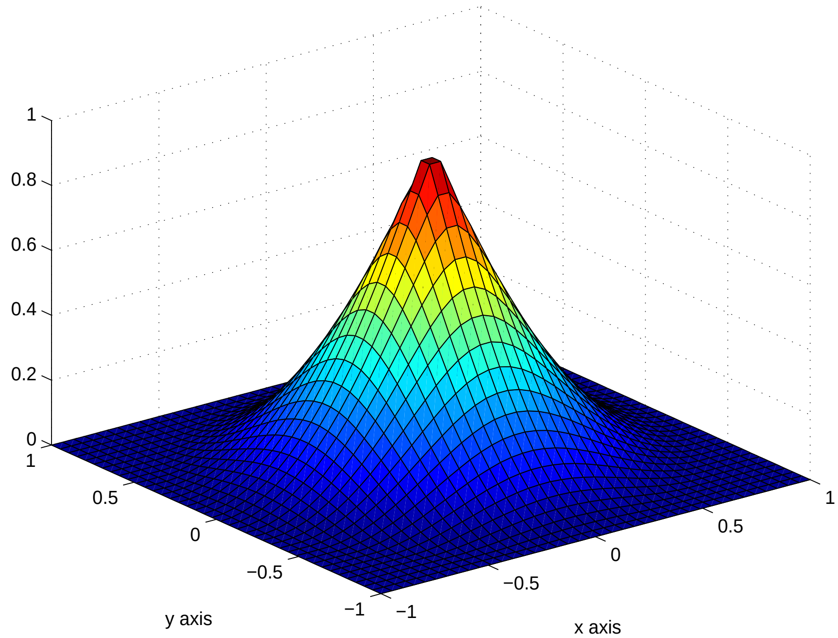

Fig. 1. The Wendland function $\phi_{3,0}$ on $\mathbb{R}^2$.

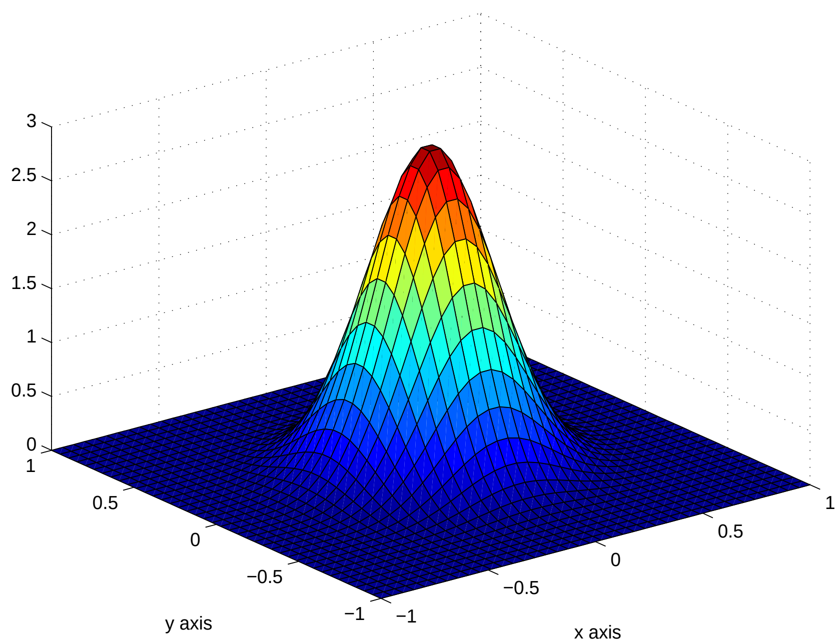

Fig. 2. The Wendland function $\phi_{3,2}$ on $\mathbb{R}^2$.

CHAPTER II

DIVERGENCE-FREE AND CURL-FREE RBFS

In this chapter we will discuss two important classes of matrix-valued RBFs. First we briefly mention divergence-free RBFs. Next we introduce matrix-valued RBFs which can be used to produce curl-free interpolants. We finish the chapter by proving that the curl-free RBFs are positive definite.

A. Divergence-free Matrix-Valued RBFs

Matrix-valued RBFs which yield C^∞ divergence-free interpolants were first introduced by Narcowich and Ward in 1994. In 2002, Lowisitzch introduced a class of these functions which are C^{2k} and compactly supported. Constructing such functions turns out to be fairly simple. If ϕ is a scalar-valued function consider

$$\Phi_{div} := \left(-\Delta I + \nabla\nabla^T\right)\phi,$$

where ∇ is the $n \times 1$ gradient operator and $\Delta = \nabla^T\nabla$ is the Laplacian operator. Then Φ_{div} is an $n \times n$ matrix-valued function with divergence-free columns. If ϕ is positive definite, then this function can be used to produce divergence-free interpolants. An example of a compactly supported divergence-free RBF is shown in Figure 3.

B. Curl-free Matrix-Valued RBFs

We now present curl-free matrix-valued RBFs. As in the divergence-free case, they are easy to produce. Again we chose a positive definite scalar-valued function and act on it with the appropriate differential operator. Given $\phi \in C^2$, define

$$\Phi_{curl} := -\nabla\nabla^T\phi.$$

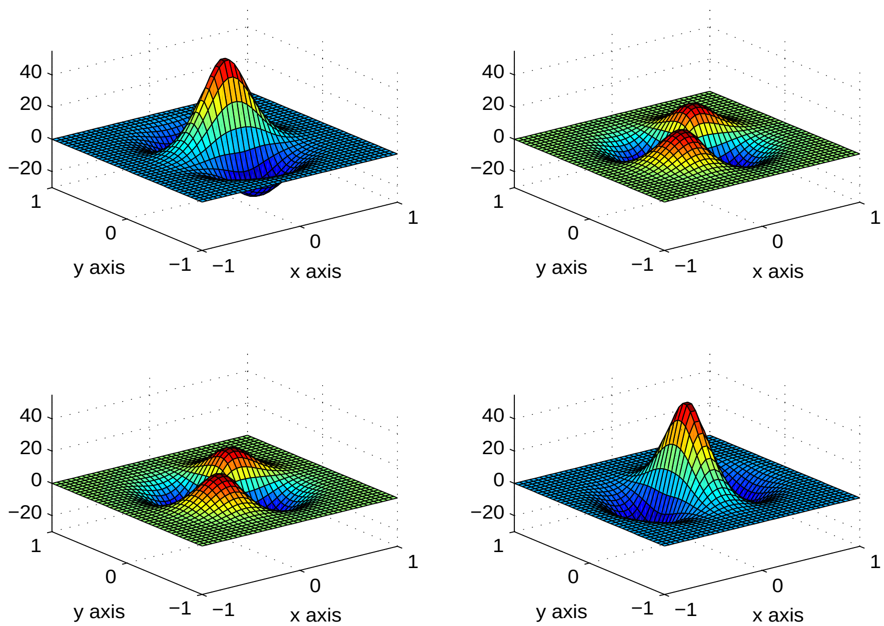

Fig. 3. A two-dimensional divergence-free RBF with $\phi = \phi_{3,2}$.

It is easy to see that the columns of this function are curl-free. The j^{th} column is given by $\Phi_{curl}e_j$, where e_j is the standard basis vector with a one in the j^{th} position. This gives us

$$\Phi_{curl}e_j = -\nabla\nabla^T\phi e_j = \nabla\left(-\nabla^T(\phi e_j)\right) = \nabla g,$$

where $g = -\partial\phi/\partial x_j$, which is a scalar function. Since the column is the gradient of a scalar, it is curl-free. An example of a curl-free RBF is shown in Figure 4.

To see that Φ_{curl} is positive definite, we will use the Fourier transform and its inverse. In order to be rigorous, we must assume that ϕ, $-\Delta\phi$, and their Fourier transforms are in $C \cap L_1$. This enables us to recover Φ from $\widehat{\Phi}$ using the inverse

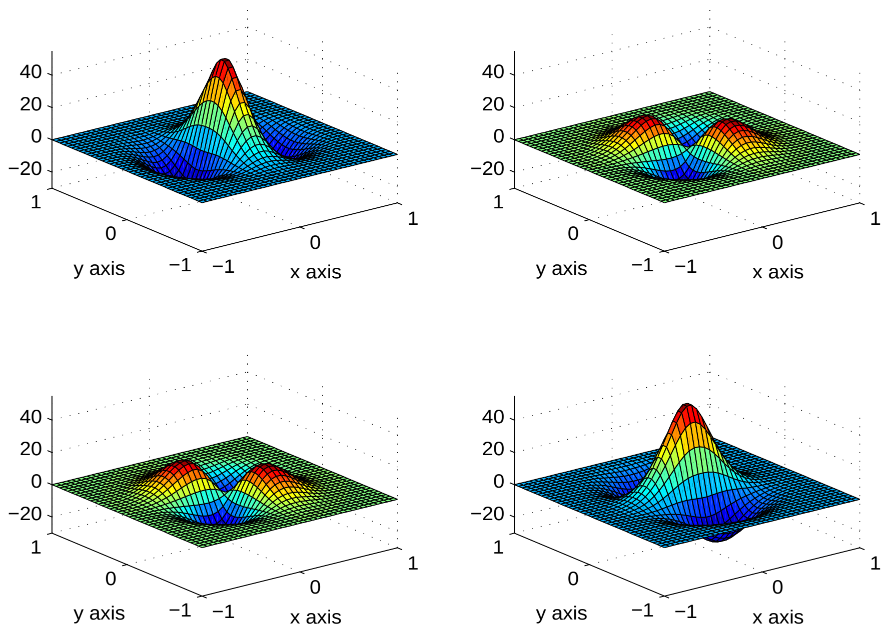

Fig. 4. A two-dimensional curl-free RBF with $\phi = \phi_{3,2}$.

Fourier transform.

$$\begin{aligned} c^T A_{X,\Phi_{curl}} c &= \sum_{j,k} c_j^T \Phi_{curl}(x_j - x_k) c_k = \sum_{j,k} c_j^T \left(\int_{\mathbb{R}^n} \widehat{\Phi_{curl}} e^{ix_j^T \xi} e^{-ix_k^T \xi} d\xi \right) c_k \\ &= \int_{\mathbb{R}^n} \left(\sum_j c_j e^{-ix_j^T \xi} \right)^* \xi \xi^T \widehat{\phi} \left(\sum_k c_k e^{-ix_k^T \xi} \right) d\xi \\ &= \int_{\mathbb{R}^n} \left| \sum_j \xi^T c_j e^{-ix_j^T \xi} \right|^2 \widehat{\phi} \, d\xi \geq 0. \end{aligned} \tag{2.1}$$

This shows that Φ_{curl} is positive definite. To see when it is strictly positive definite, we will need the following lemma.

Lemma 1. *Let* $X = \{x_j\}_{j=1}^N$ *and* $\{c_j\}_{j=1}^N$ *be finite subsets of* $\mathbb{R}^n$*, and let* U *be an open subset of* $\mathbb{R}^n$*. If the function* $f(\xi) = \sum_j \xi^T c_j e^{ix_j^T \xi}$ *is zero on* U *then* $c_j = 0$ *for all* $j = 1, \ldots, N$*.*

Proof. Note that f can be extended to an analytic function on $\mathbb{C}^n$. Since it analytic and identically zero on an open subset of $\mathbb{R}^n$, it must be zero on all of $\mathbb{R}^n$. Now let g be any function in L_1 such that g and its first derivatives can be recovered by the inverse Fourier transform and consider:

$$0 = f(\xi)\widehat{g}(\xi) = \sum_j \xi^T c_j e^{ix_j^T \xi}\widehat{g}(\xi) = \left(\sum_j \nabla^T (g(\cdot - x_j)c_j)\right)^{\widehat{}}$$

$$\Longrightarrow \qquad \sum_j \sum_i c_{ji} \frac{\partial}{\partial x_i} g(\cdot - x_j) \equiv 0,$$

where c_{ji} is the i^{th} coordinate of c_j. We will show that $c_{11} = 0$, and the rest are proved similarly. To do this, we choose g to have compact support withing the ball of radius $\epsilon < \min_{j \neq k} \|x_j - x_k\|$, and that at the origin $\partial g/\partial x_1 = 1$ and $\partial g/\partial x_j = 0$ for all $j \neq 1$. For a concrete example that such a g exists, one can use Hermite-Birkoff interpolation to find a linear combination of Wendland functions with those properties. Applying this to the above equation gives us the result. □

Applying the lemma to (2.1), we see that if $\widehat{\phi}$ is continuous, then Φ_{curl} is strictly positive definite. This almost proves the following theorem.

Theorem 1. *Let $\phi \in C^2$ be a scalar-valued strictly positive definite function on $\mathbb{R}^n$ such that ϕ and $-\Delta\phi$ are in L_1. Then Φ_{curl} is a strictly positive definite matrix-valued function with columns that are curl-free.*

Proof. We need only show that if $\phi \in C^2$ be a scalar-valued positive definite function on $\mathbb{R}^n$ such that ϕ and $-\Delta\phi$ are in L_1, then we can recover them through the inverse Fourier transform. This will happen if the Fourier transforms of ϕ and $-\Delta\phi$ are in L_1. Since ϕ is positive definite, by Bochner's theorem so is $-\Delta\phi$. Now we use [30, Corollary 6.12], which says that if a positive definite function is continuous and L_1 integrable then its nonnegative Fourier transform is in L_1. □

CHAPTER III

NATIVE SPACES FOR MATRIX-VALUED KERNELS

While native spaces were defined with distributions in [14], our presentation here follows the treatment of native spaces for scalar-valued functions found in [30]. We will begin by giving the definition of a *reproducing kernel Hilbert space* (RKHS) and define the native space for a positive definite matrix-valued function. Next we offer a uniqueness result. This feature will allow us to give a more useful characterization of the native space. In particular, we will use the Fourier transform and show that the native space of certain kernels is comprised of functions with a specific smoothness. When the Fourier transform of ϕ has algebraic decay, which is the case with Wendland functions, we will get "Sobolev-like" spaces.

In the last section we will deal with generalized interpolation on native spaces. The has already been done in the case where native spaces were defined distributionally, but it has not been shown for the definition of the native space given here. In the process of proving this we will show that the two definitions are equivalent.

A. Native Spaces as Reproducing Kernel Hilbert Spaces

An important idea in the theory of RBFs is that of a reproducing kernel Hilbert space, which we define now.

Definition 1. Let $\mathcal{F}$ be a Hilbert space of vector-valued functions $f : \Omega \to \mathbb{R}^n$. A continuous $n \times n$ matrix-valued function Φ is called a *reproducing kernel* for $\mathcal{F}$ if for all $x \in \Omega$ and $c \in \mathbb{R}^n$ we have

1. $\Phi(\cdot - x)c \in \mathcal{F}$.
2. $c^T f(x) = (f, \Phi(\cdot - x)c)_{\mathcal{F}}$ for all $f \in \mathcal{F}$.

Remark 1. Note that other than the fact that Ω should not be empty, there is no restriction on Ω in the definition. However, in the context of RBFs one usually assumes that $\Omega = \mathbb{R}^n$.

The first property of the definition tells us that such an $\mathcal{F}$ would contain all functions of the form $f = \sum_{j=1}^{N} \Phi(x - x_j)\alpha_j$, where $\alpha_j \in \mathbb{R}^n$ and $x_j \in \Omega$. The second property gives us an expression for the norm of such functions:

$$\|f\|_{\mathcal{F}}^2 = \sum_{k=1}^{N} \sum_{j=1}^{N} \alpha_j^* \Phi(x_j - x_k)\alpha_k.$$

These features will guide us in the construction of a RKHS for a given SPD matrix-valued function. Encouraged by this, we define the space

$$F_\Phi(\Omega) := \left\{ \sum_{j=1}^{N} \Phi(\cdot - x_j)\alpha_j : x_j \in \Omega,\ \alpha_j \in \mathbb{R}^n,\ \text{and } N \in \mathbb{N} \right\}.$$

We furnish this space with the bilinear form

$$\left(\sum_{j=1}^{N} \Phi(\cdot - x_j)\alpha_j, \sum_{k=1}^{M} \Phi(\cdot - y_k)\beta_k \right)_\Phi := \sum_{j=1}^{N} \sum_{k=1}^{M} \beta_k^T \Phi(y_k - x_j)\alpha_j.$$

If Φ is SPD then this bilinear form defines an inner product on $F_\Phi(\Omega)$.

We will denote the completion of $F_\Phi(\Omega)$ with respect to the $\|\cdot\|_\Phi$ norm as $\mathcal{F}_\Phi(\Omega)$. The elements of $\mathcal{F}_\Phi(\Omega)$ are abstract, and we wish to interpret them as functions. To do this we define function values for an element f by $f_j(x) := (f, \Phi(\cdot - x)e_j)_\Phi$. Will will show that this leads to an injective linear mapping $R : \mathcal{F}_\Phi(\Omega) \to C(\Omega)$ given by $R(f)_j(x) := (f, \Phi(\cdot - x)e_j)_\Phi$. The image of this map is the space of functions we are looking for.

Lemma 2. *The mapping $R : \mathcal{F}_\Phi(\Omega) \to C(\Omega)$ is an injective linear map.*

Proof. The map is obviously linear. First we show that each coordinate of $R(f)$ is continuous. We have

$$\begin{aligned}|R(f)_j(x) - R(f)_j(y)| &= \left|(f, \Phi(\cdot - x)e_j)_\Phi - (f, \Phi(\cdot - y)e_j)_\Phi\right| \\ &= \left|(f, \Phi(\cdot - x)e_j - \Phi(\cdot - y)e_j)_\Phi\right| \\ &\le \|f\|_\Phi \|\Phi(\cdot - x)e_j - \Phi(\cdot - y)e_j\|_\Phi.\end{aligned}$$

To conclude that $R(f)_j$ is continuous, we use the continuity of Φ and the fact that

$$\|\Phi(\cdot - x)e_j - \Phi(\cdot - y)e_j\|_\Phi^2 = 2e_j^T\Phi(0)e_j - e_j^T\Phi(x-y)e_j - e_j^T\Phi(y-x)e_j.$$

For injectivity, suppose $R(f) = 0$ for some $f \in \mathcal{F}_\Phi(\Omega)$. This means that $\forall x \in \Omega$ and all $c \in \mathbb{R}^n$ we have $(f, \Phi(\cdot - x)c)_\Phi = 0$. Thus f is perpendicular to F_Φ, but $\mathcal{F}_\Phi$ is the completion of F_Φ, so $f = 0$ and R is injective. □

With this result, we are able to define the native space.

