THE ROLE OF INSTRUCTIONAL REPRESENTATIONS ON STUDENTS' WRITTEN REPRESENTATIONS AND ACHIEVEMENTS

A Dissertation

by

YE SUN

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements of the degree of

DOCTOR OF PHILOSOPHY

August 2005

Major Subject: Curriculum and Instruction

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Approved as to style and content by:

Chair of Committee,	Gerald Kulm
Committee Members,	Victor Willson
	Cathleen C. Loving
	Charles Lamb
Head of Department,	Dennie Smith

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ABSTRACT

The Role of Instructional Representations on Students' Written Representations and Achievements. (August 2005)

Ye Sun, B.Ed., Beijing Normal University;

M.Ed., Beijing Normal University

Chair of Advisory Committee: Dr. Gerald Kulm

This research is based on Middle School Mathematics Project (MSMP) funded by the Interagency Educational Research Initiative through a grant to the American Association for the Advancement of Science. Both teacher's instructional representations and students' written representations were coded and analyzed to investigate the nature and structure of the representations in teaching fractions, decimals and percents in middle school classrooms in three school districts in Texas. The study further explored the relationship between both the quality and quantity of instructional representations and students' written representations, and the relationship between students' written representations and their achievements.

This dissertation used a mixed approach utilizing both quantitative and qualitative methods. The data was collected in the first two years of a five-year study. A total of 14 sixth grade mathematics teachers from three school districts in Texas were selected from the MSMP project. Before the actual videotaping procedure, a professional development focusing on multiple representations was held for the teachers. Both pretests and posttests were used to examine the relationship between the structure of students' written representations and their achievements.

The results showed that the both the quantity and quality of teachers' instructional representations varied a lot. Symbolic representations were the predominant representations in classroom teaching. Structures of instructional representations converge to content sub-constructs rather than format sub-constructs. Here sub-constructs include part-whole, measure, quotient, multiplication by one and cross product. Instead, format sub-constructs include real world, manipulatives, pictures, spoken symbolic representations and written symbolic representations, however, connections between these sub-constructs were not statistically significant. Within the three content sub-constructs (part-whole, quotient, and multiplication by one) that revealed by students' written representations, quotient and multiplication by one significantly predicated the students' posttest scores. It was also found that, among the three quality criteria (accuracy, comprehensibility and connections) of instructional representations, the comprehensibility score significantly predicated students' achievement in the posttests.

DEDICATION

To my parents for their love and care in helping me achieve my education.

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The research relayed in this dissertation has been thoroughly challenging but rewarding. That reward mainly came from the interaction with my supervisors, colleagues and the teachers who participated in the study.

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Last but not least, I would like to thank my parents for their love and caring. Without their continuous understanding and support, I could not have realized today's achievements. I am, and will always be deeply indebted to my parents during my whole lifetime. I wish I could thank them adequately. I hope that the completion of my doctoral studies will serve as a testament to their faith in my abilities, and this dissertation is dedicated to them.

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CHAPTER I

INTRODUCTION

Background

The use of representations has received researchers' attentions in the mathematic education community since the 1980s, with more and more agreement on the positive influence of representations in developing students' conceptual understanding, mathematics reasoning, problem solving, and communication (Ainsworth, 1999; Ball, 1988; Baxter & Glaser, 1998; Hiebert & Wearne, 1986; Kaput, 1989). The National Council of Teachers of Mathematics (NCTM) has strongly advocated the critical role of representation in both mathematics instruction and learning (NCTM, 2000).

There exists a large body of literature on the role of different forms of representations in facilitating students' learning (Chandler & Sweller, 1992; Garrity, 1998; Haas, 1998; Hinzman, 1997; Kalyuga, Chandler & Sweller, 1998; Leinenbach & Raymond, 1996; McClung, 1998; Post, 1981; Sharp, 1995). However, empirical results revealed inconsistency regarding whether one form of representation was better than another. For example, manipulatives was one of the most controversial forms of representations in public schools, and it was reported as both effective and ineffective in the literature. Some empirical studies stated that manipulatives improved students' learning (Garrity, 1998; Haas, 1998; Leinenbach & Raymond, 1996; Post, 1981). In contrast, other studies claimed that there was no significant correlation between

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manipulatives and students' learning improvements (Hinzman, 1997; McClung, 1998; Sharp, 1995). Another example of a conflict in the literature was the debate on multiple representations, i.e., different forms of representations. Some studies indicated that less effective learning occurred because of increased cognitive load when using multiple representations (Chandler & Sweller, 1992; Kalyuga, Chandler, & Sweller, 1998). In contrast, some other studies showed that students who used multiple representations tended to have a better understanding (Kaput, 1989; Resnick & Omanson, 1987; Schoenfeld, 1986; Sfard, 1991). A third idea claimed that there was no single best representation and that the effectiveness depended on the properties of the content that was learned (Bibby & Payne, 1993).

Multiple representations (e.g., tables, graphs, equations and symbolic representations) can facilitate students' understanding, thus they are advocated by the National Council of Teachers of Mathematics (NCTM) as a tool for learning fractions (American Association for the Advancement of Science, 2000; NCTM, 2000; Wood, 1999). Reasons to use multiple representations could be classified into the following three categories. The first one relates to the nature of the concept in reality. A concept usually consists of several representation aspects. Only one form of representation usually limits the meaning of a concept. Thus multiple representations will prevent superficial understanding of a certain concept, as Kaput (1992) stated, "and hence require multiple systems for their full expression, meaning that multiple, linked representations will grow in importance as an application of the new, dynamic, interactive media "(p. 530). Another advantage of multiple representations is the assumption that multiple forms of representation are more likely to reach the student body by appealing to various learning styles (Ainsworth, 1999). Finally, science research indicates that perception and cognition are processes that depend on and correlate with each other. Different portions of the brain are associated with comprehending a certain related representation, for example, symbolic or pictorial representations, and different knowledge is represented in different forms. Therefore, the use of multiple representations tends to capture the internal connections between the distributed forms of knowledge in a more comprehensive manner (Gazzaniga, 2000).

Researchers have repeatedly reported that middle-grade students have difficulties in developing conceptual understanding of fractions, decimals and percents (Condon & Hilton, 1999; Goldin & Passantino, 1996; Lesh, Post, & Behr, 1987; Post, Cramer, Behr, Lesh, & Harel, 1993; Watanabe, Reynolds, & Lo, 1995). Indeed, even students in junior college have difficulties dealing with fractions, which can be connected to their earlier experiences in elementary school study when they first learned fractions (Haas, 1998). Haas (1998) reported that the reason for the difficulties was that instruction on fractions was delivered neither appropriately nor adequately in order to build up the connections between manipulatives representation and symbolic representations. Taber (2001) also indicated that addressing the connection among different forms of representations was important in order to develop the conceptual understanding of fractions.

In contrast to whole numbers, there are not very many real world experiences for students to use fractions to solve problems. Thus the classroom is the major environment wherein students can learn fractions (Streefland, 1991). If students have received inadequate instruction in the early stage of their learning, it is not surprising that they may find themselves behind as they advance to the middle schools, or even as adults. Students' poor performance on fractions, decimals and percents reflects the instruction they received.

Emphasizing the importance and effectiveness of representations in learning fractions is not enough. Teachers should have the corresponding mathematics knowledge and pedagogical knowledge to construct an environment allowing students to experience different representations to facilitate learning. However, what is the reality of using representations in teaching fractions, decimals and percents in middle school classrooms? What are the concepts that are most commonly taught? How do the teaching quality and quantity relate to students' achievements? There is limited research on investigating how middle school teachers use representations in classroom practice, examining the impact of their teaching quality and quantity on students' understanding and achievements.

The Middle School Mathematics Project (MSMP) at Texas A&M University is part of a five-year longitudinal study funded by the Interagency Educational Research Initiative through a grant to the American Association for the Advancement of Science (Roseman, Kulm, & Manon, 2001). The main goal of the MSMP is to investigate the role of content-based professional development and textbooks in assisting teachers' classroom instructional practices, and further investigate how teaching practices influence students' achievements. Four professional development workshops were conducted in the first four years. Each year, three to five lessons per teacher were video-taped, and corresponding students were administered a pretest and a posttest. This dissertation used the data collected by the MSMP project by analyzing the teachers' videotapes and students' pretests and posttests to investigate the role of the quality and quantity of teachers' instructional representations on students' understanding and achievements in fractions, decimals and percents. During a pilot study, some teachers were found to have insufficient knowledge or skills, which might have led to their inability to use representations appropriately in classrooms. Teachers must be aware of the benefits and disadvantages of using different forms of representations, and their effectiveness of improving conceptual understanding. Armed with this knowledge, teachers can apply representations effectively in classroom instruction, thus better serving their students.

Statement of the Problem

Researchers in the field of cognitive psychology claimed that there were two categories of representations: external and internal, and that they were correlated with each other (Kaput, 1999; Goldin, 2003; Zelazo & Lourenco, 2003). Both internal and external representations were critical in developing children's mathematics understanding (Jonassen, Cole, & Bamford, 1992; Kaput, 2001; Lenze & Dwyer, 1993; Miura, 2001). The visualization aspect of external representations could illustrate a concept profoundly by capturing different characteristics of the concept (Goldin, 2003). Internal representations also play an important role in learning (Hall, Bailey & Tillman, 1997; Hiebert & Carpenter, 1992; Schwartz, 1993). Hiebert and Carpenter (1992) contended that knowledge represented in an internal mental network tended to enhance mathematical conceptual understanding. Zhang (1997) stated that learning occurred during the interaction between the external representations and internal representations. This dissertation aims to investigate the role of teachers' instructional representations (external representations) on students' external representations. Students' external representations are correlated to their internal representations and thus indicated their level of understanding. According to Zelazo and Lourenco (2003), "It has long been assumed that children's understanding and use of external representations, such as drawings and speech, potentially provide insight into the development of internal representations" (p. 55).

The research literature suggests that students' understandings of external symbolic representations of fractions, decimals and percents is one of the most difficult tasks facing middle school mathematic education (Condon & Hilton, 1999; Goldin & Passantino, 1996; Lesh, Post, & Behr, 1987; Post, Cramer, Behr, Lesh, & Harel, 1993; Watanabe, Reynolds, & Lo, 1995). Many middle school students have problems in translations between external symbolic representations, such as changing from fractions to decimals and from decimals to percents (Condon & Hilton, 1999; Markovits & Sowder, 1991; Thompson & Walker, 1996; Vance, 1992). They also have problems in translations between external symbolic representations and external pictorial representations and manipulatives representations, for example, find out the location of ¼ on a number line (Vance, 1992), or using a hundredths grid to represent 0.4 presents a challenge (Hiebert & Wearne, 1986).

Students' learning depends on both the quality and quantity of teachers' instruction (Aronson, Zimmerman, & Carlos, 1998; Black, 2002; Carpenter & Fennema, 1991; Simon, 1997; Smith, 2000; Walker, 1976). In terms of which is more important, some researchers argue for quality (Aronson, Zimmerman, & Carlos, 1998; Smith, 2000), while some argue for quantity (Black, 2002; Walker, 1976). The American Association of Advancement of Science (AAAS) claimed that good representations should be accurate, comprehensible and included a variety of representations, and should not allow students to develop misconceptions (AAAS, 2000). However, little research has been done to examine the quality of teachers' instructional representations in terms of accuracy and comprehensibility. Black (2002) categorized teaching time into allocated time, engaged instructional time and academic learning time. There have been few empirical studies which investigated the structure of engaged instructional time in teaching and learning fractions. There was not many empirical studies reported the influence of both the quality and quantity of instructional representations on students' written representations and the influence on student's achievement.

This study investigated both the quality and the quantity of instructional representations of fractions, decimals, and percents. The quality of representations was indicated by whether the instructional representations were accurate, comprehensible and connected. The quantity of instructional representations is investigated as to the extent to which the teachers used symbolic representations, manipulatives, pictures and real world experiences in their instruction. The influence of both quality and quantity of instructional representations on student's written representations and the effect of different forms of written representations on students' achievement on fractions, decimals and percents were explored.

Theoretical Framework

There are two components in the theoretical framework. One aspect involved different forms of representation and the other addressed the sub-constructs of fractions. Discovery learning that aimed to provide experiences for students to explore and investigate knowledge was proposed by Bruner in 1960. During the learning process, understanding of a concept was developed based on their previous knowledge and understanding (Bruner, 1960). Later in 1966, Bruner also developed a three-stage model of representations; enactive, iconic and symbolic. Through discovery learning, students make connections between enactive representations, iconic representations, and symbolic representations. It was proposed that students could then establish a symbolic representation by reconceptualizing previous knowledge (Bruner, 1966). Lesh (1979) elaborated upon Bruner's (1966) three-stage model by proposing a five-stage model including two more categories: real world representations and spoken symbolic representations.

The sub-constructs of rational numbers have been studied since the 1970s by Kieren, Freudenthal, Harel and Behr, and Lamon. Kieren (1976) first proposed six sub-constructs of the rational numbers: fractions, decimal fractions, equivalence classes of fractions, numbers of $\frac{p}{q}$, multiplicative operators, and discrete relationship. Later in the 1980s, a four sub-construct model emerged: measure relationship, part-whole relationship, discrete relationship (part of different wholes, improper fractions), and operation relationship (Freudenthal, 1983). Different from Freudenthal's four sub-construct model, Kieren (1988) used quotient and ratio to replace part-whole and discrete relationship in Freudenthal's model, which resulted in another four sub-construct model: measure, quotient, ratio and multiplicative operator. Harel and Behr (1990) discussed three sub-constructs: part-whole, quotient and operator. Most recently, Lamon (2001) claimed another five sub-construct model: part-whole, ratio and rates, operator, measure and quotient. Procedural knowledge of multiplicative operators such as multiplication by one and cross product were used to find equivalent fractions (AAAS, 2002). In this dissertation, because of the specific content area covered in the sixth grade mathematics textbooks, the three sub-constructs (viz., part-whole, measure and quotient) as well as the procedural knowledge (e.g., multiplication by one and cross product) were coded and analyzed. This study used Lesh's model (1979) in terms of different forms of representations, i.e., real world, manipulatives, pictures, spoken symbolic representations and written symbolic representations.

Research Questions

This study focused on how instructional representations were used in classroom instruction and their further influence on students' comprehension. The following questions were investigated in this study:

1. What are the nature and quality of real world, manipulatives, pictures, spoken symbolic and written symbolic representations in teaching fractions, decimals and percents? Specifically, what is the nature of classroom interactions and instructional time in the use of representations, and how are the instructional representations aligned with the textbooks?

2. What are the structures of instructional representations and students' written representations? How do these structures reflect the format sub-constructs: real world, manipulatives, pictures, spoken symbolic and written symbolic representations? How do they reflect the content sub-constructs: measure, part-whole, quotient, wonderful one, and cross product?
3. What is the relationship between the structures of representations students use and their achievement? That is, how do students use representations that reflect measure, part-whole, quotient, multiplication by one and cross product and how is this reflected in their achievements on learning fractions, decimals, and percents?
4. What is the relationship between the quality and quantity of instructional representations and student's learning of fractions, decimals, and percents? Specifically, what is the relationship between the quality of teachers' instructional representations, different forms of students' written representations, and student achievements?

Definitions of Key Terms

The following operational definitions are defined as they are used in this study:

Cross product is a procedure commonly used to confirm or find equivalent fractions. It involves finding the product of the first fraction's numerator and the second fraction's denominator, setting it equal to the product of the first fraction's denominator and the second fraction's numerator.

External representations refer to “Normative natural languages (e.g. “standard” English) include concrete manipulative materials or computer-based microworlds; and sociocultural structures, such as those of kinship, economic relationships, political hierarchies, or school systems” (Goldin, 2003, p. 277).

Internal representation is “the knowledge and structure in memory, as propositions, productions, schemas, neural networks, or other forms.” (Zhang, 1997, p. 180).

Instructional representations are the external representations that teachers use when delivering mathematical knowledge in classroom settings.

Manipulatives are concrete objects used to model corresponding mathematics ideas by providing hands-on experiences (Hynes, 1986). In this dissertation, the manipulatives include pattern blocks and fraction strips.

Measure is a content sub-construct of fraction. Fractions can be interpreted as a unit to measure something, for example, length, or area.

Multiplication by one (also referred to as Wonderful One) is a procedure commonly used to find equivalent fractions and to simplify fractions, multiplying or dividing by a fraction that has the same numerator and denominator.

Part-whole is a content sub-construct of fraction used in teaching fractions, for example, if a whole is equally divided into four parts, and three parts of the whole represent a fraction of three fourths.

Quotient is a content sub-construct of fraction. Fraction can be explained as a form of a numerator (A) divided by a denominator (B), where both A and B are whole numbers and B could be any number but zero.

Understanding "is a mental model comprises the representation that is currently active about a specific problem or concept. Students' conceptual understanding is domain specific. It is dynamic rather than static. Reorganization is an important component of understanding" (Bickerton, 2000, p. 12).

Whole means the concept of the unit "one" in teaching fractions.

Limitations of the Study

This study was limited to the experiences and practices of the volunteer teachers and students in selected samples in Texas. Therefore, the study may not be general to the whole nation except the area where the samples were selected. Another limitation is associated with the Structural Equation Modeling (Kline , 2005) analysis. The teacher sample size is small given the parameters' number in current study. The student sample size matches the sample size requirements proposed by Bentler and Chou (1987). They suggested that every parameter estimate should have at least five cases' estimation. However, larger sample size is always a preference in SEM analysis.

CHAPTER II

REVIEW OF THE LITERATURE

The literature review focuses on the following seven components. The first component dealt with research on conceptual understanding in teaching and learning of fractions, decimals and percents, for example, the sub-construct of the rational numbers and the possible misconceptions. The second aspect focused on the external representations and internal representations. The third component concentrated on role of different forms of representations in developing the conceptual understanding of fractions, decimals and percents. The fourth part addressed the relationship between different forms of representations and achievements. The fifth section emphasized on the role of the textbooks in facilitating learning fractions, decimals and percents. The sixth component discussed about the relationship between the quality and quantity of instruction and students' achievements. Finally, the last part engaged in the theoretical foundations and rationale of the dissertation.

Conceptual Understanding of Fractions, Decimals and Percents

Either conceptual understanding or procedural understanding, or both occur during the learning process. Research has indicated that the connection between conceptual understanding and procedural understanding is critical in knowledge acquisition (Hiebert, 1984; Hiebert & Lefevre, 1986; Owens & Menon, 1991). Thus both conceptual understanding and procedural understanding should be addressed appropriately in classroom instruction. Compared to experienced teachers, novice teachers tend to concentrate on the procedural understanding (Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992).

The conceptual knowledge of rational numbers has been studied for almost three decades. Three different views on the sub-constructs are stated in the literature: six sub-constructs, five sub-constructs and four sub-constructs. Kieren (1976) first proposed the six sub-constructs' theory that defined the rational number concepts into the following six parts: fractions, decimal fractions, equivalent fractions, quotient form as $\frac{p}{q}$, multiplicative operators, and discrete relationships. The five sub-constructs theory was proposed by Lamon (2001), consisting of part-whole, ratio and rates, operator, measure, and quotient. Both Freudenthal (1983) and Kieren (1988) contended four sub-constructs of rational number concepts even though the content was different. Freudenthal (1983) contrived the four sub-constructs to be measure, part-whole, discrete relationship, and operation. Alternatively, Kieren (1988) considered the four sub-constructs to be measure, quotient, ratio, and multiplicative operators. In summation, three researchers, Freudenthal (1983), Kieren (1988) and Lamon (2001), included the measure sub-construct. Two researchers, Lamon (2001) and Freudenthal (1983), included the part-whole sub-construct. Kieren (1976, 1988) mentioned the quotient sub-construct.

All of the researchers include the operators in the sub-constructs. In particular, Kieren (1976, 1988) contended that multiplicative operators exist. Both Freudenthal (1983) and Lamon (2001) proposed the operator relationship. Moreover, multiplicative operators for example, multiplication by one and cross product were used to find equivalent fractions (AAAS, 2002; Van de Walle, 2001). In this dissertation, because of the specific content area covered in the sixth grade mathematics textbooks, the three sub-constructs (viz., part-whole, measure and quotient) as well as the multiplicative relationship (e.g., multiplication by one and cross product) were coded and analyzed. These four sub-constructs are described individually in the following section.

Part-whole relationship is a fundamental sub-construct, upon which the other concepts are developed (Post, Behr, & Lesh, 1982). However, it is one of the most difficult concepts in learning fractions (Behr, Harel, Post, & Lesh, 1994; Behr & Post, 1992; Hiebert & Hiebert, 1983). In order to develop the part-whole concept, different models, such as geometric regions including length, area, volume and set models, are used to introduce the partition (Behr & Post, 1992). The concept of equal partition is critical. Post and Cramer (1987) reported that students tend to think $\frac{1}{3}$ is bigger than $\frac{1}{2}$ because the same whole was partitioned into more pieces in $\frac{1}{3}$ than $\frac{1}{2}$. Only a few children are able to understand equal partitions (Lesh, Post, & Behr, 1987). The literature has reported that misunderstanding of the part-whole relationship and units will cause a sequence of problems in later conceptual development, such as understanding addition of fractions, ordering of fractions, and finding equivalent fractions (Behr, Harel, Post, & Lesh, 1994; Behr & Post, 1992; Post, Behr, & Lesh, 1982). For example, students may not be able to realize that the whole should be equally partitioned, which may cause a further mistake that adds numerators and denominators together. Brase

(2002) and Post (1981) also reported that students believed a wrong algorithm that $\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$ due to the failure to recognize the part-whole relationship.

The literature suggested that rational numbers can also be interpreted as a measure sub-construct (Freudenthal, 1983; Kieren, 1988; Lamon, 2001). It is reported that the measure sub-construct is more difficult than the part-whole sub-construct (Gay & Aichele, 1997; Ni, 2000). Unit of measure is a critical aspect (Harel & Behr, 1988; Steffe, Cobb & Von Glasersfeld, 1988). Post, Cramer, Behr, Lesh and Harel (1993) mentioned "the flexible concept of unit", which referred to both continuous and discrete objects can be used in partition. Steffe, Cobb, and Von Glasersfeld (1988) advocated the critical relationship between units and conceptual understanding of number concepts. Lamon (2001) further suggested that the combination of part-whole and measure sub-construct has great potential in developing conceptual understanding. The number line is as the most difficult yet important model in terms of the measure sub-construct (Behr & Post, 1992). It is reported that children easily get confused with the unit (Brown, Carpenter, Kouba, Lindquist, Silver, & Swafford, 1988). A number line with a length of other than one usually causes more confusion than number line with a length of one (Bay, 2001; Ni, 2000).

Additionally, a fraction $\frac{a}{b}$ can also be interpreted as a quotient according to Kieren (1976, 1988). However, students often do not recognize that the form indicates division (Behr & Post, 1992; Siegal & Smith, 1997). They usually regard the numerator as a number and the denominator as another number (Cramer, Behr, Post, & Lesh, 1997;

Pitkethly & Hunting, 1996). This may lead to common misconceptions, for example, "multiplication always makes larger", and "division always makes smaller", (Graeber, 1993, p. 408). Not only students, but also middle school teachers tended to have similar misconceptions (Bell, Fischbein, & Greer, 1984; Fischbein, Deri, Sainatinelle, & Marino, 1985 as cited in Lacampagne, Post, Harel, & Behr, 1988). Therefore, developing meanings for the numerator and denominator is important to learn later concepts such as adding and subtracting fractions relying on this understanding (NCTM, 2000). Sometimes language could complicate the confusion to develop the correct meaning of numerators and denominators. For example, "more" and "greater" are ambiguous according to Post and Cramer (1987), because "more" can be interpreted in two ways: a bigger numerator (more pieces of units) or a bigger denominator (more partitions). If the teacher does not address the difference, some students would think $\frac{1}{5}$ is smaller than $\frac{1}{6}$ because $\frac{1}{6}$ has a bigger denominator than $\frac{1}{5}$. The misunderstanding about numerator and denominator could be carried on to percent learning. Moss and Case (1999) pointed out that middle school students did not understand that the concept of percents was related to quotient. For example, when asked to find 65% of 160, some students obtain a wrong answer by subtracting 65 from 160.

Cross product and multiplication by one are two procedure skills that are used to find equivalent fractions (AAAS, 2002; Van de Walle, 2001). There has not been very much research done in this specific area, however, teachers and textbooks used these concepts to teach equivalent fractions.

External Representations and Internal Representations

"Representation" is a term used in mathematics education, and it is classified into two types: external and internal representations (Goldin, 2003; Zhang, 1997). Zhang (1997) defined external representation as "the knowledge and structure in the environment, as physical symbols, objects, or dimensions (e.g. written symbols, beans of abacuses, dimensions of a graph, etc.), and as external rules, constraints, or relations embedded in physical configurations (e.g. spatial relations of written digits, visual and spatial layouts of diagrams, physical constraints in abacuses, etc.)" (p. 180). Internal representations are "the knowledge and structure in memory, as propositions, productions, schemas, neural networks, or other forms." (Zhang, 1997, p. 180). Both internal and external representations play important roles in facilitating mathematics learning. Internal representations and external representations can be transformed to each other. Internalization refers to the process that transforms the external representations into internal representations. The opposite process from external representations to internal representations is called externalization. Zhang (1997) also summarized that there are three different views about the relationship between the external and internal representations: (1) External representations are dominant; (2) internal representations are dominant; (3) they are interrelated. The first idea views the external representations as more important, because if no mental processes are required for perception and action, then there are no internal representations involved (Zhang, 1997). However, the second idea views internal representations as more important than external representations, because the information has to be translated into an internal model in order to be understood (Newell, 1990 as cited by Zhang, 1997). The third idea views both the external representations and the internal representations as necessary aspects when solving a distributed cognitive task. It advocates that people store information as internal representations, and external representations could stimulate internal representations if cues are provided (Zhang, 1997). The mathematics education community tends to agree with the third view that both external and internal representations are dependent on each other, and both contribute to the conceptual understanding in mathematics knowledge acquisition (Goldin & Steingold, 2001; Hiebert & Carpenter, 1992; Voutsina & Jones, 2001).

