CYLINDER KERNEL EXPANSION OF CASIMIR ENERGY WITH A ROBIN BOUNDARY

A Thesis

by

ZHONGHAI LIU

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

August 2006

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Approved by:

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ABSTRACT

Cylinder Kernel Expansion of Casimir Energy

with a Robin Boundary. (August 2006)

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Chair of Advisory Committee: Dr. Stephen A. Fulling

We compute the Casimir energy of a massless scalar field obeying the Robin boundary condition ($\frac{\partial}{\partial x}\varphi = \beta\varphi$) on one plate and the Dirichlet boundary condition ($\varphi = 0$) on another plate for two parallel plates with a separation of a. The Casimir energy densities for general dimensions ($D = d + 1$) are obtained as functions of a and β by studying the cylinder kernel. We construct an infinite-series solution as a sum over classical paths. The multiple-reflection analysis continues to apply. We show that finite Casimir energy can be obtained by subtracting from the total vacuum energy of a single plate the vacuum energy in the region $(0, \infty) \times R^{d-1}$. In comparison with the work of Romeo and Saharian(2002), the relation between Casimir energy and the coefficient β agrees well.

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CHAPTER I

INTRODUCTION

A. Zero-point oscillations and their manifestation

In 1948, Casimir and Polder [1] computed for the first time the retarded interaction energy between a neutral but polarizable atom and a perfectly conducting wall. At the same year, Casimir [2] predicted the well-known Casimir effect, that is, two extremely clean, neutral, parallel, microflat conducting surfaces, in a vacuum environment, attract one another by a very weak force that varies inversely as the fourth power of the distance between them [3].

$$F(a) = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}S \tag{1.1}$$

where a is the separation between the plates, $S \gg a^2$ is their area and c is the speed of light.

The Casimir force is widely regarded as rising from the zero-point fluctuations intrinsic to any quantum system. A harmonic oscillator has correspondingly a ground-state energy which is nonzero

$$E_n = \hbar\omega(n + \frac{1}{2}) \tag{1.2}$$

The vacuum of quantum field theory may similarly be regarded as an enormously large collection of harmonic oscillators, representing the fluctuation of, for quantum electrodynamics, the electric and magnetic fields at each point in space. Put otherwise, the QED vacuum is a sea of virtual photons. Thus the zero-point energy density

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The journal model is *Physical Review A*.

of the vacuum is

$$U = \sum_J \frac{1}{2}\hbar\omega_J = 2\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{2}\hbar c|\mathbf{k}| \tag{1.3}$$

where $\mathbf{k}$ is the wavevector of the photon, and the factor of 2 reflects the two polarization states of the photon.

In quantum field theory one is faced with the problem of ultraviolet divergences which come into play when one tries to assign a ground state energy to each mode of the field. Sum (1.3) is clearly infinite. To yield finite expressions for measurable quantities, Casimir had subtracted away from the infinite vacuum energy of (1.3) in the presence of plates, the infinite vacuum energy of quantized electromagnetic field in free Minkowski space. Both infinite quantities were regularized and after subtraction, the regularization was removed leaving the finite result. Boundaries can be considered as a concentrated external field. The vacuum energy in restricted quantization volumes is the vacuum polarization by an external field imposing Dirichlet boundary conditions. We can then say that material boundaries polarize the vacuum of a quantized field, and the force acting on the boundary is a result of this polarization.

The Casimir force has a very strong dependence on the geometry. It was a surprise that the zero-point force was repulsive for the case of a sphere [4]. The attraction between parallel uncharged conducting plates has been convincingly demonstrated by many experiments in the last few years [5]. A statistical precision of 1% was achieved with the use of the Atomic Force Microscope (AFM) [6].

B. Various approaches to Casimir effect

Various methods have been developed to evaluate Casimir energy (for review see [5, 7]) since H.B.G. Casimir published his famous paper [2] in 1948. The Casimir energy can be defined directly as the sum of half-frequencies that is interpreted via

ζ-function regularization [8]. The Green function formalism [9], multiple scattering expansion [10] and heat kernel expansion [11] are proposed in different approaches to calculating the Casimir energy. Recently, optical approximation is proposed as a new approach based on classical ray optics in [12]. In this paper, we follow a multiple scattering expansion approach based on the cylinder kernel [13, 14, 15]. We start our discussion by comparing cylinder kernel and heat kernel due to their similarity.

The local heat kernel is defined by

$$K(t,x,y) = \sum_{n=1}^{\infty} \phi_n(x)\phi_n^*(y)e^{-t\omega_n^2}, \tag{1.4}$$

here ω_n^2 and $\phi_n(x)$ are the corresponding discrete spectrum and eigenfunctions of the problem $\frac{\partial^2 u}{\partial t^2} + \nabla^2 u = 0$. The global heat kernel can be obtained formally as trace over the local one

$$K(t) = Tr(K(t,x,x)) = \sum_{n=1}^{\infty} e^{-t\omega_n^2} \tag{1.5}$$

The less known local cylinder kernel is defined by

$$T(t,x,y) = \sum_{n=1}^{\infty} \phi_n(x)\phi_n^*(y)e^{-t\omega_n} \tag{1.6}$$

Then the global cylinder kernel can be written as

$$T(t) = Tr(T(t,x,x)) = \sum_{n=1}^{\infty} e^{-t\omega_n} \tag{1.7}$$

Actually, the heat kernel and the cylinder kernel can be viewed as the Green functions of the corresponding problems: the heat kernel, $K(t,x,y)$, solves the heat equation in the sense that

$$u(t,x) = \int K(t,x,y)f(y)dy \tag{1.8}$$

is the unique solution of the initial-value problem

$$\frac{\partial u}{\partial t} - \nabla^2 u = 0; u(0,x) = f(x) \tag{1.9}$$

The well known asymptotic expansion of heat kernel is

$$K(t) = \sum_{s=0}^{\infty} b_s t^{-\frac{d}{2}+\frac{s}{2}} \tag{1.10}$$

Here d is the spatial dimension. The cylinder kernel, $T(t,x,y)$, can be defined similarly

$$u(t,x) = \int T(t,x,y) f(y) dy \tag{1.11}$$

is the unique bounded solution of the initial-value problem

$$\frac{\partial^2 u}{\partial t^2} + \nabla^2 u = 0; u(0,x) = f(x), u(0,\infty) \to 0 \tag{1.12}$$

And the counterpart of (1.10) for cylinder is [10]

$$T(t) = \sum_{s=0}^{\infty} e_s t^{-d+s} + \sum_{\substack{s=d+1 \\ s-d\ odd}}^{\infty} f_s t^{-d+s} lnt \tag{1.13}$$

Heat kernel expansion turns out to be a powerful tool to investigate the divergence structure of the vacuum energy, but it doesn't contain nonlocal geometrical information. Casimir energy is a nonlocal effect; its magnitude cannot be deduced from heat kernel expansions, even those including the integrated boundary terms. On the contrary the cylinder kernel coefficients incorporate nonlocal geometrical information. Formally, we can relate Casimir energy to the global cylinder kernel by taking the t-derivative

$$E = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n = -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t} TrT(t,x,x) = -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int T(t,x,x) \tag{1.14}$$

and the simplest definition of the vacuum energy density is

$$T_{00}(t,x) = \frac{1}{2}\sum_{n=1}^{\infty}\omega_n\phi_n(x)\phi_n^*(x) = -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}T(t,x,x) \tag{1.15}$$

In reality, the definitions of Casimir energy and vacuum energy density in (1.14) and (1.15) contain divergent terms. But as we have seen in (1.13), the coefficients of the divergent terms are simple, local objects that can be absorbed by renormalization, only the term of order t in (1.13) contributes to Casimir energy. So when we consider Casimir energy as the coefficients in the short-time asymptotics of the cylinder kernel, the universal, x-independent divergent terms should be discarded in renormalization, then the finite Casimir energy is given by

$$E = -\frac{1}{2}e_{d+1} \tag{1.16}$$

We will discuss the structure of the divergent terms in detail in Chapter IV since it depends on the dimension.

C. Why Robin boundary?

The Casimir effect, a prediction of quantum electrodynamics, can be understood as resulting from the modification of the zero point vacuum fluctuations of the electromagnetic field by the presence of boundaries [16]. Since the electromagnetic field can be separately studied as transverse electric (TE) and transverse magnetic (TM) modes, one can always reduce the electromagnetic field problem to two corresponding scalar field problems [17]. For example, for two parallel plates, one can study the Casimir energy of the electromagnetic field by using the Casimir energy of a scalar field satisfying Dirichlet boundary conditions (TE modes) and a scalar field satisfying Neumann boundaries (TM modes) [18]. That's why the existing literature paid more attention to Dirichlet and Neumann boundary other than the less known Robin boundary. Actually, Robin boundary condition can be made conformally invariant while purely-Neumann boundary condition cannot. The importance of conformal invariance is discussed in [19].

Robin boundary condition has been studied in many different contexts. The Robin boundary condition can be expressed as

$$\frac{\partial}{\partial x}\varphi(x) = \beta\varphi(x) \tag{1.17}$$

A phenomenological model for a penetrable surface was considered in for 2-D massless scalar field with β^{-1} representing the finite penetration depth [20]. One also encounters Robin boundary condition when dealing with the Dynamical Casimir effect [21]. For a D-dimensional sphere, the TM modes satisfy Robin boundary conditions on the surface and the TE modes still satisfy Dirichlet boundary conditions [22]. A very detailed calculation of the static Casimir effect with Robin boundary condition was made by Romeo and Saharian [19]. Heat kernel coefficients associated with Robin boundary were studied by Bordag et al [23].