Definition 2. The *native Hilbert function space* corresponding to the SPD kernel Φ is defined by

$$\mathcal{N}_\Phi(\Omega) := R(\mathcal{F}_\Phi(\Omega)).$$

It is equipped with the inner product

$$(f, g)_{\mathcal{N}_\Phi(\Omega)} := (R^{-1}f, R^{-1}g)_{\mathcal{F}_\Phi(\Omega)}.$$

Defined in this way, we see that the native space is indeed a Hilbert space of continuous functions with reproduction kernel Φ. To see this, since $\Phi(\cdot - x)c$ is mapped to itself through R for all $x \in \Omega$ and $c \in \mathbb{R}^n$, we get

$$f_j(x) = \left(R^{-1}f, \Phi(\cdot - x)e_j\right)_\Phi = (f, \Phi(\cdot - x)e_j)_{\mathcal{N}_\Phi}$$

for all $f \in \mathcal{N}_\Phi$. One can also use the fact that $F_\Phi(\Omega)$ is dense in $\mathcal{N}_\Phi$ with $\|f\|_\Phi = \|f\|_{\mathcal{N}_\Phi}$ for all $f \in F_\Phi$ to conclude that the native space for Φ is unique in the following way.

Theorem 2. *Suppose that Φ is a SPD matrix-valued kernel. Suppose further that $\mathcal{G}$ is a Hilbert space of functions $f : \Omega \to \mathbb{R}^n$ with reproducing kernel Φ. Then $\mathcal{G} = \mathcal{N}_\Phi(\Omega)$ and the inner products are the same.*

Proof. From the remarks after Definition 1, we know that $F_\Phi(\Omega) \subseteq \mathcal{G}$ and $\|f\|_{\mathcal{G}} = \|f\|_{\mathcal{N}_\Phi(\Omega)}$ for all $f \in F_\Phi(\Omega)$. Let $f \in \mathcal{N}_\Phi(\Omega)$. By density of $F_\Phi(\Omega)$ in the native space, there is a Cauchy sequence $\{f_n\}$ in $F_\Phi(\Omega)$ converging to f in $\mathcal{N}_\Phi(\Omega)$. One can show that if a sequence converges in a RKHS, then it converges point-wise. Indeed if Φ is the reproducing kernel for the RKHS $\mathcal{F}$, we have

$$|(f_n)_j(y) - f_j(y)| = \left|(f_n - f, \Phi(\cdot - y)e_j)_{\mathcal{F}}\right| \leq \|f_n - f\|_{\mathcal{F}} \|\Phi(\cdot - y)e_j\|_{\mathcal{F}}.$$

Thus we get $f(y) = \lim_{n\to\infty} f_n(y)$. Note that $\{f_n\}$ is also a Cauchy sequence in $\mathcal{G}$, so it also converges to a $g \in \mathcal{G}$. The reproducing-kernel property of $\mathcal{G}$ gives $g(y) = \lim_{n\to\infty} f_n(y) = f(y)$. Therefore $f = g \in \mathcal{G}$, so $\mathcal{N}_\Phi(\Omega) \subseteq \mathcal{G}$ and $\|f\|_{\mathcal{N}_\Phi(\Omega)} = \|f\|_{\mathcal{G}}$ for all $f \in \mathcal{N}_\Phi(\Omega)$.

Now suppose that $\mathcal{N}_\Phi(\Omega)$ is not equal to $\mathcal{G}$. Then we can find an element $g \in \mathcal{G} \setminus \{0\}$ orthogonal to $\mathcal{N}_\Phi(\Omega)$. However, since $\Phi(\cdot - y)e_j \in \mathcal{N}_\Phi$ this means that $g_j(y) = (g, \Phi(\cdot - y)e_j)_{\mathcal{G}} = 0$ for all $y \in \Omega$ and all $j = 1, \ldots, n$. Thus $\mathcal{N}_\Phi(\Omega) = \mathcal{G}$. To finish the proof, by polarization we know that the inner products are equal because the norms are. $\square$

This feature will give us the tools to give other characterizations of the native space in the next section. In particular, we will use the Fourier transform and show that the native space of certain kernels is comprised of functions with a specific smoothness.

B. Alternate Characterizations of Native Spaces

In the theory of scalar-valued positive definite functions, it is well-known that in the case of $\phi \in C(\mathbb{R}^n) \cap L_1(\mathbb{R}^n)$, one may characterize the native space of ϕ by

$$\mathcal{N}_\phi(\mathbb{R}^n) = \left\{ f \in L_2 \cap C : \int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)|^2}{\widehat{\phi}(\xi)} d\xi < \infty \right\},$$

with inner product given by

$$(f,g)_{\mathcal{N}_\phi(\mathbb{R}^n)} := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \frac{\overline{\widehat{g}(\xi)}\widehat{f}(\xi)}{\widehat{\phi}(\xi)} d\xi. \tag{3.1}$$

If $\widehat{\Phi}(\xi)$ is invertible for all ξ, an appropriate guess for the generalization of (3.1) would be

$$(f,g)_{\mathcal{N}_\Phi(\mathbb{R}^n)} := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{g}(\xi)^* \widehat{\Phi}(\xi)^{-1} \widehat{f}(\xi) d\xi.$$

However, $\widehat{\Phi_{div}}$ and $\widehat{\Phi_{curl}}$ are not invertible at *any* point. Nevertheless we get around this obstruction by considering the *Moore-Penrose inverse*, or *pseudo-inverse*, of $\widehat{\Phi}(\xi)$, which we denote by $\widehat{\Phi}(\xi)^+$. Thus a more appropriate inner product would be

$$(f,g)_{\mathcal{N}_\Phi(\mathbb{R}^n)} := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{g}(\xi)^* \widehat{\Phi}(\xi)^+ \widehat{f}(\xi) d\xi.$$

Also, to make sure the inner product is strictly positive definite, one should be careful about what functions are allowed in the space. In particular, we should avoid those whose Fourier transforms are perpendicular to the columns of $\widehat{\Phi}(\xi)^+$.

In the case of divergence-free and curl-free matrix-valued functions, a simple calculation gives us the following equalities:

$$\begin{aligned} \widehat{\Phi_{div}}(\xi)^+ &= \frac{1}{\|\xi\|_2^2 \widehat{\phi}(\xi)} \left(I - e_\xi e_\xi^T \right) \\ \widehat{\Phi_{curl}}(\xi)^+ &= \frac{1}{\|\xi\|_2^2 \widehat{\phi}(\xi)} \left(e_\xi e_\xi^T \right), \end{aligned}$$

where e_ξ is the unit vector in the ξ-direction. We would like for $\widehat{\Phi}(\xi)^+\widehat{\Phi}(\xi)$ to be the identity for the Fourier transforms of functions in the native space. Therefore in the divergence-free case, we will consider functions f such that $e_\xi^T\widehat{f}(\xi) = 0$. In the curl-free case we want functions of the form $\widehat{f}(\xi) = e_\xi h(\xi)$, where h is a scalar-valued function. Now we are ready to state and prove the following results.

Theorem 3. *Suppose that $\phi \in C^2$ is a positive definite function such that $\Delta\phi \in L_1$. Define*

$$\mathcal{G}_{div} := \left\{ f \in L_2 \cap C : \int_{\mathbb{R}^n} \widehat{f}(\xi)^* \widehat{\Phi_{div}}(\xi)^+ \widehat{f}(\xi) d\xi < \infty \text{ and } e_\xi^T \widehat{f}(\xi) = 0 \text{ a.e.} \right\}$$

and equip this space with the bilinear form

$$(f,g)_{\mathcal{G}_{div}} := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{g}(\xi)^* \widehat{\Phi_{div}}(\xi)^+ \widehat{f}(\xi) d\xi.$$

Then $\mathcal{G}_{div}$ is a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{G}_{div}}$ and reproducing kernel Φ_{div}. Hence $\mathcal{G}_{div} = \mathcal{N}_{\Phi_{div}}$ and the inner products are the same.

Proof. We begin by noting that since $f \in \mathcal{G}_{div}$ satisfies $e_\xi^T\widehat{f}(\xi) = 0$ for all ξ, we have

$$(f,f)_{\mathcal{G}_{div}} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \frac{\|\widehat{f}(\xi)\|_2^2}{\|\xi\|_2^2 \widehat{\phi}(\xi)} d\xi. \tag{3.2}$$

Also, since ϕ is positive definite so is $-\Delta\phi$. Then that fact that it is continuous and L_1 integrable puts its Fourier transform is in $L_1(\mathbb{R}^n)$ [30, Corollary 6.12]. This means that $\widehat{f} \in L_1$ for all $f \in \mathcal{G}_{div}$. Indeed, we have

$$\int_{\mathbb{R}^n} |\widehat{f}_j(\xi)| d\xi \leq \left(\int_{\mathbb{R}^n} \frac{|\widehat{f}_j(\xi)|_2^2}{\|\xi\|_2^2 \widehat{\phi}(\xi)} d\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} \|\xi\|_2^2 \widehat{\phi}(\xi) d\xi \right)^{1/2}.$$

This allows us to recover f point-wise from its Fourier transform by the inverse Fourier transform.

We now show that $(\cdot,\cdot)_{\mathcal{G}_{div}}$ is an inner product. The linearity and conjugate symmetry properties are obvious. The fact that $\widehat{\phi}$ is positive along with (3.2) tells us $(f,f)_{\mathcal{G}_{div}} = 0$ implies that $f = 0$. Thus $(\cdot,\cdot)_{\mathcal{G}_{div}}$ is positive definite and hence an inner product.

To show completeness of $\mathcal{G}_{div}$, suppose that $\{f_n\}$ is a Cauchy sequence in $\mathcal{G}_{div}$. This means that the sequence $\left\{\widehat{f}_n(\|\cdot\|_2^2\widehat{\phi})^{-1/2}\right\}$ is Cauchy in L_2, and so it converges to a function $g \in L_2$. Note that the function g satisfies $g\sqrt{\|\cdot\|_2^2\widehat{\phi}} \in L_1 \cap L_2$. Namely,

$$\int_{\mathbb{R}^n} \left| g_j(\xi)\sqrt{\|\xi\|_2^2\widehat{\phi}(\xi)} \right| d\xi \leq \|g_j\|_{L_2} \left\| \|\cdot\|_2^2\widehat{\phi} \right\|_{L_1}^{1/2}$$

and

$$\int_{\mathbb{R}^n} \left| g_j(\xi)\sqrt{\|\xi\|_2^2\widehat{\phi}(\xi)} \right|^2 d\xi \leq \|g_j\|_{L_2}^2 \left\| \|\cdot\|_2^2\widehat{\phi} \right\|_{L_\infty}.$$

for all $j = 1, \ldots, n$. Thus

$$f(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left(g(\xi)\sqrt{\|\xi\|_2^2\widehat{\phi}(\xi)} \right) e^{ix^T\xi} d\xi$$

is well defined, continuous, an element of L_2, and satisfies

$$\widehat{f}(\|\cdot\|_2^2\widehat{\phi})^{-1/2} = g \in L_2. \tag{3.3}$$

Note that since g is the L_2 limit of the sequence $\left\{\widehat{f}_n(\|\cdot\|_2^2\widehat{\phi})^{-1/2}\right\}$, g satisfies $e_\xi^T g(\xi) = 0$ *a.e.* This is because $e_\xi^T \widehat{f}_n(\|\cdot\|_2^2\widehat{\phi})^{-1/2}$ tends to $e_\xi^T g(\xi)$ in L_2 as n tends to ∞, and the former is a sequence of zeros. Then by (3.3), the Fourier transform of f satisfies the orthogonality condition. Therefore f resides in $\mathcal{G}_{div}$. Now we have that

$$\begin{aligned} \|f_n - f\|_{\mathcal{G}_{div}}^2 &= (2\pi)^{-n/4}\|\widehat{f}_n(\|\cdot\|_2^2\widehat{\phi})^{-1/2} - \widehat{f}(\|\cdot\|_2^2\widehat{\phi})^{-1/2}\|_{L_2} \\ &= (2\pi)^{-n/4}\|\widehat{f}_n(\|\cdot\|_2^2\widehat{\phi})^{-1/2} - g\|_{L_2} \to 0 \end{aligned}$$

as n tends to ∞. We conclude that $\mathcal{G}_{div}$ is complete.

All that is left is to show that Φ_{div} is the reproducing kernel of $\mathcal{G}_{div}$. Let c and

y be vectors in $\mathbb{R}^n$. We have

$$\begin{aligned}(f, \Phi_{div}(\cdot - y)c) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left(\widehat{\Phi_{div}}(\xi) c e^{-i\xi^T y}\right)^* \widehat{\Phi_{div}}(\xi)^+ \widehat{f}(\xi) d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} c^T \widehat{\Phi_{div}}(\xi) \widehat{\Phi_{div}}(\xi)^+ \widehat{f}(\xi) e^{i\xi^T y} d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} c^T \left(I - e_\xi e_\xi^T\right) \widehat{f}(\xi) e^{i\xi^T y} d\xi \\ &= c^T \left((2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{i\xi^T y} d\xi\right) = c^T f(x).\end{aligned}$$

□

Theorem 4. *Suppose that* $\phi \in C^2 \cap L_1$ *is a positive definite function such that* $\Delta\phi \in L_1$. *Define*

$$\mathcal{G}_{curl} := \left\{ f \in L_2 \cap C : \int_{\mathbb{R}^n} \widehat{f}(\xi)^* \widehat{\Phi_{curl}}(\xi)^+ \widehat{f}(\xi) d\xi < \infty \text{ and } \widehat{f}(\xi) = e_\xi h(\xi), h \in L_2 \right\}$$

and equip this space with the bilinear form

$$(f, g)_{\mathcal{G}_{curl}} := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{g}(\xi)^* \widehat{\Phi_{curl}}(\xi)^+ \widehat{f}(\xi) d\xi.$$

Then $\mathcal{G}_{curl}$ *is a Hilbert space with inner product* $(\cdot, \cdot)_{\mathcal{G}_{curl}}$ *and reproducing kernel* Φ_{curl}. *Hence* $\mathcal{G}_{curl} = \mathcal{N}_{\Phi_{curl}}$ *and the inner products are the same.*

Proof. The proof is the same as the previous, with minor modifications. See Appendix B. □

We have a few observations. First, the inner products for $\mathcal{N}_{\Phi_{div}}$ and $\mathcal{N}_{\Phi_{curl}}$ are exactly the same: if f and g are in one of these spaces, the inner product is given by

$$(f, g) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \frac{\widehat{g}(\xi)^* \widehat{f}(\xi)}{\|\xi\|_2^2 \widehat{\phi}(\xi)} d\xi.$$

Also, if $\Phi := -I\Delta\phi$, it can be shown that the inner product for $\mathcal{N}_\Phi$ is the same one as above and $\mathcal{N}_\Phi = \mathcal{N}_{\Phi_{div}} \oplus \mathcal{N}_{\Phi_{curl}}$.

C. Native Spaces as Sobolev Spaces

Now that our native spaces have a more useful form, we will be able to see how certain native spaces, particularly those associated with Wendland functions, are related to Sobolev spaces. We begin by defining some "Sobolev-like" spaces. If $\tau \geq 0$, we define

$$\widetilde{H}^\tau(\mathbb{R}^n) := \left\{ f \in (L_2(\mathbb{R}^n))^n : \int_{\mathbb{R}^n} \frac{\|\widehat{f}(\xi)\|_2^2}{\|\xi\|_2^2} \left(1 + \|\xi\|_2^2\right)^{\tau+1} d\xi < \infty \right\}.$$

It is not difficult to see that this space a Hilbert space with the inner product

$$(f, g)_{\widetilde{H}^\tau(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{\widehat{g}(\xi)^* \widehat{f}(\xi)}{\|\xi\|_2^2} \left(1 + \|\xi\|_2^2\right)^{\tau+1} d\xi.$$

Also, we define the corresponding spaces

$$\begin{aligned}
\widetilde{H}^\tau_{div}(\mathbb{R}^n) &:= \left\{ f \in \widetilde{H}^\tau(\mathbb{R}^n) : e_\xi^T \widehat{f}(\xi) = 0 \text{ a.e.} \right\} \\
\widetilde{H}^\tau_{curl}(\mathbb{R}^n) &:= \left\{ f \in \widetilde{H}^\tau(\mathbb{R}^n) : \widehat{f}(\xi) = e_\xi h(\xi), \text{ where } h \in L_2(\mathbb{R}^n) \right\}.
\end{aligned}$$

As in the case of Sobolev spaces, we will sometimes use $\widetilde{H}^\tau$ as shorthand for $\widetilde{H}^\tau(\mathbb{R}^n)$. From the results of the last section we see that these are native spaces when the Fourier transform of ϕ has algebraic decay, as is the case with Wendland functions.

Corollary 1. *Let $\tau > n/2$ and let ϕ be a positive definite function and let $\Phi := -\Delta\phi I$. If $\widehat{\phi}$ has algebraic decay, i.e., if there exists constants c_1 and c_2 such that*

$$c_1 \left(1 + \|\xi\|_2^2\right)^{-(\tau+1)} \leq \widehat{\phi}(\xi) \leq c_2 \left(1 + \|\xi\|_2^2\right)^{-(\tau+1)}, \tag{3.4}$$

then $\mathcal{N}_\Phi = \widetilde{H}^\tau(\mathbb{R}^n)$ with equivalent norm.

The following result will allow us to see a more precise relationship $\widetilde{H}^\tau(\mathbb{R}^n)$ has to a Sobolev space.

