BACKWARD TIME BEHAVIOR OF DISSIPATIVE PDE

A Dissertation

by

RADU DASCALIUC

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2005

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Approved by:

Chair of Committee,	Ciprian Foias
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ABSTRACT

Backward Time Behavior of Dissipative PDE. (December 2005)

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Chair of Advisory Committee: Dr. Ciprian Foias

We study behavior for negative times t of the 2D periodic Navier-Stokes equations and Burgers' original model for turbulence. Both systems are proved to have rich sets of solutions that exist for all $t \in \mathbb{R}$ and increase exponentially as $t \to -\infty$. However, our study shows that the behavior of these solutions as well as the geometrical structure of the sets of their initial data are very different. As a consequence, Burgers original model for turbulence becomes the first known dissipative system that despite possessing a rich set of backward-time exponentially growing solutions, does not display any similarities, as $t \to -\infty$, to the linear case.

To the memory of my loving mother

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CHAPTER I

INTRODUCTION

A. Motivation

Forward time behavior of the dissipative partial differential equations (PDE) is characterized by the fact that all their solutions exist for all positive times and converge uniformly to a certain compact set called the global attractor ($\mathcal{A}$). This set can be described as the biggest bounded set invariant to the flow. As a consequence a lot of the studies of the dissipative systems are concentrated on the global attractor and the ways of approximating it. It is remarkable however, that many dissipative systems possess unbounded invariant sets, in particular the set of initial data for which the solutions exist for all times $t \in \mathbb{R}$ and increase exponentially as $t \to -\infty$. Existence of such a set is trivial in the linear case; its full decription is provided later in this chapter. Surprisingly, some nonlinear systems still retain a lot of similarities in their backward time dynamics with the linear case. These studies were pioneered in [1] where the two dimensional Navier-Stokes equations were considered.

Among other results, it was proved that a solution $u(t)$ increases exponentially as $t \to -\infty$ if and only if its Dirichlet quotient $|A^{1/2}u(t)|^2/|u(t)|^2 \to \lambda_n$ as $t \to -\infty$ (here $|\cdot|$ is the L^2-norm, A is the Stokes operator, and λ_n is one of its eigenvalues - see Section B.2 for more precise definitions). The invariant set $\mathcal{M}_n$ of all the trajectories of these solutions is proved to project entirely onto the spectral space associated with the first n eigenvalues of the Stokes operator (see [1]). This fact implied the only known partial answer to the Bardos-Tartar conjecture (see [1]). This conjecture (see [2]) affirms that the set of initial data for which solutions of the 2-D space periodic

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Navier-Stokes equations exist for all times is dense in the phase space equipped with the energy norm (a.e. the L^2-norm in this case). However, in [1] the density was proved in the norm $|A^{-1/2} \cdot |$.

The paper [1] also raised a number of questions regarding the geometric structure of $\mathcal{M}_n$. For example, it would be interesting to investigate the relationship between these sets and the other invariant sets of the Navier-Stokes equations, namely the global attractor and inertial manifolds.

Another open question is whether $\cup_n \mathcal{M}_n$ is dense in the energy norm of the phase space, which, if answered affirmatively, would solve the Bardos-Tartar conjecture in the energy norm. The study of higher order quotients on the sets $\mathcal{M}_n$ is of particular interest in this respect. In fact, a good result about boundedness of the quotients of the form $|A^\alpha u|^2/|u|^\beta$ would imply the desired density result for $\cup_n \mathcal{M}_n$ via the method presented in [1].

Similar results are established for the 2-D space periodic Navier-Stokes α-model and 2-D space periodic Kelvin-filtered Navier-Stokes equations. The analogs of the sets $\mathcal{M}_n$ defined for these systems have very similar properties compared to the Navier-Stokes case. In particular for the 2-D space periodic Navier-Stokes α-model, $\cup_n \mathcal{M}_n$ is dense in the L^2 norm, which is still weaker than the energy norm for that system (see [3]). On the other hand, for the 2-D space periodic Kelvin-filtered Navier-Stokes equations the density is proved in their energy norm (see [4]). The Lorenz system, after a suitable transformation, again displayed a similar picture (see [5]).

However, not all dissipative systems have the same kind of behavior for negative times. For example, in the case of the 1-D space periodic Kuramoto-Sivashinsky equation it was established that all the solutions outside the global attractor will blow up backward in finite time (see [6], [7]). That implied that all the invariant set of the Kuramoto-Sivashinky flow are contained inside the global attractor.

This dissertation is an attempt to answer some of the questions left by the previous studies as well as to resolve more general problems of backward time dynamics of dissipative systems. In particular, we were motivated by the following questions:

- What are the other dissipative PDE that have similar invariant sets?
- What are the geometrical properties of the sets $\mathcal{M}_n$?
- How to classify dissipative PDE by their backward-time dynamics?

After providing some preliminary results in Chapter I we discuss in Chapter II further developments in the 2D Navier-Stokes case. In particular, we show that is a solution increases exponentially as $t \to -\infty$, then in does so in any Sobolev norm allowed by the force. We also give a geometric characterization of the sets $\mathcal{M}_n$.

Chapter III deals with the Burgers' original model for turbulence. We show that this system displays significant differences in backward time behavior from both Navier-Stokes and Kuramoto-Sivashinsky equations. In particular, we show that while this model has a variety of the solutions that increase exponentially as $t \to -\infty$, their structure is completely different from the linear case.

We conclude with systematizing the results in the view of classifying the dissipative PDE by their backward time behavior, as well as pose some open questions and possible directions of further research.

These studies are important for several reasons.

First, in fluid mechanics, once the driving force is fixed, the Reynolds numbers of flows on the global attractor are bounded. The study of flows with huge Reynolds numbers automatically places their dynamics far away from the global attractor. To classify these flows, one needs to find invariant sets of the dynamical system in a neighborhood of the infinity in the phase space. In analogy with the local theory near a fixed point, one looks for the sets of solutions which exist for all times and have exponential growth for $t \to -\infty$. In fact, by an involution in the phase space, the study of the behavior near the infinity becomes equivalent to the study of the transformed dynamical system near a fixed point, for which the classical theory (because of singularities) does not apply. Nevertheless, even for some of these systems one can still prove existence of a rich exponentially stable invariant "manifold".

A second reason to find out if a dissipative system has global solutions with exponential growth as $t \to -\infty$ arose in the applications of the numerical methods introduced in [8] and [9]. These methods use the classical Caratheodory and Nevanlinna-Pick interpolation algorithms to decide if the point in the phase space is on the global attractor. It turned out that the algorithms have difficulty in distinguishing the solutions on the global attractor (i.e. globally bounded) from the global solutions with low exponential growth for $t \to -\infty$. In particular, the main result of Chapter III (Theorem F.1) suggests that the approximation of the global attractor of Burgers' original model for turbulence by the methods in [8] and [9] may be not sharp, while for the Kuramoto-Sivashinky equation they give a useful characterization of the global attractor attractor (see [7]).

B. Preliminaries and background

Many of the dissipative partial differential equations have the following form:

$$\begin{cases} u_t + \nu A u + B(u,u) = f \in H \\ u(0) = u_0 \in H, \end{cases} \tag{1.1}$$

where H is a suitable Hilbert space, $u(t)$ is the unknown function, $\nu > 0$ is a positive constant, A is an unbounded positive self-adjoint linear operator with a dense domain and a compact inverse, B is a nonlinearity of a quadratic type, u_0, f are given.

We denote by $\{\lambda_n\}_{n\in\mathbb{N}}$ the eigenvalues of the operator A arranged in the increasing order. For every $n \in \mathbb{N}$ we denote by P_n the orthogonal projector of the space H onto the space generated by the eigenvectors corresponding to $\lambda_1, \ldots, \lambda_n$. We also denote $R_n = P_n - P_{n-1}$, $n > 1$ ($R_1 = P_1$) the orthogonal projector onto the eigenspace corresponding to λ_n. Note that the structure of A implies that for every $n \in \mathbb{N}$, P_nH is a finite dimensional subspace of H and $\cup_{n\in\mathbb{N}} P_nH$ is dense in H.

Among the typical PDE that have the form (1.1) are the 2D periodic Navier-Stokes equations ([10, 11, 12, 13, 14, 15]), the 1D periodic Kuramoto-Sivashinsky equation ([14, 16, 17, 18]), and the original Burgers model for turbulence ([19, 20, 21, 22, 18, 23]). For each u_0 these equations have a unique solution $u(t) = S(t)u_0$ which exists for all times $t > 0$. Moreover, as $t \to \infty$, $S(t)u_0$ converges to the global attractor $\mathcal{A}$ that can be characterized as the set of all the initial data for which the solutions exist for all times and remain bounded as $t \to -\infty$ ([24, 14, 25, 26, 27]). The supplemental regularity properties of these equations imply that the uniqueness result for (1.1) is also valid backward in time, which allows to pose a problem of extension of $S(t)u_0$ for $t < 0$.

Definition B.1. *We say that a solution $u(t)$ of the equation (1.1) is global if it exists for all times, both positive and negative.*

Note that $u(t) = S(t)u_0$ is global if and only if

$$u_0 \in \bigcap_{t\geq 0} S(t)H.$$

Notation B.1. *Denote*

$$\mathcal{G} = \bigcap_{t\geq 0} S(t)H$$

the set of initial date for which $u(t) = S(t)u_0$ is global.

In this section we will provide an overview of the previous studies of the global solutions of (1.1)

1. Linear case

The simplest case, which is instructive in all the subsequent work is $B = 0$ in (1.1). This case was presented as motivation for the similar study of the 2-D Navier-Stokes equations in [1]. The system (1.1) becomes the linear system

$$\begin{cases} u_t + \nu A u = f \in H \\ u(0) = u_0 \in H. \end{cases} \tag{1.2}$$

The solution of this equation has the form

$$u(t) = S(t)u_0 = \frac{A^{-1}f}{\nu} + e^{-At}\left(u_0 - \frac{A^{-1}f}{\nu}\right). \tag{1.3}$$

Note that

$$\lim_{t\to\infty} u(t) = \frac{A^{-1}f}{\nu},$$

And thus the global attractor is

$$\mathcal{A} = \left\{\frac{A^{-1}f}{\nu}\right\}.$$

Besides the global attractor, equation (1.2) has a whole variety of global solutions. For example, if u_0 is such that

$$|R_n(u_0 - A^{-1}f/\nu)| \le e^{-\lambda_n s_n} b_n,$$

where $s_n \to \infty$ as $n \to \infty$ and $\sum_n b_n^2 < \infty$, then $u(t)$ given by (1.3) exists for all times and

$$|u(-s_n) - A^{-1}f/\nu|^2 \le \sum_{m=1}^{n} e^{2\lambda_n(s_n - s_m)} b_m^2 + \sum_{m=n+1}^{\infty} b_m^2.$$

Some of these solutions grow exponentially as $t \to -\infty$.

Theorem B.1. *A global solution $u(t)$ of (1.2) increases at most exponentially as $t \to -\infty$ if and only if there exists $n \in \mathbb{N}$ such that $u(t) - A^{-1}f/\nu \in P_nH$ for all $t \in \mathbb{R}$.*

Proof. It is clear that if $u_0 \in P_nH$, then $u(t)$ given by (1.3) exists for all $t \in \mathbb{R}$ with $u(t) - A^{-1}f/\nu \in P_nH$ and $|u(t)| = O(e^{-\lambda_n t})$ as $t \to -\infty$.

Conversely, if $u(t)$ increases at most exponentially as $t \to \infty$, then there exists $n \in \mathbb{N}$ such that $|u(t)| = O(e^{-\lambda_n t})$ as $t \to -\infty$. In this case, if $R_m(u_0 - A^{-1}f/\nu) \neq 0$ for some $m > n$, then

$$|u(t) - A^{-1}f/\nu|^2 \geq |R_m(u(t) - A^{-1}f/\nu)|^2 = e^{-2\lambda_m t}|R_m(u_0 - A^{-1}f/\nu)|^2 = O(e^{-\lambda_m t}),$$

which contradicts the fact that $|u(t)| = O(e^{-\lambda_n t})$ as $t \to -\infty$. Thus, $R_m(u_0 - A^{-1}f/\nu) = 0$ for all $m > n$, and consequently $u(t) - A^{-1}f/\nu \in P_nH$ for all $t \in \mathbb{R}$. □

Corollary B.1. *If $u(t) \notin \mathcal{A}$ of increases at most exponentially as $t \to -\infty$ then there exists $n \in \mathbb{N}$ such that*

$$u(t) - A^{-1}f/\nu \in P_nH \backslash P_{n-1}H;$$
$$\lim_{t \to -\infty} \frac{|A^{1/2}u(t)|^2}{|u(t)|^2} = \lambda_n;$$
$$|u(t)| = O(e^{-\lambda_n t}), \quad as \quad t \to -\infty$$

We denote

$$\mathcal{M}_n = \{u_0 \in \mathcal{G} : \ |S(t)u_0| = O(e^{-\lambda_n t}) \text{ as } t \to -\infty\}. \tag{1.4}$$

Note that for (1.2), the sets $\mathcal{M}_n$ have the following properties

$$\mathcal{M}_n = \frac{A^{-1}f}{\nu} + P_nH = \mathcal{A} \cup \left\{ u_0 \in \mathcal{G} : \ \lim_{t \to -\infty} \frac{|A^{1/2}S(t)u_0|^2}{|S(t)u_0|^2} = \lambda_n \right\}.$$

Also,

$$P_n\mathcal{M}_n = P_n H.$$

Thus, $\cup_n \mathcal{M}_n$ is dense in H.

2. 2-D Navier-Stokes equations

We consider the 2-D space periodic Navier-Stokes Equations (NSE) in $\Omega = [0, L]^2$:

$$\begin{cases} \frac{d}{dt}u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \\ \nabla\cdot u = 0 \\ u, p \quad \Omega\text{−periodic}, \int\limits_\Omega u = 0, \end{cases}$$

where $u(t) : \mathbb{R}^2 \to \mathbb{R}^2$, $p(t) : \mathbb{R}^2 \to \mathbb{R}$ are unknown functions and $\nu > 0$, $f \in L^2(\Omega)$ (f is Ω-periodic, $\int_\Omega f = 0$) are given.

Let H be the closure in $L^2(\Omega)^2$ of

$$\left\{ v \in L^2(\Omega)^2 : \; v \;\; \Omega\text{−periodic trigonometric polynomial}, \; \nabla\cdot v = 0, \; \int_\Omega v = 0 \right\}.$$

We denote

$$(v, w) := \int\limits_\Omega v\cdot w$$

and

$$|v| := (v, v)^{1/2}$$

the inner product and the norm in H.

Let $A = -P_L\Delta$ be the Stokes operator (defined on $D(A) = H \cap H^2(\Omega)^2$), where P_L is the orthogonal projection from $L^2(\Omega)^2$ onto H. Observe that $A : D(A) \to H$ is an unbounded positive self-adjoint operator with a compact inverse. Its eigenvalues are $(2\pi/L)^2(k_1^2 + k_2^2)$, where $(k_1, k_2) \in \mathbb{N}^2\backslash\{0, 0\}$. We arrange them in the increasing sequence:

$$(2\pi/L)^2 = \lambda_1 < \lambda_2 < \dots$$

We will need the following fact about $\{\lambda_n\}$ (see [24]).

$$\limsup_{n\to\infty}(\lambda_{n+1} - \lambda_n) = \infty.$$

Also, it is obvious that

$$\lambda_{n+1} - \lambda_n \geq \lambda_1, \quad n \geq 1,$$

and

$$\lim_{n\to\infty} \lambda_n = \infty.$$

Next we denote $B(u,v) = P_L((v \cdot \nabla)w))$ and $b(u,v,w) = (B(u,v),w)$, $u, w \in H$, $v \in D(A)$. Observe that

$$b(u,v,w) = -b(u,w,v), \quad u \in H,\ v,w \in D(A),$$

$$b(u,u,Au) = 0, \quad u \in D(A).$$

We will also use the following inequality for b:

$$|b(u,v,w)| \leq c_0|u|^{1/2}|A^{1/2}u|^{1/2}|A^{1/2}v|\,|w|^{1/2}|A^{1/2}w|^{1/2}, \tag{1.5}$$

where $u,v,w \in D(A^{1/2})(= H \cap H^1(\Omega)^2)$.

Finally, denote $g = P_L f$.

Then the NSE can be written as

$$\frac{d}{dt}u + \nu Au + B(u,u) = g. \tag{1.6}$$

We denote by $S(t)u_0$ the solution of the NSE which is u_0 at $t = 0$.

Let

$$\mathcal{A} = \left\{ u_0 \in \mathcal{G} : \sup_{t \in \mathbb{R}} |S(t)u_0| < \infty \right\} \tag{1.7}$$

be the global attractor of the equation (1.6). Refer to [10], [12] or [13] for the comprehensive treatment of the equation (1.6).

We will study the $S(t)$-invariant sets $\mathcal{M}_n$, which can be written as

$$\mathcal{M}_n = \mathcal{A} \bigcup \left\{ u_0 \in \mathcal{G} : \limsup_{t \to -\infty} \frac{|A^{1/2} S(t) u_0|^2}{|S(t)u_0|^2} \leq \frac{\lambda_n + \lambda_{n+1}}{2} := \overline{\lambda}_n \right\}. \tag{1.8}$$

We will use the following known facts about $\mathcal{M}_n$ (see [1]).

Theorem B.2. *The set $\cup_n \mathcal{M}_n$ is dense in H with the topology of the norm $|A^{-1/2} \cdot|$.*

Also:

- *$u(t) \in \mathcal{M}_n$ if and only if*

$$|u(t)| = O(e^{-\nu \lambda_n t}), \quad as\ t \to -\infty. \tag{1.9}$$

- *$u(t) \in \mathcal{M}_n \backslash \mathcal{M}_{n-1}$ if and only if*

$$\lim_{t \to -\infty} \frac{|A^{1/2} u(t)|^2}{|u(t)|^2} = \lambda_n. \tag{1.10}$$

Moreover in this case,

$$\liminf_{t \to -\infty} \frac{|u(t)|}{e^{-\nu \lambda_n t}} > 0; \tag{1.11}$$

also, there exists $t_n \to -\infty$, and an eigenvector $w \in R_n H$, $|w| = 1$ such that

$$\lim_{n \to \infty} \frac{u(t_n)}{|u(t_n)|} = w; \tag{1.12}$$

and, finally, if

$$|u_0| \geq \gamma_0 := \max \left\{ \frac{2|g|}{\nu \lambda_1}, \nu \right\} \tag{1.13}$$

then

$$\frac{|A^{1/2}S(t)u_0|^2}{|S(t)u_0|^2} \le \overline{\lambda}_n, \tag{1.14}$$

for all $t \le 0$.

