CAUSAL EQUIVALENCE OF FRAMES

A Dissertation

by

TROY LEE HENDERSON, IV

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2005

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ABSTRACT

Causal Equivalence of Frames. (August 2005)

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Frames have recently become popular in the area of applied mathematics known as digital signal processing. Frames offer a level of redundancy that bases do not provide. In a sub-area of signal processing known as data recovery, redundancy has become increasingly useful; therefore, so have frames. Just as orthonormal bases are desirable for numerical computations, Parseval frames provide similar properties as orthonormal bases while maintaining a desired level of redundancy. This dissertation will begin with a basic background on frames and will proceed to encapsulate my research as partial fulfillment of the requirements for the Ph.D. degree in Mathematics at Texas A&M University. More specifically, in this dissertation we investigate an apparently new concept we term *causal equivalence* of frames and techniques for transforming frames into Parseval frames in a way that generalizes the Classical Gram-Schmidt process for bases. Finally, we will compare and contrast these techniques.

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CHAPTER I

INTRODUCTION AND PRELIMINARIES

A. Introduction

Merriam-Webster Online Dictionary defines a *signal* to be *a detectable physical quantity or impulse by which messages or information can be transmitted.* Signals are used as a means of communication either between people or between people and machines. Signal processing is an area of engineering and applied mathematics dealing with the representation of signals as mathematical objects as well as the transformation, manipulation, and interpretation of these objects. Signals are often categorized in one of two forms – *analog* and *digital.*

Since analog signals are the natural type of signals that people understand and digital signals are the natural type of signals for machines (in particular, computers and other forms of modern technology), it is often necessary to convert analog signals to digital ones. This process is called *sampling* or *encoding.* Once this conversion takes place, the area of *digital signal processing* (DSP) provides many techniques for manipulating this digital data. For example, techniques for providing a digital low-pass filter can be used to separate an intended signal from potential noise as well as to provide digital compression. Also, techniques for providing a digital band-pass filter can be used to isolate individual frequency components of the signal.

Once the digital signal has been processed (or analyzed), it is usually converted back into analog format. This process is known as *decoding.* Many times the decoded signal is different from the original analog signal. This is often unintentional; however, there are many cases where this difference is desired. For example, the process of

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storing sounds from an audio cassette tape onto a digital compact disc can take advantage of several DSP techniques. Since many audio cassette tapes inherently contain wanted sounds as well as an unwanted high frequency "hiss", certain DSP techniques can remove this unwanted "hiss".

Currently, the standard process for transmitting analog information from a source to a destination requires that the analog signal be encoded, processed, transmitted, received, anti-processed, and decoded. That is, once the signal is encoded, techniques are used to process the encoded signal before transmission. This is done to help guarantee that the received signal is as close to the transmitted signal as possible. Once the signal has been received, anti-processing is done to "undo" the original processing. Finally, the signal is decoded. All of this work is done to ensure the authenticity of the received signal.

One such technique for pre-transmission processing is to provide the encoded signal with a particular level of redundancy. A naive approach to accomplish this is to transmit the signal multiple times. As a consequence of this approach, a substantial increase in bandwidth is required. A more modern approach makes use of tools called *frames*. Frames [3] offer a way of providing an encoded signal with an arbitrary level of redundancy. The most desirable types of frames are called *Parseval* frames. Parseval frames have many of the same properties as classical signal processing tools while providing redundancy to the encoded signal.

A *discrete linear system* (or simply *system*) is viewed mathematically as a linear transformation (or operator) T that maps an input sequence $X = \{x_i\}_{i\in\mathbb{J}}$ into an output sequence $Y = \{y_i\}_{i\in\mathbb{J}}$ denoted by

$$Y = TX$$

where $\mathbb{J}$ is a discrete ordered set. T is often thought of by engineers as a "black box"

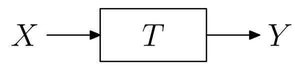


Fig. 1. Discrete Linear System.

that completely describes the input/output process and is indicated pictorially in Fig. 1. A system is said to be *causal* if the output does not depend on future input. That is, if the sequence X is the input to a causal system and the resulting output sequence is Y, then for each $i \in \mathbb{J}$, y_i does not depend on x_j when $j \in \mathbb{J}$ and $j > i$.

A goal of this dissertation is to provide a causal way of converting ordinary frames into Parseval frames. Since the Classical Gram-Schmidt process accomplishes this for a particular class of frames (namely, bases), we investigate processes needed to generalize this method to all frames. Work by Casazza and Kutyniok [2] offers an algorithm for transforming frames into Parseval frames. However, their method is not causal. Furthermore, if the initial frame is in fact Parseval, their algorithm produces a Parseval frame that is (in general) different from the original Parseval frame. In this dissertation, we investigate two additional methods for transforming frames into Parseval frames. Furthermore, with appropriate care, the use of the two methods guarantees that the resulting Parseval frame is identical to the original frame if the original frame is itself Parseval. Finally, we show that these additional methods are equivalent.

B. Necessary Fourier Analysis Preliminaries

The most fundamental “tool of the trade” for analyzing signals is Fourier analysis. Much work has been done in applied [5] and theoretical [4][9] Fourier analysis in the area of signal processing known as *wavelets*. Wavelets are most often considered in the context of orthonormal bases; however, recent work [7] provides analysis of wavelets as frames. Let H be a (separable) Hilbert space indexed by a (countable) index set $\mathbb{J}$ and let $K = \ell^2(\mathbb{J})$. If we let $f \in H$ represent our signal, then the goal of Fourier analysis is to transform f into $a \in \ell^2(\mathbb{J})$, perform the analysis on a, and then transform a back into f. If we let $E = \{e_i\}_{i\in\mathbb{J}} \subset H$ be an orthonormal basis for H, then one method of encoding f into $a \in \ell^2(\mathbb{J})$ is by defining $a_i = \langle f, e_i\rangle$ for each $i \in \mathbb{J}$. Since E is an orthonormal basis for H, it follows that

$$f = \sum_{i\in\mathbb{J}} \langle f, e_i\rangle\, e_i = \sum_{i\in\mathbb{J}} a_i\, e_i \tag{1.1}$$

for each $f \in H$. Furthermore, by Parseval's identity, we have

$$\|f\|_H = \|a\|_K. \tag{1.2}$$

However, E need not be an orthonormal basis in order for Equations 1.1 and 1.2 to hold. To illustrate this fact, consider the following example.

Example 1. Let

$$E = \left\{ \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right\}.$$

It is clear that E is not an orthonormal basis for $\mathbb{R}^2$ since E is neither a linearly independent nor an orthogonal set. However, for each $x \in \mathbb{R}^2$ we have

$$\begin{aligned} \sum_{i=1}^{3} \langle x, e_i\rangle\, e_i = &\left(x_1 \cdot \left(\frac{\sqrt{2}}{\sqrt{3}}\right) + x_2 \cdot 0\right) \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \end{pmatrix} \\ &+ \left(x_1 \cdot \left(-\frac{1}{\sqrt{6}}\right) + x_2 \cdot \left(\frac{1}{\sqrt{2}}\right)\right) \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ &+ \left(x_1 \cdot \left(-\frac{1}{\sqrt{6}}\right) + x_2 \cdot \left(-\frac{1}{\sqrt{2}}\right)\right) \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \end{aligned}$$

Expanding and simplifying, we get

$$\sum_{i=1}^{3} \langle x, e_i \rangle \, e_i = \begin{pmatrix} \left(\frac{2}{3} + \frac{1}{6} + \frac{1}{6}\right) \cdot x_1 + \left(0 - \frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{3}}\right) \cdot x_2 \\ \left(0 - \frac{1}{2\sqrt{3}} + \frac{1}{2\sqrt{3}}\right) \cdot x_1 + \left(0 + \frac{1}{2} + \frac{1}{2}\right) \cdot x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x.$$

Therefore, it is clear that Equation 1.1 (and thus Equation 1.2) is satisfied for each $x \in \mathbb{R}^2$.

Example 1 shows that a signal can be decomposed and reconstructed (as in Equations 1.1 and 1.2) without using an orthonormal basis. In fact, Equations 1.1 and 1.2 are satisfied if and only if E is a *Parseval frame* for H. We will see that Parseval frames are generalizations of orthonormal basis and that every orthonormal basis is in fact a Parseval frame.

C. A Brief Overview of Frames

As indicated in the previous section, we see that (Parseval) frames have become useful tools in signal processing. Recent work [8] provides techniques for decomposing operators as a sum of tensor products of frames for a Hilbert space H. For a finite dimensional Hilbert space H, a frame for H is simply a spanning set for H. Since spanning sets are not (in general) linearly independent, frames often contain redundant vectors. However, this redundancy can be used to aid in the authentication of signals.

To illustrate this, suppose that $E = \{e_1, e_2\}$ is an orthonormal set in a Hilbert space H. Suppose further that $x \in \text{span}\{e_1, e_2\}$ is a signal to be transmitted. Since

$$x = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2,$$

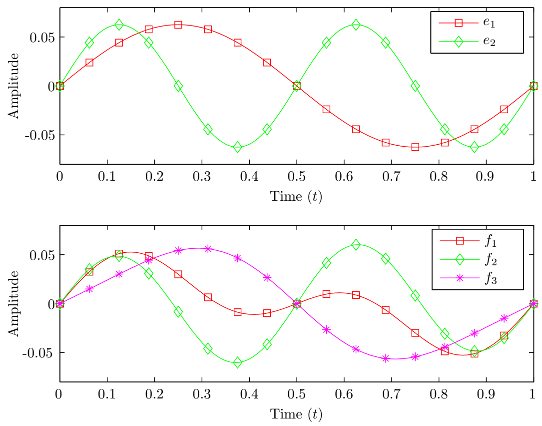
transmission and reception of the scalars $\langle x, e_1 \rangle$ and $\langle x, e_2 \rangle$ is sufficient to guarantee reconstruction provided that the receiver knows e_1 and e_2. However, if either $\langle x, e_1 \rangle$ or

$\langle x, e_2\rangle$ is lost or corrupted before it is received, then the reconstructed signal u will be different (in general) from x. To that end, let u_1 and u_2 be the reconstructed signals for which either $\langle x, e_1\rangle$ or $\langle x, e_2\rangle$, respectively, is lost completely. If $F = \{f_1, f_2, f_3\}$ is a Parseval frame for span$\{e_1, e_2\}$, then we also have that

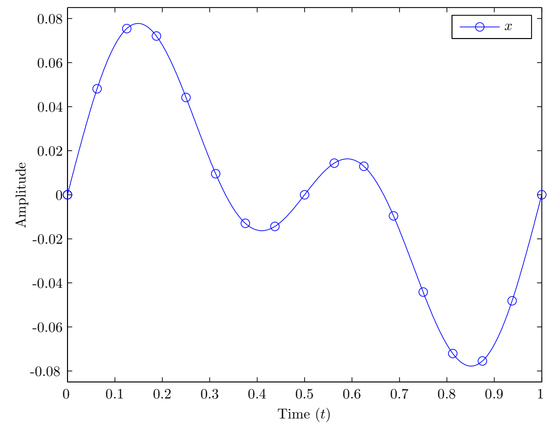
$$x = \langle x, f_1\rangle f_1 + \langle x, f_2\rangle f_2 + \langle x, f_3\rangle f_3.$$

Therefore, if we let v_1, v_2, and v_3 be the reconstructed signals for which either $\langle x, f_1\rangle$, $\langle x, f_2\rangle$, or $\langle x, f_3\rangle$, respectively, is lost completely, we would like to compare $\|x - u_i\|$ and $\|x - v_j\|$ for each i and j. Figure 2 illustrates this for a particular choice of e_1, e_2, f_1, f_2, f_3, and x. In Figure 2a, we have that $e_1 = e_1(t) = c_1 \sin(2\pi t)$ and $e_2 = e_2(t) = c_2 \sin(4\pi t)$ where c_1, c_2 are chosen for normalization purposes. Also, f_1, f_2, and f_3 is some Parseval frame for span$\{e_1, e_2\}$. In Figure 2b, we have that $x = e_1 + e_2$. Figure 2c shows the received signals (after data loss) u_1, u_2, v_1, v_2, and v_3. Finally, Figure 2d shows the error in each case. Notice that the error induced by using the Parseval frame F is approximately $\frac{2}{3}$ of the error induced by using E in $\| \cdot \|_2$. Therefore, it is clear from this example how Parseval frames can provide superior authenticity results to orthonormal bases. Furthermore, in the case illustrated by Figure 2, if either $\langle x, f_1\rangle$, $\langle x, f_2\rangle$, or $\langle x, f_3\rangle$ was known to be lost before reconstruction, then post-reception processing could be performed to **perfectly** reconstruct x. This feature is also lacking when using orthonormal bases. So, it is clear that if redundancy is acceptable, then (Parseval) frames are particularly important in signal processing in order to insure the authenticity of signals.

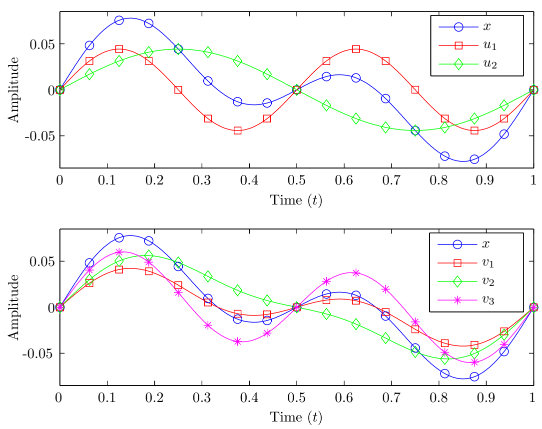
As previously stated, frames for finite dimensional Hilbert spaces H are simply spanning sets for H. However, in the case where H is not finite dimensional, then extra care must be taken to define a frame for H.