D. The structure of this thesis

The present thesis is organized as follows: in Chapter II, we set up the notation and show how to construct the cylinder kernel for a slab; in Chapter III we'll reproduce the results for two parallel plates with Dirichlet or Neumann boundary; Chapter IV is the main part of this thesis, it concentrates on the case of two parallel plates with Robin boundary condition; Chapter V is conclusion.

CHAPTER II

NOTATION AND MAIN THEOREM

A. How to construct the Green function for a single boundary condition

The cylinder kernel of the free massless scalar field in R^d is

$$T(t,x,y) = C(d)t(t^2 + |\overrightarrow{x} - \overrightarrow{y}|^2)^{-\frac{d+1}{2}} \tag{2.1}$$

where

$$C(d) = \pi^{-\frac{d+1}{2}}\Gamma(\frac{d+1}{2}) \tag{2.2}$$

The cylinder kernel (2.1) is actually the Green function of cylinder equation (1.12) for free space R^d with no boundary. We know from the method of images that the Green function associated with a Dirichlet problem ($u(t,0) = 0$) in a half space $(0,\infty) \times R^{d-1}$ is

$$G_D(t,x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) = G(t,x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) - G(t,-x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) \tag{2.3}$$

and the Green function associated with a Neumann problem ($\frac{\partial}{\partial x}u(t,0) = 0$) in a half space $(0,\infty) \times R^{d-1}$ is

$$G_N(t,x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) = G(t,x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) + G(t,-x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) \tag{2.4}$$

But when it comes to Robin problem

$$\frac{\partial}{\partial x}u(t,0) = \beta u(t,0) \qquad (\beta > 0), \tag{2.5}$$

the elementary method of images doesn't apply any more.

Here we develop a technique for constructing a solution to a differential equation with a Robin boundary condition when a solution to the same or a related equation with the Dirichlet boundary condition is available [24]. We define an operator T as

$$Tf = \frac{\partial f}{\partial x} - \beta f, \tag{2.6}$$

then if Tf satisfies the Dirichlet condition $Tf(x) = 0$ at $x = 0$, $f(x)$ will satisfy the Robin boundary condition at $x = 0$ correspondingly.

The Green functions in (2.3) and (2.4) represent operators that are functions of ∇^2 and hence commute with T. Therefore, in operator notation,

$$G_R = T^{-1} G_D T \tag{2.7}$$

should be the corresponding operator for the Robin problem. It is understood that the action of a Green function on a function is

$$Gf(x, \mathbf{x}_\perp) = \int_0^\infty dy \int_{\mathrm{R}^{d-1}} d\mathbf{x}_\perp G(x, \mathbf{x}_\perp, y, \mathbf{y}_\perp) f(y, \mathbf{y}_\perp) \tag{2.8}$$

Finally, we get that the Green function for the corresponding Robin problem is

$$\begin{aligned} G_R(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = G(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) + G(t, -x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) \\ - 2\beta \int_0^\infty e^{-\beta\varepsilon} G(t, -x-\varepsilon, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) d\varepsilon \end{aligned} \tag{2.9}$$

All these lead us to the more general problem, how to construct the Green function for a slab $(0, a) \times R^{d-1}$ with any kind of boundary conditions. It is helpful if we define three operators D, N and R and $\tilde{R}$ as below

$$D_a G(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = -G(t, 2a - x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) \tag{2.10}$$

$$N_a G(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = G(t, 2a - x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) \tag{2.11}$$

$$R_aG(t,x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) = G(t,2a-x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) \\ - 2\beta \int_0^\infty e^{-\beta\varepsilon} G(t,2a-x-\varepsilon,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp})d\varepsilon \tag{2.12}$$

$$\tilde{R}_aG(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp)G(t,2a-x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) \\ + 2\gamma \int_0^\infty e^{-\gamma\varepsilon} G(t,2a-x+\varepsilon,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp})d\varepsilon \tag{2.13}$$

The corresponding Green functions

$$G_D(t,x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) = (1+D_a)G(t,x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) \tag{2.14}$$

$$G_N(t,x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) = (1+N_a)G(t,x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) \tag{2.15}$$

$$G_R(t,x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) = (1+R_a)G(t,x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) \tag{2.16}$$

will satisfy Dirichlet, Neumann and Robin boundary condition at $x = a$ respectively. Furthermore, G_D and G_N are Green functions (in particular, they have the correct Dirac-delta boundary behavior as $t \to 0$) both in the region to the left of a and in the region to the right of a, whereas G_R has that property to the right of a.

A Robin condition at a right-hand boundary, to be physically similar to (2.5), must be of the form

$$\frac{\partial}{\partial x}u(t,0) = -\gamma u(t,0) \qquad (\gamma > 0). \tag{2.17}$$

(The *inward* normal derivative must have the positive sign.) Then

$$G_{\tilde{R}}(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) = (1+\tilde{R}_a)G(t,x,\mathbf{x}_\perp,y,\mathbf{y}_\perp) \tag{2.18}$$

is the correct Green function for the region left of a.

B. How to construct the Green function for a slab

Theorem 2.1 *Let $T(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp})$ be the cylinder kernel on all of R^d. Then the corresponding cylinder kernel of the slab $(0, a) \times R^{d-1}$with Robin boundary condition at x=0 and Dirichlet boundary condition at x=L is*

$$T_{RD}(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = \sum_{n=0}^{\infty}(D_a R_0)^n T + \sum_{n=1}^{\infty}(R_0 D_a)^n T + \sum_{n=0}^{\infty}(D_a R_0)^n D_a T + \sum_{n=1}^{\infty}(R_0 D_a)^{n-1} R_0 T. \tag{2.19}$$

Here

$$(D_a R_0)^n T(x, y) =(-1)^n T(x - 2na, y) + (-1)^{n+1}(2\beta)\int_0^\infty L_{n-1}^1(2\beta\varepsilon)e^{-\beta\varepsilon}T(x - \varepsilon - 2na, y)d\varepsilon \tag{2.20}$$

$$(R_0 D_a)^n T(x, y) =(-1)^n T(x + 2na, y) + (-1)^{n+1}(2\beta)\int_0^\infty L_{n-1}^1(2\beta\varepsilon)e^{-\beta\varepsilon}T(x + \varepsilon + 2na, y)d\varepsilon \tag{2.21}$$

$$(D_a R_0)^n D_a T(x, y) = (-1)^{n+1} T(-x + 2(n + 1)a, y) + (-1)^n(2\beta)\int_0^\infty L_{n-1}^1(2\beta\varepsilon)e^{-\beta\varepsilon}T(-x + \varepsilon + 2(n + 1)a, y)d\varepsilon \tag{2.22}$$

$$(R_0 D_a)^{n-1} R_0 T(x, y) = (-1)^{n+1} T(-x - 2(n - 1)a, y) + (-1)^n(2\beta)\int_0^\infty L_{n-1}^1(2\beta\varepsilon)e^{-\beta\varepsilon}T(-x - \varepsilon - 2(n - 1)a, y)d\varepsilon \tag{2.23}$$

where

$$L_{n-1}^1(x) = \sum_{j=1}^{n} \frac{n!}{j!(n - j)!}\frac{(-x)^{j-1}}{(j - 1)!} \tag{2.24}$$

is a Laguerre Polynomial. Two notational abbreviations have been adopted: The *variables*$(t, \overrightarrow{x_\perp}, \overrightarrow{y_\perp})$ *are suppressed because they undergo no alteration, and it is understood that the integral terms are to be omitted whenever n=0.*

We provide the proof of Theorem 2.1 in Appendix A. We now comment on the structure of the formula, which is a sum over classical paths from $(y, \overrightarrow{y_\perp})$ to $(x, \overrightarrow{x_\perp})$, including integrations over time delays at the Robin boundary. The terms can be thought of as wave pulses in a generalized sense. Terms (2.20), experience an even number of reflections, starting at the left; terms (2.21), experience an even number of reflections, starting at the right. When $y = x$ these terms are constant and are equal in pairs; these classical paths are periodic orbits (at least when $\overrightarrow{y_\perp} = \overrightarrow{x_\perp}$) and will contribute the spatially uniform Casimir energy associated with the finiteness of L. Terms (2.22), experience an odd number of reflections, starting at the right; Terms (2.23), experience an odd number of reflections, starting at the left. When $(y, \overrightarrow{y_\perp}) = (x, \overrightarrow{x_\perp})$ these paths are bounce orbits (closed but not periodic) that contribute the localized vacuum energy of interaction of a quantum field with the boundaries.

Actually, based on theorem 2.1, we can deal with any kind of boundary condition by replacing R_0 or D_a by corresponding operator, for instance, for the case with Neumann boundary condition at $x = 0$ and Dirichlet boundary condition at $x = a$, the corresponding cylinder kernel satisfying both boundary conditions is just

$$\begin{aligned} T_{ND}(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = \sum_{n=0}^{\infty} (D_a N_0)^n T + \sum_{n=1}^{\infty} (N_0 D_a)^n T \\ + \sum_{n=0}^{\infty} (D_a N_0)^n D_a T + \sum_{n=1}^{\infty} (N_0 D_a)^{n-1} N_0 T \end{aligned} \tag{2.25}$$

Because of the simplicity of the slab geometry, the series solution (2.19) is exact, in principle; no stationary-phase approximations, for instance, have been needed. In practice, it may become necessary to truncate the sum, considering only short paths.