According to the definition of the external representations, instructional representations are defined as a form of external representations in this dissertation. It refers to both the delivery of content knowledge and the interaction between the teachers and students in the class, e.g., classroom discourse between teachers and students. The ability of developing meaningful internal representations of a certain concept is a measure of conceptual understanding (Gobert & Clement, 1999). However, it is difficult to measure students' internal representations (Goldin & Steingold, 2001). Therefore, external representations usually serve as an indicator of students' internal representations.

The Role of Different Forms of External Representations in Developing Conceptual Understanding of Fractions, Decimals and Percents

There were different classifications of external representations in the literature. Bruner classified external representations into three types: enactive, iconic and symbolic representations (Bruner, 1966). Based on Bruner's categorization, Lesh (1979) proposed another model including the following five forms of representations: real world, pictures, manipulative, spoken symbols and written symbols. A third classification was contended by Lesh, Hamilton, and Landau (1981) that physical aids, verbal, pictorial, and symbolic representations are five elements forming the representational system. However, physical aids belonged to the manpulatives representations, if they were static then they belonged to the pictorial representations, which were also included in Lesh (1979)'s model. Zhang (1997) proposed that external representations include diagrams, graphs and pictures. However, these three types all belonged to pictorial representations. In this dissertation, Lesh's model (1979) was used to distinguish different forms of representations.

Different forms of representations, such as real world, manipulatives, pictures, spoken symbols and written symbols, contributed differently to conceptual understanding. Applying real world representations could motivate learning and make learning meaningful (NCTM, 2000); it also served as an intuitive foundation on which later learning could be built (Kieren, 1992; Mack, 1990; Saenz-Ludlow, 1993, 1994). Pictorial representations could be used to convey meaning (Monk, 2003) or to simplify the information processing (Stenning & Oberlander, 1995), and served as a tool to inspire connections between different concepts (Chambers & Reisberg, 1985; Higginbotham-Wheat, 1991). Manipulatives were also reported as benefiting students' learning by providing hands-on experiences to make symbolic representations more concrete (Cramer & Henry, 2002; Stix, 1997). Language was a critical factor in instructional representations. Miura (2001) pointed out some particular advantages embedded in a certain language that facilitated conceptual understanding. For example, the part-whole relationship was reflected by Japanese language, which possibly generated connections between the symbolic representations and pictorial representations. Written symbolic representations were also reported to be one important factor (Bloomfield, 1933; Donald, 1991; Lampert, 2003). Written symbolic representations stimulated reflective thinking (Lampert, 2003; Norman, 1993), which served as the bases of "logical, analytic, rational, and scientific" thoughts (Goody, 1977; Ong, 1982 as cited by Zhang, 1997, p. 183); therefore, it served not only as a result but also as a process of thinking.

Moreover, different forms of external representations contributed differently in problem solving and decision processing by highlighting some attributes over the others (Kleinmuntz & Schkade, 1993; Zhang, 1997). Mathematics concept, for example, rational numbers, usually involves at least four sub-constructs. A single representation could not address all of these sub-constructs substantially, and therefore multiple representations were necessary.

Among the representations that were most commonly used, area models (pie graphs, pattern blocks, and fraction strips), set and number lines were used to demonstrate the part-whole sub-construct. Instead, number lines and fraction strips were used to demonstrate the measure sub-construct. Pie graphs are criticized for their limitations of demonstrating fractions (Kerslake, 1986; Kieren, 1995; Mack, 1990; Nunes & Bryant, 1996; Ohlsson, 1988 as cited in Moss & Case, 1999). Number lines were reported as a most difficult form of representations (Ni, 2000; Vance, 1992).

Comparing manipulatives and pictorial representations, symbolic representations were reported as more difficult (Gay & Aichele, 1997). Orton, Post, Behr, Cramer, Harel, and Lesh (1995) reported three characteristics of students' thinking in terms of representations. The first was the translation between different forms of representations, e.g., between real world, manipulatives, pictures, and symbolic representations. The second characteristic was to translate within the same form of representations. For example, four chips out of six chips is the same representation as two chips out of three chips. The third characteristic was to use the symbolic representations without relying on either the manipulatives or picture representations. Post, Wachsmuth, Lesh, and Behr (1985) also proposed that in order to use symbolic representations without manipulatives and picture representations, students should be able to translate between manipulatives and symbolic (or picture and symbolic) representations easily. According to Lesh, Behr, and Post (1987), the translation from pictures to symbols is most difficult among the seven forms of representations they compared: "(a) symbols to written language, (b) written language to symbols, (c) pictures to pictures, (d) written language to pictures, (e) pictures to written language, (f) symbols to pictures, (g) pictures to symbols."(p. 48) They further stated that "(a) Translations to pictures is easier than translations from pictures; (b) translations involving written language (e.g., three fourths) are easier than translations involving written symbols (e.g., $\frac{3}{4}$); and (c) the easiest translations are those that only require a student to 'read' a fraction or ratio in two different written forms" (p. 48). Lesh, Post, and Behr (1987) also pointed out that spoken symbolic representations can serve as a mediator to bridge the difficulty encountered by the translation from real world representations to symbolic representations.

Most common misconceptions are related to the difficulty of translation between the manipulatives/picture representations and symbolic representations (Bay, 2001; Cramer, Behr, Post, & Lesh, 1997; Hiebert, 1985; Post, 1981; Wearne & Hiebert, 1986), e.g., the difficulty of realizing the equivalence between symbolic representation $\frac{3}{12}$ and $\frac{1}{4}$ from a pie graph (Cramer, Behr, Post, & Lesh, 1997). For example, four percent of the seventh grade students in their study knew the symbolic representation of a shaded section (Hiebert, 1985) and they were not able to generate the picture representation given the symbolic representation, e.g., to locate 0.3 on a number line (Bay, 2001), relate certain fraction to a number line (Post, 1981), or use hundredth grid to represent 0.4 (Wearne & Hiebert, 1986). The translation within the symbolic representations is also reported as problematic (Condon & Hilton, 1999; Hiebert, 1985; Thompson & Walker, 1996; Vance, 1992; Wearne & Hiebert, 1986). Translations between decimals and fractions are not correctly done (Hiebert, 1985; Wearne & Hiebert, 1986). Post (1981) reported that only half of the 9-year old students can make connections between the spoken symbolic representations and written symbolic representations.

According to the literature, there are several reasons for the above common misconceptions. The first one is that rational number is one of the most difficult topics in middle school mathematics (Millsaps & Reed, 1998). Another reason is that students do not realize the difference between rational number system and whole number system, they still apply rules from whole number system to learn rational numbers (Ball, 1993; Hiebert & Hiebert, 1983; Pitkethly & Hunting, 1996; Streefland, 1991). The third reason is that teachers do not have enough mathematics content knowledge and pedagogical content knowledge to facilitate students' conceptual understanding of rational numbers (Lacampagne, Post, Harel, & Behr, 1988; Titus, 1995). Moreover, if teachers focused on symbolic representations without developing conceptual understanding, it will generate difficulty in later learning. Because of a lack of the understanding of certain concepts, students tend to generate "buggy" algorithms, algorithms that work with whole numbers but do not always work with rational numbers (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Oliver, & Human, 1997).

Relationship between Different Forms of Representation and Achievements

Researchers agree that there were strong connections between students' understanding and the representations they use (Diezmann & English, 2001; Kaput, 1987; Friedlander & Tabach, 2001; Lamon, 2001). Post, Behr, and Lesh (1982) suggested that addressing the translations and connections between different representations as well as demonstrating the concepts from different perspectives contributed to conceptual developments. Some research contended the critical role of informal knowledge in developing fraction concepts, because the pre-experiences gained by students from real world experiences influence their later study (Kieren, 1992; Mack, 1990; Saenz-Ludlow, 1993, 1994). Saenz-Ludlow (1994) further stated that fraction concepts can be developed without symbolic representations. Dienes (1967 as cited in Post, Behr, & Lesh, 1982) stated that a concept is better developed through multiple representations and multiple sub-conceptual perspectives. However, some studies revealed that multiple representations could increase the cognitive load, which might be less effective (Chandler & Sweller, 1992; Kalyuga, Chandler, & Sweller, 1998). Bibby and Payne (1993) proposed a third idea that the effectiveness of representations depended on the content materials (Bibby & Payne, 1993). Thus the roles played by different representations in the conceptual developments of fractions, decimals and percents remained obscure and needed further investigation (Goldin, 2003).

The Role of the Textbooks in Facilitating Learning Fractions, Decimals and Percents

Standard-based textbooks with high quality enhanced students' achievement (Kulm & Capraro, 2004; Reys, Reys, Lapan, Holliday, & Wasman, 2003). Trafton, Reys, and Wasman (2001) proposed that being comprehensive, coherent, developing idea in depth, promoting sense making, engaging students, and motivating learning are six characteristics that standard-based curriculum (e.g. textbooks) features. Textbooks influence students' learning both directly and indirectly through teachers' providing mathematics content knowledge and teaching strategies (Kulm & Capraro, 2004; Reys et al., 2003). DeBoer, Morris, Roseman, Wilson, Capraro, Capraro, Kulm, Willson, and Manon (2004) proposed a linear relationship between the following four aspects: professional development together with the curriculum materials, teacher knowledge, skills and attitude, teaching behavior, and the students' learning. A study conducted by Project 2061 examined the quality of 13 textbooks based on a total of 24 criteria classified into seven categories: identifying a sense of purpose, building on student ideas about mathematics, engaging students in mathematics, developing mathematical ideas, promoting student thinking about mathematics, assessing student progress in mathematics, and enhancing the mathematics learning goal (AAAS, 2000). Three textbooks that were used in this dissertation ranked as high, medium and low according to AAAS (2000). These three textbooks included Connected Mathematics Projects (CMP) (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998), Middle School Math Thematics (Billstein, Lowery, Montoya, Williams, & Williamson, 1999) and Mathematics: Applications and connections (Collins, Howard, Dristas, McClain, Frey, Molina, Moore-Harris, Price, Ott, Smith, Pelfrey, & Wilson, 1999). Connected Mathematics was ranked as satisfactory with a score range from 2.0 to 3.0 on the corresponding criteria. Middle Grade Math Mathematics was ranked as partial satisfactory with a score ranged from 1.3 to 3.0 on the corresponding criteria, while Mathematics: Applications and Connections was graded as unsatisfactory with a score range from 0.3 to 2.6 on the corresponding criteria. Empirical findings by Kulm and Capraro (2004) reported that despite the variation of enacted curriculum delivered by the teachers, students' achievements were related to the rankings of the textbooks rated by AAAS. Spielman and Lloyd (2004) indicated that reform-oriented curriculum "may help teachers shift their sense of efficacy from teaching as telling to more effective instructional practices (Smith, 1996)" (p. 40).

Impact of Instructional Time and Teaching Quality on Students' Achievement

Teachers have been reported to be a critical factor influencing the students' achievement (Carpenter & Fennema, 1991; DeBoer et al., 2004; Fennema & Franke, 1992; Wright, Horn, & Sanders, 1997). However, a large proportion of teachers were reported as not having enough mathematics content knowledge (Ball, 1988; Post, Cramer, Behr, Lesh, & Harel, 1993; Post et al., 1988) or pedagogical content knowledge to teach effectively (An, 2000). Simon (1997) claimed that the interactions between teachers and students are a factor of teaching for understanding. Teacher-centered instruction and student-centered instruction are two models of classroom interaction. Teacher-centered instruction sometimes is referred to as traditional while student-centered instruction is referred to as constructivist.

Black (2002) categorized teaching quantity into three different categories: allocated time, engaged instructional time, and academic learning time. Meanwhile, Project 2061 indicated that the accuracy, comprehensibility and variety of representations reflect whether the textbooks are supportive in developing mathematics ideas, more specifically, representing ideas effectively (AAAS, 2000). Moreover, connections between different representations have also been identified as a key issue in developing mathematics ideas (Post, Behr, & Lesh, 1982). Because it has been repeatedly reported that the criteria used to rate the textbooks could be used to grade the quality of instruction (DeBoer, et. al. ,2004; Kulm & Capraro, 2004; Sun & Kulm 2003), these criteria and indicators are used as a measure of the teaching quality in the dissertation.

Three different opinions about the relationship between the quantity of instructional time and students' achievements have been proposed in the literature. The first one claims that there is a positive relationship between the total time engaged in instruction and students’ achievements (Black, 2002; Walker, 1976). However, some other researchers held a different opinion that quality played a more important role compared to the instructional time (Aronson, Zimmerman, & Carlos, 1998; Smith, 2000). NCTM also claimed the critical role of the quality by stating that “This is true for all students, including those with special educational needs. Many children with learning disabilities can learn when they receive high-quality, conceptually oriented instruction” (NCTM, 2000, p. 87). DeBoer et. al. (2004) proposed a third view that both teaching time and teaching quality contribute to students’ achievements. Thus in this study, both quality and quantity of instructional representations were investigated in order to investigate the relationships between the teacher instruction and students achievements.

Theoretical Foundations and Rationale

Both Bruner and Lesh contributed to representation research from different perspectives. Bruner proposed a theory of discovery learning in mathematics education in which activities were constructed for exploration and investigation (Bruner, 1960). He also developed a three-stage mode of representations: enactive, iconic and symbolic. Through discovering the matches between enactive real world and iconic representations, learners establish a symbolic representation by reorganizing previous knowledge in order to come to a better understanding (Bruner, 1966).

Lesh (1979) proposed a five-stage model based on Bruner’s (1966) three-stage model by adding two more modes: spoken symbolic representations and real-world representations. In his model, manipulative representations corresponded to enactive representations; iconic representations were connected to pictorial representations; and written symbolic representations corresponded to symbolic representations. Based on Lesh's model, the Rational Number Project conducted a series of research studies on these different forms of representations (Behr, Post, Lesh, 1981; Cramer, 2003; Post, 1988; Post, Behr, Lesh; 1982; Behr, Lesh, Post, & Silver, 1983). Lesh's model of representations was adopted and is depicted in Figure 1.

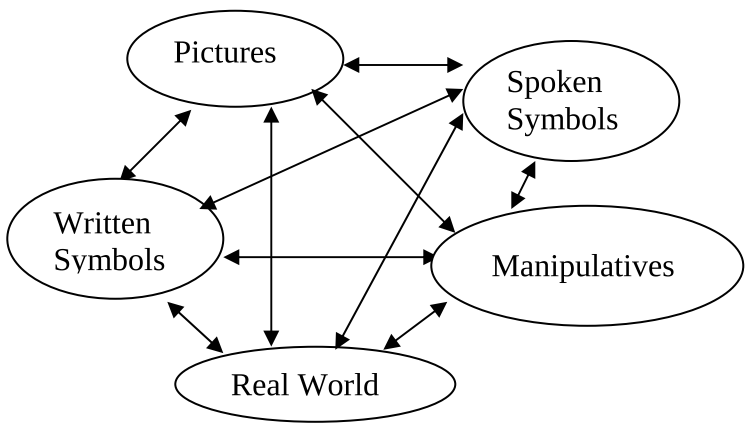

Figure 1. Lesh's model of different forms of representations (Lesh, 1979).

Table 1
Measurement of Instructional Representations of Fractions, Decimals and Percents

Types of representation	Part-whole	Measure	Quotient	Wonderful one	Cross Product	Others
Real world						
Manipulatives						
Pictures						
Spoken symbols						
Written symbols						

The literature of rational numbers revealed the three sub-constructs: part-whole, measure, and quotient (Freudenthal, 1983; Kieren, 1976; 1988; Lamon, 2001). In addition, multiplication by one and cross product are procedural knowledge taught in the middle grade curriculum materials. Therefore, the measurement of instructional representations of fractions, decimals and percents was developed (See Table 1).

CHAPTER III

METHODOLOGY

This study was designed to investigate how instructional representations influence students' written representations. The data were collected from 14 sixth grade teachers in three different school districts in Texas, who participated in the MSMP project during the 2002-2003 school year. Both qualitative and quantitative data were collected and analyzed. Videotapes of three to five lessons for each teacher were obtained. The teachers used three different textbooks: Connected Mathematics (CMP) (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998), Middle School Math Thematics (Billstein, Lowery, Montoya, Williams, & Williamson, 1999) and Mathematics: Applications and Connections (Collins, Howard, Dristas, McClain, Frey, Molina, Moore-Harris, Price, Ott, Smith, Pelfrey, & Wilson, 1999). The rankings of the quality of instruction for the textbooks, including the use of representations, were rated as high, medium, and low according to an analysis by Project 2061 of the American Association for the Advancement of Science (AAAS, 2000).

An analysis of teachers' usage of real world, manipulatives, pictures, written symbolic and spoken symbolic representations of fractions in classrooms was conducted in order to examine the teachers' classroom instruction and students' pre and posttests. Within these representations, teachers' use of strategies for representing the meaning of equivalent fractions and translations between fractions, decimals and percents were observed and coded. The participants, procedures, instrumentation, and data analysis are discussed in detail in this chapter.

Participants

The participants in this dissertation consist of 14 sixth grade teachers and their students. The teachers include four male and ten female employed at five different public schools who were participating in a larger study of the effects of professional developments on the teaching for understanding. For this study, purposeful sampling was applied to select teachers who had volunteered to be part of the project. Specifically, since the study focuses on the content of fractions, decimals and percents, the 14 sixth-grade teachers who taught lessons on this content were selected. Table 2 presents a summary of the demographic data of the teachers. Each teacher was assigned a number from one to fourteen based on their school districts.

Table 2
Teachers' Demographic Data

Years	Min	Max	Mean	Std. Deviation
T. E.	1	23	10.85	6.05
C. T. E.	1	18	5.94	5.20

Note: T. E. = teaching experience
C.T.E. = teaching experience in the middle grades, 6 – 8.

The student participants included one class from each of the 14 teachers. They were White, Hispanic, African American and Asian, ranging from 12 to 14 years old. The academic backgrounds and ethnic distribution of each of 14 classes' students were found to be similar to each other. Table 3 indicates the social economic background, ethnicity, and gender characteristics of the students.

Table 3
Students' Participants Demographic Data in the Analysis

		Frequency	Percent	Valid percent
Gender	M	109	51.2	51.2
	F	104	48.8	48.8
Ethnicity	Asian or PI	6	2.8	3.9
	A.A.	13	6.1	8.4
	Hispanic	29	13.6	18.8
	White	106	49.8	68.8
Economically Disadvantaged	Free Meal	27	12.7	17.5
	Reduced Meal	5	2.3	3.2
	Other Disad.	6	2.8	3.9
	No Disad.	116	54.5	75.3
Title -I- Part	N	154	72.3	100
Migrant	Y	1	.5	.6
	N	153	71.8	99.4
Limited English Proficient	Y	3	1.4	1.9
	N	151	70.9	98.1
Bilingual	N	154	72.3	100
English as Second Language	Y	3	1.4	1.9
	N	151	70.9	98.1
Special Education	Y	3	1.4	1.9
	N	151	70.9	98.1
Gifted & Talented	Y	36	16.9	23.4
	N	118	55.4	76.6
At Risk	Y	18	8.5	11.7
	N	136	63.8	88.3

Note: PI=Pacific Islander
AA=African American
Title -1- Part = student does not currently participant in and has not previously participated in program at current campus

Instruments

There were four instruments used for data collection in this study. The pre and posttests were developed by AAAS to measure students' performance on number skills and concepts. Based on Lesh's (1979) model and rational number concepts the second instrument was developed to collect the amount of time each individual teacher spent on classroom instruction using a particular form and strategy of representations. The third instrument assessed the quality of teachers' instruction in using representations. The fourth instrument was adapted from the second instrument to collect data from students' posttests related to written representations.

Number Test

The number tests (pre and posttests) were developed by AAAS to assess sixth grade students' knowledge and understanding of the learning goal: "Use, interpret, and compare numbers in several equivalent forms such as integers, fractions and decimals" (AAAS, 1993). The number test is designed to assess three dimensions of number constructs: multiple meanings and models of fraction, converting forms, and comparing and ordering. Both the pre and posttests consisted of the same 16 test items including nine multiple choices, six short constructed responses, and one extended response SOLO-type (structure of learning outcome) item composed of four parts (Wilson, 1990). For the current research on representations, only three items (items 14, 15 and 16e)[1] were coded and analyzed in order to study students' uses of representations because the student could choose the form of representations they preferred in these three items.

[1] There are totally five sub-questions (16a-e) for Question 16 and only 16e is relative to the current research.

Quantity Measurement of Instructional Representations

A coding system for recording the teachers' uses of representations in the classroom was developed based on Lesh's model (1979) and the content sub-constructs of fraction concepts (Kieren, 1976; 1988; Lamon, 2001). The first dimension of the coding system adapted the content sub-constructs of fractions. In particular, it included part-whole, measure, quotient, multiplication by one, and cross product according to the content covered in the videotaped lessons. An additional column others were added for the cases where the instructional representations were not related to any previous categories. The second dimension used Lesh's (1979) five-stage model of representations, i.e., real world, manipulatives, pictures, spoken symbols and written symbols. An additional format mode "calculators" were added to complement the written symbolic representations because some teachers used calculators in classroom instruction. Table 4 presents the instrument in its final adapted form.

Table 4

Instrument for Engaged Instructional Time on Representations for Teacher Measurement

Types of representation	Part-whole	Measure	Quotient	Wonderful One	Cross Product	Others
Real world						
Manipulatives						
Pictures						
Spoken symbols						
Written Symbols / Calculators						

Table 5

Instrument for Coding Quality of Teachers' Instruction on Representation

Criterion	Indicators
1. Accurately depicts the intended mathematics learning goal	No misconceptions
	Point out the limitations of the representation
2. Comprehensible to students	Meaningful to students
	Includes real world experiences
	Hands-on activity
3. Connections are made between the representation and what is being represented	Justify why it is being represented in a particular way
	Connections are made between a variety of representations

This instrument was used to provide specific information about the amount of time each teacher spent on each content sub-construct and format sub-construct. More specifically, in each cell of the table, the amount of time spent on that content using a particular form of representation was recorded for each instance of use in the classroom. For example, if the teacher used real world experience to explain a part-whole relationship, the amount of time was recorded in the cell of (real world, part-whole), i.e., the cell belonging to the first line and first column. Similarly, if the teacher used a manipulative to explain fractions using the part-whole idea, the amount of time for that instance was recorded in the cell of (manipulative, part-whole). If the teacher talked while she/he was doing this, the amount of time was also recorded in the cell of (spoken symbols, part-whole). It was common for the teachers to talk and write symbols at the same time. In these cases, the time was recorded both for spoken symbols and written symbols. For example, if the teacher wrote on the board explaining quotients while she/he was speaking, both of the cells of (spoken symbols, quotient) and (written symbols, quotient) were assigned the amount of time of this instance. If the teacher didn't address anything corresponding to a cell at anytime during the class period, that cell was assigned a zero.

Quality Measurement of Instructional Representations

A coding system for rating the quality of instruction in using representations was adapted from AAAS's criteria for evaluating instructional materials (AAAS, 2000). AAAS specifies the following three representation criteria: accuracy, comprehensibility and variety of representations. However, the quality of instruction is distinguished not only by the variety of the instructional representations, but also by the connections between different forms of representations. For this reason, the third criterion as "connection" rather than the "variety" specified by AAAS was introduced. Thus this instrument consists of three criteria: accuracy, comprehensibility and connection (See Table 5).

Each criterion further consisted of several indicators. According to AAAS (2000), no misconceptions and pointing out the limitations of the representation are two indicators reflecting accuracy criterion. For comprehensibility, AAAS (2000) only stated that the comprehensible criterion depends on the students' grade level and the content. Three indicators were proposed for the comprehensible criterion in this instrument: meaningful to students, including real world experiences, and hands-on activities. More specifically, to be meaningful to students, the instructional language should be close to the students' level of understanding, using only necessary and developed mathematical vocabularies (AAAS, 2000; Sun & Kulm, 2003). Real world experiences referred to the content were developed based on students' prior knowledge in a real world context. When students were engaged in the hands-on activity through which mathematics ideas were developed, they were likely to understand the concepts better. For the connection criterion, because connections between different forms of representations were critical to developing conceptual understanding (AAAS, 2000) and variety of representations were not observable in a pilot study (Sun & Kulm, 2003), the connection criterion was used instead of the variety criterion. And the following two indicators: justify why it is being represented in a particular way, and connections are made between a variety of representations were used as indicators of connection criterion.

Teaching quality was then evaluated using this instrument. In particular, teachers' instruction in each class was checked with each indicator. Each indicator was scored as Met (score 1) or Not Met (score 0) for the class. A teacher met a criterion requirement and thus received a score of 1 for that criterion if at least one of the indicators for the criterion was scored as 1. If all of the indicators of a criterion received a score of 0, the teacher did not meet the criterion requirement and received a 0 for that criterion. The three criterion scores were summed as a quality score for the teacher in that lesson. Thus each teacher received a quality core ranging from 0 to 3 for each lesson she/he taught.

Measurement of Students' Written Representation

The fourth instrument was adapted from the quantity measurement of instructional representations in order to code the representations students chose in their answers to the three items (14, 15 and 16e) in the posttest. This instrument used the same content sub-constructs as those used in the teacher instrument (the second instrument), i.e., part-whole, quotient, measure, multiply by one, cross product and others. However, since minipulatives and spoken symbols were not available in the posttest, a paper and pencil test, the format sub-constructs of this instrument were limited to real world, picture and written symbols. Table 6 depicts this instrument.