3. Kuramoto-Sivashinsky equation

The 1-D periodic Kuramoto-Sivashinsky equation (KSE) in the domain $\Omega = [0, L]$ has the following form

$$\begin{cases} u_t + u_{xxxx} + u_{xx} + uu_x = 0 \\ u(0) = u_0(x) \\ u_0 \text{ is } L\text{-periodic}, \ \int_0^L u_0 \, dx = 0. \end{cases}$$

We define

$$H = \{u(x) : \ u \ - \ L - \text{periodic, locally in } L^2, \ \int_0^L u = 0\},$$

which is a Hilbert space with the inner product

$$(u, v) = \int_0^L u(x)v(x)\, dx.$$

For every $u_0 \in H$, KSE has a unique solution $u(t) = S(t)u_0$ that exist for all $t > 0$. This solution, for $t > 0$, is an analytic function in time and space [14].

There is an important difference of KSE from the NSE-like dissipative systems, namely its linear part $u_{xxxx} + u_{xx}$ is not positive when $L > 2\pi$, in fact, the low modes of the solution of KSE become unstable, and, as a result, the dissipativity of this equation does not follow similarly to the NSE case. Nevertheless, the dissipativity of KSE for odd initial data was established in [16], while in [17] and [28] the oddness condition was removed.

In particular, it was proved that there exists $R_L \geq 0$ such that for every $u_0 \in H$, there exists $T(|u_0|) > 0$ such that $|S(T(|u_0|))u_0| < R_L$ and $|S(t)u_0| < \sqrt{2}R_L$ for all $t \geq T(u_0)$. Moreover, KSE has a global attractor

$$\mathcal{A} = \{u_0 \in \mathcal{G} : \sup_{t\in\mathbb{R}} |S(t)u| < \infty\}.$$

However, as was shown in [7], KSE does not have any other global solutions besides those on the global attractor.

Theorem B.3. *Any solution of the KSE which does not belong to the global attractor cannot be extended for all negative times.*

As a consequence ([6]), there are no solutions growing exponentially for $t \to -\infty$.

Corollary B.2. *For all* $n \in \mathbb{N}$,

$$\mathcal{M}_n = \mathcal{A}.$$

Also, Theorem B.3 implies the following fact about invariant sets of the KSE ([7]).

Corollary B.3. *The global attractor is the biggest invariant set of the KSE.*

The method provided in [7] can be easily adapted to prove the same results for the generalized Burgers equation

$$u_t - u_{xx} - \beta u + uu_x = f$$

with the parameter $\beta \in \mathbb{R}$ and the same boundary conditions as KSE.

CHAPTER II

2D NAVIER-STOKES EQUATION*

As we mentioned in the introduction, the NSE is a typical case of dissipative system (1.1), which for $t \to -\infty$ display many similarities to the linear case (see Section I.B.2, for the preliminary information and the summary of the known facts about the NSE system). In particular, the it was proved that $u(t)$ grows exponentially as $t \to -\infty$ if and only its Dirichlet quotient $|A^{1/2}u(t)|^2/|u(t)|^2$ is bounded as $t \to -\infty$ (see [1]).

In this chapter we prove that the quotients $|A^\alpha u|^2/|u|^{4\alpha}$ are bounded on any $\mathcal{M}_n$. (see Theorem A.1 and its Corollary A.1). Our bounds, however, are not sufficient to prove the density of $\cup_n \mathcal{M}_n$ in the energy norm of the phase space. But as a corollary we show that if a solution of the 2-D space periodic Navier-Stokes equation exists for all times and increases exponentially in the energy norm (as $t \to -\infty$), than it increases exponentially in any Sobolev norm, provided the driving force is regular (see Corollary A.2). In particular, the L^∞ norm of any derivative of such a solution grows at most exponentially as $t \to -\infty$.

It is worth mentioning that by a slight modification of the proofs given in this paper one can prove similar results for the 2-D space periodic Navier-Stokes α-model and 2-D space periodic Kelvin-filtered Navier-Stokes equations.

A. Main result

For every $\theta \geq 0$ and $g \in D(A^\theta)$ define a generalized Grashoff number as follows

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*Some of the results in this chapter are reprinted with permission from R. Dascaliuc, "On backward-time behavior of the solutions to the 2-D space periodic Navier-Stokes equations", *Ann. Inst. H. Poincaré Anal. Non Lináire* vol. 22, Iss. 4, pp. 385-519,© 2005, available on-line at www.sciencedirect.com.

$$G_\theta = \frac{|A^\theta g|}{\nu^2 \lambda_1^{\theta+1}}, \tag{2.1}$$

Our main goal is to prove the following

Theorem A.1. *Let $\theta = k/2$, $k \in \mathbb{N}\backslash\{0\}$, and $g \in D(A^\theta)$. Then for every $u_0 \in \mathcal{M}_n$ such that $|u_0| \geq \gamma_0$, there exists a positive constant $M_\theta(G_\theta)$ depending only θ, c_0 (where c_0 the constant from (1.5)), and G_θ such that*

$$\frac{|A^\theta u_0|^2}{|u_0|^{4\theta}} \leq \frac{M_\theta(G_\theta)}{\nu^{4\theta-2}} \overline{\lambda_n}^{2\theta}. \tag{2.2}$$

Moreover, if $\theta > 1$ than there exists a positive constant $N_\theta(G_{\theta-1/2})$, that depends only on θ, c_0, and $G_{\theta-1/2}$ such that

$$\int_{-\infty}^{t_0} \frac{|A^\theta u|^2}{|u|^{4\theta-2}} d\tau < \frac{N_\theta(G_{\theta-1/2})}{\nu^{4\theta-3}} \overline{\lambda_n}^{2\theta-1}, \tag{2.3}$$

where $u(t)$ is a solution of the NSE satisfying $u(t_0) = u_0$.

Also, if $\theta \geq 1/2$ then

$$\lim_{t\to-\infty} \frac{|A^\theta u(t)|^2}{|u(t)|^{4\theta}} = 0. \tag{2.4}$$

Observe that (2.2) expands the estimate (1.14) from Theorem B.2 to the quotients involving higher powers of the operator A. In fact, these estimates hold for any power of the operator.

Corollary A.1. *Let $\alpha > 1/2$ and $g \in D(A^\theta)$, where $\theta = ([2\alpha]+1)/2$. Then for every $u_0 \in \mathcal{M}_n$ with $|u_0| > \gamma_0$, there is a constant M_α (depending only on θ, $G_{([2\alpha]+1)/2}$,and c_0) such that*

$$\frac{|A^\alpha u_0|^2}{|u_0|^{4\alpha}} \leq \frac{M_\alpha}{\nu^{4\alpha-2}} \overline{\lambda_n}^{2\alpha}. \tag{2.5}$$

Proof. Let $\theta = ([2\alpha]+1)/2$. Observe that $\theta \geq \alpha$. Then, by interpolation,

$$\begin{aligned}\frac{|A^\alpha u_0|^2}{|u_0|^{4\alpha}} &\leq \left(\frac{|A^\theta u_0|^2}{|u_0|^{4\theta}}\right)^{\frac{2\alpha-1}{2\theta-1}} \left(\frac{|A^{1/2} u_0|^2}{|u_0|^2}\right)^{\frac{2\theta-2\alpha}{2\theta-1}} \\ &\leq \left(\frac{M_\theta}{\nu^{4\theta-2}} \overline{\lambda_n}^{2\theta}\right)^{\frac{2\alpha-1}{2\theta-1}} \overline{\lambda_n}^{\frac{2\theta-2\alpha}{2\theta-1}} = \frac{M_\theta^{\frac{2\alpha-1}{2\theta-1}}}{\nu^{4\alpha-2}} \overline{\lambda_n}^{2\alpha},\end{aligned}$$

and thus, (2.5) holds with $M_\alpha = M_\theta^{\frac{2\alpha-1}{2\theta-1}}$.

□

Another consequence of Theorem A.1 is that on $\mathcal{M}_n$ any Sobolev norm of a solution will grow exponentially for negative time.

Corollary A.2. *Suppose $u(t) \in \mathcal{M}_n \backslash \mathcal{A}$ and $g \in D(A^{m/2})$ then*

$$|A^{m/2} u(t)|^2 \leq O(e^{-2m\nu\lambda_n t}), \ t \to -\infty.$$

Moreover, if $m \geq 2$, then for any $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 \leq m-2$, we have

$$|D^\alpha u|_{L^\infty} = O(e^{-(\alpha_1+\alpha_2+2)\nu\lambda_n t}), \ t \to -\infty,$$

where

$$D^\alpha u(x_1, x_2) = \frac{\partial^{\alpha_1+\alpha_2} u}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2}.$$

In particular, when $g \in C^\infty(\Omega)$, any solution u of the NSE which exists for all times and increases exponentially as $t \to -\infty$ in the phase space H, will also increase exponentially as $t \to -\infty$ in any Sobolev space $H^m_{per}(\Omega)^2 = W^{2,m}_{per}(\Omega)^2$ ($m \geq 0$). Moreover, the L^∞ norm of any (space) derivative of u will also increase exponentially as $t \to -\infty$.

Proof. Recall that the Sobolev norm in $H^m_{per}(\Omega)$ is equivalent to the norm

$$|\cdot|_m := \left(|A^{m/2} \cdot |^2\right)^{1/2}.$$

Note that by Theorem B.2 $u(t) \in \mathcal{M}_n \backslash \mathcal{A}$ implies that $|u(t)|_m$ grows at least exponentially as $t \to -\infty$, and $|u(t)|^2 = O(e^{-\nu\lambda_n t})$ as $t \to -\infty$.

On the other hand, according to Theorem A.1,

$$|A^{m/2}u(t)|^2 \leq \frac{M_{m/2}}{\nu^{2m-2}} \lambda_n^k |u|^{2m} = O(e^{-2m\nu\lambda_n t}).$$

Thus, $u(t)$ increases exponentially in $H^m_{per}(\Omega)^2$ as $t \to -\infty$.

To prove the second part of the corollary we apply the Sobolev Embedding Theorem to obtain that

$$|D^\alpha u|_\infty \leq C|D^\alpha u|_{H^2(\Omega)},$$

for any multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$. Here we are writing

$$D^\alpha u(x_1, x_2) = \frac{\partial^{\alpha_1+\alpha_2} u}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2}.$$

Observe that by the first part of the corollary, $|D^\alpha u(t)|_{H^2(\Omega)} = O(e^{-(\alpha_1+\alpha_2+2)\nu\lambda_n t})$ as $t \to -\infty$. Consequently, we also have that $|D^\alpha u|_\infty = O(e^{-(\alpha_1+\alpha_2+2)\nu\lambda_n t})$ as $t \to -\infty$.

□

B. The proof of the main result

For convenience we will use the following notation:

Notation B.1.

$$\lambda := \frac{|A^{1/2}u|^2}{|u|^2},$$

$$\mu := \frac{|Au|^2}{|u|^4},$$

$$\xi := (A - \lambda)\frac{u}{|u|},$$

$$\sigma := (A - \frac{3}{2}\lambda)\frac{A^{1/2}u}{|u|^2},$$

$$\overline{\lambda}_n := \frac{\lambda_{n+1} + \lambda_n}{2},$$
$$\mu_{\theta,m} := \frac{|A^\theta u|^2}{|u|^m}.$$

First, we will prove the following useful lemma.

Lemma B.1. *Let u be a solution of the NSE that exists for all times and satisfies $|u(t_0)| > \gamma_0$ for some t_0. Then for any $t \leq t_0$ and any $m \geq 1$,*

$$\frac{2}{3m}\frac{1}{|u(t)|^m} \leq \nu \int_{-\infty}^{t} \frac{\lambda(\tau)}{|u(\tau)|^m} d\tau \leq \frac{2}{m}\frac{1}{|u(t)|^m}. \tag{2.6}$$

Also, if $u(t_0) \in \mathcal{M}_n \backslash \mathcal{A}$, then

$$\nu \int_{-\infty}^{t} \lambda(\tau)|\xi(\tau)|^2 d\tau \leq \frac{1}{2}(\lambda_n^2 - \lambda^2(t)) + \frac{|g|^2}{\nu^2|u(t)|^2}. \tag{2.7}$$

and

$$\nu \int_{-\infty}^{t} \mu(\tau)\, d\tau \leq O\left(\frac{1}{|u(t)|^2}\right), \text{ for } t \to -\infty. \tag{2.8}$$

Proof. From (1.6) we obtain

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \nu\lambda|u|^2 = (g, u), \tag{2.9}$$

from which we get

$$\frac{1}{|u|^{m+1}}\frac{1}{2}\frac{d}{dt}|u| + \nu\frac{\lambda}{|u|^m} = \left(g, \frac{u}{|u|^{m+2}}\right).$$

Thus

$$\nu \int_{-\infty}^{t} \frac{\lambda}{|u|^m} d\tau - \int_{-\infty}^{t} \left(g, \frac{u}{|u|^{m+2}}\right) d\tau = \frac{1}{m}\frac{1}{|u(t)|^m}. \tag{2.10}$$

Notice that for $t \leq t_0$

$$\left|\int_{-\infty}^{t} \left(g, \frac{u}{|u|^{m+2}}\right) d\tau\right| \leq \int_{-\infty}^{t} \frac{g}{|u|^{m+1}} d\tau \leq \int_{-\infty}^{t} \left(\frac{g}{\nu\lambda_1}\frac{1}{|u|}\right)\frac{1}{|u|^m} d\tau$$

$$\leq \frac{1}{2}\nu \int_{-\infty}^{t} \frac{\lambda_1}{|u|^m} d\tau \leq \frac{1}{2}\nu \int_{-\infty}^{t} \frac{\lambda}{|u|^m} d\tau,$$

since $|u(t)| \geq \gamma_0 (\geq 2g/(\nu\lambda_1))$ and $\lambda(t) \geq \lambda_1$ for all $t \leq t_0$. Thus, returning to (2.10) we get

$$\nu \int_{-\infty}^{t} \frac{\lambda}{|u|^m} d\tau - \frac{1}{2}\nu \int_{-\infty}^{t} \frac{\lambda}{|u|^m} d\tau \leq \frac{1}{m} \frac{1}{|u(t)|^m}$$

and

$$\nu \int_{-\infty}^{t} \frac{\lambda}{|u|^m} d\tau + \frac{1}{2}\nu \int_{-\infty}^{t} \frac{\lambda}{|u|^m} d\tau \geq \frac{1}{m} \frac{1}{|u(t)|^m}$$

for all $t \leq t_0$, from which the relation (2.6) readily follows.

In order to prove (2.7), we observe that

$$\frac{1}{2}\frac{d}{dt}|A^{1/2}u|^2 + \nu|Au|^2 = (g, Au), \tag{2.11}$$

which, together with (2.9), implies that

$$\frac{1}{2}\frac{d}{dt}\lambda + \nu|\xi|^2 = (\frac{g}{|u|}, \xi),$$

from which we obtain

$$\frac{1}{2}\frac{d}{dt}\lambda^2 + \nu\lambda|\xi|^2 \leq \frac{\lambda|g|^2}{\nu|u|^2}.$$

By integrating the relation above, using (2.6) for $m = 2$ as well as the fact that $\lambda(t) \to \lambda_n$ as $t \to -\infty$ (see results from [1] summarized in Theorem B.2), we get

$$\frac{1}{2}(\lambda^2(t) - \lambda_n^2) + \nu \int_{-\infty}^{t} \lambda|\xi|^2 d\tau \leq \frac{|g|^2}{\nu^2|u(t)|^2},$$

which implies the inequality (2.7) from the statement of the lemma.

Finally, to prove (2.8) consider

$$\frac{1}{2}\frac{d}{dt}\frac{\lambda}{|u|^2} = \frac{-\nu|Au|^2 + (g, Au)}{|u|^4} - 2\frac{\lambda}{|u|^2}\frac{-\nu|A^{1/2}u|^2 + (g, u)}{|u|^2},$$

from where

$$\frac{1}{2}\frac{d}{dt}\frac{\lambda}{|u|^2} \leq -\nu\mu + \frac{|g|^2}{|u|^2}\mu^{1/2} + 2\nu\frac{\lambda^2}{|u|^2} + 2\frac{\lambda|g|}{|u|^3}.$$

Consequently

$$\frac{d}{dt}\frac{\lambda}{|u|^2} + \nu\mu \leq \frac{|g|^2}{|u|^2} + 4\nu\frac{\lambda^2}{|u|^2} + 4\frac{\lambda|g|}{|u|^3}.$$

Thus, by itegrating the previous inequality and using (2.6) we obtain

$$\nu\int_{-\infty}^{t}\mu(\tau)\,d\tau \leq O\left(\frac{1}{|u(t)|^2}\right) \text{ for } t \to -\infty.$$

□

Let $u(t)$ be a solution of the NSE such that $u(t) \in \mathcal{M}_n$. Our first result is

Proposition B.1. *If $g \in D(A)$ and*

$$|u(0)| \geq \gamma_0,$$

then for every $t \leq 0$ we have

$$\mu(t) + e^{-3}\nu\int_{-\infty}^{t}\lambda(\tau)\mu(\tau)\,d\tau \leq \frac{e^4}{2\nu^2}(\lambda_n^2 - \lambda^2(t)) + \frac{K_1\lambda_1^2 + (13/4)e^4\overline{\lambda}_n^2}{|u(t)|^2},$$

where $K_1 = e^4(c_0G_0 + G_1)$ with c_0- the constant from the inequality (1.5). Moreover,

$$\frac{\nu}{4}e^{-3}\int_{-\infty}^{t}\frac{|A^{3/2}u|^2}{|u|^4}d\tau \leq \frac{e^4}{2\nu^2}(\lambda_n^2 - \lambda^2(t)) + \frac{K_1\lambda_1^2 + (13/4)e^4\overline{\lambda}_n^2}{|u(t)|^2},$$

for any $t \leq 0$.