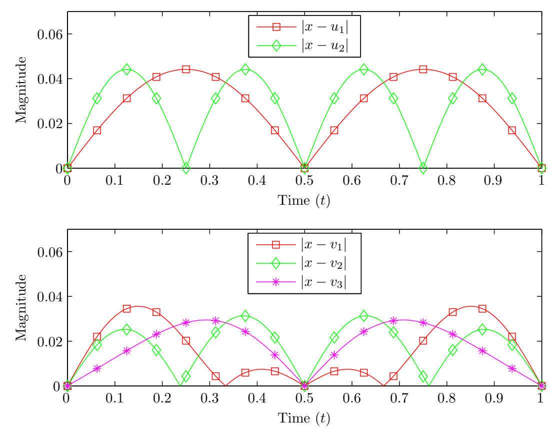
(a) Basis and Frame



(b) Transmitted Signal



(c) Received Signals



(d) Error

Fig. 2. Error Analysis Using Bases and Frames.

Definition 1. Let H be a separable Hilbert space, and let $\mathbb{J}$ be a countable index set. We say that a sequence $X = \{x_i\}_{i\in\mathbb{J}} \subset H$ is a *frame* for H if there exists real numbers $0 < A \leq B$ such that

$$A\|x\|^2 \leq \sum_{i\in\mathbb{J}} |\langle x, x_i\rangle|^2 \leq B\|x\|^2 \tag{1.3}$$

for each x in H.

We call the largest A and the smallest B for which Equation 1.3 holds the lower and upper frame bounds for X, respectively. If $A = B$, we say that X is a *tight* frame, and if $A = 1 = B$ we say that X is a *Parseval* frame. If we let $K = \ell^2(\mathbb{J})$ and $E = \{e_i\}_{i\in\mathbb{J}}$ be the standard orthornormal basis for K, then the linear operator $\theta : H \to K$ defined by $\theta x = \sum_{i\in\mathbb{J}} \langle x, x_i\rangle e_i$ for each $x \in H$ is called the *analysis operator* of X, and its adjoint θ^* is called the *synthesis operator* of X formulated by $\theta^* a = \sum_{i\in\mathbb{J}} \langle a, e_i\rangle x_i$ for each $a \in K$. Using the analysis operator of X, we see that Equation 1.3 is equivalent to

$$\sqrt{A}\|x\| \leq \|\theta x\| \leq \sqrt{B}\|x\|$$

for each $x \in H$. Furthermore, we have that

$$A = \inf_{x\neq 0} \frac{\|\theta x\|^2}{\|x\|^2} = \frac{1}{\sup\limits_{x\neq 0} \frac{\|x\|^2}{\|\theta x\|^2}} = \frac{1}{\sup\limits_{x\neq 0} \frac{\left\|(\theta^*\theta)^{-\frac{1}{2}} x\right\|^2}{\|x\|^2}} = \left\|(\theta^*\theta)^{-\frac{1}{2}}\right\|^{-2} = \left\|(\theta^*\theta)^{-1}\right\|^{-1} \tag{1.4}$$

and

$$B = \sup_{x\neq 0} \frac{\|\theta x\|^2}{\|x\|^2} = \|\theta\|^2 = \left\|(\theta^*\theta)^{\frac{1}{2}}\right\|^2 = \|\theta^*\theta\|. \tag{1.5}$$

If $A = B$, we have that

$$\kappa_H(\theta^*\theta) = \|\theta^*\theta\| \cdot \|(\theta^*\theta)^{-1}\| = \frac{B}{A} = 1.$$

Furthermore, since $\theta^*\theta$ is positive (and self-adjoint), it follows that $\theta^*\theta = \lambda I$ for some $\lambda > 0$. If $A = 1 = B$, we have that $\lambda = 1$ and thus $\theta^*\theta = I$. Conversely, if $\theta^*\theta = \lambda I$ for some $\lambda > 0$, we have that $A = \|(\theta^*\theta)^{-1}\|^{-1} = \lambda$ and $B = \|\theta^*\theta\| = \lambda$. That is, $A = B$. If $\lambda = 1$, we have $A = 1 = B$. Therefore, θ is the analysis operator of a tight frame if and only if $\theta^*\theta = \lambda I$ for some $\lambda > 0$, and in particular, θ is the analysis operator of a Parseval frame if and only if $\theta^*\theta = I$.

Since a Riesz basis is precisely a frame for which the frame vectors are linearly independent, we have the following lemma.

Lemma 1. *Let H be a separable Hilbert space and let $X = \{x_i\}_{i\in\mathbb{J}}$ be a Riesz basis for H where $\mathbb{J}$ is a countable index set. If $V \subset H$ is a subspace of H and P_V is the orthogonal projection of H onto V, then $Y = \{y_i\}_{i\in\mathbb{J}} = \{P_V x_i\}_{i\in\mathbb{J}}$ is a frame for V which is no looser than X. In particular, if X is orthonormal, then Y is Parseval.*

Proof. Suppose that A and B are the lower and upper Riesz bounds for X, respectively. Let $v \in V$ be arbitrary. Then,

$$\sum_{i\in\mathbb{J}} |\langle v, P_V x_i\rangle|^2 = \sum_{i\in\mathbb{J}} |\langle P_V v, x_i\rangle|^2 = \sum_{i\in\mathbb{J}} |\langle v, x_i\rangle|^2.$$

Therefore, we have that

$$A\|v\|^2 \leq \sum_{i\in\mathbb{J}} |\langle v, P_V x_i\rangle|^2 \leq B\|v\|^2.$$

That is, Y is a frame for V with lower frame bound at least A and upper frame bound at most B. If X is orthonormal, the $A = 1 = B$ and so Y is Parseval. □

The previous Lemma states that the projection of a Riesz basis onto a subspace is a frame for the subspace (which is no looser than the original Riesz basis). We will also show that every frame is the projection of Riesz basis, and in particular, every Parseval frame is the projection of an orthornormal basis. Since the proof is constructive, it provides a method for lifting (or dilating) a frame to a basis.

Proposition 1. *Let H be a separable Hilbert space, and let $X = \{x_i\}_{i\in\mathbb{J}}$ be a frame for H with lower and upper frame bounds A and B, respectively and $\mathbb{J}$ a countable index set. Then there exists a complementary Hilbert space $\widetilde{H}$ and a complementary tight frame $\widetilde{X} = \{\widetilde{x}_i\}_{i\in\mathbb{J}}$ for $\widetilde{H}$ (i.e. $X \oplus \widetilde{X}$ is a Riesz basis for $H \oplus \widetilde{H}$) such that $X \oplus \widetilde{X}$ has lower and upper frame bounds A and B, respectively. In particular, if X is Parseval, then $X \oplus \widetilde{X}$ is orthonormal.*

Proof. Let $K = \ell^2(\mathbb{J})$, $\widetilde{H} = (\operatorname{ran}\theta_X)^\perp \subset K$, $E = \{e_i\}_{i\in\mathbb{J}}$ be the standard orthonormal basis for K, and $P_{\widetilde{H}}$ be the orthogonal projection of K onto $\widetilde{H}$. By Lemma 1 we have that $\widetilde{X}' = \{P_{\widetilde{H}} e_i\}_{i\in\mathbb{J}}$ is a Parseval frame for $\widetilde{H}$. Therefore, $\widetilde{X} = \sqrt[4]{AB}\,\widetilde{X}' = \{\sqrt[4]{AB}\,P_{\widetilde{H}} e_i\}_{i\in\mathbb{J}}$ is a tight frame (with frame bound $\sqrt{AB}$) for $\widetilde{H}$. So, let $v \in H \oplus \widetilde{H}$ be arbitrary. Then, there exists $x \in H$ and $\widetilde{x} \in \widetilde{H}$ such that $v = x \oplus \widetilde{x}$. So, we have

$$\begin{aligned}\sum_{i\in\mathbb{J}} |\langle v, x_i \oplus \widetilde{x}_i\rangle|^2 &= \sum_{i\in\mathbb{J}} |\langle x \oplus \widetilde{x}, x_i \oplus \widetilde{x}_i\rangle|^2 = \sum_{i\in\mathbb{J}} |\langle x, x_i\rangle|^2 + \sum_{i\in\mathbb{J}} |\langle \widetilde{x}, \widetilde{x}_i\rangle|^2 \\ &= \sum_{i\in\mathbb{J}} |\langle x, x_i\rangle|^2 + \sqrt{AB}\,\|\widetilde{x}\|^2\end{aligned}$$

and therefore

$$A\|v\|^2 = A\left(\|x\|^2 + \|\widetilde{x}\|^2\right) \le \sum_{i\in\mathbb{J}} |\langle v, x_i \oplus \widetilde{x}_i\rangle|^2 \le B\left(\|x\|^2 + \|\widetilde{x}\|^2\right) = B\|v\|^2.$$

So, we see that $X \oplus \widetilde{X}$ is a frame for $H \oplus \widetilde{H}$ with lower and upper frame bounds A and B, respectively. We will now show that it is in fact a (Riesz) basis. Let Λ be any finite subset of $\mathbb{J}$ and let $\{\alpha_i\}_{i\in\Lambda}$ be a (finite) sequence of scalars. Suppose that $\sum_{i\in\Lambda} \alpha_i(x_i \oplus \widetilde{x}_i) = 0$. Then, both $\sum_{i\in\Lambda} \alpha_i x_i = 0$ and $\sum_{i\in\Lambda} \alpha_i \widetilde{x}_i = 0$. Since $\sum_{i\in\Lambda} \alpha_i \widetilde{x}_i = 0$, and since $\widetilde{x}_i = \sqrt[4]{AB}\,P_{\widetilde{H}} e_i$ for each $i \in \mathbb{J}$, it follows that $P_{\widetilde{H}} \sum_{i\in\Lambda} \alpha_i e_i = 0$. That is,

$\sum_{i\in\Lambda} \alpha_i e_i \in \operatorname{ran}\theta_X$. Furthermore, since $x_i = \theta_X^* e_i$ for each $i \in \mathbb{J}$, we have that

$$\theta_X^* \sum_{i\in\Lambda} \alpha_i e_i = \sum_{i\in\Lambda} \alpha_i \, \theta_X^* e_i = \sum_{i\in\Lambda} \alpha_i x_i = 0.$$

This implies that $\sum_{i\in\Lambda} \alpha_i e_i \in (\operatorname{ran}\theta_X)^\perp$. Therefore, $\sum_{i\in\Lambda} \alpha_i e_i = 0$. Since $\{e_i\}_{i\in\mathbb{J}}$ is a basis for K, it follows that $\alpha_i = 0$ for each $i \in \Lambda$ which implies that $X \oplus \widetilde{X}$ is a Resiz basis for $H \oplus \widetilde{H}$. Now if X is Parseval, then $\widetilde{X}$ is Parseval and $X \oplus \widetilde{X}$ is Parseval. Since $X \oplus \widetilde{X}$ is a linearly independent set, it follows that $X \oplus \widetilde{X}$ is an orthonormal basis for $H \oplus \widetilde{H}$. □

Let $P : K \to \operatorname{ran}\theta_X \subset K$ be defined by $P = \theta_X(\theta_X^*\theta_X)^{-1}\theta_X^*$, and let $a \in \operatorname{ran}\theta_X$ be arbitrary. Then, $P^2 = P$ and $a = \theta_X x$ for some $x \in H$. Furthermore,

$$Pa = P\theta_X x = \theta_X(\theta_X^*\theta_X)^{-1}\theta_X^*\theta_X x = \theta_X x = a,$$

and thus P is surjective. So, P is the orthogonal projection of K onto $\operatorname{ran}\theta_X \subset K$, and therefore

$$P^\perp = I_K - P = I_K - \theta_X(\theta_X^*\theta_X)^{-1}\theta_X^* \tag{1.6}$$

is the orthogonal projection of K onto $\widetilde{H} = (\operatorname{ran}\theta_X)^\perp \subset K$. If X is Parseval, then $\theta_X^*\theta_X = I_H$, and so $\theta_X\theta_X^*$ is the orthogonal projection of K onto $\operatorname{ran}\theta_X \subset K$.

If X, Y are frames for H, we say that X and Y are *similar* if there exists an invertible $S : H \to H$ such that $y_i = Sx_i$ for each $i \in \mathbb{J}$. If we let θ_X and θ_Y be the analysis operators of X and Y, respectively, then X and Y being similar is equivalent to $\theta_Y = \theta_X S$. Also, if X is a frame that is similar to a Parseval frame Y by S (i.e. S is invertible with $\theta_Y = \theta_X S$), then $S = (\theta_X^*\theta_X)^{-\frac{1}{2}}U$ where U is unitary.

If X is a Parseval frame for H, then $\theta^*\theta = I$ and thus there is a straightforward formulation for reconstructing x from θx (that is, simply applying θ^*). However, if X is not (in general) Parseval, the reconstruction formulation is less straightforward.

If we let $y_i = (\theta^*\theta)^{-1}x_i$ for each $i \in \mathbb{J}$, then

$$\sum_{i\in\mathbb{J}}\langle x, x_i\rangle y_i = \sum_{i\in\mathbb{J}}\langle x, x_i\rangle(\theta^*\theta)^{-1}x_i = (\theta^*\theta)^{-1}\sum_{i\in\mathbb{J}}\langle x, x_i\rangle x_i = (\theta^*\theta)^{-1}\theta^*\theta x = x.$$

That is, $Y = \{y_i\}_{i\in\mathbb{J}}$ is the canonical dual frame for X and $\theta_X^*\theta_Y = I = \theta_Y^*\theta_X$.

Example 2. Let

$$X = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}$$

be an ordered set in $\mathbb{R}^2$. Then, X is a frame for $\mathbb{R}^2$ with lower and upper frame bounds 2 and 3, respectively. Furthermore, the canonical dual frame of X is

$$\left\{\begin{pmatrix}\frac{1}{3}\\0\end{pmatrix}, \begin{pmatrix}-\frac{1}{3}\\\frac{1}{2}\end{pmatrix}, \begin{pmatrix}-\frac{1}{3}\\-\frac{1}{2}\end{pmatrix}\right\}$$

with lower and upper frame bound 2 and 3, respectively. Finally, the canonical Parseval frame similar to X is

$$\left\{\begin{pmatrix}\frac{1}{\sqrt{3}}\\0\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{2}}\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}\right\}.$$

From the standpoint of computational cost, determining both $(\theta^*\theta)^{-1}$ and $(\theta^*\theta)^{-\frac{1}{2}}$ can be expensive. Therefore, it is clear that we would like another method for transforming frames into Parseval frames. When we restrict to bases, the standard method for transforming bases into orthonormal bases is the Gram-Schmidt process. Therefore, part of this dissertation will address a technique for transforming frames into Parseval frames via a Causal Generalized Gram-Schmidt (CGGS) process. One of the most important properties of such a procedure is that if X is a frame for H and Y is a Parseval frame obtained via the CGGS, then X and Y are causally equivalent. This property is desirable since the Classical Gram-Schmidt process for bases possesses it.