Table 6
Instrument for Students' Written Representations Measurement

Types	Part Whole	Measure	Quotient	Wonderful One	Cross Product	Others
Real World						
Picture						
Written Symbol						

Students' posttests were first checked by scorers trained by AAAS according to a rubric developed by the MSMP project and each student received a total score of the posttest. The instrument shown in Table 6 was used to code different forms of the representations students used for questions 14, 15 and 16e in the posttest. The instrument was applied to each question for every student, based on the representations they used. For example, if a student drew a picture to answer question 14 using the idea of part-whole, the cell of (picture, part-whole) is assigned a score of 1 for question 14. A cell was assigned a score of 0 if no corresponding representation form is used. In this study, since only three questions were considered, each student was coded with three copies of Table 6, one for each question.

Procedures

This study was a part of the Middle School Mathematics Project (MSMP) at Texas A&M University. Professional developments focusing on identifying the learning goal for the teacher participants were conducted in the summer of year 2001, followed by classroom visits and observations from fall 2001 to summer 2002 before actual videotaping in fall 2002. In summer 2002, a workshop was held for the teachers who were going to be videotaped in the fall in order to develop their understanding of the mathematics learning goal, "use, interpret, and compare numbers in several equivalent forms such as integers, fractions and decimals" (AAAS, 1993). The content-based workshop focused on the following three aspects: relevant literature on multiple representations, prerequisite knowledge as well as common misconceptions related to the learning goal. Teachers experienced the learning process by working in group activities to use multiple representations to solve real world problems.

In September 2002, the paper and pencil number test (pretest) was administered by each teacher during the regularly scheduled mathematics classes in all three participating school districts. During the school year, each teacher taught several lessons that addressed the target number benchmark on fractions, decimals, and percents. Three to five lessons that focused most directly on the benchmark were videotaped for each teacher. In addition, copies of handouts, transparencies, and materials used in the class were collected. Major variations from the intended textbook lessons (e.g. sequences, gaps, additions, students work and materials used) were noted, and major contextual or management issues (e.g. schedule, interruptions and other difficulties) were described.

At the end of the spring semester 2002, a posttest was administered by teachers during regular class time. Students were told that the test did not affect their scores at school.

Data Coding

A total of 58 teachers' videotapes on the topic of fractions, decimals and percents were coded and analyzed according to the instruments of how teachers use instructional representations in class (Table 4) as well as the quality of their teaching on such topics (Table 5). Students' written representations in answering the three questions in the posttest were also coded based on Table 6. The instrument shown in Table 4 was used to collect data for the teachers' engaged instructional time on representations. More specifically, whenever a teacher used a representation, the starting and ending time was coded with the corresponding content sub-construct and format sub-construct such as part-whole, quotient, multiplication by one, measure , cross product , and others . The category of others meant that if the engaged instructional time did not fall into any of the previous five sub-constructs, it belonged to the category of others . Since each school district has different time length for a lesson, the total time of a lesson was also recorded. For the students' data, analysis of the responses to the three questions were scored and coded into different forms of representations (based on Table 6) in order to distinguish between different levels of understanding as well as different types of representations, e.g., symbolic, picture and real world representations.

Two independent researchers watched and coded all fifty-eight videotapes as well as the 213 students' posttests used in this study based on the criteria and the indicators. When there were disagreements, the researchers watched the videotape together or read the posttest answers together, discussed it, and reached agreement on the same coding. Triangulation from observations, videotapes, and researcher notes ensure the dependability (Anfara, Brown, & Mangione, 2002), while a code-recode strategy was adopted to ensure the credibility (Lincoln & Guba, 1985).

Data Analysis

Both quantitative and qualitative analysis methods were used to analyze the video tapes and students' written responses to the test items. The following statistical procedures were used to analyze the data:

1. Descriptive statistics, such as frequencies, means and standard deviations, were used to summarize both the measure of teachers' teaching quality and quantity.
2. Factor analysis techniques were conducted using the teaching quantity data to investigate whether variables converge to the sub-constructs of rational number concepts in the study.
3. Structural equation modeling techniques were applied to investigate the relationships among the teachers' instructional representations and the relationships between students' representations and their understanding of the concepts of fractions, decimals, and percents in this study. Both measurement models and path models were conducted.
4. Two-way Analysis of Variances (ANOVA) was used to investigate the impact of differences between the forms of instructional representations and students' written representations.

CHAPTER IV

RESULTS

This chapter presents the data analysis and addresses the research questions of the study. A mix of quantitative and qualitative analysis was applied to investigate the nature and quality of instructional representations, the types of classroom interactions, and the alignment with the textbooks. Structural equation modeling (SEM) was the primary method employed to investigate the structure of the instructional representations, the relationship between the structures of the instructional representations and students' written representations, the relationship between the structures of representations students use and their achievements, and the relationships between the quality and quantity of the instructional representations and student's achievements of fractions, decimals, and percents. The four research questions are addressed separately below.

Research Question 1

What are the nature and quality of real world, manipulatives, pictures, spoken symbolic and written symbolic representations in teaching fractions, decimals and percents? Specifically, what is the nature of classroom interactions and instructional time in the use of representations, and how are the instructional representations aligned with the textbooks?

The qualitative analysis of the videotaped lessons produced three sets of data for each teacher: 1) whether or not the teacher addressed the mathematical learning goal including how her/his approach differed from the textbook approach, 2) the nature of the classroom interaction, with attention to teacher-centered (traditional) or student-centered (constructive) interactions, and 3) a description of how the lessons developed students' representations of fraction ideas, with attention to the use of different forms of representations.

Addressing the Mathematics Learning Goals

Teachers 1, 2, 3, 4 and 5 were from a suburban school district that had adopted Middle School *Math Thematics* (Billstein et al., 1999) as their textbook. Middle School *Math Thematics* was developed to address mathematics reform goals and was ranked as acceptable in its instructional quality by the AAAS analysis (AAAS, 2000). The textbook uses manipulatives (called pattern blocks) and technology in teaching fractions, decimals and percents. All of the teachers who used Middle School *Math Thematics* addressed the concept of equivalent fractions in their lessons. However, Teacher 1 and Teacher 5 differed from the approach suggested by the textbooks. Instead of using pattern blocks, teacher 1 asked the students to read the textbook while she taught them the concept of equivalent fractions. She asked factual questions such as "How many trapezoids are there in the first picture?" or "How many triangles does it take to replace the trapezoid?" or "How many triangles are in that new picture?" or "What if I want to replace the whole thing? I have got two more trapezoids there to replace. How many more triangles would I need?" or "So how many do I need altogether?"

Teacher 1 did not explain in the lesson why the picture on the left in Figure 2 represented $\frac{4}{6}$, and why the picture on the right represented $\frac{12}{18}$, nor how $\frac{4}{6}$ was equivalent to $\frac{12}{18}$ from the pattern blocks. Teacher 1 then jumped to the conclusion: "So we start off with 4 out of 6, and then we move to the eighteenth, see that over the top of page 111. (writing on the board, $\frac{4}{6} = \frac{?}{18}$) how could I do that? What can I multiply to do that? Three (answering her own question). We say those two fractions are equivalent. Because we multiply by the same number."

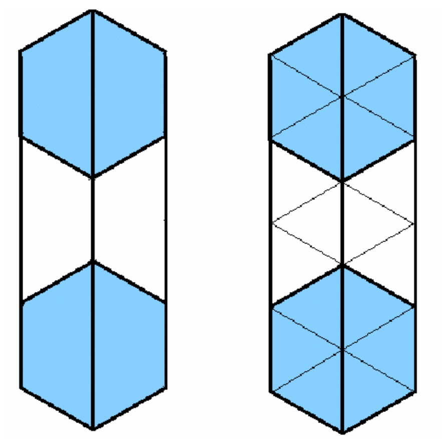

Figure 2. An example from *Math Mathematics*, illustrating that $\frac{4}{6}$ equals to $\frac{12}{18}$.

Teacher 5 did use the pattern blocks. However, she wanted the students to follow her while using the manipulatives and there was no time for students to think or investigate the problem. She asked "Each of the red ones was what part of the shape? How many red shapes were there all together?" "One sixths." She answered herself quickly, then proceeded to say "How many triangles would you have to put on top of, or replace it? Three of them. How many little green shapes will be altogether on that second shape? You got six on the top, you would have six in the middle, and you will have six green on the bottom. Eighteen of them… so this would be how many of altogether, yes, this would show you that how $\frac{1}{6}$ would be equal to $\frac{3}{18}$". Though teacher 5 mentioned that $\frac{1}{6}$ actually meant one trapezoid out of the six trapezoids, she did not explain clearly how $\frac{3}{18}$ represented the triangles, nor did she explain why $\frac{1}{6}$ would equal to $\frac{3}{18}$.

The lessons of teachers 2, 3 and 4 met the learning goal and were aligned with the textbook strategies. Teacher 2 asked the questions to guide students as they worked to build the figures with pattern blocks. For example, she asked "How many total trapezoids does it take to build the figure?" while students were building a hexagonal window with trapezoids. And then she asked, "How many trapezoids did we use to replace those triangles? Four out of how many trapezoids totally together? What would my fraction be? What if I turn them all into green triangles? How many trapezoids do we use to replace those triangles?" She also used higher order questions like "How did you come up with 18?" or "Do you have any different ideas?" or "Are those two equivalent? Are those two fractions equivalent", and "How do I know then?" to facilitate the process of learning.

Teacher 3 addressed the learning goal through demonstrating the whole process on the projector. After students finished building the hexagonal window with pattern blocks, she said "How many trapezoids did it take you to make that window? When I took away one trapezoid, that one trapezoid would be equal to what? One sixth of the window, right?" In order to build students' understanding of where $\frac{12}{18}$ comes from, teacher 3 first asked, "How many triangles do you think it is going to take to replace what I just took away?" while she removed the top two trapezoids with six triangles. Then she removed the bottom two trapezoids with another six triangles, asking "If I want to remove two more, the two on the ends here, how many would that be? Which would give us a total of how many triangles?" It turned out to be 12. Then she asked, "How many triangles you think it would take to make the whole thing? What would my fraction be?" The third issue is why $\frac{4}{6}$ equals $\frac{12}{18}$. Teacher 3 asked, "How do we get to $\frac{4}{6}$ is equal to $\frac{12}{18}$ numerically?"

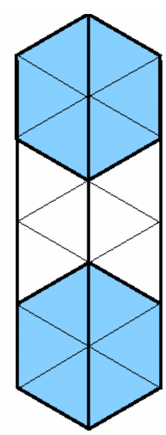

Figure 3. An example of student's misconceptions. (Note: Four trapezoids equal to twelve triangles so one student thought the fraction was $\frac{4}{12}$).

Teacher 4 addressed the learning goal by correcting students' misconceptions of fractions. She asked, "What fraction of the original window is replaced with the green triangles?" One student came up with a wrong answer four-twelfths (Figure 3). The teacher did not ignore the wrong answer, but continued to ask "How did you get four-twelfths? But is this four-twelfths of this would change (pointing to the trapezoids), four of these pieces, will twelve of them make up the whole thing? That would mean four trapezoids out of twelve trapezoids," she noted, "We have to use the same unit, so changing your unit will work, which one do you want to use, you can leave your 4 or you can leave your 12, it is up to you, which one do you want to use? Say with the trapezoids, four trapezoids changed out of …? Another way you can describe the fraction that has been changed." She also asked questions such as, "How come it is twelve eighteenths?" "Why is $\frac{4}{6}$ equal $\frac{12}{18}$? How do they make them equal?"

Table 7 provides a summary about whether the lessons of these five teachers addressed the main learning goal and whether the lessons were aligned with the textbooks.

Table 7

Summary of Qualitative Analysis of Lessons of Teachers 1, 2, 3, 4 and 5

Qualitative Measure of Instruction		1	2	3	4	5
1	Addressed the learning goal	Yes	Yes	Yes	Yes	Yes
2	Aligned with textbook approach	No	Yes	Yes	Yes	No

Both teachers 6 and 7 were from a rural school district that had adopted *Mathematics: Applications and Connections* (MAC) textbook (Collins et al., 1999), a commercially successful textbook that has been used by the district for several years. The textbook lesson started by describing the grip size of a tennis racket, and then introduced the concept of "mixed numbers", followed by a hands-on activity of grid manipulatives representing five fourths.

Both Teachers 6 and 7 addressed the learning goal, but with different methods. Teacher 6 did not use the approach the textbook suggested. She used manipulatives to introduce how to translate between mixed numbers and improper fractions. For example, she asked one student to come to the board to form a hexagon with seven triangles (Figure 4). She asked, "We have a problem here, we have one more. How many sixths do we have in here?" "Seven sixths, what fraction do we come up with?" "How can I change the improper fraction to one whole and one left over? Seven divided by six equals $1\frac{1}{6}$, does this make sense to you?" Later on, she asked students to change $3\frac{2}{3}$ into an improper fraction. One student said "use 3 multiply 3 and then add 2 (the teacher wrote $\frac{11}{3}$ on board)." The teacher picked out three hexagons and two rhombuses. She did the translations from symbolic representations to manipulatives by herself and said "Let us see if this is right…so if there are three on each one (hexagon), and we have two left over, I am not going to have enough blue (rhombuses) to go in these (hexagons), but if there are three in this one, how many are there that went in here? (pointing to the second hexagon) How many are there that went in here? (pointing to the third hexagon), and we have two left over. Three plus three plus three is nine and plus that two is eleven. Eleven thirds, does that make sense?" Then a student disagreed, saying: "Multiply three and two first and then add three." Teacher 6 did not clarify the student's misconception but rather concluded, "I think you got confused with something else. Because that won't work at all". "I think you got confused on that one."

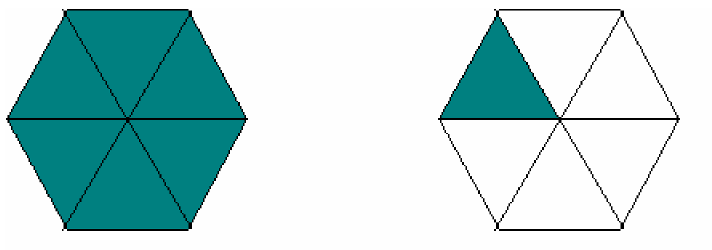

Figure 4. An example illustrating that $\frac{7}{6}$ equals $1\frac{1}{6}$.

Teacher 7 mainly used spoken language to address the learning goals. He used the examples suggested by the textbooks. He drew three circles (representing three pies) on the board with one of them divided into five parts. He asked, "What is a mixed number? How do I come up with a mixed number? Ok, first thing I want to do is to look for my wholes, how many wholes do I have? I don't have a whole third pie myself, do I? I have a partial, I have a part of my third pie. How many parts of the third pie do I have?" He then asked, "Then what is that pie broken into? Fifths, so fifths would be there (denominator, wrote $2\frac{2}{5}$ on board), so I have two and two-fifths of a pie, right?" "Can it be expressed as an improper fraction? Excuse me… as a mixed number … how?"

During the whole questioning process, there was not very much interaction going on among the students, as he talked through the concept of mixed numbers.

Table 8 provides a summary of whether the lessons of these two teachers addressed the main learning goal and whether their instructions were aligned with the textbooks.

Table 8

Summary of Qualitative Analysis of Lessons of Teachers 6 and 7

Qualitative Measure of Instruction		6	7
1	Addressed the learning goal	Yes	Yes
2	Aligned with textbook approach	No	Yes

Seven teachers used the CMP textbook. The lesson addressed the ideas “to use the ‘out of 100’ interpretation of fractions and decimals to develop an understanding of percent” and “investigating the relationships among fractions, decimals, and percents and to move flexibly among representations (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998, p. 66i).

All of these seven teachers addressed the mathematics learning goal. They varied somewhat in their approaches by asking questions of the students, by direct presentation or by a combination of these approaches. Teachers 8, 11, and 12 used direct instruction during a whole class session. Teacher 8 asked the question, “What does percent mean?” A student answered “47 percent, because percent means out of one hundred.” She then asked, “Do you know what is fifty three hundredths mean? Fifty three out of a hundred.” Teacher 9 asked higher order questions aimed at understanding the meaning of the percent, like “We are told that 2000 dollars on and up, we have 18% of the people, 18% meaning what? ” However, instead of letting the students answer the questions, she answered them herself: “No, no, no, eighteen percent meaning what? Eighteen out of what”? “Now what we have learned about, is that percent what?” Teacher 10 asked and answered the questions herself, “Fifty four, remember, what does this mean? Fifty-four what? What does fifty-four mean? Fifty-four out of a hundred.” Teacher 11 used open questions such as, “Does any one know what this ‘out of a hundred’ stuff is about?” or “Then what do we know about that 78 percent then?” Teacher 12 employed a series of questions through the whole class, for example, “The first thing I want you to know is what does percent mean?” and “That is absolutely right, out of one hundred. So 78 percent is same as 78 out of one hundred, do you agree with that?” “(56 percent) means that 56 out of a hundred.” After a few exercises, she elaborated more on the concept of percent in a real world situation, “You told me that you come up with some great examples… we are talking about how a juice provides 125% of the daily recommended allowance, what do you think that means? It is greater than the actual percent that you are supposed to have, do you agree? It mixed the actual percent that you suppose to have and then exceeded by what percent?” Teacher 13 started with a statement “I want you to think about out of one hundred, there is a specific word we are using that will be fit into what we are talking about. You know what I am thinking about, percent. Percent can be a substitute to mean out of a hundred. Turn to your neighbor partner and tell them what percent means.” “Out of one hundred. Ok, very nice.” Later on, he revisited the concept again, “What do we know a percent to be?” Teacher 14 combined the symbolic representation of the percent sign (%) with something meaningful, “The percent sign % … it helps you to remember that it means out of one hundred”. And later in the class she asked students “What does percent look like? 78 percent because it is out of one hundredth.”

These seven teachers can be categorized into three levels in terms of addressing the idea of investigating the “relationship among fractions, decimals and percents and to move flexibly among representations”(Lappan et al., 1998, p66i). The first level included teacher 9 who did not ask the students to investigate the relationships between the different representations. Teachers 8, 10, 12, 13, 14 belonged to the second level. They mentioned the idea, but did not elaborate on it. Teacher 8 said “Isn’t that (fractions, decimals, and percents) said the same way, it is, isn’t it?” Teacher 10 pointed out that “They are all the same numbers.” Teacher 12 said, “Do you agree that there is a connection between decimals and percents? Where there were connections between decimals and percents, there must also be connections between…factions and percents.” Teacher 13 also mentioned: “now you have graph (percent), fraction and decimal, three different ways of showing me the same thing.” Teacher 14 said that, “Two ways you have already studied are with decimals and with hundredths grids. Another useful way to express a fraction with a denominator of 100 is to use a special symbol: the percent symbol. ” Teacher 11 belonged to the third level, who built up students’ understanding through investigation. She asked the students to investigate the relationship among fractions, decimals and percents by asking probing questions like: “Tell me what they are talking about over here. They showed us the grid, they are talking about all these different ways we have already discussed. What are these different ways? What do we know about that?" "What do we know about these three different representations? Ok, awesome, we have the fraction, 56 hundredths, the decimal 56 hundredths and the percent 56 percent. What do we know about all three of these things? What could I write over here? " After a student simplified $\frac{56}{100}$ into $\frac{14}{25}$, the teacher asked the question again, "Fifty six one hundredths and fourteen twenty fifths, what do we know about it? What do we now know about these two fractions? So if it is equal to fourteen twenty fifths, and what do we know about all these other stuff over here?" (pointing to fraction, decimal and percent form of $\frac{56}{100}$) . She went deeper by pointing out that sometimes even fractions that do not look equal to each other could actually be equal to each other by asking "Now does this ($\frac{14}{25}$) look like equal to this ($\frac{56}{100}$)? But are they equal?"

The procedures of moving from one representation to the other were addressed by all of these seven teachers who used CMP as textbooks. However, there were variations in their approaches. Teachers 8, 9 and 13 all mentioned some general strategies. Teacher 8 emphasized phonological understanding. He asked the students to read correctly the fraction when translating between fractions and decimals. For example, he said "write that fraction as a decimal and percent… this is the fraction, read it, read it correctly, now write that as a decimal, what does fifty three hundredths look like as a decimal?" Teacher 9 relied more on memorization. She said "Come on, talk to me. When we have a percent sign, what do we do? We move into the what? Into the left, and we take out what? Our percent sign, let us move to the left and take out our percent sign." Teacher 13 discussed more thoroughly the general strategies of translating among fractions, decimals and percents. He drew a triangle on the board with fractions, decimals and percents as three vertexes. On each of the three edges, he asked the students to write some words describing the strategies of converting between the two vertexes as shown in Figure 5.

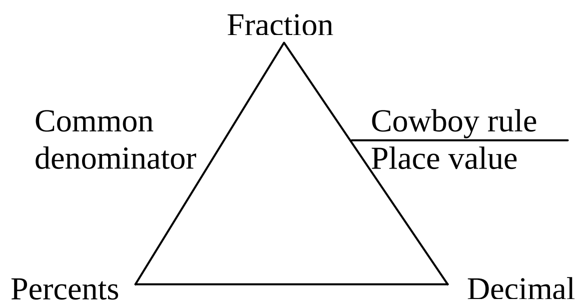

Figure 5. Representation of converting between fractions, decimals and percents.

Teacher 10 used an exercise of translating among the fraction $\frac{46}{100}$, the decimal 0.46 and the percent 46%. She asked students to "put out the number of male cats of the fraction, decimal and percents" on the board. After students finished writing their answers on the board, the teacher said: "Ok, here are our male cats, they are 46 out of a hundred (as a fraction), 46 hundredths (as a decimal) and 46 percent. " Teacher 11 used exercises of translating among the fraction $\frac{56}{100}$, the decimal 0.56 and the percent 56%. However, the teacher did not address the transformation among these three different representations as much as the relationship between them. Teacher 12 asked students to translate 25% into a fraction and then into a decimal, "(25%) in a fraction", "One fourth, very good. And into a decimal? How do you say that? 25 hundredths, excellent! ” In the activity of Raining Cats, she said “Look at his age, (.25 means) 25 months old? 25 years old? This is his age in years. Oh, JJ says that the cat is 25 hundredths years old.” When students came up with the right answer, she explored deeper into students’ understanding by asking probing questions like, “(.25 years) which means that he is 3 month old, JJ, how do you come up with that?” “Where did you get the four from? (.25 is $\frac{1}{4}$).” Teacher 14 focused on the procedures of translating between representations. She asked students to change $\frac{54}{100}$ to a decimal, and to a percent. For example, she asked, “Did everybody get 56%? Which is point five six, which is 56 over one hundred.” She also discussed about how to change 8 months to $\frac{2}{3}$ of a year. She used a number line, and she further changed $\frac{2}{3}$ of a year into .66 of a year using division.

Table 9 summarized about whether the lessons of these seven teachers addressed the main learning goal and whether they were aligned with the textbooks.

Table 9
Summary of Qualitative Analysis of Lessons of Teachers 8, 9, 10, 11, 12, 13 and 14

Qualitative Measures of Instruction		8	9	10	11	12	13	14
1	Addressed the learning goal	Y	Y	Y	Y	Y	Y	Y
2	Aligned with the textbook approach	Y	N	Y	Y	Y	Y	Y

Classroom Interactions

Teacher-student interactions can be clustered into two categories: teacher-centered interaction with few student interactions, and constructive interaction with opportunities for student work and interactions. Teachers 1, 5, 6, 7 and 9 belonged to the first category, while teachers 2, 3, 4, 8, 10, 11, 12, 13 and 14 belonged to the second one.

Teachers belonging to the first category usually did not have student activities in class. If a student activity existed, the time was short, and students were not highly involved in the activity. These teachers asked mainly factual questions. When potentially, higher order questions were asked, students were not given a chance to answer. Instead, the teachers answered the question themselves. These teachers focused on the rules and facts, rather than on a learning environment for students to experience the mathematics. There was an emphasis on "correctness", that is, answers which match what the teacher thinks, anything that was not correct was either ignored or negated.

There were no activities in teacher 1's class even though the textbook suggested using manipulatives to teach equivalent fractions. The majority of questions teacher 1 asked in the class were factual. For example, she asked, " $\frac{4}{9}$ equals something over the18 ($\frac{4}{9} = \frac{?}{18}$)? Right, 9 times 2, so what do I do on the top? Times 2, so what would it be?" The only higher order question she asked in lesson two was "Does anybody know how to find out what goes above 18? (converts $\frac{4}{9}$ to $\frac{?}{18}$)." When no students answered the question, she answered it herself instead of asking guiding questions. When students answered the question wrong, she either ignored it or asked the question again. For example, “How many triangles are in that new picture?” Some students said six, some students said three and some said twelve. She agreed with twelve, saying “ See, six on the top and six on the bottom.”

Teacher 5 used about three minutes for an activity the textbook suggested. She handed out the manipulatives to the students; however, she gave orders to students rather than letting them explore the mathematics. For example, she walked past a student, saying “Make that one on the left first please… now replace the top red with those 6 greens, you can either put it on top or move it and take their place.” Most of her questions were also factual questions. For example “Each one of the red ones was what part of the triangle?” or “How many red shapes were there all together?” She asked one higher order question “How did you figure out $\frac{3}{18}$?” There was little or no waiting time for the students to answer any of the questions before she answered them herself.