Proof. Observe that since $g \in D(A)$, we have

$$\frac{1}{2}\frac{d}{dt}\mu = \frac{-\nu|A^{3/2}u|^2 - b(Au,u,Au) + (Ag,Au)}{|u|^4} - 2\mu\frac{-\nu|A^{1/2}u|^2 + (g,u)}{|u|^2},$$

so,

$$\frac{1}{2}\frac{d}{dt}\mu = -\nu\left(\mu_{3/2,4} - 2\mu\lambda\right) - \frac{b(Au,u,Au)}{|u|^4} + \frac{(Ag,Au)}{|u|^4} - 2\mu\frac{(g,u)}{|u|^2}.$$

Thus,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mu + \nu\lambda\mu &= -\nu|\sigma|^2 + \frac{9\nu}{4}\frac{\lambda^3}{|u|^2} - \frac{b(Au,u,Au)}{|u|^4} \\ &+ \left(\frac{Ag}{|u|^2}, \frac{Au}{|u|^2}\right) - 2\mu\left(g, \frac{u}{|u|^2}\right).\end{aligned} \tag{2.12}$$

Note that

$$\begin{aligned}\frac{|b(Au,u,Au)|}{|u|^4} &= \frac{b(Au-\lambda u,u,Au-\lambda u)}{|u|^4} \leq \frac{c_0|\xi|\,|A^{1/2}\xi|\,|A^{1/2}u|}{|u|^2} \\ &= c_0\lambda^{1/2}|\xi|\frac{|A^{1/2}\xi|}{|u|} = c_0\lambda^{1/2}|\xi|\left(|\sigma|^2 + \lambda\mu - \frac{5}{4}\frac{\lambda^3}{|u|^2}\right)^{1/2},\end{aligned}$$

and thus

$$\frac{|b(Au,u,Au)|}{|u|^4} \leq \frac{c_0^2}{2\nu}\lambda|\xi|^2 + \frac{\nu}{2}\left(|\sigma|^2 + \lambda\mu - \frac{5}{4}\frac{\lambda^3}{|u|^2}\right).$$

Now, going back to (2.12) we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\mu + \nu\lambda\mu &\le -\frac{\nu}{2}|\sigma|^2 + \frac{\nu}{2}\lambda\mu - \frac{5\nu}{8}\frac{\lambda^3}{|u|^2} + \frac{9\nu}{4}\frac{\lambda^3}{|u|^2} \\ &+ \frac{c_0\lambda}{2\nu}|\xi|^2 + \frac{|Ag|}{|u|^2}\mu^{1/2} - 2\mu\left(g, \frac{u}{|u|^2}\right). \end{aligned}$$

Observe that

$$\frac{|Ag|}{|u|^2}\mu^{1/2} \le \frac{1}{2\nu^3\lambda_1}\frac{|Ag|^2}{|u|^2} + \frac{\nu^3\lambda_1}{2}\frac{\mu}{|u|^2}.$$

Consequently

$$\begin{aligned} \frac{d}{dt}\mu + \nu\lambda\mu &\le \left[\frac{\nu^3\lambda_1}{|u|^2} - 4\left(g, \frac{u}{|u|^2}\right)\right]\mu \\ &- \nu|\sigma|^2 + \frac{c_0\lambda}{\nu}|\xi|^2 + \frac{1}{|u|^2}\left(\frac{13\nu}{4}\lambda^3 + \frac{|Ag|^2}{\nu^3\lambda_1}\right). \end{aligned}$$

Note that the conditions of the proposition imply that $\lambda(t) \le \overline{\lambda}_n$, for all $t \le 0$. Let us denote

$$\Gamma_n := \frac{13\nu}{4}\overline{\lambda}_n^2 + \frac{|Ag|^2}{\nu^3\lambda_1^2},$$

$$\beta := \frac{\nu^3\lambda_1}{|u|^2} - 4\left(g, \frac{u}{|u|^2}\right).$$

Then, by the Gronwall inequality,

$$\mu(t) \le \mu(t_0)e^{\int_{t_0}^t \beta} + \int_{t_0}^{t} \left(-\nu|\sigma|^2 - \nu\lambda\mu + \frac{c_0}{\nu}\lambda|\xi|^2 + \frac{\Gamma_n\lambda}{|u|^2}\right) e^{\int_\tau^t \beta} d\tau. \tag{2.13}$$

Observe that see Theorem B.2,

$$\liminf_{t\to-\infty}\frac{|u(t)|^2}{e^{-\nu\lambda_1 t}}>0,$$

and so $\beta(\tau)$ is bounded and absolutely integrable on the interval $(-\infty,t]$. Moreover, by Lemma B.1, $(c_0\nu)\lambda|\xi|^2+\Gamma_n\lambda/|u|^2$ is also absolutely integrable on $(-\infty,t]$. On the other hand, from (2.8) we conclude that there exists a sequence $t_n^0\to-\infty$ such that $\mu(t_n^0)\to 0$. Thus, by taking $t_0=t_n^0$ and letting $n\to\infty$, the inequality (2.13) yields:

$$\mu(t)+c_1\nu\int_{-\infty}^{t}(|\sigma|^2+\lambda\mu)d\tau\le c_2\int_{-\infty}^{t}\left(\frac{c_0}{\nu}\lambda|\xi|^2+\frac{\Gamma_n\lambda}{|u|^2}\right)d\tau, \tag{2.14}$$

where

$$c_1(t)=\inf_{\tau\le t}e^{\int_{-\infty}^{\tau}\beta},$$

and

$$c_2(t)=\sup_{\tau\le t}e^{\int_{-\infty}^{\tau}\beta}.$$

Observe that the relation (2.6) from Lemma B.1 implies that

$$\int_{-\infty}^{t}\frac{\Gamma_n\lambda}{|u|^2}d\tau\le\frac{\Gamma_n}{\nu|u(t)|^2}.$$

Using this, together with (2.7), in the inequality (2.14), we obtain

$$\mu(t)+c_1\nu\int_{-\infty}^{t}(|\sigma|^2+\lambda\mu)d\tau\le\frac{c_2c_0}{\nu^2}\left(\frac{1}{2}\left(\lambda_n^2-\lambda^2(t)\right)+\frac{|g|^2}{\nu^2|u(t)|^2}\right)+\frac{c_2\Gamma_n}{\nu|u(t)|^2}. \tag{2.15}$$

Observe that from (2.15) we can infer that $\mu(t)$ is bounded, while $|\sigma|^2$ and $\lambda\mu$ are integrable on $(-\infty, t]$, and thus

$$c_1\nu \int_{-\infty}^{t} \frac{|A^{3/2}u|^2}{|u|^4} d\tau \leq c_1\nu \int_{-\infty}^{t} 3\mu\lambda \, d\tau + \frac{c_2 c_0}{2\nu^2}\left(\lambda_n^2 - \lambda^2(t)\right) + \left(\frac{c_0|g|^2}{\nu^3} + \Gamma_n\right)\frac{c_2}{\nu|u(t)|^2}$$

Using (2.15) again to estimate $c_1\nu \int_{-\infty}^{t} 3\mu\lambda \, d\tau$, we obtain

$$c_1\nu \int_{-\infty}^{t} \frac{|A^{3/2}u|^2}{|u|^4} d\tau \leq \frac{2c_2 c_0}{\nu^2}\left(\lambda_n^2 - \lambda^2(t)\right) + \left(\frac{c_0|g|^2}{\nu^3} + \Gamma_n\right)\frac{4c_2}{\nu|u(t)|^2} < \infty. \tag{2.16}$$

Observe that from (2.6) we obtain that

$$\nu \int_{-\infty}^{t} \frac{\lambda}{|u|} d\tau \leq \frac{3}{2|u(t)|}.$$

Hence,

$$c_1 > e^{-\int_{-\infty}^{0} \frac{4|g|}{|u|}} \geq e^{\frac{-6|g|}{\nu\lambda_1|u(0)|}} \geq e^{-3}$$

and, since $|u| \geq \gamma_0 \geq \nu$,

$$c_2 \leq e^{\int_{-\infty}^{t}\left(\frac{\nu^3\lambda_1}{|u|^2} + \frac{4|g|}{|u|}\right)} \leq e^{\frac{\nu^2}{|u(t)|^2} + \frac{6|g|}{\nu\lambda_1|u(t)|}} \leq e^{1+3} = e^4.$$

Finally, if we define

$$K_1(G_0, G_1) := e^4(c_0 G_0 + G_1),$$

and use (2.15) and (2.16) we will obtain the desired estimates from the proposition.

□

The following proposition allows us to deduce the boundedness of higher order quotients based on the boundedness on the lower ones.

Proposition B.2. *Let $g \in D(A^{\theta+1/2})$ with $\theta = k/2$ and $k \in \mathbb{N}$. Suppose that for every $u(t) \in \mathcal{M}_n$, $|u(t_0)| \geq \gamma_0$ we have*

$$\int_{-\infty}^{t_0} \frac{|A^\theta u(t)|^2}{|u(t)|^m} dt < \frac{C_{\theta,m}(G_{\theta-1/2})}{\nu^{m-1}} \overline{\lambda_n}^{k-1},$$

where $C_{\theta,m}(\cdot)$ is a positive increasing function. Then there exist positive increasing functions $C'_{\theta,m}(\cdot), K_{\theta,m}(\cdot)$, such that

$$\frac{|A^\theta u(t_0)|^2}{|u(t_0)|^{m+2}} + \nu \int_{-\infty}^{t_0} \frac{|A^{\theta+1/2} u(t)|^2}{|u(t)|^{(m+2)}} dt < \frac{C'_{\theta,m}(G_\theta)}{\nu^m} \overline{\lambda_n}^{k}$$

and

$$\frac{|A^{\theta+1/2} u(t)|^2}{|u(t)|^{(m+4)}} \leq \frac{K_{\theta,m}(G_{\theta+1/2})}{\nu^{m+2}} \overline{\lambda_n}^{k+1}.$$

Moreover,

$$\lim_{t\to-\infty} \frac{|A^{\theta+1/2} u(t)|^2}{|u(t)|^{(m+4)}} = \lim_{t\to-\infty} \frac{|A^\theta u(t)|^2}{|u(t)|^{(m+2)}} = 0.$$

Proof. Using (1.6) we get the following equation for the Galerkin approximations u^N (see [10] for the facts about the Galerkin approximations for the NSE)

$$\frac{1}{2}\frac{d}{dt}\mu^N_{\theta,m+2} = \frac{-\nu|A^{\theta+1/2}u^N|^2 + (g, A^{2\theta}u^N) - b(u^N, u^N, A^{2\theta}u^N)}{|u^N|^{(m+2)}} + \frac{m+2}{2}\mu^N_{\theta,m+2}\frac{\nu|A^{1/2}u^N|^2 - (g, u^N)}{|u^N|^2}.$$

Applying Theorem 1 from the Appendix as well as the Cauchy-Schwarz inequality, we get

$$\frac{1}{2}\frac{d}{dt}\mu^N_{\theta,m+2} \leq -\nu\mu^N_{\theta+1/2,m+2} + \frac{|A^\theta g|}{|u^N|^{(m+2)/2}}{\mu^N_{\theta,m+2}}^{1/2}$$
$$+c_0 c_{2\theta}\frac{|A^{\theta+1/2}u^N|\,|A^\theta u^N|\,|A^{1/2}u^N|}{|u^N|^{(m+2)}} + \nu\frac{m+2}{2}\lambda\mu^N_{\theta,m+2} + \frac{m+2}{2}\frac{|g|}{|u^N|}\mu^N_{\theta,m+2}$$

(here $c_{2\theta} = 6([\theta] + (2\theta - [\theta])2^{2\theta-2})$ is the constant from Theorem 1). Now, using the

Jensen inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\mu^N_{\theta,m+2} \leq -\frac{\nu}{2}\mu^N_{\theta+1/2,m+2} + \frac{1}{2\nu\lambda_1}\frac{|A^\theta g|^2}{|u^N|^{(m+2)}} + \frac{\nu\lambda_1}{2}\mu^N_{\theta,m+2}$$
$$+\frac{c_0^2c_{2\theta}^2}{2\nu}\lambda^N\mu^N_{\theta,m} + \nu\frac{m+2}{2}\lambda^N\mu^N_{\theta,m+2} + \frac{m+2}{2}\frac{|g|}{|u^N|}\mu^N_{\theta,m+2}.$$

Hence,

$$\frac{d}{dt}\mu^N_{\theta,m+2} + \nu\mu^N_{\theta+1/2,m+2} \leq \frac{|A^\theta g|^2}{\nu\lambda_1|u^N|^{(m+2)}}$$
$$+\left(\frac{\nu\lambda_1}{|u^N|^2} + \frac{c_0^2c_{2\theta}^2}{\nu}\lambda^N + \frac{\nu(m+2)\lambda^N}{|u^N|^2} + \frac{(m+2)|g|}{|u^N|^3}\right)\mu^N_{\theta,m}.$$

Since $g \in D(A^{\theta+1/2})$, we can integrate from t to t_0 ($t < t_0$) and pass to the limit $N \to \infty$. Taking into the account that $\lambda(t) \leq \overline{\lambda}_n$ we get

$$\mu_{\theta,m+2}(t_0) + \nu\int\limits_t^{t_0}\mu_{\theta+1/2,m+2}d\tau \leq \mu_{\theta,m+2}(t) + \frac{|A^\theta g|^2}{\nu\lambda_1}\int_t^{t_0}\frac{d\tau}{|u|^{m+2}}$$
$$+\left[\frac{c_0^2c_{2\theta}^2}{\nu}\overline{\lambda}_n + \frac{1}{|u(t_0)|^2}\left(\nu\lambda_1 + \nu(m+2)\overline{\lambda}_n + \frac{(m+2)|g|}{|u(t_0)|}\right)\right]\int\limits_t^{t_0}\mu_{\theta,m}d\tau.$$

Since
$$\int\limits_{-\infty}^{t_0}\mu_{\theta,m}d\tau < \frac{C_{\theta,m}(G_{\theta-1/2})}{\nu^{m-1}}\overline{\lambda_n}^{k-1},$$
there exists a sequence $t_l \to -\infty$ such that

$$\lim_{l\to\infty}\mu_{\theta,m}(t_l) = 0 (= \lim_{l\to\infty}\mu_{\theta,m+2}(t_l)).$$

Thus, by letting $t = t_l \to -\infty$, we get

$$\mu_{\theta,m+2}(t_0) + \nu\int\limits_{-\infty}^{t_0}\mu_{\theta+1/2,m+2}d\tau \leq \frac{|A^\theta g|^2}{\nu\lambda_1}\int_{-\infty}^{t_0}\frac{d\tau}{|u|^{m+2}}$$

$$+\left[\frac{c_0^2 c_{2\theta}^2}{\nu}\overline{\lambda}_n + \frac{1}{|u(t_0)|^2}\left(\nu\lambda_1 + \nu(m+2)\overline{\lambda}_n + \frac{(m+2)|g|}{|u(t_0)|}\right)\right]\int_{-\infty}^{t_0} \mu_{\theta,m} d\tau.$$

Hence,

$$\lim_{t\to-\infty} \mu_{\theta,m+2}(t) = 0.$$

Moreover, since according to Lemma B.1,

$$\nu\int_{-\infty}^{t_0} \frac{\lambda}{|u|^{m+2}} d\tau \le \frac{2}{m+2}\frac{1}{|u(t_0)|^{m+2}},$$

we obtain

$$\mu_{\theta,m+2}(t_0) + \int_{-\infty}^{t_0} \mu_{\theta+1/2,m+2} d\tau \le \frac{|A^\theta g|^2}{\nu^2\lambda_1^2}\frac{2}{m+2}\frac{1}{\gamma_0^{m+2}}$$
$$+\left[\frac{c_0^2 c_{2\theta}^2}{\nu}\overline{\lambda}_n + \frac{1}{\gamma_0^2}\left(\nu\lambda_1 + \nu(m+2)\overline{\lambda}_n + \frac{(m+2)|g|}{\gamma_0}\right)\right]\frac{C_{\theta,m}(G_{\theta-1/2})}{\nu^{m-1}}\overline{\lambda}_n^{k-1}.$$

Observe that by the Poincaré inequality, $G_\theta > G_{\theta-1/2}$. Thus $C_{\theta,m}(G_{\theta-1/2}) \le C_{\theta,m}(G_\theta)$. Using this fact, together with the definition of γ_0, we can define the positive increasing functions $C'_{\theta,m}(G_\theta)$ from the statement of the proposition as follows:

$$\frac{2|A^\theta g|^2\nu^m}{(m+2)\nu^2\lambda_1^{2+k}\gamma_0^{m+2}} + \left[c_0^2 c_{2\theta}^2 + \frac{\nu^2}{\gamma_0^2}\left(1 + (m+2)(1 + \frac{|g|}{\nu\lambda_1\gamma_0})\right)\right] C_{\theta,m}(G_{\theta-1/2})$$
$$= \frac{2}{m+2}G_\theta^2 + \left(c_0^2 c_{2\theta}^2 + \frac{3}{2}m + 4\right) C_{\theta,m}(G_{\theta-1/2})$$
$$\le \frac{2}{m+2}G_\theta^2 + \left(c_0^2 c_{2\theta}^2 + \frac{3}{2}m + 4\right) C_{\theta,m}(G_\theta) := C'_{\theta,m}(G_\theta).$$

On the other hand, again for the Galerkin approximaximations, we have

$$\frac{1}{2}\frac{d}{dt}\mu^N_{\theta+1/2,m+4} = \frac{-\nu|A^{\theta+1}u^N|^2 + (g, A^{2\theta+1}u^N) - b(u^N, u^N, A^{2\theta+1}u^N)}{|u^N|^{(m+4)}}$$
$$+\frac{m+4}{2}\mu^N_{\theta+1/2,m+4}\frac{\nu|A^{1/2}u^N|^2 - (g, u^N)}{|u^N|^2},$$

and similarly to what was done above we get

$$\mu_{\theta+1/2,m+4}(t_0) + \nu \int_t^{t_0} \mu_{\theta+1,m+4} d\tau \leq \mu_{\theta+1/2,m+4}(t) + \frac{|A^{\theta+1/2}g|^2}{\nu\lambda_1} \int_t^{t_0} \frac{d\tau}{|u|^{(m+4)}}$$

$$+ \left[\frac{c_0^2 c_{2\theta+1}^2}{\nu} \overline{\lambda}_n + \frac{1}{|u(t_0)|^2} \left(\nu\lambda_1 + (m+4)(\nu\overline{\lambda}_n + \frac{|g|}{|u(t_0)|}) \right) \right] \int_t^{t_0} \mu_{\theta+1/2,m+2} d\tau$$

(here again $c_{2\theta+1}$ is the constant from Theorem 1) By the same argument as in the previous case, when $t \to -\infty$ we obtain

$$\mu_{\theta+1/2,m+4}(t_0) + \nu \int_{-\infty}^{t_0} \mu_{\theta+1,m+4} d\tau \leq \frac{|A^{\theta+1/2}g|^2}{\nu\lambda_1} \int_{-\infty}^{t_0} \frac{d\tau}{|u|^{(m+4)}}$$

$$+ \left[\frac{c_0^2 c_{2\theta+1}^2}{\nu} \overline{\lambda}_n + \frac{1}{|u(t_0)|^2} \left(\nu\lambda_1 + (m+4)(\nu\overline{\lambda}_n + \frac{|g|}{|u(t_0)|}) \right) \right] \int_{-\infty}^{t_0} \mu_{\theta+1/2,m+2} d\tau.$$

Thus

$$\lim_{t \to -\infty} \mu_{\theta+1/2,m+4}(t) = 0,$$

and

$$\mu_{\theta+1/2,m+4}(t) \leq \frac{K_{\theta,m}}{\nu^{m+2}} \overline{\lambda}_n^{k+1},$$

where

$$K_{\theta,m}(G_{\theta+1/2}) := \frac{2}{m+4} G_{\theta+1/2}^2 + \left(c_0^2 c_{2\theta+1}^2 + \frac{3}{2}m + 7 \right) C'_{\theta,m}(G_{\theta+1/2}).$$

Observe that $K_{\theta,m}$ satisfies conditions from the proposition, since

$$\frac{2|A^{\theta+1/2}g|^2 \nu^{m+4}}{(m+4)\nu^4 \lambda_1^{k+3} \gamma_0^{m+4}} + \left[c_0^2 c_{2\theta+1}^2 + \frac{\nu^2}{\gamma_0^2} \left(1 + (m+4)(1 + \frac{|g|}{\lambda_1 \nu \gamma_0}) \right) \right] C'_{\theta,m}(G_\theta)$$
$$= \frac{2}{m+4} G_{\theta+1/2}^2 + \left(c_0^2 c_{2\theta+1}^2 + \frac{3}{2}m + 7 \right) C'_{\theta,m}(G_\theta) \leq K_{\theta,m}(G_{\theta+1/2}).$$

$\square$

Proof of the main theorem.