Proposition 1. *The norm for the space* $\widetilde{H}^\tau(\mathbb{R}^n)$ *is equivalent to the norm defined by*

$$\|f\|_*^2 := \int_{\mathbb{R}^n} \frac{\|\widehat{f}(\xi)\|_2^2}{\|\xi\|_2^2} d\xi + \|f\|_{H^\tau(\mathbb{R}^n)}^2. \tag{3.5}$$

Proof. To get the above equivalence, it is enough to show the existence of positive constants c_1 and c_2 such that

$$c_1 \left(\frac{1}{\|\xi\|_2^2} + (1 + \|\xi\|_2^2)^\tau \right) \leq \frac{(1 + \|\xi\|_2^2)^{\tau+1}}{\|\xi\|_2^2} \leq c_2 \left(\frac{1}{\|\xi\|_2^2} + (1 + \|\xi\|_2^2)^\tau \right).$$

The first part of this inequality can be done easily with $c_1 = 1$. We have

$$\frac{(1 + \|\xi\|_2^2)^{\tau+1}}{\|\xi\|_2^2} = \frac{(1 + \|\xi\|_2^2)^\tau}{\|\xi\|_2^2} + (1 + \|\xi\|_2^2)^\tau \geq \frac{1}{\|\xi\|_2^2} + (1 + \|\xi\|_2^2)^\tau.$$

For the reverse inequality we consider $\|\xi\|_2 \leq 1$ and $\|\xi\|_2 > 1$ separately. If $\|\xi\|_2 > 1$, we have

$$\frac{(1 + \|\xi\|_2^2)^{\tau+1}}{\|\xi\|_2^2} = \frac{(1 + \|\xi\|_2^2)^\tau}{\|\xi\|_2^2} + (1 + \|\xi\|_2^2)^\tau \leq 2(1 + \|\xi\|_2^2)^\tau.$$

If $\|\xi\|_2 \leq 1$ we have $(1 + \|\xi\|_2^2) \leq 2$ which gives us

$$\frac{(1 + \|\xi\|_2^2)^{\tau+1}}{\|\xi\|_2^2} \leq \frac{2^{\tau+1}}{\|\xi\|_2^2}.$$

This shows that we can take $c_2 = 2^{\tau+1}$. □

This "decomposition" of the norm tells us that $\widetilde{H}^\tau(\mathbb{R}^n)$ is the subspace of $H^\tau(\mathbb{R}^n)$ consisting of functions f such that there is a potential $g \in H^{\tau+1}(\mathbb{R}^n)$ with $\sqrt{-\Delta} g = f$. Indeed, if $f \in \widetilde{H}^\tau(\mathbb{R}^n)$ then the Fourier transform of the appropriate potential would be $\widehat{g} = \widehat{f}/\|\cdot\|_2$. Conversely, if $\sqrt{-\Delta} g = f$ with $g \in H^{\tau+1}(\mathbb{R}^n)$, then $\|f\|_{\widetilde{H}^\tau(\mathbb{R}^n)}^2 = \|g\|_{H^{\tau+1}(\mathbb{R}^n)}^2 < \infty$. This makes sense since we differentiated ϕ to get our kernel Φ, that is, one may expect that differentiating the kernel would result in "differentiating" the native space. It also shows us that when we measure functions in the native space, we are essentially measuring their "anti-derivative". Furthermore, it is important to note that when restricted to a bounded Lipschitz domain Ω, these native spaces *are* Sobolev spaces.

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Then we have*

$$H^\tau(\Omega) = \left\{ f|_\Omega,\ f \in \widetilde{H}^\tau(\mathbb{R}^n) \right\}.$$

Furthermore, the norms are equivalent.

Proof. See Appendix B. □

One can also use Proposition 3.5 to show that the spaces $\widetilde{H}^\tau_{div}(\mathbb{R}^n)$ and $\widetilde{H}^\tau_{curl}(\mathbb{R}^n)$ consist of functions in $\widetilde{H}^\tau(\mathbb{R}^n)$ that have potentials in $H^{\tau+1}(\mathbb{R}^n)$ in the following sense.

Proposition 2. *The function spaces $\widetilde{H}^\tau_{div}(\mathbb{R}^n)$ and $\widetilde{H}^\tau_{curl}(\mathbb{R}^n)$ can be characterized as follows:*

$$\widetilde{H}^\tau_{div}(\mathbb{R}^n) = \left\{ f \in H^\tau \mid \exists\ g \in H^{\tau+1} \text{ with } \nabla \times g = f \right\}$$
$$\widetilde{H}^\tau_{curl}(\mathbb{R}^n) = \left\{ f \in H^\tau \mid \exists\ g \in H^{\tau+1} \text{ with } \nabla g = f \right\},$$

where $n = 2$ or 3 in the divergence-free case.

Proof. See Appendix B. □

D. Generalized Interpolation on Native Spaces

In this section we investigate the situation where we are reconstructing a vector-valued function from generalized data, that is, data coming from any continuous linear functional (not just point evaluations). This has already been done in [14], but it was proved for a slightly different definition of the native space. In our definition, we "built" the native space out of translates of the kernel, whereas in [14] the native space was constructed with convolutions of the kernel with *any* compactly supported distribution. This fact gives the older native space definition an advantage: one gets the existence of generalized interpolants basically for free. In our case, there is something to prove, which we do now.

Theorem 6. *Suppose that $\mathcal{N}_\Phi$ is a real Hilbert space of vector-valued functions with matrix-valued reproducing kernel Φ. Let $\mu, \nu \in \mathcal{N}_\Phi^*$ and let $\nu^y \Phi(\cdot - y)$ denote the vector-valued function whose i^{th} coordinate is given by ν^y acting on the i^{th} column of $\Phi(\cdot - y)$. Then $\nu^y \Phi(\cdot - y) \in \mathcal{N}_\Phi$ and*

$$\nu(f) = (f, \nu^y \Phi(\cdot - y))_{\mathcal{N}_\Phi} \text{ for all } f \in \mathcal{N}_\Phi.$$

Also, we have $(\nu, \mu)_{\mathcal{N}_\Phi^*} = \mu^x \nu^y \Phi(x - y)$.

Proof. Given $\nu \in \mathcal{N}_\Phi^*$, the Riesz Representation Theorem guarantees the existence of $h_\nu \in \mathcal{N}_\Phi$ such that $\nu(f) = (f, h_\nu)_{\mathcal{N}_\Phi}$ for all f in the native space. Using the reproducing kernel properties of Φ, we get:

$$\nu(\Phi(\cdot - x)e_i) = (\Phi(\cdot - x)e_i, h_\nu)_{\mathcal{N}_\Phi} = (h_\nu, \Phi(\cdot - x)e_i)_{\mathcal{N}_\Phi} = e_i^T h_\nu(x).$$

Since x was arbitrary, it follows that $h_\nu = \nu^y \Phi(\cdot - y)$ and the first property is proved. For the last result, we have

$$(\nu, \mu)_{\mathcal{N}_\Phi^*} = (h_\mu, h_\nu)_{\mathcal{N}_\Phi} = \mu(h_\nu) = \mu^x \nu^y \Phi(x - y).$$

□

This result shows that the definitions of native spaces are equivalent. As you will recall, we constructed the native space out of the dense subspace

$$F_\Phi(\Omega) = \left\{ \sum_{j=1}^{N} \Phi(\cdot - x_j)\alpha_j : x_j \in \Omega,\ \alpha_j \in \mathbb{R}^n,\ \text{and } N \in \mathbb{N} \right\}.$$

Our approach is the same in [14], but instead of $F_\Phi(\Omega)$ the space used contains all functions of the form $\nu^y\Phi(\cdot - y)$, where ν is a compactly supported linear functional. This space was then closed in the norm given by $\|\nu^y\Phi(\cdot - y)\|^2 = \nu^x\nu^y\Phi(x-y)$, which coincides with our norm if one restricts the linear functionals to be point evaluations. Thus our native space is contained in the other one. Theorem 6 shows that this space is contained in our definition of the native space, and the norms are the same.

This theorem also guarantees the existence of generalized interpolants. Suppose that $\{\nu_i\}_{i=1}^N$ is a linearly independent subset of $\mathcal{N}_\Phi^*$. Let the matrix A be defined by $A_{i,j} = \nu_i^x\nu_j^y\Phi(x-y)$. From the above result we immediately see that A is symmetric, and we will soon see that it is positive definite. Let $c \in \mathbb{R}^N$ be nonzero and let $\mathcal{L} = \sum c_i\nu_i$. We have

$$\begin{aligned} c^T Ac &= \sum_{i,j=1}^{N} c_i \left(\nu_i^x \nu_j^y \Phi(x-y)\right) c_j \\ &= \sum_{i=1}^{N} c_i \nu_i^x \left(\sum_{j=1}^{N} \nu_j^y \Phi(x-y) c_j \right) \\ &= \left(\sum_{i=1}^{N} c_i \nu_i \right)^x \left(\sum_{j=1}^{N} c_j \nu_j \right)^y \Phi(x-y) \\ &= \mathcal{L}^x \mathcal{L}^y \Phi(x-y) = \|\mathcal{L}\|^2_{\mathcal{N}_\Phi^*} > 0, \end{aligned}$$

where the last inequality follows from the fact that the functionals are linearly independent. This shows us that given linearly-independent functionals $\{\nu_i\}_{i=1}^N$ and data $\{d_i\}_{i=1}^N$, one can find an interpolant s_ν of the form

$$s_\nu(\cdot) = \sum_{j=1}^{N} c_j \nu_j^y \Phi(\cdot - y)$$

such that

$$\nu_k s_\nu(\cdot) = \sum_{j=1}^{N} c_j \nu_k \nu_j^y \Phi(\cdot - y) = d_k \quad \forall k = 1, \ldots, N.$$

Thus matrix-valued RBFs can be used not only to interpolate point evaluations, but data coming from any continuous linear functional on the native space. In particular, one can use Hermite-Birkoff data to approximate solutions to differential equations. Again, this was already known, but since we used a different definition of the native space we needed to show it.

CHAPTER IV

STABILITY

Knowing the stability of the interpolation process turns out to be very important in the escape process. Let X be a finite subset of $\mathbb{R}^n$. In this section we explore the stability of the interpolation matrix, $A_{X,\Phi}$, through its spectral condition number. As done in [16] and [22], we do this by estimating the norm of the inverse of $A_{X,\Phi}$. Since the interpolation matrix is symmetric and positive definite, this amounts to bounding its lowest eigenvalue, $\lambda_{min}(A_{X,\Phi})$, from below. The way this is usually done is by finding a matrix-valued SPD function Ψ such that

$$\sum_{j,k=1}^{N} \alpha_j^* \Phi(x_j - x_k)\alpha_k \geq \sum_{j,k=1}^{N} \alpha_j^* \Psi(x_j - x_k)\alpha_k \geq \lambda \|\alpha\|_2,$$

where $\alpha_j \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}^{nN}$ with the j^{th} n elements of α given by α_j. Such a λ is obviously a lower bound for $\lambda_{min}(A_{X,\Phi})$.

In [16] and [22], this was done for divergence-free matrix valued functions and λ was found not to depend on N, but only on the dimension n and the minimum separation radius of X, denoted q_X. We will choose a Ψ different than that used in these papers and obtain slightly improved results.

A. Lower Bounds For $\lambda_{\min}(A_{X,\Phi_{div}})$

Let $\chi_{\frac{\sigma}{2},n}(\xi)$ be the characteristic function for the ball $B(0,\sigma/2) \subset \mathbb{R}^n$. It was shown in [21, Eq. 3.9] that its Fourier transform is given by

$$\check{\chi}_{\frac{\sigma}{2},n}(x) = \left(\frac{\sigma^2}{8\pi}\right)^{n/2} \left(\frac{\|x\|_2\sigma}{2}\right)^{-n/2} J_{n/2}\left(\frac{\|x\|_2\sigma}{2}\right),$$

where $J_{n/2}$ is the Bessel function of the first kind of order $n/2$. We know from [23, page 6] that $\check{\chi}_{\frac{\sigma}{2},n}(0)$ is given by

$$\check{\chi}_{\frac{\sigma}{2},n}(0) = \frac{(\sigma/(4\sqrt{\pi}))^n}{\Gamma\left((n+2)/2\right)}. \tag{4.1}$$

We define ϕ^σ by

$$\phi^\sigma := \check{\chi}^2_{\frac{\sigma}{2},n}.$$

This function is band-limited. More precisely, $\text{supp}(\widehat{\phi^\sigma}) \subseteq B(0,\sigma)$ [23, page 6]. This is the same function used in [21] to bound the lower eigenvalue in the scalar case. We define Φ^σ_{div} via

$$\Phi^\sigma_{div} := \left(-\Delta I + \nabla\nabla^t\right)\phi^\sigma.$$

Now we calculate the derivatives of ϕ^σ. The derivatives of the Bessel function satisfy the identity (see [28, page 66])

$$\frac{d}{dz} z^{-\nu} J_\nu(z) = -z^{-\nu} J_{\nu+1}(z).$$

From this we have

$$\begin{aligned}
\frac{\partial}{\partial x_k}\check{\chi}_{\frac{\sigma}{2},n} &= \left(\frac{\sigma^2}{8\pi}\right)^{n/2} \frac{\partial}{\partial x_k}\left(\left(\frac{\|x\|_2\sigma}{2}\right)^{-n/2} J_{n/2}\left(\frac{\|x\|_2\sigma}{2}\right)\right) \\
&= \left(\frac{\sigma^2}{8\pi}\right)^{n/2}\left(-\left(\frac{\|x\|_2\sigma}{2}\right)^{-n/2} J_{(n+2)/2}\left(\frac{\|x\|_2\sigma}{2}\right)\frac{x_k\sigma}{2\|x\|_2}\right) \\
&= -x_k\frac{\sigma^2}{4}\left(\frac{\sigma^2}{8\pi}\right)^{n/2}\left(\left(\frac{\|x\|_2\sigma}{2}\right)^{-(n+2)/2} J_{(n+2)/2}\left(\frac{\|x\|_2\sigma}{2}\right)\right) \\
&= -x_k 2\pi\left(\frac{\sigma^2}{8\pi}\right)^{(n+2)/2}\left(\left(\frac{\|x\|_2\sigma}{2}\right)^{-(n+2)/2} J_{(n+2)/2}\left(\frac{\|x\|_2\sigma}{2}\right)\right) \\
&= -x_k 2\pi\check{\chi}_{\frac{\sigma}{2},n+2}.
\end{aligned} \tag{4.2}$$

Using (4.2), we have

$$\begin{aligned}\frac{\partial}{\partial x_k}\phi^\sigma &= \frac{\partial}{\partial x_k}\left(\check{\chi}^2_{\frac{\sigma}{2},n}\right) = 2\check{\chi}_{\frac{\sigma}{2},n}\left(\frac{\partial}{\partial x_k}\check{\chi}_{\frac{\sigma}{2},n}\right)\\ &= 2\check{\chi}_{\frac{\sigma}{2},n}\left(-x_k 2\pi\check{\chi}_{\frac{\sigma}{2},n+2}\right)\\ &= -x_k 4\pi\check{\chi}_{\frac{\sigma}{2},n}\left(\check{\chi}_{\frac{\sigma}{2},n+2}\right).\end{aligned}$$

We now consider second order derivatives of ϕ^σ. We have

$$\begin{aligned}\frac{\partial^2}{\partial^2 x_k}\phi^\sigma &= -4\pi\check{\chi}_{\frac{\sigma}{2},n}\left(\check{\chi}_{\frac{\sigma}{2},n+2}\right) - x_k 4\pi\frac{\partial}{\partial x_k}\left(\check{\chi}_{\frac{\sigma}{2},n}\left(\check{\chi}_{\frac{\sigma}{2},n+2}\right)\right)\\ &= -4\pi\check{\chi}_{\frac{\sigma}{2},n}\left(\check{\chi}_{\frac{\sigma}{2},n+2}\right) - x_k 4\pi\left(\left(\check{\chi}_{\frac{\sigma}{2},n+2}\right)\frac{\partial}{\partial x_k}\check{\chi}_{\frac{\sigma}{2},n} + \left(\check{\chi}_{\frac{\sigma}{2},n}\right)\frac{\partial}{\partial x_k}\check{\chi}_{\frac{\sigma}{2},n+2}\right)\\ &= -4\pi\check{\chi}_{\frac{\sigma}{2},n}\left(\check{\chi}_{\frac{\sigma}{2},n+2}\right) + x_k^2 8\pi^2\left(\left(\check{\chi}_{\frac{\sigma}{2},n+2}\right)^2 + \check{\chi}_{\frac{\sigma}{2},n}\left(\check{\chi}_{\frac{\sigma}{2},n+4}\right)\right).\end{aligned}$$

Let $j \neq k$. Then we have

$$\begin{aligned}\frac{\partial^2}{\partial x_j \partial x_k}\phi^\sigma &= -x_k 4\pi\frac{\partial}{\partial x_j}\left(\check{\chi}_{\frac{\sigma}{2},n}\left(\check{\chi}_{\frac{\sigma}{2},n+2}\right)\right)\\ &= -x_k 4\pi\left(\left(\check{\chi}_{\frac{\sigma}{2},n+2}\right)\frac{\partial}{\partial x_j}\check{\chi}_{\frac{\sigma}{2},n} + \left(\check{\chi}_{\frac{\sigma}{2},n}\right)\frac{\partial}{\partial x_j}\check{\chi}_{\frac{\sigma}{2},n+2}\right)\\ &= x_j x_k 8\pi^2\left(\check{\chi}^2_{\frac{\sigma}{2},n+2} + \check{\chi}_{\frac{\sigma}{2},n}\left(\check{\chi}_{\frac{\sigma}{2},n+4}\right)\right).\end{aligned}$$

Now we define

$$A := 4\pi\check{\chi}_{\frac{\sigma}{2},n}\left(\check{\chi}_{\frac{\sigma}{2},n+2}\right), \tag{4.3}$$

$$B := 8\pi^2\left(\check{\chi}^2_{\frac{\sigma}{2},n+2} + \check{\chi}_{\frac{\sigma}{2},n}\left(\check{\chi}_{\frac{\sigma}{2},n+4}\right)\right). \tag{4.4}$$

With these definitions and our formulas for the derivatives of ϕ^σ, we get the following equality:

$$\Phi^\sigma_{div}(x) = (n-1)A(x)I + B(x)\left(-\|x\|_2^2 I + xx^t\right).$$

Note that this is a symmetric matrix-valued function. Also, the eigenvalues of $\Phi^\sigma_{div}(x)$ are $(n-1)A(x) - \|x\|_2^2 B(x)$ with multiplicity $(n-1)$ and $(n-1)A(x)$ with multiplicity

1. The value $A(0)$ turns out to be important, so we compute it now: using (4.1) we have

$$A(0) = 4\pi \frac{(\sigma/4\sqrt{\pi})^n}{\Gamma\left((n+2)/2\right)} \frac{(\sigma/4\sqrt{\pi})^{n+2}}{\Gamma\left((n+4)/2\right)} = \left(\frac{\sigma^2}{16\pi}\right)^{n+1} \frac{8\pi}{(n+2)\Gamma^2\left((n+2)/2\right)}.$$

Here we have used the fact that $\Gamma(z+1) = z\Gamma(z)$.

Now we get a bound on the eigenvalues of $\Phi(x)$. Recall that the eigenvalues of $\Phi(x)$ are $(n-1)A(x) - \|x\|_2^2 B(x)$ with multiplicity $(n-1)$ and $(n-1)A(x)$ with multiplicity 1. Therefore they are bounded by

$$\Lambda(x) := (n-1)|A(x)| + \|x\|_2^2 |B(x)|. \tag{4.5}$$

We bound $\Lambda(x)$ with the following lemma.