CHAPTER II

PRIMARY RESULTS

A. Causal Equivalence

Let V be a vector space over $\mathbb{C}$ or $\mathbb{R}$ and let $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ be finite (or infinite) ordered sequences in V. We say that Y is *causally related* to X if there exists scalars $\{\alpha_{i,j}\}_{i,j=1}^{k}$ such that

$$\begin{aligned} y_1 &= \alpha_{1,1}x_1 \\ y_2 &= \alpha_{2,1}x_1 + \alpha_{2,2}x_2 \\ &\vdots \\ y_k &= \alpha_{k,1}x_1 + \alpha_{k,2}x_2 + \cdots + \alpha_{k,k}x_k. \end{aligned}$$

In general, this is not an equivalence relation. To illustrate this, let $X = \{x_1, x_2\}$ be an arbitrary ordered sequence in V with x_1 and x_2 linearly independent, and let $Y = \{y_1, y_2\}$ with $y_1 = y_2 = x_1$. Then Y is causally related to X. Suppose there exist scalars $\{\beta_{i,j}\}_{i,j=1}^{2}$ such that

$$\begin{aligned} x_1 &= \beta_{1,1}y_1 \\ x_2 &= \beta_{2,1}y_1 + \beta_{2,2}y_2. \end{aligned}$$

Then we have that $\beta_{1,1} = 1$ and therefore $x_2 = (\beta_{2,1} + \beta_{2,2})x_1$ which contradicts the fact that x_1 and x_2 are linearly independent. Thus, X is **not** causally related to Y. However, if we require the "diagonal elements" to be nonzero (i.e. $\alpha_{i,i} \neq 0$ for each i), then it becomes an equivalence relation. To see this fact, we notice that every sequence is causally related to itself by choosing $\alpha_{i,j} = \delta_{i,j}$ where

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

is the Kronecker delta. Furthermore, if

$$y_i = \sum_{j=1}^{i} \alpha_{i,j} x_j$$

for each i, then we can recursively write the x_i's in terms of the y_i's by

$$x_1 = \alpha_{1,1}^{-1} y_1$$
$$x_i = \alpha_{i,i}^{-1} \left(y_i - \sum_{j=1}^{i-1} \alpha_{i,j} x_j \right), \quad \text{for each } i > 1.$$

Finally, if $y_i = \sum_{j=1}^{i} \alpha_{i,j} x_j$ for each i and $z_m = \sum_{i=1}^{m} \beta_{m,i} y_i$ for each m, then $z_m = \sum_{j=1}^{m} \gamma_{m,j} x_j$ for each m where

$$\gamma_{m,j} = \sum_{i=j}^{m} \alpha_{i,j} \beta_{m,i}$$

for each m and j. We see that $\gamma_{m,j} = 0$ if $m < j$ and $\gamma_{m,m} = \alpha_{m,m} \beta_{m,m} \neq 0$ for each m.

Definition 2. We say that two ordered sequences $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ are *causally equivalent* if there exists scalars $\{\alpha_{i,j}\}_{i,j=1}^{k}$ such that $\alpha_{i,i} \neq 0$ and

$$y_i = \sum_{j=1}^{i} \alpha_{i,j} x_j$$

for each i, and we write $X \sim_{\mathrm{c}} Y$.

If X and Y are sequences of vectors in a vector space V with $X \sim_{\mathrm{C}} Y$, then X

and Y have the same "partial spans." That is,

$$\operatorname{span}\{x_1, \ldots, x_i\} = \operatorname{span}\{y_1, \ldots, y_i\}$$

for each $1 \leq i \leq k$. This property appears in the Classical Gram-Schmidt process and is demonstrated in the following example.

Example 3. If $X = \{x_1, \ldots, x_n\}$ is a basis for $\mathbb{C}^n$ (or $\mathbb{R}^n$) and if $Y = \{y_1, \ldots, y_n\}$ is the orthonormal basis derived from the Classical Gram-Schmidt process, then $X \sim_{\mathrm{c}} Y$. Moreover, if $Z = \{z_1, \ldots, z_n\}$ is any other orthonormal basis for $\mathbb{C}^n$ (or $\mathbb{R}^n$) with $X \sim_{\mathrm{c}} Z$, then there exists uni-modular scalars $\{\alpha_i\}_{i=1}^n$ such that $z_i = \alpha_i y_i$ for each i.

Example 3 affirms that there is essentially one orthonormal basis that is causally equivalent to a given basis and that this orthonormal basis is obtained by the Classical Gram-Schmidt process.

B. Causal Generalized Gram-Schmidt (CGGS) Process

Suppose $X = \{x_i\}_{i\in\mathbb{J}}$ and $Y = \{y_i\}_{i\in\mathbb{J}}$ are (ordered) frames for H. Let $K = \ell^2(\mathbb{J})$ and $E = \{e_i\}_{i\in\mathbb{J}}$ be the standard orthonormal basis for K. We will now show that X and Y are causally equivalent if and only if there is an invertible lower triangular (with respect to E) operator $L : K \to K$ such that $\theta_Y = L\theta_X$. Let $L : K \to K$ with $L = (\overline{\alpha}_{i,j})$ with respect to E where $\{\alpha_{i,j}\}_{i,j\in\mathbb{J}} \subset \mathbb{C}$. Then,

$$\begin{aligned} L\theta_X x &= L\sum_{j\in\mathbb{J}}\langle x, x_j\rangle e_j = \sum_{j\in\mathbb{J}}\langle x, x_j\rangle Le_j = \sum_{j\in\mathbb{J}}\langle x, x_j\rangle \sum_{i\in\mathbb{J}}\overline{\alpha}_{i,j}e_i \\ &= \sum_{i\in\mathbb{J}}\left\langle x, \sum_{j\in\mathbb{J}}\alpha_{i,j}x_j\right\rangle e_i \end{aligned}$$

and $\theta_Y x = \sum_{i \in \mathbb{J}} \langle x, y_i \rangle e_i$. Therefore, since E is an orthonormal basis for K, we have that $\theta_Y = L\theta_X$ if and only if

$$y_i = \sum_{j \in \mathbb{J}} \alpha_{i,j} x_j \tag{2.1}$$

for each $i \in \mathbb{J}$. Notice that whether or not X and Y are causally equivalent, there exists an invertible operator $L : K \to K$ such that $\theta_Y = L\theta_X$. However, by Definition 2 and Equation 2.1, we have that X and Y are causally equivalent if and only if $\theta_Y = L\theta_X$ with $L : K \to K$ invertible and lower triangular (with respect to E).

The above result provides a convenient method for characterizing causal equivalence. We see in Example 3 that if X is a basis for H and if Y, Z are orthonormal bases for H such that $X \sim_{\mathrm{c}} Y$ and $X \sim_{\mathrm{c}} Z$, then $\theta_Z = D\theta_Y$ where $D : K \to K$ is diagonal (with respect to E) and unitary. Therefore, given a basis X for H, there is a unique (up to rescaling by uni-modular constants) orthonormal basis Y such that $Y \sim_{\mathrm{c}} X$. Furthermore, this Y is obtained by performing the Classical Gram-Schmidt algorithm to X. We will now show that every frame X for H is casually equivalent to a Parseval frame Y for H.

Proposition 2. *Let X be a frame for H. Then there exists a Parseval frame Y for H with $X \sim_C Y$.*

Proof. Let $\widetilde{H} = (\operatorname{ran} \theta_X)^{\perp} \subset K$. By Proposition 1, there exists a frame $\widetilde{X}$ for $\widetilde{H}$ such that $X \oplus \widetilde{X}$ is a Riesz basis for $H \oplus \widetilde{H}$. Then by the Classical Gram-Schmidt algorithm, there exists an orthonormal basis of the form $Y \oplus \widetilde{Y}$ for $H \oplus \widetilde{H}$ and an invertible lower triangular (with respect to E) L such that $\theta_{Y \oplus \widetilde{Y}} = L\theta_{X \oplus \widetilde{X}}$. Therefore we have that $\theta_Y = L\theta_X$ and $\theta_{\widetilde{Y}} = L\theta_{\widetilde{X}}$. Since Y and $\widetilde{Y}$ are the projections of $Y \oplus \widetilde{Y}$ onto H and $\widetilde{H}$, respectively, and since $Y \oplus \widetilde{Y}$ is an orthonormal basis for $H \oplus \widetilde{H}$, we have that Y and $\widetilde{Y}$ are Parseval frames for H and $\widetilde{H}$, respectively, that are causally equivalent to X and $\widetilde{X}$. $\square$

The following example illustrates the procedure described in Proposition 2

Example 4. Let

$$X = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix} \right\}.$$

Then, X is a frame for $H = \mathbb{R}^2$ with lower and upper frame bounds $A = \frac{1}{2}$ and $B = 2$, respectively. According to Proposition 1, we have that

$$\widetilde{X} = \left\{ \left(\tfrac{1}{\sqrt{2}}\right), \left(\tfrac{1}{2}\right), \left(\tfrac{1}{2}\right) \right\}.$$

is a tight frame (with frame bound $\sqrt{AB} = 1$, i.e. Parseval) for $\widetilde{H} = \operatorname{ran}\left(\theta_X^* \theta_X\right)^{\perp} \subset \mathbb{R}^3$. Furthermore, we have that $X \oplus \widetilde{X}$ is a basis for $H \oplus \widetilde{H}$ (with lower and upper frames bounds $\frac{1}{2}$ and 2, respectively). Performing the Classical Gram-Schmidt algorithm on $X \oplus \widetilde{X}$, we obtain the orthonormal basis

$$Y \oplus \widetilde{Y} = \left\{ \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{2\sqrt{2}}{\sqrt{33}} \\ \frac{\sqrt{3}}{\sqrt{11}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{11}} \\ -\frac{2\sqrt{2}}{\sqrt{11}} \end{pmatrix} \right\} \oplus \left\{ \left(\tfrac{1}{\sqrt{3}}\right), \left(\tfrac{4}{\sqrt{33}}\right), \left(\tfrac{\sqrt{2}}{\sqrt{11}}\right) \right\}$$

for $H \oplus \widetilde{H}$. By projecting $Y \oplus \widetilde{Y}$ onto H, we obtain a Parseval frame Y for H with

$$Y = \left\{ \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{2\sqrt{2}}{\sqrt{33}} \\ \frac{\sqrt{3}}{\sqrt{11}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{11}} \\ -\frac{2\sqrt{2}}{\sqrt{11}} \end{pmatrix} \right\}$$

and

$$\begin{aligned} y_1 &= \tfrac{\sqrt{2}}{\sqrt{3}} x_1 \\ y_2 &= \tfrac{\sqrt{2}}{\sqrt{33}} x_1 \ + \tfrac{2\sqrt{3}}{\sqrt{11}} x_2 \\ y_3 &= \tfrac{1}{2\sqrt{11}} x_1 - \tfrac{5}{2\sqrt{22}} x_2 + \tfrac{\sqrt{22}}{4} x_3. \end{aligned}$$

Figure 3 graphically depicts the process described in Example 4. Figure 3a shows the original frame X and Figure 3b shows the same frame embedded in $\mathbb{R}^3$. Figure 3c shows the frame lifted to a basis $X \oplus \widetilde{X}$ for $\mathbb{R}^3$. Figure 3d shows the orthonormal basis $Y \oplus \widetilde{Y}$ for $\mathbb{R}^3$ obtained by applying the classical Gram-Schmidt process to $X \oplus \widetilde{X}$. Finally, Figure 3e shows Y as the projection of $Y \oplus \widetilde{Y}$ back onto the xy-plane, and

Figure 3f shows both the original frame and the transformed Parseval frame.

Remark 1. Many of the numerical examples formulated in this dissertation (including Example 4) were developed in conjunction with computer simulations using the workstation provided by a supplement to NSF grant DMS-0070796 ("Operator Algebras and Wavelet Theory") of Professor David Larson and Texas A&M University.

The above method provides a way of transforming frames into Parseval frames in a causal fashion (similar to the Classical Gram-Schmidt algorithm). However, the method allows for many "lifts" $\widetilde{X}$ for which $X \oplus \widetilde{X}$ is a Riesz basis for $H \oplus \widetilde{H}$. We will define an optimal lift $\widetilde{X}$ to be one that minimizes $\kappa(L) = \|L\|\|L^{-1}\|$ (i.e. the condition number of L). However, in order to determine a L for which $\kappa(L)$ is minimal, we must first find a lower bound for such a $\kappa(L)$.