We will prove Theorem A.1 by induction on $k = 2\theta$.

When $k = 1$ the Theorem holds (see (1.14)).

When $k = 2$ the Theorem is valid via Proposition B.1. Observe that this proposition allows us to choose, for example,

$$M_2(G_1) = \left((c_0 + 1)G_1 + \frac{15}{4}\right) e^4.$$

Moreover, Proposition B.1 gives us that

$$N_{3/2} = 4e^3 M_2.$$

Thus, applying Proposition B.2, we conclude that the theorem holds when $k = 3$.

Suppose now that the theorem is true for some integer $k \geq 3$. Then there exists a positive increasing function $N_\theta(\cdot)$, such that

$$\int_{-\infty}^{t_0} \frac{|A^\theta u|^2}{|u|^{4\theta-2}} d\tau < \frac{N_\theta(G_{\theta-1/2})}{\nu^{4\theta-3}} \overline{\lambda}_n^{2\theta-1},$$

where $\theta = k/2$. But according to Proposition B.2, if $g \in D(A^{\theta+1/2})$, then there exist positive increasing functions $M_{\theta+1/2}(\cdot)$ and $N_{\theta+1/2}(\cdot)$ such that

$$\int_{-\infty}^{t_0} \frac{|A^{\theta+1/2} u(t)|^2}{|u(t)|^{4\theta}} dt < \frac{N_{\theta+1/2}(G_\theta)}{\nu^{4\theta-1}} \overline{\lambda}_n^{2\theta},$$

$$\frac{|A^{\theta+1/2} u_0|^2}{|u_0|^{4\theta+2}} \leq \frac{M_{\theta+1/2}(G_{\theta+1/2})}{\nu^{4\theta}} \overline{\lambda}_n^{2\theta+1},$$

and

$$\lim_{t \to -\infty} \frac{|A^{\theta+1/2} u(t)|^2}{|u(t)|^{4\theta+2}} = 0,$$

which shows that the theorem is true for the integer $k + 1$, and by induction, the proof is complete. $\square$

CHAPTER III

BURGERS' ORIGINAL MODEL FOR TURBULENCE*

A. Background

In the 1940's Burgers wrote a sequence of papers (see [19, 20, 21]) in which a model for turbulence in hydrodynamical systems was presented. In particular, in [21] he introduced a simplified model for turbulent flow in a channel. This model consists of a coupled system of differential equations:

$$\begin{cases} b\frac{d}{dt}U(t) = P - \frac{\nu}{b}U(t) - \frac{1}{b}\int\limits_0^b v^2(t,y)\,dy \\ \frac{\partial}{\partial t}v(t,y) = \frac{1}{b}U(t)v(t,y) + \nu\frac{\partial^2}{\partial y^2}v(t,y) - 2v(t,y)\frac{\partial}{\partial y}v(t,y)\,. \end{cases} \tag{3.1}$$

The functions $U(t)$ and $v(y,t)$ represent velocities (U is the mean motion of the fluid and v is the secondary motion, modeling random fluctuations of the velocity around the mean motion), t is the time variable, and y is the space variable perpendicular to the channel, which has constant width b. The function v is assumed to be zero on the walls $y = 0$ and $y = b$. The constant ν is the viscosity coefficient. Finally, the constant P represents the energy per unit of mass provided by the driving force. The term Uv/b in the second equation results from the energy transmitted from the primary motion U to the secondary motion v.

Such a system is a typical case of an abstract dissipative evolution PDE (see e.g. [25], [24], [26], [14]), for which many of the standard results, like existence and uniqueness of solutions, as well as existence of the global attractor are valid. It is important to stress that unlike the classical Burgers equation (which is the

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second equation of (3.1) with $U = 0$), Burgers' original model for turbulence has nontrivial dynamics. In particular, (3.1) has a global attractor of fractal dimension of order $\sqrt{Pb^2/\nu^2}$ (see [22]), an exponential attractor of fractal dimension of order $(Pb^2/\nu^2)^{15/8}$ (see [18]), as well as an inertial manifold (see [23]). Moreover, there is some numerical evidence that the global attractor may have complicated geometrical structure (see [29]).

Our interest in Burgers' original model for turbulence was motivated by the supposition that as an analog to the Navier-Stokes equations (both being dissipative with a quadratic nonlocal nonlinearity), it may have similar backward-time behavior, including, eventually, validity of some form of the Bardos-Tartar conjecture. (Observe that we can place (3.1) in the same settings as problems in [1, 5, 6, 7, 3, 4], if we give it the suitable functional framework (see Section A)).

However, our results show significant differences in backward-time dynamics between Burgers' original model for turbulence and the space-periodic 2-D Navier-Stokes equations (see Section H for the detailed discussion of these differences).

Note that the classical 1-D periodic Burgers equation displays the same backward-time behavior as the Kuramoto-Sivashinsky equation, namely: all of its solutions outside the global attractor blow up backward in finite time as was mentioned in Chapter I (see also [6], [7]). But from our results presented later in this chapter, it follows that the system (3.1) has a rich (invariant) set of global solutions that grow exponentially for $t \to -\infty$.

B. Preliminaries

Let us establish precisely the mathematical problem to be treated. Equations (3.1) become, after a simple scaling transformation:

$$\begin{cases} \frac{d}{dt}U(t) = R - U(t) - \int\limits_0^1 v^2(t,\xi)\,d\xi \\ v_t(t,x) = U(t)v(t,x) + v_{xx}(t,x) - 2v(t,x)v_x(t,x)\,, \end{cases} \tag{3.2}$$

where v_t, v_x, and v_{xx} stand for $\partial v/\partial t$, $\partial v/\partial x$, and $\partial^2 v/\partial x^2$ respectively; $x \in [0,1]$, $v(t,0) = v(t,1) = 0$, and $R = Pb^2/\nu^2$ - the equivalent of the Reynolds number for this model. If we extend v on $[-1,0]$ by the formula $v(t,x) = -v(t,-x)$, we obtain the solution for the system

$$\begin{cases} \frac{d}{dt}U(t) = R - U(t) - \frac{1}{2}\int\limits_{-1}^1 v^2(t,\xi)\,d\xi \\ v_t(t,x) = U(t)v(t,x) + v_{xx}(t,x) - (v^2(t,x))_x \end{cases} \tag{3.3}$$

satisfying

$$v(t,-1) = v(t,1) \tag{3.4}$$

and

$$\int_{-1}^1 v(t,\xi)\,d\xi = 0. \tag{3.5}$$

In the present paper we will consider the more general problem in which (U,v) is a solution of (3.3) satisfying periodic boundary condition (3.4), together with condition (3.5). To represent this problem in functional settings, we will need the Hilbert space

$$L_0^2(-1,1) = \left\{ w \in L^2(-1,1) : \int_{-1}^1 w(x)\,dx = 0 \right\}$$

(we will denote by $|\cdot|$ the L^2-norm on $L_0^2(-1,1)$). and the direct sum

$$H = \mathbb{R} \oplus L_0^2(-1,1)$$

equipped with the inner product

$$\langle (r_1,w_1),(r_2,w_2)\rangle_H = r_1 r_2 + \frac{1}{2}\langle w_1,w_2\rangle_{L^2(-1,1)}. \tag{3.6}$$

We will also need the the space

$$H^1_p(-1,1) = \left\{ w \in H^1(-1,1) : \; w(-1) = w(1), \; \int_{-1}^{1} w(x)\,dx = 0 \right\}$$

(equipped with the norm $||w||^2 = |\frac{\partial w}{\partial x}|^2$), and the direct sum

$$V = \mathbb{R} \oplus H^1_p(-1,1)$$

equipped with the inner product

$$\langle\langle (r_1, w_1), (r_2, w_2) \rangle\rangle_V = r_1 r_2 + \frac{1}{2} \langle \frac{\partial w_1}{\partial x}, \frac{\partial w_2}{\partial x} \rangle_{L^2(-1,1)}. \tag{3.7}$$

We will denote by $|\cdot|_H$ and $||\cdot||_V$ the corresponding norms in H and V. It is easy to check that Poincaré inequalities

$$|u|_H \le ||u||_V, \; u \in V \tag{3.8}$$

and

$$\pi |w| \le ||w||, \; w \in H^1_p(-1,1) \tag{3.9}$$

are valid, and hence $V \subset H$ and $H^1_p(-1,1) \subset L^1_0(-1,1)$.

Let $A : D(A) \subset H \to H$, with $D(A) = V \cap (\mathbb{R} \oplus H^2(-1,1))$, be the unbounded operator given by the matrix

$$A = \begin{pmatrix} I & 0 \\ 0 & A_0 \end{pmatrix},$$

where $A_0 = -\frac{\partial^2}{\partial x^2}$.

Also, let $B : V \oplus V \to H$ be the bounded bilinear operator given by:

$$B\left((r_1, w_1), (r_2, w_2)\right) = \begin{pmatrix} \frac{1}{2} \langle w_1, w_2 \rangle_{L^2(-1,1)} \\ -r_2 w_1 + \frac{4}{3} w_1 (w_2)_x + \frac{2}{3} w_2 (w_1)_x \end{pmatrix}.$$

Moreover, denote

$$f = \begin{pmatrix} R \\ 0 \end{pmatrix}.$$

Then (3.3) can be written in a general form as an evolution equation:

$$\begin{cases} \frac{d}{dt}u + Au + B(u,u) = f \\ u(0) = u_0 \in H \end{cases}. \tag{3.10}$$

As we mentioned before, this type of evolution equation enjoys existence, uniqueness, as well as some important regularity properties of its solutions (see [26, section 7.4]). Moreover, by using the method given in [15], one can easily see that the solutions of (3.10) are analytic functions in space and time for $t > 0$.

C. Some general results about backward-time dynamics

Let $u = (U, v)$ be a solution of (3.10). Observe that if we take the scalar product in H of both sides of (3.10) with $u = (U, v)$, we obtain:

$$\frac{1}{2}\frac{d}{dt}|u|_H^2 + ||u||_V^2 = RU \leq \frac{R^2}{2} + \frac{U^2}{2} \leq \frac{R^2}{2} + \frac{||u||_V^2}{2}. \tag{3.11}$$

Whence,

$$\frac{d}{dt}|u|_H^2 + ||u||_V^2 \leq R^2.$$

Thus by (3.8),

$$\frac{d}{dt}|u|_H^2 + |u|_H^2 \leq R^2.$$

So,

$$|u|_H^2 \leq |u_0|_H^2 e^{-t} + R^2(1 - e^{-t}).$$

Hence given $u_0 \in H$, there is a time $t_0(|u_0|)$ such that for all $t \geq t_0$

$$|u(t)|_H^2 \leq 2R^2. \tag{3.12}$$

For convenience we will call a solution $u(\cdot)$ a global solution, if it satisfies (3.10) for all times $t \in (-\infty, \infty)$.

Let us recall that the global attractor $\mathcal{A}$ for the system (3.10) consists of all $u_0 \in H$ such that the solution $u(\cdot)$ with $u(0) = u_0$ is global and bounded in H on the whole $(-\infty, \infty)$. In fact, in the definition of $\mathcal{A}$, H can be replaced with V or $D(A)$ (see e.g. [14, section III.1] for details). Therefore we can conclude that on the global attractor, estimate (3.12) holds for all $t \in (-\infty, \infty)$.

First, we will treat the trivial case of $v = 0$ in (3.3):

Proposition C.1. *If $v(0) = 0$ in $L_0^2(-1, 1)$, then the solution of (3.10) has the form $u(x, t) = (R+(U_0-R)e^{-t}, 0)$. It converges as $t \to \infty$ to the stationary solution $(R, 0)$ exponentially and $\lim_{t\to-\infty} \ln |u(t)|_H/t = 1$.*

Proof. Note that if $u_0 = (U_0, 0)$, then $u(x, t) = (R + (U_0 - R)e^{-t}, 0)$ will solve (3.10). Obviously, $\lim_{t\to\infty} u(t) = (R, 0)$ and $\lim_{t\to-\infty} \ln |u(t)|_H/t = 1$. □

Because of the proposition above, in what follows we can and will assume that $v(t)$ is not zero in $L_0^2(-1, 1)$ for all t (Note that a sufficient condition for this is $v(t_0) \neq 0$ in $L_0^2(-1, 1)$ for some t_0).

Next we will present several results that will be useful in the describing backward-time behavior of the solutions of (3.10).

Lemma C.1. *Suppose that $U(t_1) < 0$. Then there exists a $\tau_1 < t_1$ (depending on $U(t_1)$ and $|v(t_1)|$) such that, if u is defined on $[\tau_1, t_1]$, then $U(\tau) > 0$ for some $\tau \in [\tau_1, t_1]$.*

Proof. Suppose $U \leq 0$ on $[t_0, t_1]$. Then, the second equation of (3.3) will imply that

$$\frac{1}{2}\frac{d}{dt}|v|^2 \leq -||v||^2 \leq -\pi^2|v|^2.$$

So, for any $s \in [t_0,\, t_1]$:

$$|v(s)|^2 \geq |v(t_1)|^2 e^{2\pi^2(t_1-s)}.$$

After integrating the first equation of (3.3) from t_0 to t_1, we will have

$$\begin{aligned}
0 \;\geq\;& U(t_0) = R + (U(t_1) - R)e^{t_1-t_0} + \frac{1}{2}\int_{t_0}^{t_1} |v(s)|^2 e^{-(t_0-s)}\, ds \\
\geq\;& R + (U(t_1) - R)e^{t_1-t_0} + \frac{1}{2}|v(t_1)|^2 e^{2\pi^2 t_1 - t_0} \int_{t_0}^{t_1} e^{-(2\pi^2-1)s}\, ds \\
=\;& R + (U(t_1) - R)e^{t_1-t_0} + \frac{|v(t_1)|^2 e^{2\pi^2 t_1 - t_0}}{2(2\pi^2-1)}(e^{-(2\pi^2-1)t_0} - e^{-(2\pi^2-1)t_1}) \\
=\;& R - \left(R - U(t_1) + \frac{|v(t_1)|^2}{2(2\pi^2-1)}\right) e^{t_1-t_0} + \frac{1}{2(2\pi^2-1)}|v(t_1)|^2 e^{2\pi^2(t_1-t_0)}.
\end{aligned}$$

Then

$$0 \geq -\left(R - U(t_1) + \frac{1}{2(2\pi^2-1)}|v(t_1)|^2\right) + \frac{1}{2(2\pi^2-1)}|v(t_1)|^2 e^{(2\pi^2-1)(t_1-t_0)}.$$

So,

$$t_0 \geq T_1 := t_1 - \frac{1}{2\pi^2-1}\ln\left(\frac{2(2\pi^2-1)(R-U(t_1)) + |v(t_1)|^2}{|v(t_1)|^2}\right),$$

and the lemma holds for any $\tau_1 < T_1$.

□

As an immediate corollary we have the following:

Proposition C.2. *If $u = (U, v)$ is an arbitrary global solution of (3.10), then $\limsup_{t\to-\infty} U(t) \geq 0$ and for any T_1 there exists a $t_1 < T_1$ such that $U(t_1) > 0$.*

The key result of this section is the next theorem:

Theorem C.1. *Suppose $u = (U, v)$ is a global solution for (3.3) with $u(0) \notin \mathcal{A}$ and $v(0) \neq 0$. Then $\liminf_{t\to-\infty} U(t)e^t > 0$.*

Proof. Observe that the first equation of (3.3) implies that

$$U(t) > R + (U(t_0) - R)e^{(t_0 - t)}$$

for any $t < t_0$, and so, if there exists t_0 such that $U(t_0) \geq R$, then $U(t) > R$ for $t < t_0$ and $\liminf_{t \to -\infty} U(t)e^t > 0$. Thus if the theorem is false, then $U(t) < R$ for all t. In this case, equation (3.11) implies that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u(t)|_H^2 &= -||u(t)||_V^2 + RU(t) \leq -|u(t)|_H^2 + RU(t) \\ &< -2R^2 + R^2 = -R^2, \end{aligned} \tag{3.13}$$

as long as $|u(t)|_H^2 > 2R^2$. Since the solution u is not in the global attractor, there exists a t_0 with $|u(t_0)|_H^2 > 2R^2$. Because estimate (3.13) holds for t_0, it follows that $|u(t)|_H^2 \geq |u(t_0)|_H^2 > 2R^2$ for all $t \leq t_0$, and hence, (3.13) holds for all $t \leq t_0$, which implies that

$$\lim_{t \to -\infty} |u(t)|_H^2 = \infty. \tag{3.14}$$

In order to continue, we will establish the following claim.