Proposition 3. *Let $\Lambda(x) = (n-1)|A(x)| + \|x\|_2^2|B(x)|$, then we have the following bound*

$$\Lambda(x) \leq 2^{n+5} \left(\frac{\sigma^2}{8\pi}\right)^{n+1} \left((n-1)\left(\frac{\|x\|_2\sigma}{2}\right)^{-n-2} + 4\left(\frac{\|x\|_2\sigma}{2}\right)^{-n-1}\right). \tag{4.6}$$

Proof. We will make use of [21, Lemma 3.3], which states that for $s = 1, 2, \dots,$ and for all $z > 0$,

$$J_{s/2}^2(z) \leq 2^{s+2}/z\pi.$$

First we concentrate on $|A(x)|$.

$$\begin{aligned} |A(x)| &= 4\pi \left(\frac{\sigma^2}{8\pi}\right)^{n+1} \left(\frac{\|x\|_2\sigma}{2}\right)^{-n-1} \left| J_{n/2}\left(\frac{\|x\|_2\sigma}{2}\right) J_{(n+2)/2}\left(\frac{\|x\|_2\sigma}{2}\right)\right| \\ &\leq 4\pi \left(\frac{\sigma^2}{8\pi}\right)^{n+1} \left(\frac{\|x\|_2\sigma}{2}\right)^{-n-1} \left(\frac{2^{n+3}}{(\pi\|x\|_2\sigma/2)}\right) \\ &= 2^{n+5} \left(\frac{\sigma^2}{8\pi}\right)^{n+1} \left(\frac{\|x\|_2\sigma}{2}\right)^{-n-2}. \end{aligned} \tag{4.7}$$

Now we bound the terms of $\|x\|_2^2|B(x)|$. We have

$$\begin{aligned}
8\pi^2\|x\|_2^2\check{\chi}^2_{\frac{\sigma}{2},n+2} &= 8\pi^2\|x\|_2^2\left(\frac{\sigma^2}{8\pi}\right)^{n+2}\left(\frac{\|x\|_2\sigma}{2}\right)^{-n-2}J^2_{(n+2)/2}\left(\frac{\|x\|_2\sigma}{2}\right) \\
&\leq 8\pi^2\|x\|_2^2\left(\frac{\sigma^2}{8\pi}\right)^{n+2}\left(\frac{\|x\|_2\sigma}{2}\right)^{-n-2}\left(\frac{2^{n+4}}{(\pi\|x\|_2\sigma/2)}\right) \\
&= 2^{n+7}\pi\|x\|_2^2\left(\frac{\sigma^2}{8\pi}\right)^{n+2}\left(\frac{\|x\|_2\sigma}{2}\right)^{-n-3} \\
&= 2^{n+7}\pi\|x\|_2^2\frac{4}{\|x\|_2^2\sigma^2}\left(\frac{\sigma^2}{8\pi}\right)^{n+2}\left(\frac{\|x\|_2\sigma}{2}\right)^{-n-1} \\
&= 2^{n+6}\left(\frac{\sigma^2}{8\pi}\right)^{n+1}\left(\frac{\|x\|_2\sigma}{2}\right)^{-n-1},
\end{aligned}$$

$$\begin{aligned}
8\pi^2\|x\|_2^2\check{\chi}_{\frac{\sigma}{2},n}\check{\chi}_{\frac{\sigma}{2},n+4} &= 8\pi^2\|x\|_2^2\left(\frac{\sigma^2}{8\pi}\right)^{n+2}\left(\frac{\|x\|_2\sigma}{2}\right)^{-n-2}J_{n/2}\left(\frac{\|x\|_2\sigma}{2}\right)J_{(n+4)/2}\left(\frac{\|x\|_2\sigma}{2}\right) \\
&\leq 8\pi^2\|x\|_2^2\left(\frac{\sigma^2}{8\pi}\right)^{n+2}\left(\frac{\|x\|_2\sigma}{2}\right)^{-n-2}\left(\frac{2^{n+4}}{(\pi\|x\|_2\sigma/2)}\right) \\
&= 2^{n+6}\left(\frac{\sigma^2}{8\pi}\right)^{n+1}\left(\frac{\|x\|_2\sigma}{2}\right)^{-n-1}
\end{aligned}$$

This gives us that

$$\|x\|_2^2B(x) \leq 2^{n+7}\left(\frac{\sigma^2}{8\pi}\right)^{n+1}\left(\frac{\|x\|_2\sigma}{2}\right)^{-n-1}. \tag{4.8}$$

Adding the above inequality to our bound on $(n-1)|A(x)|$ gives us the result. □

In [20], it was shown that we have the estimate

$$\sum_{j\neq k} f(|x_j - x_k|) \leq 3^n\sum_{m=1}^{\infty} m^{n-1}\kappa_{f,m}, \tag{4.9}$$

where f is a scalar valued function on $\mathbb{R}^n$ and $\kappa_{f,m}$ is given by

$$\kappa_{f,m} := \sup\left\{|f(\|x\|_2)| : mq_X \leq \|x\|_2 \leq (m+1)q_X\right\}$$

and q_X is the separation radius of X. We will use this fact with the above estimate on $\Lambda(x)$ to prove the following.

Lemma 3. *Choose σ such that $\sigma \geq \max\left\{2/q_X, \widetilde{C}/q_X\right\}$, where*

$$\widetilde{C} := 24\left(\frac{\pi(n+2)(n+3)n}{4(n-1)}\Gamma^2\left(\frac{n+2}{2}\right)\right)^{1/(n+1)}.$$

Then we have

$$\max_k \sum_{j\neq k} \Lambda(x_j - x_k) \leq \frac{n-1}{2n} A(0). \tag{4.10}$$

Proof. Note that that fact that $\sigma \geq q_X/2$ combined with (4.6) gives us

$$\begin{aligned}
\kappa_{\Lambda,m} &\leq 2^{n+5}\left(\frac{\sigma^2}{8\pi}\right)^{n+1}\left((n-1)\left(\frac{mq_X\sigma}{2}\right)^{-n-2} + 4\left(\frac{mq_X\sigma}{2}\right)^{-n-1}\right) \\
&= 2^{n+5}\left(\frac{\sigma^2}{8\pi}\right)^{n+1}\left(\frac{q_X\sigma}{2}\right)^{-n-1}\left((n-1)\left(\frac{q_X\sigma}{2}\right)^{-1} m^{-n-2} + 4m^{-n-1}\right) \\
&\leq 2^{n+5}\left(\frac{\sigma^2}{8\pi}\right)^{n+1}\left(\frac{q_X\sigma}{2}\right)^{-n-1}\left((n-1)m^{-n-1} + 4m^{-n-1}\right) \\
&= 2^{n+5}(n+3)\left(\frac{\sigma^2}{8\pi}\right)^{n+1}\left(\frac{q_X\sigma}{2}\right)^{-n-1} m^{-n-1}.
\end{aligned}$$

Using this with (4.9) we have

$$\begin{aligned}
\max_k \sum_{j\neq k} \Lambda(x_j - x_k) &\leq 3^n \sum_{n=1}^{\infty} m^{n-1}\kappa_{\Lambda,m} \\
&\leq 3^n 2^{n+5}(n+3)\left(\frac{\sigma^2}{8\pi}\right)^{n+1}\left(\frac{q_X\sigma}{2}\right)^{-n-1}\sum_{m=1}^{\infty} m^{n-1}m^{-n-1} \\
&= 3^n 2^{n+5}(n+3)\left(\frac{\sigma^2}{16\pi}\right)^{n+1}\left(\frac{q_X\sigma}{4}\right)^{-n-1}\sum_{m=1}^{\infty} m^{-2} \\
&= \pi^2 3^{n-1} 2^{n+4}(n+3)\left(\frac{\sigma^2}{16\pi}\right)^{n+1}\left(\frac{q_X\sigma}{4}\right)^{-n-1} \\
&\leq \pi^2(n+3)\left(\frac{\sigma^2}{16\pi}\right)^{n+1}\left(\frac{q_X\sigma}{24}\right)^{-n-1}
\end{aligned}$$

Here we have used the well known fact that $\sum_{m=1}^{\infty} m^{-2} = \pi^2/6$. Using the fact that

$\sigma > \widetilde{C}/q_X$, we continue with the inequality to get

$$\begin{aligned}\max_k \sum_{j\neq k} \Lambda(x_j - x_k) &\leq \pi^2(n+3)\left(\frac{\sigma^2}{16\pi}\right)^{n+1}\left(\frac{\widetilde{C}}{24}\right)^{-n-1} \\ &= \pi^2(n+3)\left(\frac{\sigma^2}{16\pi}\right)^{n+1}\left(\frac{4(n-1)}{\pi(n+2)(n+3)n\Gamma^2\left((n+2)/2\right)}\right) \\ &= \frac{1}{2n}\left(\frac{\sigma^2}{16\pi}\right)^{n+1}\frac{8\pi(n-1)}{(n+2)\Gamma^2\left((n+2)/2\right)} = \frac{n-1}{2n}A(0).\end{aligned}$$

□

With these results we are now ready to prove the following theorem.

Theorem 7. *Let ϕ be an even positive definite function, which possesses a positive Fourier transform $\widehat{\phi} \in C(\mathbb{R}^n/0)$. With the function*

$$M(\sigma) := \inf_{\|\xi\|_2 \leq \sigma} \widehat{\phi}(\xi)$$

a lower bound on the smallest eigenvalue of the interpolation matrix is given by

$$\lambda_{\min}(A_{X,\Phi_{div}}) \geq \left(\frac{\sigma^2}{16\pi}\right)^{(n+2)/2} \frac{M(\sigma)\pi}{(4\pi)^n\Gamma\left((n+2)/2\right)}$$

for any $\sigma > 0$ satisfying

$$\sigma \geq \widetilde{C}/q_X.$$

Proof. Define ψ by

$$\psi := \frac{M(\sigma)\Gamma\left((n+2)/2\right)}{\sigma^n\pi^{n/2}}\phi^\sigma.$$

Note that ψ is positive definite and the support of $\widehat{\psi}$ is $B(0,\sigma)$. Then $\widehat{\phi}(\xi) \geq \widehat{\psi}(\xi)$ for $\|\xi\|_2 > \sigma$. If $\|\xi\|_2 \leq \sigma$ we have

$$\widehat{\psi}(\xi) \leq \frac{M(\sigma)\Gamma\left((n+2)/2\right)}{\sigma^n\pi^{n/2}} vol(B(0,\sigma)) \leq M(\sigma) \leq \widehat{\phi}(\xi).$$

This shows us that

$$\sum_{j,k=1}^{N} \alpha_j^* \Phi(x_j - x_k)\alpha_k = \int_{\mathbb{R}^n} \left\| (I - e_\xi e_\xi^T) \left(\sum \alpha_j e^{i\xi^T x_j} \right) \right\|_2^2 \|\xi\|^2 \widehat{\phi}(\xi) d\xi$$
$$\geq \int_{\mathbb{R}^n} \left\| (I - e_\xi e_\xi^T) \left(\sum \alpha_j e^{i\xi^T x_j} \right) \right\|_2^2 \|\xi\|^2 \widehat{\psi}(\xi) d\xi$$
$$= \sum_{j,k=1}^{N} \alpha_j^* \Psi_{div}(x_j - x_k)\alpha_k,$$

where $\alpha_j \in \mathbb{R}^n$ and Ψ_{div} is defined by

$$\Psi_{div} := (-\Delta + \nabla\nabla^t)\psi.$$

Next we use Lemma 3 to get:

$$\sum_{j,k=1}^{N} \alpha_j^* \Psi_{div}(x_j - x_k)\alpha_k \geq \|\alpha\|_2^2 \psi(0) - \sum_{j \neq k} |\alpha_j^* \Psi_{div}(x_j - x_k)\alpha_k|$$
$$\geq \|\alpha\|_2^2 \psi(0) - n \max_{1 \leq j \leq N} \sum_{j \neq k} \frac{M(\sigma)\Gamma\left((n+2)/2\right)}{\sigma^n \pi^{n/2}} \Lambda(x_j - x_k)$$
$$= \|\alpha\|_2^2 \psi(0) - n\|\alpha\|_2^2 \max_{1 \leq j \leq N} \sum_{j \neq k} \frac{M(\sigma)\Gamma\left((n+2)/2\right)}{\sigma^n \pi^{n/2}} \Lambda(x_j - x_k)$$
$$= \|\alpha\|_2^2 \left(\psi(0) - n \frac{M(\sigma)\Gamma\left((n+2)/2\right)}{\sigma^n \pi^{n/2}} \frac{n-1}{2n} A(0) \right)$$
$$= \|\alpha\|_2^2 \left(\psi(0) - \frac{M(\sigma)\Gamma\left((n+2)/2\right)}{\sigma^n \pi^{n/2}} \frac{n-1}{2} A(0) \right)$$
$$= \|\alpha\|_2^2 \frac{\psi(0)}{2}.$$

Plugging in the value of $\psi(0)$ gives us

$$\lambda_{\min}(A_{X,\Phi_{div}}) \geq (1/2) \frac{M(\sigma)(n-1)}{\sigma^n \pi^{n/2}} \left(\frac{\sigma^2}{16\pi} \right)^{n+1} \frac{8\pi}{(n+2)\Gamma\left((n+2)/2\right)}$$
$$\geq \frac{M(\sigma)}{\sigma^n \pi^{n/2}} \left(\frac{\sigma^2}{16\pi} \right)^{n+1} \frac{\pi}{\Gamma\left((n+2)/2\right)}$$
$$= \left(\frac{\sigma^2}{16\pi} \right)^{(n+2)/2} \frac{M(\sigma)\pi}{(4\pi)^n \Gamma^2\left((n+2)/2\right)}.$$

Here we have used the fact that $(n-1)/n+2) \geq 1/4$ for $n \geq 2$. □

Corollary 2. *In the case of the compactly supported Wendland functions $\phi = \phi_{n,k}$ the smallest eigenvalue of the interpolation matrix $A_{X,\Phi_{div}}$ can be bounded by*

$$\lambda_{\min}(A_{X,\Phi_{div}}) \geq c_n q_X^{2k-1},$$

where c_n is a constant depending only on n.

To see the improvement, we compare the above result to the older estimates. For $\phi_{n,k}$, the previous estimate was

$$\lambda_{min}(A_{X,\Phi_{div}}) \geq C q_X^{(2k+1)(n+1)/n},$$

where C depends only on n. Note that the new estimates are better in that the power of q_X is smaller and no longer depends on n. It is worthy to note that the result in Corollary 2 is exactly what one would expect. In the scalar theory the kernel $\phi_{n,k}$ is in C^{2k} and the resulting estimate is

$$\lambda_{min}(A_{X,\phi}) \geq C q_X^{2k+1}. \tag{4.11}$$

Thus a reduction in smoothness of the kernel should reduce the power of q_X in a precise way. That is, one should be able to just replace the $2k$ with the new smoothness. To get Φ_{div} we differentiate $\phi_{n,k}$ twice, so it is in C^{2k-2}. Replacing the $2k$ in 4.11 with $2k-2$ gives us the same estimate derived in Corollary 2. Furthermore, the orders of q_X in the scalar estimates are sharp, so we expect that the bounds presented here are sharp as well.

B. Lower Bounds For $\lambda_{\min}(A_{X,\Phi_{curl}})$

To get the stability estimates in the curl-free case we go through the same basic steps as in the previous section. Using the same ϕ^σ as before, we begin by defining

$$\Phi^\sigma_{curl} := -\left(\nabla\nabla^t\right)\phi^\sigma.$$

With our formulas for the derivatives of ϕ^σ, we get the following equality:

$$\Phi^\sigma_{curl} = A(x)I - B(x)\left(xx^t\right),$$

Where $A(x)$ and $B(x)$ are given by (4.3) and (4.4), respectively. Note that the eigenvalues of $\Phi^\sigma_{curl}(x)$ are $A(x)$ with multiplicity $n-1$ and $A(x) - B(x)\|x\|_2^2$ with multiplicity 1. Furthermore, $\Lambda(x)$ from the previous section bounds the eigenvalues of $\Phi^\sigma_{curl}(x)$. Therefore we may use Lemma 3 to prove the following theorem:

Theorem 8. *Let ϕ be an even positive definite function, which possesses a positive Fourier transform $\widehat{\phi} \in C(\mathbb{R}^n/0)$. With the function*

$$M(\sigma) := \inf_{\|\xi\|_2 \le \sigma} \widehat{\phi}(\xi)$$

a lower bound on $\lambda_{\min}(A_{X,\Phi_{curl}})$ is given by

$$\lambda_{\min}(A_{X,\Phi_{curl}}) \ge \left(\frac{\sigma^2}{16\pi}\right)^{(n+2)/2} \frac{M(\sigma)\pi}{(4\pi)^n\Gamma\left((n+2)/2\right)}$$

for any $\sigma > 0$ satisfying

$$\sigma \ge \widetilde{C}/q_X.$$

Sketch of Proof. Define ψ by

$$\psi := \frac{M(\sigma)\Gamma\left((n+2)/2\right)}{\sigma^n\pi^{n/2}}\phi^\sigma.$$

As shown in the proof of Theorem 7, we have $\widehat{\psi}(\xi) \le \widehat{\phi}(\xi)$ for all $\xi \in \mathbb{R}^n$. This shows us that

$$\sum_{j,k=1}^{N} \alpha_j^* \Phi(x_j - x_k)\alpha_k = \int_{\mathbb{R}^n} \left\| (I - e_\xi e_\xi^T)\left(\sum \alpha_j e^{i\xi^T x_j}\right) \right\|_2^2 \|\xi\|^2 \widehat{\phi}(\xi) d\xi$$
$$\geq \int_{\mathbb{R}^n} \left\| (I - e_\xi e_\xi^T)\left(\sum \alpha_j e^{i\xi^T x_j}\right) \right\|_2^2 \|\xi\|^2 \widehat{\psi}(\xi) d\xi$$
$$= \sum_{j,k=1}^{N} \alpha_j^* \Psi_{div}(x_j - x_k)\alpha_k,$$

where $\alpha_j \in \mathbb{R}^n$ and Ψ_{div} is defined by

$$\Psi_{div} := (-\Delta + \nabla\nabla^t)\psi.$$

Next we use Lemma 3 and follow the same steps in the proof of Theorem 7, replacing Ψ_{div} with Ψ_{curl}. □

Corollary 3. *In the case of the compactly supported Wendland functions $\phi = \phi_{n,k}$ the smallest eigenvalue of the interpolation matrix $A_{X,\Phi_{curl}}$ can be bounded by*

$$\lambda_{\min}(A_{X,\Phi_{curl}}) \geq c_n q_X^{2k-1},$$

where c_n is a constant depending only on n.

CHAPTER V

BAND-LIMITED INTERPOLATION AND APPROXIMATION

In the scalar theory, escaping the native space involves using the approximation properties of band-limited functions, which are functions in L_2 whose Fourier transforms are compactly supported. These functions are analytic, and their smoothness puts them in most native spaces. In fact, they are contained in *every* native space of the scalar RBFs mentioned in this paper. It turns out that band-limited functions approximate both functions in the native space and rougher functions, enabling one to eventually use a triangle inequality to escape the native space.