Teacher 6 did use manipulatives and had some activities in class which deviated from the textbook suggestions. However, not every student had the opportunity to work with the manipulatives. Only those who were asked to go to the board were able to use the materials. The questions she asked were factual. For example, “How many (triangles) does it take him to make (the hexagon)? How many sixths do we have over here? Does it make sense to you?” Some higher order questions were asked such as “How are we going to figure how many square feet that are going to melt?” or “How can I change improper fraction to one whole and one left over?” After students said “divide,” she failed to ask students to justify their answers. Teacher 6 did not investigate students’ misconceptions. She emphasized on the “right answer.” For example, one student got confused in changing a mixed number $3\frac{2}{3}$ to an improper fraction; he got $\frac{9}{3}$ by using 3 times 2 then adding 3, which is totally wrong. Instead of explaining why it would not work, she just said, “That won’t work; I thought you got confused somewhere else.”

Teacher 7 followed the textbook approach. He used no manipulatives or student activities in class. The questions were all factual. For example, “What is a mixed number?” or “How many wholes do I have?” and “I do not have a whole third pie myself, do I?” or “Then what is that third pie broke into?”

Even though teacher 9 used many of the activities suggested by the textbook, her teaching emphasized memorizing the procedures. For example, when students were asked to convert percents to decimals, she said, “When we have a percent sign, what do we do? We move to the what? To the left, and we talk about what? Our percent sign. Let us move to the left and take our percent sign”. She asked many factual questions and sometimes she answered them herself. For example, “18% meaning what? 18% meaning what? 18 out of what? When we simplify, what do we take? Find out what? And we could go back to our divisibility rules and do what? We find the greatest common factor! And we do what? Reduce by it, what is the greatest common factor in this? So another way of writing 18% in fractional form would be what? Nine fiftieths, ok? What would our chart look like for nine fifties?” Even though she asked some higher order questions, she ignored student answers. For example she asked, “Why do you think the grid would end up completely shaded?” Some students answered, but she did not make any comments and switched to talking about another question, “Ok, all right, let us look at the chart over here, we are told that 2000 dollars on and up, we have 18% of the people, 18% meaning what? 18% meaning what?"

Constructivist teachers often used activities in which students were directly involved. They asked factual questions as well as guiding and probing questions and allowed waiting time for students to think and respond to these questions. When students came up with wrong answers, the teachers followed up to investigate the misconceptions. Students experienced mathematics through a constructive learning environment where understanding how students came up with the conclusion was the main focus in the class. Teachers 2, 3, 4, 8, 10, 11, 12, 13, 14 tended to be more constructive.

Teacher 2 followed the textbook suggestions to let students investigate the concept of equivalent fractions by building the figure in the textbook with the pattern blocks. Her strategy of teaching was to let the students play first, and then she guided students' learning through probing questions. For example she asked, "How do you come up with 18?" and "Do you have any different ideas?" or "Are they equivalent? How do we know that? " followed by a discussion of the textbook problem which emphasized both the manipulatives representations and symbolic representations.

Teacher 3 followed the textbook suggestions too. Her teaching style was to let the students use pattern blocks first, then asked some probing questions to challenge students. For example, "But how do we get there? How do we get there numerically?" She emphasized on both the manipulatives representations and symbolic representations.

Teacher 4 also used the activity the textbook suggested, making figures with different pattern blocks. Her teaching style was to guide students' understanding through probing the misconceptions they indicated in the class. For example, one student thought that $\frac{4}{12}$ of the original window was changed because four trapezoids were equal to twelve triangles, which was incorrect. She used probing questions like "How did you get four twelfths? Will twelve of them make up the whole thing? ($\frac{4}{12}$) means four trapezoids out of twelve trapezoids, we have to use the same unit, so change your unit it will work, you can leave your four or leave your twelve, it is up to you, which one do you want to use? Say with the trapezoids, four trapezoids changed out of …" and "got to be equal pieces too." or "You have to say trapezoids are changed out of trapezoids and triangles are changed out of triangles. You have to keep the same unit, you cannot flop around." She constructed a learning environment by asking factual questions as well as probing questions to guide students learning. For example she asked, "What fraction of that window was changed? Another way you can describe the fraction that has been changed?" or "How come it is twelve eighteenths?" and she also asked, "So what do we know about the trapezoids?...How many triangles changed?" "Why $\frac{4}{6} = \frac{4*3}{6*3} = \frac{12}{18}$? They are equal the same, how do they make them equal?" and "We are thinking about the magic power of one. Think of any number in your head, multiple by one, multiply by one again, what did you find?" or "Can anyone tell me any fraction dressed up like one? It is not a number one, it is a fraction dressed up like one… when we used that fraction ($\frac{3}{3}$), it makes $\frac{4}{6}$ equal to $\frac{12}{18}$." Teacher 4 used both manipulatives and symbolic representations.

Teachers 8, 10, 11, 12, 13, and 14 all followed the *Connected Mathematics* textbook, which suggested a variety of activities and real world examples for teaching fractions. They each asked many probing questions in class. However, they varied in the process of teaching and their focuses were also different. Teachers 8, 11, and 13 all focused on conceptual understanding but through different approaches.

Teacher 8 emphasized the phonological connections between different representations. For example, he asked questions like “Oh, what is that the percent of? 53 percent because? What does percent mean?” and “What does 100% mean? 100 out of 100, that means?” He also asked, “Do you know what does fifty three hundredths mean? How do I write it as a decimal?”, and “This is a fraction, read it, read it correctly; now write that as a decimal, what does fifty three hundredth look like a decimal?” Teacher 11 used a series of guiding and probing questions to get the whole class involved in the process of developing the conceptual understanding of relationships among different representations. For instance, she asked “What do we know about all of these things? Why they are equal? Why is she doing that? Can anybody explain that to me why she did that? What are these different ways?” she also asked, “What do we know about that? What do we know about these three different representations?” and “What about the grid filled in? How does that have 78?” Teacher 13 developed students’ conceptual understanding through a picture of triangular relationships among different representations generally. He asked, “If I want to change a fraction into decimal, how am I going to do that? What is going to happen if I want to change decimal back to fraction?" "Based on what you have already known, what do you think should be put in the blank from fraction to percent?"

Teachers 8, 10, 12, 13, and 14 focused on the strategies that students used to work out the problem. Teacher 8 asked "What is the first step? Why does that give me how many are not kittens? What do you notice about the combinations of cats and kittens?" she stated, "I want you to express your reasoning. I do not want just the numbers. What do you mean by that? That is what we are doing on the board, we wrote down the number and I said how you got that." Teacher 10 asked, "Did you need a key for this? Why not? What was your observation? And how did you find it? From what? What did you know about the combined percentage? 54 and 46 are one hundred percent, why? There is one danger, what is the danger if you did it that way rather than counting?" Teacher 12 asked, "What do you think that means? What do you notice? Where did you get the four from?" and "How did you come up with that?" Teacher 12 also asked, "Tell me what you could do if you know how many females there are." "Why do you do that? They add up to 100, why is that?" Teacher 12 also elaborated a little bit more, "Ok, why you think it is easy to move from fraction to decimal to percent with this example? What do you think? Will it be easier if we deal with 150 cats? What about 50? Could you still figure out?" Teacher 13 asked, "Do you know any specific things about those two fractions? You add the numerators together, you get 100. Why?" Teacher 14 asked, "What did you use? Where did you get 54 from? What do you think of these?"

Both teachers 13 and teacher 14 pointed out the advantages of some representations over the others. For example, teacher 13 asked, “So what do you think we can do with the database? How could we use that chart? Is it easier to read the database to find which one is female and male? Or is it easier to look at your graph? Why? Why it is easier to look at the graph?” Teacher 14 asked “What is beneficial about making a chart and not making a chart?”

Quantity of Instructional Representations

Five forms of representations were found in teachers’ instruction from the videotapes. There were 1) real world representations, 2) manipulatives representations, 3) pictorial representations, 4) spoken symbolic representations, and 5) written symbolic representations that included calculators. Tables 10 and 11 present summaries of the percentage of time each teacher engaged in different forms of representations.

Table 10

Percentage of Time on Different Forms of Representations

Forms of represent-ations	Teacher													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Real World	0.0	0.1	0.0	2.9	1.4	2.0	0.0	0.9	3.0	5.6	3.1	8.2	5.8	2.2
Manupu-latives	5.9	16.9	19.2	10.4	13.7	1.8	0.0	8.1	6.3	0.1	16.6	18.7	7.4	13.6
Picture	13.2	17.1	8.1	3.9	10.9	15.8	16.1	9.1	0.3	17.5	9.5	7.4	23.9	8.8
Spoken Symbolic	30.9	39.5	39.6	32.1	33.3	46.1	44.3	39.2	64.7	38.6	41.7	28.5	35.1	56.0
Written Symbolic	33.2	23.3	32.3	38.7	21.8	31.6	31.9	23.0	57.0	25.5	19.4	14.2	16.8	43.6

Table 11

Descriptive Statistics of Instructional Representations

Forms of representations	N	Minimum	Maximum	Mean	Std. Deviation
Real World	14	0.0	8.2	2.5	2.5
Manipulatives	14	0.0	19.2	9.9	6.7
Picture	14	.30	23.9	11.5	6.2
Spoken Symbols	14	28.5	64.7	40.7	9.9
Written Symbols	14	14.2	57.0	29.5	11.5

The mean percentage of time the teachers spent on real world representations was 2.5% with a minimum of 0%, a maximum of 8.2%, and a standard deviation of 1.8%. Five teachers, 1, 2, 3, 7 and 8 used less than 1% of time on real world representations. The other nine teachers spent only 1% up to 9% of their class time using real world representations.

The mean percentage of time spent on manipulative representations was 9.9% with a minimum of 0%, a maximum of 19.2%, and a standard deviation of 6.7%. Seven teachers, 1, 6, 7, 8, 9, 10, and13, spent less than 9.9% time on manipulatives representations. Four teachers, teacher 2, teacher 3, teacher 11, and teacher 12 spent more than 15% of their instructional time on real world representations.

The mean percentage of time spent on picture representations was 11.5% with the minimum of 0.3% , a maximum of 64.7%, and a standard deviation of 6.2%. Five teachers, teacher 2, 6, 7, 10, and 13 spent more than 15% of the instructional time on picture representations while the other nine teachers, teacher 1, 3, 4, 5, 8, 9, 11, 12 and 14 spent less than 15% of the time on spoken symbols.

The mean percentage of time spent on spoken symbolic representations was 40.69% with the minimum of 28.5%, a maximum of 64.7% , and a standard deviation of 9.87%. Two teachers, teacher 9 and teacher 14 spent more than 50% of the instructional time on spoken symbolic representations while the other 12 teachers, teacher 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, and 13 spent less than 45% of the time on spoken symbols.

The mean percentage of time spent on written symbolic representations was 29.45%, with a minimum of 14.2% , a maximum of 57.0% , and a standard deviation of 11.5%. Two teachers, teacher 1 and teacher 4, tended to use more time on written symbolic representations than spoken symbolic representations, while the other teachers, teacher 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, and 14 tended to spend more time on spoken symbolic representations.

Teachers 1, 3 and 7 did not use real world representations, while the other eleven teachers 2, 4, 5, 6, 8, 9,10,11,12,13 and 14 all used five forms of representations. Teacher 1 spent 5.9% of time on manipulative representations, 13.2 on picture representations, 30.9% on spoken symbolic representations, and 33.2% on written symbolic representations. Teacher 2 spent 0.1% on real world representations, 16.9% on manipulative representations, 17.1% on picture representations, 39.5% on spoken symbolic representations and 23.3% on written symbolic representations. Teacher 3 did not use real world representations, she spent 16.9% time on manipulative representations, 17.1% on picture representations, 39.6% on spoken symbolic representations and 32.3% on written symbolic representations. Teacher 4 spent 2.9% on real world representations, 19.2% on manipulatives representations, 8.1% on picture representations, 32.1% on spoken symbolic representations and 38.7% on written symbolic representations. Teacher 5 spent 1.4 % on real world representations, 13.7% on manipulatives representations, 10.9% on picture representations, 33.3% on spoken symbolic representations, and 21.8% on written symbolic representations. Teacher 6 spent 2.0% on enactive representations, 1.8% on manipulatives representations, 15.8% on picture representations, 46.1% on spoken symbolic representations and .31.6% on written symbolic representations. Teacher 7 did not use real world representations, however, he spent 16.1% on picture representations, 44.3% on spoken symbolic representations and 31.9% on written symbolic representations. Teacher 8 spent 0.9% on real world representations, 8.1% on manipulatives representations, 9.1% on picture representations, 39.2% on spoken symbolic representations and 23% on written symbolic representations. Teacher 9 spent 3.0% on real world representations, 6.3% on manipulatives representations, 0.3% on picture representations, 64.7% on spoken symbolic representations and 57.0% on written symbolic representations. Teacher 10 spent 5.6% on real world representations, 0.1% on manipulatives representations, 17.5% on picture representations, 38.6% on spoken symbolic representations and 25.5% on written symbolic representations. Teacher 11 spent 3.1% on real world representations, 16.6% on manipulative representations, 9.5% on picture representations, 41.7% on spoken symbolic representations and 19.4% on written symbolic representations. Teacher 12 spent 8.2% on real world representations, 18.7% on manipulatives representations, 7.4% on picture representations, 28.5% on spoken symbolic representations and 14.2% on written symbolic representations. Teacher 13 spent 5.8% on real world representations, 7.4% on manipulatives representations, 23.9% on picture representations, 35.1% on spoken symbolic representations and 16.8% on written symbolic representations. Teacher 14 spent 2.2% on real world representations, 13.6% on manipulatives representations, 8.8% on picture representations, 56.0% on spoken symbolic representations and 43.6% on written symbolic representations.

Case Example of Quality of Teacher Representation

The quality of teachers' instruction regarding representations was assessed using three criteria; accuracy, comprehensibility and connections adapted from the AAAS textbook study (AAAS, 2000). Each teacher received a score of 1 (met) or 0 (not met) on accuracy, comprehensibility and connection for each lesson. Each of the three criteria was scored based on the presence or absence in the lesson using the indicators described in Table 12[2].

Table 12

Criteria and Indicators of Instructional Quality

Criterion	Indicators
1. Accurately depicts the intended mathematics learning goal	No misconceptions
	Point out the limitations of the representations
2. Comprehensible to students	Meaningful to students
	Includes real world experiences
	Hands-on activity
3. Connections are made between the representations and what is being represented	Justify why it is being represented in a particular way
	Connections are made between a variety of representations

[2] Table 12 is exactly the same as Table 5. It is presented here again for ease of reading.

The following illustration provides an example of how the indicators were applied in scoring the Instructional Quality criteria. Assume that a teacher used a pie graph to illustrate that $\frac{3}{4}$ equals $\frac{9}{12}$ by first dividing a pie into 4 parts, and then dividing each quarter of a pie into 3 parts to get 12 parts (see Figure 6). If she did not explain that it should be 12 equal parts and the pie was not divided evenly in the drawing, this representation may result in a student's misconception of part-whole, and consequently, the teacher received a zero on the misconception indicator. If the teacher mentioned the limitations of the pie representation, she/he received a score of 1; if limitations were not mentioned, it was scored zero. Because the pie is not divided equally, it is graded as zero for accuracy criterion.

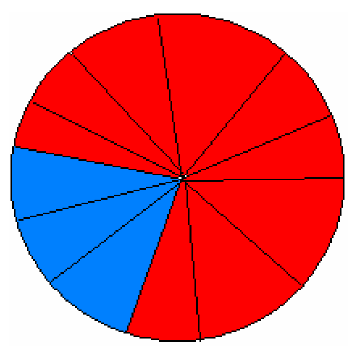

Figure 6. A pie chart diagram for fractions.

In order to be comprehensible and meaningful to students, the instructional language must be close to the students' level of understanding, using only necessary and developed mathematical vocabularies. For example, when using a hexagon as a representation to introduce how to convert $\frac{7}{6}$ to $1\frac{1}{6}$, only after the students have already gained some direct experiences based on hands-on activities, the term "mixed number" was introduced. Real world experiences should be within students' realm of prior knowledge. For instance, one teacher told the class, "It is cold in the Dallas area and people want to melt the ice. A four ounce bag of salt could melt nine square feet of ice. Suppose there is only one ounce of salt left in the bag. How many square feet of ice could be melted?" This is counted as a real world situation where students could apply their fraction knowledge. Concrete hands-on activities means students were doing an activity, often with the use of a manipulative to support their learning. For example, students were asked to create a fraction strip of $\frac{1}{5}$. If the teacher addressed any one of the three indicators, she/he was considered as meeting the requirement of comprehensibility.

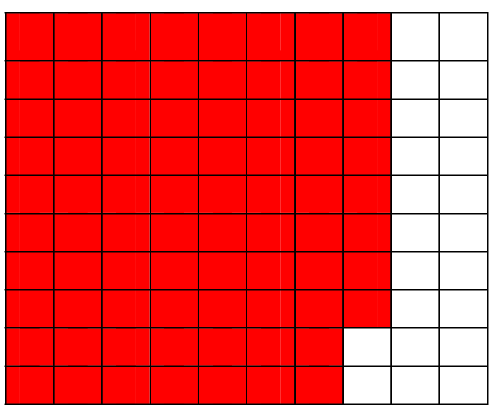

Figure 7. A hundreds grid representing 78%.

The connection criterion had two indicators. If the teacher addressed why an idea was being represented in a particular way, she/he received credits for this indicator. For example, when the teacher was trying to describe that the proportion of the shaded area of the picture in Figure 7 was 78%, she/he should have emphasized that 78 out of 100 squares were shaded, so it was 78%. The second indicator was "Connections are made between varieties of representations." In demonstrating how $\frac{4}{6}$ equals $\frac{12}{18}$ by using trapezoids and triangles, the teacher could point out that by looking at a picture in terms of 6 trapezoids (See Figure 7), one will think the shaded area is $\frac{4}{6}$, however, one will get $\frac{12}{18}$ if he/she looks at it in terms of triangles.

Quality of Instructional Representation

The quality score for each of the criteria was computed by averaging the scores across the lessons that were videotaped. The total of the three criteria averages provided an overall instructional quality score for representations. The teachers' individual scores for each criterion and the overall scores are listed in Table 13.

The mean for all teachers' accuracy was .84 with a minimum of .67 and a maximum of 1.00. Six teachers scored higher than the mean. Teachers 1, 2, 4, 6, 11, and 13 scored 1.00. Eight teachers, teachers 3, 5, 7, 8, 9, 10, 12 and 14 scored lower than the mean. Teachers 1, 2, 4, 6, 11, and 13 all used the instructional representations correctly. Both teacher 13 and teacher 14 mentioned the advantages of some representations over other representations.

Table 13
Instructional Quality of Representation Scores for Teachers

Teacher	Accuracy	Comprehensibility	Connection	Total
T1	1.00	.00	.00	1.00
T2	1.00	1.00	.66	2.66
T3	.80	.40	.20	1.40
T4	1.00	1.00	.80	2.80
T5	.67	.33	.00	1.00
T6	1.00	.33	.00	1.33
T7	.67	.00	.00	0.67
T8	.80	1.00	.80	2.60
T9	.67	.33	.00	1.00
T10	.67	.67	.33	1.67
T11	1.00	1.00	1.00	3.00
T12	.80	1.00	.80	2.60
T13	1.00	1.00	1.00	3.00
T14	.80	.50	.20	1.50
Means	.84	.61	.41	1.87

The mean of comprehensibility was 0.61 with a minimum of .00 and a maximum of 1.00. Seven teachers scored higher than the mean, among which six teachers, teacher 2, 4, 8, 11, 12, and 13 scored 1.00, and the other teacher, teacher 10 scored .67. Seven teachers, teacher 1, 3, 5, 6, 7, 9 and 14 scored lower than the mean.

The mean of connections was .41 with a minimum of 0, and a maximum of 1.00. Six teachers, teacher 2, 4, 8, 11, 12, and 13 scored higher than the mean. However, only two teachers, teacher 11 and 13 scored 1. Eight teachers, teacher 1, 3, 5, 6, 7, 9, 10 and 14 scored lower than the mean.

Summary of Results for Research Question 1

Both qualitative and quantitative analysis methods were used to investigate the quality and quantity of the teachers' instructional representations. Various levels of teacher performance were found based on three themes: textbook alignments, classroom interactions, and quality and quantity of teachers' instructional representations. Teachers who followed the teaching approaches suggested by CMP (Bits and Pieces unit) and Middle School Math *Thematics* were found to have interactive classrooms, while teachers who either followed or varied from MAC tended to have teacher-centered classrooms. Furthermore, teachers' instructional representations varied in terms of the time they engaged in delivering the content as well as the quality of representations. Symbolic representations tended to be the predominant representations in the classrooms that were analyzed. Quantity of instructional representations (for example, the time each teacher engaged in using the real world, manipulatives, pictures, spoken symbolic and written symbolic representations) varied as much as the quality of the representations (i.e., accuracy, comprehensibility and connections). These results are presented in Tables 14 and 15, and Figures 8 and 9.

Table 14
Summary Table of Teachers Textbook Alignments and Learning Goals

Qualitative Measure of Instruction	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Address learning goal	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
Aligned with textbooks	N	Y	Y	Y	N	N	Y	Y	N	Y	Y	Y	Y	Y

Table 15

Summary of Classroom Interactions and Alignment with the Textbook

	Aligned with textbooks	Yes			No		
		Math-*Thematics*	MAC	CMP	Math-*Thematics*	MAC	CMP
Constructive Classroom	Yes	2,3,4	7	8, 10, 11, 12, 13, 14			
	No				1, 5	6	9

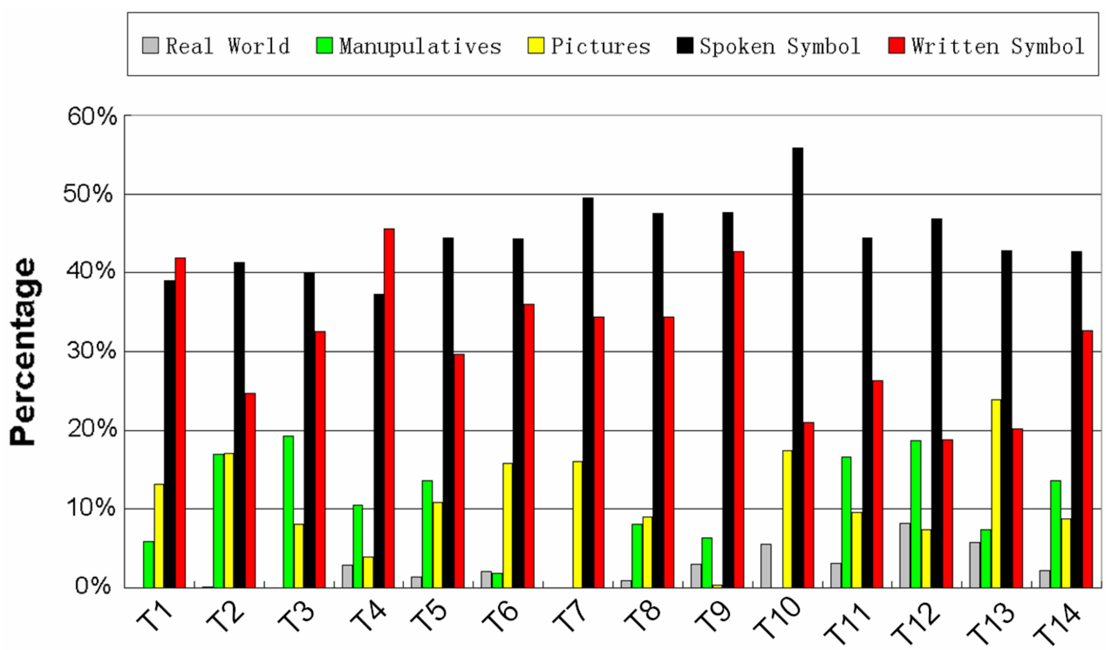

Figure 8. Teaching quantity results (percentage of time).

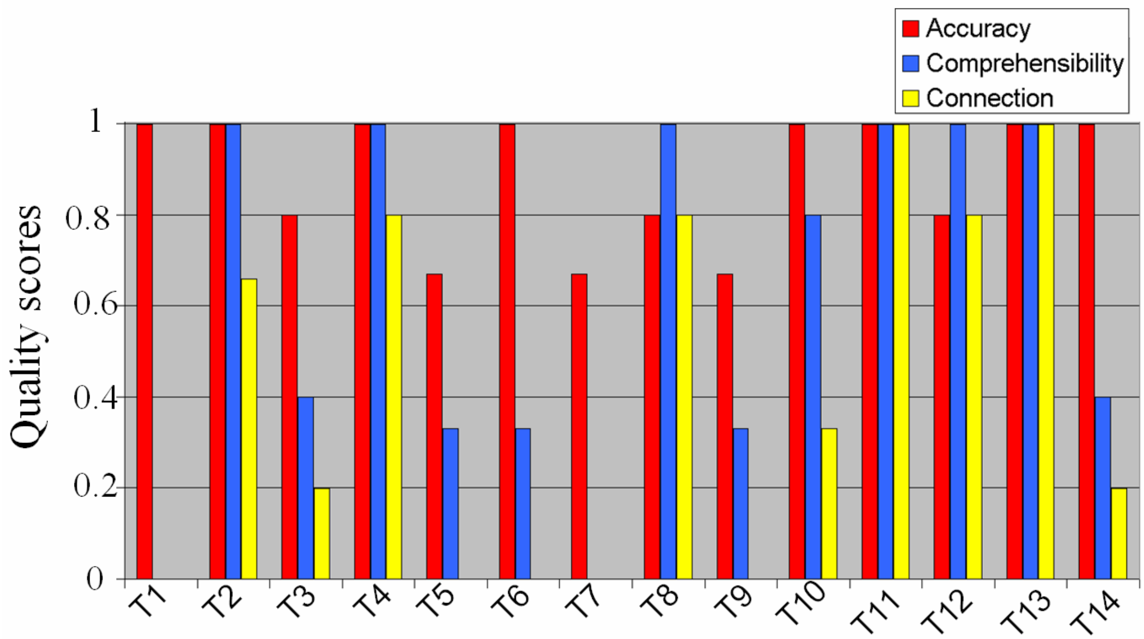

Figure 9. Teaching quality results.