Lemma 2. *Suppose that H,M, and N are Hilbert spaces. Suppose further that $T : H \to M$ is bijective, $S : M \to N$ is bijective, and ST is unitary (i.e. bijective and isometric). Then, $\|S\|_M = \|T^{-1}\|_M$ and $\|S^{-1}\|_N = \|T\|_N$.*

Proof. We first notice that

$$\|S\|_M = \left\|STT^{-1}\right\|_M \leq \|ST\|_H \cdot \left\|T^{-1}\right\|_M = \left\|T^{-1}\right\|_M .$$

Similarly, we have

$$\left\|T^{-1}\right\|_M = \left\|T^{-1}S^{-1}S\right\|_M \leq \left\|(ST)^{-1}\right\|_N \cdot \|S\|_M = \|S\|_M .$$

Likewise, we have

$$\|T\|_H = \left\|S^{-1}ST\right\|_H \leq \left\|S^{-1}\right\|_N \cdot \|ST\|_H = \left\|S^{-1}\right\|_N$$

and

$$\left\|S^{-1}\right\|_N = \left\|TT^{-1}S^{-1}\right\|_N \leq \|T\|_H \cdot \left\|(ST)^{-1}\right\|_N = \|T\|_H .$$

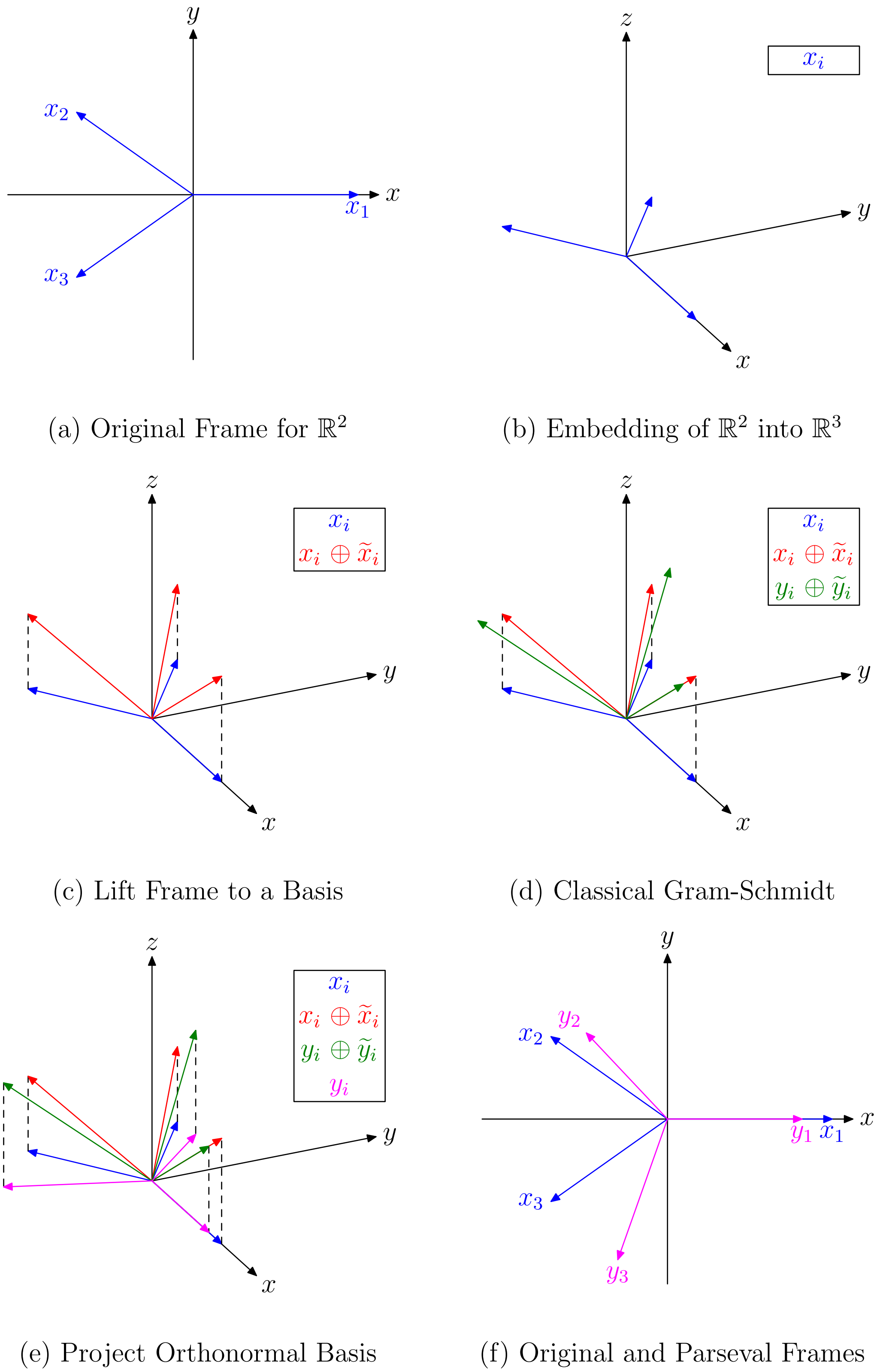


Fig. 3. Causal Generalized Gram-Schmidt Process.

The first two inequalities show that $\|S\|_M = \|T^{-1}\|_M$ and the last two inequalities show that $\|S^{-1}\|_N = \|T\|_N$. □

We will now use Lemma 2 to prove a result about condition numbers.

Lemma 3. *Suppose H and K are Hilbert spaces. Suppose further that $T : H \to K$ is injective, $S : K \to K$ is bijective, and ST is an isometry. Then $\|S\|_K \geq \left\|(T^*T)^{-\frac{1}{2}}\right\|_H$ and $\|S^{-1}\|_K \geq \left\|(T^*T)^{\frac{1}{2}}\right\|_H$. In particular, $\kappa_K(S) \geq \kappa_H((T^*T)^{1/2})$.*

Proof. Let $M = \operatorname{ran} T \subset K$ and $N = SM \subset K$. Let $T' : H \to M$ be defined by $T'x = Tx$ for all $x \in H$ and $S' : M \to N$ be defined by $S' = S\big|_M$. Then $T' : H \to M$ is bijective and $S' : M \to N$ is bijective. Furthermore, for each $x \in H$ we have that $S'T'x = S'Tx = STx$ and so $S'T'$ is isometric. So, by Lemma 2, we have that $\|S'\|_M = \|(T')^{-1}\|_M$. However, it is clear (by the definition of S') that $\|S\|_K \geq \|S'\|_M = \|(T')^{-1}\|_M = \|((T')^*T')^{-1}\|_H^{1/2}$. Furthermore, for each $x, y \in H$, we have that

$$\langle (T')^* T'x, y\rangle_H = \langle T'x, T'y\rangle_H = \langle Tx, Ty\rangle_H = \langle T^*Tx, y\rangle_H.$$

Therefore, $(T')^* T' = T^*T$. Since T is injective, $(T^*T)^{-1}$ exists, and thus $(T^*T)^{-1} = ((T')^* T')^{-1}$. So, we have

$$\|S\|_K \geq \left\|((T')^* T')^{-1}\right\|_H^{1/2} = \left\|(T^*T)^{-1}\right\|_H^{1/2} = \left\|(T^*T)^{-\frac{1}{2}}\right\|_H.$$

This establishes the first inequality. To show the second inequality, simply notice that

$$\left\|S^{-1}\right\|_K = \left\|S^{-1}\right\|_K \cdot \|ST\|_H \geq \left\|S^{-1}ST\right\|_H = \|T\|_H = \left\|(T^*T)^{\frac{1}{2}}\right\|_H.$$

Therefore, we have $\kappa_K(S) \geq \kappa_H((T^*T)^{1/2})$ as desired. □

Now, if X is a frame for H with lower and upper frame bound A and B, respectively and Y is a Parseval frame causally equivalent to X, then there exists an invertible lower triangular (with respect to E) L such that $\theta_Y = L\theta_X$. Since Y is Parseval, θ_Y is an isometry; hence, by Lemma 3, we have that $\kappa_K(L) \geq \kappa_H((\theta_X^*\theta_X)^{\frac{1}{2}})$. By Equations 1.4 and 1.5, we have that $\left\|(\theta_X^*\theta_X)^{\frac{1}{2}}\right\|_H = \sqrt{B}$ and $\left\|(\theta_X^*\theta_X)^{-\frac{1}{2}}\right\|_H^{-1} = \sqrt{A}$. Therefore, we have $\kappa_K(L) \geq \sqrt{\frac{B}{A}}$.

Theorem 1. *Let X be a frame for H with lower and upper frame bounds A and B, respectively. Then, there exists an invertible and lower triangular (with respect to E) $L : K \to K$ such that $\theta_X^* L^* L\theta_X = I_H$ and $\kappa_K(L) = \sqrt{\frac{B}{A}}$.*

Proof. Let

$$T = \theta_X(\theta_X^*\theta_X)^{-\frac{3}{2}}\theta_X^* \oplus \frac{1}{\sqrt[4]{AB}}\, P^\perp$$

where $P^\perp$ is defined by Equation 1.6. Then,

$$T^{-1} = \theta_X(\theta_X^*\theta_X)^{-\frac{1}{2}}\theta_X^* \oplus \sqrt[4]{AB}\, P^\perp.$$

Notice that $\|\theta_X(\theta_X^*\theta_X)^{-\frac{3}{2}}\theta_X^*\|_K = \|(\theta_X^*\theta_X)^{-\frac{1}{2}}\|_H = \frac{1}{\sqrt{A}}$ and $\|\theta_X(\theta_X^*\theta_X)^{-\frac{1}{2}}\theta_X^*\|_K = \|(\theta_X^*\theta_X)^{\frac{1}{2}}\|_H = \sqrt{B}$. Furthermore, since $(I_K - \theta_X(\theta_X^*\theta_X)^{-1}\theta_X^*)$ is the orthogonal projection of K onto $(\operatorname{ran}\theta_X)^\perp \subset K$, we have that $\|I_K - \theta_X(\theta_X^*\theta_X)^{-1}\theta_X^*\| = 1$. Therefore, we have

$$\|T\|_K = \max\left\{\left\|\theta_X(\theta_X^*\theta_X)^{-\frac{3}{2}}\theta_X^*\right\|, \left\|\frac{1}{\sqrt[4]{AB}}\, P^\perp\right\|\right\} = \max\left\{\frac{1}{\sqrt{A}}, \frac{1}{\sqrt[4]{AB}}\right\} = \frac{1}{\sqrt{A}}$$

and

$$\|T^{-1}\|_K = \max\left\{\left\|\theta_X(\theta_X^*\theta_X)^{-\frac{1}{2}}\theta_X^*\right\|, \left\|\sqrt[4]{AB}\, P^\perp\right\|\right\} = \max\left\{\sqrt{B}, \sqrt[4]{AB}\right\} = \sqrt{B}.$$

Since T is positive and invertible, we have that

$$T^*T = T^2 = \theta_X(\theta_X^*\theta_X)^{-2}\theta_X^* \oplus \frac{1}{\sqrt{AB}}\, P^\perp$$

is positive and invertible and therefore there exists an invertible and lower triangular (with respect to E) L such that $T^*T = L^*L$ (via a Cholesky-like factorization of T^*T). Furthermore, $\|L\|_K = \|T\|_K = \frac{1}{\sqrt{A}}$ and $\|L^{-1}\|_K = \|T^{-1}\|_K = \sqrt{B}$. That is, $\kappa_K(L) = \sqrt{\frac{B}{A}}$. Finally, since $\theta_X(\theta_X^*\theta_X)^{-\frac{3}{2}}\theta_X^* = P\left(\theta_X(\theta_X^*\theta_X)^{-\frac{3}{2}}\theta_X^*\right)P$, we have

$$\theta_X^* L^* L\theta_X = \theta_X^* T^* T\theta_X = \theta_X^*\left(\theta_X(\theta_X^*\theta_X)^{-2}\theta_X^*\right)\theta_X = I_K.$$

That is, $L\theta_X$ is the analysis operator of a Parseval frame for H. □

Corollary 1. *There exists an invertible and lower triangular (with respect to E) L such that $L\theta_X$ is the analysis operator for a Parseval frame for H and $\kappa_K(L)$ is minimized.*

Proof. Since $\kappa_K(L) \geq \sqrt{\frac{B}{A}}$ whenever L is invertible and lower triangular (with respect to E) and $L\theta_X$ is the analysis operator for a Parseval frame for H, and since Theorem 1 provides a method for constructing such an L with $\kappa_K(L) = \sqrt{\frac{B}{A}}$, it follows that such an L is optimal (in the sense that $\kappa_K(L)$ is minimized). □

Notice that there are many choices for T in Theorem 1. However, for each choice we must have $TP = U\theta_X(\theta_X^*\theta_X)^{-\frac{3}{2}}\theta_X^*$ where $U : K \to K$ is unitary. The following example illustrates the construction in Theorem 1 for a specific frame.

Example 5. Let

$$X = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix} \right\}.$$

Then, X is a frame for $\mathbb{R}^2$ with lower and upper frame bounds $A = \frac{1}{2}$ and $B = 2$, respectively. Following the process described in Theorem 1, we obtain $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T = \begin{pmatrix} \frac{\sqrt{2}+2}{4} & \frac{\sqrt{2}-1}{4} & \frac{\sqrt{2}-1}{4} \\ \frac{\sqrt{2}-1}{4} & \frac{5\sqrt{2}+2}{8} & \frac{-3\sqrt{2}+2}{8} \\ \frac{\sqrt{2}-1}{4} & \frac{-3\sqrt{2}+2}{8} & \frac{5\sqrt{2}+2}{8} \end{pmatrix}$$

with respect to the standard orthonormal basis for $\mathbb{R}^3$, and so

$$L = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 \\ \frac{\sqrt{2}}{\sqrt{33}} & \frac{2\sqrt{3}}{\sqrt{11}} & 0 \\ \frac{1}{2\sqrt{11}} & -\frac{5}{2\sqrt{22}} & \frac{\sqrt{22}}{4} \end{pmatrix}.$$

Therefore, defining $\theta_Y = L\theta_X$, we obtain

$$Y = \left\{ \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{2\sqrt{2}}{\sqrt{33}} \\ \frac{\sqrt{3}}{\sqrt{11}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{11}} \\ -\frac{2\sqrt{2}}{\sqrt{11}} \end{pmatrix} \right\}.$$

Furthermore, we have that Y is a Parseval frame for $\mathbb{R}^2$, $Y \sim_C X$, and $\kappa(L) = 2$ is minimized.

C. A Lifting Method for the CGGS Process

The proof of Proposition 2 uses a lift $\widetilde{X}$ of a frame X for H to a Riesz basis for $H \oplus \widetilde{H}$. However, determination of a best Parseval frame Y for which $Y \sim_C X$ (as constructed in Theorem 1) appears to be independent of such a lift. That is, the construction of Y is the result of defining $\theta_Y = L\theta_X$ where L is determined by performing a Cholesky-like factorization of a positive and invertible operator on K. We will now show a correlation between $\widetilde{X}$ and L and investigate a minimization of $\kappa_{\text{HS}}(L)$ where κ_{HS} is the Hilbert-Schmidt (or Frobenius) condition number.