To make use of band-limited functions for matrix-valued RBFs, we must ensure that they live within the native spaces Φ_{div} and Φ_{curl}. As seen in chapter III, functions in these native spaces must satisfy

$$\int_{\mathbb{R}^n} \frac{\|\widehat{f}(\xi)\|^2}{\|\xi\|_2^2} d\xi < \infty.$$

Therefore we will work with the following band-limited spaces:

$$\begin{aligned}
\mathcal{B}^\sigma &:= \left\{ f \in (L_2)^n : \mathtt{supp}(\widehat{f}) \subset B(0,\sigma) \right\} \\
\widetilde{\mathcal{B}}^\sigma &:= \left\{ f \in \mathcal{B}^\sigma : \int_{\mathbb{R}^n} \frac{\|\widehat{f}(\xi)\|_2^2}{\|\xi\|_2^2} d\xi < \infty \right\} \\
\widetilde{\mathcal{B}}^\sigma_{div} &:= \left\{ f \in \widetilde{\mathcal{B}}^\sigma : e_\xi^T \widehat{f}(\xi) = 0 \right\} \\
\widetilde{\mathcal{B}}^\sigma_{curl} &:= \left\{ f \in \widetilde{\mathcal{B}}^\sigma : \widehat{f}(\xi) = e_\xi h(\xi), \ h \in L_2 \right\}.
\end{aligned}$$

A. Divergence-Free and Curl-Free Approximation

First we show that a divergence-free function in $\widetilde{H}^t$ can be approximated by a band-limited divergence-free function in $\widetilde{\mathcal{B}}^\sigma_{div}$. The proof is simple; one only has to chop off the Fourier transform of the function. The same proof works in the curl-free case, with the obvious modifications.

Proposition 4. *Let $t \geq r \geq 0$. If $f \in \widetilde{H}^t$ is divergence-free then for every $\sigma > 0$ we have a function $g_\sigma \in \widetilde{\mathcal{B}}^\sigma_{div}$ with*

$$\|f - g_\sigma\|_{\widetilde{H}^r} \leq \sigma^{r-t}\|f\|_{\widetilde{H}^t}.$$

Proof. To do this, we simply multiply the Fourier transform of f with a cut off function. Define g_σ by $\widehat{g_\sigma} := \widehat{f}\chi_\sigma$, where χ_σ is the characteristic function of the ball $B(0,\sigma)$. Since $t \geq r$, for all $\|\xi\|_2 \geq \sigma$ we have the inequality

$$\left(1 + \|\xi\|_2^2\right)^{r-t} \leq \sigma^{2(r-t)}.$$

This gives us

$$\begin{aligned}
\|f - g_\sigma\|^2_{\widetilde{H}^r} &= \int_{\|\xi\| \geq \sigma} \frac{\|\widehat{f}(\xi)\|_2^2}{\|\xi\|_2^2}\left(1 + \|\xi\|_2^2\right)^{r+1} d\xi \\
&= \int_{\|\xi\| \geq \sigma} \frac{\|\widehat{f}(\xi)\|_2^2}{\|\xi\|_2^2}\left(1 + \|\xi\|_2^2\right)^{t+1}\left(1 + \|\xi\|_2^2\right)^{r-t} d\xi \\
&\leq \sigma^{2(r-t)} \int_{\|\xi\| \geq \sigma} \frac{\|\widehat{f}(\xi)\|_2^2}{\|\xi\|_2^2}\left(1 + \|\xi\|_2^2\right)^{t+1} d\xi \\
&\leq \sigma^{2(r-t)}\|f\|^2_{\widetilde{H}^t}.
\end{aligned}$$

We still need to check that g_σ is divergence-free. Note that $\widehat{g}_\sigma$ is equal to a scalar function times $\widehat{f}$, so any relation $\widehat{f}$ has with ξ is inherited by $\widehat{g}_\sigma$. Thus g_σ is divergence-free as long as f is. This also shows that if f were curl-free, g_σ would be curl free. □

B. Divergence-Free Band-limited Interpolation

In the previous section we showed we can approximate a divergence-free function in $\widetilde{H}^\tau$ with a band-limited divergence-free function in $\widetilde{\mathcal{B}}^\sigma_{div}$. Now it is our aim to show that we can simultaneously approximate *and* interpolate with divergence-free band-limited functions. We will follow the approach used for the scalar-valued case in [26]. There it was shown that if one chooses $\sigma \sim 1/q_X$, for every continuous function f there exists a band-limited interpolant f_σ whose Fourier transform is supported in $B(0,\sigma)$. We will prove something similar, and to do this we will need the following result from [24, Prop. 3.1],

Proposition 5. *Let $\mathcal{Y}$ be a (possibly complex) Banach Space, $\mathcal{V}$ be a subspace of $\mathcal{Y}$, and Z^* be a finite dimensional subspace of $\mathcal{Y}^*$, the dual of $\mathcal{Y}$. If for every $z^* \in Z^*$ and some $\beta > 1$, β independent of z^*,*

$$\|z^*\|_{\mathcal{Y}^*} \leq \beta \|z^*|_{\mathcal{V}}\|_{\mathcal{V}^*}, \tag{5.1}$$

then for $y \in \mathcal{Y}$ there exists $v \in \mathcal{V}$ such that v interpolates y on Z^*; that is, $z^*(y) = z^*(v)$ for all $z^* \in Z^*$. In addition, v approximates y in the sense that $\|y - v\|_{\mathcal{Y}} \leq (1+2\beta)\, dist(y, \mathcal{V})$.

We will also need some results involving the space $\widetilde{H}^\tau_{div}$. This space can be characterized as a reproducing kernel Hilbert space for $\tau > n/2$. The kernel $\widetilde{\mathcal{K}}^\tau_{div}$ is defined by its Fourier transform:

$$\widehat{\widetilde{\mathcal{K}}^\tau_{div}}(\xi) = \left(\|\xi\|_2^2 I - \xi\xi^T\right)\left(1 + \|\xi\|_2^2\right)^{-(\tau+1)}.$$

The inverse Fourier transform of $(1 + \|\xi\|_2^2)^{-(\tau+1)}$ is given by

$$\mathcal{K}^\tau := c_\tau \|x\|_2^{\tau+1-n/2} K_{\tau+1-n/2}(\|x\|_2),$$

where K_ν is the modified Bessel function of the second kind and c_τ is a constant [30, Theorem 6.13]. Taking the inverse Fourier transform gives us that $\widetilde{\mathcal{K}}^\tau_{div}$ can be written as

$$\widetilde{\mathcal{K}}^{\tau}_{div}(x) = c_\tau \left(-\Delta I + \nabla\nabla^T\right) \|x\|_2^{\tau+1-n/2} K_{\tau+1-n/2}(\|x\|_2). \tag{5.2}$$

Suppose that $X = \{x_1, \ldots, x_N\} \subset \Omega$ is a set of distinct points from a bounded set $\Omega \subset \mathbb{R}^n$, and that $c_1, \ldots, c_N$ are vectors in $\mathbb{R}^n$. If $g := \sum_{j=1}^{N} \widetilde{\mathcal{K}}^{\tau}_{div}(\cdot - x_j)c_j$, using the fact that $\widetilde{H}^{\tau}_{div}$ is the native space for $\widetilde{\mathcal{K}}^{\tau}_{div}$ gives us

$$\|g\|^2_{\widetilde{H}^\tau} = (2\pi)^n \sum_{j,k} c_j^* \widetilde{\mathcal{K}}^{\tau}_{div}(x_j - x_k)c_k. \tag{5.3}$$

As a result, we have that

$$(2\pi)^n \lambda_X \|c\|_2^2 \leq \|g\|^2_{\widetilde{H}^\tau} \leq (2\pi)^n \Lambda_X \|c\|_2^2, \tag{5.4}$$

where λ_X and Λ_X are the minimum and maximum eigenvalues of the $nN \times nN$ matrix $A_{X,\widetilde{\mathcal{K}}^{\tau}_{div}}$, and c is the vector in $\mathbb{R}^{nN}$ whose j^{th} n-components are given by c_j.

We can get a lower bound to the minimum eigenvalues using the stability estimates of Theorem 7. The result is

$$\lambda_X \geq c_{\tau,n} q_X^{2\tau-n}, \tag{5.5}$$

where q_X is the separations radius for X and $c_{\tau,n}$ is a constant depending on its subscripts. To get upper bounds for Λ_X, we need to calculate the kernel explicitly. The function K_ν satisfies (see [28, page 79])

$$\frac{d}{dz}(z^{-\nu} K_\nu(z)) = -z^\nu K_{\nu+1}(z).$$

Note that this is the same formula for J_ν as in (4.2). The calculations following (4.2) show us that that the kernel has the form:

$$\widetilde{\mathcal{K}}^{\tau}_{div} = A(x)I + B(x)\left(-\|x\|_2^2 + xx^T\right), \tag{5.6}$$

where $A(x)$ and $B(x)$ are given by

$$A(x) := c_\tau n \|x\|_2^{-(\nu+1)} K_{\nu+1}(\|x\|_2)$$
$$B(x) := c_\tau \|x\|_2^{-(\nu+2)} K_{\nu+2}(\|x\|_2). \tag{5.7}$$

Here $\nu = \tau + 1 - n/2$. From 5.6, we see that the eigenvalues of $\widetilde{\mathcal{K}}^\tau_{div}(x)$ are $A(x) - \|x\|_2^2 B(x)$ with multiplicity $n-1$ and $A(x)$ with multiplicity 1. This gives us that the modulus of the largest eigenvalue of the $n \times n$ matrix $\widetilde{\mathcal{K}}^\tau_{div}(x)$ is bounded by

$$\Lambda_{\widetilde{\mathcal{K}}^\tau_{div}}(x) := |A(x)| + \|x\|_2^2 |B(x)|.$$

We will also need the following estimate:

Proposition 6. *The function $\|x\|_2^{-\nu} K_\nu(\|x\|_2)$ is positive, decreasing on $[0, \infty)$, and has the bound*

$$\|x\|_2^{-\nu} K_\nu(\|x\|_2) \leq \sqrt{2\pi} \|x\|_2^{\nu - 1/2} e^{-\|x\|_2 + \frac{\nu^2}{2\|x\|_2}}. \tag{5.8}$$

Proof. See Corollary 5.12 and Lemma 5.13 in [30]. □

Note that if $\|x\|_2 > 1$, we have:

$$|\Lambda_{\widetilde{\mathcal{K}}^\tau_{div}}(x)| \leq c_\tau n \|x\|_2^{-(\nu+1)} K_{\nu+1}(\|x\|_2) + c_\tau \|x\|_2^2 \|x\|_2^{-(\nu+2)} K_{\nu+2}(\|x\|_2)$$
$$\leq C_{\tau,n} \|x\|_2^{\nu+7/2} e^{-\|x\|_2} =: \Gamma(x), \tag{5.9}$$

where $\nu = \tau + 1 - n/2$. Furthermore, $\Gamma(x)$ will be decreasing with $\|x\|_2$ if $\|x\|_2 > \nu + 7/2$. This can be shown by simple calculus.

Lemma 4. *Let $g := \sum_{j=1}^N \widetilde{\mathcal{K}}^\tau_{div}(\cdot - x_j) c_j$ and define g_σ by $\widehat{g_\sigma} = \widehat{g}\chi_\sigma$, where χ_σ is the characteristic function of the ball $B(0, \sigma)$. Then, there exists a constant $\kappa > 0$, which is independent of X and the c_j's, such that for $\sigma = \kappa/q_X$ the following inequality* *holds*

$$I_\sigma := \|g - g_\sigma\|_{\widetilde{H}^\tau} \leq \frac{1}{2}\|g\|_{\widetilde{H}^\tau}. \tag{5.10}$$

Proof. From the definition of I_σ and the change of variables to $\xi = \sigma\xi$, we have

$$\begin{aligned}
I_\sigma^2 &= \int_{\|\xi\|_2\geq\sigma} \widehat{g}^*(\xi)\widehat{\widetilde{\mathcal{K}}_{div}}^{+}(\xi)\widehat{g}(\xi)d\xi = \sigma^n \int_{\|\xi\|_2\geq 1} \widehat{g}^*(\sigma\xi)\widehat{\widetilde{\mathcal{K}}_{div}}^{+}(\sigma\xi)\widehat{g}(\sigma\xi)d\xi \\
&= \sigma^n \int_{\|\xi\|_2\geq 1} \widehat{g}^*(\sigma\xi)\left(I - \frac{\xi\xi^T}{\|\xi\|_2^2}\right)\widehat{g}(\sigma\xi)\frac{(1+\sigma^2\|\xi\|_2^2)^{\tau+1}}{|\sigma\xi|^2}d\xi \\
&= \sigma^{n+2}\int_{\|\xi\|_2\geq 1}\left(\sum_j c_j e^{-ix_j^T\sigma\xi}\right)^* \frac{\left(\|\xi\|_2^2 I - \xi\xi^T\right)}{(1+\sigma^2\|\xi\|_2^2)^{(\tau+1)}}\left(\sum_k c_k e^{-ix_k^T\sigma\xi}\right)d\xi
\end{aligned}$$

Note that the matrix $(I\|\xi\|_2^2 - \xi\xi^T)$ is a scaled projection, so the integrand is positive. Now since $\|\xi\|_2 \geq 1$, we have the inequality

$$\frac{1}{(1+\sigma^2\|\xi\|_2^2)^{\tau+1}} \leq \frac{2^{\tau+1}}{\sigma^{2\tau+2}}\frac{1}{(1+\|\xi\|_2^2)^{\tau+1}},$$

so that

$$\begin{aligned}
I_\sigma^2 &\leq 2^{2\tau+2}\sigma^{n-2\tau}\int_{\mathbb{R}^n}\left(\sum_j c_j e^{-i(\sigma x_j^T)\xi}\right)^* \frac{\left(\|\xi\|_2^2 I - \xi\xi^T\right)}{(1+\|\xi\|_2^2)^{(\tau+1)}}\left(\sum_k c_k e^{-i(\sigma x_k^T)\xi}\right)d\xi \\
&= 2^{2\tau+2}\sigma^{n-2\tau}\int_{\mathbb{R}^n}\left(\sum_j c_j e^{-i(\sigma x_j^T)\xi}\right)^* \widehat{\widetilde{\mathcal{K}}^\tau_{div}}(\xi)\left(\sum_k c_k e^{-i(\sigma x_k^T)\xi}\right)d\xi \\
&= 2^{2\tau+2}\sigma^{n-2\tau}\int_{\mathbb{R}^n}\left(\sum_j c_j e^{-i(\sigma x_j^T)\xi}\right)^* \widehat{\widetilde{\mathcal{K}}^\tau_{div}}^{*}(\xi)\widehat{\widetilde{\mathcal{K}}^\tau_{div}}^{+}(\xi)\widehat{\widetilde{\mathcal{K}}^\tau_{div}}(\xi)\left(\sum_k c_k e^{-i(\sigma x_k^T)\xi}\right)d\xi \\
&= (2\pi)^n 2^{2\tau+2}\sigma^{n-2\tau}\sum_{j,k} c_j^*\widetilde{\mathcal{K}}^\tau_{div}(\sigma x_j - \sigma x_k)c_k \leq (2\pi)^n 2^{2\tau+2}\sigma^{n-2\tau}\Lambda_{\sigma X}\|c\|_2^2.
\end{aligned} \tag{5.11}$$

In the third line of the inequality we have used the fact that $\widetilde{\mathcal{K}}^\tau_{div}$ is Hermitian, and the last line follows from the fact that $\widetilde{\mathcal{K}}^\tau_{div}$ is a reproducing kernel. From (5.4) and (5.5), we also have the estimate

$$(2\pi)^n c_{\tau,n} q^{2\tau-n}\|c\|_2^2 \leq \|g\|^2_{\widetilde{H}^\tau},$$

so we obtain

$$I_\sigma^2 \le 2^{2\tau+2} c_{\tau,n}^{-1} (\sigma q_X)^{n-2\tau} \Lambda_{\sigma X} \|g\|^2_{\widetilde{H}^\tau(\mathbb{R}^n)}. \tag{5.12}$$

Now note that the set σX has separation distance $q_{\sigma X} = \sigma q_X$. This will enable us to choose σ so that we get a uniform bound on $\Lambda_{\sigma X}$. Let c be the unit eigenvector associated with $\Lambda_{\sigma X}$ and let $c_j \in \mathbb{R}^n$ be the j^{th} n-components of c. We choose σ so that $\sigma q_X \ge \nu + 7/2$ and use (4.9) to get:

$$\begin{aligned}
\Lambda_{\sigma X} &= \sum_{j,k} c_j^T \widetilde{\mathcal{K}}^\tau_{div}(\sigma x_j - \sigma x_k) c_k = A(0) + \sum_{j \ne k} c_j^T \widetilde{\mathcal{K}}^\tau_{div}(\sigma x_j - \sigma x_k) c_k \\
&\le A(0) + n \sum_{j \ne k} |\Lambda_{\widetilde{\mathcal{K}}^\tau_{div}}(\sigma x_j - \sigma x_k)| \le A(0) + n \sum_{j \ne k} \Gamma(\sigma x_j - \sigma x_k) \\
&\le A(0) + n 3^n C_{\tau,n} \sum_{m=1}^\infty m^{n-1} \Gamma(m \sigma q_X) \\
&\le A(0) + n 3^n C_{\tau,n} \sum_{m=1}^\infty m^{n-1} \Gamma(m) := C_{1,\tau,n},
\end{aligned} \tag{5.13}$$

Here we have used the fact that such a choice of σ allows us to use (5.9), such a choice causes Γ to be decreasing, and the above series is convergent since Γ is rapidly decreasing. From this bound it follows that

$$I_\sigma^2 \le 2^{2\tau+2} C_{1,\tau,n} c_{\tau,n}^{-1} (\sigma q_X)^{n-2\tau} \|g\|^2_{\widetilde{H}^\tau}.$$

Now choose $\sigma q_x = \kappa$ so large that the factor multiplying $\|g\|^2_{\widetilde{H}^\tau}$ is less than 1/4. Taking square roots gives us the result. □

Now we describe the scenario for proving the main result of this section. The result will follow from Proposition 5, and we will apply it to the following scheme:

$$\begin{aligned}
\mathcal{Y} &= \widetilde{H}^\tau_{div} \\
\mathcal{V} &= \widetilde{\mathcal{B}}^\sigma_{div} \\
Z^* &= \operatorname{span}\left\{ c^T \delta_{x_j} : c \in \mathbb{R}^n,\ x_j \in X \right\}.
\end{aligned}$$

Theorem 9. *Let $\tau, t \in \mathbb{R}$ such that $\tau > n/2$ and $t > 0$. Given $f \in \widetilde{H}^{\tau+t}_{div}$ and a point set $X \subset \mathbb{R}^n$ with separation distance q_X, there exists a function $f_\sigma \in \widetilde{\mathcal{B}}^\sigma_{div}$ such that*

$$f|_X = f_\sigma|_X \text{ and } \|f - f_\sigma\|_{\widetilde{H}^\tau} \le 5 \cdot dist_{\widetilde{H}^\tau}(f, \widetilde{\mathcal{B}}^\sigma_{div}) \le 5\kappa^{-t} q_X^t \|f\|_{\widetilde{H}^{\tau+t}}.$$

with $\sigma = \kappa/q_X$, where $\kappa \ge 1$ depends on only τ and n.