CHAPTER III

CASIMIR ENERGY OF A SLAB WITH DIRICHLET OR NEUMANN BOUNDARY CONDITIONS

A. Both Dirichlet boundary conditions

In Theorem 2.1, we replace R_0 with D_0, then the cylinder kernel satisfying both Dirichlet boundary conditions at $x = 0$ and $x = a$ is

$$\begin{aligned} T_{DD}(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = &\sum_{n=0}^{\infty}(D_a D_0)^n T + \sum_{n=1}^{\infty}(D_0 D_a)^n T \\ &+ \sum_{n=0}^{\infty}(D_a D_0)^n D_a T + \sum_{n=1}^{\infty}(D_0 D_a)^{n-1} D_0 T \end{aligned} \tag{3.1}$$

it can be divided into four parts

$$\begin{aligned} (D_a D_0)^n T =& \frac{C(d)t}{(t^2 + (2na)^2)^{\frac{d+1}{2}}}, n \geq 0 \\ (D_0 D_a)^n T =& \frac{C(d)t}{(t^2 + (2na)^2)^{\frac{d+1}{2}}}, n \geq 1 \\ (D_a D_0)^n D_a T =& -\frac{C(d)t}{(t^2 + (2na + 2a - 2x)^2)^{\frac{d+1}{2}}}, n \geq 0 \\ (D_0 D_a)^{n-1} D_0 T =& -\frac{C(d)t}{(t^2 + (2na - 2a + 2x)^2)^{\frac{d+1}{2}}}, n \geq 1 \end{aligned} \tag{3.2}$$

In the view of sum over classical paths that experience a number of reflections on the boundary, the cylinder kernel in (2.19) can be reorganized by number of reflections

$$T_{DD}(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = T + (D_0 T + D_a T) + (D_0 D_a T + D_a D_0 T) + \ldots \tag{3.3}$$

The first term T experiences no reflection; the contribution of this term to vacuum energy density is

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}T = -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\frac{C(d)}{t^d} = \frac{C(d)}{2}\frac{d}{t^{d+1}}|_{t\to 0} \tag{3.4}$$

This is the anticipated leading divergent term. It is the universal, x-independent formal vacuum energy of infinite empty flat space; it should be discarded in renormalization. The second term D_aT experiences only one reflection on the boundary, it contributes

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}D_aT=\frac{C(d)}{2}\frac{1}{(2a-2x)^{d+1}} \tag{3.5}$$

The third term D_0T experiences one reflection on the boundary too, it contributes

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}D_0T=\frac{C(d)}{2}\frac{1}{(2x)^{d+1}} \tag{3.6}$$

These two terms are *dangerous* since we can see that the energy density contributed by D_aT is divergent near the Dirichlet boundary ($x\to a$) and the energy density contributed by D_0T is divergent near the Dirichlet boundary ($x\to 0$). Next we write down the subcontributions to vacuum energy density for general n terms which are reflected at least twice on the boundary.

$$(D_aD_0)^nT:-\frac{C(d)}{2(2na)^{d+1}},n\geq 1 \tag{3.7}$$

$$(D_0D_a)^nT:-\frac{C(d)}{2(2na)^{d+1}},n\geq 1 \tag{3.8}$$

$$(D_aD_0)^nD_aT:\frac{C(d)}{2(2na+2a-2x)^{d+1}},n\geq 1 \tag{3.9}$$

$$(D_0D_a)^{n-1}D_0T:\frac{C(d)}{2(2na-2a+2x)^{d+1}},n\geq 2 \tag{3.10}$$

The subcontributions from $(D_aD_0)^nT$ and $(D_0D_a)^nT$ are constant terms and independent of x, they correspond to the periodic orbits. The subcontributions from $(D_aD_0)^nD_aT$ and $(D_0D_a)^{n-1}D_0T$ are dependent of x, they correspond to the bounce orbits. All terms with at least twice reflections are finite; so after we discard the universal divergent term T itself, the *dangerous* terms with only one reflection, D_aT and D_0T, are the only two terms which contain divergence. They can be related to the situation of a single plate. We consider a single plate with Dirichlet boundary condition at $x = a$ and a single plate with Dirichlet boundary condition at $x = 0$ respectively. The corresponding cylinder kernel can be constructed based on (2.19).

The cylinder kernel satisfying Dirichlet boundary condition for a single plate placed at $x = a$ is

$$T_D(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = T + D_aT \tag{3.11}$$

and cylinder kernel satisfying Dirichlet boundary condition for a single plate placed at $x = 0$ is

$$T_D(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = T + D_0T \tag{3.12}$$

Each cylinder kernel contains the trivial term T; it's the universal divergent term too so we discard it. The corresponding vacuum energy densities are

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}D_aT = \frac{C(d)}{2}\frac{1}{(2a-2x)^{d+1}} \tag{3.13}$$

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}D_0T = \frac{C(d)}{2}\frac{1}{(2x)^{d+1}} \tag{3.14}$$

They are precisely the *dangerous* terms of the slab case. We conclude that the *dangerous* terms can be removed by renormalization and thus finite Casimir energy can be obtained.

Integrating the corresponding energy densities for the total energy in the region $(0, a) \times R^{d-1}$ we obtain that the contributions of the two *dangerous* terms D_aT and

D_0T are

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_0^a D_aT = \int_0^a \frac{C(d)}{2}\frac{1}{(2a-2x)^{d+1}}dx = \frac{C(d)}{4d}\frac{1}{(2a-2x)^d}|_0^a \quad (3.15)$$

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_0^a D_0T = \int_0^a \frac{C(d)}{2}\frac{1}{(2x)^{d+1}}dx = -\frac{C(d)}{4d}\frac{1}{(2x)^d}|_0^a \quad (3.16)$$

They still contain divergent terms, but we can get the renormalized total energy by subtracting the vacuum energy of a single plate. For the Dirichlet plate at $x = a$, we subtract the vacuum energy of a single plate in the region $(-\infty, a) \times R^{d-1}$

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}(\int_0^a D_aTdx - \int_{-\infty}^a D_aTdx) = \frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_{-\infty}^0 D_aTdx = \frac{C(d)}{4d}\frac{1}{(2a)^d} \quad (3.17)$$

The same way, for the Dirichlet plate at $x = 0$, we subtract the vacuum energy of a single plate in the region $(0, \infty) \times R^{d-1}$

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}(\int_0^a D_0Tdx - \int_0^\infty D_0Tdx) = \frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_a^\infty D_0Tdx = \frac{C(d)}{4d}\frac{1}{(2a)^d} \quad (3.18)$$

For general $n \geq 1$, no divergent terms are involved. $(D_aD_0)^nT$ and $(D_0D_a)^nT$ each contribute to the total energy

$$-\frac{C(d)}{(2)^{d+2}n^{d+1}a^d} \quad (3.19)$$

$(D_aD_0)^nD_aT$ contributes

$$-\frac{C(d)}{4d}[\frac{1}{(2na)^d} - \frac{1}{(2na+2a)^d}] \quad (3.20)$$

For general $n \geq 2$, $(D_0D_a)^{n-1}D_0T$ contributes

$$-\frac{C(d)}{4d}[\frac{1}{(2na-2a)^d} - \frac{1}{(2na)^d}] \quad (3.21)$$

Now we sum up all terms after renormalization to obtain the finite total energy

$$E_{DD} = -\frac{C(d)}{2^{d+1}a^d}\sum_{n=1}^{\infty}\frac{1}{n^{d+1}} = -\frac{C(d)}{2^{d+1}a^d}\zeta(d+1) \tag{3.22}$$

Then we reproduce the known results $E_{DD} = -\frac{\pi}{24a}$ for d=1 and $E_{DD} = -\frac{\pi^2}{1440a^3}$ for d=3.

B. One Dirichlet and one Neumann boundary conditions

In Theorem 2.1, we replace R_0 with N_0, then the cylinder kernel satisfying Neumann boundary condition at $x=0$ and Dirichlet boundary condition at $x=a$ is

$$\begin{aligned} T_{ND}(t,x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) = &\sum_{n=0}^{\infty}(D_aN_0)^nT + \sum_{n=1}^{\infty}(N_0D_a)^nT \\ &+ \sum_{n=0}^{\infty}(D_aN_0)^nD_aT + \sum_{n=1}^{\infty}(N_0D_a)^{n-1}N_0T \end{aligned} \tag{3.23}$$

It can be divided into four parts

$$(D_aN_0)^nT = (-1)^n\frac{C(d)t}{(t^2+(2na)^2)^{\frac{d+1}{2}}}, \quad n \geq 0 \tag{3.24}$$

$$(N_0D_a)^nT = (-1)^n\frac{C(d)t}{(t^2+(2na)^2)^{\frac{d+1}{2}}}, \quad n \geq 1 \tag{3.25}$$

$$(D_aN_0)^nD_aT = (-1)^{n+1}\frac{C(d)t}{(t^2+(2na+2a-2x)^2)^{\frac{d+1}{2}}}, \quad n \geq 0 \tag{3.26}$$

$$(N_0D_a)^{n-1}N_0T = (-1)^{n+1}\frac{C(d)t}{(t^2+(2na-2a+2x)^2)^{\frac{d+1}{2}}}, \quad n \geq 1 \tag{3.27}$$