Claim C.1. *Under the hypothesis of Theorem C.1, suppose $U(t) < R$ for all t. Then there exists t_1 such that $U(t) > -R$ for all $t < t_1$.*

Proof. By Proposition C.2, $\limsup_{t \to -\infty} U(t) \geq 0 > -R$. Assume that we also have

$$\liminf_{t \to -\infty} U(t) < -\frac{3}{4}R. \tag{3.15}$$

Then there exists a decreasing sequence $\{t_n\}$ with $\lim\limits_{n \to \infty} t_n = -\infty$, such that $U(t_n) = -\frac{1}{2}R$ and $\frac{dU}{dt}(t_n) \geq 0$ for any n. Observe that

$$|u(t_n)|_H^2 = U^2(t_n) + \frac{1}{2}|v(t_n)|^2 = \frac{1}{4}R^2 + \frac{1}{2}|v(t_n)|^2.$$

So, by (3.14), $\lim_{n \to \infty} |v(t_n)|^2 = \infty$. Thus there exists n_0 for which $|v(t_{n_0})|^2 > 5R$.

But

$$\frac{dU}{dt}(t_{n_0}) = R - U(t_{n_0}) - \frac{1}{2}|v(t_{n_0})|^2 \leq \frac{3}{2}R - \frac{1}{2}|v(t_{n_0})|^2 \leq \frac{3}{2}R - \frac{5}{2}R = -R < 0.$$

This contradiction proves that $\liminf_{t\to-\infty} U(t) \geq -(3/4)R$, which obviously implies the claim.

□

Let us return to the proof of Theorem C.1. Under the assumption of the previous claim, for $t < t_1$, $|u(t)|_H^2 = U^2(t) + \frac{1}{2}|v(t)|^2 < R^2 + \frac{1}{2}|v(t)|^2$. Then (3.14) forces $|v(t)|^2 \to \infty$ as $t \to -\infty$. So, we can choose $t_2 < t_1$ such that $|v(t)|^2 > 6R$ for $t < t_2$. Then, for $t < t_2$,

$$\frac{dU}{dt}(t) = R - U(t) - \frac{1}{2}|v(t)|^2 \leq 2R - \frac{1}{2}|v(t)|^2 < 2R - 3R = -R.$$

Consequently $\lim_{t\to-\infty} U(t) = \infty$, which contradicts our initial assumption that $\limsup_{t\to-\infty} U(t) \leq R$ and concludes the proof.

□

Corollary C.1. *A global solution $u = (U, v)$ (with $v \neq 0$) is outside the global attractor if and only if there exists a time t_0 such that $U(t_0) \geq R$.*

Proof. The "if" part follows directly from the first equation in (3.3). The "only if" part is a direct consequence of the previous theorem. □

D. Dirichlet quotients and backward-time blow up

Backward-time behavior of u depends in part on the nature of Dirichlet quotients $\lambda = ||v||^2/|v|^2$ and $\Lambda = ||u||_V^2/|u|_H^2$. In fact, for the space periodic 2-D Navier-Stokes equations and their α-model, the set of initial data for which $\limsup_{t\to-\infty} \lambda < \infty$ is a rich set in the space of all initial data (see [1], [3], and [4]). Namely it is dense in an appropriate topology in H. On the other hand, in the case of the space periodic 1-D Kuramoto-Sivashinsky equation, all the solutions that backward in time have no more than exponential rate of growth are globally bounded (see [6]). We should note that in both [1] and [6] $\Lambda \equiv \lambda$ and so, there is only one Dirichlet quotient for these equations.

In this and in the following three sections we will concern ourselves with the case when Dirichlet quotients for (3.3) are bounded.

Proposition D.1. *Suppose u is a global solution of (3.10) and there exists $\overline{\Lambda}$ and t_0 such that $\Lambda(t) \leq \overline{\Lambda}$ for all $t < t_0$. Then $\limsup_{t\to-\infty} |u(t)|_H e^{\overline{\Lambda}t} < \infty$.*

Moreover, if $u(t) \notin \mathcal{A}$ and there exists $\overline{\lambda}$ such that $\lambda(t) \leq \overline{\lambda}$, for all $t < t_0$, then

$$0 < \liminf_{t\to-\infty} |u(t)|_H e^t \leq \limsup_{t\to-\infty} |u(t)|_H e^t < \infty. \tag{3.16}$$

More precisely, there exist positive constants K_1, K_2, C_1, C_2 and a time $\tau_0 < t_0$ such that

$$K_1 e^{-t} \leq U(t) \leq K_2 e^{-t} \tag{3.17}$$

and

$$0 < |v(t)|^2 \leq C_1 e^{2\overline{\lambda}t - C_2 e^{-t}} \tag{3.18}$$

for all $t < \tau_0$.

Proof. Note that if $u(t) \in \mathcal{A}$ then u is bounded, and hence in this case we will obviously have $\limsup_{t\to-\infty} |u(t)|_H e^{\overline{\Lambda}t} = 0 < \infty$. If $u(t) \notin \mathcal{A}$, then by Theorem C.1, without loss of generality we can assume that $U(t_0) > R$.

Under this assumption, using (3.11) we obtain

$$\frac{1}{2}\frac{d}{dt}|u|_H^2 = -||u||_V^2 + RU \geq -\overline{\Lambda}|u|_H^2.$$

Thus $\limsup_{t\to-\infty} |u(t)|_H e^{\overline{\Lambda}t} < \infty$.

If moreover $\lambda(t) \leq \overline{\lambda}$, then by taking the scalar product in $L^2(-1,1)$ of the second equation of (3.3) with v, we obtain

$$\frac{1}{2}\frac{d}{dt}|v|^2 = U|v|^2 - ||v||^2 \geq (U - \overline{\lambda})|v|^2. \tag{3.19}$$

Thus for $t < t_0$,

$$|v(t)|^2 \leq |v(t_0)|^2 e^{2\left(\overline{\lambda}^2 (t_0 - t) - \int_t^{t_0} U(t)\, dt\right)}. \tag{3.20}$$

Also, from the first equation of (3.3), for $t < t_0$, we get

$$U(t) \geq R + (U(t_0) - R)e^{t_0 - t}. \tag{3.21}$$

Hence there exists $K_1 > 0$ and $t_1 < t_0$ such that for any $t < t_1$ we have $U(t) \geq K_1 e^{-t}$. Then (3.20) implies the existence of positive constants C_1 and C_2 such that $|v(t)|^2 \leq C_1 e^{2\overline{\lambda}t - C_2 e^{-t}}$ for all $t < t_1$. Going back to the first equation of (3.3), it is easy to see that there must exist $K_2 > 0$ and $\tau_0 \leq t_1$ such that $U(t) \leq K_2 e^{-t}$ for all $t < \tau_0$. These estimates for U and v imply that $0 < \liminf_{t\to-\infty} |u(t)|_H e^t \leq \limsup_{t\to-\infty} |u(t)|_H e^t < \infty$, which concludes the proof.

□

By the previous proposition, the fact that a Dirichlet quotient is bounded, implies backward-time exponential growth of the solution. When the quotient λ is bounded, the estimate (3.18) allows us to establish more facts about this case which we will discuss in the next two sections.

E. Properties of global solutions with bounded quotient λ

Observe that from (3.3),

$$\frac{1}{2}\frac{d||v(t)||^2}{dt} = U(t)||v(t)||^2 - ||v_x(t)||^2 - \int_{-1}^{1} (v_x(t,\xi))^3\, d\xi. \tag{3.22}$$

Then using (3.19), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\frac{||v||^2}{|v|^2} &= \frac{U||v||^2 - ||v_x||^2 - \int_{-1}^{1}(v_x)^3\,d\xi}{|v|^2} - \frac{||v||^2(U|v|^2 - ||v||^2)}{|v|^4} \\ &= -\frac{||v_x||^2}{|v|^2} + \frac{||v||^4}{|v|^4} - \frac{\int_{-1}^{1}(v_x)^3\,d\xi}{|v|^2}. \end{aligned} \tag{3.23}$$

Now we estimate the last term of (3.23) as follows:

$$\begin{aligned}\left|\frac{\int_{-1}^{1}(v_x)^3\,d\xi}{|v|^2}\right| &\le |v_x|_\infty\frac{||v||^2}{|v|^2} \le c_0||v||^{1/2}||v_x||^{1/2}\frac{||v||^2}{|v|^2} = c_0\left(\frac{||v||^2}{|v|^2}\right)^{5/4}\left(\frac{||v_x||^2}{|v|^2}\right)^{1/4}|v| \\ &= \left(\frac{c_0}{(4\epsilon)^{1/4}}|v|\left(\frac{||v||^2}{|v|^2}\right)^{5/4}\right)\left((4\epsilon)^{1/4}\left(\frac{||v_x||^2}{|v|^2}\right)^{1/4}\right) \\ &\le \frac{3}{4}\left(\frac{c_0}{(4\epsilon)^{1/4}}|v|\left(\frac{||v||^2}{|v|^2}\right)^{5/4}\right)^{4/3} + \frac{1}{4}\left((4\epsilon)^{1/4}\left(\frac{||v_x||^2}{|v|^2}\right)^{1/4}\right)^4 \\ &= \frac{3c_0^{4/3}}{4(4\epsilon)^{1/3}}|v|^{4/3}\left(\frac{||v||^2}{|v|^2}\right)^{5/3} + \epsilon\frac{||v_x||^2}{|v|^2},\end{aligned}$$

with any $\epsilon \in (0,1)$ (above we used Agmon's inequality $|v|_\infty \le c_0|v|^{1/2}||v||^{1/2}$). Observe that

$$\frac{3c_0^{4/3}}{4(4\epsilon)^{1/3}}|v|^{4/3} \le \epsilon$$

if

$$|v| \le \gamma_0\epsilon, \tag{3.24}$$

where $\gamma_0 = 4/(c_0 3^{3/4})$.

Returning to (3.23), if (3.24) holds, then, because $\frac{||v||^2}{|v|^2} > 1$,

$$\frac{1}{2}\frac{d}{dt}\frac{||v||^2}{|v|^2} \le -(1-\epsilon)\left(\frac{||v_x||^2}{|v|^2} - \frac{||v||^4}{|v|^4}\right) + 2\epsilon\frac{||v||^4}{|v|^4}, \tag{3.25}$$

or equivalently,

$$\frac{1}{2}\lambda' \le -(1-\epsilon)\left|(A_0-\lambda)\frac{v}{|v|}\right|^2 + 2\epsilon\lambda^2,$$

where $A_0 = -\frac{\partial^2}{\partial x^2}$.

Thus we have proved

Lemma E.1. *There exists a universal constant $\gamma_0 > 0$ for equations (3.3), such that for any $\epsilon > 0$, if*

$$|v(t)| \leq \gamma_0 \epsilon, \tag{3.26}$$

for all t in some interval $J \subset \mathbb{R}$, then

$$\frac{1}{2}\lambda'(t) \leq -(1-\epsilon)\left|(A_0 - \lambda(t))\frac{v(t)}{|v(t)|}\right|^2 + 2\epsilon\lambda^2(t) \tag{3.27}$$

for all $t \in J$.

In what follows we will denote $\lambda_i = (i\pi)^2$ - the eigenvalues of A_0.

Proposition E.1. *Suppose $u = (U, v)$ is a global solution of equations (3.3) such that $\lambda_i \leq \limsup_{t\to-\infty} \lambda(t) < \lambda_{i+1}$. Then $\lim_{t\to-\infty} \lambda(t) = \lambda_i$.*

Proof. Let $\bar{l} := \limsup_{t\to-\infty} \lambda(t)$.

Suppose $\bar{l} > \lambda_i$. Choose $\gamma > 0$ small enough such that $[\bar{l} - \gamma, \bar{l} + \gamma] \subset (\lambda_i, \lambda_{i+1})$. Observe that there exists t_γ for which $\lambda(t) < \bar{l} + \gamma$ for all $t < t_\gamma$. Also, by Proposition D.1, for any $\epsilon > 0$ there exists a $t_\epsilon < t_\gamma$ such that inequality (3.26) from Lemma E.1 holds for all $t < t_\epsilon$. Note that whenever $\lambda(t) \in [\bar{l} - \gamma, \bar{l} + \gamma]$ and $t < t_\epsilon$, we have

$$\left|(A_0 - \lambda(t))\frac{v(t)}{|v(t)|}\right|^2 \geq \delta,$$

where $\delta := \min\{(\bar{l} - \gamma - \lambda_i)^2, (\lambda_{i+1} - \bar{l} - \gamma)^2\}$. Thus

$$\frac{1}{2}\lambda'(t) \leq -(1-\epsilon)\delta + 2\epsilon\lambda_{i+1}^2. \tag{3.28}$$

Now choose $\epsilon = \delta/(2{\lambda_{i+1}}^2 + \delta + 1)$. Then

$$\frac{1}{2}\lambda'(t) \le -\epsilon, \tag{3.29}$$

whenever $\lambda(t) \in [\bar{l} - \gamma, \bar{l} + \gamma]$ and $t < t_\epsilon$. Since $\bar{l} = \limsup_{t\to-\infty} \lambda(t)$, there exists a $\bar{t} < t_{\epsilon_0}$ so that $\lambda(\bar{t}) \in [\bar{l} - \gamma, \bar{l} + \gamma]$. Hence inequality (3.29) holds for all $t \le \bar{t}$ and

$$\bar{l} + \gamma \ge \lambda(t) \ge \bar{l} - \gamma + 2\epsilon(\bar{t} - t)$$

for all $t \le \bar{t}$, which is impossible. This contradiction forces $\bar{l} = \lambda_i$.

Suppose now that $\underline{l} := \liminf_{t\to-\infty} \lambda(t) < \bar{l} = \lambda_i$. Then we can choose a $\gamma \in (0, 1/2)$ such that $\underline{l} < \lambda_i - 2\gamma$. Under these conditions, for any $\underline{\lambda} \in (\lambda_i - 2\gamma, \lambda_i - \gamma)$ there exists a $t_{\underline{\lambda}} < t_\epsilon$ with $\lambda(t_{\underline{\lambda}}) = \underline{\lambda}$. Repeating the argument that led to inequality (3.28), we obtain

$$\frac{1}{2}\lambda'(t) \le -(1-\epsilon)\gamma^2 + 2\epsilon\underline{\lambda}^2 < 0, \tag{3.30}$$

whenever $\lambda(t) \in [\lambda_i - 2\gamma, \lambda_i - \gamma]$. We choose now $\epsilon < \gamma^2/(2{\lambda_i}^2 + \gamma^2)$. Then inequality (3.30) implies that $\lambda(t) \ge \underline{\lambda}$ for $t \le t_{\underline{\lambda}}$, and thus $\liminf_{t\to-\infty} \lambda(t) \ge \lambda_i - \delta > \underline{l}$, which is a contradiction. Hence we must have $\liminf_{t\to-\infty} \lambda(t) = \limsup_{t\to\ -\infty} \lambda(t) = \lambda_i$.

□

We are now ready to present the main theorem of this section.

Theorem E.1. *Suppose $u = (U, v)$ is a global solution of (3.10) with the bounded Dirichlet quotient λ. Then there exists an eigenvalue λ_i of the operator A_0 such that*

$$\lim_{t\to-\infty} \lambda(t) = \lambda_i \tag{3.31}$$

and, moreover, there exists a unit eigenvector w_i that corresponds to λ_i, and a sequence $t_n \to -\infty$ such that

$$\lim_{n\to\infty} \frac{v(t_n)}{|v(t_n)|} = w_i. \tag{3.32}$$

Proof. Identity (3.31) follows immediately from Proposition E.1. What is left to prove is that (3.32) holds for some sequence $t_n \to -\infty$ and a unit vector w_i with $A_0 w_i = \lambda_i w_i$.

Observe that by Proposition D.1, for any $\epsilon > 0$ there exists t_ϵ such that (3.27) holds for all $t < t_\epsilon$. Thus,

$$\frac{1}{2}(\lambda(t_0+1) - \lambda(t_0)) \leq -(1-\epsilon) \int_{t_0}^{t_0+1} \left| (A_0 - \lambda(t)) \frac{v(t)}{|v(t)|} \right| dt + 2\epsilon \lambda_{i+1}^2,$$

provided $t_0 + 1 < t_\epsilon$. Actually, because of (3.31), we can choose t_ϵ so that

$$\int_{t_0}^{t_0+1} \left| (A_0 - \lambda(t)) \frac{v(t)}{|v(t)|} \right| dt \leq \left(\frac{2\lambda_{i+1}}{1-\epsilon} + 1 \right) \epsilon.$$

Thus

$$\int_{t_0}^{t_0+1} \left| (A_0 - \lambda(t)) \frac{v(t)}{|v(t)|} \right| dt \to 0 \qquad \text{as } t_0 \to -\infty.$$

Then there exists a sequence $t_n \searrow -\infty$ such that

$$\left| (A_0 - \lambda(t_n)) \frac{v(t_n)}{|v(t_n)|} \right| \to 0 \qquad \text{as } n \to \infty.$$

Let Q_i be an orthogonal projection on the eigenspace associated with the eigenvalue λ_i. Then for n large enough,

$$\left(\frac{\lambda_{i+1} - \lambda_i}{2} \right) \left| (I - Q_i) \frac{v(t_n)}{|v(t_n)|} \right| \leq \left| (A_0 - \lambda(t_n))(I - Q_i) \frac{v(t_n)}{|v(t_n)|} \right| \to 0,$$

as $n \to \infty$. That is

$$(I - Q_i) \frac{v(t_n)}{|v(t_n)|} \to 0.$$

Then by passing to a subsequence if necessary, we can assume that $v(t_n)/|v(t_n)|$ converges to some unit vector w_i satisfying $A_0 w_i = \lambda_i w_i$.