Proof. The proof will follow from Proposition 5 once we establish the following. Given $z^* \in Z^*$, we need to show that

$$\|z^*\|_{(\widetilde{H}^\tau_{div})^*} \le 2\|z^*|_{\widetilde{\mathcal{B}}^\sigma_{div}}\|_{(\widetilde{H}^\tau_{div})^*}.$$

Since $\widetilde{H}^\tau_{div}$ is a reproducing kernel Hilbert space with kernel $\widetilde{\mathcal{K}}_{div}$, then by Theorem 6 $\widetilde{\mathcal{K}}^\tau_{div}(x - x_j)c$ is the Riesz representer of the functional $c^T\delta_{x_j}$. It follows that if $z^* = \sum c_j^T \delta_{x_j}$ and $g = \sum \widetilde{\mathcal{K}}^\tau_{div}(\cdot - x_j)c_j$ we have $\|z^*\|_{(\widetilde{H}^\tau_{div})^*} = \|g\|_{\widetilde{H}^\tau_{div}}$. Also, note that $\|z^*|_{\widetilde{\mathcal{B}}^\sigma_{div}}\|_{(\widetilde{H}^\tau_{div})^*} = \|g_\sigma\|_{\widetilde{H}^\tau_{div}}$, where g_σ is defined by $\widehat{g_\sigma} = \widehat{g}\chi_\sigma$. This can be seen by

$$\begin{aligned}
\|z^*|_{\widetilde{\mathcal{B}}_\sigma}\|_{(\widetilde{H}^\tau_{div})^*} &= \sup_{\substack{f \in \widetilde{\mathcal{B}}^\sigma_{div} \\ \|f\|_{\widetilde{H}^\tau_{div}} = 1}} z^*(f) = \sup_{\substack{f \in \widetilde{\mathcal{B}}^\sigma_{div} \\ \|f\|_{\widetilde{H}^\tau_{div}} = 1}} (f, g)_{\widetilde{H}^\tau_{div}} \\
&= \sup_{\substack{f \in \widetilde{\mathcal{B}}^\sigma_{div} \\ \|f\|_{\widetilde{H}^\tau_{div}} = 1}} \int_{\mathbb{R}^n} \widehat{g}^*(\xi) \widehat{\widetilde{\mathcal{K}}_{div}}(\xi)^+ \widehat{f}(\xi) d\xi \\
&= \sup_{\substack{f \in \widetilde{\mathcal{B}}^\sigma_{div} \\ \|f\|_{\widetilde{H}^\tau_{div}} = 1}} \int_{B(0,\sigma)} \widehat{g}^*(\xi) \widehat{\widetilde{\mathcal{K}}_{div}}(\xi)^+ \widehat{f}(\xi) d\xi \\
&= \sup_{\substack{f \in \widetilde{\mathcal{B}}^\sigma_{div} \\ \|f\|_{\widetilde{H}^\tau_{div}} = 1}} \int_{\mathbb{R}^n} \widehat{g_\sigma}^*(\xi) \widehat{\widetilde{\mathcal{K}}_{div}}(\xi)^+ \widehat{f}(\xi) d\xi \\
&= \sup_{\substack{f \in \widetilde{\mathcal{B}}^\sigma_{div} \\ \|f\|_{\widetilde{H}^\tau_{div}} = 1}} (f, g_\sigma)_{\widetilde{H}^\tau_{div}} = \|g_\sigma\|_{\widetilde{H}^\tau_{div}}.
\end{aligned}$$

Now, by Lemma 4 for a sufficiently large κ and all $\sigma = \kappa/q_X$ we have the estimate

$$\begin{aligned}\|g_\sigma\|_{\widetilde{H}^\tau} &\geq \|g\|_{\widetilde{H}^\tau} - \|g_\sigma - g\|_{\widetilde{H}^\tau} \\ &\geq \|g\|_{\widetilde{H}^\tau} - \frac{1}{2}\|g\|_{\widetilde{H}^\tau} = \frac{1}{2}\|g\|_{\widetilde{H}^\tau}.\end{aligned}$$

Thus the conditions of Proposition 5 are satisfied with $\beta = 2$. The proposition tells us that for any divergence-free function $f \in \widetilde{H}^{\tau+t}_{div}(\mathbb{R}^n)$ there exists a divergence-free function $f_\sigma \in \widetilde{\mathcal{B}}_\sigma$, with $\sigma = \kappa/q_X$, such that f_σ interpolates f on X and approximates it, in the sense that

$$\|f - f_\sigma\|_{\widetilde{H}^\tau_{div}(\mathbb{R}^n)} \leq 5 \cdot \operatorname{dist}_{\widetilde{H}^\tau}(f, \widetilde{\mathcal{B}}^\sigma_{div}).$$

For the last inequality, note that if $f \in \widetilde{H}^{\tau+t}_{div}$, then we have

$$\begin{aligned}\operatorname{dist}_{\widetilde{H}^\tau}(f, \widetilde{\mathcal{B}}^\sigma_{div})^2 &= \int_{\|\xi\|_2 \geq \sigma} \widehat{f}(\xi)^* \left(I - \frac{\xi\xi^T}{\|\xi\|_2^2}\right) \widehat{f}(\xi) \frac{(1 + \|\xi\|_2^2)^{\tau+1}}{\|\xi\|_2^2} d\xi \\ &= \int_{\|\xi\|_2 \geq \sigma} \widehat{f}(\xi)^* \left(I - \frac{\xi\xi^T}{\|\xi\|_2^2}\right) \widehat{f}(\xi) \frac{(1 + \|\xi\|_2^2)^{\tau+t+1}}{\|\xi\|_2^2 (1 + \|\xi\|_2^2)^t} d\xi \\ &= \sigma^{-2t} \int_{\|\xi\|_2 \geq \sigma} \widehat{f}(\xi)^* \left(I - \frac{\xi\xi^T}{\|\xi\|_2^2}\right) \widehat{f}(\xi) \frac{(1 + \|\xi\|_2^2)^{\tau+t+1}}{\|\xi\|_2^2} d\xi \\ &= \sigma^{-2t} \|f\|^2_{\widetilde{H}^{\tau+t}}.\end{aligned}$$

Taking square roots and using the fact that $\sigma = \kappa/q_X$ gives us the result. □

The estimates in Theorem 9 are exactly what we expected based the on the ones proved in [26, Theorem 3.4]. There it was shown that a function f in $H^{\tau+t}$ can be approximated by a band-limited interpolant with the estimate

$$\|f - f_\sigma\|_{H^\tau} \leq 5 \cdot \operatorname{dist}_{H^\tau}(f, B_\sigma) \leq 5\kappa^{-t} q_X^t \|f\|_{H^{\tau+t}}.$$

In other words, if the error is being measured in a space that is t degrees less smooth than the space the target function resides, then the approximation rate should be given by q_X^t. Also, we would like to point out just how crucial the new stability estimates were in proving this result. In order to achieve the proper approximation rates, the power of q_X in (5.5) had to *exactly* match that of σ in in the proof of Lemma 4, so that σ could be chosen to "control" q_X.

C. Curl-Free Band-limited Interpolation

One can also interpolate a curl-free function with a band-limited curl-free function. The next result follows by using the same steps as in the proof of Theorem 9, replacing any divergence-free functions with the corresponding curl-free function. It also requires an estimate similar to that of Lemma 4, which is easy to show by mimicking its proof.

Theorem 10. *Let $\tau, t \in \mathbb{R}$ such that $\tau > n/2$ and $t > 0$. Given a function $f \in \widetilde{H}^{\tau+t}_{curl}$ and a point set $X \subset \mathbb{R}^n$ with separation distance q_X, there exists a curl-free function $f_\sigma \in \widetilde{\mathcal{B}}^\sigma_{curl}$ such that*

$$f|_X = f_\sigma|_X \text{ and } \|f - f_\sigma\|_{\widetilde{H}^\tau_n} \leq 5 \cdot dist_{\widetilde{H}^\tau}(f, \widetilde{\mathcal{B}}^\sigma_{curl}) \leq 5\kappa^{-t} q_X^t \|f\|_{\widetilde{H}^{\tau+t}}.$$

with $\sigma = \kappa/q_X$, where $\kappa \geq 1$ depends on only τ and n.

CHAPTER VI

ERROR ESTIMATES FOR FUNCTIONS OUTSIDE THE NATIVE SPACE

In this chapter we present the main result of the paper, which is to show that vector-valued RBFs can approximate functions that are more rough than those in the native space. The error estimates we give are in terms of the *mesh norm*. Given a compact set $\Omega \subset \mathbb{R}^n$ and a finite set $X \subset \Omega$, the mesh norm is given by

$$h_{X,\Omega} := \sup_{x\in\Omega} \inf_{x_j \in X} \|x - x_j\|_2.$$

Another important value in the estimates is the *mesh ratio*, given by $\rho_{X,\Omega} := h_{X,\Omega}/q_X$. In what follows, we will let $I_X f$ be the divergence-free RBF interpolant to f on X if f is divergence-free, and the curl-free RBF interpolant to f on X if f is curl-free. This should cause no confusion. Also, to be able to work on Sobolev spaces, we must assume that the Fourier transform of ϕ has algebraic decay, i.e.,

$$c_1 \left(1 + \|\xi\|_2^2\right)^{-(\tau+1)} \leq \widehat{\phi}(\xi) \leq c_2 \left(1 + \|\xi\|_2^2\right)^{-(\tau+1)}. \tag{6.1}$$

Our goal is to introduce error estimates in terms of Sobolev norms.

A. Extensions of Sobolev Spaces to the Native Space

To estimate error in the scalar-valued theory, one is able to start with a Sobolev function on a bounded domain and then extended the function continuously to $\mathbb{R}^n$ in a way that puts it inside of a native space, at least in the case where the native space is a Sobolev space. Once in the native space, best approximation properties of interpolants can be used to help estimate the error. It is our wish to do something similar here; that is, we want to extend divergence-free or curl-free Sobolev functions defined on a domain Ω to $\mathcal{N}_{\Phi_{div}}$ or $\mathcal{N}_{\Phi_{curl}}$. In the scalar-valued case, one has the advantage that the native spaces are Sobolev spaces, so well-known extension operators can be used. As we have seen in Corollary 1, native spaces for divergence-free and curl-free kernels are almost, but not quite Sobolev spaces, even though they are closely related. However, when working on a bounded domain we will see that it is still possible to begin with a Sobolev function and extend it to the native space in a continuous manner. The ability to do this will depend upon the geometry of Ω. In what follows we will work on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ that satisfies an interior cone condition. We will also assume Ω is simply connected, i.e., it has no "holes".

Let $m \geq 0$ be an integer. We will require our extension operators to extend functions continuously from $H^m(\Omega)$ to $\widetilde{H}^m(\mathbb{R}^n)$. Further, if a function is divergence-free or curl-free on Ω then the extensions should also be divergence-free or curl-free, respectively. We will be able to construct such operators for functions that can be written as the gradient or curl of a potential function. Most results involving vector potentials on Sobolev spaces are only proved in small dimensions, so in what follows $n = 2$ or 3. Once we obtain a potential, we will extend the original function by using Stein's extension operator.

We will begin with the more simple case of curl-free functions. By definition of $H^m_{curl}(\Omega)$, if Ω is a simply connected domain, $\nabla \times u = 0$ for $u \in L_2(\Omega)$ if and only if there is a potential function ϕ such that $u = \nabla\phi$. Furthermore, by choosing ϕ to have zero average value, it is unique up to a constant. Also, note that when $u \in H^m(\Omega)$ we automatically get that $\phi \in H^{m+1}(\Omega)$. To see this, we need to check that derivatives of order $m + 1$ or less of ϕ are in $L_2(\Omega)$. If α is a multi-index with $|\alpha| \leq m + 1$, then $D^{\widetilde{\alpha}}$ is a differential operator of order $|\widetilde{\alpha}| \leq m$, where $\alpha_j = \widetilde{\alpha}_j$ for all $j \neq i$ and $\widetilde{\alpha}_i = \alpha_i - 1$. Since $u = \nabla\phi$ we get $u_i = \partial\phi/\partial x_i$ and

$$D^\alpha \phi = D^{\widetilde{\alpha}} u_i \in L_2(\Omega).$$

In order for the extension we construct to be continuous, we need an estimate of the form $\|\phi\|_{H^{m+1}(\Omega)} \leq C\|u\|_{H^m(\Omega)}$. Proving this is an easy application of the Closed Graph Theorem.

Lemma 5. *Let $m \geq 0$ be an integer and let $\Omega \subset \mathbb{R}^n$ be a simply connected domain with Lipschitz boundary. There exists a continuous operator $T : H^m_{curl}(\Omega) \to H^{m+1}(\Omega)$ such that $u = \nabla(Tu)$.*

Proof. For each curl-free function u, we will let Tu be one of its potential functions. To be sure that T is well-defined, Tu will be the potential with minimum norm in $H^{m+1}(\Omega)$. Using the fact that all potentials of u differ by a constant, it is easy to show that if ϕ is any function such that $u = \nabla\phi$, then

$$Tu = \phi - \frac{1}{|\Omega|}\langle 1, \phi\rangle_{L_2(\Omega)},$$

where $|\Omega|$ is the Lebesgue measure of Ω. From this we get that T is a well-defined linear operator.

Now we show that T is a closed map. Suppose that $u_n \to u$ in $H^m(\Omega)$ and $Tu_n \to \phi$ in $H^{m+1}(\Omega)$. We need to show that $Tu = \phi$. This will follow if ϕ satisfies $u = \nabla\phi$ and $\langle 1, \phi\rangle_{L_2(\Omega)} = 0$. We have

$$\begin{aligned} \|u - \nabla\phi\|_{H^m(\Omega)} &\leq \|u - u_n\|_{H^m(\Omega)} + \|\nabla(Tu_n) - \nabla\phi\|_{H^m(\Omega)} \\ &\leq \|u - u_n\|_{H^m(\Omega)} + \|Tu_n - \phi\|_{H^{m+1}(\Omega)} \to 0. \end{aligned}$$

Similarly, we have

$$|\langle 1, \phi\rangle_{L_2(\Omega)}| = |\langle 1, \phi - Tu_n\rangle_{L_2(\Omega)}| \leq |\Omega|\|Tu_n - \phi\|_{L_2(\Omega)} \to 0. \tag{6.2}$$

Thus $Tu = \phi$ and T is closed. By the Closed Graph Theorem, T is continuous, and we get the bound $\|Tu\|_{H^{m+1}(\Omega)} \leq C\|u\|_{H^m(\Omega)}$. □

With this result we are able to construct our extension operator. Let $\mathfrak{E}$ denote the Stein's extension operator on Ω. We extend u by

$$\widetilde{\mathfrak{E}}_{curl} u := \nabla(\mathfrak{E} T u).$$

Note that $\widetilde{\mathfrak{E}}_{curl} u$ is automatically curl-free since it is the gradient of a scalar function. To show that $\widetilde{\mathfrak{E}}_{curl} : H^m_{curl}(\Omega) \to \widetilde{H}^m_{curl}(\mathbb{R}^n)$ is continuous, consider

$$\begin{aligned}
\|\widetilde{\mathfrak{E}}_{curl} u\|^2_{\widetilde{H}^m_{curl}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \frac{\|\widehat{\widetilde{\mathfrak{E}}_{curl} u}\|_2^2}{\|\xi\|_2^2} \left(1 + \|\xi\|_2^2\right)^{m+1} d\xi \\
&= \int_{\mathbb{R}^n} \frac{\|\xi \widehat{\mathfrak{E}(Tu)}\|_2^2}{\|\xi\|_2^2} \left(1 + \|\xi\|_2^2\right)^{m+1} d\xi \\
&\leq \int_{\mathbb{R}^n} \|\widehat{\mathfrak{E}(Tu)}\|_2^2 \left(1 + \|\xi\|_2^2\right)^{m+1} d\xi = \|\mathfrak{E} T u\|^2_{H^{m+1}(\mathbb{R}^n)} \\
&\leq C \|Tu\|^2_{H^{m+1}(\Omega)} \leq C \|u\|^2_{H^m(\Omega)}.
\end{aligned}$$

This proves the following theorem.

Theorem 11. *Let $m \geq 0$ be an integer and let $\Omega \subset \mathbb{R}^n$ be a simply-connected bounded Lipschitz domain satisfying an interior cone condition. Then there exists a continuous operator $\widetilde{\mathfrak{E}}_{curl} : H^m_{curl}(\Omega) \to \widetilde{H}^m_{curl}(\mathbb{R}^n)$ such that $\widetilde{\mathfrak{E}}_{curl} u|_\Omega = u$ for all $u \in H^m_{curl}(\Omega)$.*

Our strategy for the divergence-free case is the same: first we work on domains that allow for potential functions, construct a continuous operator T that assigns a potential to each divergence-free function, and use Stein's operator $\mathfrak{E}$ to construct a continuous extension. The divergence-free case is slightly more difficult because it is not immediately obvious how to construct the operator T so that it is well-defined. Nevertheless, it is possible to get around this.

We will need several results from Appendix A. First, Proposition 9 tells us that given a divergence-free vector field u on Ω, we can find a vector potential such that $\nabla \times \phi = u$. Using Proposition 11, we see that when $u \in H^k(\Omega)$ and the boundary of Ω is $\mathcal{C}^{k+1,1}$, then the potential ϕ is in $H^{k+1}(\Omega)$. Furthermore, by Proposition 10, when Ω is simply connected, there is a *unique* vector potential ϕ satisfying

$$\nabla \times \phi = 0, \quad \nabla \cdot \phi = 0, \quad \phi \cdot n = 0 \quad \text{on } \partial\Omega. \tag{6.3}$$

Now we can prove the following.

Lemma 6. *Let $m \geq 0$ be an integer and let Ω be a simply-connected domain of $\mathbb{R}^n$ with $\mathcal{C}^{k+1,1}$ boundary, where $k \geq m$ is an integer. Then there exists a continuous operator $T : H^m_{div}(\Omega) \to H^{m+1}(\Omega)$ such that $u = \nabla \times (Tu)$.*

Proof. For each divergence-free function u, we will let Tu be the unique potential satisfying (6.3). From this we get that T is well-defined, and we can easily check that it is linear.