Lemma 4. *If X is a frame for H and $\widetilde{X}$ is a complimentary frame of X for $\widetilde{H}$, then*

$$\theta_{X\oplus\widetilde{X}}\theta^*_{X\oplus\widetilde{X}} = \theta_X\theta^*_X + \theta_{\widetilde{X}}\theta^*_{\widetilde{X}}.$$

Proof. Let $a \in K$ be arbitrary. Then,

$$\theta_{X\oplus\widetilde{X}}\,\theta^*_{X\oplus\widetilde{X}}\,a = \theta_{X\oplus\widetilde{X}}\sum_{i\in\mathbb{J}}\langle a, e_i\rangle\,(x_i \oplus \widetilde{x}_i) = \sum_{i\in\mathbb{J}}\langle a, e_i\rangle\,\theta_{X\oplus\widetilde{X}}\,x_i \oplus \widetilde{x}_i$$

Using the definition of $\theta_{X\oplus\widetilde{X}}$ we have

$$\begin{aligned}\theta_{X\oplus\widetilde{X}}\,\theta^*_{X\oplus\widetilde{X}}\,a &= \sum_{i\in\mathbb{J}}\langle a,e_i\rangle\sum_{j\in\mathbb{J}}\langle x_i\oplus\widetilde{x}_i, x_j\oplus\widetilde{x}_j\rangle e_j\\ &= \sum_{i\in\mathbb{J}}\langle a,e_i\rangle\sum_{j\in\mathbb{J}}\langle x_i,x_j\rangle e_j + \sum_{i\in\mathbb{J}}\langle a,e_i\rangle\sum_{j\in\mathbb{J}}\langle \widetilde{x}_i,\widetilde{x}_j\rangle e_j\end{aligned}$$

Using the definition of θ_X and $\theta_{\widetilde{X}}$ we have

$$\begin{aligned}\theta_{X\oplus\widetilde{X}}\,\theta^*_{X\oplus\widetilde{X}}\,a &= \sum_{i\in\mathbb{J}}\langle a,e_i\rangle\,\theta_X x_i + \sum_{i\in\mathbb{J}}\langle a,e_i\rangle\,\theta_{\widetilde{X}}\widetilde{x}_i\\ &= \theta_X\sum_{i\in\mathbb{J}}\langle a,e_i\rangle\,x_i + \theta_{\widetilde{X}}\sum_{i\in\mathbb{J}}\langle a,e_i\rangle\,\widetilde{x}_i\\ &= \theta_X\theta_X^*a + \theta_{\widetilde{X}}\theta^*_{\widetilde{X}}a = \left(\theta_X\theta_X^* + \theta_{\widetilde{X}}\theta^*_{\widetilde{X}}\right)a\end{aligned}$$

Since a was chosen arbitrarily, we have

$$\theta_{X\oplus\widetilde{X}}\,\theta^*_{X\oplus\widetilde{X}} = \theta_X\theta_X^* + \theta_{\widetilde{X}}\theta^*_{\widetilde{X}}.$$

□

Proposition 3. *Let X be a frame for H and let $\widetilde{X}$ be a frame for $\widetilde{H}$ that is a strong complement of X (i.e. $X\oplus\widetilde{X}$ is a Riesz basis for $H\oplus\widetilde{H}$ and $\theta^*_{\widetilde{X}}\theta_X = 0$). Then,*

$$\left(\theta_{X\oplus\widetilde{X}}\,\theta^*_{X\oplus\widetilde{X}}\right)^{\frac{1}{2}} = \theta_X\left(\theta_X^*\theta_X\right)^{-\frac{1}{2}}\theta_X^* + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{1}{2}}\theta^*_{\widetilde{X}}.$$

Furthermore, $\theta_{X\oplus\widetilde{X}}\,\theta^*_{X\oplus\widetilde{X}}$ is invertible and

$$\left(\theta_{X\oplus\widetilde{X}}\,\theta^*_{X\oplus\widetilde{X}}\right)^{-\frac{1}{2}} = \theta_X\left(\theta_X^*\theta_X\right)^{-\frac{3}{2}}\theta_X^* + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{3}{2}}\theta^*_{\widetilde{X}}.$$

Proof. By direct computation, we notice that

$$\left(\theta_X\left(\theta_X^*\theta_X\right)^{-\frac{1}{2}}\theta_X^* + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{1}{2}}\theta^*_{\widetilde{X}}\right)^2 = \theta_X\theta_X^* + \theta_{\widetilde{X}}\theta^*_{\widetilde{X}}$$

since $\theta^*_{\widetilde{X}}\theta_X = 0$ $\left(\text{and } \theta^*_X\theta_{\widetilde{X}} = 0\right)$. Therefore, by Lemma 4, we have

$$\left(\theta_{X\oplus\widetilde{X}}\,\theta^*_{X\oplus\widetilde{X}}\right)^{\frac{1}{2}} = \theta_X\left(\theta^*_X\theta_X\right)^{-\frac{1}{2}}\theta^*_X + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{1}{2}}\theta^*_{\widetilde{X}}$$

as desired. Next, we also notice that

$$\left(\theta_X\left(\theta^*_X\theta_X\right)^{-\frac{3}{2}}\theta^*_X + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{3}{2}}\theta^*_{\widetilde{X}}\right)^2 = \theta_X\left(\theta^*_X\theta_X\right)^{-2}\theta^*_X + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-2}\theta^*_{\widetilde{X}}.$$

Therefore,

$$\begin{aligned}
&\left(\theta_X\left(\theta^*_X\theta_X\right)^{-\frac{3}{2}}\theta^*_X + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{3}{2}}\theta^*_{\widetilde{X}}\right)^2\left(\theta_{X\oplus\widetilde{X}}\,\theta^*_{X\oplus\widetilde{X}}\right)\\
&\qquad= \left(\theta_X\left(\theta^*_X\theta_X\right)^{-2}\theta^*_X + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-2}\theta^*_{\widetilde{X}}\right)\left(\theta_X\theta^*_X + \theta_{\widetilde{X}}\theta^*_{\widetilde{X}}\right)\\
&\qquad= \theta_X\left(\theta^*_X\theta_X\right)^{-1}\theta^*_X + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-1}\theta^*_{\widetilde{X}}
\end{aligned}$$

and

$$\begin{aligned}
&\left(\theta_X\theta^*_X + \theta_{\widetilde{X}}\theta^*_{\widetilde{X}}\right)\left(\theta_X\left(\theta^*_X\theta_X\right)^{-2}\theta^*_X + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-2}\theta^*_{\widetilde{X}}\right)\\
&\qquad= \left(\theta_{X\oplus\widetilde{X}}\,\theta^*_{X\oplus\widetilde{X}}\right)\left(\theta_X\left(\theta^*_X\theta_X\right)^{-\frac{3}{2}}\theta^*_X + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{3}{2}}\theta^*_{\widetilde{X}}\right)^2\\
&\qquad= \theta_X\left(\theta^*_X\theta_X\right)^{-1}\theta^*_X + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-1}\theta^*_{\widetilde{X}}
\end{aligned}$$

Furthermore, since $X\oplus\widetilde{X}$ is a Riesz basis for $H\oplus\widetilde{H}$ and since $\theta^*_{\widetilde{X}}\theta_X = 0$, we have that $\widetilde{H} = \left(\operatorname{ran}\theta_X\right)^{\perp} = \operatorname{ran}\theta_{\widetilde{X}}$ and therefore

$$K = \left(\operatorname{ran}\theta_X\right)\oplus\left(\operatorname{ran}\theta_{\widetilde{X}}\right).$$

Also, since $\theta_X\left(\theta^*_X\theta_X\right)^{-1}\theta^*_X$ is the orthogonal projection of K onto $\operatorname{ran}\theta_X\subset K$ and

$\theta_{\widetilde{X}} \left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-1} \theta^*_{\widetilde{X}}$ is the orthogonal projection of K onto $\widetilde{H} \subset K$, we have that

$$\begin{aligned}\left(\theta_X \left(\theta^*_X\theta_X\right)^{-\frac{3}{2}} \theta^*_X + \theta_{\widetilde{X}} \left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{3}{2}} \theta^*_{\widetilde{X}}\right)^2 \theta_{X\oplus\widetilde{X}} \theta^*_{X\oplus\widetilde{X}} \\ &= I_K \\ &= \theta_{X\oplus\widetilde{X}} \theta^*_{X\oplus\widetilde{X}} \left(\theta_X \left(\theta^*_X\theta_X\right)^{-\frac{3}{2}} \theta^*_X + \theta_{\widetilde{X}} \left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{3}{2}} \theta^*_{\widetilde{X}}\right)^2\end{aligned}$$

Therefore, we have

$$\left(\theta_{X\oplus\widetilde{X}} \theta^*_{X\oplus\widetilde{X}}\right)^{-\frac{1}{2}} = \theta_X \left(\theta^*_X\theta_X\right)^{-\frac{3}{2}} \theta^*_X + \theta_{\widetilde{X}} \left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{3}{2}} \theta^*_{\widetilde{X}}.$$

□

If $\widetilde{X}$ is a frame for $\widetilde{H}$ such that $X \oplus \widetilde{X}$ is a Riesz basis for $H \oplus \widetilde{H}$ and if

$$\theta^*_{X\oplus\widetilde{X}} L^* L \theta_{X\oplus\widetilde{X}} = I_K,$$

then

$$L^* L = \left(\theta_{X\oplus\widetilde{X}} \theta^*_{X\oplus\widetilde{X}}\right)^{-1}.$$

In particular,

$$(L^* L)^{\frac{1}{2}} = \left(\theta_{X\oplus\widetilde{X}} \theta^*_{X\oplus\widetilde{X}}\right)^{-\frac{1}{2}}. \tag{2.2}$$

Analogous to Equations 1.4 and 1.5, we define

$$C = \left\| \left(\theta^*_X\theta_X\right)^{-\frac{1}{2}} \right\|_{\mathrm{HS}}^{-2} \tag{2.3}$$

and

$$D = \left\| \left(\theta^*_X\theta_X\right)^{\frac{1}{2}} \right\|_{\mathrm{HS}}^{2} \tag{2.4}$$

to be the Hilbert-Schmidt lower and upper frame bounds for X, respectively. We will now prove a theorem regarding the minimization of $\kappa_{\mathrm{HS}}(L)$ for a certain class of lifts $\widetilde{X}$.

Theorem 2. *Let $X = \{x_i\}_{i\in\mathbb{J}}$ be a frame for H where $k = \operatorname{card}\mathbb{J} < \infty$. For each frame $\widetilde{X}$ for $\widetilde{H}$ that is a strong complement of X and each lower triangular (with respect to E) and invertible operator L such that $\theta^*_{X\oplus\widetilde{X}} L^* L \theta_{X\oplus\widetilde{X}} = I_{H\oplus\widetilde{H}}$, then*

$$\kappa_{HS}(L) \geq \kappa_{HS}\left((\theta_X^*\theta_X)^{\frac{1}{2}}\right) + (k-n). \tag{2.5}$$

Furthermore, equality in Equation 2.5 holds if and only if $\widetilde{X}$ is tight with frame bound $\sqrt{CD}$ where C and D are the lower and upper Hilbert-Schmidt frame bounds for X, respectively.

Proof. Since $k < \infty$, we have that $n = \dim H \leq k < \infty$. Let $\widetilde{X}$ be an arbitrary frame for $\widetilde{H}$ that is a strong compliment of X, and let $\widetilde{C}$ and $\widetilde{D}$ be the lower and upper Hilbert-Schmidt frame bounds for $\widetilde{X}$. By Equation 2.2 we have that

$$\begin{aligned}\kappa_{\mathrm{HS}}(L) = \|L\|_{\mathrm{HS}} \cdot \|L^{-1}\|_{\mathrm{HS}} &= \left\|(L^*L)^{\frac{1}{2}}\right\|_{\mathrm{HS}} \cdot \left\|(L^*L)^{-\frac{1}{2}}\right\|_{\mathrm{HS}} \\ &= \left\|\left(\theta_{X\oplus\widetilde{X}}\,\theta^*_{X\oplus\widetilde{X}}\right)^{-\frac{1}{2}}\right\|_{\mathrm{HS}} \cdot \left\|\left(\theta_{X\oplus\widetilde{X}}\,\theta^*_{X\oplus\widetilde{X}}\right)^{\frac{1}{2}}\right\|_{\mathrm{HS}}\end{aligned}$$

and thus by Proposition 3 we have

$$\begin{aligned}\kappa_{\mathrm{HS}}(L) = &\left\|\theta_X\,(\theta_X^*\theta_X)^{-\frac{3}{2}}\,\theta_X^* + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{3}{2}}\theta^*_{\widetilde{X}}\right\|_{\mathrm{HS}} \\ &\cdot \left\|\theta_X\,(\theta_X^*\theta_X)^{-\frac{1}{2}}\,\theta_X^* + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{1}{2}}\theta^*_{\widetilde{X}}\right\|_{\mathrm{HS}}\end{aligned}$$

Since X and $\widetilde{X}$ are strongly disjoint, we have

$$\begin{aligned}\kappa^2_{\mathrm{HS}}(L) &= \left(\left\|\theta_X\,(\theta_X^*\theta_X)^{-\frac{3}{2}}\,\theta_X^*\right\|^2_{\mathrm{HS}} + \left\|\theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{3}{2}}\theta^*_{\widetilde{X}}\right\|^2_{\mathrm{HS}}\right) \\ &\quad\cdot \left(\left\|\theta_X\,(\theta_X^*\theta_X)^{-\frac{1}{2}}\,\theta_X^*\right\|^2_{\mathrm{HS}} + \left\|\theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{1}{2}}\theta^*_{\widetilde{X}}\right\|^2_{\mathrm{HS}}\right) \\ &= \left(\left\|(\theta_X^*\theta_X)^{-\frac{1}{2}}\right\|^2_{\mathrm{HS}} + \left\|\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{1}{2}}\right\|^2_{\mathrm{HS}}\right) \cdot \left(\left\|(\theta_X^*\theta_X)^{\frac{1}{2}}\right\|^2_{\mathrm{HS}} + \left\|\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{\frac{1}{2}}\right\|^2_{\mathrm{HS}}\right)\end{aligned}$$