In the view of sum over classical paths that experience a number of reflections on the boundary, the cylinder kernel in (2.25) can be reorganized by number of reflection

$$T_{ND}(t,x,\overrightarrow{x_\perp},y,\overrightarrow{y_\perp}) = T + (N_0T + D_aT) + (N_0D_aT + D_aN_0T) + \ldots \tag{3.28}$$

The first term T experiences no reflection; the contribution of this term to vacuum energy density is

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}T = -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\frac{C(d)}{t^d} = \frac{C(d)}{2}\frac{d}{t^{d+1}}|_{t\to 0} \tag{3.29}$$

This is the anticipated leading divergent term. It is the universal, x-independent formal vacuum energy of infinite empty flat space; it should be discarded in renormalization. The second term D_aT experiences only one reflection on the boundary, it contributes

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}D_aT = \frac{C(d)}{2}\frac{1}{(2a-2x)^{d+1}} \tag{3.30}$$

The third term N_0T experiences one reflection on the boundary too, it contributes

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}N_0T = -\frac{C(d)}{2}\frac{1}{(2x)^{d+1}} \tag{3.31}$$

These two terms are *dangerous* since we can see that the energy density contributed by D_aT is divergent near the Dirichlet boundary ($x \to a$) and the energy density contributed by N_0T is divergent near the Neumann boundary ($x \to 0$). Next we write down the subcontributions to vacuum energy density for general n terms which are reflected at least twice on the boundary.

$$(D_aN_0)^nT : (-1)^n\frac{C(d)}{2(2na)^{d+1}}, n \geq 1 \tag{3.32}$$

$$(N_0D_a)^nT : (-1)^n\frac{C(d)}{2(2na)^{d+1}}, n \geq 1 \tag{3.33}$$

$$(D_aN_0)^nD_aT : \quad (-1)^{n+1}\frac{C(d)}{2(2na+2a-2x)^{d+1}}, n \geq 1 \tag{3.34}$$

$$(N_0D_a)^{n-1}N_0T : \quad (-1)^n\frac{C(d)}{2(2na-2a+2x)^{d+1}}, n \geq 2 \tag{3.35}$$

The subcontributions from $(D_aN_0)^nT$ and $(N_0D_a)^nT$ are constant terms and independent of x, they correspond to the periodic orbits. The subcontributions from $(D_aN_0)^nD_aT$ and $(N_0D_a)^{n-1}N_0T$ are dependent of x, they correspond to the bounce orbits. All terms with at least twice reflections are finite; so after we discard the universal divergent term T itself, the *dangerous* terms with only one reflection, D_aT and N_0T, are the only two terms which contain divergences. They can be related to the situation of a single plate.

We consider a single plate with Dirichlet boundary condition at $x = a$ and a single plate with Neumann boundary condition at $x = 0$ respectively. The corresponding cylinder kernels can be constructed based on (2.19).

The cylinder kernel satisfying Dirichlet boundary condition for a single plate placed at $x = a$ is

$$T_D(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = T + D_aT \tag{3.36}$$

and cylinder kernel satisfying Neumann boundary condition for a single plate placed at $x = 0$ is

$$T_N(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = T + N_0T \tag{3.37}$$

Each cylinder kernel contains the trivial term T; it's the universal divergent term too so we discard it. The corresponding vacuum energy densities are

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}D_aT = \frac{C(d)}{2}\frac{1}{(2a-2x)^{d+1}} \tag{3.38}$$

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}N_0T = -\frac{C(d)}{2}\frac{1}{(2x)^{d+1}} \tag{3.39}$$

They are precisely the *dangerous* terms of the slab case. We conclude that the *dangerous* terms can be removed by renormalization then finite Casimir energy can be obtained.

Integrating the corresponding energy densities for the total energy in the region $(0, a) \times R^{d-1}$ we obtain the contributions of the two *dangerous* terms $D_a T$ and $N_0 T$ are

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_0^a D_a T = \int_0^a \frac{C(d)}{2}\frac{1}{(2a-2x)^{d+1}}dx = \frac{C(d)}{4d}\frac{1}{(2a-2x)^d}|_0^a \tag{3.40}$$

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_0^a N_0 T = \int_0^a -\frac{C(d)}{2}\frac{1}{(2x)^{d+1}}dx = \frac{C(d)}{4d}\frac{1}{(2x)^d}|_0^a \tag{3.41}$$

They still contain divergent terms, but we can get the renormalized total energy by subtracting the vacuum energy of a single plate. For the Dirichlet plate at $x = a$, we subtract the vacuum energy of a single plate in the region $(-\infty, a) \times R^{d-1}$

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}(\int_0^a D_a T dx - \int_{-\infty}^a D_a T dx) = \frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_{-\infty}^0 D_a T dx = -\frac{C(d)}{4d}\frac{1}{(2a)^d} \tag{3.42}$$

The same way, for the Neumann plate at $x = 0$, we subtract the vacuum energy of a single plate in the region $(0, \infty) \times R^{d-1}$

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}(\int_0^a N_0 T dx - \int_0^\infty N_0 T dx) = \frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_a^\infty N_0 T dx = \frac{C(d)}{4d}\frac{1}{(2a)^d} \tag{3.43}$$

For general $n \geq 1$, no divergent terms are involved. $(D_a N_0)^n T$ and $(N_0 D_a)^n T$ each contribute to the total energy

$$(-1)^{n+1}\frac{C(d)}{(2)^{d+2}n^{d+1}a^d} \tag{3.44}$$

$(D_a N_0)^n D_a T$ contributes

$$(-1)^{n+1}\frac{C(d)}{4d}[\frac{1}{(2na)^d} - \frac{1}{(2na+2a)^d}] \tag{3.45}$$

For general $n \geq 2$, $(N_0 D_a)^{n-1} N_0 T$ contributes

$$(-1)^{n+1} \frac{C(d)}{4d} [\frac{1}{(2na-2a)^d} - \frac{1}{(2na)^d}] \tag{3.46}$$

Now we sum up all terms after renormalization to obtain the finite total energy

$$E_{ND} = \frac{C(d)}{2^{d+1} a^d} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{d+1}} = \frac{C(d)}{2^{d+1} a^d} \eta(d+1) \tag{3.47}$$

Again we reproduce the result $E_{ND} = \frac{\pi}{48a}$ for d=1 and $E = \frac{7\pi^2}{11520 a^3}$ for d=3.

CHAPTER IV

VACUUM ENERGY DENSITIES OF A SLAB WITH ROBIN OR DIRICHLET BOUNDARY CONDITIONS

In this section we calculate vacuum energy density of a slab with Robin boundary at $x = 0$ and Dirichlet boundary at $x = a$ for general spatial dimension d. We'll discuss the divergent structure of the vacuum energy density. We also consider scalar field satisfying Robin boundary condition or Dirichlet boundary condition for a single plate geometry. In Theorem 2.1 we have constructed the corresponding cylinder kernel; with the definition of vacuum energy density in (1.15), vacuum energy density can be obtained as an infinite summation too.

A. Vacuum energy density for two parallel plates

In Theorem 2.1, we replace T with the expression in (2.1), then the cylinder kernel as a summation can be viewed as four parts

$$(D_aR_0)^nT = \frac{(-1)^nC(d)t}{(t^2+(2na)^2)^{\frac{d+1}{2}}} + \int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\frac{(-1)^{n+1}(2\beta)C(d)t}{(t^2+(2na+\varepsilon)^2)^{\frac{d+1}{2}}}d\varepsilon, n \geq 0 \tag{4.1}$$

$$(R_0D_a)^nT = \frac{(-1)^nC(d)t}{(t^2+(2na)^2)^{\frac{d+1}{2}}} + \int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\frac{(-1)^{n+1}(2\beta)C(d)t}{(t^2+(2na+\varepsilon)^2)^{\frac{d+1}{2}}}d\varepsilon, n \geq 1 \tag{4.2}$$

$$\begin{aligned}(D_aR_0)^nD_aT =& \frac{(-1)^{n+1}C(d)t}{(t^2+(2na+2a-2x)^2)^{\frac{d+1}{2}}} \\ &+ \int_0^\infty L^1_{n-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\frac{(-1)^n(2\beta)C(d)t}{(t^2+(2na+2a+\varepsilon-2x)^2)^{\frac{d+1}{2}}}d\varepsilon, n \geq 0\end{aligned} \tag{4.3}$$

$$(R_0D_a)^{n-1}R_0T = \frac{(-1)^{n+1}C(d)t}{(t^2+(2na-2a+2x)^2)^{\frac{d+1}{2}}} + \int_0^\infty L_{n-1}^1(2\beta\varepsilon)e^{-\beta\varepsilon}\frac{(-1)^n(2\beta)C(d)t}{(t^2+(2na-2a+\varepsilon+2x)^2)^{\frac{d+1}{2}}}d\varepsilon, n \geq 1 \tag{4.4}$$

In the view of sum over classical paths that experience a number of reflections on the boundary, the cylinder kernel in (2.19) can be reorganized by number of reflection

$$T_{RD}(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = T + (R_0T + D_aT) + (R_0D_aT + D_aR_0T) + \dots \tag{4.5}$$

The first term T experiences no reflection; the contribution of this term to vacuum energy density is

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}T = -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\frac{C(d)}{t^d} = \frac{C(d)}{2}\frac{d}{t^{d+1}}|_{t\to 0} \tag{4.6}$$