As in the curl-free case, we show that T is a closed map. Suppose that $u_n \to u$ in $H^m(\Omega)$ and $Tu_n \to \phi$ in $H^{m+1}(\Omega)$. We need to show that $Tu = \phi$. This will follow if ϕ satisfies (6.3). We have

$$\begin{aligned}\|u - \nabla \times \phi\|_{H^m(\Omega)} &\leq \|u - u_n\|_{H^m(\Omega)} + \|\nabla \times (Tu_n) - \nabla \times \phi\|_{H^m(\Omega)} \\ &\leq \|u - u_n\|_{H^m(\Omega)} + C\|Tu_n - \phi\|_{H^{m+1}(\Omega)} \to 0.\end{aligned}$$

Similarly, we have

$$\|\nabla \cdot \phi\|_{L_2(\Omega)} = \|\nabla \cdot \phi - \nabla \cdot Tu_n\|_{L_2(\Omega)} \leq \|Tu_n - \phi\|_{H^m(\Omega)} \to 0.$$

Also, the Trace theorem gives us

$$\begin{aligned}\|\phi \cdot n\|_{L_2(\Gamma)} = \|\phi \cdot n - Tu_n \cdot n\|_{L_2(\Omega)} &\leq \|\phi - Tu_n\|_{L_2(\Gamma)} \\ &\leq C\|\phi - Tu_n\|_{H^{1/2}(\Gamma)} \to 0.\end{aligned}$$

Thus ϕ satisfies (6.3) and T is closed. By the Closed Graph Theorem, T is continuous, and we get the bound $\|Tu\|_{H^{m+1}(\Omega)} \leq C\|u\|_{H^m(\Omega)}$. $\square$

With this result we are able to construct our divergence-free extension operator. We extend u by

$$\widetilde{\mathfrak{E}}_{div} u := \nabla \times (\mathfrak{E}Tu),$$

where $\mathfrak{E}Tu$ represents Stein's extension operating on each coordinate of the function Tu. Note that $\widetilde{\mathfrak{E}}_{div} u$ is automatically divergence-free since it is the curl of a vector field. To show that $\widetilde{\mathfrak{E}}_{div} : H^m_{div}(\Omega) \to \widetilde{H}^m_{div}(\mathbb{R}^n)$ is continuous, consider

$$\begin{aligned}
\|\widetilde{\mathfrak{E}}_{div} u\|^2_{\widetilde{H}^m_{div}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \frac{\|\widehat{\widetilde{\mathfrak{E}}_{div} u}\|_2^2}{\|\xi\|_2^2} \left(1 + \|\xi\|_2^2\right)^{m+1} d\xi \\
&= \int_{\mathbb{R}^n} \frac{\|\xi \times \widehat{\mathfrak{E}(Tu)}\|_2^2}{\|\xi\|_2^2} \left(1 + \|\xi\|_2^2\right)^{m+1} d\xi \\
&\leq C \int_{\mathbb{R}^n} \|\widehat{\mathfrak{E}(Tu)}\|_2^2 \left(1 + \|\xi\|_2^2\right)^{m+1} d\xi \\
&= C\|\mathfrak{E}Tu\|^2_{H^{m+1}(\mathbb{R}^n)} \leq C\|Tu\|^2_{H^{m+1}(\Omega)} \leq C\|u\|^2_{H^m(\Omega)}.
\end{aligned}$$

This proves the following theorem for $n = 2$ or 3.

Theorem 12. *Let $m \geq 0$ be an integer and let $\Omega \subset \mathbb{R}^n$ be a simply-connected domain with $\mathcal{C}^{k+1,1}$ boundary, where $k \geq m$ is an integer. Then there exists a continuous operator $\widetilde{\mathfrak{E}}_{div} : H^m_{div}(\Omega) \to \widetilde{H}^m_{div}(\mathbb{R}^n)$ such that $\widetilde{\mathfrak{E}}_{div} u|_\Omega = u$ for all $u \in H^m_{div}(\Omega)$.*

Remark 2. Note that we only showed that our extension operators were continuous for integer-ordered Sobolev spaces. We expect the same to be true for fractional-ordered Sobolev spaces.

B. Interpolation Error Estimates

With our extensions defined, we are now able to begin estimating interpolation approximation rates. In what follows we assume that Ω is a simply-connected domain satisfying an interior cone condition and has a $\mathcal{C}^{k+1,1}$ boundary. Also, we require ϕ to satisfy (6.1) for some τ. The proofs will only address the divergence-free case because are essentially the same in the curl-free case. We will begin by making use of a recent result from Narcowich, Ward, and Wendland concerning Sobolev estimates of functions with many zeros [25, Theorem 2.12].

Proposition 7. *Let k be a positive integer, $0 < s \leq 1$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, and let α be a multi-index satisfying $k > |\alpha| + n/p$, or $p = 1$ and $k \geq |\alpha| + n$. Also, let $X \subset \Omega$ be a discrete set with mesh norm $h_{X,\Omega}$. Then there is a constant depending only on Ω such that if $h_{X,\Omega} \leq C_\Omega$ and if $u \in W_p^{k+s}(\Omega)$ satisfies $u|_X = 0$, then*

$$|u|_{W_q^{|\alpha|}(\Omega)} \leq C h_X^{k+s-|\alpha|-n(1/p-1/q)_+} |u|_{W_p^{k+s}(\Omega)}, \tag{6.4}$$

where $(x)_+ = x$ is $x \geq 0$ and is 0 otherwise. Here the constant C is independent of $h_{X,\Omega}$ and u.

One can use the relation between p and q norms to get the same result for $u \in W_p^{k+s}(\Omega)^n$.

$$\begin{aligned} |u|_{W_q^{|\alpha|}(\Omega)^n} &= \left(\sum_{j=1}^n |u_j|^q_{W_q^{|\alpha|}(\Omega)} \right)^{1/q} \leq C h^{k+s-|\alpha|-n(1/p-1/q)_+} \left(\sum_{j=1}^n |u_j|^q_{W_p^{k+s}(\Omega)} \right)^{1/q} \\ &\leq C n^{(1/q-1/p)_+} h^{k+s-|\alpha|-n(1/p-1/q)_+} \left(\sum_{j=1}^n |u_j|^p_{W_p^{k+s}(\Omega)} \right)^{1/p} \\ &= C h^{k+s-|\alpha|-n(1/p-1/q)_+} |u|_{W_p^{k+s}(\Omega)^n}. \end{aligned}$$

We will use this to prove our first error estimate, which bounds the error for a class of functions in the native space.

Theorem 13. *Let $m = \lceil \tau \rceil$, and let p and q be as in Proposition 7. If $f \in H^m(\Omega)$ is* *divergence-free, we have*

$$\|f - I_X f\|_{W_q^\beta(\Omega)} \le C h_{X,\Omega}^{\tau-\beta-n(1/2-1/q)_+} \|f\|_{H^m},$$

for all β *satisfying* $0 \le \beta \le \tau$.

Proof. The remarks after the previous lemma gives us that

$$\|f - I_X f\|_{W_q^\beta(\Omega)} \le C h_{X,\Omega}^{\tau-\beta-n(1/2-1/q)_+} \|f - I_X f\|_{H^\tau(\Omega)}.$$

Recall that in our case the native space is equivalent to $\widetilde{H}^\tau(\mathbb{R}^n)$. Now we continuously extend f to $\widetilde{H}^m(\mathbb{R}^n)$ using $\widetilde{\mathfrak{E}}_{div}$ from Theorem 12. Since $m \ge \tau$, we have that $\widetilde{\mathfrak{E}}_{div} f \in \widetilde{H}^\tau(\mathbb{R}^n)$. Once in $\widetilde{H}^\tau(\mathbb{R}^n)$ we can use the best approximation property of the interpolant in the native space to get

$$\begin{aligned}\|f - I_X f\|_{H^\tau(\Omega)} &\le \|\widetilde{\mathfrak{E}}_{div} f - I_X f\|_{H^\tau(\mathbb{R}^n)} \le C\|\widetilde{\mathfrak{E}}_{div} f - I_X f\|_{\widetilde{H}^\tau(\mathbb{R}^n)} \\ &\le C\|\widetilde{\mathfrak{E}}_{div} f\|_{\widetilde{H}^\tau(\mathbb{R}^n)} \le C\|\widetilde{\mathfrak{E}}_{div} f\|_{\widetilde{H}^m(\mathbb{R}^n)} \\ &\le C\|f\|_{H^m(\Omega)}\end{aligned}$$

Putting the above inequalities together finishes the proof. □

We will also need to measure the error of the band-limited interpolant to the RBF interpolant of f. Note that in this case f may not be in the native space.

Lemma 7. *Let* m*,* p*, and* q *be as in Theorem 13, and let* β *be an integer such that* $\beta \le \tau$*. If* $f \in H^\beta(\Omega)$ *is divergence-free, let* $f_\sigma \in \widetilde{\mathcal{B}}_\sigma$ *be the interpolant to* $\widetilde{\mathfrak{E}}_{div} f$ *on* X *from Theorem 9 with* $t = 0$*. Then we have*

$$\|f_\sigma - I_X f_\sigma\|_{W_q^\mu(\Omega)} \le C h_{X,\Omega}^{\beta-\mu-n(1/2-1/q)_+} \rho_{X,\Omega}^{\beta-\tau} \|f\|_{H^\beta(\Omega)},$$

for all μ *satisfying* $0 \le \mu \le \beta$.

Proof. Note that since f_σ is band-limited, it is in $\widetilde{H}^t(\Omega)$ for all t. Since $f_\sigma - I_X f_\sigma$ is a function with zeroes on X, we get

$$\|f_\sigma - I_X f_\sigma\|_{W_q^\beta(\Omega)} \le C h_{X,\Omega}^{\tau-\beta-n(1/2-1/q)_+} \|f_\sigma - I_X f_\sigma\|_{H^\tau(\Omega)}.$$

We can estimate the right hand side using the best approximation property of $I_X f_\sigma$ in $\widetilde{H}^\tau(\Omega)$:

$$\|f_\sigma - I_X f_\sigma\|_{H^\tau(\Omega)} \le \|f_\sigma - I_X f_\sigma\|_{\widetilde{H}^\tau(\mathbb{R}^n)} \le \|f_\sigma\|_{\widetilde{H}^\tau(\mathbb{R}^n)}.$$

Since $(1+\|\xi\|_2^2)^{\tau+1} \le \sigma^{\tau-\beta}(1+\|\xi\|_2^2)^{\beta+1}$ for $\|\xi\|_2 \le \sigma$, we have the Bernstein inequality

$$\|f_\sigma\|_{\widetilde{H}^\tau(\mathbb{R}^n)} \le C q_X^{\beta-\tau} \|f_\sigma\|_{\widetilde{H}^\beta(\mathbb{R}^n)},$$

which implies that

$$\|f_\sigma - I_X f_\sigma\|_{\widetilde{H}^\tau(\mathbb{R}^n)} \le C h_{X,\Omega}^{\tau-\beta} q_X^{\beta-\tau} \|f_\sigma\|_{\widetilde{H}^\beta(\mathbb{R}^n)} = C \rho_{X,\Omega}^{\tau-\beta} \|f_\sigma\|_{\widetilde{H}^\beta(\mathbb{R}^n)}.$$

All we have left to show is that $\|f_\sigma\|_{\widetilde{H}^\beta(\mathbb{R}^n)} \le C\|f\|_{H^\beta(\Omega)}$. Using the approximation property of f_σ and the continuity of $\widetilde{\mathfrak{E}}_{div}$ gives us:

$$\begin{aligned}
\|f_\sigma\|_{\widetilde{H}^\beta(\mathbb{R}^n)} &\le \|f_\sigma - \widetilde{\mathfrak{E}}_{div} f\|_{\widetilde{H}^\beta(\mathbb{R}^n)} + \|\widetilde{\mathfrak{E}}_{div} f\|_{\widetilde{H}^\beta(\mathbb{R}^n)} \\
&\le C_1 \|\widetilde{\mathfrak{E}}_{div} f\|_{\widetilde{H}^\beta(\mathbb{R}^n)} + C_2 \|f\|_{H^\beta(\Omega)} \\
&\le C\|f\|_{H^\beta(\Omega)}.
\end{aligned}$$

The result thus follows. □

Now we come to our main result, which is to estimate the RBF approximation error for divergence-free functions less smooth than those in the native space.

Theorem 14. *Let m, p, and q be as in Proposition 7 and let β be a positive integer* *such that $\beta \le \tau$. If $f \in H^\beta(\Omega)$ is a divergence-free function, then*

$$\|f - I_X f\|_{W_q^\mu(\Omega)} \le C h_{X,\Omega}^{\beta-\mu-n(1/2-1/q)_+} \rho_{X,\Omega}^{\tau-\beta} \|f\|_{H^\beta(\Omega)},$$

where μ is any real number such that $0 \le \mu \le \beta$.

Proof. Using the fact that $f - I_X f$ is a function with many zeros, we get

$$\|f - I_X f\|_{W_q^\mu(\Omega)} \le C h_{X,\Omega}^{\beta-\mu-n(1/2-1/q)_+} \|f - I_X f\|_{H^\beta(\Omega)}. \tag{6.5}$$

The rest of the proof will be to estimate $\|f - I_X f\|_{H^\beta(\Omega)}$. We extend f to $\widetilde{H}^\beta(\mathbb{R}^n)$ by $\widetilde{\mathfrak{E}}_{div} f$. According to Theorem 9 we may select a divergence-free function $f_\sigma \in \widetilde{\mathcal{B}}_\sigma$ with $\sigma = \kappa/q_X$ so that $f_\sigma|_X = \widetilde{\mathfrak{E}}_{div} f|_X$ and

$$\|\widetilde{\mathfrak{E}}_{div} f - f_\sigma\|_{\widetilde{H}^\beta(\mathbb{R}^n)} \le \|\widetilde{\mathfrak{E}}_{div} f\|_{\widetilde{H}^\beta(\mathbb{R}^n)}.$$

Since f_σ interpolates f on X implies that $I_X f = I_X f_\sigma$. This gives us

$$\begin{aligned}
\|f - I_X f\|_{H^\beta(\Omega)} &\le \|f - f_\sigma\|_{H^\beta(\Omega)} + \|f_\sigma - I_X f_\sigma\|_{H^\beta(\Omega)} \\
&\le \|\widetilde{\mathfrak{E}}_{div} f - f_\sigma\|_{\widetilde{H}^\beta(\mathbb{R}^n)} + \|f_\sigma - I_X f_\sigma\|_{H^\beta(\Omega)} \\
&\le C\|\widetilde{\mathfrak{E}}_{div} f\|_{\widetilde{H}^\beta(\mathbb{R}^n)} + \|f_\sigma - I_X f_\sigma\|_{H^\beta(\Omega)}. \\
&\le C\|f\|_{H^\beta(\Omega)} + \|f_\sigma - I_X f_\sigma\|_{H^\beta(\Omega)}.
\end{aligned}$$

Now we use Lemma 7 to get

$$\|f_\sigma - I_X f_\sigma\|_{H^\beta(\Omega)} \le C \rho_{X,\Omega}^{\beta-\tau} \|f\|_{H^\beta(\Omega)}.$$

Using the fact that $\rho_{X,\Omega} \ge 1$, we get

$$\|f - I_X f\|_{H^\beta(\Omega)} \le C \rho_{X,\Omega}^{\beta-\tau} \|f\|_{H^\beta(\Omega)}.$$

We plug the above inequality into (6.5) to complete the proof. □

Of course, there is an analogous theorem in the curl-free case. The proof follows from exactly the same arguments in this section. However, we can strengthen the result a bit by relaxing the smoothness of $\partial\Omega$. Here we only need to assume that the boundary is Lipschitz, while in Theorem 14 the boundary had to be $\mathcal{C}^{k+1,1}$. The reason for this is to ensure that the we can use the extension operator $\widetilde{\mathfrak{E}}_{div}$.

Theorem 15. *Let m, p, and q be as in Proposition 7 and let β be an integer such that $\beta \leq \tau$. If $f \in H^{\beta}(\Omega)$ is a curl-free function, then*

$$\|f - I_X f\|_{W_q^{\mu}(\Omega)} \leq C h_{X,\Omega}^{\beta-\mu-n(1/2-1/q)_+} \rho_{X,\Omega}^{\tau-\beta} \|f\|_{H^{\beta}(\Omega)},$$

where μ is any real number such that $0 \leq \mu \leq \beta$.

CHAPTER VII

CONCLUSIONS AND FUTURE RESEARCH

We have introduced a new class of matrix-valued RBFs which yield curl-free interpolants. We have offered a new characterization of the native space for both divergence-free and curl-free RBFs, based on the Fourier transform of the kernel as in the scalar theory. This led us to the fact that when the Fourier transform of the kernel decays algebraically, as is the case of compactly-supported Wendland functions, we get native spaces that are subspaces of $\widetilde{H}^{\tau}(\mathbb{R}^n)$. Also, we derived new upper bounds on the stability of the interpolation process by estimating the norm of the inverse of the interpolation matrix. We found that the bounds depend on the separation radius, q_X, and the smoothness of the kernel. Our results coincide with the stability estimates of the scalar-valued theory. We proved an approximation result using band-limited functions, and then showed that one can simultaneously approximate and interpolate a divergence-free function with a band-limited divergence-free function. Finally, we developed Sobolev-type error estimates when the divergence-free target function is less smooth than functions in the native space. Also, as one might expect, for every divergence-free result there was an analogous curl-free result.

There are several possibilities for future research in this area. One is to escape the native space for other RBFs. Here we only dealt with native spaces that are Sobolev spaces, such as those arising from Wendland functions. What about divergence-free RBFs arising from other popular RBFs, such as the Gaussian or the Hardy-multiquadric? This question has not even been answered in the case of scalar-valued RBFs and would be of much interest.

We could also extend our results to more general domains. Our error estimates were proved only for simply connected domains. The proofs relied heavily on the fact that domains without holes always have the property that divergence-free fields have a vector potential that can be used to extend the function to the native space. However, domains with holes *never* have this property. Take for example the vector field $v(x) = x/\|x\|_2^3$ on $\mathbb{R}^3$. Away from the origin, this function is divergence-free. If one restricts this function to an annulus around the origin, it is impossible to extend it to a divergence-free function on $\mathbb{R}^3$ (such an extension would violate Green's Theorem). Numerical tests confirm that on simply connected domains away from the origin, one gets good approximation, but on domains surrounding the origin the approximation properties of divergence-free RBFs break down. It should be possible to modify the process on domains with holes in such a way that the interpolant has good approximation properties.

As we have seen in Chapter IV, as one adds more nodes the interpolation matrix becomes more unstable. This makes things difficult when dealing with large data sets. To make the numerical implementation more realistic, some preconditioning is in order. Preconditioners have been already successfully constructed for some classes of scalar-valued RBFs [7, 12], and some of these ideas might extend to matrix-valued RBFs.