Research Question 2

What are the structures of instructional representations and students' written representations? How do these structures reflect the format sub-constructs: real world, manipulatives, pictures, spoken symbolic and written symbolic representations? How do they reflect the content sub-constructs: measure, part-whole, quotient, wonderful one, and cross product?

The structural equation modeling (SEM) technique was used to answer research questions two through four. Because statistical techniques such as factor analysis, measurement theory, path analysis, multiple regression, and general linear modeling of relations are all included in SEM analysis, it is regarded as one of the most up-to-date and advanced statistical techniques (Kline, 2005). SEM is usually preferred rather than the traditional techniques because of its capability to approximate the measurement error

Table 16
Definitions of Instructional Representation Variables

Variable Name	Description
RWM	Real World Manipulatives
RWQ	Real World Quotient
MPW	Manipulatives Part-whole
MM	Manipulatives Measure
MQ	Manipulatives Quotient
PPW	Picture Part-whole
PQ	Picture Quotient
PM	Picture Manipulatives
PWO	Picture Wonderful One
SSPW	Spoken Symbol Part-whole
SSM	Spoken Symbol Measure
SSQ	Spoken Symbol Quotient
SSWO	Spoken Symbol Wonderful One
SSCP	Spoken Symbol Cross Product
WSPW	Written Symbol Part-whole
WSM	Written Symbol Measure
WSQ	Written Symbol Quotient
WSWO	Written Symbol Wonderful One
WSCP	Written Symbol Cross Product
SCQ	Symbolic Calculator Quotient

thus generating more precise estimates (Kline, 2005). Measurement models and path models are two major steps typically involved in conducting a SEM analysis. The measurement models aim at investigating whether the proposed theoretical model fits the data by engaging confirmative factor analysis (CFA). Then path models are developed to investigate the causal relationships (Kline, 2005). Therefore, a measurement model of teacher’s instructional representations and a measurement model of students’ written representations were developed in this study. A set of variables that describes both the sub-constructs and forms of instructional representations was defined in order to develop the theoretical model. Table 16 presents the variable names and their brief descriptions.

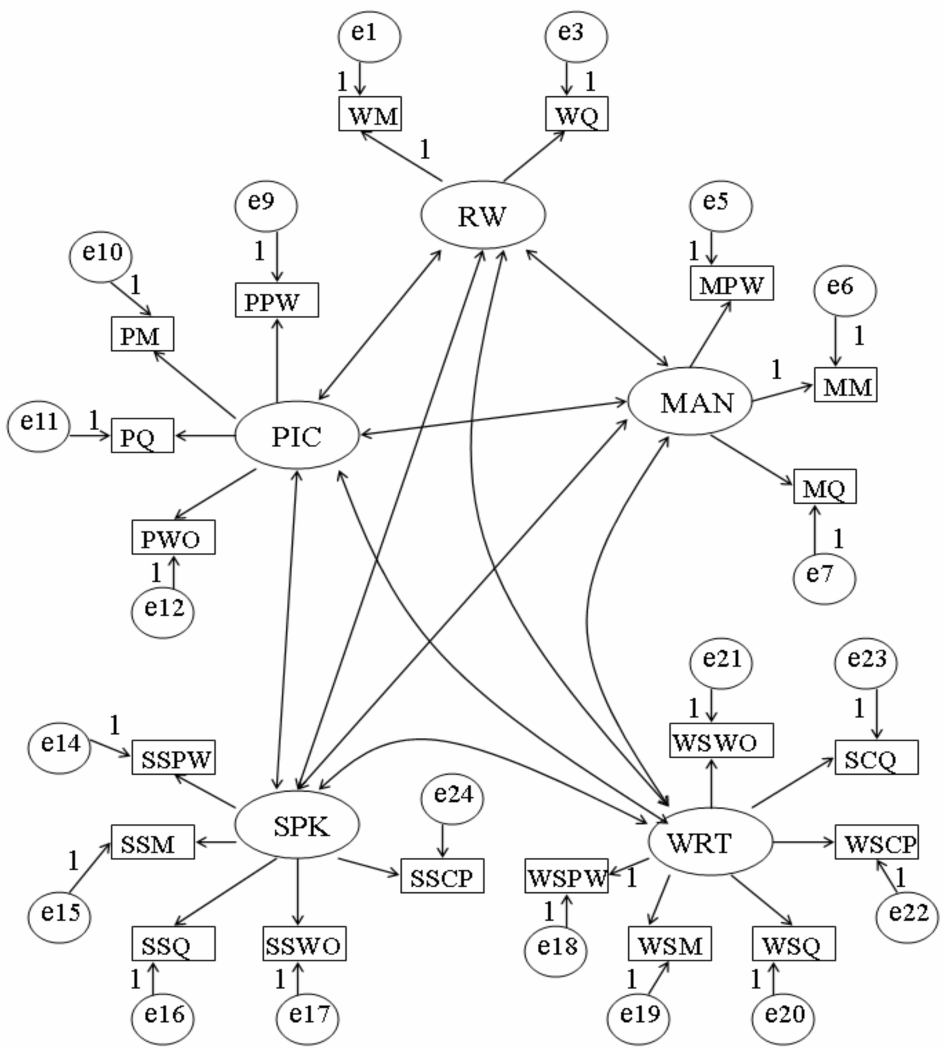

Figure 10. Initial theoretical measurement model.

The proposed theoretical measurement model for the relationship among the real world, manipulatives, pictorial, spoken symbolic and written symbolic representations used by the teachers is shown in Figure 10.

The circles in Figure 10 illustrated the endogenous (latent) variables and the rectangles illustrated the exogenous (observed) variables. A two-headed arrow connecting the endogenous variables showed that connections are assumed to exist between each variable. A one-headed arrow directed from the endogenous variable to the exogenous variable illustrated that the endogenous variable is predicted by the exogenous variable. The endogenous variables, the exogenous variables, the errors and the arrows comprise a measurement model. If the initial measurement model is rejected, the model is revised by reorganizing, deleting or adding the variables based on modification indexes in order to achieve a better fit. Several issues regarding background knowledge and limitations of the SEM should be mentioned before the results of SEM analysis for this study are discussed.

1. The issue of small sample sizes. The literature suggested that SEM applied in any sample size less than 250 may cause problems (Hu & Bentler, 1999). Kline (2005) stated that a sample size less than 100 is considered as small, between 100 and 200 is considered as medium, and greater than 200 is considered as large. In this study, only 58 videotapes were coded and analyzed in the teacher measurement model and 213 students' pretests and posttest data set were analyzed. In this case, the students' measurement is based on a large data set according to Kline (2005), however, the teachers' measurement model requires improvement in terms of sample size.

2. The issue of multicollinearity, which is an assumption of the SEM analysis that extremely high correlations should not exist between observable variables (Kline, 2005). However, the spoken symbolic representations and the written symbolic representations are highly correlated in this study, for the simple reason that teachers tended to use verbal communication while they were writing on the board.

Two programs, AMOS (Arbuckle & Wothke, 1999) and M-plus (Muthen & Muthen, 2004) were employed. Lagrange Multipliers were evaluated by revealing that adding certain paths will reduce corresponding amount of chi-square, thus improving the total model fit. Four consecutive proposed models were constructed in order to determine the model with the best fit. The chi-square statistics, *Normed Fit Index* (NFI), *Comparative Fit Index* (CFI), and *Root Mean Square Error of Approximation* (RMSEA) were reviewed for the following models.

Table 17
Fit Indices for the Initial Model

Model	χ^2	Df	CFI	NFI	RMSEA	CI	
						Lo 90%	Hi 90%
Initial model	N/A	N/A	N/A	N/A	N/A	N/A	N/A

Note: N=58

In Figure 10 it can be seen that real world, manipulatives, pictures, spoken symbolic representation, and written symbolic representations were measured by two, three, four, five and six variables, respectively. The proposed model terminated at the 49th iteration, yielding a negative error variance with a value of -74.98, therefore, no fit indexes were reported because of the limit of the iteration were reached as indicated by

Table 17. The model could not be improved to generate any estimates, so that the five latent variables (real world, manipulatives, pictures, spoken symbolic representations, and written symbolic representations) were not able to be verified in current study.

Table 18
Pattern Coefficient Matrix from Promax with Kaiser Normalization Rotation

	Factor					
	1	2	3	4	5	6
RWM	0.05	0.15	0.81	-0.02	-0.16	0.01
RWQ	-0.17	-0.13	0.55	0.01	0.24	0.05
MPW	-0.03	0.52	0.09	0.02	-0.13	0.02
MM	0.95	0.05	0.01	0.01	0.06	0.04
MQ	0.05	0.12	-0.07	-0.20	0.44	0.05
PPW	-0.26	0.48	-0.05	0.27	-0.13	-0.07
PQ	-0.04	0.01	0.98	-0.01	0.06	0.07
PM	0.22	-0.01	0.48	-0.01	-0.34	-0.21
PWO	0.06	0.15	0.03	-0.17	0.11	0.68
SSPW	0.06	0.93	0.01	0.07	0.05	0.01
SSM	0.98	-0.05	-0.03	0.04	-0.02	-0.03
SSQ	0.01	-0.07	0.11	-0.09	0.92	-0.01
SSWO	0.00	-0.07	0.03	0.14	-0.08	0.88
SSCP	0.05	-0.03	-0.02	0.95	-0.08	0.05
WSPW	0.04	0.90	0.00	-0.12	0.19	0.03
WSM	1.00	-0.03	-0.04	0.04	0.06	0.03
WSQ	0.03	-0.01	0.07	0.40	0.75	-0.12
WSWO	-0.01	-0.04	0.02	0.09	-0.07	0.82
WSCP	0.04	0.08	-0.01	0.94	-0.03	-0.01
SCQ	0.05	0.03	-0.09	-0.05	0.42	-0.01

A factor analysis was conducted to investigate any other potential relationships between these variables. Principle component analysis was used as an extraction method and Promax with Kaiser Normalization was used as a rotation method. The pattern coefficient matrix is summarized in Table 18. Instead of factors like real world, manipulatives, pictures, spoken symbolic representation, and written symbolic representations, the empirical data in this study is shown according to mathematics content factors such as part-whole, measure, quotient, wonderful one and cross product.

All three variables in factor one showed the sub-concepts of measurement in understanding fractions. Manipulative Measurement (MM), Spoken Symbols Measurement (SSM) and Written Symbols Measurement (WSM) loaded .954, .982 and 1.000 on the factor, respectively. Since all these three variables are connected to the concept of measurement, the first factor is called the Measurement factor.

All four variables in factor two indicate the sub-concepts of part-whole in understanding of fractions. Manipulative Part-Whole (MPW), Picture Part-Whole (PPW), Spoken Symbols Part-Whole (SSPW), Written Symbols Part-Whole (WSPW) loaded .523, .475, .932 and .901 on factor two, respectively. Since all these four variables are connected to the concept of Part-Whole, the second factor is called the Part-whole factor.

Both variables in factor three contribute to the procedural knowledge of cross product. Spoken Symbols Cross Product (SSCP) loaded .946 and Written Symbols Cross Product (WSCP) loaded .944 on this factor. Since both of these two variables are connected to procedure of cross product, the third factor is called Cross Product factor.

All four variables on factor four are connected to the concept of quotient. Manipulative Quotient (MQ), Spoken Symbols Quotient (SSQ), Written Symbols Quotient (WSQ), and Symbols Calculator Quotient (SCQ) loaded .453, .921, .749 and .421 on the fourth factor, respectively. Since these fourth factors are connected to the concept of quotient, the fourth factor is called the Quotient factor.

All three variables loaded on factor five address the procedural knowledge of wonderful one. Picture Wonderful One (PWO), Spoken Symbols Wonderful One (SSWO) and Written Symbols Wonderful One (WSWO) loaded .682, .878 and .822 on the fifth factor, respectively. Since the fifth factor is connected to the procedure knowledge of wonderful one, it is called the Wonderful One factor.

Revised Model 1

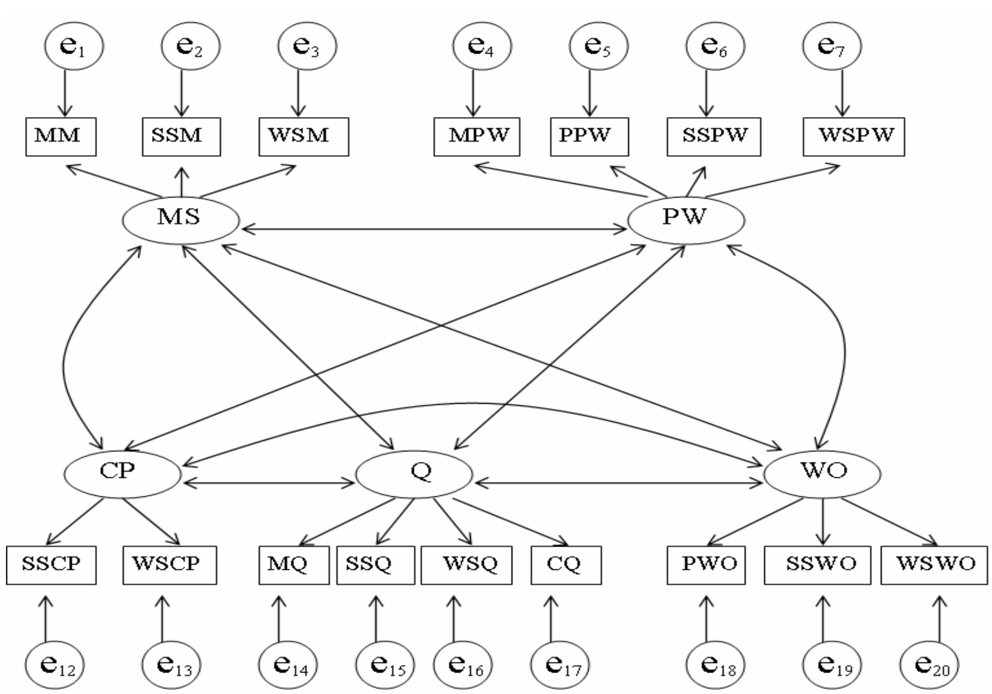

Figure 11. Measurement model of engaged instructional time.

Figure 11 shows the revised theoretical model 1 based on the results from the exploratory factor analysis. Fit indices for the revised models 1 to 4 listed in Table 19 evaluate how the model reproduced the data. The CFA results indicated the endogenous variables of part-whole, quotient, wonderful one, measurement, and cross product matched the data. These fit indices are listed in Table 19.

Table 19
Fit Indices for Teacher Measurement Model

Model	χ^2	Df	CFI	NFI	RMSEA	CI	
						Lo90%	Hi 90%
Initial model	N/A	N/A	N/A	N/A	N/A	N/A	N/A
Model 1	164.1	98	.89	.76	.11	.08	.13
Model 2	117.2	97	.97	.83	.06	.00	.10
Model 3	87.4	96	1.00	.87	.00	.00	.06
Model 4	65.8	96	1.00	.91	.00	.00	.00

Note: N=58

The chi-square for the revised model 1 was 164.1 with 98 degrees of freedom, which gave a ratio of 1.67. This ratio is less than 3 and is acceptable (Kline, 2005). The Comparative Fit Index (CFI) was .89, which is close to the cut off criterion of .90. Thus it suggested that the proposed model closely matches the data. The *Normed Fit Index* (NFI) was .76. It suggested how close the model reproduced the data. Since Hu and Bentler (1999) recommended that the cut off criterion for NFI is .90, this NFI index did not represent a good fit. The Root Mean Square Error of Approximation (RMSEA) was .11, which is reasonable to be below .05 (Hu & Bentler, 1999). This fit index did not show a reasonable estimate of the data. Furthermore, the RMSEA does not depend on the sample size, so .11 indicated a bad fit of the data.

The error variance of the observed variable “spoken symbols quotient” correlated with the latent variable cross product , which reflected part of the quotient concept that solving $\frac{1}{2} = \frac{?}{6}$, is to first compute the product of 1 times 6, and then to divide 2 to get 3. So the exogenous variable “spoken symbols quotient” and endogenous variable cross product actually correlated with each other. Therefore, a path between the error terms of “spoken symbols quotient” and cross product was added in the revised Model 2 which is shown in Figure 12.

Revised Model 2

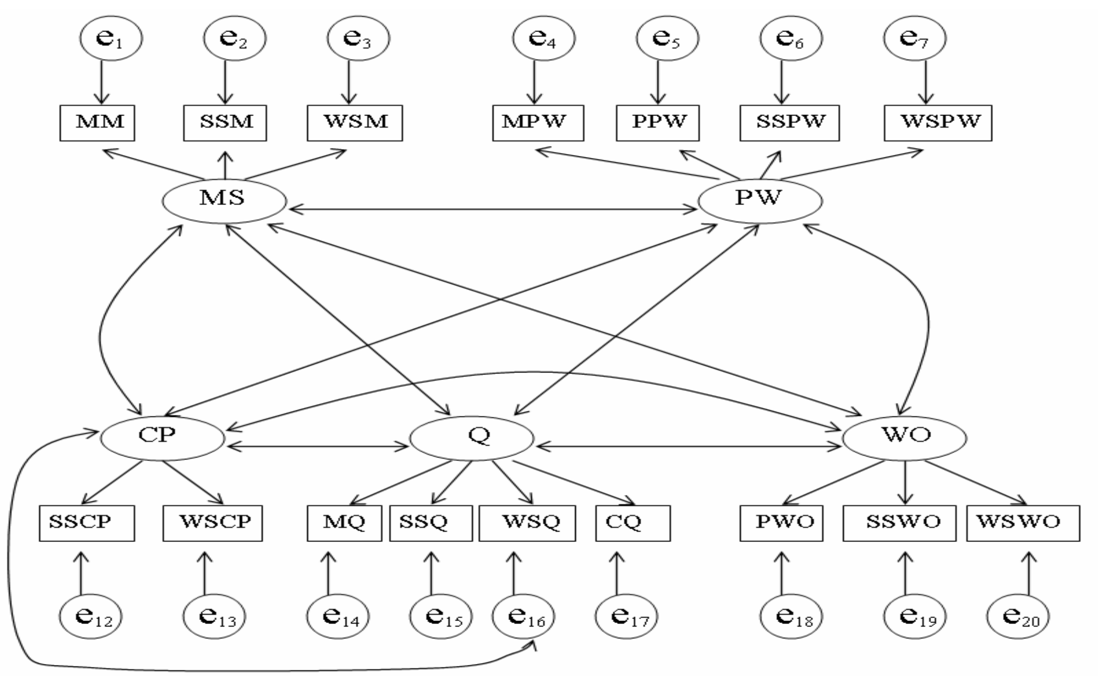

Figure 12. Revised measurement model 2 of engaged instructional time.

The chi-square for the revised Model 2 was reduced by 46.9 (164.1-117.2) with 97 degrees of freedom, giving a ratio of 1.21, which is acceptable (Kline, 2005). The *Comparative Fit Index* (CFI) was .97, meeting the cut off criterion of .90. Thus it suggested that the proposed model closely matched the data. The *Normed Fit Index* (NFI) was .83, which demonstrated that to what extent the model reproduced the data. Because .83 is much less than .95, this NFI is not an acceptable fit (Hu & Bentler, 1999). The *Root Mean Square Error of Approximation* (RMSEA) was .06 and barely met the cut off criterion of .05 suggested by Hu and Bentler (1999), and it showed a reasonable estimate of the data.

The error variances of the exogenous variable "manipulatives quotient" correlated with the variances of exogenous variable "written symbol quotient". It matched the literature that the written symbols were correlated with manupulatives in terms of quotient because the verbal communication of quotient went along the manipulatives in CMP textbook, so a path from e14 to e16 was added as suggested by modification indexes. Revised Model 3 is shown with this modification in Figure 13.

Revised Model 3

The chi-square for the revised model 3 was reduced by 29.8 from model 2 (117.2-87.4) with 96 degrees of freedom, giving an acceptable ratio of 0.91 (Kline, 2005). The *Normed Fit Index* (NFI) was .87, and it did not meet the .90 cut off criterion. So it does not reflect a good estimate of the proposed model with the independent model where no variables were correlated with each other. The *Comparative Fit Index* (CFI) was 1.00, and it met the .90 cut off criterion. Thus it suggested that the proposed model closely matched the data. The *Root Mean Square Error of Approximation* (RMSEA) was .00 and it met the cut off criterion of .05, which showed that the data was reasonably reproduced.

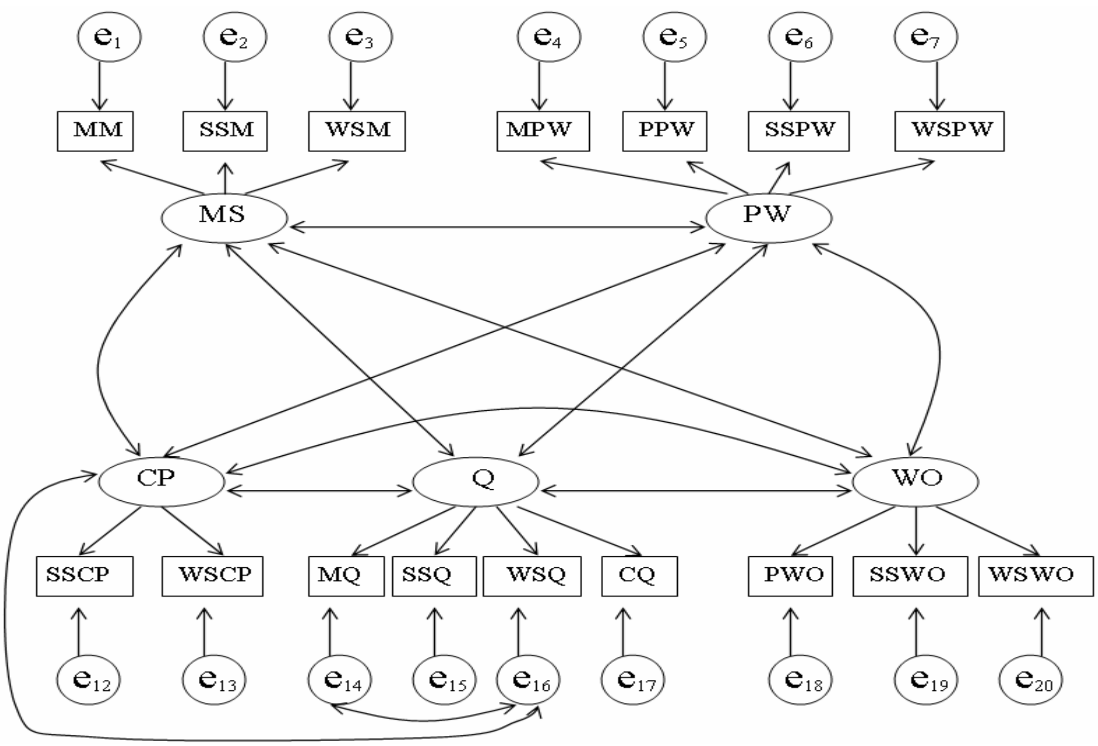

Figure 13. Revised measurement model 3 of engaged instructional time

The error variances of the exogenous variable “written symbols measure” and the error variance of exogenous viable “manipulatives part-whole” correlated to each other, because one of the textbooks asked students to use “fraction strips” to measure the funds raising represented by a thermometer (Lappan et al., 1998, p. 19). So a path from e3 to e4 was added as suggested by the modification indexes and was constrained at a negative value to obtain revised model 4 for the simple reason that given a certain amount of time, if teachers spend more time using manipulatives to demonstrate part-whole sub-construct, then the time spent on the "written symbolic measure" is automatically shortened. Another reason to constrain the path between e3 and e4 is to improve the estimates of the parameters, otherwise it would be biased. And the model is shown in Figure 14.