This is the anticipated leading divergent term. It is the universal, x-independent formal vacuum energy of infinite empty flat space; it should be discarded in renormalization. The second term D_aT experiences only one reflection on the boundary, it contributes

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}D_aT = \frac{C(d)}{2}\frac{1}{(2a-2x)^{d+1}} \tag{4.7}$$

The third term R_0T experiences one reflection on the boundary too, it contributes

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}R_0T = -\frac{C(d)}{2}\frac{1}{(2x)^{d+1}} + \int_0^\infty e^{-\beta\varepsilon}\frac{\beta C(d)}{(\varepsilon+2x)^{d+1}}d\varepsilon \tag{4.8}$$

These two terms are *dangerous* since we can see that the energy density contributed by D_aT is divergent near the Dirichlet boundary ($x \to a$) and the energy density contributed by R_0T is divergent near the Robin boundary ($x \to 0$). Next we write down the subcontributions to vacuum energy density for general n terms which are reflected at least twice on the boundary.

$$(D_aR_0)^nT: \frac{(-1)^{n+1}C(d)}{2(2na)^{d+1}} + (-1)^n \int_0^\infty L_{n-1}^1(\beta\varepsilon)e^{-\beta\varepsilon}\frac{(2\beta)C(d)}{(2na+\varepsilon)^{d+1}}d\varepsilon, n \geq 1 \quad (4.9)$$

$$(R_0D_a)^nT: \frac{(-1)^{n+1}C(d)}{2(2na)^{d+1}} + (-1)^n \int_0^\infty L_{n-1}^1(\beta\varepsilon)e^{-\beta\varepsilon}\frac{(2\beta)C(d)}{(2na+\varepsilon)^{d+1}}d\varepsilon, n \geq 1 \quad (4.10)$$

$$\begin{aligned}(D_aR_0)^nD_aT :&(-1)^n\frac{C(d)}{2(2na+2a-2x)^{d+1}} \\ &+ (-1)^{n+1}\int_0^\infty L_{n-1}^1(\beta\varepsilon)e^{-\beta\varepsilon}\frac{2\beta C(d)}{(2na+2a+\varepsilon-2x)^{d+1}}d\varepsilon, n \geq 1\end{aligned} \quad (4.11)$$

$$\begin{aligned}(R_0D_a)^{n-1}R_0T :&\frac{(-1)^nC(d)}{2(2na-2a+2x)^{d+1}} \\ &+ (-1)^{n+1}\int_0^\infty L_{n-1}^1(\beta\varepsilon)e^{-\beta\varepsilon}\frac{2\beta C(d)}{(2na-2a+\varepsilon+2x)^{d+1}}d\varepsilon, n \geq 2\end{aligned} \quad (4.12)$$

The subcontributions from $(D_aR_0)^nT$ and $(R_0D_a)^nT$ are constant terms and independent of x, they correspond to the periodic orbits. The subcontributions from $(D_aR_0)^nD_aT$ and $(R_0D_a)^{n-1}R_0T$ are dependent of x, they correspond to the bounce orbits. All terms with at least twice reflections are finite; so after we discard the universal divergent term T itself, the *dangerous* terms with only one reflection, D_aT and R_0T, are the only two terms which contain divergence. They can be related to the situation of a single plate.

B. Vacuum energy density for a single plate

We consider a single plate with Dirichlet boundary condition at $x = a$ and a single plate with Robin boundary condition at $x = 0$ respectively. The corresponding cylinder kernel can be constructed based on (2.19).

The cylinder kernel satisfying Dirichlet boundary condition for a single plate placed at $x = a$ is

$$T_D(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = T + D_a T \quad (4.13)$$

and cylinder kernel satisfying Robin boundary condition for a single plate placed at $x = 0$ is

$$T_R(t, x, \overrightarrow{x_\perp}, y, \overrightarrow{y_\perp}) = T + R_0 T \quad (4.14)$$

Each cylinder kernel contains the trivial term T; it's the universal divergent term too so we discard it. The corresponding vacuum energy densities are

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t} D_a T = \frac{C(d)}{2}\frac{1}{(2a-2x)^{d+1}} \quad (4.15)$$

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t} R_0 T = -\frac{C(d)}{2}\frac{1}{(2x)^{d+1}} + \int_0^\infty e^{-\beta\varepsilon}\frac{\beta C(d)}{(\varepsilon+2x)^{d+1}} d\varepsilon \quad (4.16)$$

They are precisely the *dangerous* terms of the slab case. We conclude that the *dangerous* terms can be removed by renormalization then finite Casimir energy can be obtained.

C. Total vacuum energy of a slab with Robin or Dirichlet boundary conditions

In this section we will consider the total vacuum energy of a slab with Robin boundary at $x = 0$ and Dirichlet boundary at $x = a$ for general spatial dimension d. Integrating the corresponding energy densities for the total energy in the region $(0, a) \times R^{d-1}$ we obtain the contributions of the two *dangerous* terms $D_a T$ and $R_0 T$ as

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_0^a D_a T = \int_0^a \frac{C(d)}{2}\frac{1}{(2a-2x)^{d+1}} dx = \frac{C(d)}{4d}\frac{1}{(2a-2x)^d}\Big|_0^a \quad (4.17)$$

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_0^a R_0 T = \int_0^a \Big[-\frac{C(d)}{2}\frac{1}{(2x)^{d+1}} + \int_0^\infty e^{-\beta\varepsilon}\frac{\beta C(d)}{(\varepsilon+2x)^{d+1}} d\varepsilon\Big] dx \quad (4.18)$$

They still contain divergent terms, but we can get the renormalized total energy by subtracting the vacuum energy of a single plate. For the Dirichlet plate at $x = a$, we subtract the vacuum energy of a single plate in the region $(-\infty, a) \times R^{d-1}$

$$-\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\left(\int_0^a D_a T dx - \int_{-\infty}^a D_a T dx\right) = \frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_{-\infty}^0 D_a T dx = -\frac{C(d)}{4d}\frac{1}{(2a)^d} \quad (4.19)$$

The same way, for the Robin plate at $x = 0$, we subtract the vacuum energy of a single plate in the region $(0, \infty) \times R^{d-1}$

$$\begin{aligned} -\frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\left(\int_0^a R_0 T dx - \int_0^\infty R_0 T dx\right) &= \frac{1}{2}\lim_{t\to 0}\frac{\partial}{\partial t}\int_a^\infty R_0 T dx \\ &= \frac{C(d)}{4d}\frac{1}{(2a)^d} - \int_0^\infty e^{-\beta\varepsilon}\frac{\beta C(d)}{2d(\varepsilon+2a)^d}d\varepsilon \end{aligned} \quad (4.20)$$

For general $n \geq 1$, no divergent terms are involved. $(D_a R_0)^n T$ and $(R_0 D_a)^n T$ each contribute to the total energy

$$(-1)^{n+1}\frac{C(d)}{(2)^{d+2}n^{d+1}a^d} + (-1)^n\int_0^\infty L_{n-1}^1(2\beta\varepsilon)e^{-\beta\varepsilon}\frac{\beta a C(d)}{(2na+\varepsilon)^{d+1}}d\varepsilon \quad (4.21)$$

$(D_a R_0)^n D_a T$ contributes

$$\begin{aligned} &(-1)^{n+1}\frac{C(d)}{4d}\left[\frac{1}{(2na)^d} - \frac{1}{(2na+2a)^d}\right] \\ &+ (-1)^{n+1}\frac{\beta C(d)}{2d}\int_0^\infty L_{n-1}^1(2\beta\varepsilon)e^{-\beta\varepsilon}\left[\frac{1}{(\varepsilon+2na)^d} - \frac{1}{(\varepsilon+2na+2a)^d}\right]d\varepsilon \end{aligned} \quad (4.22)$$

For general $n \geq 2$, $(R_0 D_a)^{n-1} R_0 T$ contributes

$$\begin{aligned} &(-1)^{n+1}\frac{C(d)}{4d}\left[\frac{1}{(2na-2a)^d} - \frac{1}{(2na)^d}\right] \\ &+ (-1)^{n+1}\frac{\beta C(d)}{2d}\int_0^\infty L_{n-1}^1(2\beta\varepsilon)e^{-\beta\varepsilon}\left[\frac{1}{(\varepsilon+2na-2a)^d} - \frac{1}{(\varepsilon+2na)^d}\right]d\varepsilon \end{aligned} \quad (4.23)$$

Now we sum up all terms after renormalization to obtain the finite total energy

$$E = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}C(d)}{2^{d+1}n^{d+1}a^d} - \int_0^\infty e^{-\beta\varepsilon} \frac{2\beta a C(d)}{(\varepsilon+2a)^{d+1}} d\varepsilon - \int_0^\infty e^{-\beta\varepsilon} \frac{\beta C(d)}{2d(\varepsilon+4a)^d} d\varepsilon - \sum_{n=2}^{\infty} \int_0^\infty L_{n-1}^1(2\beta\varepsilon) e^{-\beta\varepsilon} [\frac{(-1)^n \beta C(d)}{2d(\varepsilon+2na-2a)^d} - \frac{(-1)^n \beta C(d)}{2d(\varepsilon+2na+2a)^d} - \frac{(-1)^n 2\beta a C(d)}{(2na+\varepsilon)^{d+1}}] d\varepsilon \tag{4.24}$$