Another avenue of research, and probably the most important, is to test the approximation properties of these functions in some real-world applications. In many physical applications vector fields need to be divergence-free or curl-free, such as those arising from fluids. One possible application is to apply matrix-valued RBFs to fluids by solving the Navier-Stokes equations. This has been done in [15], where Lowitzsch used a collocation approach to numerically solve a simple two-dimensional driven cavity problem, and more examples would be valuable. Another possible application is the Maxwell eigenvalue problem, which is used to solve problems in electro-magnetics. Also, only collocation methods have been used to test vector-valued RBF approximants. In the scalar theory one can also use RBFs to solve problems variationally (see [29]), and this idea extends easily to matrix-valued RBFs. It would be interesting to see how these methods compare on various applications.

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APPENDIX A

DIVERGENCE-FREE AND CURL-FREE FACTS

In the following we assume that Ω is a connected, bounded region of $\mathbb{R}^n$ with Lipschitz boundary with $n = 2$ or 3.

Proposition 8. *Let Ω be simply-connected. A function $f \in L_2(\Omega)^n$ satisfies*

$$\nabla \times f = 0$$

if and only if there exists a function $\phi \in H^1(\Omega)^n$ such that $f = \nabla\phi$. Furthermore, if we require that the average value of ϕ on Ω is zero, then ϕ is unique.

Proof. See [8, Theorem 2.9]. □

Proposition 9. *Suppose $\partial\Omega$ has p connected components, and denote the connected components of $\partial\Omega$ by Γ_i, $0 \leq i \leq p$. A vector field $v \in L_2(\Omega)$ satisfies*

$$\nabla \cdot v = 0, \ \langle v \cdot n, 1\rangle_{\Gamma_i} = 0 \text{ for } 0 \leq i \leq p \tag{A.1}$$

if and only if there exists a vector potential ϕ in $H^1(\Omega)$ such that

$$v = \nabla \times \phi.$$

Furthermore,

$$\nabla \cdot \phi = 0.$$

Proof. See [8, Theorem 3.4]. □

Proposition 10. *Let Ω be as in Proposition 9 and let $v \in L_2(\Omega)$ satisfy A.1. Among the vector potentials satisfying $v = \nabla \times \phi$ and $\nabla \cdot \phi = 0$, we can choose ϕ such that $\phi \cdot n = 0$. When Ω is simply-connected, such a ϕ is unique.*

Proof. See [8, Theorem 3.5]. □

Proposition 11. *Let Ω be a bounded open subset of $\mathbb{R}^n$ ($n = 2$ or 3) with $\mathcal{C}^{k+1,1}$ boundary. Suppose $u \in L_2(\Omega)$, $\nabla \times u \in H^k(\Omega)$, $\nabla \cdot u \in H^k(\Omega)$, and $n \cdot u|_{\partial\Omega} \in H^{k+1/2}(\partial\Omega)$. Then $u \in H^{k+1}(\Omega)$.*

Proof. See [3, Chapter 9] □

APPENDIX B

VARIOUS PROOFS

Proof of Theorem 4

Proof. We begin by noting that since $f \in \mathcal{G}_{curl}$ satisfies $\widehat{f}(\xi) = e_\xi h(\xi)$ for some $h \in L_2$, we have

$$(f,f)_{\mathcal{G}_{curl}} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \frac{\|\widehat{f}(\xi)\|_2^2}{\|\xi\|_2^2 \widehat{\phi}(\xi)} d\xi. \tag{B.1}$$

Also, since ϕ is positive definite so is $-\Delta\phi$. Then that fact that it is continuous and L_1 integrable puts its Fourier transform is in $L_1(\mathbb{R}^n)$ [30, Corollary 6.12]. This means that $\widehat{f} \in L_1$ for all $f \in \mathcal{G}_{curl}$. Indeed, we have

$$\int_{\mathbb{R}^n} |\widehat{f}_j(\xi)| d\xi \leq \left(\int_{\mathbb{R}^n} \frac{|\widehat{f}_j(\xi)|_2^2}{\|\xi\|_2^2 \widehat{\phi}(\xi)} d\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} \|\xi\|_2^2 \widehat{\phi}(\xi) d\xi \right)^{1/2}.$$

This allows us to recover f point-wise from its Fourier transform by the inverse Fourier transform.

We now show that $(\cdot,\cdot)_{\mathcal{G}_{curl}}$ is an inner product. The linearity and conjugate symmetry properties are obvious. The fact that $\widehat{\phi}$ is positive along with (B.1) tells us $(f,f)_{\mathcal{G}_{curl}} = 0$ implies that $f = 0$. Thus $(\cdot,\cdot)_{\mathcal{G}_{curl}}$ is positive definite and hence an inner product.

To show completeness of $\mathcal{G}_{curl}$, suppose that $\{f_n\}$ is a Cauchy sequence in $\mathcal{G}_{curl}$. This means that the sequence $\left\{ \widehat{f}_n (\| \cdot \|_2^2 \widehat{\phi})^{-1/2} \right\}$ is Cauchy in L_2, and so it converges to a function $g \in L_2$. Note that the function g satisfies $g\sqrt{\| \cdot \|_2^2 \widehat{\phi}} \in L_1 \cap L_2$. Namely,

$$\int_{\mathbb{R}^n} \left| g_j(\xi) \sqrt{\|\xi\|_2^2 \widehat{\phi}(\xi)} \right| d\xi \leq \|g_j\|_{L_2} \left\| \| \cdot \|_2^2 \widehat{\phi} \right\|_{L_1}^{1/2}$$

and

$$\int_{\mathbb{R}^n} \left| g_j(\xi) \sqrt{\|\xi\|_2^2 \widehat{\phi}(\xi)} \right|^2 d\xi \leq \|g_j\|_{L_2}^2 \left\| \| \cdot \|_2^2 \widehat{\phi} \right\|_{L_\infty}.$$

for all $j = 1, \dots, n$. Thus

$$f(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left(g(\xi) \sqrt{\|\xi\|_2^2 \widehat{\phi}(\xi)} \right) e^{ix^T\xi} d\xi$$

is well defined, continuous, an element of L_2, and satisfies

$$\widehat{f}(\| \cdot \|_2^2 \widehat{\phi})^{-1/2} = g \in L_2. \tag{B.2}$$

To conclude that $f \in \mathcal{G}_{curl}$, we need to show that there is a scalar-valued L_2 function h such that $\widehat{f} = e_\xi \widehat{h}(\xi)$. Each f_n satisfies $\widehat{f_n}(\xi) = e_\xi h_n(\xi)$ for some scalar-valued $h_n \in L_2$. Since the sequence $\left\{ \widehat{f_n}(\| \cdot \|_2^2 \widehat{\phi})^{-1/2} \right\}$ converges to g in L_2, it follows that the sequence $\left\{ h_n(\| \cdot \|_2^2 \widehat{\phi})^{-1/2} \right\}$ converges to $e_\xi^T g(\xi)$ in L_2. This gives us that

$$h(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left(e_{xi}^T g(\xi) \sqrt{\|\xi\|_2^2 \widehat{\phi}(\xi)} \right) e^{ix^T\xi} d\xi$$

is well defined, continuous, an element of L_2, and satisfies $\widehat{h}(\xi)(\|\xi\|_2^2 \widehat{\phi}(\xi))^{-1/2} = e_\xi^T g(\xi)$. Now we have

$$\begin{aligned}
\|g - e_\xi e_\xi^T g\|_{L_2} &\le \|g - \widehat{f_n}(\| \cdot \|_2^2 \widehat{\phi})^{-1/2}\|_{L_2} + \|\widehat{f_n}(\| \cdot \|_2^2 \widehat{\phi})^{-1/2} - e_\xi \widehat{h}(\| \cdot \|_2^2 \widehat{\phi})^{-1/2})\|_{L_2} \\
&= \|g - \widehat{f_n}(\| \cdot \|_2^2 \widehat{\phi})^{-1/2})\|_{L_2} + \|e_\xi \widehat{h_n}(\| \cdot \|_2^2 \widehat{\phi})^{-1/2} - e_\xi \widehat{h}(\| \cdot \|_2^2 \widehat{\phi})^{-1/2})\|_{L_2} \\
&\le \|g - \widehat{f_n}(\| \cdot \|_2^2 \widehat{\phi})^{-1/2})\|_{L_2} + \|\widehat{h_n}(\| \cdot \|_2^2 \widehat{\phi})^{-1/2} - \widehat{h}(\| \cdot \|_2^2 \widehat{\phi})^{-1/2})\|_{L_2} \to 0.
\end{aligned}$$

Thus $g = e_\xi e_\xi^T g$, which implies that

$$g(\xi) = e_\xi \widehat{h}(\xi)(\|\xi\|_2^2 \widehat{\phi}(\xi))^{-1/2}.$$

Putting this together with (B.2) gives us that

$$\widehat{f}(\xi) = e_\xi \widehat{h}(\xi),$$

Which means that f is curl-free and $f \in \mathcal{G}_{curl}$. Finally, we show that $\{f_n\}_{n=1}^\infty$ converges to f in $\mathcal{G}_{curl}$.

$$\begin{aligned}\|f_n - f\|^2_{\mathcal{G}_{curl}} &= (2\pi)^{-n/4}\|\widehat{f}_n(\|\cdot\|_2^2\widehat{\phi})^{-1/2} - \widehat{f}(\|\cdot\|_2^2\widehat{\phi})^{-1/2}\|_{L_2} \\ &= (2\pi)^{-n/4}\|\widehat{f}_n(\|\cdot\|_2^2\widehat{\phi})^{-1/2} - g\|_{L_2} \to 0\end{aligned}$$

for $n \to \infty$. We conclude that $\mathcal{G}_{curl}$ is complete.

All that is left is to show that Φ_{curl} is the reproducing kernel of $\mathcal{G}_{curl}$. This follows from

$$\begin{aligned}(f, \Phi_{curl}(\cdot - y)c) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\left(\widehat{\Phi_{curl}}(\xi)ce^{-i\xi^T y}\right)^* \widehat{\Phi_{curl}}(\xi)^+\widehat{f}(\xi)d\xi \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} c^T\widehat{\Phi_{curl}}(\xi)\widehat{\Phi_{curl}}(\xi)^+\widehat{f}(\xi)e^{i\xi^T y}d\xi \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} c^T\left(e_\xi e_\xi^T\right)\widehat{f}(\xi)e^{i\xi^T y}d\xi \\ &= c^T\left((2\pi)^{-n/2}\int_{\mathbb{R}^n}\widehat{f}(\xi)e^{i\xi^T y}d\xi\right) = c^T f(x).\end{aligned}$$

□

Proof of Theorem 5

First we will need the following Lemma.

Lemma 8. *Let $g \in L_1(\mathbb{R}^n)$ such that supp$(g) \subset \Omega$, where Ω is a compact subset of $\mathbb{R}^n$. Then there exists a constant c depending only on Ω such that for all $\|\xi\|_2 \le 1$ we have*

$$|\widehat{g}(\xi) - \widehat{g}(0)| \le c\|\xi\|_2\|g\|_{L_1}. \tag{B.3}$$

Proof. A quick application of the Mean Value Theorem gives us the estimate $|1 - e^{-i\xi^T x}| \le \|\xi\|_2\|x\|_2$. Now we have

$$\begin{aligned}|\widehat{g}(\xi) - \widehat{g}(0)| &= \left|\int_\Omega g(x)\left(1 - e^{-i\xi^T x}\right)dx\right| \le \int_\Omega |g(x)|\|\xi\|_2\|x\|_2 dx \\ &= \|\xi\|_2\int_\Omega |g(x)|\|x\|_2 dx \le \left(\sup_{x\in\Omega}\|x\|_2\right)\|\xi\|_2\|g\|_{L_1}.\end{aligned}$$

Putting $c = \sup_{x\in\Omega} \|x\|_2$ finishes the proof. □

Now we can prove Theorem 5.

Proof. Let $\widetilde{H}^\tau(\Omega)$ be defined by

$$\widetilde{H}^\tau(\Omega) := \left\{ f|_\Omega,\ f \in \widetilde{H}^\tau(\mathbb{R}^n) \right\}.$$

By Proposition 1, if $g \in \widetilde{H}^\tau(\mathbb{R}^n)$ then $g \in H^\tau(\mathbb{R}^n)$. Thus $g|_\Omega \in H^\tau(\Omega)$ and $\widetilde{H}^\tau(\Omega) \subset H^\tau(\Omega)$. Also, we have

$$\|g\|_{H^\tau(\Omega)} \leq \|g\|_{H^\tau(\mathbb{R}^n)} \leq \|g\|_{\widetilde{H}^\tau(\mathbb{R}^n)}.$$

For the reverse direction, we must be able to extend every function $g \in H^\tau(\Omega)$ to a function $\widetilde{\mathfrak{E}}g \in \widetilde{H}^\tau(\mathbb{R}^n)$. For the norms to be equivalent, this extension needs to be continuous. We will be able to construct such an extension using the above Lemma and Stein's extension.

A useful fact about Stein's extension operator is that if V is any neighborhood of Ω so that $\bar{\Omega}$ is compact in V, then $\mathfrak{E}$ can be chosen so that the support of any extended function is contained in V. Since Ω is bounded we can choose V to be a large ball. Now fix a point x_0 so that if f is any function supported in V, then the supports of $f(x)$ and $f(x - x_0)$ do not intersect. For $g \in H^\tau(\Omega)$, we define our linear extension by

$$\widetilde{\mathfrak{E}}g(x) := \mathfrak{E}g(x) - \mathfrak{E}g(x - x_0).$$

By our choices of V and x_0 we have that $\widetilde{\mathfrak{E}}g|\Omega = g$, so $\widetilde{\mathfrak{E}}$ is an extension. It is linear since $\mathfrak{E}$ is. To show continuity, instead of the $\widetilde{H}^\tau$ norm we work with the equivalent norm given in (3.5). Since the support of g is compact and $g \in L_2(\mathbb{R}^n)$, then $g \in L_1(\mathbb{R}^n)$ and so is $\widetilde{\mathfrak{E}}g$. This makes its Fourier transform continuous, so $\widehat{\widetilde{\mathfrak{E}}g}(0)$ is well-defined. Our construction of $\widetilde{\mathfrak{E}}g$ shows that $\widehat{\widetilde{\mathfrak{E}}g}(0) = 0$. Now we may use the

Lemma to get

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\widehat{\widetilde{\mathfrak{E}}g}|^2}{\|\xi\|_2^2} d\xi &= \int_{\|\xi\|_2\le 1} \frac{|\widehat{\widetilde{\mathfrak{E}}g}|^2}{\|\xi\|_2^2} d\xi + \int_{\|\xi\|_2>1} \frac{|\widehat{\widetilde{\mathfrak{E}}g}|^2}{\|\xi\|_2^2} d\xi \\ &\le c \int_{\|\xi\|_2\le 1} \frac{\|\widetilde{\mathfrak{E}}g\|_{L_1}^2 \|\xi\|_2^2}{\|\xi\|_2^2} d\xi + \int_{\|\xi\|_2>1} |\widehat{\widetilde{\mathfrak{E}}g}|^2 d\xi \\ &\le c\|\widetilde{\mathfrak{E}}g\|_{L_1}^2 + \|\widetilde{\mathfrak{E}}g\|_{H^\tau(\mathbb{R}^n)}^2 \\ &\le c\|\widetilde{\mathfrak{E}}g\|_{H^\tau(\mathbb{R}^n)}^2.\end{aligned}$$

With this and the fact that $\mathfrak{E}$ is continuous in the H^τ-norm we get

$$\begin{aligned}\|\widetilde{\mathfrak{E}}g\|_{\widetilde{H}^\tau(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} \frac{\|\widehat{\widetilde{\mathfrak{E}}g}(\xi)\|_2^2}{\|\xi\|_2^2} d\xi + \|\widetilde{\mathfrak{E}}g\|_{H^\tau(\mathbb{R}^n)}^2 \\ &\le C\|\widetilde{\mathfrak{E}}g\|_{H^\tau(\mathbb{R}^n)}^2 = 4C\|\mathfrak{E}g\|_{H^\tau(\mathbb{R}^n)}^2 \le \tilde{C}\|g\|_{H^\tau(\Omega)}^2.\end{aligned}$$

Thus $\widetilde{\mathfrak{E}}$ is continuous and the result follows. □

Proof of Proposition 2

Proof. Containment in one direction is simple. If $f = \nabla \times g$ for some $g \in H^{\tau+1}$, we get $f \in H^\tau$, it is divergence-free, and

$$\int_{\mathbb{R}^n} \frac{\|\widehat{f}(\xi)\|_2^2}{\|\xi\|_2^2} d\xi \le C \int_{\mathbb{R}^n} \frac{\|\xi\|_2^2 \|\widehat{g}(\xi)\|_2^2}{\|\xi\|_2^2} d\xi = \|g\|_2^2 < \infty.$$

By Proposition 1, f is in $\widetilde{H}_{div}^\tau(\mathbb{R}^n)$. The curl-free case is similar.

To show the reverse direction, suppose that $f \in \widetilde{H}_{div}^\tau(\mathbb{R}^n)$. Using the fact that f is divergence-free, we get the equality

$$\xi \times \xi \times \widehat{f}(\xi) = \|\xi\|_2^2 \widehat{f}(\xi). \tag{B.4}$$

Thus we define g by

$$\widehat{g} := \frac{-i}{\|\xi\|_2} \left(\xi \times \widehat{f}(\xi)\right).$$

This function is in $H^{\tau+1}$. To see this, consider

$$\|g\|_{H^{\tau+1}}^2 = \int_{\mathbb{R}^n} \|\widehat{g}(\xi)\|_2^2 \left(1+\|\xi\|_2^2\right)^{\tau+1} d\xi \leq \int_{\mathbb{R}^n} \frac{\|\widehat{f}(\xi)\|_2^2}{\|\xi\|_2^2} \left(1+\|\xi\|_2^2\right)^{\tau+1} d\xi < \infty.$$

Also, using (B.4), we see that g satisfies $\nabla \times g = f$. This completes the divergence-free case.

For the curl-free case, $f \in \widetilde{H}^{\tau}_{curl}(\mathbb{R}^n)$ means there is a function $h \in H^\tau$ such that $\widehat{f}(\xi) = e_\xi \widehat{h}(\xi)$. We define g by

$$\widehat{g} := \frac{-i}{\|\xi\|_2} \widehat{h}(\xi).$$

To see that g is in $H^{\tau+1}$, consider

$$\|g\|_{H^{\tau+1}}^2 = \int_{\mathbb{R}^n} |\widehat{g}(\xi)|^2 \left(1+\|\xi\|_2^2\right)^{\tau+1} d\xi = \int_{\mathbb{R}^n} \frac{\|\widehat{f}(\xi)\|_2^2}{\|\xi\|_2^2} \left(1+\|\xi\|_2^2\right)^{\tau+1} d\xi < \infty.$$

Also, it is easily seen that $f = \nabla g$. Therefore we get containment both ways, and the proof is finished. $\square$

VITA

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