Using the definition of the Hilbert-Schmidt frame bounds, we have

$$\kappa_{\mathrm{HS}}^2(L) = \left(\frac{1}{C} + \frac{1}{\widetilde{C}}\right) \cdot \left(D + \widetilde{D}\right).$$

Since X and $\widetilde{X}$ are complementary, we also have that $X \oplus \widetilde{X}$ is a (Riesz) basis for $H \oplus \widetilde{H}$. By definition, $\widetilde{H} = (\operatorname{ran}\theta_X)^\perp \subset K$. Since X and $\widetilde{X}$ are strong complements, we have that $\widetilde{H} = (\operatorname{ran}\theta_X)^\perp = \operatorname{ran}\theta_{\widetilde{X}} \subset K$. Thus $\dim\left(\operatorname{ran}\theta_{\widetilde{X}}\right) = \dim\left((\operatorname{ran}\theta_X)^\perp\right) = (k-n)$. So, it follows that

$$\kappa_{\mathrm{HS}}\left((\theta_{\widetilde{X}}^*\theta_{\widetilde{X}})^{\frac{1}{2}}\right) = \sqrt{\frac{\widetilde{D}}{\widetilde{C}}} \geq (k-n).$$

with equality if and only if $\theta_{\widetilde{X}}^*\theta_{\widetilde{X}} = c\, I_{\widetilde{H}}$ for some $c > 0$ (i.e. $\widetilde{X}$ is a tight frame for $\widetilde{H}$ with frame bound c). Thus, we have

$$\begin{aligned}\kappa_{\mathrm{HS}}^2(L) &= \frac{D}{C} + \frac{\widetilde{D}}{\widetilde{C}} + \frac{D}{\widetilde{C}} + \frac{\widetilde{D}}{C}\\ &\geq \kappa_{\mathrm{HS}}^2\left((\theta_X^*\theta_X)^{\frac{1}{2}}\right) + (k-n)^2 + \left(\frac{D(k-n)^2}{\widetilde{D}} + \frac{\widetilde{D}}{C}\right)\end{aligned}$$

A straightforward calculus exercise shows that if $f(x) = \dfrac{\alpha}{x} + \beta x$ with $\alpha, \beta > 0$, then $f(x) \geq 2\sqrt{\alpha\beta}$ for all $x > 0$. Furthermore, $f(x)$ attains its minimum on $(0, \infty)$ if and only if $x = \sqrt{\frac{\alpha}{\beta}}$. Thus $\kappa(L)$ attains its minimum if and only if $\widetilde{D} = (k-n)\sqrt{CD}$. Therefore, we have that

$$\begin{aligned}\kappa_{\mathrm{HS}}^2(L) &\geq \kappa_{\mathrm{HS}}^2\left((\theta_X^*\theta_X)^{\frac{1}{2}}\right)^2 + (k-n)^2 + 2\,(k-n)\,\kappa_{\mathrm{HS}}\left((\theta_X^*\theta_X)^{\frac{1}{2}}\right)\\ &= \left(\kappa_{\mathrm{HS}}\left((\theta_X^*\theta_X)^{\frac{1}{2}}\right) + (k-n)\right)^2.\end{aligned}$$

Thus,

$$\kappa_{\mathrm{HS}}(L) \geq \kappa_{\mathrm{HS}}\left((\theta_X^*\theta_X)^{\frac{1}{2}}\right) + (k-n)$$

with equality if and only if $\widetilde{X}$ is a tight frame for $\widetilde{H}$ with Hilbert-Schmidt frame bound $(k-n)\sqrt{CD}$ or equivalently (natural) frame bound $\sqrt{CD}$. Furthermore, in the special case that X in Theorem 2 is tight, then the lifting tight frame $\widetilde{X}$ for $\widetilde{H}$ must have the same frame bound as X. □

Theorem 1 provides an operator factorization method for computing a best Parseval frame Y that is causally equivalent to a given frame X. That is, Theorem 1 provides a constructive method for computing a lower triangular (with respect to E) $L : K \to K$ for which $\theta_Y = L\theta_X$ and $\kappa(L)$ is minimized. As previously noted, there are many choices for L in Theorem 1, but for each choice we must have that

$$(L^*L)^{\frac{1}{2}}\, P = U\theta_X\, (\theta_X^*\theta_X)^{-\frac{3}{2}}\, \theta_X^*$$

where P is the projection of K onto $\operatorname{ran}\theta_X \subset K$ and $U : K \to K$ is unitary.

It is a straightforward exercise [6] to show that if A is a positive and invertible operator on an n dimensional Hilbert space, then

$$\frac{\sqrt{n}}{\|A^{-1}\|} \leq \|A\|_{\mathrm{HS}} \leq \sqrt{n}\|A\|.$$

So, it immediately follows that

$$\frac{1}{\sqrt{n}\|A^{-1}\|} \leq \frac{1}{\|A^{-1}\|_{\mathrm{HS}}} \leq \frac{\|A\|}{\sqrt{n}},$$

and therefore

$$\frac{1}{\|A^{-1}\|^2} \leq \frac{\|A\|_{\mathrm{HS}}}{\|A^{-1}\|_{\mathrm{HS}}} \leq \|A\|^2. \tag{2.6}$$

We will now show that the tight frames that minimize κ_{HS} in Theorem 2 can be used to construct a best Parseval frame that is causally equivalent to a given frame as in Theorem 1.

Proposition 4. *Let H be a Hilbert space such that $\dim H = n < \infty$. Let X be* *a frame for H with* $\operatorname{card} X = k < \infty$ *and lower and upper frame bounds A and B, respectively, and lower and upper Hilbert-Schmidt frame bounds C and D, respectively. Let $\widetilde{X}$ be a tight frame for $\widetilde{H}$ with frame bound $\sqrt{CD}$ that is a strong complement of X. Then, there exists an invertible and lower triangular (with respect to E) operator $L : K \to K$ such that*

$$\theta^*_{X\oplus\widetilde{X}} L^* L \theta_{X\oplus\widetilde{X}} = I_K \tag{2.7}$$

and $\kappa(L) = \sqrt{\frac{B}{A}}$ and $\kappa_{HS}(L) = \sqrt{\frac{D}{C}} + (k-n)$ are minimal. Furthermore, $L\theta_X$ is the analysis operator for a Parseval frame.

Proof. Since $n < \infty$, it follows that $\left\|(\theta_X^*\theta_X)^{\frac{1}{2}}\right\|_{\mathrm{HS}} < \infty$ and $\left\|(\theta_X^*\theta_X)^{-\frac{1}{2}}\right\|_{\mathrm{HS}} < \infty$. Also, since $k < \infty$ it follows that $k - n < \infty$ and thus $\left\|\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{\frac{1}{2}}\right\|_{\mathrm{HS}} < \infty$ and $\left\|\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{1}{2}}\right\|_{\mathrm{HS}} < \infty$. By Equations 2.3 and 2.4 we have that

$$\sqrt{CD} = \frac{\left\|(\theta_X^*\theta_X)^{\frac{1}{2}}\right\|_{\mathrm{HS}}}{\left\|(\theta_X^*\theta_X)^{-\frac{1}{2}}\right\|_{\mathrm{HS}}},$$

and therefore, by Equations 1.4, 1.5, and 2.6, it follows that

$$A = \frac{1}{\left\|(\theta_X^*\theta_X)^{-\frac{1}{2}}\right\|^2} \le \sqrt{CD} \le \left\|(\theta_X^*\theta_X)^{\frac{1}{2}}\right\|^2 = B. \tag{2.8}$$

We solve Equation 2.7 for L^*L and then perform a Cholesky-like factorization on L^*L to obtain L. Since $X \oplus \widetilde{X}$ is a (Riesz) basis for $H \oplus \widetilde{H}$, it follows that $\theta_{X\oplus\widetilde{X}} : H \oplus \widetilde{H} \to K$ is invertible, and therefore by Equation 2.2 we have

$$\begin{aligned}\|L\| = \left\|(L^*L)^{\frac{1}{2}}\right\| &= \left\|\left(\theta_{X\oplus\widetilde{X}}\theta^*_{X\oplus\widetilde{X}}\right)^{-\frac{1}{2}}\right\| \\ &= \left\|\theta_X\left(\theta_X^*\theta_X\right)^{-\frac{3}{2}}\theta_X^* + \theta_{\widetilde{X}}\left(\theta^*_{\widetilde{X}}\theta_{\widetilde{X}}\right)^{-\frac{3}{2}}\theta^*_{\widetilde{X}}\right\|,\end{aligned}$$

and similarly

$$\|L\|_{\mathrm{HS}} = \left\| \theta_X \left(\theta_X^* \theta_X\right)^{-\frac{3}{2}} \theta_X^* + \theta_{\widetilde{X}} \left(\theta_{\widetilde{X}}^* \theta_{\widetilde{X}}\right)^{-\frac{3}{2}} \theta_{\widetilde{X}}^* \right\|_{\mathrm{HS}}.$$

Since X and $\widetilde{X}$ are strongly disjoint, and since $\widetilde{X}$ is tight with frame bound $\sqrt{CD}$, it follows by Equation 2.8 that

$$\begin{aligned}\|L\| &= \max\left\{ \left\| \theta_X \left(\theta_X^* \theta_X\right)^{-\frac{3}{2}} \theta_X^* \right\|, \left\| \theta_{\widetilde{X}} \left(\theta_{\widetilde{X}}^* \theta_{\widetilde{X}}\right)^{-\frac{3}{2}} \theta_{\widetilde{X}}^* \right\| \right\} \\ &= \max\left\{ \left\| \left(\theta_X^* \theta_X\right)^{-\frac{1}{2}} \right\|, \left\| \left(\theta_{\widetilde{X}}^* \theta_{\widetilde{X}}\right)^{-\frac{1}{2}} \right\| \right\} = \max\left\{ \frac{1}{\sqrt{A}}, \frac{1}{\sqrt[4]{CD}} \right\} \\ &= \frac{1}{\sqrt{A}}\end{aligned}$$

and

$$\|L\|_{\mathrm{HS}} = \left(\left\| \left(\theta_X^* \theta_X\right)^{-\frac{1}{2}} \right\|^2 + \left\| \left(\theta_{\widetilde{X}}^* \theta_{\widetilde{X}}\right)^{-\frac{1}{2}} \right\|^2 \right)^{\frac{1}{2}} = \left(\frac{1}{C} + \frac{k-n}{\sqrt{CD}} \right)^{\frac{1}{2}}.$$

Also, we have that

$$\begin{aligned}\|L^{-1}\| = \left\| (L^* L)^{-\frac{1}{2}} \right\| &= \left\| \left(\theta_{X \oplus \widetilde{X}} \theta_{X \oplus \widetilde{X}}^* \right)^{\frac{1}{2}} \right\| \\ &= \left\| \theta_X \left(\theta_X^* \theta_X\right)^{-\frac{1}{2}} \theta_X^* + \theta_{\widetilde{X}} \left(\theta_{\widetilde{X}}^* \theta_{\widetilde{X}}\right)^{-\frac{1}{2}} \theta_{\widetilde{X}}^* \right\|,\end{aligned}$$

and also

$$\|L^{-1}\|_{\mathrm{HS}} = \left\| \theta_X \left(\theta_X^* \theta_X\right)^{-\frac{1}{2}} \theta_X^* + \theta_{\widetilde{X}} \left(\theta_{\widetilde{X}}^* \theta_{\widetilde{X}}\right)^{-\frac{1}{2}} \theta_{\widetilde{X}}^* \right\|_{\mathrm{HS}}.$$

Similar to above, we have that

$$\begin{aligned}\|L^{-1}\| &= \max\left\{ \left\| \theta_X \left(\theta_X^* \theta_X\right)^{-\frac{1}{2}} \theta_X^* \right\|, \left\| \theta_{\widetilde{X}} \left(\theta_{\widetilde{X}}^* \theta_{\widetilde{X}}\right)^{-\frac{1}{2}} \theta_{\widetilde{X}}^* \right\| \right\} \\ &= \max\left\{ \left\| \left(\theta_X^* \theta_X\right)^{\frac{1}{2}} \right\|, \left\| \left(\theta_{\widetilde{X}}^* \theta_{\widetilde{X}}\right)^{\frac{1}{2}} \right\| \right\} = \max\left\{ \sqrt{B}, \sqrt[4]{CD} \right\} \\ &= \sqrt{B}\end{aligned}$$

and also

$$\|L^{-1}\|_{\text{HS}} = \left(\left\|(\theta_X^*\theta_X)^{\frac{1}{2}}\right\|^2 + \left\|(\theta_{\widetilde{X}}^*\theta_{\widetilde{X}})^{\frac{1}{2}}\right\|^2\right)^{\frac{1}{2}} = \left(D + (k-n)\sqrt{CD}\right)^{\frac{1}{2}}.$$

Therefore, $\kappa(L) = \sqrt{\frac{B}{A}}$ and $\kappa_{\text{HS}}(L) = \sqrt{\frac{D}{C} + (k-n)}$ as desired. We then define the Parseval frame Y for H by $\theta_Y = L\theta_X$. □

Theorem 2 provides a "lifting method" for determining a CGGS algorithm to transform a frame X into a Parseval frame in a causal fashion. It demonstrates how choosing a tight lifting frame $\widetilde{X}$ for $\widetilde{H}$ that is strongly disjoint to X followed by the Classical Gram-Schmidt process and the orthogonal projection onto H yields a Parseval frame Y that is causally equivalent to X by L with minimal $\kappa_{\text{HS}}(L) = \sqrt{\frac{D}{C} + (k-n)}$. Theorem 1 provides an "operator factorization method" for determining a similar CGGS algorithm. Proposition 4 shows that the lifting in Theorem 2 yield a lower triangular (with respect to E) operator L as in Theorem 1 for which $\kappa(L) = \sqrt{\frac{B}{A}}$ and $\kappa_{\text{HS}}(L) = \sqrt{\frac{D}{C} + (k-n)}$ are minimal. Appendix A provides the CGGS methods discussed in Theorem 1 and Theorem 2 in pseudo-algorithm form, and Appendix B provides several MATLAB routines used to numerically compute and verify these results.