The first term in E can be expressed by the Riemann η-function:

$$E_{ND} = \frac{C(d)}{2^{d+1}a^d} \eta(d+1)\,; \tag{4.25}$$

it is the known result for one Neumann and one Dirichlet plate. The integrals in (4.24) can be evaluated in terms of the incomplete gamma function [25, 26]. The resulting infinite summation presumably can't be converted to a closed form. However, the terms starting with $n = 4$ are relatively small and almost cancel each other, so the expression truncated to $n \leq 3$ is a good approximation for the total energy. (The proof of this assertion is in Appendix C.) Explicitly, the total energy for $d = 3$ as a function of $b = \beta a$ (through order $n = 3$) is

$$\begin{aligned} E_{RD} = \frac{7\pi^2}{11520a^3} + \frac{1}{\pi^2 a^3} [& -b^3 e^{2b}\Gamma(-2, 2b) + 3b^3 e^{4b}\Gamma(-2, 4b) + b^3 e^{6b}\Gamma(-2, 6b) \\ & - (19b^3/6)e^{8b}\Gamma(-2, 8b) - 12b^4 e^{4b}\Gamma(-3, 4b) - 72b^4 e^{6b}\Gamma(-3, 6b) \\ & + 56b^4 e^{8b}\Gamma(-3, 8b) + 864b^5 e^{6b}\Gamma(-4, 6b) - 256b^5 e^{8b}\Gamma(-4, 8b) \\ & - 2880b^6 e^{6b}\Gamma(-5, 6b)]. \end{aligned} \tag{4.26}$$

(The b of Romeo and Saharian [19] is the negative reciprocal of our b.)

Note that at $\beta = 0$ the Robin boundary becomes a Neumann boundary and one recovers

$$a^3 E_{RD}\big|_{\beta\to 0} = a^3 E_{ND} = \frac{7\pi^2}{11520} = 0.00599. \tag{4.27}$$

When $\beta \to \infty$, the Robin boundary becomes a Dirichlet boundary, so we expect to

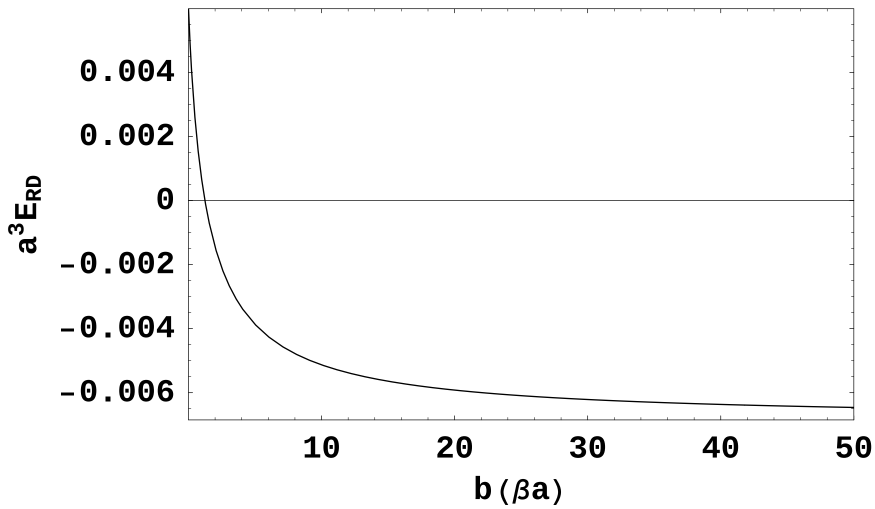

Fig. 1. Total integrated Casimir energy per unit area multiplied by a^3, for $d = 3$ and $0 \leq b = \beta a \leq 5$. The graph of $E(a)$ itself for fixed $\beta \neq 0$ or $+\infty$ would have a minimum somewhere to the right of $a = 1.237/\beta$ and a singularity at the origin.

recover the familiar result

$$a^3 E_{RD}\big|_{\beta\to\infty} = a^3 E_{DD} = -\frac{\pi^2}{1440} = -0.00685. \tag{4.28}$$

The graph of $a^3 E_{RD}$ as a function of $b = \beta a$ is given in Fig. 1, which (together with numerical calculations for larger b) confirms ND and DD cases. The crossover from positive to negative energy occurs near $b = 1.237$, or $-\frac{1}{b} \approx -0.81$, in agreement with [19]. That reference states that this value marks a change from repulsive to attractive Casimir force, but that is incorrect: The zero of the force function $-\frac{\partial}{\partial a}E_{RD}$ occurs at some larger value of a. Our numerical results agree with those of Romeo and Saharian [19] to the extent that they have been compared. Because we use different notations to express the Robin boundary condition, the counterpart of b_2 in their notation is our $-1/b$. For a more direct comparison, we plot in Fig. 2 the total energy E_{RD} with respect to $-1/b$. The result matches [19, Fig. 3] very well, including the location of

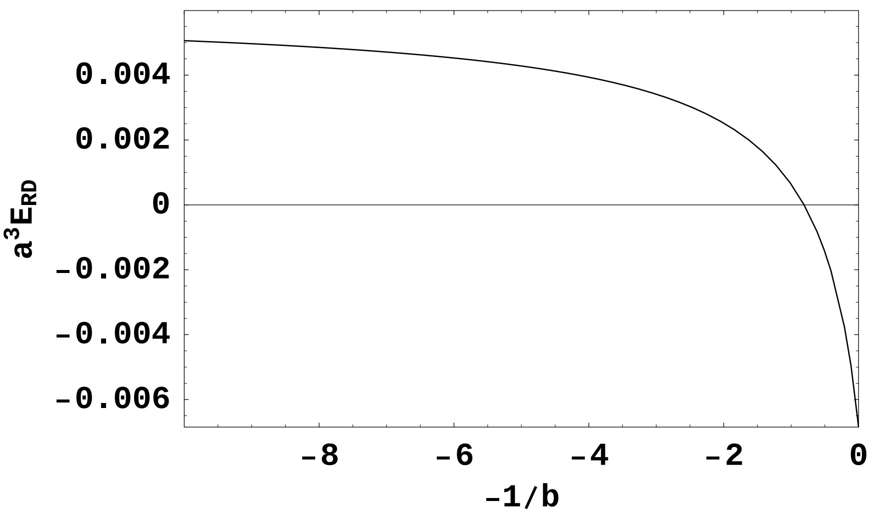

Fig. 2. Total integrated Casimir energy per unit area multiplied by a^3, for $d = 3$ and $-10 \leq -1/b = -1/\beta a \leq 0$. Our $-1/b$ here is the counterpart of b_2 in [19]. The zero of the total energy is $-1/b \approx -0.81$.

the zero.

CHAPTER V

CONCLUSION

In Casimir theory — and in the general study of partial differential equations and the spectral theory of differential operators — the Robin boundary condition is of theoretical interest as the simplest step beyond the standard Dirichlet and Neumann problems for any particular geometrical configuration. The Robin condition also has physical applications: it arises naturally in place of the Neumann condition for half of the modes of the electromagnetic field in the presence of a curved boundary, it mocks up in a simple way the effect of a boundary between two media, and it may have cosmological significance in the brane-world scenario [27].

Their formula in [19] for the total energy is a rather complicated integral. Ours is an infinite sum whose terms fall off fairly rapidly, so reasonable accuracy can be attained by truncating the series. At least in the case where only one of the boundaries is Robin, the individual terms in the series can be evaluated in terms of known special functions, the Laguerre polynomials. The scope of this paper has not allowed us to tackle the case of two Robin boundaries in such detail, nor to study in much depth the questions of how the signs of the energy and the force depend on the parameters. Finally, we have restricted attention to positive Robin constants; the negative case is of more dubious physical significance, and the construction of the cylinder kernel in that case requires different mathematics.

We have taken pains to calculate the local energy density (albeit for only the easiest choice of the conformal coupling parameter, $\xi = \frac{1}{4}$) and to conduct the calculation of the total energy in the same framework. It has been known for many years [28] that vacuum energy densities in flat space are pointwise finite (apart from the ubiquitous zero-point energy of every quantized field) but nonintegrable near boundaries. The Robin condition introduces a new (less singular) divergent term in addition to those familiar from the more elementary conditions. Direct calculations of total energy lead immediately to formal divergences. When an ultraviolet cutoff (in particular, the cylinder-kernel approach) is used, the divergent terms depend on the cutoff parameter t polynomially or logarithmically, and these terms have a close relation to the divergent integrals of the energy density [14]. The divergence associated with the Robin constant is of the logarithmic class when $d = 1$. "Analytic" regularization schemes (dimensional and zeta functions) automatically remove the polynomial terms. However, it is not clear that this nonchalance is physically justified. The energy density serves as a source in the gravitational field equation, so its singular behavior at boundaries cannot just be ignored [28, 29]. Also, the traditional approach to Casimir forces, while plausible for predicting attractions between rigid bodies, has been strongly criticized when applied to deformations of bodies [30, 31, 32, 33]. It may be that the divergent terms in the vacuum energy (or the related divergent integrals of the energy density) can be absorbed into terms in the equations of motion representing the mechanical response of the materials in the bodies, but there is generally no justification for simply setting those terms to zero. In the end a successful physical analysis of a particular system of experimental relevance must be based on a more realistic and complete model, but in the meantime a clear understanding of the (relatively tractable) vacuum-energy calculations is needed in order to diagnose the problems and to determine the limits of validity of the theory.