Revised Model 4

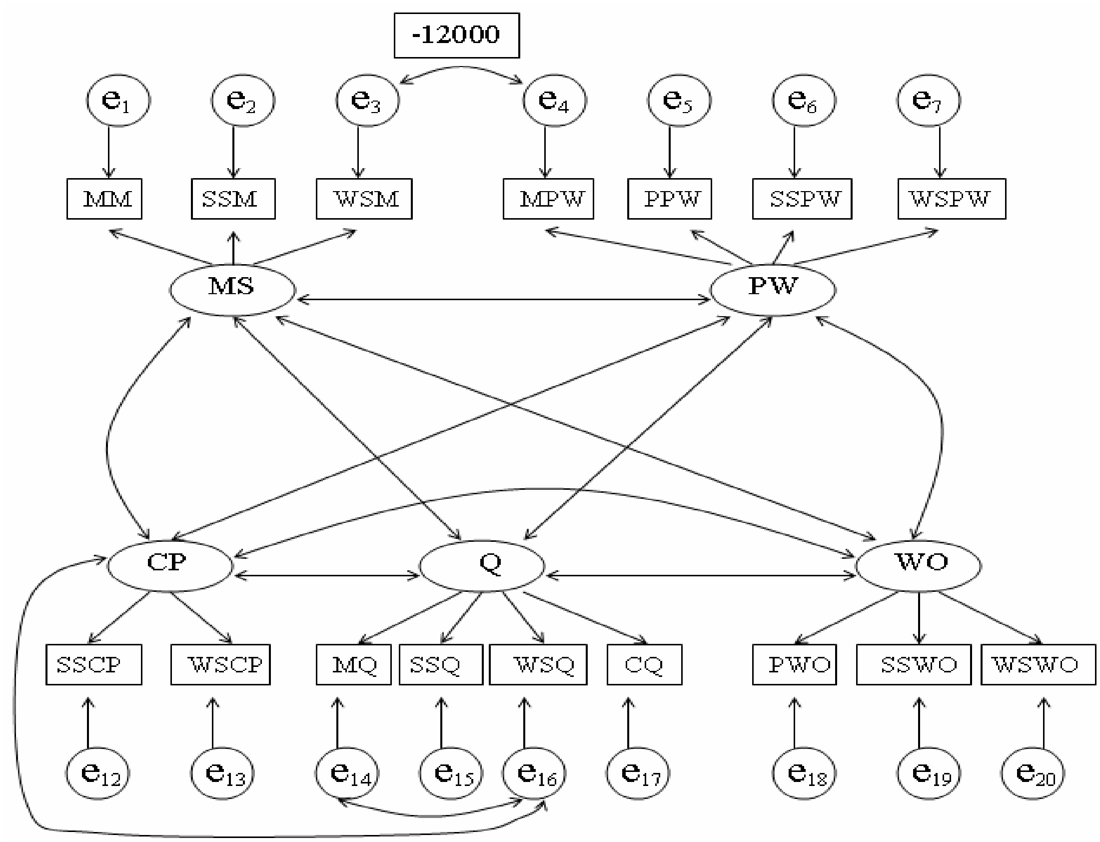

Figure 14. Revised measurement model 4 of engaged instructional time.

The chi-square for the revised Model 4 was reduced by 21.6 from model 3 (87.4-65.8) with 96 degrees of freedom, resulting a ratio of 0.69. This ratio indicates that this is an acceptable measurement model (Kline, 2005). The *Normed Fit Index* (NFI) was .91 comparing with a measure where a perfect fit is 1. The NFI indicated the variation between the proposed model and the independent model where no variables are correlated to each other. A measure of .91met the .90 cut off criterion and it showed that proposed model is likely to be stable. The *Comparative Fit Index* (CFI) was 1.00 and it met the .90 cut off criterion. Basically in order to conclude that the proposed model closely matches the data, CFI should be bigger than .90. So this CFI index represented a fairly good fit. The *Root Mean Square Error of Approximation* (RMSEA) was .00 and is less than the normally agreed .05 cut-off criterion, illustrating that the proposed model yield a fairly good estimation of the data.

The regression estimates for Model 4 are shown in the Table 20. The bolded numbers mean that the loadings are statistically significant. According to Anderson and Gerbing (1988) convergent validity was achieved based on the statistical significance revealed by the critical ratio.

It appears that the latent variable of part-whole was mainly predicted by SSPW (spoken symbols) and WSPW (written symbols). The regression weight is 1.00 and .80 respectively. The latent variable of measurement was mainly predicted by manipulatives representation, spoken symbols and written symbols. The regression weights are .90, .98, .99 respectively. The latent variable quotient was predicted by spoken symbols and written symbols. The regression weights are .78 and 1.00 respectively. The latent variable multiplication by one was mainly predicted by spoken symbols and written symbols. The regression weight is 1.00 and .71, respectively. The latent

Table 20

Regression Estimates for Model 4 of Teacher Instructional Representations

PATH	Stand. Estimate	Estimates	Stand. Error	Critical Ratio
IMPW ← PW	0.49	0.62	0.12	5.26
IPPW←PW	0.47	0.40	0.10	4.03
SSSPW ← PW	**1.00**	1.00		
SWSPW← PW	**0.80**	0.69	0.07	10.09
I MM ← MS	**0.90**	0.33	0.02	15.03
SSSM ← MS	**0.98**	1.12	0.04	30.31
SWSM ← MS	**0.99**	1.00		
I M Q ← Q	0.36	0.03	0.01	2.91
SWSQ ← Q	**0.78**	0.84	0.09	9.40
SSSQ ← Q	**1.00**	1.00		
SCQ ← Q	0.25	0.03	0.02	1.95
IPWO ← WO	0.40	21.70	6.83	3.18
SSSWO ←WO	**1.00**	335.61	31.43	10.68
SWSWO←WO	**0.71**	318.89	51.33	6.21
SSSCP←CP	**0.86**	1.13	0.09	12.72
SWSCP ← CP	**1.00**	1.00		
PW ↔ MS	0.05	5743.49	16629.76	0.35
Q ↔ WO	-0.18	-39.03	27.96	-1.40
Q ↔ CP	-0.09	-650.33	921.74	-0.71
PW ↔ Q	-0.11	-8638.99	10475.09	-0.83
MS ↔ WO	-0.13	-42.21	44.47	-0.95
WO ↔ CP	0.21	6.81	2.42	2.81
PW ↔ CP	0.03	366.93	905.37	0.41
MS ↔CP	-0.01	-55.08	822.18	-0.07
PW ↔ WO	-0.06	-20.23	48.88	-0.41
MS ↔ Q	-0.09	-6666.64	9545.42	-0.70

variable cross product was predicted by spoken symbols and written symbols. The regression weights are .86 and 1.00, respectively. Therefore, the symbolic representations were the best predictors of classroom instructions. In order to determine the relationships between each of the latent variables, a correlation between each of the variables was also examined and listed in Table 20.

The only significant relationship between any of the latent variables was between wonderful one and cross product. The estimate of the correlation between these two variables is 6.81, which is significant at the .01 level. This result indicated that teachers only make connections between wonderful one and cross product in terms of engaged instructional representations, but not with any other latent constructs.

Measurement Model of Students' Written Representations

Table 21
Definitions of Written Representation Variables

Variable Name	Description
U5	Picture Part-whole (question14)
U6	Written Symbol Part-whole (question14)
U7	Written Symbol Quotient (question 15)
U8	Written Symbol Multiplication by One (question 15)
U11	Written Symbol Quotient (question 16e)
U12	Written Symbol Multiplication by One (question 16e)

Students' posttests were analyzed to identify the types of representations they used. Because of the void of two latent constructs of measurement and cross product, a three-latent-construct measurement model of students' representations was developed. Table 21 shows the observed variables and their brief definition that were used to describe students' written representations. The types of representations used in the students' measurement models contained three open ended questions in the posttests. Question 15 asked students to explain why $\frac{4}{5} = 0.8$. Both U7 and U8 are answers to question 15, which indicated a correlation between these two exogenous variables.

The overall fit indices are a measure that reflects how well the model reproduced the data. Two measurement models were compared and the fit indexes were reported in Table 22. The results generated by CFA indicated that part-whole, quotient and wonderful one existed in students' written representations.

Table 22
Fit Indices of Students' Written Representations on Posttests

Model	χ^2	Df	CFI	NFI	RMSEA	CI	
						Lo 90%	Hi 90%
Model 1	14.39	6	.95	.88	.08	.03	.14
Model 2	3.48	5	1.00	1.03	.00	.00	.08

Model 1

The chi-square is 14.39 with 5 degrees of freedom, and the ratio is 2.40 (less than 3), and this is an acceptable measurement model according to Kline (2005). The *Comparative Fit Index* (CFI) was .95 and met the cut-off criterion of .90. This indicated the proposed model closely matched the data. So this CFI index represented a good fit. The *Tucker-Lewis index* (TLI) was .88. It illustrated how close the model reproduced the covariance matrix and it penalized for model complexity. Hu and Bentler (1999) recommended that the cut off criterion for TLI is .95. So the TLI index represented a poor fit. The *Root Mean Square Error of Approximation* (RMSEA) was .08 and it is greater than the cut-off criterion of .05. This index of .08 with 90 percent of confidence interval range from .03 to .14 indicating a bad fit of the data.

The error variances of observable variable U7 (written symbols quotient) and U8 (written symbols wonderful one) are correlated with each other, because high correlations between observable variables U7 and U8 could lead to correlations between the error terms, a path between U7 and U8 was added as suggested by modification indexes to obtain Model 2, otherwise, the estimates of the parameters would be biased.

Model 2

In Model 2, the chi-square went down by10.91 (14.39-3.48), and 5 degrees of freedom were left. The ratio of the chi-square and degrees of freedom is 2.18 (less than 3) indicated that this is an acceptable measurement model (Kline, 2005). The *Comparative Fit Index* (CFI) was 1.00 and met the cut-off criterion of .90, thus the proposed model closely matched the data. So the CFI index represented a perfect fit. The *Tucker-Lewis index* (TLI) was 1.03 and it illustrated how close the model reproduced the covariance matrix and it penalized for model complexity. Since Hu and Bentler (1999) recommended that the cut off criterion for TLI is .95, this TLI index represented an acceptable fit. The *Root Mean Square Error of Approximation* (RMSEA) was .00 and is less than .05 cut off criterion. This index of .00 with 90 percent confidence interval ranged from .00 to .08 indicating a reasonable fit of the data.

CFA model of students' written representations as shown in Figure 15 indicated that the latent variables of part-whole, quotient, and wonderful one existed in students' posttests. Factor one is the part-whole factor, predicted by written symbolic representation and picture representation of question 14 on students' posttests. Factor two is the quotient factor, represented by written symbols in question 15 and question 16 of the posttests. Factor three is the wonderful one, which is a latent factor represented by written symbols in both question 15 and 16 of posttests.

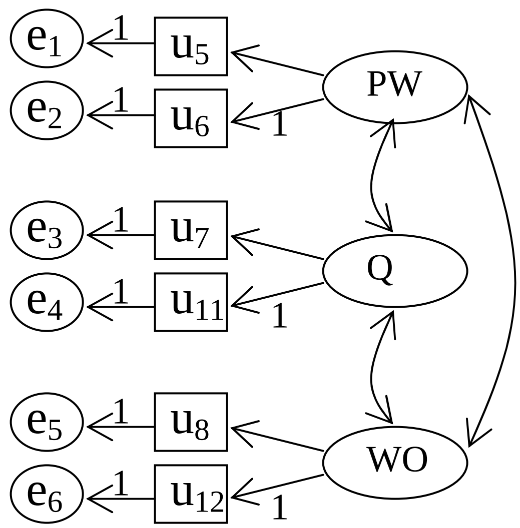

Figure 15. Measurement model of students' representations with three latent constructs

The regression estimates are shown in Table 23. The bolded numbers mean that the loadings are statistically significant. According to Anderson and Gerbing (1988) convergent validity was achieved based on the statistical significance revealed by the critical ratio.

The latent variable of part-whole was mainly predicted by SSPW and PPW, written symbols and pictures. The regression weights are 1.00 and .94, respectively. The latent variable quotient was predicted by written symbols of question 15 and question

Table 23
Regression Estimates for Students' Written Representations

Path	Standard Estimates	Standard Error	Critical Ratio
u_5 ← PW	1.00	.00	.00
u_6← PW	**.94**	**.16**	**5.78**
u_7 ← Q	1.00	.00	.00
u_{11} ← Q	.86	.51	1.68
u_{12} ← WO	1.00	.00	.00
u_8 ← WO	**.79**	**.34**	**2.32**
PW ↔ Q	**.13**	**.06**	**2.23**
WO ↔ Q	.02	.03	.74
PW ↔ WO	**.23**	**.06**	**3.68**

Note: N=213

16 respectively. The regression weights are 1.00 and 0.86 respectively. The latent variable wonderful one is mainly predicted by written symbols of answers of question 15 and 16, and the regression weights are 1.00 and .79, respectively. Therefore, the written symbolic representations and the picture representations are mainly used by the students among the three latent variables in posttests.

A statistically significant relationship was found between the latent variable part-whole and quotient. The correlation coefficient is .13 ($p<.05$). A statistically significant relationship between latent variables part-whole and wonderful one was also found, with a correlation coefficient of .23 ($p<.01$). It indicated that students who used written symbols to represent the construct of part-whole also tended to use written symbols to represent the construct quotient , whereas students who represented the construct part-whole symbolically also tended to represent the construct of wonderful one symbolically. No statistically significant correlation was found between quotient and wonderful one.

Research Question 3

What is the relationship between the structures of representations students use and their achievement? That is, how do students use representations that reflect measure, part-whole, quotient, multiplication by one and cross product and how is this reflected in their achievements on learning fractions, decimals, and percents?

Structural equation modeling with continuous factor indicators was used to investigate the relationship between the types of representation students used and their achievements on fraction, decimal and percents. Two models were developed. The fit indices reflect whether the model interpreted the data, how well the model fits the data, and the results are reported in Table 24.

Table 24
Fit Indices of Student's Written Representations and Achievements

Model	χ^2	Df	CFI	NFI	RMSEA	CI	
						Lo 90%	Hi 90%
Model 1	39.39	10	0.92	0.83	0.12	.08	.16
Model 2	8.51	9	1.00	1.00	.00	.00	.07

Model 1

The chi-square was 39.39 with 10 degrees of freedom yielding a ratio of 3.94, however, it failed to meet the criterion of less than 3 (Kline, 2005). The *Comparative Fit Index* (CFI) was .92 and met the cut-off criterion of .90 suggested by Hu and Bentler (1999), so the proposed model closely matches the data. The *Tucker-Lewis index* (TLI) was .83. It penalized for model complexity. Hu and Bentler (1999) recommended that the cut off criterion for TLI is .95. So the TLI index represented a poor fit. The *Root Mean Square Error of Approximation* (RMSEA) was .12 and was much larger than the cut-off criterion of .05, which indicated a poor fit of the data.

Based on similar reasons provided for students' measurement model, a path between error variances of U7 (written symbols quotient) and U8 (written symbols multiplication by one) was added in order to improve the parameter estimates. Thus a revised model two was developed, which is included as Figure 16.

Revised Model 2

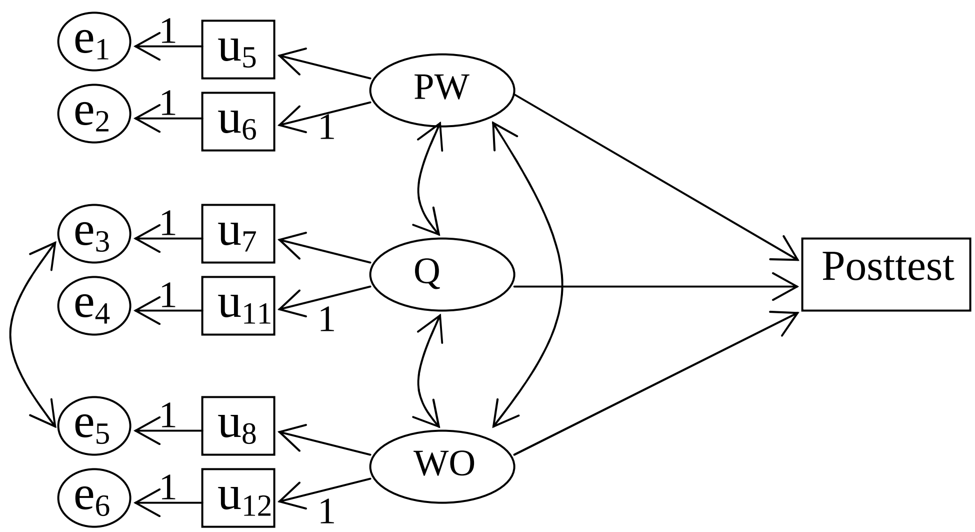

Figure 16. Revised model 2 of student representations and achievements.

Chi-square was reduced by 30.88 (39.39-8.51) for 9 degrees of freedom, generating a ratio of 0.48 which is an acceptable SEM model (Kline, 2005). The *Comparative Fit Index* (CFI) was 1.00. Generally if CFI is greater than .90, then the proposed model closely matches the data. So the CFI index represented a perfect fit. The

Tucker-Lewis index (TLI) was 1.00 and met the .95 cut-off criterion recommended by Hu and Bentler (1999). So the TLI index represented a perfect fit. The *Root Mean Square Error of Approximation* (RMSEA) was .00 and met the .05 cut-off criterion. It is independent of sample size, this index of .00 indicated a good fit of the data. Model 2 is shown in Figure 16.

Table 25
Regression Estimates for Students' Representations and Achievements

Path	Standard Estimate	Standard Error	Critical Ratio
posttest ← PW	1.32*	.66	2.01*
posttest ← Q	11.13**	3.12	3.57**
posttest← WO	11.95**	2.37	5.05**
PW↔Q	.06	.03	1.85
PW ↔ WO	.24**	.06	3.82**
Q↔WO	.01	.02	.57

Note: N=213

The SEM results as shown in Table 25 indicated that all three latent variables: part-whole, quotient , and wonderful one significantly predicted the students' posttest scores. It also showed that the estimated standard regression weights of latent variables of part-whole, quotient and multiplication by one were 1.32 ($p<.05$), 11.13($p<.01$), and 11.95 ($p<.01$) respectively. It meant that an increase of one standard score on part-whole resulted in an increase of 1.32 standard score on students' posttests. An increase of one standard score on quotient resulted in an increase of 11.13 score on students' posttests. An increase of one standard score on wonderful one resulted in an increase of 11.95

score on students' post tests. Factor wonderful one contributed most to students' posttest scores among the three factors in magnitude, and part-whole contributed least.

As shown in Table 25, the correlations between the latent variables in the model varied from the students' measurement model. Correlation between the part-whole and the quotient is only marginally significant (C.R. =1.85). Correlation between the part-whole and wonderful one was still significant at the .01 level and it matched the measurement model of the students' written representations.

Research Question 4

What is the relationship between the quality and quantity of instructional representations and student's learning of fractions, decimals, and percents? Specifically, what is the relationship between the quality of teachers' instructional representations, different forms of students' written representations, and student achievements?

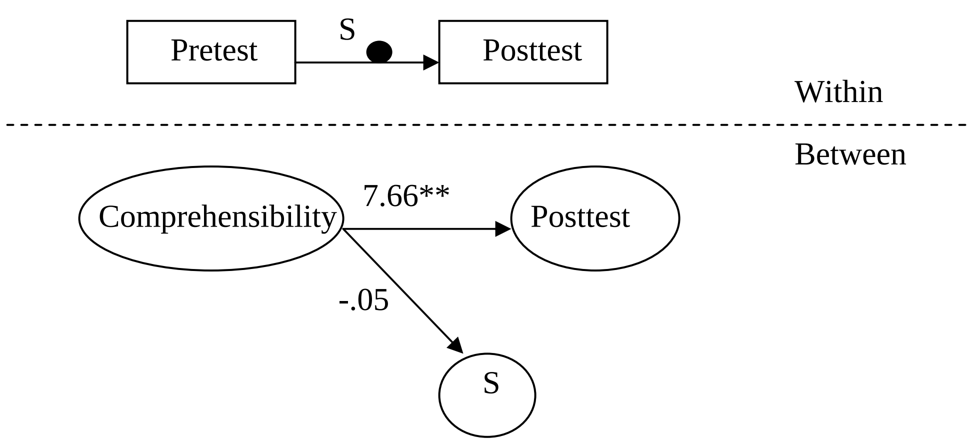

Figure 17. A two-level regression model.

There were two different measurements of teachers' instructional representation. One was the quality of teachers' instructional representation, and the other is the quantity of teachers' instructional representation. Research indicated that teaching quality is the key factor that influenced students' performance (Aronson, Zimmerman, & Carlos, 1998). So the relationship between the quality of instructional representations and students' achievement was analyzed by using a two level regression analysis as indicated by Figure 17. However, the total quality scores were not used because the model failed to converge. Neither did the accuracy nor the connection score work. Therefore, only comprehensibility scores were used in this two-level regression model.

Akaike Information Criterion (AIC) indicated the variation of the covariance matrix between the proposed model and the data. It penalized model complexity. The closer the AIC is to zero, the better the fit of the model. This model's AIC is 1957.41, However, since this is the simplest model given the variables, the comparison between alternative models cannot be conducted. *Bayesian Information Criterion* (BIC) used the log of a Bayes factor, and it considered both sample sizes and model complexity. The BIC in this model is 1377.47, and the Adjusted BIC is 1987.67.

Results indicated that the random slopes on students' scores of posttests on pretests were not statistically significant within classrooms. Neither was the random slope on the comprehension score of instructional representation. However, at the classroom level the regression weight of students' posttests scores on the comprehension scores of instructional representation is 7.66 ($p<.01$), which meant that the higher score of the comprehension of instructional representation, the higher the students' posttest scores tended to be. An increase of one standard score of the comprehension on instructional representation resulted in the increase of 7.66 standard scores of the students' posttests.

The second question of interest is the relationship between the latent constructs of instructional representation and students' posttest scores. A two-level measurement model of teachers' instructional representations on students' cogitative representations failed to converge. This is illustrated in figure 18.

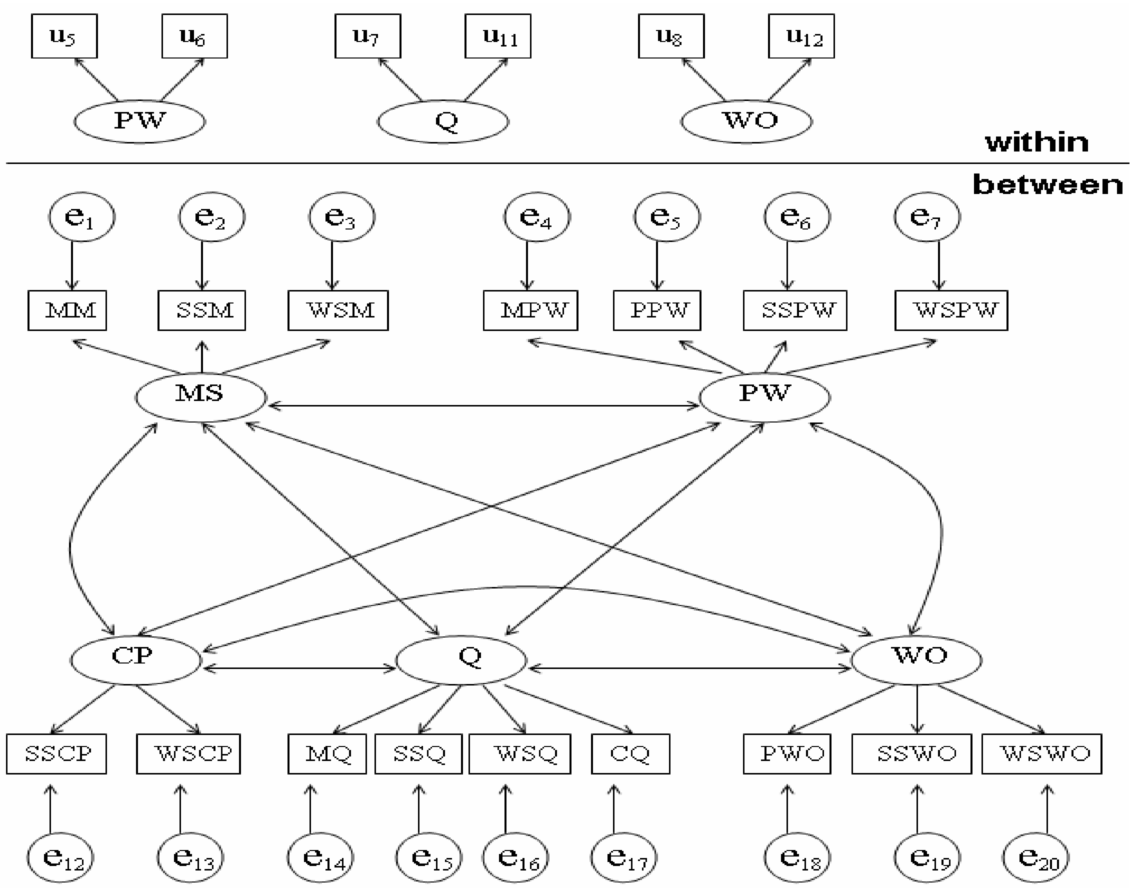

Figure 18. A two-level model of teacher and students' representations.

A factor analysis was run to generate an instructional quantity factor score on all of the 16 observed variables. A two level regression model described in Figure 19 was proposed to investigate the relationship between the teachers' teaching quantity and students' achievement, considering the effect of students' pretests on their posttests. However, the model was not acceptable because of a negative residual variance on students' posttests' scores.

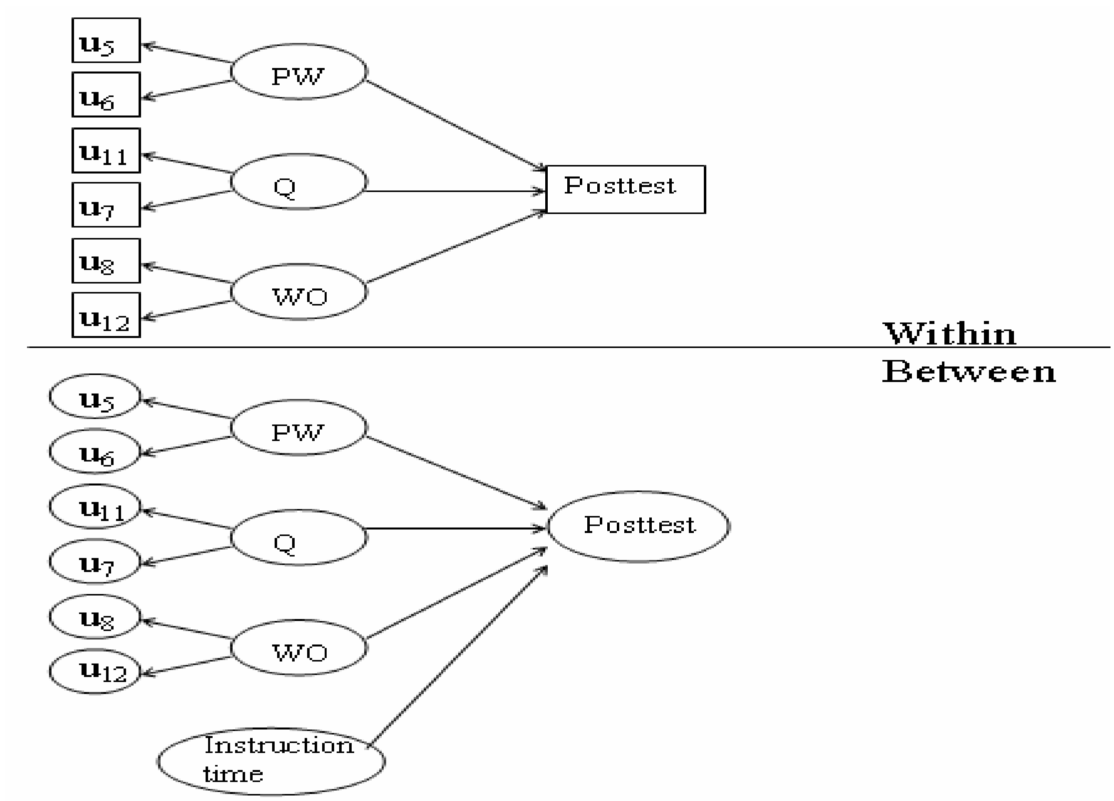

Figure 19. A two-level model of the effect of teaching quantity on students' pretests and posttests.