□

Observe that a similar result is valid for the 2-D space periodic Navier-Stokes equations (see [1]). However in that case, the convergence holds for the entire scaled solution $u/|u|$, and not for a part of it. In our case, as we will see later, if $\Lambda(t)$ is bounded for $t < 0$, then $u(t)/|u(t)|_H$ will always converge in H to $(1, 0)$ (as $t \to -\infty$) and never to any of the other eigenvectors of A.

F. Existence of global solutions with bounded quotient λ

Let $S(t)(U_0, v_0) = (S_1(v_0, t)U_0, S_2(U_0, t)v_0)$ be the continuous semigroup of solutions of (3.10). Denote by P_i the orthogonal projection onto the spectral space of A_0 corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_i\}$.

The key step to proving existence is the following preliminary result:

Proposition F.1. *For any eigenvalue λ_i there exist $\epsilon_i \in (0, 1)$ and $M_i > 0$ such that for any $n \in \mathbb{N}$ there are $U_0^n \in \mathbb{R}, v_0^n \in P_i L_0^2(-1, 1)$, for which $(U_n(t), v_n(t)) = S(t)(U_0^n, v_0^n)$ has the following properties:*

1. *$|v_n(t)| \le \epsilon_i$, $\lambda^n(t) := \frac{||v_n(t)||^2}{|v_n(t)|^2} \le \lambda_i + 1/2$, $t \in [0, n]$;*
2. *$R \le U_n(n) \le M_i$, $(2/3)\epsilon_i \le |v_n(n)| \le \epsilon_i$, $\lambda^n(n) \ge \lambda_i$.*

We will start with the following useful lemma:

Lemma F.1. *For any eigenvalue λ_i of the operator A_0 there exists a positive $\epsilon_i <$* $\min\{1, R/2\}$ *(ϵ_i depends only on λ_i), such that if a solution $(U(t), v(t))$ satisfies*

1. *$\lambda(t_0) < \lambda_i + 1/2$;*
2. *$|v(t)| \le \epsilon_i$, $t \in [t_0, t_1]$,*

then $\lambda(t) < \lambda_i + 1/2$ for all $t \in [t_0, t_1]$.

If on the other hand,

3. *$\lambda(t_1) > \lambda_i - 1/2$;*

4. $|v(t)| \leq \epsilon_i$, $t \in [t_0, t_1]$,

then $\lambda(t) > \lambda_i - 1/2$ *for all* $t \in [t_0, t_1]$.

Proof. Observe that under the conditions of the lemma, inequality (3.27) holds with $\epsilon = \epsilon_i/\gamma_0$.

Suppose that the first part of the lemma is not true. This means that although Conditions 1 and 2 from the lemma are satisfied, there exists a $\tilde{t} \in [t_0, t_1]$ such that $\lambda(\tilde{t}) \geq \lambda_i + 1/2$. Then we can find a $\bar{t} \in [t_0, \tilde{t}]$ for which $\lambda(\bar{t}) = \lambda_i + 1/2$. Observe that for any $t \in [t_0, t_1]$ such that $\lambda(t) = \lambda_i + 1/2$, the inequality (3.27) yields:

$$\frac{1}{2}\lambda'(t) \leq -(1-\epsilon)\left(\frac{1}{2}\right)^2 + 2\epsilon\left(\lambda_i + \frac{1}{2}\right)^2 < 0, \tag{3.33}$$

if $0 < \epsilon_i < \min\left\{\frac{(1/2)^2\gamma_0}{2(\lambda_i+1/2)^2+(1/2)^2}, \frac{R}{2}\right\}$ and $\epsilon = \epsilon_i/\gamma_0$. With this choice of ϵ_i, Inequality (3.33) forces $\lambda(t) \geq \lambda_i + 1/2$ for all $t \in [t_0, \bar{t}\,]$. This contradicts Condition 1: $\lambda(t_0) < \lambda_i + 1/2$. Thus $\lambda(t) < \lambda_i + 1/2$ for all $t \in [t_0, t_1]$.

The proof of the second part of the lemma is similar and is omitted. □

Now we are ready to start the proof of Proposition F.1.

Proof of Proposition F.1. Let $\epsilon_i \in (0, 1)$ be from Lemma F.1, and let $U_0^n = (3/2 + \lambda_i)e^n + R_0$, where $R_0 = R - {\epsilon_i}^2/2$.

Observe that if $|v_n| \leq \epsilon_i$ on $[0, t_n]$, $t_n \leq n$, then by the first equation of (3.3), it follows that

$$(U_0^n - R_0)e^{-t} + R_0 \leq U_n(t) \; t \in [0, t_n], \tag{3.34}$$

which with our choice of the U_0^n implies that

$$U_n(t) \geq \max\{\lambda_i + 3/2, R\}, \; t \in [0, t_n], \; t_n \leq n.$$

Moreover, if $|v_n(t)| \leq \epsilon_i$ on $[0, t_n]$ $(t_n \leq n)$ with $\lambda^n(0) \leq \lambda_i + 1/2$, then by Lemma

F.1,

$$\lambda^n(t) < \lambda_i + 1/2,$$

and consequently, from (3.19),

$$\frac{d}{dt}|v_n(t)| = U_n(t)|v_n(t)| - \lambda^n(t)|v_n(t)| > |v_n(t)| \tag{3.35}$$

for all $t \in [0, t_n]$.

Define a function $\theta_n : B_i \to [0,1]$ (B_i is the closed ball in $P_iL_0^2(-1,1)$, centered in zero of radius ϵ_i), $\theta_n(v_0) = \sup\{t_0 \in [0,n] : \ |v(t)| < \epsilon_i \ \forall t \in [0,t_0)\}$, where $v(t) = S_2(U_0^n, t)v_0$.

Claim F.1. *θ_n is a continuous function.*

Proof. Let $w_0^m \to v_0$ in $P_iL_0^2(-1,1)$ (v_0 and w_0^m are from B_i) and let $t_0 = \theta_n(v_0)$, $t_m = \theta_n(w_0^m)$. Then by the continuity of the semigroup S, we obtain that $w_m(t) := S_2(U_0^n, t)w_0^m \to v(t)$, as $m \to \infty$, for all $t \geq 0$. Observe that by (3.35), $|v(t)|$ is increasing for $t \in [t_0, t_0 + \delta]$ and a $\delta > 0$ small enough. Then using the convergence of the sequence $\{w_m(t)\}$ to $v(t)$ for all t, we infer that there exists m_δ such that $|w_n(t_0 + \delta/2)| > |v(t_0)|$ for any $m > m_\delta$. This inequality implies that $\theta_n(w_0^m) < t_0 + \delta/2$. By making $\delta \to 0$, we obtain that $\limsup_{m\to\infty} \theta_n(w_0^m) \leq t_0$.

If $t_0 = 0$ the proof is complete, otherwise suppose that $\liminf_{m\to\infty} \theta_n(w_0^m) := \tau_0 < t_0$. By passing to a subsequence we can assume that $\lim_{m\to\infty} \theta_n(w_0^m) = \tau_0$. Moreover, from the definition of θ_n combined with the assumption that $\tau_0 < t_0$, we see that the subsequence can be chosen so that $|w_m(t_m)| = \epsilon_i$ with $t_m < t_0 \leq n$ for any m.

From the fact that the function $|v(t)|$ increases on $[0, t_0]$ (see (3.35)), it follows that $|v(t)| < |v(t_0)| \leq \epsilon_i$ for any $t < t_0$. Choose a $\gamma \in (0, t_0 - \tau_0)$. Let $\mu = \epsilon_i - |v(\tau_0 + \gamma)|$. Then by continuity of the solutions of (3.3) with respect to the initial data, there exists an m_μ such that $|w_m(t) - v(t)| < \mu$ for any $m > m_\mu$ and for any $t \in [0, t_0]$. Observe that

$$|w_m(t)| \leq |w_m(t) - v(t)| + |v(t)| < \mu + |v(t)| \leq \epsilon_i - |v(\tau_0 + \gamma)| + |v(\tau_0 + \gamma)| = \epsilon_i$$

for any $t \in [0, \tau_0 + \gamma]$. Then because of (3.35), it follows that $|w_m(t)|$ is increasing on $[0, \tau_0 + \gamma]$. Also, $\lim_{m\to\infty} t_m = \tau_0$ implies that there exists $m_\gamma > m_\mu$ such that $t_m < \tau_0 + \gamma$ for all $m > m_\delta$. Then $\epsilon_i = |w_m(t_m)| < |w_m(\tau_0 + \gamma)| < \epsilon_i$ for all $m > m_\delta$, a contradiction. Thus, $\liminf_{m\to\infty} \theta_n(w_0^m) \geq t_0$.

In this way, for any $v_0 \in B_i$ and any sequence $\{w_0^m\} \subset B_i$ with $\lim_{m\to\infty} w_0^m = v_0$, $\lim_{m\to\infty} \theta_n(w_0^m) = \theta_n(v_0)$. This concludes the proof of the claim. □

Resuming the proof of Proposition F.1, we define the function $\Psi_n : B_i \to B_i$, $\Psi_n(v_0) = P_i(S_2(U_0^n, \theta_n(v_0))v_0)$. Observe that Ψ_n is continuous and $\Psi_n(v_0) = v_0$ if $|v_0| = \epsilon_i$. Then by Brouwer's Theorem, Ψ_n is an onto map. It follows then, that for any $p_0 \in (P_i - P_{i-1})L_0^2(-1,1)$, $|p_0| = (2/3)\epsilon_i$, there exists a $v_0^n \in P_i L_0^2(-1,1)$, $|v_0^n| \leq \epsilon_i$ such that $\Psi_n(v_0^n) = p_0$.

Since $|p_0| < \epsilon_i$, it follows that $t_n := \theta_n(v_0^n) > 0$ and $|P_i v_n(t_n)| = |p_0| = (2/3)\epsilon_i$. Clearly $\lambda^n(0)(= \frac{||v_n(0)||^2}{|v_n(0)|^2}) < \lambda_i + 1/2$, and by the definition of θ_n, $|v(t)| \leq \epsilon_i$ for all $t \in [0, t_n]$. Lemma F.1 implies that $\lambda^n(t) < \lambda_i + 1/2$ for all $t \in [0, t_n]$.

Observe that

$$\begin{aligned} ||P_i v_n(t_n)||^2 + ||(I - P_i)v_n(t_n)||^2 &= ||v_n(t_n)||^2 < (\lambda_i + 1/2)|v_n(t_n)|^2 \\ &= (\lambda_i + 1/2)(|P_i v_n(t_n)|^2 + |(I - P_i)v_n(t_n)|^2), \end{aligned}$$

and since $P_i v_n(t_n) \in (P_i - P_{i-1})L_0^2(-1,1)$, it follows that

$$\lambda_i |P_i v_n(t_n)|^2 + \lambda_{i+1}|(I - P_i)v_n(t_n)|^2$$
$$< (\lambda_i + 1/2)(|P_i v_n(t_n)|^2 + |(I - P_i)v_n(t_n)|^2),$$

and thus

$$|(I - P_i)v_n(t_n)|^2 < (1/2)|P_i v_n(t_n)|^2 = (1/2)(4/9)(\epsilon_i)^2.$$

Then

$$|v_n(t_n)|^2 = |P_i v_n(t_n)|^2 + |(I - P_i)v_n(t_n)|^2 < (1 + 1/2)(4/9)(\epsilon_i)^2 < (\epsilon_i)^2.$$

This means that $\theta_n(v_0^n) = n$ and $(2/3)\epsilon_i \leq |v_n(n)| \leq \epsilon_i$. Observe that we also have $\lambda^n(n) \geq \lambda_i$ (from the fact that $P_i v_0^n = p_0 \in (P_i - P_{i-1})L_0^2(-1, 1))$. Moreover, from the first equation of (3.3),

$$\begin{aligned} U_n(n) &\leq (U_0^n - R)e^{-n} + R = ((3/2 + \lambda_i)e^n + R_0 - R)e^{-n} + R \\ &< (\lambda_i + (3/2)) + R := M_i. \end{aligned}$$

In this way, (U_0^n, v_0^n) will satisfy the conclusions of the proposition.

□

Now we are ready to prove existence of global solutions of system (3.3) with bounded quotient λ.

Theorem F.1. *For every λ_i - eigenvalue of A_0, there exists a nontrivial unbounded global solution $u(t) = (U(t), v(t))$ of the (3.3) such that* $\lim_{t \to -\infty} \lambda(t) = \lambda_i$.

Proof. Obviously, Proposition F.1 with the appropriate time translations implies that there exist constants $M_i, \epsilon_i > 0$ and a sequence of solutions $u_n(t) = (U_n(t), v_n(t))$ such that $u_n(t)$ is defined at least for $t \in [-n, \infty)$, $(2/3)\epsilon_i \leq |v_n(0)| \leq \epsilon_i$, $R \leq U_n(0) \leq M_i$, $|v(t)| \leq \epsilon_i$ and $\lambda^n(t) = \frac{||v_n(t)||^2}{|v_n(t)|^2} < \lambda_i + 1/2$ for $t \in [-n, 0]$ with $\lambda^n(0) \geq \lambda_i$. Then there is a subsequence $u_{n_k}(0)$ convergent to some $u_0 \neq 0$ in H. It is clear that $u(t) = S(t)u_0$ is a solution for (3.3) such that $u_{n_k}(t) \to u(t)$ for any $t > 0$.

Let $\tau < 0$. By eliminating a finite number of members of $\{n_k\}$ we can suppose that $\tau > -n_k$ for all k. From the conditions above, the sequence $\{u_{n_k}(\tau)\}$ is relatively compact in H, and if a subsequence $u_{n_{k_m}}(\tau)$ converges to a u_τ in H, then $S(t)u_{n_{k_m}}(0) \to S(t)u_0 = S(t-\tau)u_\tau$ for any $t > \tau$. This means that the only limit point of $\{u_{n_k}(\tau)\}$ is u_τ, and consequently, $\lim_{k\to\infty} u_{n_k}(\tau) = u_\tau$. Thus $u(t)$ can be extended to $[\tau, \infty]$ for any $\tau < 0$ with $\lim_{k\to\infty} u_{n_k} = u(t)$ for all $t \in [\tau, \infty]$. By making $\tau \to -\infty$, we obtain a global solution $u(t)$ such that $u_{n_k}(t) \to u(t)$ for any $t \in \mathbb{R}$. From this convergence it follows that $|v(t)| \leq \epsilon_i$ and $\lambda(t) = \frac{||v(t)||^2}{|v(t)|^2} \leq \lambda_i + 1/2$ for any $t \leq 0$; and, moreover, $(2/3)\epsilon_i \leq |v(0)| \leq \epsilon_i$, $U(0) \geq R$, and $\lambda(0) \geq \lambda_i$. The second part of Lemma F.1 yields that $\lambda(t) \geq \lambda_i - 1/2$ for any $t \leq 0$. Then by Theorem E.1, $\lambda(t) \to \lambda_i$ as $t \to -\infty$. Finally, $(2/3)\epsilon_i \leq |v(0)| \leq \epsilon_i$ means that $v \neq 0$, while $U(0) \geq R$ assures the unboundedness of u (see Corollary C.1). □

This theorem shows that in fact there is a variety of backward-time exponentially growing solutions, which is a big contrast with the 1-D space periodic Kuramoto-Sivashinsky case (see [6]).

G. Properties of the global solutions with bounded quotient Λ

Now we will turn to the study of the quotient Λ, as well as to the link that exists between Λ and λ.

First, observe that if λ is bounded then Λ is bounded as well. In fact, in this case $\Lambda(t) \to 1$ as $t \to -\infty$ very fast:

Proposition G.1. *Suppose $\lambda(t) \leq \overline{\lambda}$ for $t \leq 0$. Then there exist some positive constants M and C, as well as a time $\tau_0 \leq 0$ such that*

$$\Lambda(t) - 1 \leq M e^{2(\overline{\lambda}-1)t - Ce^{-t}}$$

for all $t \leq \tau_0$.

Proof. By the definition of Λ,

$$\Lambda = \frac{U^2 + \frac{\lambda}{2}|v|^2}{U^2 + \frac{1}{2}|v|^2} = 1 + \frac{\lambda - 1}{2}\frac{|v|^2}{U^2 + \frac{1}{2}|v|^2} \leq 1 + \frac{\overline{\lambda} - 1}{2}\frac{|v|^2}{U^2 + \frac{1}{2}|v|^2}.$$

If we choose τ_0 from Proposition D.1, then for $t \leq \tau_0$,

$$\Lambda - 1 \leq \frac{\overline{\lambda} - 1}{2}\frac{C_1 e^{2\overline{\lambda}t - C_2 e^{-t}}}{K_1 e^{-t}} = M e^{2(\overline{\lambda}-1)t - C_2 e^{-t}},$$

where $M = \frac{\overline{\lambda}-1}{2}\frac{C_1}{K_1}$ and $C = C_2$.

$\square$

Proposition D.1 shows that if λ is bounded, the possible backward behaviors of the solutions of (3.3) are quite different from the case treated in [1]. However in our problem, the true correspondent of the Dirichlet quotient studied in [1] is the quotient Λ. The next proposition treats the case of $\limsup_{t\to-\infty} \Lambda < \infty$.