CHAPTER III

PARTIAL RESULTS AND CONCLUSIONS

A. Compressions of Positive Operators

Theorem 2 provides a lower bound for the Hilbert Schmidt condition number of L given that the lifting frame $\widetilde{X}$ for $\widetilde{H}$ is strongly disjoint from X. That is

$$\kappa_{\mathrm{HS}}(L) \geq \kappa_{\mathrm{HS}}\left((\theta_X^*\theta_X)^{\frac{1}{2}}\right) + (k-n)$$

with equality if and only if $\widetilde{X}$ is tight for $\widetilde{H}$ with frame bound

$$\frac{\left\|(\theta_X^*\theta_X)^{\frac{1}{2}}\right\|_{\mathrm{HS}}}{\left\|(\theta_X^*\theta_X)^{-\frac{1}{2}}\right\|_{\mathrm{HS}}}.$$

In order to satisfy $\theta_X^*L^*L\theta_X = I_H$, we must have that $PL^*LP = \theta_X\left(\theta_X^*\theta_X\right)^{-2}\theta_X^*$ and $P(L^*L)^{-1}P = \theta_X\theta_X^*$ where P is the orthogonal projection of K onto $\operatorname{ran}\theta_X \subset K$.

However, Theorem 2 requires that $\widetilde{X}$ be strongly disjoint from X. We will now show some analysis of relaxing the condition that X and $\widetilde{X}$ be strongly disjoint. If T is a positive and invertible operator on a Hilbert space K and P is a (non-trivial) self-adjoint projection of K onto as subspace H of K, then $R = PTP$ is positive but not invertible. However, R can be viewed as a positive and invertible operator from the Hilbert space H to H. Therefore, we will view R^{-1} as the operator $R' : H \to H$ such that $R'R = P$ and $RR' = I_H$. Since the compression of a positive operator T to a subspace is non-negative and since $T \geq PTP$ (i.e. $T - PTP$ is non-negative), we have that

$$\|T\|_{\mathrm{HS}} \geq \|PTP\|_{\mathrm{HS}}$$

and

$$\|T^{-1}\|_{\mathrm{HS}} \geq \|PT^{-1}P\|_{\mathrm{HS}}.$$

However, there is no relationship (in general) between $\|PT^{-1}P\|_{\mathrm{HS}}$ and $\|R'\|_{\mathrm{HS}} = \left\|(PTP)^{\text{“}-1\text{”}}\right\|_{\mathrm{HS}}$. To illustrate this fact, consider the following two examples.

Example 6. Let $T_1, T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T_1 = \begin{pmatrix} 5 & -3 & -2 \\ -3 & 5 & 2 \\ -2 & 2 & 3 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 5 & 3 & 3 \\ 3 & 3 & 4 \\ 3 & 4 & 4 \end{pmatrix}$$

Then T_1 and T_2 are positive and invertible. Let H be the subspace of $\mathbb{R}^3$ defined by

$$H = \operatorname{span}\left\{ \begin{pmatrix} -4 \\ -9 \\ 2 \end{pmatrix}, \begin{pmatrix} 121 \\ -56 \\ -10 \end{pmatrix} \right\}.$$

Then, $\left\|PT_1^{-1}P\right\|_{\mathrm{HS}} = \frac{\sqrt{2{,}309{,}673}}{2{,}832} \approx 0.537$ and $\left\|(PT_1P)^{\text{“}-1\text{”}}\right\|_{\mathrm{HS}} = \frac{\sqrt{539{,}689}}{1{,}416} \approx 0.519$. We also have that $\left\|PT_2^{-1}P\right\|_{\mathrm{HS}} = \frac{\sqrt{7{,}073{,}449}}{1{,}947} \approx 1.366$ and $\left\|(PT_2P)^{\text{“}-1\text{”}}\right\|_{\mathrm{HS}} = \frac{\sqrt{20{,}455{,}437{,}355}}{97{,}969} \approx 1.460$. In particular, we have that

$$\left\|PT_1^{-1}P\right\|_{\mathrm{HS}} > \left\|(PT_1P)^{\text{“}-1\text{”}}\right\|_{\mathrm{HS}}$$

and

$$\left\|PT_2^{-1}P\right\|_{\mathrm{HS}} < \left\|(PT_2P)^{\text{“}-1\text{”}}\right\|_{\mathrm{HS}}.$$

B. Confluent Equivalence Relations

A relation is any subset of a Cartesian product. For example, if X and Y are sets, then any subset of $X \times Y$ is a *binary relation* from X to Y. In particular, a subset of $X \times X$ is called a binary relation on X. For a binary relation R, one often writes xRy to mean that $(x, y) \in R$. A binary relation R is an *equivalence relation* if R is reflexive, symmetric, and transitive. For an equivalence relation R, we write $x \sim_R y$ to mean xRy.

If R and S are binary relations on X, we define the *confluence* of R with S denoted $R \diamond S$ by

$$R \diamond S = \{(x, y) \in X \times X : \exists\, z \in X \text{ with } xRz \text{ and } zSy\}\,.$$

It is clear that $R \subset R \diamond S$ and $S \subset R \diamond S$. Also, notice that, in general, $R \diamond S \neq S \diamond R$. Furthermore, the following example shows that if R and S are equivalence relations on X, then, in general, neither $R \diamond S$ nor $S \diamond R$ is an equivalence relation on X.

Example 7. Let X be a set of three elements, namely $X = \{a, b, c\}$. Let $R, S \subset X \times X$ defined by

$$R = \{(a, a), (a, c), (b, b), (c, a), (c, c)\}$$

and

$$S = \{(a, a), (b, b), (b, c), (c, b), (c, c)\}.$$

It is clear that both R and S are equivalence relations on X. If we let $A = \{a, c\} \subset X$ and $B = \{b, c\} \subset X$, then we see that $R = (A \times A) \cup \left(A^{\mathsf{C}} \times A^{\mathsf{C}}\right)$ and $S = (B \times B) \cup \left(B^{\mathsf{C}} \times B^{\mathsf{C}}\right)$. Notice that

$$R \diamond S = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)\}$$

and

$$S \diamond R = \{(a, a), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}.$$

It is clear that $R \diamond S$ is not symmetric since $(a, b) \in R \diamond S$ but $(b, a) \notin R \diamond S$. Similarly, $S \diamond R$ is not symmetric since $(b, a) \in S \diamond R$ but $(a, b) \notin S \diamond R$. Also, neither $R \diamond S$ nor $S \diamond R$ is transitive.

Example 7 shows that the confluence of equivalence relations is not necessarily an equivalence relation. Thus, it may be surprising that the confluence of similarity and causality of frames of length k for H **does** yield an equivalence relation. To see this, let

$$\mathcal{H}(n,k) = \{X = \{x_i\}_{i=1}^k \subset H : X \text{ is a frame for } H, \dim H = n\}.$$

If we let S be the equivalence relation on $\mathcal{H}(n,k)$ representing similarity and C be the equivalence relation on $\mathcal{H}(n,k)$ representing causality, we have that

$$S = \{(X,Y) \in \mathcal{H}(n,k) \times \mathcal{H}(n,k) : \theta_X = \theta_Y T, T \in \mathcal{B}(H), T^{-1} \in \mathcal{B}(H)\}$$

and

$$\begin{aligned} C = \{&(X,Y) \in \mathcal{H}(n,k) \times \mathcal{H}(n,k) : \\ &\theta_X = L\theta_Y,\ L \in \mathcal{B}(K),\ L^{-1} \in \mathcal{B}(K),\ L \text{ lower triangular}\} \end{aligned}$$

Therefore,

$$\begin{aligned} S \diamond C &= \{(X,Y) \in \mathcal{H}(n,k) \times \mathcal{H}(n,k) : \\ &\qquad \exists\ Z \in \mathcal{H}(n,k) \text{ with } (X,Z) \in S \text{ and } (Z,Y) \in C\} \\ &= \{(X,Y) \in \mathcal{H}(n,k) \times \mathcal{H}(n,k) : \theta_X = L\theta_Y T,\ T \in \mathcal{B}(H),\ T^{-1} \in \mathcal{B}(H), \\ &\qquad L \in \mathcal{B}(K),\ L^{-1} \in \mathcal{B}(K),\ L \text{ lower triangular}\} \\ &= \{(X,Y) \in \mathcal{H}(n,k) \times \mathcal{H}(n,k) : \\ &\qquad \exists\ Z \in \mathcal{H}(n,k) \text{ with } (X,Z) \in C \text{ and } (Z,Y) \in S\} \\ &= C \diamond S. \end{aligned}$$

It is a straightforward exercise to show that $S \diamond C$ is also both reflexive and transitive.

C. Future Research Directions

While Chapter II establishes two methods for transforming frames into Parseval frames in a causal fashion, both methods still have several unanswered and/or unstudied questions. Of particular importance is the restriction that the lifting frame in Theorem 2 be strongly disjoint from the original frame. Numerical computations indicate that Theorem 2 still holds if the requirement that the lifting frame be strongly disjoint from the original frame is relaxed. However, Section A of this chapter shows that examining the compression of positive and invertible operators to subspaces is not sufficient in gaining a complete understanding of this problem. Therefore, we now formally outline a future research problem.

Question 1. *Let* $n = \dim H$ *and let* X *be a frame for* H *with* $k = \operatorname{card} X < \infty$. *If we let* $\widetilde{H} = (\operatorname{ran}\theta_X)^{\perp} \subset K$, *then each frame* $\widetilde{X}$ *for* $\widetilde{H}$ *that is complimentary to* X *induces a (Riesz) basis* $X \oplus \widetilde{X}$ *for* $H \oplus \widetilde{H}$ *and therefore induces an orthonormal basis (by the Classical Gram-Schmidt process) of the form* $Y \oplus \widetilde{Y}$ *for* $H \oplus \widetilde{H}$ *and a lower triangular (with respect to* E*) and invertible operator* $L : K \to K$ *such that* $\theta_{Y\oplus\widetilde{Y}} = L\theta_{X\oplus\widetilde{X}}$. *By Theorem 2, we have that*

$$\alpha = \inf_{\widetilde{X}} \kappa_{HS}(L) \leq \sqrt{\frac{D}{C}} + (k-n).$$

Since $\operatorname{rank} L = k$, *we know that* $\alpha \geq k$. *What is* α*? In particular, is it true that* $\alpha = \sqrt{\frac{D}{C}} + (k-n)$*?*

Since recent work [1] and current research is in the area of finite frames, it is clear why the restriction of Theorem 2 to finite frames is of particular importance. However, in order to generalize the Classical Gram-Schmidt process to all frames in a causal fashion, the restriction to finite frames may be inadequate. Equation 2.8 shows that if X is a (finite) frame with lower and upper frame bounds A and B, respectively, then $A \leq \sqrt{CD} \leq B$, where C and D are the lower and upper Hilbert-Schmidt frame bounds for X, respectively. Notice that even though both C and D depend on the dimension of H, the quantity $\sqrt{CD}$ does not since it is bound below and above by constants that do not depend on $\dim H$. If H is not finite dimensional, then $\theta_X^* \theta_X : H \to H$ is not necessarily a Hilbert-Schmidt class operator in $\mathcal{B}(H)$. Therefore, neither C nor D is necessarily finite.

Question 2. *If* $\operatorname{card} X = +\infty$*, then it is clear that* $\dim K$ *is not finite. Therefore,* $\kappa_{HS}(L)$ *in Theorem 2 may or may not be finite. Furthermore, the best lifting frame in Theorem 2 is a tight frame for* $\widetilde{H}$ *(that is strongly disjoint of* X*) with frame bound* $\sqrt{CD}$ *which is bound between the frame bounds of* X*. What can be said about extending the analysis of Theorem 2 beyond finite frames (i.e. to frames* X *with* $\operatorname{card} X = +\infty$*).*

A natural question regarding the CGGS process described in Theorem 1 and Theorem 2 is whether the process is continuous. That is, if X and X' are frames which are close (in some sense of "close"), then is it true also that the corresponding Parseval frames Y and Y' obtained by the CGGS, respectively, are also close? One measurement of distance between frames, which is used in [2], is

$$\operatorname{dist}(X, X') = \left(\sum_{i \in \mathbb{J}} \|x_i - x_i'\|^2 \right)^{\frac{1}{2}}.$$

Another measurement of distance between frames is

$$\operatorname{dist}(X, X') = \|\theta_X - \theta_{X'}\|$$

where $\|\theta_X - \theta_{X'}\|$ is the Hilbert space operator norm or Hilbert-Schmidt norm. Other measurements of distances between frames are sometimes used, but these listed are the most prevalent. Due to numerical imprecision and other forms of round off error, it is often the case that construction and/or reconstruction of frames is not completely accurate. That is, if X is the intended frame and X' is the numerically computed frame, then it may be that $\text{dist}(X, X') \neq 0$. Therefore, we consider the following problem regarding the continuity of the CGGS algorithms described in Theoremis 1 and 2.

Question 3. *Let X be a frame for H, and let $\text{dist}(\cdot,\cdot)$ be a metric on the set of frames for H with the same cardinality as X. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that whenever $\text{dist}(X, X') < \delta$, then $\text{dist}(\text{CGGS}(X), \text{CGGS}(X'))$? That is, is the CGGS algorithm in Theorems 1 and 2 continuous?*

The process of lifting a frame to a basis followed by the Classical Gram-Schmidt algorithm and compression to the original Hilbert space requires **no optimality** in the lifting frame. Theorem 2 provides optimal results, and as a consequence, requires the computation of $(\theta_X^* \theta_X)^{-1}$ among other quantities. While the previously described procedure has the same level of computational complexity as the Classical Gram-Schmidt process, it is clear that this complexity is increased if optimality is required. That is, finding a Parseval frame that is causally equivalent to a given frame requires the same number of operations as the Classical Gram-Schmidt process. However, finding the best (as described in Theorems 1 and 2) Parseval frame that is causally equivalent to a given frame requires moderately more computations.