Next, a two-way Analysis of Variances (ANOVA) was applied three times in order to investigate the relationship between the quantity of instructional representations, written representations used by the students and the students' specific achievements on item 14, 15 and 16e. Teachers were categorized into two groups: one including three teachers who did not use real world representations at all when teaching fractions, decimals and percents; the other including eleven teachers who used real world representations. Also students' open ended questions were categorized into different ways according to what kind of constructs (part-whole, quotient, and wonderful one) they preferred to use in terms of answering the short constructed questions 14, 15 and 16e.

Several issues should be addressed before the results of the two-way ANOVA are presented. One of the assumptions of ANOVA is the homogeneity of variances. Levene's tests for all three ANOVA tests in this study are all significant. However, violation of homogeneity of variances of ANOVA as the dependent variable generated a conservative α, because the variances and sample sizes are paired as Glass and Hopkins (1996) suggested.

Question 14

A statistically significant main effect of teacher category ($F= 4.97$, $p<.05$) as well as the main effect of different representation types students used ($F=42.08$, $p<.01$) were found in question 14 as indicated by Table 26. Students who received all five types of representations significantly differ from those who did not use all five types of representations. The Eta-square is .02 which meant the teacher difference explained 2%

of the total variances. Another main effect is the students' representations. Students who used written symbol representations of part-whole differed significantly from those students who did not receive the representations of part-whole, where the Eta-square is .17 which meant the students' difference in representations explained 17% of the total variances.

Table 26

Tests of between-Subjects Effects on Dependent Variable: S14Post

Source	Type III Sum of Squares	Df	Mean Square	F	P	Eta Square	Observed Power
Corrected Model	97.15		32.38	26.51	.00	.28	1.00
Intercept	70.53		70.53	57.74	.00	.22	1.00
TEACHCAT	6.07		6.07	4.97	.03	.02	.60
Q14	51.40		51.40	42.08	.00	.17	1.00
TEACHCAT * Q14	1.52		1.52	1.25	.27	.01	.20
Error	255.29		1.22				
Total	1161.00						
Corrected Total	352.43						

Note: a Computed using alpha = .05
b R Squared = .276 (Adjusted R Squared = .265)

Question 15

A statistically significant main effect of teachers' category (F=17.13, p<.01) as well as the students' difference in the written representations (F=19.29, p<.01) was found in posttest question 15 as indicated by Table 27. Students who received all five types of representations (real world representations, manipulative representations, picture representations, spoken symbolic representations, written symbolic

Table 27
Tests of between-Subjects Effects on Dependent Variable: S15Post

Source	Type III Sum of Squares	Df	Mean Square	F	P	Eta Square	Observed Power
Corrected Model	54.03	7	7.72	20.09	0.00	0.41	1.00
Intercept	59.01	1	59.01	153.62	0.00	0.43	1.00
TEACHCAT	6.58	1	6.58	17.13	0.00	0.08	0.99
Q15	22.23	3	7.41	19.29	0.00	0.22	1.00
TEACHCAT * Q15	0.54	3	0.18	0.47	0.70	0.01	0.14
Error	78.74	205	0.38				
Total	332	213					
Corrected Total	132.77	212					

*Note***:** a Computed using alpha = .05
b R Squared = .407 (Adjusted R Squared = .387)

representations) significantly differed from those who did not receive all five forms of representations. The Eta-square was .08, which meant the teacher difference explained 8% of the total variances. Another main effect is the students' representations. Students who used written symbol representation of nothing, part-whole, wonderful one and quotient significantly differ from each other, and the Eta-square was .22 which meant the students' difference in representations explained 22% of the total variances. A post hoc test ("Scheffe") was run in order to investigate the further differences. Table 28 showed that students who represented nothing were not significantly different from students who used symbolic representations of part-whole (p=.56). However, they were significantly different from the students who used quotient(p<.01), also significantly differs from students who used wonderful one (p<.01). Students who represented part-whole is not significant different from the students who represented (p=.08) and wonderful one (p=.23). Students who represented quotient were not significantly different from the students who represented wonderful one (p=.69).

Table 28
Multiple Comparisons on Dependent Variable: S15Post Scheffe

		Mean			95% Confidence Interval	
		Difference			Lower	Upper
(I) Q15	(J) Q15	(I-J)	Std. Error	P	Bound	Bound
0.00	1.00	-0.35	0.24	0.56	-1.03	0.34
	2.00	-1.02	0.12	0.00	-1.37	-0.67
	3.00	-0.86	0.10	0.00	-1.13	-0.59
1.00	0.00	0.35	0.24	0.56	-0.34	1.03
	2.00	-0.67	0.26	0.08	-1.40	0.05
	3.00	-0.51	0.25	0.23	-1.21	0.18
2.00	0.00	1.02	0.12	0.00	0.67	1.37
	1.00	0.67	0.26	0.08	-0.05	1.40
	3.00	0.16	0.13	0.69	-0.21	0.52
3.00	0.00	0.86	0.10	0.00	0.59	1.13
	1.00	0.51	0.25	0.23	-0.18	1.21
	2.00	-0.16	0.13	0.69	-0.52	0.21

*Note***:** Based on observed means.
* The mean difference is significant at the .05 level.

Question 16e

Statistically significant main effects for both teachers' difference (F=6.69, p<.05) and the students' types of representations (F=14.60, p<.01) were found in question 16e (Table 29). And a statistically significant interaction was also found between the teachers' categories and the students' types of representations (F= 4.12, p<.01) as indicated by Table 29. The partial Eta-square was .057, which meant the interaction explained 5.7% of the total variances. Students who received all five types of representations in class (real world, manipulatives, pictures, spoken symbolic representation, and written symbolic representations) significantly differed from those who did not receive all five types of representations during instruction. The Eta-square was .03, which meant the teacher difference explained 3% of the total variances. Another main effect was the students' representations. Statistical significance was found among students who used written symbol representations of nothing, part-whole, wonderful one and quotient with an Eta-square of .18, which meant the students' difference in representations, explained 18% of the total variances. However, since a significant interaction was found, judgment was suspended regarding the simple main effect associated with the significant interaction.

Table 29

Tests of between-Subjects Effects on Dependent Variable: S16EPost

Source	Type III Sum of Squares	df	Mean Square	F	P	Eta Square	Observed Power
Corrected Model	56.66	7	8.09	12.67	0.00	0.30	1.00
Intercept	45.62	1	45.62	71.38	0.00	0.26	1.00
TEACHCAT	4.27	1	4.27	6.69	0.01	0.03	0.73
Q16E	27.98	3	9.33	14.60	0.00	0.18	1.00
TEACHCAT * Q16E	7.89	3	2.63	4.12	0.01	0.06	0.84
Error	131.00	205	0.64				
Total	281.00	213					
Corrected Total	187.66	212					

Note: a Computed using alpha = .05
b R Squared = .302 (Adjusted R Squared = .278)

CHAPTER V

DISCUSSION AND CONCLUSIONS

This study investigated the structures of teachers' instructional representations, students' written representations, and the relationship between the instructional representations and students' written representations. First, this study examined instructional representations involved alignment with the textbook and classroom interactions using a qualitative method, followed by the descriptive statistics of the quantity and quality of instructional representations. Second, it investigated the structures of instructional representations as well as the structures of written representations. Thirdly, the relationship between students' written representations and their achievement was investigated. Last, the relationship between instructional representations and students' achievements was examined. The participants included fourteen sixth grade mathematics teachers as well as 213 sixth grade students.

This chapter discusses the major findings in the results, which addressed the four research questions.

Quantity and Quality of Instructional Representations

The results of the study showed that teachers varied in their use of instructional representations. A mixed method has been used to analyze research question one. All of the 14 teachers addressed the mathematics learning goal: "use, interpret, and compare numbers in several equivalent forms such as integers, fractions and decimal" (AAAS number atlas, 2002). The possible reason could be that the professional development workshop that they attended in summer 2001 was mainly aimed at asking teachers to identify the learning goal themselves. However, four teachers, teachers 1, 5, 6, and 9 did not follow the textbooks' suggestions to teach the learning goal. This might due to several factors, one is that teachers might have already developed their own teaching strategies so that they are not limited to the textbooks' approach. Another possible reason is that they were not familiar with the textbooks' approach so that they relied on their previous teaching experiences.

Two types of classroom interactions were also found in the study, teacher-centered and constructive classrooms. Constructivist classrooms featured less direct instruction, aimed at creating an environment to help students to learn independently. Teachers acted as facilitators to monitor the student's learning and thinking in groups using hands-on activity. In contrast, teacher-centered classrooms featured more direct instruction, did not emphasize the students' motivation and interests, and students' ability to remember the content was more important than the ability to think. Teachers in teacher-centered classroom acted as leaders of students' learning process, thus the group activities and group learning are restricted (Chall, 2000). The findings of this study are able to identify these features proposed by Chall (2000) through investigating the classroom interactions, especially the questions the teacher posed during instruction. Teacher-centered classrooms usually had no activities, or very short activities. More emphasis was placed on factual questions than probing questions in order to ensure that students remembered the mathematics content. Teacher-centered classrooms revealed that memorizing the mathematics was more important than thinking, which is similar with the findings by Chall (2000). Among five out of fourteen teachers who belong to the teacher-centered approach, three of them spent a short amount of time on hands-on activities, and four of them posed higher order questions somehow aimed at promoting student's thinking. Because there was not enough time for students to work on activities and answer the higher order questions, they were categorized as teacher centered. However, the phenomenon that they included some student-centered approach indicated that these traditional teachers were influenced by the student-centered learning theory somehow and were incorporating some of those approaches, which is similar as Chall's findings(2000).

Tyson and Woodward (1989) and Woodward and Elliot (1990) found that 75% to 95% of instruction came from the textbooks. Hudson, McMahon, and Overstreet (2002) reported in their study that 36% of the instructional time was spent on whole class lecture and discussion, 11% of instructional time was spent on hands-on activity or manipulatives. This study revealed that 9.9% time was spent by using manipulatives representations, which is a little bit lower than the Hudson et al. report. This might be because some traditional teachers did not use manipulatives.

Textbooks seemed to be a major factor that influenced instructional representations. Those teachers who followed the textbooks Middle School Math *Thematics* and Connected Mathematics (Bits and Pieces unit) suggestions were actually constructivist classrooms, while those teachers who did not follow the textbooks' suggestions were teacher-centered. However, both teachers who used the textbook, Mathematics: Applications and Connections were teacher-centered classrooms. This is congruent with the textbooks analysis that was done by AAAS that Connected

Mathematics and Middle School Math *Thematics* ranked higher than Mathematics: Applications and Connections (AAAS, 2000). Generating more constructive teacher-student interactions requires the support from a highly ranked textbook. Textbooks also contributed to the quantity of instructional representations; teachers who used CMP tended to have more real world representations than the rest of the teachers if they followed the textbook's suggestions. This is congruent with other research suggesting that highly ranked textbooks influenced the teachers' content knowledge and instructional knowledge (Kulm & Capraro, 2004; Reys et al., 2003). For those teachers who have already used a highly ranked textbook, creating closer alignment with the textbooks approaches should be addressed in order to conduct a more constructive learning environment for the students.

It was found that teachers do not vary a lot in the accuracy criteria. Except for a few teachers, most teachers used instructional representations correctly to address the learning goal. Comprehensibility scores and connections varied considerably. Only six teachers out of total fourteen teachers fully addressed the comprehensibility criterion. The comprehensibility score is also related to the textbooks ranking, which also supported Trafton's (et al., 2001) findings that standard-based textbooks developed ideas completely and deeply. Moreover, only two teachers fully addressed the connections criterion through all lessons that were videotaped, which may explain the phenomenon that middle school students' have difficulty in translating between different forms of representations that is repeated reported by the research (Bay, 2001; Cramer et al., 1997; Hiebert, 1985; Post, 1981; Post et al., 1985; Wearne & Hiebert, 1986).

Structures of Instructional Representations and Students' Written Representations

The results showed five sub-constructs, part-whole, measure, quotient, wonderful one and cross product in the teachers' instructional representations. This study was able to empirically identify three sub-constructs (part-whole, measure, and quotient) proposed in the literature (Behr, Lesh, Post, & Silver, 1983; Freudenthal, 1983; Kieren 1976, 1988; Lamon, 2001) and two other sub-constructs, multiplication by one and cross product (AAAS, 2002) in teachers' instructional representations. However, the students' work showed that only three sub-constructs reflected in their representations, part-whole, quotient and multiplication by one. An interesting phenomenon is that both instructional representations and students' written representations converge on the mathematics sub-constructs of fractions, rather than the representational forms proposed as real world, manipulatives, pictures`, spoken symbolic representations and written symbolic representations (Lesh, 1979), even though the sample sizes are small. A possible reason could be that in public schools, many teachers emphasize merely teaching the mathematics learning goal rather than developing students' understanding by using different forms of representations. This may be done in part to the pressures generated by the standards-based assessments.

Even though teachers in this study used measure sub-construct in their instructional representations, no students used it in their written representations, this may be because the measure sub-constructs are used most often with manipulatives, for example, fraction strips. In a paper and pencil test, students would not have access to such kind of manipulatives and it is plausible that they would therefore not choose this form of representations on the exam's open-ended questions. Another possible reason could be that measure is a more difficult construct to use in terms of answering questions relating to how to convert fractions into decimals and then percents.

It is not uncommon for symbolic representations to dominate classroom instructions. Except for the measurement construct, the other four constructs, (part-whole, quotient, multiplication by one and cross product) were all strongly predicted by two symbolic representations; spoken symbols and written symbols. Regardless of the research suggestion that teachers use real world examples and hands-on activities, spoken symbols (mainly English language) are the most common way to convey knowledge in current public schools because of the critical power of language. Part of the reasons might be, as pointed out by Lesh, Post and Behr (1987), that in order to establish a link between different forms of representations, language (spoken symbolic representations) functioned as reconciliation. Moreover some other types, such as real world representations, were not used at all by some teachers. Significant connections were not found between the latent constructs except multiplication by one and cross product within the five sub-constructs. One possible reason could be that the teachers had considerable pressure to conform their practices in an effort to help students earn high scores in state-mandated exams, so getting the content delivered to the students already takes a significant portion of available time so that not enough time is left to make connections. In addition, another possible reason could be that teachers do not have enough pedagogical content knowledge to generate connections between the sub-constructs.

The results from students' written representations on posttests showed significant correlations between the latent constructs of part-whole and quotient, part-whole and multiplication by one. The discrepancy revealed that even though there were no correlations between the teachers' engaged instructional time in delivering the four sub-constructs of fractions, the connections between these sub-constructs were so strong that they were reflected in the students' written representations. This result empirically verified that the part-whole relationship is a critical concept, as proposed in the literature (e.g., Behr et al., 1983; Post, Behr, & Lesh, 1982).

Relationship between the Forms of Representations Students Use and Their Achievements

The results of this study suggested that all three latent variables -- part-whole, quotient and multiplication by one -- significantly predicted the students' posttest scores. It also demonstrated that students' written representations on these three sub-constructs significantly predicted students' achievement on the posttests. Significant correlations were also found between part-whole and multiplication by one. This study provided empirical data to support the research findings reported by Lesh, Post and Behr (1987) that translations between different forms of representations are critical in developing students' understanding. part-whole was the basic sub-construct in developing the ideas of fractions (Post, Behr, & Lesh, 1982). This study was not able to generate a similar representation model as that referred by Lesh, Post and Behr (1987) using SEM techniques. However, further deconstruction of the mathematics sub-constructs revealed that three major sub-constructs of part-whole, quotient , and multiplication by one, and all three factors significantly predicted students' achievements.

Relationship between Instructional Representations and Students' Achievements

Research in the literature indicated that teachers and teaching were the major factors related to students' mathematical achievements (Mullis et al., 2000; Stigler & Hiebert, 1999; TIMSS, 1999). Teaching quality and quantity were the two major foci.

This study supported the idea that effective teaching required teaching for understanding by investigating the instructional quality: accuracy, comprehensibility and connections. However, only comprehensibility was identified as the predictor of students' performance. A two level regression analysis aimed at investigating the relationship between the quality of instructional representations and students' achievements indicated that the teachers' instructional quality, mainly the comprehensibility of representations, significantly predicted students' posttest scores. The SEM model revealed a significant path coefficient value between the two variables, which was congruent with the suggestions proposed by Aronson, Zimmerman, and Carlos (1998) that quality of teaching is the key factor influencing students' performance.

An interesting finding was that students' pretest scores were not correlated with their posttest scores. This may due to the fact that students did not know very much about the concept of "fractions" at the beginning year of sixth grade. Another possible reason is that because of the teachers' focus on the low level cognitive processes which "engages students with mathematical ideas in a superficial rather than deep way" (Silver, 1998), it is not uncommon for students to have already forgotten the knowledge about fractions learned in the early sixth grade.

Though the literature reported a positive relationship between the instructional quantity (mainly engaged instructional time), and students' mathematics performance (Fisher, 1977; Grouws & Cebulla, 2000; McKnight, Crosswhite, Dossey, Kifer, Swaffor, Travers & Cooney, 1987; Mirel, 1994; Purvis & Levine, 1975; Suarez et al., 1991), this positive relationship was not be able to be identified because a two-level regression model failed to converge. The sample size may not have been large enough to investigate such a relationship. Thus three two-way ANOVA models were constructed to investigate whether students' achievement differed in terms of different representations they preferred and the teachers' representations that were used in class. Results revealed that students who received all five types of representations (real world, manipulatives, pictures, spoken symbolic and written symbolic representations) significantly differed from those who did not receive all five types of representations. Thus, this study provided empirical evidence that those students who received instruction using multiple representations differed greatly from those who did not, implying that there was a possibility that multiple representations could improve students' learning (Garrity, 1998; Haas, 1998; Leinenbach & Raymond, 1996; Post, 1981; Post, Behr, & Lesh, 1982). The study also revealed that students varied in choosing particular forms of representations in the open-ended questions on number posttests. The three questions (14, 15 and 16e) aimed at evaluating students' understanding of fraction ideas, with each question focusing on a specific perspective. For example, question 14 focused on the idea of part-whole whereas question 15 and 16e focused on quotient and multiplication by one. Thus students' achievements (i.e., their scores) varied, depending on the particular form of representation they chose. The study revealed that students who used the appropriate form of representation (the one intended by the question) achieved higher scores than those who did not.

Implications for Future Research

The current standards-based curriculum called for "teaching for understanding" rather than "teaching to the test". How can classroom practices address understanding? This study looked into both the quantity and quality of instructional representations, the students' written representations, and the relationship between instructional representations and students' achievements. First, the results provided qualitative analysis of the nature and quality of representations in teaching fractions, decimals and percents by looking into the classroom interactions, the variations across teachers and the alignment with textbooks. It was found that teachers who followed high quality textbooks tend to have constructivist classrooms. Those teachers who did not follow the high quality textbooks tended to be more teacher-centered in terms of classroom interactions. The results from this study bridged the gap between the theory and practices with rich empirical evidence. It provided quantitative data on how instructional representations converge with five sub-constructs of fractions mentioned in the literature rather than Lesh's (1979) five representational model, and how different forms of students' written representations contribute to students' achievements. This study also provided empirical evidence that the comprehensibility of the instructional representations significantly correlated to the students' posttest scores, which adds to the literature arguing that improving teachers' teaching quality rather than extending instructional time positively contributes to student's achievement (Aronson, Zimmerman, & Carlos, 1998; Smith, 2000). Teachers could also build effective teaching strategies by scaffolding sub-constructs of part-whole, quotient and multiplication by one because these three factors significantly predicted students' posttest scores.

Implications and recommendations for future research could be mainly categorized into the following three aspects; classroom practices, curriculum developers and teacher education and professional development programs. The model conducted in this dissertation could be used to investigate the teaching of other mathematics concepts or even in other disciplines, for example, science education or social studies. How do those teachers address different learning goals and how are these concepts developed via instructional representations? For classroom practice, given the reality of the quantity and quality of instructional representations, alignment with high quality textbook materials are important in empowering teachers in conducting constructivist classroom. Compared to real world, manipulatives, and picture representations, a large proportion of instructional representations were symbolic representations, which reveal the predominant role of symbolic representations in fractions teaching practices. Teachers' lack of addressing the connections between different representations should also be noted.

Given the power of representations of part-whole, quotient and wonderful one, an emphasis on these sub-constructs is critical for developing middle school student's conceptual understanding of fractions, decimals and percents. With the support of high quality teaching, mainly comprehensible instructional representations, students are able to have higher achievement scores. Curriculum designers and textbook developers could improve the curriculum and textbooks by providing more activities that emphasize different sub-constructs of fractions, especially the sub-constructs of part-whole, quotient and multiplication by one as well as some of the activities which address the connections between these different sub-constructs. For teacher education programs and professional developments, more emphasis on how to address the connections between the different representations and improve teaching comprehensibility should be addressed.

Recommendations for Future Research

Based on the results of this study, the following suggestions are recommended:

- The relationship between the instructional time and student's achievement was not able to be verified in this study. A possible reason could be the limitation of the small sample size. Future research should explore the effects of time spent on instructional representations on the students' understanding and achievements using a larger sample.
- Similar research should be done to investigate the external validity of the model. In particular, how teachers address different sub-constructs using instructional representations, how students' representations varied from the instructional representations, and how the quality (especially comprehensibility) of instructional representations influence students' achievement should be explored.

- The effectiveness of instructional representations may be different in different ages and with a diverse students' background preferring various learning styles, thus both the developmental stages and learning styles need to be investigated in future studies.

REFERENCES

Ainsworth, S. (1999). The functions of multiple representations. *Computers and Education, 33*, 131-152.

American Association for the Advancement of Science. (1993). *Benchmarks for science literacy*. New York: Oxford University Press.

American Association for the Advancement of Science. (2000). *Middle grades mathematics textbooks: A benchmarks-based evaluation.* Washington, DC: American Association for the Advancement of Science.

American Association for the Advancement of Science. (2002). *Number: Diagram to aid assessment task design*. Project 2061. Unpublished document.

An, S. (2000). *A comparative study of mathematics programs in the U.S. and China: The pedagogical content knowledge of middle school mathematics teachers in the U.S. and China.* Unpublished doctoral dissertation, Department of Teaching, Learning and Culture. Texas A&M University, College Station.

Anderson, J. C. & Gerbing, D.W. (1988). Structural equation modeling in practice: A review and recommended two-step approach. *Psychological Bulletin*, *103*, 411-423.

Anfara, V. A., Brown, K. M., & Mangione, T. L. (2002). Qualitative analysis on stage: Making the research process more public. *Educational Researcher*, *31*(7), 28-38.

Arbuckle, J. L. & Wothke, W. (1999). *Amos 4.0 users' guide.* Chicago, IL: Small Waters Corporation.

Aronson, J., Zimmerman, J., & Carlos, L. (1998). *Improving student achievement by extending school: Is it just a matter of time?* San Francisco: WestEd.

Ball, D. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education*. Unpublished doctoral dissertation. East Lansing, MI: Michigan State University.

Ball, D. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*(4), 373-397.

Baxter, G. P. & Glaser, R. (1998). Investigating the cognitive complexity of science assessments. *Educational Measurement: Issues and Practices, 17*(3), 37-45.

Bay, J. (2001). Developing number sense on the number line. *Mathematics Teaching in the Middle school, 6,* 448-452.

Behr, M., Harel, G., Post, T., & Lesh, R. (1994). Units of quantity: A conceptual basis common to additive and multiplicative structures. In G. Harel & J. Confrey (Eds.) *The development of multiplicative reasoning in the learning of mathematics* (pp. 123-180). Albany, NY: SUNY Press.

Behr, M., Lesh, R., Post, T., & Silver E. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes*, (pp. 91-125). New York: Academic Press.

Behr, M. & Post, T. (1992). Teaching rational number and decimal concepts. In T. Post (Ed.), *Teaching mathematics in grades K-8: Research based methods.* (2nd ed.) (pp. 201-248). Boston: Allyn and Bacon.

Behr, M., Post, T., & Lesh R. (1981, July). Construct Analyses, Manipulative Aids, Representational Systems and the Learning of Rational Numbers. In *Proceedings of the Fifth Conference of the International Group for the Psychology of Mathematics Education.* (pp. 203-209). Grenoble, France: PME.

Bell, A., Fischbein, E., & Greer, B. (1984). Choice of operation in verbal arithmetic problems: The effect of number size, problem structure, and context. *Educational studies in Mathematics*, *15*, 129-147.