Proposition G.2. *Suppose that $u = (U, v)$ is a global solution of (3.3) for which $\Lambda(t) \leq \overline{\Lambda}$ and $\Lambda(t) \geq \underline{\Lambda}(\geq 1)$ for $t \leq 0$. Then*

$$\limsup_{t\to-\infty} \frac{|v(t)|^2}{2U(t)} \leq \overline{\Lambda} - 1$$

and

$$\liminf_{t\to-\infty} \frac{|v(t)|^2}{2U(t)} \geq \underline{\Lambda} - 1.$$

Proof. Let us denote

$$\mu = \frac{|v|^2}{2U}.$$

Then

$$\begin{aligned}
\mu_t &= \frac{(|v|^2)_t}{2U} - \frac{|v|^2}{2U^2}U_t = \frac{1}{U}(U|v|^2 - ||v||^2) - \frac{|v|^2}{2U^2}(R - U - \frac{1}{2}|v|^2) \\
&= \frac{1}{U}(U|v|^2 - 2||u||_V^2 + 2U^2) - \frac{R}{U}\mu + \mu + \mu^2 \\
&= \frac{1}{U}(U|v|^2 - 2\Lambda|u|_H^2 + 2U^2) - \frac{R}{U}\mu + \mu + \mu^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{U}(U|v|^2 - 2(\Lambda - 1)U^2 - \Lambda|v|^2) - \frac{R}{U}\mu + \mu + \mu^2 \\
&= 2U\mu - 2(\Lambda - 1)U - 2\Lambda\mu - \frac{R}{U}\mu + \mu + \mu^2.
\end{aligned}
$$

Observe that if $\Lambda \leq \overline{\Lambda}$, then

$$
\mu_t \geq 2U\mu - 2(\overline{\Lambda} - 1)U - 2\overline{\Lambda}\mu - \frac{R}{U}\mu + \mu + \mu^2.
$$

Hence,

$$
\begin{aligned}
\left(\mu - \overline{\Lambda}\right)_t \;\geq\; & \left(\mu + 2U - \overline{\Lambda} - \frac{R}{U}\right)\left(\mu - \overline{\Lambda}\right) + \left(\mu + 2U - \overline{\Lambda} - \frac{R}{U}\right) \\
+ \; & \left(\frac{R}{U} - \left(\overline{\Lambda} - 1 + \frac{R}{U}\right)\overline{\Lambda}\right).
\end{aligned}
$$

By Theorem C.1, for any $\alpha < 1$ there exists t_1 such that

$$
(1 - \alpha)\left(\mu + 2U - \overline{\Lambda} - \frac{R}{U}\right) \geq -\left(\frac{R}{U} - \left(\overline{\Lambda} - 1 + \frac{R}{U}\right)\overline{\Lambda}\right)
$$

for all $t < t_1$. Thus for $t < t_1$,

$$
(\mu - \overline{\Lambda})_t \geq \left(\mu + 2U - \overline{\Lambda} - \frac{R}{U}\right)(\mu - \overline{\Lambda}) + \alpha\left(\mu + 2U - \overline{\Lambda} - \frac{R}{U}\right).
$$

This shows that for $t < t_1$,

$$
(\mu - \overline{\Lambda} + \alpha)_t \geq \left(\mu + 2U - \overline{\Lambda} - \frac{R}{U}\right)(\mu - \overline{\Lambda} + \alpha).
$$

Thus

$$
\mu(t) - \overline{\Lambda} + \alpha \leq (\mu(t_1) - \overline{\Lambda} + \alpha)e^{-\int_t^{t_1}(\mu + 2U - \overline{\Lambda} - \frac{R}{U})\,d\tau}.
$$

Consequently, by Theorem C.1,

$$
\limsup_{t \to -\infty} \mu(t) - \overline{\Lambda} \leq -\alpha
$$

for any $\alpha < 1$, which readily implies the first part of the proposition.

If $\Lambda \geq \underline{\Lambda}$, then

$$\mu_t \leq 2U\mu - 2(\underline{\Lambda} - 1)U - 2\underline{\Lambda}\mu - \frac{R}{U}\mu + \mu + \mu^2.$$

And analogously to the previous case, we get

$$\liminf_{t\to-\infty} \mu(t) - \underline{\Lambda} \geq -\alpha$$

for every $\alpha > 1$, which concludes the proof.

□

Remark G.1. *The conclusions of the proposition above hold if* $\underline{\Lambda} = \liminf_{t\to-\infty} \Lambda(t)$ *and* $\overline{\Lambda} = \limsup_{t\to-\infty} \Lambda(t)$.

As an immediate consequence, we obtain

Corollary G.1. *If* $\overline{\Lambda} = \limsup_{t\to-\infty} \Lambda(t)$ *then for any* $M > \overline{\Lambda} - 1$ *there exists a time* $t_1 < 0$ *such that* $|v(t)|^2 < 2MU(t)$ *for all* $t < t_1$. *Likewise, If* $\underline{\Lambda} = \liminf_{t\to-\infty} \Lambda(t)$ *then for any* $m < \underline{\Lambda} - 1$ *there exists a time* $t_0 < 0$ *such that* $|v(t)|^2 \geq 2mU(t)$ *for all* $t < t_0$.

Proposition G.1 shows that when $\lambda(t)$ is bounded for $t < 0$, then so is $\Lambda(t)$. However if $\Lambda(t)$ is bounded, then $\lambda(t)$ might be unbounded.

Proposition G.3. *Suppose* $\underline{\Lambda} = \liminf_{t\to-\infty} \Lambda(t)$ *and* $\overline{\Lambda} = \limsup_{t\to-\infty} \Lambda(t)$. *If* $\overline{\Lambda} > 1$ *then* $\limsup_{t\to-\infty} \lambda(t)/U(t) \geq 1$. *If moreover,* $\underline{\Lambda} > 1$ *then*

$$1 \leq \limsup_{t\to-\infty} \frac{\lambda(t)}{U(t)} \leq \frac{\overline{\Lambda} - 1}{\underline{\Lambda} - 1}.$$

Also, in this case

$$\frac{\underline{\Lambda} - 1}{\overline{\Lambda} - 1} \leq \liminf_{t\to-\infty} \frac{\lambda(t)}{U(t)} \leq 1.$$

Proof. From the definition of Λ,

$$\Lambda = \frac{1 + \frac{\lambda}{U}\frac{|v|^2}{2U}}{1 + \frac{1}{U}\frac{|v|^2}{2U}}.$$

Thus

$$\Lambda \left(1 + \frac{1}{U}\frac{|v|^2}{2U}\right) - 1 = \frac{\lambda}{U}\frac{|v|^2}{2U}.$$

Taking upper limit of the previous equation as $t \to -\infty$, using Proposition G.2 we obtain

$$\begin{aligned} \overline{\Lambda} - 1 &= \limsup_{t\to-\infty} \Lambda \left(1 + \frac{1}{U}\frac{|v|^2}{2U}\right) - 1 = \limsup_{t\to-\infty} \frac{\lambda}{U}\frac{|v|^2}{2U} \\ &\leq \limsup_{t\to-\infty} \frac{\lambda}{U} \limsup_{t\to-\infty} \frac{|v|^2}{2U} \leq \limsup_{t\to-\infty} \frac{\lambda}{U}(\overline{\Lambda} - 1). \end{aligned}$$

If $\overline{\Lambda} > 1$, then the preceding inequality yields

$$1 \leq \limsup_{t\to-\infty} \frac{\lambda}{U}.$$

Now, if we consider a sequence $\{t_n\}$, such that $t_n \to -\infty$ and $\lim_{n\to\infty} \lambda(t_n)/U(t_n) = \limsup_{t\to-\infty} \lambda(t)/U(t)$, then

$$\begin{aligned} \overline{\Lambda} - 1 &= \limsup_{t\to-\infty} \Lambda(1 - \frac{1}{U}\frac{|v|^2}{2U}) - 1 = \limsup_{t\to-\infty} \frac{\lambda}{U}\frac{|v|^2}{2U} \\ &\geq \lim_{n\to\infty} \frac{\lambda(t_n)}{U(t_n)} \liminf_{t\to-\infty} \frac{|v|^2}{2U} \geq \limsup_{t\to-\infty} \frac{\lambda}{U}(\underline{\Lambda} - 1). \end{aligned}$$

Thus, if $\underline{\Lambda} > 1$, then

$$\limsup_{t\to-\infty} \frac{\lambda}{U} \leq \frac{\overline{\Lambda} - 1}{\underline{\Lambda} - 1}.$$

The identities for the lower limits can be established in a similar way.

□

Remark G.2. Observe that when $\limsup_{t\to\ -\infty} \Lambda(t) = \overline{\Lambda} < \infty$, most of the backward growth of $|u|_H$ is carried by the U component of u (see Corollary G.1). This means that $\lim_{t\to-\infty} u(t)/|u(t)|_H = (1,0)$, which is an eigenvector of the operator A corresponding to the eigenvalue 1 (which may not coincide with $\overline{\Lambda}$). In contrast, for the 2-D space periodic Navier-Stokes equations, if $\limsup_{t\to\ -\infty} \Lambda(t) = \overline{\Lambda}$ then $\overline{\Lambda}$ is an eigenvalue for the Stokes operator and, for some $t_n \to -\infty$, $u(t_n)/|u(t_n)|_H$ will converge, at least in $L^2_{loc}(\mathbb{R}, H)$, to an eigenvector corresponding to the eigenvalue $\overline{\Lambda}$.

H. L^∞ estimates and backward-time exponential growth

Consider again $u = (U, v)$ - a global solution of the system (3.3), with $v(0) \neq 0$ and $u(0) \notin \mathcal{A}$. By Theorem C.1, U increases backward in time at least exponentially (i.e. bounded below by an exponential). Thus without loss of generality, we can assume that $U(0) > R$. Also, the fact that $U(t)$ increases at least exponentially for $t \to -\infty$ implies that $|u(t)|_H$ increases at least exponentially for $t \to -\infty$ as well. Moreover, if $|v|$ grows backward in time at most exponentially (i.e bounded above by an exponential), the first equation of (3.3) implies that so does U, which means that $|u|_H$ must also increase at most exponentially backward in time.

The results in Section G suggest that backward in time, the growth of $|v|$ (and consequently, of $|u|_H$) might be controlled by the backward-time growth of U for any unbounded global solution u. This would imply that if U grows backward in time (at most) exponentially, than $|v|$ will grow backward in time at most exponentially. The following two theorems address precisely these questions.

Theorem H.1. *For any $u(t) = (U(t), v(t))$ - unbounded global solution of (3.3) and any $\alpha > 1$,*

$$\lim_{t\to-\infty} \frac{|v(t)|^2}{U(t)^\alpha} = 0. \tag{3.36}$$

Proof. Since we can assume $U(t) > R$ for all $t \le 0$, the first equation of (3.3) implies

$$U(t) \ge (U(0) - R)e^{-t} + R > (U(0) - R)e^{-t}. \tag{3.37}$$

Divide both sides of the first equation of (3.3) by U^α:

$$\frac{U'(t)}{U(t)^\alpha} = \frac{R}{U(t)^\alpha} - \frac{1}{U(t)^{\alpha-1}} - \frac{|v(t)|^2}{2U(t)^\alpha}.$$

Thus

$$\frac{d}{dt}\frac{1}{(\alpha-1)U(t)^{\alpha-1}} + \frac{1}{U(t)^{\alpha-1}} - \frac{R}{U(t)^\alpha} = \frac{|v(t)|^2}{2U(t)^\alpha}.$$

Using (3.37) we obtain

$$\frac{d}{dt}\frac{1}{(\alpha-1)U(t)^{\alpha-1}} + \frac{1}{(U(0)-R)e^{-(\alpha-1)t}} \geq \frac{|v(t)|^2}{2U(t)^\alpha}.$$

Integrate from $t < 0$ to 0 to obtain

$$\frac{1}{(\alpha-1)U(0)^{(\alpha-1)}} - \frac{1}{(\alpha-1)U(t)^{(\alpha-1)}} + \frac{1-e^{(\alpha-1)t}}{(\alpha-1)(U(0)-R)} \geq \int_t^0 \frac{|v(\tau)|^2}{2U(\tau)^\alpha}d\tau.$$

It follows that

$$\int_{-\infty}^0 \frac{|v(\tau)|^2}{2U(\tau)^\alpha}d\tau < \infty,$$

and thus, (3.36) holds. □

Using the last result we can prove the following theorem.

Theorem H.2. *let $u = (U, v)$ be a global solution of (3.3) which does not lie inside the global attractor $\mathcal{A}$. Then $|u(t)|_H$ will grow backward in time at least exponentially, and the following are equivalent as $t \to -\infty$:*

(i) *$|u(t)|_H$ grows at most exponentially;*

(ii) *$U(t)$ grows at most exponentially;*

(iii) *$|v(t)|_{L^2}$ grows at most exponentially.*

Proof. Observe that by time translation, we can assume $U(0) > R$. As we saw in Theorem C.1, $U(t)$ grows, backward in time, at least exponentially, which will force $|u(t)|_H$ to grow at least exponentially too. At the beginning of this section, we discussed the case (iii)⇒(ii)⇒(i). It is clear that (i)⇒(ii) and (i)⇒(iii). To complete the proof of the equivalence, it is sufficient to prove (ii)⇒(iii). But, if (ii) holds, there exist positive constants γ, C such that $U(t) \leq Ce^{-\gamma t}$ for $t < 0$. By Theorem H.1, for $\alpha > 1$ there exists $t_0 < 0$, such that

$$|v(t)|_{L^2} \quad < \quad U(t)^{\alpha} \leq \left(Ce^{-\gamma t}\right)^{\alpha} = C^{\alpha} e^{-\alpha\gamma t}$$

for any $t < t_0$. Thus (iii) is true.

□

It turns out that even the L^∞ norm of the local term v of (3.3) is bounded by the nonlocal term U for negative times. To show this we will need some additional estimates for v.

If we take a derivative with respect to x of the second equation of (3.3), we get

$$(v_x)_t = Uv_x + (v_x)_{xx} - 2v(v_x)_x - 2(v_x)^2. \tag{3.38}$$

Denote $m(t) = \max_x v_x(t,x)$. Observe that $m(t) \geq 0$ for any t. For every t let x_t be such that $v_x(t,x_t) = m(t)$. Then

$$(\overline{D}_- m)(t) := \limsup_{\tau \nearrow t} \frac{m(t) - m(\tau)}{t - \tau} \leq \lim_{\tau \nearrow t} \frac{v_x(t,x_t) - v_x(\tau,x_t)}{t-\tau} = \frac{\partial v_x}{\partial t}(t,x_t).$$

Thus (3.38) implies that

$$(\overline{D}_- m)(t) \leq \frac{\partial v_x}{\partial t}(s,x_t) = U(t)m(t) + (v_x)_{xx}(t,x_t) - 2m^2(t).$$

Since in the point (t,x_t) the value of $(v_x)_{xx}$ is nonpositive, we obtain:

$$(\overline{D}_- m)(t) \leq U(t)m(t) - 2m^2(t).$$

Assume that $m(t) > U(t)$ for $t_0 \leq t \leq t_1$. Then

$$(\overline{D}_{-}m)(t) \leq -m^2(t) \tag{3.39}$$

for $t_0 \leq t \leq t_1$.

To obtain an estimate for m we will need the following lemma.

Lemma H.1. *Let $M \in C^1([t_0, t_1])$ be the function satisfying*

$$\begin{cases} M'(t) = -M^2(t) \\ M(m_1) = m(t_1) \end{cases}. \tag{3.40}$$

Then $m(t) \geq M(t)$ for every $t \in [t_0, t_1]$.

Proof. Observe that if a function $\phi \in C([t_0, t_1])$ with $\overline{D}_{-}\phi \leq 0$, then ϕ is nonincreasing (the above conditions on ϕ imply that for any $\epsilon > 0$, $\phi(t) \leq \phi(\tau) + \epsilon(t - \tau)$ with $t_0 \leq \tau \leq t \leq t_1$ which forces ϕ to be nondecreasing).

Now consider the following IVP:

$$\begin{cases} \psi'(t) = -m^2(t) \\ \psi(t_1) = m(t_1) \end{cases}.$$

Then $\overline{D}_{-}(m - \psi) \leq 0$. Hence by the above, $m(t) - \psi(t) \geq m(t_1) - \psi(t_1) = 0$ for $t_0 \leq t \leq t_1$. Consequently, $m(t) \geq \psi(t) \geq m(t_1) \geq 0$ for $t_0 \leq t \leq t_1$. Then

$$\begin{cases} \psi'(t) \leq -\psi^2(t) \\ \psi(t_1) = m(t_1) \end{cases}.$$

We combine this system with (3.40) to obtain

$$\begin{cases} (\psi - M)'(t) \leq -(\psi(t) - M(t))(\psi(t) + M(t)) \\ \psi(t_1) - M(t_1) = 0 \end{cases}.$$

The Gronwall inequality implies that

$$(\psi - M)(t) \geq (\psi - M)(t_1)e^{\int_t^{t_1} (\psi(\tau)+M(\tau))\, d\tau}.$$

Thus $\psi(t) \geq M(t)$ on $[t_0, t_1]$. Because we also have $m(t) \geq \psi(t)$, the conclusion of the lemma follows.

□

Solving (3.40), we will obtain

$$\frac{1}{M(t_1)} - \frac{1}{M(t)} = t_1 - t.$$

So, by the lemma above,

$$\frac{1}{m(t_1)} - \frac{1}{m(t)} \geq t_1 - t \tag{3.41}$$

for $t \in [t_0, t_1]$. Obviously we must have

$$t_0 > t_1 - \frac{1}{m(t_1)} \geq t_1 - \frac{1}{U(t_1)} =: \theta(t_1).$$

In this way we have established the following result.

Proposition H.1. *If (U, v) is a global solution of (3.3) with $U(0) > R$, then for all t there is $\tau_t \in [\theta(t), t]$ such that $v_x(\tau_t, x) \leq U(\tau_t)$, for all $x \in [-1, 1]$.*

This proposition has two interesting consequences.

Corollary H.1. *If (U, v) is a global solution of (3.3) with $U(0) > R$, then for every $t \leq 0$, $m(t) = \max_x v_x(t, x) < U(\theta(t))$.*

Proof. Assume that for some $t < 0$, $m(t) \geq U(\theta(t))$. Since $m(\cdot)$ and $U(\cdot)$ are decreasing, we see that $m(\tau) \geq U(\theta(t)) \geq U(\tau)$ for all τ in $[\theta(t), t]$, which contradicts the previous proposition. Therefore $m(t) < U(\theta(t))$ holds for all $t < 0$.