In the parallel-plate problem we have shown that the only divergent terms are directly associated with the individual plates. Therefore, they are not functions of the plate separation and do not contribute to the force between the plates. (This was, of course, known already, but our treatment of the total energy in the same framework as the energy density removes a certain mysticism from the renormalization and promises to elucidate the physics in more complicated situations in the future.) The remaining finite energy and force are Casimir quantities in the strictest sense; they are associated with the discretization of modes and with periodic orbits of the underlying classical system (not short nonperiodic orbits that bounce off the boundary).

The construction of the cylinder kernel as a multiple-scattering expansion is a powerful method for calculating local spectral and vacuum effects, which demands further development.

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APPENDIX A

PROOF OF THEOREM 2.1

After rearrangement the proposed series in (2.19) is

$$\begin{aligned} T_{RD}(x,y) &= T + \sum_{n=0}^{\infty}(R_0D_a)^n R_0 T + \sum_{n=0}^{\infty}(D_aR_0)^n D_a T + \sum_{n=0}^{\infty}(D_aR_0)^n D_a R_0 T \\ &\qquad + \sum_{n=1}^{\infty}(R_0D_a)^{n-1} R_0 D_a T \\ &= (1+R_0)\sum_{n=0}^{\infty}(D_aR_0)^n T + (1+R_0)\sum_{n=0}^{\infty}(D_aR_0)^n D_a T \\ &= (1+D_a)\sum_{n=0}^{\infty}(R_0D_a)^n T + (1+D_a)\sum_{n=0}^{\infty}(R_0D_a)^n R_0 T. \end{aligned} \tag{A.1}$$

Because of the falloff of T as a function of x (see (2.1)) the series converges (absolutely). Therefore, it is easy to see that it satisfies the cylinder equation (1.12) inside the slab and the proper boundary condition at $t = 0$. Finally, by virtue of (2.10) and (2.12), it satisfies both the Dirichlet condition at $x = L$ and the Robin condition at $x = 0$.

When $n = 1$,

$$D_aR_0T(x,y) = -T(x-2a,y) + (2\beta)\int_0^{\infty} e^{-\beta\varepsilon}T(x-\varepsilon-2a,y)\,d\varepsilon, \tag{A.2}$$

so (2.20) is satisfied when $n = 1$. Suppose that when $n = m$ (2.20) is satisfied:

$$\begin{aligned} (D_aR_0)^m T(x,y) =& (-1)^m T(x-2ma,y) \\ &+ (-1)^{m+1}(2\beta)\int_0^{\infty} L_{m-1}^1(2\beta\varepsilon)e^{-\beta\varepsilon}T(x-\varepsilon-2ma,y)\,d\varepsilon. \end{aligned} \tag{A.3}$$

Then when $n = m + 1$,

$$(D_aR_0)^{m+1}T(x, y) = (-1)^{m+1}T(x - 2ma - 2a, y)$$
$$+ (-1)^{m+2}(2\beta)\left[\int_0^{\infty}(1 + L^1_{m-1}(2\beta\varepsilon))e^{-\beta\varepsilon}T(x - \varepsilon - 2ma - 2a, y)\, d\varepsilon\right.$$
$$\left. - \int_0^{\infty} L^1_{m-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\, d\varepsilon \int_0^{\infty} e^{-\beta\eta}T(x - \varepsilon - \eta - 2ma - 2a, y)\, d\eta\right]. \tag{A.4}$$

Let $\theta = \varepsilon + \eta$; then

$$\int_0^{\infty} L^1_{m-1}(2\beta\varepsilon)e^{-\beta\varepsilon}\, d\varepsilon \int_0^{\infty} e^{-\beta\eta}T(x - \varepsilon - \eta - 2ma - 2a, y)\, d\eta$$
$$= \int_0^{\infty} L^1_{m-1}(2\beta\varepsilon)\, d\varepsilon \int_0^{\infty} e^{-\beta\theta}T(x - \theta - 2ma - 2a, y)\, d\theta$$
$$\equiv -\int_0^{\infty}\sum_{j=1}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{(-2\beta\theta)^{j-1}}{j!}\, e^{-\beta\theta}T(x - \theta - 2ma - 2a, y)\, d\theta. \tag{A.5}$$

Thus

$$(D_aR_0)^{m+1}T(x, y) = (-1)^{m+1}T(x - 2ma - 2a, y)$$
$$+ (-1)^{m+2}(2\beta)\int_0^{\infty}\left[1 + L^1_{m-1}(2\beta\varepsilon) + \sum_{j=1}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\frac{(-2\beta\theta)^{j-1}}{j!}\right]$$
$$\times\, e^{-\beta\varepsilon}T(x - \varepsilon - 2ma - 2a, y)\, d\varepsilon$$
$$= (-1)^{m+1}T(x - 2ma - 2a, y)$$
$$+ (-1)^{m+2}(2\beta)\int_0^{\infty} L^1_{m+1-1}(2\beta\varepsilon)e^{-\beta\varepsilon}T(x - \varepsilon - 2ma - 2a, y)\, d\varepsilon. \tag{A.6}$$

That means that (2.20) is satisfied also when $n = m + 1$. The formulas (2.21)–(2.23) can be proved by induction in the same way.

APPENDIX B

BOUNDARY DIVERGENCES IN THE TOTAL ENERGY

Here we analyze the total energy by the global approach. That is, we integrate $T_{RD}(t,x,x)$ to get the global cylinder kernel $T_{RD}(t)$ before taking its t derivative and examining the limit $t \to 0$. We concentrate on the case $d = 3$ (hence $C(d) = \pi^{-2}$), and we discard from the outset the universal divergent term T of (4.5).

For the infinite space to the right of a Robin plate at $x = 0$ the integrated cylinder kernel is, from (3.12),

$$\begin{aligned} R_0T(t) &= \int_0^\infty R_0T(t,x,x)\,dx \\ &= \int_0^\infty \frac{1}{\pi^2}\frac{t}{(t^2+(2x)^2)^2}\,dx - \int_0^\infty e^{-\beta\varepsilon}d\varepsilon \int_0^\infty \frac{2\beta}{\pi^2}\frac{t}{(t^2+(2x+\varepsilon)^2)^2}\,dx \\ &= \frac{1}{4\pi^2t^2}\left[\frac{2tx}{t^2+4x^2}+\arctan\frac{2x}{t}\right]_0^\infty \\ &\quad - \frac{\beta}{2\pi^2t^2}\int_0^\infty e^{-\beta\varepsilon}d\varepsilon\left[\frac{t(2x+\varepsilon)}{t^2+(2x+\varepsilon)^2}+\arctan\frac{2x+\varepsilon}{t}\right]_0^\infty . \end{aligned} \tag{B.1}$$

For later comparison with the case of two plates, it is convenient to keep the lower-limit and upper-limit contributions separate.

From the upper limit at ∞ one gets (for $\beta \neq 0$)

$$\frac{1}{4\pi^2t^2}\,\frac{\pi}{2} - \frac{\beta}{2\pi^2t^2}\int_0^\infty e^{-\beta\varepsilon}\frac{\pi}{2}\,d\varepsilon = +\frac{1}{8\pi t^2} - \frac{1}{4\pi t^2} = -\frac{1}{8\pi t^2}\,. \tag{B.2}$$

The discontinuity at $\beta = 0$ is only apparent, because we shall now see that the contribution from the ε integral is cancelled by a like term from the lower limit.

From the lower limit 0 one gets

$$
\begin{aligned}
&\frac{\beta}{2\pi^2 t^2}\int_0^\infty e^{-\beta\varepsilon}d\varepsilon\left[\frac{t\varepsilon}{t^2+\varepsilon^2}+\arctan\frac{\varepsilon}{t}\right] \\
&=\frac{\beta}{2\pi^2 t}[\sin\beta t\,(\tfrac{\pi}{2}-\operatorname{Si}\beta t)-\cos\beta t\operatorname{Ci}\beta t]+\frac{1}{2\pi^2 t^2}[\cos\beta t\,(\tfrac{\pi}{2}-\operatorname{Si}\beta t)+\sin\beta t\operatorname{Ci}\beta t].
\end{aligned}
\tag{B.3}
$$

The sine integral function Si and cosine integral function Ci have Taylor expansions

$$
\operatorname{Si}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{(2k+1)!(2k+1)}\,, \tag{B.4}
$$

$$
\operatorname{Ci}(z)=\gamma+\ln z+\sum_{k=1}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!(2k)}\,, \tag{B.5}
$$

where γ is Euler's constant. Therefore, the expansion of (B.3) at small t is

$$
\frac{1}{4\pi t^2}-\frac{\beta}{2\pi^2 t}+\frac{\beta^2}{8\pi^2}+\frac{\beta^3}{6\pi^2}\,t\ln(\beta t)+\frac{(3\gamma-4)\beta^3}{18\pi^2}\,t+O(t^2). \tag{B.6}
$$

The total regularized energy from (B.2) and (B.6) is

$$
E_R(t)=+\frac{1}{8\pi t^3}-\frac{\beta}{4\pi^2 t^2}-\frac{\beta^3}{12\pi^2}\ln(\beta t)-\frac{(3\gamma-1)\beta^3}{36\pi^2}+O(t^1). \tag{B.7}
$$

Similarly, the integrated cylinder kernel to the left of an isolated Dirichlet plate at $x=a$ is

$$
\begin{aligned}
D_aT(t) &= \int_{-\infty}^{a} D_aT(t,x,x)\,dx \\
&= -\frac{t}{\pi^2}\int_{-\infty}^{a}\frac{dx}{(t^2+(2a-2x)^2)^2} = -\frac{t}{\pi^2}\int_0^\infty\frac{du}{(t^2+4u^2)^2} \\
&= -\frac{1}{4\pi^2 t^2}\left[\frac{2tu}{(t^2+4u^2)}+\arctan\frac{2u}{t}\right]_0^\infty \\
&= -\frac{1}{8\pi t^2}\,,
\end{aligned}
\tag{B.8}
$$

which corresponds to a regularized energy

$$
E_D(t)=-\frac{1}{8\pi t^3}\,. \tag{B.9}
$$

We must integrate the terms (4.1)–4.4 from 0 to a. Recall that only the terms D_aT and R_0T contain divergences. In all the other terms the denominator of the integrand remains nonzero even when both t and ε are zero, and therefore one can differentiate and pass to the limit $t \to 0$ before integrating; that is, their contributions are precisely those already presented in (4.21)–(4.23).