Question 4. *Let X be a frame for H. Given $\varepsilon > 0$, is there a lifting frame $\widetilde{X}$ for $\widetilde{H}$ that is strongly disjoint from X and which is considerably easier (in terms of computational complexity) to compute than that presented in Theorem 2 such that the Parseval frame obtained by performing the Classical Gram-Schmidt algorithm to $X \oplus \widetilde{X}$ followed by the compression to H has $\kappa_{HS}(L) < \sqrt{\frac{D}{C}} + (k-n) + \varepsilon$? That is, if we allow $\kappa_{HS}(L)$ to be slightly more than optimal, can we compute $\widetilde{X}$ (and thus the* *resulting Parseval frame) more efficiently?*

Both Theorems 1 and 2 consider minimizing $\kappa(L)$ (in either the Hilbert space operator norm or the Hilbert-Schmidt norm) as the measurement of "best" Parseval frame that is causally equivalent to a given frame. However, there are other natural choices for measuring the best Parseval frame that is causally equivalent to a given frame. For example, if X is a Parseval frame for H and $D : K \to K$ is an invertible and **diagonal** (with respect to E) operator, then Y defined by $\theta_Y = D\theta_X$ is a frame for H which is not necessarily Parseval. We say, though, that such a Y is a *scalable* frame for H. Since X is Parseval, there exists a frame $\widetilde{X}$ for $\widetilde{H}$ such that $X \oplus \widetilde{X}$ is an orthonormal basis for $H \oplus \widetilde{H}$. Therefore, if we let define $\widetilde{Y}$ by $\theta_{\widetilde{Y}} = D\theta_{\widetilde{X}}$, then $Y \oplus \widetilde{Y}$ is a *scalable* basis for $H \oplus \widetilde{H}$. So, applying the Classical Gram-Schmidt algorithm to $Y \oplus \widetilde{Y}$ yields the orthonormal basis $Z \oplus \widetilde{Z}$ with

$$\theta_{Z\oplus\widetilde{Z}} = D^{-1}\theta_{Y\oplus\widetilde{Y}} = D^{-1}D\theta_{X\oplus\widetilde{X}} = \theta_{X\oplus\widetilde{X}}.$$

Notice that $\kappa(D^{-1}) = \kappa(D)$ (in either of the two norms discussed) is not necessarily minimal. However, the choice of $\widetilde{Y}$ is (in some sense) the most natural choice in that it forces the lower triangular operator from the CGGS to be "as diagonal as possible". We say that an operator $T : K \to K$ is a *band operator* [6] with *lower bandwidth* b_L and *upper bandwidth* b_U if $\langle Te_i, e_j\rangle = 0$ whenever $i > j + b_L$ or $i < j - b_U$. Furthermore, if $b_L = b_U$, we simply call this common value the *bandwidth* of T.

Question 5. *Among all frames $\widetilde{X}$ for $\widetilde{H}$ for which $X \oplus \widetilde{X}$ is a (Riesz) basis for $H \oplus \widetilde{H}$, which ones yield an invertible and lower triangular (with respect to E) operator $L : K \to K$ (via the Classical Gram-Schmidt algorithm) such that $L\theta_X$ is an isometry from H to K and the (lower) bandwidth of L is minimized? That is, instead of minimizing $\kappa(L)$ with some operator norm, we instead minimize the bandwidth of L*

to define the "best" Parseval frame that is causally equivalent to a given frame.

It is clear that the results of Theorem 1 and Theorem 2 provide definite answers toward determining a CGGS algorithm. However, we see that there are still open questions on this topic. These open questions provide opportunities for future research in the area of frame theory.

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APPENDIX A

CGGS PSEUDO-ALGORITHMS

CGGS (Operator Factorization Method)

1. If they are not already known, compute the lower and upper frame bounds A and B for X, respectively

1′. Alternatively, compute the lower and upper Hilbert-Schmidt frame bounds C and D for X, respectively

2. Compute the orthogonal projection of K onto $\widetilde{H} = (\operatorname{ran}\theta_X)^{\perp} \subset K$ by

$$P_{\widetilde{H}} = I_K - \theta_X \left(\theta_X^* \theta_X\right)^{-1} \theta_X^*$$

3. Let $T = \theta_X \left(\theta_X^* \theta_X\right)^{-2} \theta_X^* + \frac{1}{\sqrt{AB}} P_{\widetilde{H}}$

3′. Alternatively, let $T = \theta_X \left(\theta_X^* \theta_X\right)^{-2} \theta_X^* + \frac{1}{\sqrt{CD}} P_{\widetilde{H}}$

4. Factor $T = L^* L$ via a Cholesky-like factorization

5. Define Y by $\theta_Y = L\theta_X$

CGGS (Lifting Method)

1. If they are not already known, compute the lower and upper frame bounds A and B for X, respectively

1′. Alternatively, compute the lower and upper Hilbert-Schmidt frame bounds C and D for X, respectively

2. Compute the orthogonal projection of K onto $\widetilde{H} = (\operatorname{ran} \theta_X)^{\perp} \subset K$ by

$$P_{\widetilde{H}} = I_K - \theta_X \left(\theta_X^* \theta_X\right)^{-1} \theta_X^*$$

3. Define $\widetilde{X} = \frac{1}{\sqrt[4]{AB}} P_{\widetilde{H}} E$

3′. Alternatively, define $\widetilde{X} = \frac{1}{\sqrt[4]{CD}} P_{\widetilde{H}} E$

4. Perform the Classical Gram-Schmidt algorithm on the (Riesz) basis $X \oplus \widetilde{X}$ for $H \oplus \widetilde{H}$ to obtain the orthonormal basis $Y \oplus \widetilde{Y}$ for $H \oplus \widetilde{H}$

5. Project $Y \oplus \widetilde{Y}$ onto H to obtain Y

APPENDIX B

MATLAB ROUTINES

Compare "Inverse" of Compression to Compression of Inverse

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%                                                                       %
%%  Program Name: compression.m                                          %
%%                                                                       %
%%  This routine compares the "inverse" of a compression of a positive   %
%%  operator to the compression of the inverse of the positive operator. %
%%  In particular, the Hilbert Schmidt (Frobenius) norms are compared.   %
%%                                                                       %
%%  Programmer: Troy Henderson                                           %
%%  Contact: thenders@math.tamu.edu                                      %
%%                                                                       %
%%  Date: May 8, 2005                                                    %
%%                                                                       %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%%%%%%%%%%%%%%%%%%
% Initialize MATLAB %
%%%%%%%%%%%%%%%%%%%%%
clear all
clc

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% X will be the analysis operator for a frame of   %
% k vectors for an n dimensional space (with k>=n) %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
k=3;
n=2;
if (k<n)
    disp('Number of frame vectors must be at');
    disp('least the dimension of the space!');
    error(' ');
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Build the analysis operator X for a frame %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
r=0;
```

```
while(r < n)
    X=2*rand(k,n)-1;
    X=round(10*X)/10;
    r=rank(X);
end
X=[-2/5 4/5;-9/10 -4/5;1/5 0];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Find an orthonormal basis for the range of X %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Q=mgs(X);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Build a positive operator A %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
r=0;
while(r < k)
    A=4*rand(k,k)-2;
    A=A'*A;A=(A+A')/2;
    A=round(A);
    r=rank(A);
end
%A=[5 -3 -2;-3 5 2;-2 2 3];
%A=[4 -3 2;-3 2 3;-2 3 9];
%A=[6 3 -3;3 2 -3;-3 -3 3];
A=[5 3 3;3 3 4;3 4 4];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Invert the compression of A to range of X %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
A1=inv(Q'*A*Q);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Compress the inverse of A to the range of X %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
A2=Q'*inv(A)*Q;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Compare the condition numbers of A1 and A2 %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
norm(A2,'fro')-norm(A1,'fro')
```

Convert Frames to Parseval Frames Causally

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
%%                                                                          %
%%  Program Name: f2p.m                                                     %
%%                                                                          %
%%  This routine transforms a frame X into a "best" Parseval frame Y that %
%%  is causally equivalent to X.                                            %
%%                                                                          %
%%  Programmer: Troy Henderson                                              %
%%  Contact: thenders@math.tamu.edu                                         %
%%                                                                          %
%%  Date: May 8, 2005                                                       %
%%                                                                          %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [Y,G] = f2p(X);
[k,n] = size(X);
r=rank(X);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Make sure X represents a frame %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if (n>k | r<n)
   error('X must represent the analysis operator of a frame');
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Compute the inverse of the frame operators and its eigenvalues %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
S=X'*X;
d=eig(S);
SI=inv(S);
SI=(SI+SI')/2; % Done to force SI to be symmetric

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Compute the "normal" and Hilbert Schmidt frame bounds %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
A=min(d); % = 1/norm(sqrtm(SI),2)^2
B=max(d); % = norm(sqrtm(S),2)^2;
C=1/norm(chol(SI),'fro')^2;
D=norm(chol(S),'fro')^2;

%%%%%%%%%%%%%%%%%%%%%
% Lift X to a basis %
%%%%%%%%%%%%%%%%%%%%%
T=X*(SI^2)*X' + 1/sqrt(A*B)*(eye(k,k)-X*SI*X');
%T=X*(SI^2)*X' + 1/sqrt(C*D)*(eye(k,k)-X*SI*X');
```

```
T=(T+T')/2; % Done to force T to be symmetric

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Perform a Cholesky-like factorization of T %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
G=lohc(T);
Y=G*X;
```

Lift frame to a Riesz Basis

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%                                                                          %
%%  Program Name: liftframe.m                                               %
%%                                                                          %
%%  If the input matrix X is the analysis operator for a frame, then this  %
%%  routine outputs the analysis operator F for a complementary tight      %
%%  frame such that the direct sum of the two frames form a (Riesz) basis  %
%%  frame bounds the same as the original frame.                           %
%%                                                                          %
%%  Programmer: Troy Henderson                                              %
%%  Contact: thenders@math.tamu.edu                                         %
%%                                                                          %
%%  Date: May 8, 2005                                                       %
%%                                                                          %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function F = liftframe(X);
[k,n]=size(X);
r=rank(X);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Make sure X represents a frame %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if (n>k | r<n)
   error('X must be the analysis operator of a frame')'
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Compute the inverse of the frame operator and %
% an orthonormal basis for the range of X       %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
S=inv(X'*X);S=(S+S')/2;
Q=mgs(X,S);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% Compute the tight frame bound %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%A=1/norm(S);
%B=norm(X'*X);
C=1/norm(chol(S),'fro')^2;
D=norm(chol(X'*X),'fro')^2;

%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Compute the tight frame %
%%%%%%%%%%%%%%%%%%%%%%%%%%%
F=sqrt(sqrt(C*D))*Q;
```

Perform Cholesky-like Factorization

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%                                                                        %
%%  Program Name: lohc.m                                                  %
%%                                                                        %
%%  Instead of performing the canonical Cholesky factorization on a       %
%%  positive definite matrix A with A=L*L' for some lower triangular L,   %
%%  this routine performs a Cholesky-like factorization of A=L'*L for     %
%%  some lower triangular L.                                              %
%%                                                                        %
%%  Programmer: Troy Henderson                                            %
%%  Contact: thenders@math.tamu.edu                                       %
%%                                                                        %
%%  Date: May 8, 2005                                                     %
%%                                                                        %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [L,p] = lohc(A);
[L p]=chol(rot90(A,2));
L=rot90(L,2);
```

Gram-Schmidt Algorithm

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%                                                                        %
%%  Program Name: mgs.m                                                   %
%%                                                                        %
%%  If the input matrix X is taller than it is wide, this routine outputs %
%%  a matrix Y with columns obtained by performing the Gram-Schmidt       %
%%  algorithm on the columns of X.  If X is wider than it is tall, then   %
%%  this routine outputs a matrix Y whose rows are obtained by performing %
%%  the Gram-Schmidt algorithm on the rows of X.  An error will occur if  %
%%  X does not have full rank.  If X has full rank and X is square, the   %
```

```
%%  Gram-Schmidt process is applied to the columns of X.                    %
%%                                                                          %
%%  In particular, the output Y is either an isometry, a partial            %
%%  isometry, or a unitary.                                                 %
%%                                                                          %
%%  If a second matrix S is input into MGS, S is assumed to be the          %
%%  inverse of X'*X.  This is done to conserve computations if S has        %
%%  already been computed.                                                  %
%%                                                                          %
%%  Programmer: Troy Henderson                                              %
%%  Contact: thenders@math.tamu.edu                                         %
%%                                                                          %
%%  Date: May 8, 2005                                                       %
%%                                                                          %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function Y = mgs(X,S);
[k,n]=size(X);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Check to see if X has full rank %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
r=rank(X);
if ( r < min([k,n]) )
    MSG=['Input matrix has size ',num2str(k),'x',num2str(n)];
    MSG=[MSG,' but has rank ',num2str(r),'.'];
    disp(MSG);
    disp(['Input matrix should have rank ',num2str(min([k,n]))]);
   error(' ');
else
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % If X is wide, transpose, MGS, and transpose back %
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
   if ( k < n )
      Y=mgs(X')';
   else
        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
        % Compute S (if needed) and force it to be positive definite %
        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
        if ( nargin < 2 )
            S=inv(X'*X);
            S=(S+S')/2;
        end

        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
        % Do the magic taking advantage of an already computed S %
        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
      Y=X*rot90(chol(rot90(S,2)),2)';
   end
end
```

VITA

Troy Lee Henderson, IV was born in Selma, Alabama on June 6, 1974. He received his Bachelor of Science in Electrical Engineering from The University of Alabama in May, 1997. He received his Master of Arts in Mathematics from The University of Alabama in August, 1999. He has been a graduate assistant pursuing his Ph.D. in the Department of Mathematics at Texas A&M University from September 1999 to the present. His permanant address is 214 Scott Avenue, Linden, AL 36748.