□

Corollary H.2. *If (U, v) is a global solution of (3.3) with $U(0) > R$, then for every $t < 0$,*

$$|v(t)|_\infty := \sup_x |v(t,x)| < 2U(\theta(t)).$$

Proof. Fix $t < 0$. Because the x-average of v is assumed to be zero, there is a point x_t such that $v(t, x_t) = 0$. By a translation of the solution in x we can assume $x_t = -1$. Then using the previous corollary, we obtain

$$v(t,x) = \int_{-1}^{x} v_x(t,\xi)\, d\xi < U(\theta(t))\,(x+1) < 2U(\theta(t)).$$

Observe, that if $(U(t), v(t,x))$ solves (3.3), then $(U(t), -v(t,-x))$ will also solve (3.3). By the above, $-v(t,-x) < 2U(\theta(t))$. Consequently

$$|v(t,x)| < 2U(\theta(t)).$$

Thus (i) is true. □

Corollary H.1 gives us a bound on v_x in terms of U. Actually, $v_x(t,x)$ is bounded above by $U(t)$ "most" of the time as $t \to -\infty$:

Proposition H.2. *For a global solution (U, v), with $U(0) > R$, the Lebesgue measure of the set $S(t_1) = \{t \geq t_1 : \; m(t) > U(t)\}$ does not exceed $1/U(t_1)$.*

Proof. Since m and U are continuous, it follows that $S(t_1) = \bigcup_{n=1}^{\infty} I_n$, where $I_n = (t_{0n}, t_{1n})$ is a sequence of disjoint intervals with $t_1 \geq t_{1n} > t_{0n} \geq t_{1(n+1)}$ for any n. The measure of $S(t_1)$ is $\sum_{n=1}^{\infty} l(I_n)$, where $l(I_n) = (t_{1n} - t_{0n})$ - the length of the interval I_n. Observe that $m(t_{in}) = U(t_{in})$ for any $i = 0, 1$ and $n \in \mathbb{N}$. Inequality (3.41) written on I_n gives us

$$\frac{1}{U(t_{1n})} - \frac{1}{U(t_{0n})} \geq t_{1n} - t_{0n} = l(I_n).$$

Thus, for any natural number N,

$$\begin{aligned}\sum_{n=1}^{N} l(I_n) &\leq \sum_{n=1}^{N}\left(\frac{1}{U(t_{1n})}-\frac{1}{U(t_{0n})}\right)\\ &= \frac{1}{U(t_{11})}-\sum_{n=1}^{N-1}\left(\frac{1}{U(t_{0n})}-\frac{1}{U(t_{1(n+1)})}\right)-\frac{1}{U(t_{0N})}\leq\frac{1}{U(t_{11})},\end{aligned}$$

because $\frac{1}{U(t_{0n})}-\frac{1}{U(t_{1(n+1)})}\geq 0$. This means that

$$\sum_{n=1}^{\infty} l(I_n)\leq\frac{1}{U(t_{11})}\leq\frac{1}{U(t_1)},$$

and the proof is complete. □

Proposition H.2 has the following immediate consequence.

Corollary H.3. *Let $u=(U,v)$ be an unbounded global solution of (3.3). Then the Lebesgue measure of the set $S(t)$ goes to 0 as $t\to-\infty$ at least exponentially.*

Proof. Proposition H.2 implies that $\mu(S(t))$ - the Lebesgue measure of the set $S(t)$, satisfies $\mu(S(t))\leq 1/U(t)$ for any $t\leq 0$. But by Theorem C.1, $U(t)\geq Ce^{-t}$ for some $C>0$ and every $t<0$. Thus $\mu(S(t))\leq(1/C)e^t$ for all $t\leq 0$, which concludes the proof.

□

I. Comparison with the other dissipative PDE

To compare Burgers' original model for turbulence to similar systems of differential equations, we discern three possible behaviors of a solution as $t\to-\infty$: it can be globally bounded, blow up exponentially, or, perhaps, blow up faster than any exponential. It is interesting that there are no global solutions that blow up with a rate less than e^{-t} as $t\to-\infty$ (as was proved is Theorem C.1). The existence of a variety of globally bounded solutions follows from the fact that the global attractor of (3.3) is quite a rich set (see [22]).

In this chapter we were able to prove existence of a family of global solutions that blow up exponentially backward in time, namely those for which the Dirichlet quotient $\lambda(t)$ is bounded for $t < 0$ (see Theorem F.1). This fact shows a clear difference in backward-time behavior form the 1-D space periodic Kuramoto-Sivashinsky case studied in [6, 7]. Moreover, similar to the 2-D space periodic Navier-Stokes equations (see [1]), for any eigenvalue λ_i of $A_0 = -\frac{\partial^2}{\partial x^2}$ there exist solutions $u = (U, v)$ with $\lim_{t\to-\infty} \lambda(t) = \lambda_i$, and $\lim_{n\to\infty} \frac{v(t_n)}{|v(t_n)|} = w_i$ for some sequence $\{t_n\}$ decreasing to $-\infty$ and some unit vector w_i satisfying $A_0 w_i = \lambda_i w_i$ (see Theorems E.1 and F.1). However unlike in the 2-D Navier-Stokes case, for a suitable $c > 0$, $|v(t)| = o(e^{-ce^{-t}})$ as $t \to -\infty$ (see Proposition D.1).

The behavior of global solutions of system (3.3) with bounded Dirichlet quotient Λ is also interesting. While we were unable to prove existence of any global solutions with $\limsup_{t\to-\infty} \Lambda(t) > 1$, the fact that if Λ is bounded, then $\lim_{t\to-\infty} \frac{u(t)}{|u(t)|_H} = (1, 0)$ (see Remark G.2), establishes another important difference with the 2-D Navier-Stokes equations studied in [1]. In that case, for any eigenvalue of the Stokes operator there exists a solution u such that for a sequence $\{t_n\}$ decreasing to $-\infty$, $\frac{u(t_n)}{|u(t_n)|_H}$ converges in a specific space H to an eigenvector corresponding to that eigenvalue.

The results presented above make Burgers' original model for turbulence a very peculiar example of a dynamical system with a rich set of backward-time exponentially blowing up solutions which, unlike the 2-D space-periodic Navier-Stokes equations and their α-model, does not display any similarity to the linear case.

CHAPTER IV

CONCLUSIONS

Since the introduction in 1995 in [1] of the attractor-like sets M_n, there is still very little known about their geometrical structure and the relationship between these sets and other invariants of the dissipative PDE, in particular, global and exponential attractors, and inertial manifolds. For the 2-D periodic Navier-Stokes equations (NSE) we were able to obtain new geometric properties of these sets and show that the solutions on this sets will grow exponentially in any Sobolev norm (see Chapter II). A further study is needed in order to resolve whether in this case the sets $\mathcal{M}_n$ have a manifold structure and whether their union is dense in H. As we mentioned in Chapter I, a remarkable similarity in the properties of $\mathcal{M}_n$ between the NSE and the linear case, are in the sharp contrast with the Kuramoto-Sivashinsky equation (KSE), for which there are no unbounded invariant sets at all. On the other hand, various NSE-like systems still displayed NSE-like behavior for negative-times. The Burgers' original model for turbulence (BOMT) is the first known dissipative PDE that has a completely different backward-time dynamics from both NSE and KSE cases (as is described at the end of Chapter III).

We can in fact characterize dissipative PDE by their backward-time behavior as follows:

1. Linear-like systems, with $\mathcal{M}_n$ rich for all n, and all the rates of exponential growth present (NSE, Kelvin-filtered NSE);

2. Equations with the fast backward-time growth, i.e. satisfying $\mathcal{M}_n = \mathcal{A}$ for all n (KSE);

3. Equations which admit some backward-time exponentially growing solutions,

but with a very non-linear negative-time dynamics (BOMT).

Additionally, I am working on several remaining problems related to the better understanding the backward time behavior of the PDEs described above. In particular, for the Burgers' original model for turbulence more studies are needed to answer the following questions:

1. Are there any $\mathcal{M}_n \neq \mathcal{M}_1$?
2. Are there any solutions on $\mathcal{M}_1$ that have unbounded $\lambda(t)$?
3. Is $\cup_n \mathcal{M}_n$ dense, at least in some sense, in H?
4. What are the geometrical properties of $\mathcal{M}_n$?

The questions about the NSE are:

1. What is the geometrical structure of $\mathcal{M}_n$?
2. Can the density property of $\cup_n \mathcal{M}_n$ be improved?

The answers to the questions above will help better understanding of the structure of the sets $\mathcal{M}_n$ and make the classification more precise.

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APPENDIX A

ESTIMATE FOR THE NONLINEAR TERM (SEE [30])

Lemma 1. *For each* $n \in \mathbb{N}$ $(n \geq 2)$ *and every* $u \in D(A^n)$*:*

$$b(u, u, A^n u) = -\sum_{h=1}^{n-1} b(A^h u, u, A^{n-h} u). \tag{A.1}$$

Proof. Observe first that

$$A(B(u, v) + B(v, u)) = B(u, Av) + B(v, Au) - B(Au, v) - B(Av, u).$$

Thus, if n is odd, then

$$\begin{aligned} A\sum_{h=0}^{n} B(A^h u, A^{n-h} u) &= \sum_{h=0}^{(n+1)/2} A\left(B(A^h u, A^{n-h} u) + B(A^{n-h} u, A^h u)\right) \\ &= \sum_{h=0}^{(n+1)/2} B(A^h u, A^{n-h+1} u) + \sum_{h=0}^{(n+1)/2} B(A^{n-h} u, A^{h+1} u)) \\ &- \sum_{h=0}^{(n+1)/2} B(A^{h+1} u, A^{n-h} u) - \sum_{h=0}^{(n+1)/2} B(A^{n-h+1} u, A^h u)) \\ &= B(u, A^{n+1} u) - B(A^{n+1} u, u). \end{aligned}$$

Consequently, in this case,

$$\begin{aligned} \sum_{h=0}^{n} b(A^h u, u, A^{n-h} u) = -\sum_{h=0}^{n} b(A^h u, A^{n-h} u, u) = \\ -\left(A\sum_{h=0}^{n} B(A^h u, A^{n-h} u), A^{-1} u\right) = \\ -b(u, A^{n+1}, A^{-1} u) + b(A^{n+1} u, u, A^{-1} u) \\ = b(u, A^{-1} u, A^{n+1} u) - b(A^{n+1} u, A^{-1} u, u) = 0, \end{aligned}$$

since

$$b(Av, v, w) = b(w, v, Av)$$

for every $v \in D(A)$ and $w \in H$.

If n is even, we have

$$\begin{aligned} A\sum_{h=0}^{n} B(A^h u, A^{n-h}u) = AB(A^n u, u) + \sum_{h=0}^{n/2} A\left(B(A^h u, A^{n-h}u) + B(A^{n-h}u, A^h u)\right) \\ = AB(A^n u, u) + B(u, A^n u) - B(A^n u, u). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{h=0}^{n} b(A^h u, u, A^{n-h}u) = -\sum_{h=0}^{n} b(A^h u, A^{n-h}u, u) = \\ -\left(A\sum_{h=0}^{n} B(A^h u, A^{n-h}u), A^{-1}u\right) = \\ -\left(AB(A^n u, u), A^{-1}u\right) - b(u, A^{n+1}u, A^{-1}u) + b(A^{n+1}u, u, A^{-1}u) \\ = b(A^n u, u, u) + 0 = 0. \end{aligned}$$

Consequently the identity from the lemma holds for all n.

□

Theorem 1. *For each $n \in \mathbb{N}$ ($n \geq 2$) and every $u \in D(A^n)$:*

$$|b(u, u, A^n u)| \leq c_0 c_n |A^{n/2}u|\,|A^{(n+1)/2}u|\,|A^{1/2}u|, \tag{A.2}$$

where $c_n := 6([n/2] + (n - [n/2])2^{n-2})$.

Proof. Observe that going to the Fourier coefficients:

$$|b(u,v,w)| = \left| \sum_{j+k+l=0\in\mathbb{Z}^2} (a_k \cdot j)(b_j \cdot c_l) \right| \le \sum_{j+k+l=0\in\mathbb{Z}^2} |a_k|\,|j|\,|b_j|\,|c_l| := \tilde{b}(u,v,w).$$

where

$$u(x) = \sum_{k\in\mathbb{Z}^2} a_k e^{(2\pi i/L)(k\cdot x)}, \quad v(x) = \sum_{j\in\mathbb{Z}^2} b_j e^{(2\pi i/L)(j\cdot x)}, \quad w(x) = \sum_{l\in\mathbb{Z}^2} c_l e^{(2\pi i/L)(l\cdot x)},$$

with $u, w \in H$, $v \in V$.

Using the previous lemma we get

$$|b(u,u,A^n u)| \le \sum_{h=1}^{n-1} |b(A^h u, u, A^{n-h} u)| \le \sum_{h=1}^{n-1} \tilde{b}(A^h u, u, A^{n-h} u).$$

Observe that

$$\begin{aligned} \sum_{h=1}^{n-1} \tilde{b}(A^h u, u, A^{n-h} u) &= \sum_{h=1}^{n-1} \sum_{j+k+l=0\in\mathbb{Z}^2} |k|^{2h} |a_k|\,|j|\,|a_j|\,|a_l|\,|l|^{2(n-h)} \\ &\le A + B + C, \end{aligned}$$

where

$$A := \sum_{h=1}^{n-1} \sum_{j+k+l=0\in\mathbb{Z}^2, |k|\le\min\{|j|,|l|\}} |k|^{2h} |a_k|\,|j|\,|a_j|\,|a_l|\,|l|^{2(n-h)},$$

$$B := \sum_{h=1}^{n-1} \sum_{j+k+l=0\in\mathbb{Z}^2, |j|\le\min\{|k|,|l|\}} |k|^{2h} |a_k|\,|j|\,|a_j|\,|a_l|\,|l|^{2(n-h)},$$

and

$$C := \sum_{h=1}^{n-1} \sum_{j+k+l=0\in\mathbb{Z}^2, |l|\le\min\{|j|,|k|\}} |k|^{2h} |a_k|\,|j|\,|a_j|\,|a_l|\,|l|^{2(n-h)}.$$

Because of the symmetry we have that $A = C$.

Also,

$$B \geq \sum_{h=1}^{n-1} \sum_{j+k+l=0\in\mathbb{Z}^2, |k|\leq \min\{|j|,|l|\}} |k|^{2h}|a_k|\,|j|^{2(n-h)}\,|a_j|\,|a_l|\,|l| = C(=A),$$

and thus we get

$$\begin{aligned}
|b(u,u,A^n u)| \leq 3B &\leq 3\sum_{h=1}^{n-1} 2 \sum_{j+k+l=0\in\mathbb{Z}^2, |j|\leq|l|\leq|k|} |k|^{2h}|a_k|\,|j|\,|a_j|\,|a_l|\,|l|^{2(n-h)} \\
&= 6\sum_{h=1}^{[n/2]} \sum_{j+k+l=0\in\mathbb{Z}^2, |j|\leq|l|\leq|k|} |k|^{2h}|a_k|\,|j|\,|a_j|\,|a_l|\,|l|^{2(n-h)} \\
&+6\sum_{h=[n/2]+1}^{n-1} \sum_{j+k+l=0\in\mathbb{Z}^2, |j|\leq|l|\leq|k|} |k|^{2h}|a_k|\,|j|\,|a_j|\,|a_l|\,|l|^{2(n-h)} \\
&\leq 6\sum_{h=1}^{[n/2]} \sum_{j+k+l=0\in\mathbb{Z}^2, |j|\leq|l|\leq|k|} |k|^{n}|a_k|\,|j|\,|a_j|\,|a_l|\,|l|^{n} \\
&+6\sum_{h=[n/2]+1}^{n-1} \sum_{j+k+l=0\in\mathbb{Z}^2, |j|\leq|l|\leq|k|} |k|^{2h}|a_k|\,|j|\,|a_j|\,|a_l|\,|l|^{2(n-h)}.
\end{aligned}$$

Observe that in the previous sums, $|k| = |j+l| \leq |j| + |l| \leq 2|l|$. Thus,

$$\begin{aligned}
&6\sum_{h=[n/2]+1}^{n-1} \sum_{j+k+l=0\in\mathbb{Z}^2, |j|\leq|l|\leq|k|} |k|^{2h}|a_k|\,|j|\,|a_j|\,|a_l|\,|l|^{2(n-h)} \\
\leq 6 &\sum_{h=[n/2]+1}^{n-1} \sum_{j+k+l=0\in\mathbb{Z}^2, |j|\leq|l|\leq|k|} |k|^{n}(2|l|)^{2h-n}|a_k|\,|j|\,|a_j|\,|a_l|\,|l|^{2(n-h)} \\
&\leq 6(n-[n/2])2^{n-2} \sum_{j+k+l=0\in\mathbb{Z}^2, |j|\leq|l|\leq|k|} |k|^{n}|a_k|\,|j|\,|a_j|\,|a_l|\,|l|^{n}.
\end{aligned}$$

Also,

$$6\sum_{h=1}^{[n/2]} \sum_{j+k+l=0\in\mathbb{Z}^2, |j|\leq|l|\leq|k|} |k|^{n}|a_k|\,|j|\,|a_j|\,|a_l|\,|l|^{n}$$

$$\leq 6([n/2]) \sum_{j+k+l=0\in\mathbb{Z}^2, |j|\leq|l|\leq|k|} |k|^n |a_k|\,|j|\,|a_j|\,|a_l|\,|l|^n.$$

Consequently,

$$|b(u,u,A^n u)| \leq 6([n/2] + (n-[n/2])2^{n-2}) \sum_{j+k+l=0\in\mathbb{Z}^2, |j|\leq|l|\leq|k|} |k|^n |a_k|\,|j|\,|a_j|\,|a_l|\,|l|^n.$$

Let $c_n = 6([n/2] + (n-[n/2])2^{n-2})$. From the above we conclude that

$$|b(u,u,A^n u)| \leq c_n \tilde{b}(A^{n/2}u, u, A^{n/2}u) = c_n \int_\Omega \phi(x)\psi(x)\zeta(x)\,dx,$$

where we denote $\phi(x) = \sum_k e^{(2\pi i/L)k\cdot x}|a_k|\,|k|^n$, $\psi(x) = \sum_j e^{(2\pi i/L)j\cdot x}|a_j|\,|j|$, and $\zeta(x) = \sum_l e^{(2\pi i/L)l\cdot x}|a_l|\,|l|^n$. Applying Schwartz inequality we get

$$|b(u,u,A^n u)| \leq c_n |\phi|_{L^4} |\psi|_{L^2} |\zeta|_{L^4}.$$

Now apply Ladyzhenskaya inequality

$$|w|_{L^4}^2 \leq c_0 |w|_{H^1} |w|_{L^2}$$

to estimate $|\phi|_{L^4}$ and $|\zeta|_{L^4}$ and obtain

$$|b(u,u,A^n u)| \leq c_0 c_n |A^{n/2}u|\,|A^{(n+1)/2}u|\,|A^{1/2}u|.$$

□

VITA

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