Bentler, P. M. & Chou, C. (1987). Practical issues in structural modeling. *Sociological Methods and Research*, *16,* 78-187.

Bibby, P. A. & Payne, S. J. (1993). Internalization and the use specificity of device knowledge. *Human-Computer Interaction*, *8*(1), 25-56.

Bickerton, G. (2000). *Children's understanding of scientific concepts: A developmental study*. Dissertation. The University of British Columbia. UMI.

Billstein, R., Lowery, J., Montoya, P., Williams, D., & Williamson, J. (1999). *Middle grades MathThematics*. Evanston, IL: McDougal Littell.

Black, S. (2002). Time and learning: Education takes time – not just for coursework, but for reflection. *American School Board Journal*, *189* (9), 1-6.

Bloomfield, L. (1933). *Language*. New York: Holt, Rinehart and Winston.

Borko, H., Eisenhart, M., Brown, C. A., Underhill R. G., Jones, D., & Agard P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education,* *23*(3), 194-222.

Brase, G. L. (2002). “Bugs” built into the system: How privileged representations influence mathematical reasoning across the lifespan. *Learning and Individual Differences*, *12,* 391- 409.

Brown, C. A., Carpenter, T. P., Kouba, V. L., Lindquist, M. M., Silver, E. A., & Swafford, J. O. (1988). Secondary school results for the fourth NAEP mathematics assessment: Discrete mathematics, data organization and interpretation, measurement, number and operations, *Mathematics Teacher, 81*, 241-48.

Bruner, J. S. (1960). *The process of education*. New York: Vintage Books.

Bruner, J. S. (1966). *Toward a theory of instruction.* Cambridge, MA: The Belknap Press of Harvard University Press.

Carpenter, T. & Fennema, E. (1991). Research and cognitively guided instruction. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp.1–16). Albany: State University of New York Press.

Chall, J. S. (2000). *The academic achievement challenge: What really works in the classroom?* NY: New York, Guilford Press.

Chambers, D. & Reisberg, D. (1985). Can mental images be ambiguous? *Journal of Experimental Psychology: Human Perception & Performance*, *11*, 317-328.

Chandler, P. & Sweller, J. (1992). The split-attention effect as a factor in the design of instruction. *British Journal of Educational Psychology, 62*(2), 233-246.

Collins, W., Dritsas, L., Frey-Mason, P., Howard, A. C., McClain, K., Molina, D. D., Moore-Harris, B., Ott, J., Pelfrey, R. S., Price, J., Smith, B., & Wilson, P. S. (1999). *Mathematics applications and connections*. Columbus, OH: Glencoe/McGraw-Hill.

Condon, C. & Hilton, S. (1999). Decimal dilemmas. *Australia Primary Classroom, 4*(3), 26-31.

Cramer, K. (2003). Using a translation model for curriculum development and classroom instruction. In R. Lesh & H. Doerr (Eds.) *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*. Mahwah, NJ: Lawrence Erlbaum Associates.

Cramer, K., Behr, M., Post, T., & Lesh, R. (1997). *Rational number project: Fraction lessons for the middle grades-Level 2*. Dubuque, Iowa: Kendall/Hunt Publishing Co..

Cramer, K. & Henry, A. (2002). Using manipulative models to build number sense for addition of fractions. *National Council of Teachers of Mathematics 2002 yearbook: Making sense of fractions, ratios, and proportions* (pp. 41-48). Reston, VA: National Council of Teachers of Mathematics.

DeBoer, G., Morris, K., Roseman, J. E., Wilson, L., Capraro, M. M., Capraro, R., Kulm, G., Willson, V., & Manon, J. (2004, April). *Research issues in the improvement of mathematics teaching and learning through professional development.* Paper presented at American Educational Research Association annual meeting, San Diego, CA.

Diezmann, C. & English, L. (2001). Promoting the use of diagrams as tolls for thinking. In A. Cuoco (Ed.), *National Council of Teachers of Mathematics 2001 yearbook: The roles of representation in school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Donald, M. (1991). *Origins of the modern mind: Three stages in the evolution of culture and cognition.* Cambridge, MA: Harvard University Press.

Fennema, E. & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.) *Handbook of research on mathematics teaching and learning* (pp.147-164). New York: Macmillan Publishing Company.

Fischbein, E., Deri, M., Sainatinello, M., & Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, *16*, 3-16.

Fisher, C. (1977). *Instructional time and student achievement in second grade reading and mathematics.* (ERIC Document Reproduction Service No. ED137293).

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Boston: D. Reidel.

Friedlander, A. & Tabach, M. (2001). Promoting multiple representations in algebra. In A. Cuoco (Ed.), *National Council of Teachers of Mathematics 2001 yearbook: The roles of representation in school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Garrity, C. (1998). *Does the use of hands-on learning with manipulatives improve the test scores of secondary education geometry students?* Master's Program Action Research Project, St. Xavier University and IRI/Skylight.

Gay, A. S. & Aichele, D. B. (1997). Middle school students' understanding of number sense related to percent. *School Science and Mathematics, 97*(1), 27-36.

Gazzaniga, M. (2000). *The cognitive neurosciences.* Cambridge, MA: MIT Press.

Glass, V. G. & Hopkins, K. D. (1995). *Statistical methods in education and psychology* (3rd Ed.), Needham, MA: Allyn & Bacon.

Gobert, J. D. & Clement, J. J. (1999). Effects of student generated diagrams versus student generated summaries on conceptual understanding of casual and dynamic knowledge in plate tectonics. *Journal of Research in Science Teaching. 36*(1), 39-53.

Goldin, G. A. (2003). Representation in school mathematics: A unifying research perspective. *A research companion to principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Goldin, G. & Passantino, C. (1996). A longitudinal study of children's fraction representations and problem solving behavior. *Proceedings of the 20th PME conferences, Valencia, Spain, 3*, 3-10.

Goldin, G. & Steingold, N. (2001). System of mathematical representation and development of mathematical concepts. In F. R. Curcio (Ed.), *National Council of Teachers of Mathematics 2001 yearbook: The roles of representation in school mathematics*, Reston, VA: National Council of teachers of Mathematics.

Goody, J. (1977). *The domestication of the savage mind*. Cambridge (UK): Cambridge University Press.

Graeber, A. O. (1993). Misconception about multiplication and division. *Arithmetic Teacher, 40* (7), 408-411.

Grouws, D. A. & Cebulla, K. J. (2000). *Improving student achievement in mathematics* (Vol. 4). Genf, Switzerland: International Academy of Education/International Bureau of Education.

Haas, S. M. (1998). *A comparison of fraction concept development in selected junior college students and young learners*. Ph.D dissertation, Texas A&M University.

Hall, V., Bailey, J., & Tillman, C. (1997). Can student-generated illustrations be worth ten thousand words? *Journal of Educational Psychology, 89*(4), 677-681.

Harel, G. & Behr, M. (1988). Structures and hierarchy of missing value proportion problems and their representations. *The Journal of Mathematics Behavior, 8*, 77-119.

Harel, G. & Behr, M. (1990). The construct theory of rational numbers: Toward a semantic analysis. In G. Booker, P. Cobb, & T. N. de Merldicutti (Eds.) *Proceedings of the PME XIV Conference* (pp. 3-10). Mexico: Consejo Nacional de Cienciay Technologia, Gobierno del Estado de Morelos.

Hiebert, D.C. Hiebert, J. (1983). Junior high school students' understanding of fractions. *School Science and Mathematics, 83*(2), 97 – 106.

Hiebert, J. (1984). Children's mathematics learning: The struggle to link form and understanding. *The Elementary School Journal, 84*(5), 497-510.

Hiebert, J. (1985). Children's knowledge of common and decimal fractions. *Education and Urban Society, 17*(4), 427-437.

Hiebert, J. & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *A handbook of research on mathematics teaching and learning* (pp. 65-100). New York: Macmillan Library Reference Simon & Schuster Macmillan.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Wearne, D., Murray, H., Oliver, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding.* Portsmouth, NH: Heinemann.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum.

Hiebert, J. & Wearne, D. (1986). Procedures over concepts: The acquisition of decimal number knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 199-223). Hillsdale, NJ: Erlbaum.

Higginbotham-Wheat, N. L. (1991). *Use of generative learning strategies in computer-based instruction: Effects of learner-generated visual and verbal elaborations.* Unpublished doctoral dissertation, Memphis, TN: Memphis State University.

Hinzman, K. P. (1997). *Use of manipulatives in mathematics at the middle school level and their effects on students' grades and attitudes.* Unpublished master's thesis, Salem-Teikyo University, Salem, West Virginia.

Hu, L. & Bentler, P. M. (1999). Cutoff criteria for fit indices in covariance structure analysis: Conventional criteria versus new alternatives. *Structural Equation Modeling, 6*(1), 1-55.

Hudson, S., McMahon, K., & Overstreet, C. (2002). *The 2000 national survey of science and mathematics educators: Compendium of tables.* Chapel Hill NC: Horizon Research, Inc.

Hynes, M. (1986). Selection criteria. *Arithmetic Teacher, 33* (6), 11-13.

Jonassen, D. H., Cole, P., & Bamford, C. (1992). *Leaner-generated vs. instructor-provided analysis of semantic relationships.* Paper presented at the Convention of the Association for Educational Communications and Technology, 15th, New Orleans, LA. (ERIC Document Reproduction Service No. ED 362 170).

Kalyuga, S., Chandler, P., & Sweller, J. (1998). Levels of expertise and instructional design. *Human Factors, 40*(1), 1-17.

Kaput, J. (1987). Representation systems and mathematics. In C. Janvier (Ed.), *Problems of representation in the teaching and learning mathematics* (pp. 19-26). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kaput, J. (1989). Linking representations in the symbol systems of algebra. In S. Wagner (Ed.), *Research agenda for the teaching and learning mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Kaput, J. (1992). Technology and mathematics education. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp.515-556). New York: Macmillan.

Kaput, J. (1999). Representations, inscriptions, descriptions and learning: A kaleidoscope of windows. *Journal of Mathematical Behavior, 17*(2), 256-281.

Kaput, J. (2001). *Learning algebra using dynamic simulations and visually editable graphs of rate and totals quantities*. Paper presented at the QCA International Seminar on "Reasoning, Explanation and Proof in School Mathematics and Their Place in the Intended Curriculum". London, England.

Kerslake, D. (1986). *Fractions: Children's strategies and errors.* Windsor, England: NFER-Nelson.

Kieren, T. E. (1976) On the mathematical, cognitive and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and measurement: Papers from a research workshop* (pp. 101-144). Columbus, Ohio: ERIC/SMEAC.

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. J. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kieren, T. E. (1992). Rational and fractional numbers as mathematical and personal knowledge: Implications for curriculum and instruction. In R. Leinhardt, R. Putnam, & R. A. Hartrup (Eds.). *Analysis of arithmetic for mathematics teaching* (pp. 323-371). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kieren, T. E. (1995). Creating spaces for learning fractions. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 31-65). Albany: State University of New York Press.

Kleinmuntz, D.N. & Schkade, D.A. (1993). Information displays and decision processes. *Psychological Science, 4*(4), 221-227.

Kline, R. B. (2005). *Principles and practices of structural equation modeling* (2nd ed.). New York: The Guilford Press.

Kulm, G. & Capraro, R. (2004). *Relationship between textbook use and student learning of number and algebra ideas in middle grades.* Paper presented at the research pre-session of the National Council of Teachers of Mathematics annual meeting, Philadelphia, PA, April, 2004.

Lacampagne, C., Post, T., Harel, G., & Behr, M. (1988). A model for the development of leadership and the assessment of mathematical and pedagogical knowledge of middle school teachers. In M. Behr, C. Lacampagne, & M. Wheeler (Eds.), *Proceedings of the ninth annual conference of PME-NA* (pp. 418-425). DeKalb, IL: PME.

Lamon, S. J. (2001). Presenting and representing: From fractions to rational number. In A. Cuoco (Ed.), *National Council of Teachers of Mathematics 2001 yearbook: The roles of representation in school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Lampert, M. (2003). Communication and language. *A research companion to principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillip, E. D. (1998). *Connected mathematics. Bits and Pieces I: Understanding rational number.* (TE) Menlo Park, CA: Dale Seymour Publications.

Leinenbach, M. & Raymond, A. (1996). *A two-year collaborative action research study on the effects of a "hands-on" approach to learning algebra.* Paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Panama City, FL.

Lenze, J. S. & Dwyer, F. (1993). The instructional effect of learner vs. instructor generated visuals. *International Journal of Instructional Media, 20*(4), 373-381.

Lesh, R. (1979). Mathematical learning disabilities: Considerations for identification, diagnosis, remediation. In R. Lesh (Ed.), *Applied mathematical problem solving* (pp. 111-180). Columbus, Ohio: ERIC/SMEAC.

Lesh, R., Behr, M., & Post, T. (1987). Rational number relations and proportions. In C. Janiver (Ed.), *Problems of representations in the teaching and learning of mathematics* (pp. 41-58). Hillsdale, NJ: Lawrence Erlbaum.

Lesh, R., Hamilton, E., & Landau, M. (1981). Applied mathematical problem solving processes needed by middle school students in the solution of everyday problems. In T. Post & M. P. Roberts (Eds.), *Proceedings of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp.118-121). Minneapolis: University of Minnesota.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representations in the teaching and learning of mathematics* (pp. 33-40). NJ: Lawrence Erlbaum.

Lincoln, Y. S. & Guba, E. G. (1985). *Naturalistic inquiry.* Newbury Park, NJ: Sage.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21,* 16-32.

Markovits, Z. & Sowder, J. (1991). Students understanding of the relationship between fractions and decimals. *Focus on Learning Problems in Mathematics 13* (January 1991): 3-11.

McClung, L. W. (1998). *A study on the manipulatives and their effects on student achievement in high school algebra class.* Unpublished master's thesis, Salem-Teikyo University, Salem, West Virginia.

McKnight, C. C., Crosswhite, F. J., Dossey, J. A., Kifer, E., Swafford, J. O., Travers, K. J., & Cooney, T. J. (1987). *The underachieving curriculum: Assessing U.S. school mathematics from an international perspective.* Champain, IL: Stipes.

Millsaps, G. & Reed, M. (1998). *Curricula for teaching about fractions.* ERIC Digest. ED433184.

Mirel, J. (1994). High standards for all: The struggle for equality in the American high school curriculum, 1890-1990. *American Educator, 18*(2), 4-9, 40-42.

Miura, I. T. (2001). The influence of language on mathematical representations. In F. R. Curcio (Ed.), *National Council of Teachers of Mathematics 2001 yearbook: The roles of representation in school mathematics.* Reston, VA: National Council of teachers of Mathematics.

Monk, S. (2003). Representations in school mathematics: Learning to graph and graph to learn. *A research companion to principles and standards for school mathematics*. Reston: National Council of Teachers of Mathematics.

Moss, J. & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and experimental curriculum. *Journal for Research in Mathematics Education, 30*(2), 122-147, 119.

Mullis, I., Martin, M., Gonzalez, E., Gregory, K., Garden, R., O'Connor, K, Chrostowski, S., & Smith, T. (2000). *TIMSS international report: Findings from IEA's repeat of the third international mathematics and science study of eighth grade.* Retrieved February 6, 2003 from the World Wide Web http://timss.bc.edu/timss1999i/math_achievement_report.html.

Muthen, B. & Muthen, B. (2004). *Mplus user's guide* (3rd ed.). Los Angeles, CA: Muthén & Muthén.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Newell, A. (1990). *Unified theories of cognition.* Cambridge, MA: Harvard University Press.

Ni, Y. (2000). How valid is it to use number lines to measure children's conceptual knowledge about rational number? *Educational Psychology, 20*(2), 139-152.

Norman, D. A. (1993). *Things that make us smart.* Reading, MA: Addison-Wesley.

Nunes, T. & Bryant, P. (1996). *Children doing mathematics.* Cambridge, MA: Blackwell.

Ohlsson, S. (1988). Mathematics meaning and applicational meaning in the semantics of fractions and related concepts. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (*Vol. 2*, pp. 53-92). Hillsdale, NJ: Erlbaum and Reston, VA: National Council of Teachers of Mathematics.

Ong, W. J. (1982). *Orality and literacy: The technologizing of the word.* London: Methuen.

Orton, R. E., Post, T. R., Behr, M. J., Cramer, K., Harel, G., & Lesh, R. (1995). Logical and psychological aspects of rational number pedagogical reasoning. *Hiroshima Journal of Mathematics Education, 3,* 63-67.

Owens, D. T. & Menon, R. (1991). *Understanding children's development of rational number concepts: Final research report.* British Columbia University, Vancouver. Dept. of Mathematics and Science Education. (ERIC Document Reproduction Service No. ED379153).

Pitkethly, A. & Hunting, R. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics, 30,* 5-38, 1996.

Post, T. (1981). The role of manipulative materials in the learning of mathematical concepts. In M. Lindquist (Ed.), *Selected issues in mathematics education* (pp. 109-131). Berkeley, CA: McCutchan.

Post, T. (1988). Some notes on the nature of mathematics learning. In T. Post (Ed.), *Teaching mathematics in grades k-8: Research based methods* (pp. 1-19). Boston: Allyn & Bacon.

Post, T., Behr, M., & Lesh, R. (1982). Interpretations of rational number concepts. In L. Silvey & J. Smart (Eds.), *National Council of Teachers of Mathematics 1982 yearbook: Mathematics for grades 5-9* (pp. 59-72). Reston, Virginia: NCTM.

Post, T. & Cramer, K. (1987). Children's strategies when ordering rational numbers. *Arithmetic Teacher*, *35*(2), 33-35.

Post, T., Cramer, K., Behr, M., Lesh, R., & Harel, G. (1993). Curriculum implications of research on the learning, teaching, and assessing of rational number concepts. In T. Carpenter & E. Fennema (Eds.), *Research on the learning, teaching, and assessing of rational number concepts*. Hillsdale, NJ: Lawrence Erlbaum and Associates.

Post, T., Harel, G., Behr, M., & Lesh, R. (1988). Intermediate teachers knowledge of rational number concepts. In Fennema, et, al. (Eds.), *Papers from First Wisconsin Symposium for Research on Teaching and Learning Mathematics* (pp. 194-219). Madison, WI: Wisconsin Center for Education Research.

Post, T., Wachsmuth, I., Lesh, R., & Behr, M. (1985). Order and equivalence of rational number: A cognitive Analysis. *Journal for Research in Mathematics Education*, *16*(1), 18-36.

Purvis, A. & Levine, D. (1975). *Educational policy and international assessment: Implications of the IEA survey of achievement.* Berkeley, CA: McCutchan.

Resnick, L. B. & Omanson, S. F. (1987). Learning to understand arithmetic. In R. Glaser (Ed.), *Advances in instructional psychology*, *3* (41-95). Hillsdale, New Jersey: Lawrence Erlbaum.

Reys, R., Reys, B., Lapan, R., Holliday, G., & Wasman, D. (2003). Assessing the impact of standard-based middle grades mathematics curriculum materials on student achievement. *Journal for Research in Mathematics Education, 34*(1), 74-95.

Roseman, J. E., Kulm, G., & Manon, J. (2001). Improving Mathematics Teacher Practice and Student Learning through Professional Development. National Science Foundation IERI (Interagency Educational Research Initiative) grant #REC-0129398. Retrieved June, 2005 from the World Wide Web http://msmp.tamu.edu/PDF/Grant.pdf

Saenz-Ludlow, A. (1993). *Ann's fraction scheme*. Paper presented at the International Study Group on the Rational Numbers of Arithmetic, University of Georgia, Athens, GA.

Saenz-Ludlow, A. (1994). Michael's fraction schemes. *Journal for Research in Mathematics Education, 25*(1), 50-85.

Schoenfeld, A. (1986). On having and using geometric knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 225-265). Hillsdale, NJ: LEA

Schwartz, D. L. (1993). The construction and analogical transfer of symbolic visualization. *Journal of Research in Science Teaching, 30*(10), 1309-1325.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22,* 1-36.

Sharp, J. M. (1995). *Results of using algebra tiles as meaningful representations of algebra concepts.* Paper presented at the Annual Meeting of the Mid-Western Education Research Association, Chicago, IL.

Siegal, M. & Smith, J.A. (1997). Toward making representation count in children's conceptions of fractions. *Contemporary Educational Psychology, 22*(1), 1-22.

Silver, E. A. (1998). *Improving mathematics in middle school: Lesson from TIMSS and related research.* Washington, DC: U.S. Department of Education.

Simon, A. M. (1997). Developing new models of mathematics teaching: An imperative for research on mathematics teacher development. In E. Fennema & B. Scott-Nelson (Eds.), *Mathematics teachers in transition* (pp. 55-86). Mahwah, NJ: Erlbaum.

Smith, B. (2000). Quantity matters: Annual instruction time in an urban school system. *Educational Administration Quarterly, 36*(5), 652-82.

Smith, J. P. (1996). Efficacy and teaching mathematics by telling: A challenge for reform. *Journal for Research in Mathematics Education, 27*, 387-402.

Spielman, L. J. & Lloyd, G. M. (2004). The impact of enacted mathematics curriculum models on prospective elementary teachers' course perceptions and beliefs. *School Science and Mathematics, 104*(1), 32-44.

Steffe, L. P., Cobb, P., & Von Glasersfeld, E. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Stenning, K. & Oberlander, J. (1995). A cognitive theory of graphical and linguistic reasoning: Logic and implementation, *Cognitive Science, 19*: 97-140.

Stigler, J. W. & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* USA: The Free Press.

Stix, A. (1997). *Teaching fractions and decimals: Fun with picture grids*. (ERIC Document Reproduction Service No. ED 408 158)

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research.* Dordrecht, Netherlands:Kluwer.

Suarez, T. M., Torlone, T. J., McGrath, S. T., & Clark, D. L. (1991). Enhancing effective instructional time: A review of research. *Policy Brief, vol. 1,* no. 2. Chapel Hill, NC, North Carolina Educational Policy Research Center.

Sun, Y. & Kulm, G. (2003). Developing understanding of fraction concepts: Lessons from two teachers. *Texas Mathematics Teacher, L*(2), 6-9.

Taber, S. B. (2001). *Students making connections among different representations: The case of multiplication of fractions.* Paper presented at the annual meeting of the American Educational Research Association, Seattle, WA.

Thompson, C. & Walker, V. (1996). Connecting decimals and other mathematical content. *Teaching Children Mathematics, 2*(8), 496-502.

TIMSS. (1999). *The trends in international mathematics and science study*. Retrieved December, 2004 from the World Wide Web http://timss.bc.edu/timss1999.html.

Titus, J. (1995). The concept of fractional number among deaf and hard of hearing students. *American Annals of the Deaf, 140*(3), 255-263.

Trafton, P. R., Reys, B. J., & Wasman, D. G. (2001). Standards-based mathematics curriculum materials: A phrase in search of a definition. *Phi Delta Kappan*, *83*(3), 259-263.

Tyson, H. & Woodward, A. (1989). Why students aren't learning very much from textbooks. *Educational Leadership, 47,* 14-17.

Vance, J. H. (1992). Understanding equivalence: A number by any other name. *School Science and Mathematics, 92*, No. 5, 263-66.

Van de Walle, J. A. (2001). *Elementary and middle school mathematics: Teaching developmentally* (4th ed.). New York: Addison Wesley Longman, Inc.

Voutsina, C. & Jones, K. (2001). The micro development of young children's problem solving strategies when tackling addition tasks. *Psychology of Mathematics education*, *25*(4), 391-398.

Walker, D. A. (1976). *The IEA six subject survey: An empirical study of education* in *twenty-one countries*. New York: Wiley.

Watanabe, T., Reynolds, A., & Lo, J. J. (1995). A fifth grader's understanding of fractions and proportions. *PME 19 Proceedings of the Eighteenth International Conference*, Recife, Brazil, 3, 200-207.

Wearne, D. & Hiebert, J. (1986). *Learning decimal numbers: A study of knowledge acquisition.* Final report. University of Delaware, Newark. (ERIC Document Reproduction Service No. ED267973).

Wilson, M. (1990). Investigation of structured-problem solving items. In G. Kulm.(Ed.), *Assessing higher order thinking mathematics* (pp. 137-203). Washington, DC: American Association for the Advancement of Science.

Wood, T. (1999). Creating a context for argument in mathematics class. *Journal for Research in Mathematics Education, 30*(2), 171-191.

Woodward, A. & Elliot, D. L. (1990). Textbook use and teacher professionalism. In D. L. Elliot & A. Woodward (Eds.), *National Society for the Study of Education 89th yearbook: Textbooks and schooling in the United States* (Part 1, pp. 178-193). Chicago; National Society for the Study of Education.

Wright, S. P., Horn, S. P., & Sanders, W. L. (1997). Teacher and classroom context effects on student achievement: Implications for teacher evaluation. *Journal of Personnel Evaluation in Education, 11*(1), 57-67.

Zelazo, P. D. & Lourenco, S. F. (2003). Imitation and the dialectic of representation. *Developmental Review 23,* 55-78.

Zhang, J. J. (1997). The nature of external representations in problem solving. *Cognitive Science, 21*(2), 179-217.

VITA

Ye Sun received her Bachelor of Education and her Master of Education from Beijing Normal University at Beijing, China in 1998 and 2001, respectively. She received her Ph.D. degree from the Department of Teaching, Learning and Culture at Texas A&M University in 2005. Her research interests include investigating the structures of instructional representations and written representations to improve mathematics teaching and learning in middle schools.

Ms. Sun may be reached at Prete Building, 3040 University Avenue, Morgantown, WV 26505. Her email address is yesuntamu@yahoo.com.