For the divergent terms we could recycle the calculations (B.1) and (B.8), replacing the upper limit ∞ with a. However, the difference would be the negatives of the integrals from a to ∞, and to them the same argument as above applies: these are perfectly finite contributions to the energy, even when $t = 0$, and they have already been computed in (4.19)–(4.20).

All that remains to be considered is the sum of the regularized energies (B.7) and (B.9). (Recall that we have already discarded the ubiquitous t^{-4} term.) The terms of order t^{-3} cancel, but this is an artifact of our model, since Dirichlet and Neumann plates have divergent surface energies that are equal and opposite. According to the prescription (1.16) we should discard *all* the terms in the series that diverge as $t \to 0$. In the present case, because there is a logarithmic term in (B.7), we encounter the well known scale ambiguity: because the numerical factor inside the argument of the logarithm is arbitrary, the "finite part" of E_R, hence that of E_{RD}, is defined only up to an arbitrary numerical multiple of β^3. Ignoring E_R entirely in calculating E_{RD} yields the prescription of SecIV. The ambiguous β^3 term does not depend on a and hence does not affect the force between the plates. It does, of course, depend on β; one must feel some trepidation in ignoring it (or even the power-law divergent terms) in situations where β is allowed to vary.

APPENDIX C

WHY WE CAN DISCARD TERMS WITH $N \geq 4$

The expression of the total energy in (4.24) is an infinite summation, but we shall prove for $d = 3$ case that all terms after $n = 3$ are quite small, so it's reasonable to discard them. Note that when $d = 3$, $C(d) = 1/\pi^2$ and hence the β-dependent part of the remainder is

$$\sum_{n=4}^{\infty} \frac{(-1)^n}{\pi^2} \int_0^{\infty} L_{n-1}^1(2\beta\varepsilon) e^{-\beta\varepsilon} f_n(2\beta\varepsilon)\, d(\beta\varepsilon) \tag{C.1}$$

where

$$f_n(2\beta\varepsilon) = \frac{(2\beta)^3}{6(2\beta\varepsilon + 4(n-1)\beta a)^3} - \frac{(2\beta)^3}{6(2\beta\varepsilon + 4(n+1)\beta a)^3} - \frac{2a(2\beta)^4}{(2\beta\varepsilon + 4n\beta a)^4}\,. \tag{C.2}$$

Let $2\beta\varepsilon = x$; then the summation can be written as

$$\sum_{n=4}^{\infty} \frac{(-1)^n}{2\pi^2} \int_0^{\infty} L_{n-1}^1(x) e^{-\frac{x}{2}} f_n(x)\, dx \tag{C.3}$$

and

$$f_n(x) = \frac{(2\beta)^3}{6(x + 4(n-1)\beta a)^3} - \frac{(2\beta)^3}{6(x + 4(n+1)\beta a)^3} - \frac{2a(2\beta)^4}{(x + 4n\beta a)^4}\,. \tag{C.4}$$

It's easy to show in Mathematica that $f_n(x)$ is a decreasing function and $f_n(x) \geq 0$ for any $x \geq 0$, so

$$f_n(x) \leq f_n(0) = \frac{1}{a^3}\left(\frac{1}{6(2n-2)^3} - \frac{1}{6(2n+2)^3} - \frac{2}{(2n)^4}\right). \tag{C.5}$$

It follows that

$$\left| \sum_{n=4}^{\infty} \frac{(-1)^n}{2\pi^2} \int_0^{\infty} L_{n-1}^1(x) e^{-\frac{x}{2}} f_n(x)\, dx \right| \leq \frac{1}{2\pi^2} \sum_{n=4}^{\infty} \left| \int_0^{\infty} L_{n-1}^1(x) e^{-\frac{x}{2}} f_n(x)\, dx \right| \tag{C.6}$$

From the mean value theorem for integrals,

$$\int_0^\infty L_{n-1}^1(x)e^{-\frac{x}{2}}f_n(x)\,dx = f_n(0)\int_0^\eta L_{n-1}^1(x)e^{-\frac{x}{2}}dx, \text{where } 0<\eta<\infty \tag{C.7}$$

then

$$\left|\sum_{n=4}^\infty \frac{(-1)^n}{2\pi^2}\int_0^\infty L_{n-1}^1(x)e^{-\frac{x}{2}}f_n(x)\,dx\right| \le \frac{1}{2\pi^2}\sum_{n=4}^\infty f_n(0)\left|\int_0^\infty L_{n-1}^1(x)e^{-\frac{x}{2}}\,dx\right| \tag{C.8}$$

From (8.971.2) and (8.971.5) in [25] we get the integral

$$\int_0^\eta L_{n-1}^1(x)e^{-\frac{x}{2}}\,dx + \int_0^\eta L_{n-2}^1(x)e^{-\frac{x}{2}}\,dx = -2e^{-\frac{x}{2}}L_{n-1}^0(x)|_0^\eta \tag{C.9}$$

For our purpose we calculate

$$\begin{aligned}&\int_0^\eta L_{n-1}^1(x)e^{-\frac{x}{2}}\,dx + (-1)^{n-2}\int_0^\eta L_0^1(x)e^{-\frac{x}{2}}\,dx\\ &= \sum_{m=2}^n(-1)^{n-m}(\int_0^\eta L_{m-1}^1(x)e^{-\frac{x}{2}}\,dx + \int_0^\eta L_{m-2}^1(x)e^{-\frac{x}{2}}\,dx)\\ &= \sum_{m=2}^n(-1)^{n-m}(-2e^{-\frac{x}{2}}L_{m-1}^0(x)|_0^\eta)\end{aligned} \tag{C.10}$$

Note that $L_0^1(x) = 1$, then

$$\begin{aligned}&\int_0^\eta L_0^1(x)e^{-\frac{x}{2}}\,dx = -2e^{-\frac{x}{2}}|_0^\eta\\ &\int_0^\eta L_{n-1}^1(x)e^{-\frac{x}{2}}\,dx = \sum_{m=2}^n(-1)^{n-m}(-2e^{-\frac{x}{2}}L_{m-1}^0(x)|_0^\eta) + (-1)^{n-2}2e^{-\frac{x}{2}}|_0^\eta\\ &\left|\int_0^\eta L_{n-1}^1(x)e^{-\frac{x}{2}}\,dx\right| \le 2\sum_{m=1}^n\left|(e^{-\frac{x}{2}}L_{m-1}^0(x)|_0^\eta)\right| \le 2\sum_{m=1}^n(\left|e^{-\frac{\eta}{2}}L_{m-1}^0(\eta)\right| + \left|L_{m-1}^0(0)\right|)\end{aligned} \tag{C.11}$$

From([26],(2.14.13))

$$\left|e^{-\frac{\eta}{2}}L_{m-1}^0(\eta)\right| \le 1 \tag{C.12}$$

then

$$\left|\int_0^\eta L_{n-1}^1(x)e^{-\frac{x}{2}}\,dx\right| \le 4n \tag{C.13}$$

Now we continue (C.8)

$$\left|\sum_{n=4}^{\infty} \frac{(-1)^n}{2\pi^2} \int_0^{\infty} L_{n-1}^1(x) e^{-\frac{x}{2}} f_n(x)\, dx\right| \leq \frac{1}{2\pi^2} \sum_{n=4}^{\infty} 4n f_n(0) \tag{C.14}$$

$$\begin{aligned} \frac{1}{2\pi^2} \sum_{n=4}^{\infty} 4n f_n(0) & \\ &= \frac{2}{\pi^2 a^3} \sum_{n=4}^{\infty} n\left(\frac{1}{6(2n-2)^3} - \frac{1}{6(2n+2)^3} - \frac{2}{(2n)^4}\right) \\ &= \frac{1}{4\pi^2 a^3} \sum_{n=4}^{\infty} n\, \frac{10n^4 - 9n^2 + 3}{3n^4(n-1)^3(n+1)^3} \\ &\leq \frac{1}{4\pi^2 a^3} \sum_{n=4}^{\infty} \frac{10n}{3(n-1)^3(n+1)^3} = 1.5 \times 10^{-4} \frac{1}{a^3} \end{aligned} \tag{C.15}$$

which is roughly 2 percent of $|E_{DD}| = \pi^2/1440a^3$. (The actual error in our numerical calculations is at most 0.1%.)

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